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Wave suppression by nonlinear finite-dimensional control

Antonios Armaou, Panagiotis D. Christofides*

Department of Chemical Engineering, School of Engineering and Applied Sciences, University of California, 405 Hilgard Avenue, P.O. Box 951592, Los Angeles, CA 90095-1592, USA

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Abstract

Korteweg–de Vries–Burgers (KdVB) and Kuramoto–Sivashinsky (KS) equations are two nonlinear partial differential equations (PDEs) which can adequately describe motion of waves in a variety of fluid flow processes. We synthesize nonlinear low-dimensional output feedback controllers for the KdVB and KS equations that enhance convergence rate and achieve stabilization to spatially uniform steady states, respectively. The approach used for controller synthesis employs nonlinear Galerkin's method to derive low-dimensional approximations of the PDEs, which are subsequently used for controller synthesis via geometric control methods. The controllers use measurements obtained by point sensors and are implemented through point control actuators. The performance of the proposed controllers is successfully tested through simulations. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Nonlinear model reduction; Nonlinear control; Korteweg–de Vries–Burgers equation; Kuramoto–Sivashinsky equation

1. Introduction

The development of general and practical control algorithms for nonlinear partial differential equations (PDEs) that describe fluid flow processes (e.g., Navier–Stokes equations, KdVB, KS) is a fundamental problem whose practical significance ranges from feedback control of turbulence for drag reduction, to suppression of fluid mechanical instabilities in coating processes and suppression of waves exhibited by falling liquid films. For example, drag reduction through active feedback control may have a very significant impact on the design and operation of underwater vehicles, airplanes and automobiles, since according to some estimates keeping the flow over the surface of a vehicle laminar could yield up to 30% reduction in fuel consumption. Active control of fluid flow can be achieved by injection of polymers (Lumley, 1973), mass transport through porous walls (e.g., blowing/suction) (Carlson & Lumley, 1996; D'Andrea, Behnken & Murray, 1999) and application of electro-magnetic forcing (Singh & Bandyopadhyay, 1997).

The development of an efficient control system for a fluid flow should be based on the underlying PDE model that describes the flow in order to exploit its ability to accurately predict the spatiotemporal behavior of the flow field. The main obstacle for the synthesis of practically implementable output feedback controllers for fluid flow processes is the infinite-dimensional nature of the underlying PDE models, which does not allow their direct use for the design of low-order (and therefore, easy-to-implement) controllers. This fact, together with the need to develop efficient numerical methods for nonlinear PDEs, has motivated significant research on the understanding of the dynamic behavior of various fluid flows. This research has led to the discovery that turbulent flows involve coherent structures (e.g., Bakewell & Lumley, 1967; Lumley, 1981; Cantwell, 1981; Sirovich, 1987) which implies that the dominant dynamic behavior of such flows can in principle be approximated by low-dimensional systems. These results have motivated the development of model reduction methods for the derivation of low-dimensional systems that accurately describe the dominant dynamics of PDEs which model fluid flows including nonlinear Galerkin's method (e.g., Titi, 1990; Deane, Kevrekidis, Karniadakis & Orszag, 1991; Bangia, Batcho, Kevrekidis & Karniadakis, 1997) and reduced basis methods (e.g., Ito & Ravindran, 1997).

In the area of control of fluid flow processes, previous research has mainly focused on: (a) control of

* Corresponding author. Tel.: + 1-310-794-1015; fax: + 1-310-206-4107.

E-mail address: pdc@seas.ucla.edu (P.D. Christofides).

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Navier–Stokes equations via proportional-integral (Beringen, 1984; Choi, Témam, Moin & Kim, 1993), linear optimal (Desai & Ito, 1994; Joshi, Speyer & Kim, 1995; Hou & Yan, 1997), H_∞ optimal (Burns & Ou, 1994; Burns & King, 1994; King & Qu, 1995), Lyapunov-based (Kang & Ito, 1992) and nonlinear (Christofides & Armaou, 1998) control, (b) linear distributed (Russell & Zhang, 1995) and point (Russell & Zhang, 1996) control of the Korteweg–de Vries equation and nonlinear boundary control of the KdVB equation (Liu & Krstić, 1999a) and (c) linear distributed control (Christofides, 1998) and nonlinear boundary control (Liu & Krstić, 1999b) of the KS equation. At this stage, not much work has been done on the utilization of the aforementioned nonlinear model reduction techniques for the design of nonlinear low-order feedback controllers for PDEs that describe fluid flows.

In this work, we focus on nonlinear finite-dimensional control of the KdVB and KS equations which can adequately describe motion of waves in a variety of fluid flow processes (see, for example, Russell and Zhang (1996) and Chang (1986) for appropriate discussions). We initially propose a nonlinear controller design method for quasi-linear evolutionary PDEs. The method employs nonlinear Galerkin's method to derive low-dimensional ODE models, which are subsequently used for the synthesis of low-order nonlinear output feedback controllers that enforce local exponential stability in the closed-loop system. We then use this method to synthesize nonlinear low-order controllers for the KdVB and KS equations that enhance convergence rate and achieve stabilization to spatially uniform steady states, respectively. The controllers use measurements obtained by point sensors and are implemented via point control actuators. The performance of the proposed controllers is successfully tested through simulations.

2. Nonlinear controller design

2.1. Preliminaries

We consider quasi-linear infinite-dimensional systems with the following state-space description:

$$\dot{x} = A\,x + B\,u + R(x), \quad x(0) = x_0,$$
$$y_m = S\,x, \quad y_c = Cx, \tag{1}$$

where $x \in H$ is the state of the system (H is an appropriate Hilbert space), $u \in R^l$ is the vector of manipulated inputs, $y_m \in R^q$ is the vector of measured outputs, $y_c \in R^l$ is the vector of controlled outputs and $x_0 \in H$ is the initial condition. $A : D(A) \subset H \to H$ is a densely defined unbounded operator with compact inverse, $R : D(R) \subset H \to H$ is a nonlinear map which satisfies $R(0) = 0$, and $B : R^l \to H$, $S : H \to R^q$, $C : H \to R^l$ are operators describing the spatial distribution of the manipulated inputs, measured outputs and controlled outputs, respectively.

The eigenvalue problem for A is defined as

$$A\,\phi_j = \lambda_j \phi_j, \quad j = 1, \ldots, \infty, \tag{2}$$

where λ_j and ϕ_j denote the jth eigenvalue and eigenfunction, respectively. The eigenvalues of A satisfy $R(\lambda_1) \geqslant R(\lambda_2) \geqslant, \ldots$ (where $R(\,)$ denotes the real part of a complex number), while the eigenfunctions $\{\phi_1, \phi_2, \ldots\}$ form a complete orthonormal set in H.

The inner product and norm in the Hilbert space H are defined as

$$(u_1, u_2) = \int_X (u_1(z), u_2(z))_{R^n}\, dz, \quad \|u_1\| = (u_1, u_1)^{1/2} \tag{3}$$

where u_1, u_2 are the two elements of H, X is the domain of definition of the process, and the notation $(\,),\,)_{R^n}$ denotes the standard inner product in R^n.

Finally, we recall the Lie derivative notation which will be used in our development: $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f, $L_f^k h$ denotes the kth order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.

2.2. Nonlinear model reduction

We apply nonlinear Galerkin's method (Titi, 1990; Deane et al., 1991; Bangia et al., 1997) to the system of Eq. (1) to derive an approximate finite-dimensional system. Let H_s, H_f be modal subspaces, defined as $H_s = \mathrm{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $H_f = \mathrm{span}\{\phi_{m+1}, \phi_{m+2}, \ldots\}$ where ϕ_j, $j = 1, \ldots, \infty$ are eigenfunctions of A. Clearly, $H_s \oplus H_f = H$. Defining the orthogonal projection operators $P_s : H \to H_s$ and $P_f : H \to H_f$ such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq. (1) can be decomposed to

$$x = x_s + x_f = P_s x + P_f x. \tag{4}$$

Applying P_s and P_f to the system of Eq. (1) and using the above decomposition for x, the system of Eq. (1) can be written in the following form:

$$\frac{dx_s}{dt} = A_s x_s + B_s u + R_s(x_s, x_f), \quad x_s \in H_s,$$

$$\frac{\partial x_f}{\partial t} = A_f x_f + B_f u + R_f(x_s, x_f), \quad x_f \in H_f,$$

$$y_m = S\,x_s + S\,x_f, \; y_c = Cx_s + Cx_f,$$
$$x_s(0) = P_s x(0) = P_s x_0, \; x_f(0) = P_f x(0) = P_f x_0, \tag{5}$$

where $A_s = P_s A$, $B_s = P_s B$, $R_s = P_s R$, $A_f = P_f A$, $B_f = P_f B$ and $R_f = P_f R$ and the notation $\partial x_f / \partial t$ is

used to denote that the state x_f belongs in an infinite-dimensional space. In the above system, A_s is a diagonal matrix of dimension $m] m$ of the form $A_s = \mathrm{diag}\{\lambda_1, \ldots, \lambda_m\}$, $R_s(x_s, x_f)$ and $R_f(x_s, x_f)$ are Lipschitz vector functions, and A_f is a stable unbounded differential operator. The standard Galerkin's method is to approximate the solution $x(t)$ of the system of Eq. (1) by $\hat{x}_s \in H_s$ which is given by the following m-dimensional system:

$$\frac{d\hat{x}_s}{dt} = A_s \hat{x}_s + B_s u + R_s(\hat{x}_s, 0),$$

$$y_m = S_s \hat{x}_s, \quad y_c = C_s \hat{x}_s. \tag{6}$$

A finite-dimensional system of order m which yields solutions which are closer to the one of the system of Eq. (1) than the ones obtained by the system of Eq. (6) can be obtained by using the concept of inertial manifold (IM). If it exists, an inertial manifold M for the system of Eq. (1) is a subset of H, which satisfies the following properties (Témam, 1988): (i) M is a finite-dimensional Lipschitz manifold, (ii) M is a graph of a Lipschitz function $R(x_s, u)$ mapping $H_s] \mathbb{R}$ into H_f and for every solution $x_s(t), x_f(t)$ of Eq. (5) with $x_f(0) = R(x_s(0), u)$, then

$$x_f(t) = R(x_s(t), u) \quad \forall\, t \geqslant 0 \tag{7}$$

and (iii) M attracts every trajectory exponentially. The evolution of the state x_f on M is given by Eq. (7), while the evolution of the state x_s is governed by the following finite-dimensional inertial form:

$$\frac{dx_s}{dt} = A_s x_s + B_s u + R_s(x_s, R(x_s, u)),$$

$$y_m = S_s x_s + S_s R(x_s, u),$$

$$y_c = C_s x_s + C_s R(x_s, u). \tag{8}$$

Assuming that $u(t)$ is smooth, differentiating Eq. (7) and utilizing Eq. (5), $R(x_s, u)$ can be computed as the solution of the following partial differential equation:

$$\frac{\partial R}{\partial x_s}[A_s x_s + B_s u + R_s(x_s, R(x_s, u))] + \frac{\partial R}{\partial u}\dot{u}$$
$$= A_f R(x_s, u) + B_f u + R_f(x_s, R(x_s, u)) \tag{9}$$

which $R(x_s, u)$ has to satisfy for all $x_s \in H_s$, $u \in \mathbb{R}$. Unfortunately, even for PDEs for which an IM is known to exist, the derivation of an explicit analytic form of $R(x_s, u)$ is an almost impossible task. The nonlinear Galerkin method attempts to overcome the problems associated with the construction of inertial manifolds by replacing $R(x_s, u)$ with an approximate relation $R_{\mathrm{AIM}}(x_s, u)$ (called approximate inertial manifold (AIM)). In this case, the solution x of the system of Eq. (1) is approximated by $x_s + R_{\mathrm{AIM}}(x_s, u)$ which is given by the following m-dimensional system:

$$\frac{dx_s}{dt} = A_s x_s + B_s u + R_s(x_s, R_{\mathrm{AIM}}(x_s, u)),$$

$$y_m = S_s x_s + S_s R_{\mathrm{AIM}}(x_s, u),$$

$$y_c = C_s x_s + C_s R_{\mathrm{AIM}}(x_s, u). \tag{10}$$

A function $R_{\mathrm{AIM}}(x_s, u)$ which is frequently used in the literature (see, for example, Jones & Titi, 1994) is

$$R_{\mathrm{AIM}}(x_s, u) = (A_f)^{-1}[-B_f u - R_f(x_s, 0)]. \tag{11}$$

The reader may also refer to Christofides and Daoutidis (1997), Foias and Témam (1988) and Shvartsman and Kevrekidis, 1998) for alternative approaches for the construction of AIMs. The AIM of Eq. (11) leads to the following m-dimensional system:

$$\frac{dx_s}{dt} = A_s x_s + B_s u + R_s(x_s, (A_f)^{-1}$$
$$] \ [-B_f u - R_f(x_s, 0)]),$$

$$y_m = S_s x_s + S_s (A_f)^{-1}[-B_f u - R_f(x_s, 0)],$$

$$y_c = C_s x_s + C_s(A_f)^{-1}[-B_f u - R_f(x_s, 0)]. \tag{12}$$

Remark 1. We note that the above model reduction procedure which led to the approximate ODE system of Eq. (12) can be also used, when empirical eigenfunctions of the system of Eq. (1) computed through Karhunen–Loève expansion (known also as proper orthogonal decomposition) are used as basis functions in H_s and H_f, instead of the eigenfunctions of A.

2.3. Nonlinear control

In this section, we use the system of Eq. (12) to synthesize a nonlinear finite-dimensional output feedback controller that guarantees stability and enforces output tracking in the closed-loop ODE system, and establish that the same controller exponentially stabilizes the closed-loop infinite-dimensional system. The finite-dimensional output feedback controller which achieves the desired objectives for the system of Eq. (12) is constructed through a standard combination of a state feedback controller with a state observer.

Since the manipulated input in the system of Eq. (12) appears in a nonlinear fashion, we initially apply the following preliminary dynamic feedback law:

$$\frac{dm}{dt} = \dot{u},$$

$$u = m, \tag{13}$$

where $\tilde{u}$ is an auxiliary input, to the system of Eq. (2) to obtain

$$\frac{d\tilde{m}}{dt} = \tilde{u},$$

$$\frac{d\tilde{x}_s}{dt} = A_s\tilde{x}_s + B_s\tilde{m} + R_s(\tilde{x}_s, (A_f)^{-1}[-B_f\tilde{m} - R_f(\tilde{x}_s, 0)]),$$

$$y_m = S_s\tilde{x}_s + S_f(A_f)^{-1}[-B_f\tilde{m} - R_f(\tilde{x}_s, 0)],$$

$$y_c = C_s\tilde{x}_s + C_f(A_f)^{-1}[-B_f\tilde{m} - R_f(\tilde{x}_s, 0)] \tag{14}$$

which can be written in the following compact form:

$$\frac{d\hat{x}_s}{dt} = f(\hat{x}_s) + g(\hat{x}_s)\tilde{u},$$

$$y_m = h_m(\hat{x}_s), \quad y_c = h_c(\hat{x}_s), \tag{15}$$

where $\hat{x}_s = [\tilde{m}^T, \tilde{x}_s^T]^T$ and the specific form of f, g, h_m, h_c can be readily obtained by comparing Eqs. (14) and (15). On the basis of the system of Eq. (15), one can utilize geometric control methods (Isidori, 1989) to synthesize a nonlinear state feedback control law of the general form

$$\tilde{u} = p(\hat{x}_s) + Q(\hat{x}_s)v, \tag{16}$$

where $p(\hat{x}_s)$ is a smooth vector function, $Q(\hat{x}_s)$ is a smooth matrix, and $v \in \mathbb{R}^l$ is the constant reference input vector. Under the hypothesis that the system of Eq. (15) is locally observable (i.e., its linearization around the zero solution is observable), the practical implementation of the state feedback law of Eq. (16) will be achieved by employing the following nonlinear state observer:

$$\frac{d\hat{g}}{dt} = f(\hat{g}) + g(\hat{g})\tilde{u} + L(y_m - h_m(\hat{g})), \tag{17}$$

where $\hat{g}$ denotes the observer state vector (the dimension of the vector $\hat{g}$ is equal to the dimension of $\hat{x}_s$ in the system of Eq. (15)), and L is a matrix chosen so that the eigenvalues of the matrix $C_L = (L_f/L_g)_{(\hat{g}=0)} - L(L_{h_m}/L_g)_{(\hat{g}=0)}$ lie in the open left-half of the complex plane.

The dynamic control law of Eq. (13), the state feedback law of Eq. (16) and the state observer of Eq. (17) can be combined to yield the following nonlinear output feedback controller:

$$\frac{d\hat{g}}{dt} = f(\hat{g}) + g(\hat{g})(p(\hat{g}) + Q(\hat{g})v) + L(y_m - h_m(\hat{g})),$$

$$\frac{d\hat{m}}{dt} = p(\hat{g}) + Q(\hat{g})v,$$

$$u = \hat{m}. \tag{18}$$

Theorem 1 below provides an explicit synthesis formula of the output feedback controller and conditions that guarantee closed-loop stability. In order to state the result of the theorem, referring to the system of Eq. (15), we define the relative order of the output y_{c_i} with respect to the vector of manipulated inputs $\tilde{u}$ as the smallest integer r_i for which

$$[L_{g_1} L_f^{r_i-1} h_{c_i}(\hat{x}_s) \cdots L_{g_l} L_f^{r_i-1} h_{c_i}(\hat{x}_s)] \not\equiv [0 \cdots 0] \tag{19}$$

or $r_i = \infty$ if such an integer does not exist, and the characteristic matrix

$$C(\hat{x}_s) = \begin{pmatrix} L_{g_1} L_f^{r_1-1} h_{c_1}(\hat{x}_s) & \cdots & L_{g_l} L_f^{r_1-1} h_{c_1}(\hat{x}_s) \\ L_{g_1} L_f^{r_2-1} h_{c_2}(\hat{x}_s) & \cdots & L_{g_l} L_f^{r_2-1} h_{c_2}(\hat{x}_s) \\ \vdots & \cdots & \vdots \\ L_{g_1} L_f^{r_l-1} h_{c_l}(\hat{x}_s) & \cdots & L_{g_l} L_f^{r_l-1} h_{c_l}(\hat{x}_s) \end{pmatrix}, \tag{20}$$

where h_{c_i} is the ith element of the vector h_c and g_i is the ith vector of the matrix g.

Theorem 1. *Consider the system of Eq.* (15) *and assume that*: (1) *it is locally observable and* $C_L = (1/\epsilon)A$ *where* ϵ *is a small positive parameter and* A *is a Hurwitz matrix*, (2) *its characteristic matrix is nonsingular* $\forall\, \hat{x}_s \in D \subset \mathbb{R}^{m+l}$, *and* (3) *its unforced* ($v \equiv 0$) *zero dynamics are locally exponentially stable. Consider also the system of Eq.* (1) *under the nonlinear output feedback controller*

$$\frac{d\hat{g}}{dt} = f(\hat{g}) + L(y_m - h_m(\hat{g})) + g(\hat{g})M[b_{1r_1} \cdots b_{lr_l}]C(\hat{g})^{-1}\left[v - \left(\sum_{i=1}^{l}\sum_{k=0}^{r_i} b_{ik} L_f^k h_{c_i}(\hat{g})\right)\right],$$

$$\frac{d\hat{m}}{dt} = M[b_{1r_1} \cdots b_{lr_l}]C(\hat{g})^{-1}\left(v - \sum_{i=1}^{l}\sum_{k=0}^{r_i} b_{ik} L_f^k h_{c_i}(\hat{g})\right),$$

$$u = \hat{m}, \tag{21}$$

where the l*-dimensional vectors of the parameters* b_{ik} *are chosen so that the roots of the equation* $\det(B(s)) = 0$ *are in the open left-half of the complex plane* ($B(s)$ *is a* $l \times l$ *matrix, whose* (i,j)*th element is of the form* $\sum_{k=0}^{r_i} b_{ik} s^k$). *Then, there exist positive real numbers* k, ϵ_H *such that if* $\|\hat{x}_s(0)\| \leq k$ *and* $\epsilon \in (0, \epsilon_H]$, *the zero solution of the closed-loop system* (*Eqs.* (1) and (21)) *is exponentially stable.*

Remark 2. The assumption that the zero dynamics of the system of Eq. (15) are locally exponentially stable is standard in geometric control (see Isidori, 1989 for details), while the assumption $C_L = (1/\epsilon)A$, where ϵ is a small positive parameter and A is a Hurwitz matrix, is needed to ensure that the presence of closed-loop system states, which are not included in the model used for controller synthesis, in the state observer does not lead to closed-loop instability. Finally, the assumption that the characteristic matrix $C(\hat{x}_s)$ is nonsingular is made to simplify the presentation of the controller synthesis results and can be relaxed (see Isidori, 1989).

Remark 3. The exponential stability property of the closed-loop system guarantees a degree of robustness with respect to sufficiently small disturbances and uncertainty in the process parameters. The design of robust controllers is beyond the scope of this paper.

Remark 4. It is important to note that even though the presence of feedback control in the system of Eq. (1) makes very difficult (if not impossible) to answer the mathematical question of existence of an inertial manifold (i.e., exact solution of Eq. (9)), the use of the proposed method for synthesizing nonlinear controllers with guaranteed closed-loop stability for dissipative PDEs is based on the concept of approximate inertial manifold and does not require knowledge of the existence of an inertial manifold for the PDE. Moreover, referring to the expression for the approximate inertial manifold of Eq. (11), we note that it constitutes a direct generalization to the one presented in Foias, Sell and Titi (1989a) and Foias, Jolly, Kevrekidis, Sell and Titi (1989b) for dissipative PDEs without time-varying inputs, to systems with time-varying inputs. This approximation for $\mathcal{R}$ is clearly not unique; the reader may also refer to Jones and Titi (1994) and Shvartsman and Kevrekidis (1998) for other approximations of $\mathcal{R}$ for systems with time-varying inputs.

Remark 5. The implementation of the controller of Eq. (21) requires to explicitly compute the vector function $\mathcal{R}_{11}(\eta, u)$. However, $\mathcal{R}_{11}(\eta, u)$ has an infinite-dimensional range and therefore cannot be implemented in practice. Instead a finite-dimensional approximation of $\mathcal{R}_{11}(\eta, u)$, say $\mathcal{R}_{11}(\eta, u)$, can be derived by keeping the first m elements of $\mathcal{R}_{11}(\eta, u)$ and neglecting the remaining infinite ones. Clearly, as $m \to \infty$, $\mathcal{R}_{11}(\eta, u)$ approaches $\mathcal{R}_{11}(\eta, u)$. This implies that by picking m to be sufficiently large, the controller of Eq. (21) with $\mathcal{R}_{11}(\eta, u)$ instead of $\mathcal{R}_{11}(\eta, u)$ enforces local exponential stability in the closed-loop infinite-dimensional system.

Remark 6. It is instructive to compare the approach proposed in this work for finite-dimensional nonlinear controller design for dissipative infinite-dimensional systems with the method developed in Chen and Chang (1992), Chen, Wolf and Chang (1993) and Qin, Wolf and Chang (1994) for the design of nonlinear controllers for diffusion-reaction processes with unknown dynamics using Karhunen-Loève-based Galerkin's method and center manifold theory. The central difference is that the proposed approach can be applied to all dissipative PDEs (including hyperbolic-type PDEs like Korteweg–de Vries–Burgers equation and parabolic-type PDEs like Kuramoto–Sivashinsky equation), while the method in Chen and Chang (1992), Chen et al. (1993) and Qin et al. (1994) is mainly suitable for dissipative PDEs that include spatial differential operators whose eigenspectrum can be partitioned into a finite-dimensional set of eigenvalues which are close to the imaginary axis and an infinite-dimensional stable complement of eigenvalues which are far in the left-half of the complex plane (a typical feature of parabolic PDEs modeling diffusion-reaction processes). We have been able to relax this restriction by imposing additional requirements on the observability and the design of the gain of the state observer used to provide estimates of the states of the finite-dimensional model used for controller design. In contrast, the approach in Chen and Chang (1992), Chen et al. (1993) and Qin et al. (1994) motivated by the objective to control diffusion-reaction processes with unknown dynamics, employs a sufficiently large number of measurements to reconstruct the eigenmodes/eigenfunctions corresponding to the slow dynamics of the closed-loop system using Karhunen-Loève expansion.

Proof of Theorem 1. Under the controller of Eq. (21), the closed-loop system takes the form

$$\frac{d\eta}{dt} = f(\eta) + L(y_m - h_m(\eta)) + g(\eta)[b_{1r_1} \cdots b_{lr_l}]C(\eta)^{-1}$$

$$\left[v - \sum_{i=1}^{l} \sum_{k=0}^{r_i} b_{ik} L_f^k h_{ci}(\eta) \right],$$

$$\frac{d\mu}{dt} = [b_{1r_1} \cdots b_{lr_l}]C(\eta)^{-1} \left[v - \sum_{i=1}^{l} \sum_{k=0}^{r_i} b_{ik} L_f^k h_{ci}(\eta) \right],$$

$$\dot{x} = A x + B\mu + \mathcal{R}(x). \tag{22}$$

Using the decomposition of x of Eq. (4) and applying P_s and P_f to the x-subsystem of Eq. (22) we obtain

$$\frac{d\eta}{dt} = f(\eta) + L(y_m - h_m(\eta)) + g(\eta)[b_{1r_1} \cdots b_{lr_l}]C(\eta)^{-1}$$

$$\left[v - \sum_{i=1}^{l} \sum_{k=0}^{r_i} b_{ik} L_f^k h_{ci}(\eta) \right]$$

$$\frac{d\mu}{dt} = [b_{1r_1} \cdots b_{lr_l}]C(\eta)^{-1} \left[v - \sum_{i=1}^{l} \sum_{k=0}^{r_i} b_{ik} L_f^k h_{ci}(\eta) \right],$$

$$\frac{dx_s}{dt} = A_s x_s + B_s \mu + \mathcal{R}_s(x_s, x_f),$$

$$\frac{dx_f}{dt} = A_f x_f + B_f \mu + \mathcal{R}_f(x_s, x_f). \tag{23}$$

Performing a linearization of the above system around the zero solution, defining the variable $e_o = \eta - \hat{x}_s$, introducing the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, the fast dynamics of the above system are described by the system

$$\frac{de_o}{d\tau} = Ae_o \tag{24}$$

which is clearly exponentially stable. On the other hand, the system which describes the slow dynamics (i.e. $\epsilon \to 0$) of the system of Eq. (23) takes the form:

$$\frac{dm}{dt} = M[b_{11} \cdots b_{1n}]C(\hat{x}_s)N^{-1}\left(v - \sum_{i=1}^{l} + \sum_{k=0}^{n} b_{ik} L_f^k h_{ci}(\hat{x}_s)\right),$$

$$\frac{dx_s}{dt} = A_s x_s + B_s m + R_s(x_s, x_f),$$

$$\frac{Lx_f}{Lt} = A_f x_f + B_f m + R_f(x_s, x_f). \tag{25}$$

From assumptions 2 and 3 and the fact that b_{ik} are chosen so that the roots of the equation $\det(B(s)) = 0$ are in the open left-half of the complex plane, we have that the linearization of the above system around the zero solution is also exponentially stable. Therefore, there exists (Kokotovic, Khalil & O'Reilly, 1986) a real positive number ϵ_H such that $\forall\, \epsilon \in (0, \epsilon_H]$, the linearization of Eq. (23) is exponentially stable. Using Theorem 5.1.1 in Henry (1981), we then have that there exists a real positive number k such that if $\|\Delta\|_{0,2} \leqslant k$, the zero solution of the closed-loop system of Eq. (23) (and thus of Eq. (22)) is exponentially stable.

3. Korteweg–de Vries–Burgers equation

We consider the one-dimensional Korteweg–de Vries–Burgers equation with distributed control

$$\frac{L\psi}{Lt} = -\frac{L^3\psi}{Lz^3} + a_1\frac{L^2\psi}{Lz^2} - \psi\frac{L\psi}{Lz} + b(z)u(t) \tag{26}$$

subject to the periodic boundary conditions

$$\frac{L^j\psi}{Lz^j}(0,t) = \frac{L^j\psi}{Lz^j}(\pi, t), \quad j = 0, \ldots, 2 \tag{27}$$

and the initial condition

$$\psi(z,0) = \psi_0(z), \tag{28}$$

where $\psi(z,t)$ is the state, $z \in [0,\pi]$ is the spatial coordinate, a_1 is a positive real number and $b(z)$ is the vector of the actuator distribution functions.

Introducing the Hilbert space H of functions defined on $[0,\pi]$ that satisfy the boundary conditions of Eq. (27) and defining the state function x on H as

$$x(t) = \psi(z,t), \quad z \in [0,\pi] \tag{29}$$

the system of Eqs. (26)–(28) can be written in the form of Eq. (1), where the spatial differential operator takes the form

$$Ax = -\frac{L^3x}{Lz^3} + a_1\frac{L^2x}{Lz^2},$$

$$x \in D(A) = \left\{x \in H([0,\pi]; R);\ \frac{L^jx}{Lz^j}(0) = \frac{L^jx}{Lz^j}(\pi),\ j = 0, \ldots, 2\right\}. \tag{30}$$

The eigenvalue problem for A can be solved analytically and its solution is of the form

$$\lambda_j = -4a_1 j^2 + I8j^3, \quad \phi_j(z) = \sqrt{\frac{1}{\pi}}e^{(I2jz)},$$

$$j = 0, \pm 1, \ldots, \pm\infty, \tag{31}$$

where λ_j denotes the jth eigenvalue, $\phi_j(z)$ denotes the jth eigenfunction and $I = \sqrt{-1}$.

For the system of Eq. (26), the spatiotemporal evolution of the open-loop $\psi(z,t)$ is shown in Fig. 1. For all the simulations presented in this section, we use $a_1 = 0.05$ and $\psi_0(z) = 2.5 + \sum_{j=-5}^{5} [\cos(2jz) - \sin(2jz)] - 2.5$, while a 21st-order nonlinear ODE system obtained by applying Galerkin's method to the system of Eqs. (26)–(28) is used in the simulation (use of higher-order approximations led to identical numerical results). Clearly, the time required for the $\psi(z,t)$ to approach the stable zero solution is significant (i.e., more than 12 time units). To enhance the convergence rate to the steady state, we use three control actuators and the methodology of Section 2 to design a nonlinear multivariable finite-dimensional output feedback controller. We assume that a single measurement of $\psi(z,t)$ at $z = \pi/2$ (i.e., $y_m = \int_0^\pi \delta(z - \pi/2)\psi(z,t)\,dz$; see also Remark 5) is available. The three controlled outputs are defined as

$$y_a(t) = \int_0^\pi \sqrt{\frac{2}{\pi}}\cos(2z)\psi(z,t)\,dz,$$

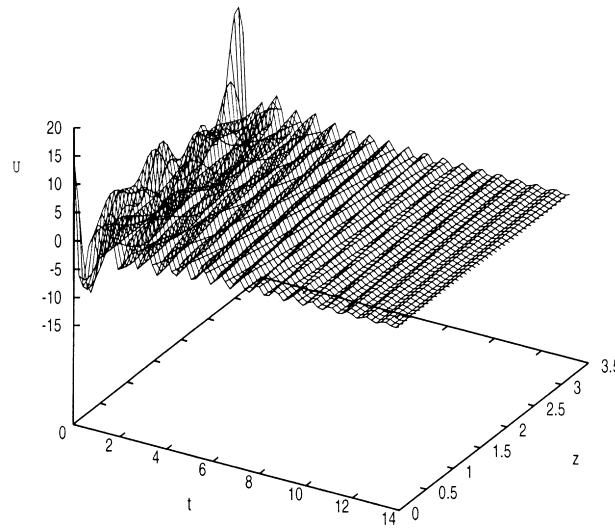

Fig. 1. Open-loop spatiotemporal profile of $\psi(z,t)$ (KdVB).

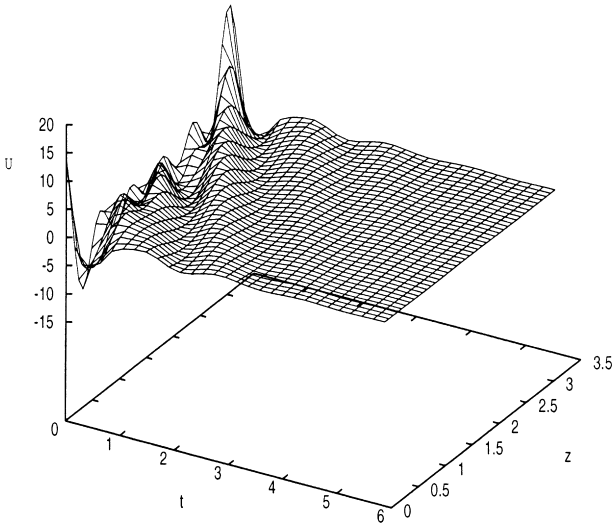

Fig. 2. Closed-loop spatiotemporal profile of ; (z,t) (KdVB).

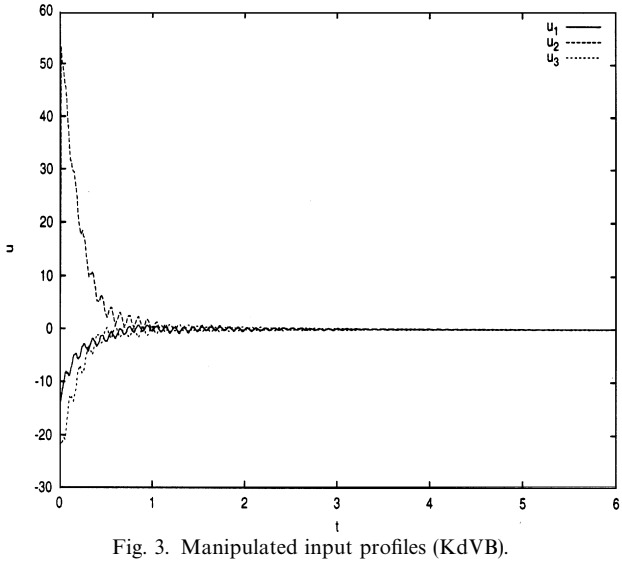

Fig. 3. Manipulated input profiles (KdVB).

$$y_{c1}(t) = \int_0^{\pi} \bar{S}_1 \sin(2z);\ (z,t)\,\mathrm{d}z,$$

$$y_{c3}(t) = \int_0^{\pi} \bar{S}_1;\ (z,t)\,\mathrm{d}z \tag{32}$$

and the actuator distribution functions are taken to be

$$b_1(z) = \delta\left(z - \frac{\pi}{2}\right),$$

$$b_2(z) = \delta\left(z - \frac{\pi}{4}\right),$$

$$b_3(z) = \bar{S}_1. \tag{33}$$

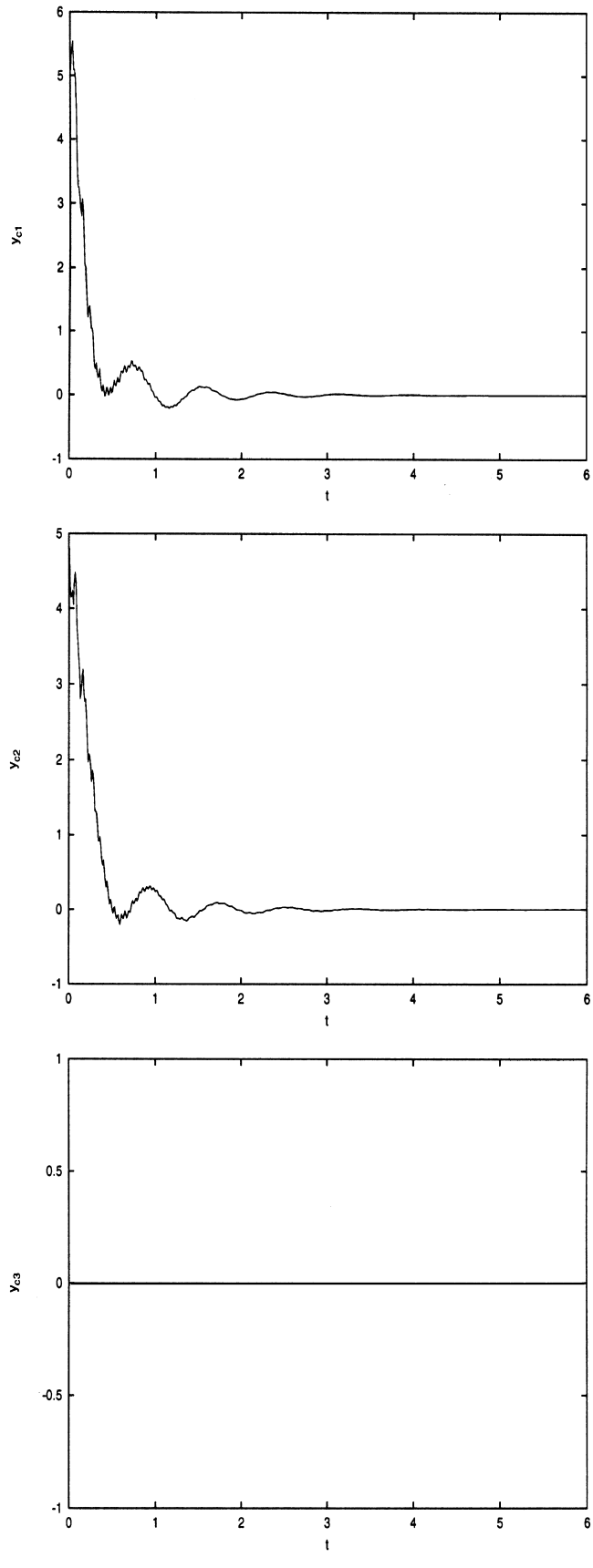

Fig. 4. Closed-loop output profiles (KdVB).

The use of two point actuators and one distributed actuator ensures that $\int_{-\pi}^{\pi} \psi(z,t)\,dz = 0$, for all times (e.g., mass conservation, see also, Russell & Zhang, 1996). For the system of Eq. (26), Galerkin's method was used to derive an approximate three-dimensional ODE model which was employed for controller design. Improvement of the accuracy of the three-dimensional model through approximate inertial manifolds was found to be insignificant. The ODE model was used for the synthesis of a nonlinear controller of the form of Eq. (21) which was implemented with

$$b_{10} = \begin{pmatrix} 1.00 \\ 0.00 \\ 0.00 \end{pmatrix}, \quad b_{20} = \begin{pmatrix} 0.00 \\ 1.00 \\ 0.00 \end{pmatrix}, \quad b_{30} = \begin{pmatrix} 0.00 \\ 0.00 \\ 1.00 \end{pmatrix},$$

$$b_{11} = \begin{pmatrix} 0.25 \\ 0.00 \\ 0.00 \end{pmatrix}, \quad b_{21} = \begin{pmatrix} 0.00 \\ 0.25 \\ 0.00 \end{pmatrix}, \quad b_{31} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.25 \end{pmatrix},$$

$$\Phi = \begin{pmatrix} 2.00 \\ 0.00 \\ 0.00 \end{pmatrix}.$$

Fig. 2 shows the evolution of $\psi(z,t)$, while Fig. 3 shows the manipulated input profiles and Fig. 4 displays the closed-loop output profiles. The nonlinear controller clearly enhances the convergence rate to the steady state; this can be also seen in Fig. 5 where the open- and closed-loop system profiles of $\|\psi\|_2$ are presented.

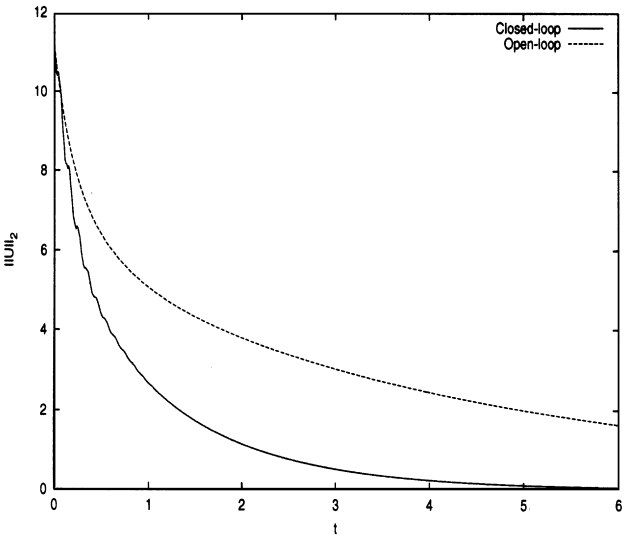

Fig. 5. Profiles of $\|\psi\|_2$ for open-loop (dashed line) and closed-loop (solid line) systems (KdVB).

4. Kuramoto–Sivashinsky equation

We consider the one-dimensional Kuramoto–Sivashinsky equation with distributed control

$$\frac{\partial \psi}{\partial t} = -\lambda \frac{\partial^4 \psi}{\partial z^4} - \frac{\partial^2 \psi}{\partial z^2} - \psi \frac{\partial \psi}{\partial z} + b(z)u(t) \tag{34}$$

subject to the periodic boundary conditions

$$\frac{\partial^j \psi}{\partial z^j}(-\pi,t) = \frac{\partial^j \psi}{\partial z^j}(+\pi,t), \quad j = 0,\ldots,3 \tag{35}$$

and the initial condition

$$\psi(z,0) = \psi_0(z), \tag{36}$$

where $\psi(z,t)$ is the state of the system, $z \in [-\pi,\pi]$ is the spatial coordinate and λ is the so-called instability parameter.

Introducing the Hilbert space H of square integrable odd functions that satisfy the boundary conditions of Eq. (35) and have spatial zero mean (i.e., $\forall u \in H$, $\int_{-\pi}^{\pi} u(z)\,dz = 0$) and defining the state function $x \in H$ as

$$x(t) = \psi(z,t) \quad \forall z \in [-\pi,\pi], \tag{37}$$

the system of Eqs. (34)–(36) can be written in the form of Eq. (1), where the spatial differential operator takes the form

$$\begin{aligned} Ax &= -\lambda \frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2}, \quad x \in D(A) \\ &= \left\{ x \in H([-\pi,\pi];R);\ \frac{\partial^j x}{\partial z^j}(-\pi) = \frac{\partial^j x}{\partial z^j}(\pi), \right. \\ &\qquad \left. j = 0,\ldots,3 \right\} \end{aligned} \tag{38}$$

A direct computation of the solution of the eigenvalue problem for A yields $\lambda_j = -\lambda j^4 + j^2$, $\phi_j(z) = \sqrt{(1/\pi)}\sin(jz)$, $j = 1,\ldots,\infty$ (note that $\phi_j(z) = \sqrt{(1/2\pi)}$ and $\phi_j(z) = \sqrt{(1/\pi)}\cos(jz)$, $j = 1,\ldots,\infty$ are also eigenfunctions of A, but are not considered here since we focus on odd functions with spatial zero mean). Clearly, a real eigenvalue of the system of Eq. (34) crosses the imaginary axis when

$$\lambda = \frac{1}{j^2}, \quad j = 1,\ldots,\infty. \tag{39}$$

Apparently, the smallest value of λ for which the $\psi(z,t) = 0$ solution of the system of Eq. (34) is about to become unstable is $\lambda = 1$ (the reader may refer to Chen and Chang (1986) for a detailed stability analysis; see also Chang, Demekhin, Kopelevich and Ye (1997) and Chang, Demekhin and Kopelevich (1996)). For example, when $\lambda = 0.3$, the operator of Eq. (38) possesses one unstable eigenvalue. The spatiotemporal evolution of $\psi(z,t)$ for $\lambda = 0.3$ is shown in Fig. 6, for the initial condition $\psi_0(z) = 5.0 + \sum_{j=1}^{5} [\sin(jz)]$ (for the simulation of Eq. (34)

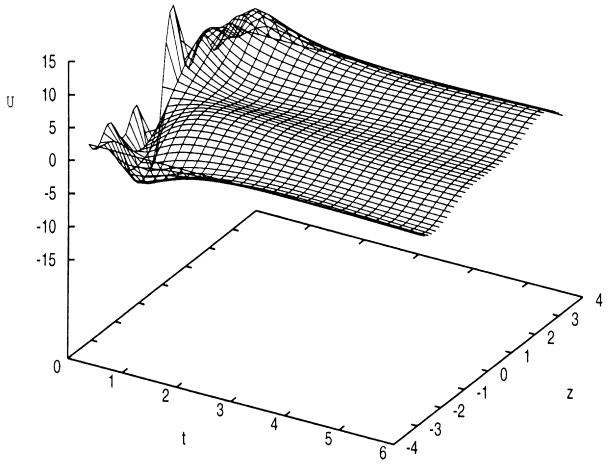

Fig. 6. Open-loop spatiotemporal profile of ; (z,t) for l = 0.3 (KS).

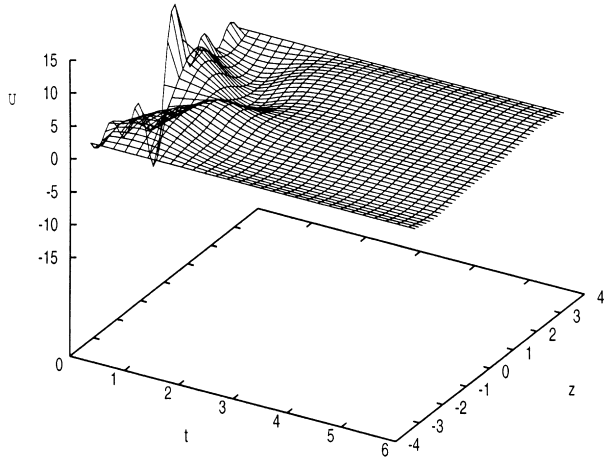

Fig. 7. Closed-loop spatiotemporal profile of ; (z,t) for l = 0.3 and (1,0) controller (KS).

a 30th-order ODE system derived using Galerkin's method is used; higher-order approximations led to identical numerical results). It is clear that for l = 0.3, the spatially uniform steady state ; $(z,t) = 0$ is unstable. Therefore, we use the proposed method to design a nonlinear finite-dimensional controller which uses one measurement of ; (z,t) at $z = p/2$ to stabilize the system at ; $(z,t) = 0$, for l = 0.3. To achieve this control objective, the controlled output is defined as

$$y_c(t) = \int_{-p}^{p} \sin(z);\ (z,t)\,dz \tag{40}$$

and the actuator distribution function is taken to be $b_1(z) = d(z - p/2)$ (i.e., point control actuation at $z = p/2$). To show the improvement in closed-loop performance achieved when nonlinear Galerkin's method is used for model reduction, standard Galerkin's method was initially used to derive a first-order model which was employed for the synthesis of a nonlinear controller using the formula of Eq. (21). This controller is denoted as (1,0) where 1 denotes the order of the controller and 0 denotes the order of the approximate inertial manifold. The controller parameters are $b_{10} = 1.00$, $b_{11} = 0.20$ and $L = 3.80$. Fig. 7 shows the evolution of ; (z,t), while Figs. 9 and 10 show the corresponding manipulated input and controlled output profiles (dashed lines), respectively. Note that this controller requires more than 15 time units to stabilize the system at ; $(z,t) = 0$. To achieve stabilization with better convergence rate to the steady state, the proposed combination of Galerkin's method with approximate inertial manifolds (i.e., nonlinear Galerkin's method) was used to derive a first-order model with $m = 2$ (dimension of AIM), which was subsequently employed for the construction of a nonlinear controller. This controller is denoted as (1,2) and is implemented with $b_{10} = 1.00$, $b_{11} = 0.20$ and $L = 3.80$. Fig. 8 shows the evolution of ; (z,t), while Figs. 9 and 10

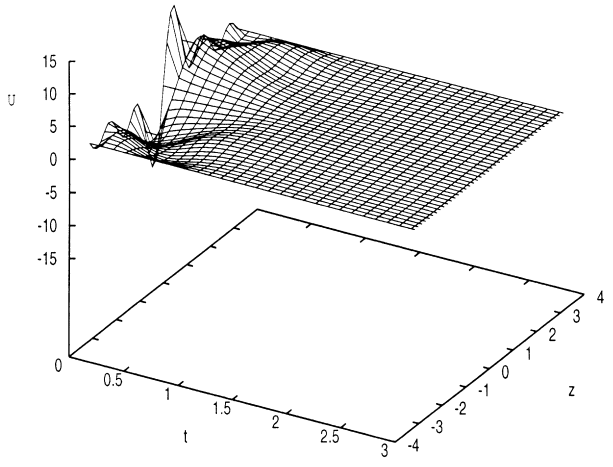

Fig. 8. Closed-loop spatiotemporal profile of ; (z,t) for l = 0.3 and (1,2) controller (KS).

show the corresponding manipulated input and output profiles (solid lines), respectively. Note that this controller stabilizes the system at ; $(z,t) = 0$ after 3 time units, while achieving a smooth transient profile. The improvement on the convergence rate to the steady state, which is achieved by using nonlinear Galerkin's method to derive the model used for controller design, is also shown in Fig. 11, where the closed-loop profiles of D_1, D_2, D_3 under the (1,0) and (1,2) controllers are presented.

Finally, in order to illustrate the value of proposed method when multiple eigenvalues are unstable, we also consider the stabilization of the KS equation with l = 0.1; in this case, the open-loop system possesses three positive (unstable) eigenvalues. The spatiotemporal evolution of ; (z,t) for l = 0.1 and ; $_0(z) = 5.0 + \sum_{j=1}^{5} [\sin(jz)]$ is shown in Fig. 12. The proposed method is used to design a nonlinear multivariable finite-dimensional controller which uses three point measurements of ; (z,t) at $z = -p/2$, $z = -p/4$,

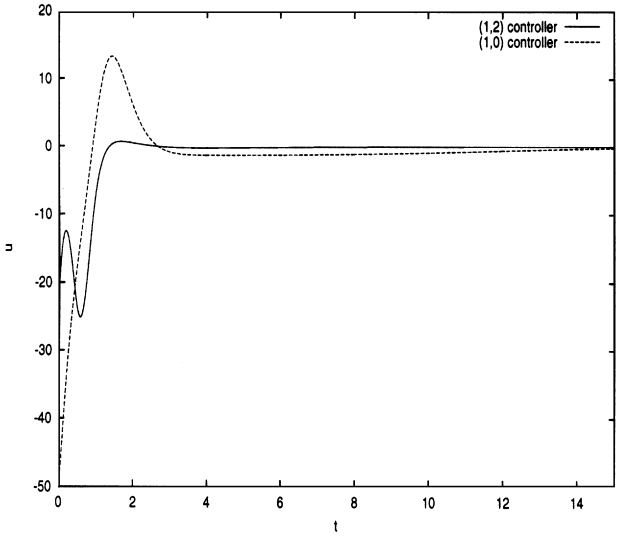

Fig. 9. Manipulated input profiles for (1, 0) controller (dashed line) and (1, 2) (solid line) controller (KS, $\lambda = 0.3$).

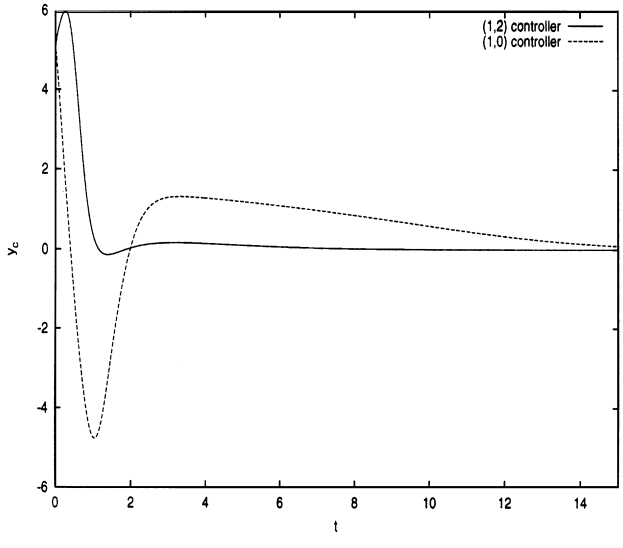

Fig. 10. Closed-loop output profiles for (1, 0) controller (dashed line) and (1, 2) (solid line) controller (KS, $\lambda = 0.3$).

$z = \pi/3$ to stabilize the system at ; $(z, t) = 0$. To achieve this control objective, the three controlled outputs are defined as

$$y_a(t) = \int_{-\pi}^{\pi} \int_{-1}^{1} \sin(z);\ (z, t)\, dz,$$

$$y_a(t) = \int_{-\pi}^{\pi} \int_{-1}^{1} \sin(2z);\ (z, t)\, dz,$$

$$y_a(t) = \int_{-\pi}^{\pi} \int_{-1}^{1} \sin(3z);\ (z, t)\, dz \tag{41}$$

and three point actuators are used with the following actuator distribution functions: $b_1(z) = \delta(z + \pi/3)$, $b_2(z) = \delta(z - \pi/4)$ and $b_3(z) = \delta(z - \pi/2)$ (i.e., point control actuation is used at $z = -\pi/3$, $z = \pi/4$ and $z = \pi/2$, respectively). Two different controllers are employed for the stabilization of the system to the spatially uniform steady state ; $(z, t) = 0.0$. Initially, a nonlinear output feedback controller was designed on the basis of an ODE model which was derived through standard Galerkin's method. It was found that the lowest order nonlinear controller that achieves stabilization of the zero solution with a reasonable convergence rate is nine (a controller of order eight achieves stabilization with a very slow convergence rate, and controllers of order seven and lower cannot stabilize the closed-loop system, for the chosen initial conditions, controller parameters and $\lambda = 0.1$). Based on this ninth-order model, a nonlinear controller was synthesized using the formula of Eq. (2.1) and is denoted as (9, 0). The controller parameters are as follows:

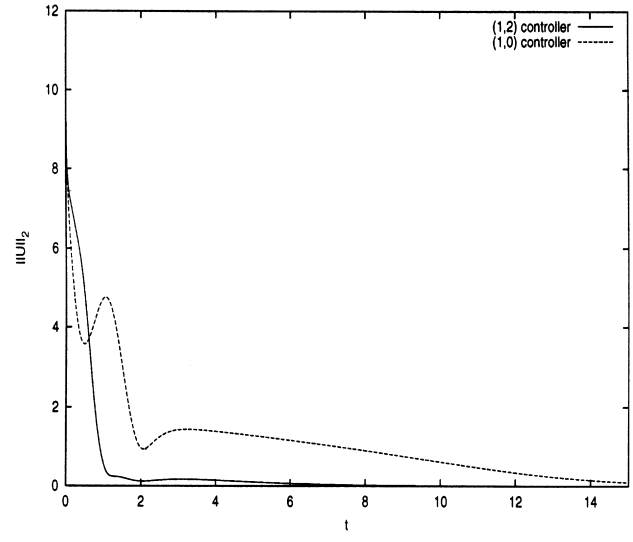

Fig. 11. Closed-loop profiles of $\|U\|_2$ for (1, 0) controller (dashed line) and (1, 2) (solid line) controller (KS, $\lambda = 0.3$).

$$b_{10} = \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ \vdots \\ 0.0 \end{bmatrix}, \quad b_{20} = \begin{bmatrix} 0.0 \\ 1.0 \\ 0.0 \\ 0.0 \\ \vdots \\ 0.0 \end{bmatrix}, \quad b_{30} = \begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \\ 0.0 \\ \vdots \\ 0.0 \end{bmatrix},$$

$$b_{11} = \begin{bmatrix} 0.5 \\ 0.0 \\ 0.0 \\ 0.0 \\ \vdots \\ 0.0 \end{bmatrix}, \quad b_{21} = \begin{bmatrix} 0.0 \\ 0.5 \\ 0.0 \\ 0.0 \\ \vdots \\ 0.0 \end{bmatrix}, \quad b_{31} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.5 \\ 0.0 \\ \vdots \\ 0.0 \end{bmatrix},$$

$$\%=\begin{bmatrix} 13.3 & -21.8 & 4.89 \\ 18.8 & -30.8 & 16.7 \\ 21.7 & -50.3 & 19.3 \\ 0.00 & 0.00 & 0.00 \\ F & F & F \\ 0.00 & 0.00 & 0.00 \end{bmatrix}.$$

Figs. 13, 15 and 16 show the evolution of ; (z,t), the manipulated input profiles (dashed lines) and the corresponding controlled output profiles (dashed lines). Clearly, this ninth-order controller achieves stabilization of the closed-loop system after 12 time units. Subsequently, we employ the proposed approach to design a stabilizing controller. Using a combination of Galerkin's method with approximate inertial manifolds (i.e., nonlinear Galerkin's method), we derived a fourth-order model with $m8 = 6$ (dimension of AIM), which was subsequently employed for the construction of a nonlinear controller (denoted as (4, 6)) (Fig. 15). This controller was implemented with the following parameters:

$$b_{10}=\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}, \quad b_{20}=\begin{bmatrix} 0.0 \\ 1.0 \\ 0.0 \\ 0.0 \end{bmatrix}, \quad b_{30}=\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \\ 0.0 \end{bmatrix},$$

$$b_{11}=\begin{bmatrix} 0.5 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}, \quad b_{21}=\begin{bmatrix} 0.0 \\ 0.5 \\ 0.0 \\ 0.0 \end{bmatrix}, \quad b_{31}=\begin{bmatrix} 0.0 \\ 0.0 \\ 0.5 \\ 0.0 \end{bmatrix},$$

$$\%=\begin{bmatrix} 13.3 & -21.8 & 4.89 \\ 18.8 & -30.8 & 16.7 \\ 21.7 & -50.3 & 19.3 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}.$$

Figs. 14, 15 and 16 show the evolution of ; (z,t), the manipulated input and controlled output profiles, respectively. Clearly, this controller stabilizes the closed-loop system at ; $(z,t) = 0$ after 4 time units, while achieving a smooth transient profile. Finally, Fig. 17 shows the closed-loop profiles of the `energy' of the system $\|\cdot\|_2$ under the (9, 0) and (4, 6) controllers. We can clearly see that the nonlinear controller based on the model obtained by nonlinear Galerkin's method outperforms the nonlinear controller based on the model obtained by standard Galerkin's method.

Remark 7. We note that in the case of point actuation (sensing) which influences (measures) the system at z_0, the function $d(z - z_0)$ is assumed to have the finite value $1/2e$ in the interval $[z_0 - e, z_0 + e]$ (where e is a small positive

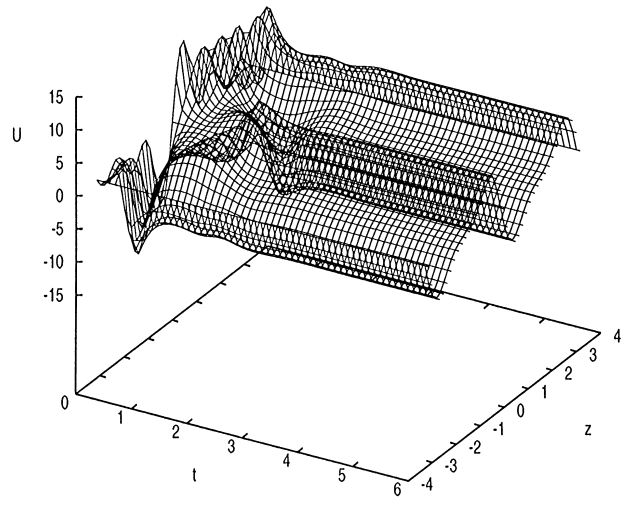

Fig. 12. Open-loop spatiotemporal profile of ; (z,t) for $l = 0.1$ (KS).

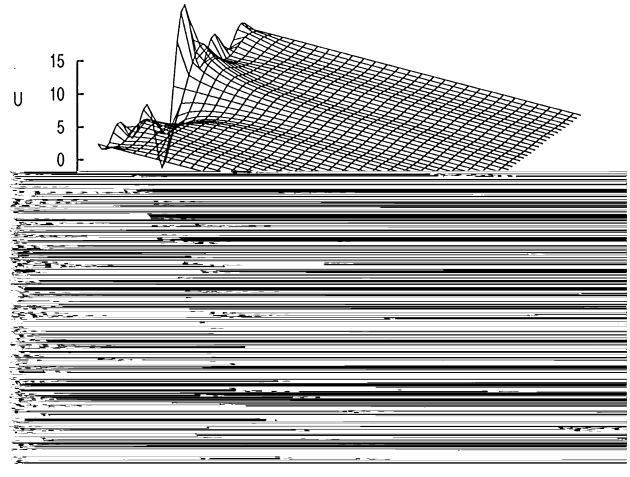

Fig. 13. Closed-loop spatiotemporal profile of $u(z,t)$ for $l = 0.1$ and (9, 0) controller (KS).

Fig. 14. Closed-loop spatiotemporal profile of $u(z,t)$ for $l = 0.1$ and (4, 6) controller (KS).