Physica D 137 (2000) 49–61
PHYSICA D
www.elsevier.com/locate/physd

Feedback control of the Kuramoto–Sivashinsky equation ☆

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Received 19 February 1998; received in revised form 11 June 1999; accepted 15 July 1999
Communicated by C.K.R.T. Jones

Abstract

This work focuses on linear finite-dimensional output feedback control of the Kuramoto–Sivashinsky equation (KSE) with periodic boundary conditions. Under the assumption that the linearization of the KSE around the zero solution is controllable and observable, linear finite-dimensional output feedback controllers are synthesized that achieve stabilization of the zero solution, for any value of the instability parameter. The controllers are synthesized on the basis of finite-dimensional approximations of the KSE which are obtained through Galerkin's method. The performance of the controllers is successfully tested through computer simulations.

1. Introduction

Nonlinear dissipative partial differential equations (PDEs) arise naturally in the modeling of many physical and chemical systems and are known to exhibit complex dynamic behavior (see, e.g. [14,31] and the references therein). This complex behavior has motivated extensive theoretical and computational studies on the determination and characterization (in terms of stability) of the steady-state solutions, attractors and manifolds that appear in various dissipative PDEs. In this direction, several nonlinear dissipative PDEs arising in the modeling of diffusion–reaction processes including the Brusselator reaction scheme [1], the Fitz-Hugh–Nagumo system [3] and the λ–ω system [13], and of fluid flows [32] have been studied.

A nonlinear dissipative PDE, which is known to exhibit temporarily complex but spatially coherent patterns, is the Kuramoto–Sivashinsky equation:

$$\frac{\partial x}{\partial t} = -\nu \frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z}, \tag{1}$$

where $\nu > 0$ is the instability parameter. Eq. (1) can adequately describe incipient instabilities arising in a variety of physico-chemical systems including falling liquid films [8], unstable flame fronts [25,29,30], Belouzov –Zabotinskii

☆ Financial support from an NSF CAREER award, CTS-9733509, is gratefully acknowledged.
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0167-2789/00/$ – see front matter
PII: S0167-2789(99)00175-X

reaction patterns [22,23], interfacial instabilities between two viscous fluids [18], etc. Numerical studies on the dynamics of Eq. (1) (e.g. [6,10,15,16,20]) have revealed the existence of steady and periodic wave solutions, as well as chaotic behavior for very small values of ν.

The studies on the dynamics of various nonlinear dissipative PDEs have revealed that the dominant dynamics of such PDEs are usually characterized by a small number of degrees of freedom (e.g. [9,31]). This feature has motivated addressing the controller design problem for dissipative PDEs on the basis of ODE approximations that accurately describe the dominant dynamics of the PDE, which are typically obtained through linear/nonlinear Galerkin's method (e.g. [2,9,11,12]). This approach has been successfully used to control complex dynamics of several diffusion–reaction processes including processes with Gray-Scott [27] and Brusselator [7,19] kinetics. An alternative approach to control of PDE systems is to directly address the controller design problem on the basis of the PDE system (see, e.g. [4,26,28]) and has been explored for the control of optical turbulence [24,33].

In this paper, we consider the problem of stabilization of the zero solution, $x(z,t)=0$, of the Kuramoto–Sivashinsky equation with periodic boundary conditions, for any value of the instability parameter ν, via dynamic output feedback control. Initially, given the value of ν, linear state feedback controllers are designed that achieve global (i.e., for every initial condition) stabilization of the zero solution of Eq. (1), provided that the linearization of the KSE around the zero solution is controllable. Then, linear finite-dimensional dynamic output feedback controllers that use a finite number of measurements (for which the linearized KSE is observable) to exponentially stabilize the system of Eq. (1) at $x(z,t)=0$ are derived. The proposed output feedback controllers are designed on the basis of ODE approximations of the Kuramoto–Sivashinsky equation obtained through Galerkin's method, which capture the dynamics of the unstable modes. Numerical simulations of the closed-loop system, for different values of the instability parameter, are shown indicating the effectiveness of the proposed control method.

2. Preliminaries

2.1. Controlled Kuramoto–Sivashinsky equation

We consider the integrated form of the *controlled* Kuramoto–Sivashinsky equation:

$$\frac{\partial x}{\partial t}=-\nu\frac{\partial^4 x}{\partial z^4}-\frac{\partial^2 x}{\partial z^2}-x\frac{\partial x}{\partial z}+\sum_{i=1}^{m}b_i u_i(t) \tag{2}$$

subject to the periodic boundary conditions:

$$\frac{\partial^j x}{\partial z^j}(-\pi,t)=\frac{\partial^j x}{\partial z^j}(+\pi,t), \quad j=0,\dots,3 \tag{3}$$

and the initial condition:

$$x(z,0)=x_0(z), \tag{4}$$

where $x\in(L^2([-\pi,\pi]),\mathbb{R})$ is the state of the system, $L^2([-\pi,\pi])$ is the Hilbert space of square integrable functions that satisfy the boundary conditions of Eq. (3), z is the spatial coordinate, t is the time and 2π is the length of the spatial domain. m is the number of manipulated inputs, $u_i(t)$ is the ith manipulated input (i.e., variable that can be manipulated externally in order to modify the dynamics of the equation in a desired fashion), $b_i(z)$ is the actuator distribution function (i.e., $b_i(z)$ determines how the control action computed by the ith control actuator, $u_i(t)$, is distributed (point or distributed actuation) in the spatial interval $[-\pi,\pi]$), and $x_0(z)\in L^2([-\pi,\pi])$ is the initial condition. In $L^2([-\pi,\pi])$, we define the inner product and norm: $(\omega_1,\omega_2)=\int_{-\pi}^{\pi}\omega_1(z)\omega_2(z)\,\mathrm{d}z$, $\|\omega_1\|_2=$

$(\omega_1, \omega_1)^{1/2}$, where ω_1, ω_2 are two elements of $L^2([-\pi, \pi])$. Finally, various control theoretic concepts used in our development are defined in Appendix A.

2.2. Stability analysis of the linearized Kuramoto–Sivashinsky equation

The objective of this subsection is to compute the values of ν, for which the eigenvalues of the linearization of the system of Eqs. (1)–(3) around $x(z, t) = 0$ cross the imaginary axis (see also [10] for a similar analysis). To this end, we compute the linearization of the system of Eq. (1) around $x(z, t) = 0$, which takes the form:

$$\frac{\partial x}{\partial t} = -\nu \frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} \tag{5}$$

and consider the corresponding eigenvalue problem:

$$\mathcal{A}\phi_n = -\nu \frac{\partial^4 \phi_n}{\partial z^4} - \frac{\partial^2 \phi_n}{\partial z^2} = \mu_n \phi_n, \quad n = 1, \dots, \infty \tag{6}$$

subject to

$$\frac{\partial^j \phi_n}{\partial z^j}(-\pi, t) = \frac{\partial^j \phi_n}{\partial z^j}(+\pi, t), \quad j = 0, \dots, 3, \tag{7}$$

where μ_n denotes an eigenvalue and ϕ_n denotes an eigenfunction. A direct computation of the solution of the above eigenvalue problem yields $f_0 = 0$ and $f_n = -\nu n^4 + n^2$ (f_n is an eigenvalue of multiplicity two) with eigenfunctions $\psi_0(z) = 1/\sqrt{2\pi}$, and $\phi_n(z) = (1/\sqrt{\pi})\sin(nz)$ and $\psi_n(z) = (1/\sqrt{\pi})\cos(nz)$, $n = 1, \dots, \infty$. Clearly, a pair of eigenvalues of the system of Eq. (5) crosses the imaginary axis when

$$\nu = \frac{1}{n^2}, \quad n = 1, \dots, \infty. \tag{8}$$

Apparently, the smallest value of ν, for which the $x(z, t) = 0$ solution of the system of Eq. (5) is about to become unstable is $\nu = 1$. Clearly, when $1/n^2 > \nu > 1/(n+1)^2$, the system of Eq. (5) has $2n$ positive eigenvalues.

The above stability analysis implies that the spatially uniform steady-state, $x(z, t) = 0$, of the nonlinear system of Eqs. (1)–(3) is locally unstable when $\nu < 1$. Instead, there is a generation of *stable* spatially non-uniform stationary solutions as well as spatially non-uniform periodic solutions, while for very small values of ν no stable solutions exist and the system of Eq. (2) exhibits chaotic behavior (the reader may refer to [10,20] for detailed characterizations of the solution patterns for various ranges of values of ν).

3. Feedback control

In this section, we synthesize state and output feedback controllers that stabilize the system of Eqs. (2) and (3) at $x(z, t) = 0$. We begin with state feedback control, and we will continue with dynamic output feedback control.

3.1. State feedback control

We assume that measurements of the state $x(z, t)$ are available at all positions and times (this assumption will be removed below) and use Galerkin's method to derive ODE approximations of the system of Eq. (2) that capture the dynamics of the unstable modes. Expanding the solution of the system of Eq. (2) in an infinite series in terms of the eigenfunctions of the operator of Eq.(6), we obtain

$$x(z,t)=\sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z)+\sum_{n=0}^{\infty}\beta_n(t)\psi_n(z), \tag{9}$$

where $\alpha_n(t)$, $\beta_n(t)$ are time-varying coefficients and $\phi_n(z)=(1/\sqrt{\pi})\sin(nz)$, $\psi_0(z)=1/\sqrt{2\pi}$ and $\psi_n(z)=(1/\sqrt{\pi})\cos(nz)$. Substituting the above expansion for the solution, $x(z,t)$, into the system of Eq. (2) and taking the inner product in $L^2([-\pi,\pi])$ with the adjoint eigenfunctions, $\phi_n^*(z)=(1/\sqrt{\pi})\sin(nz)$, $\psi_0^*(z)=1/\sqrt{2\pi}$, $\psi_n^*(z)=(1/\sqrt{\pi})\cos(nz)$ of the operator of Eq. (6) (note that the operator of Eq. (6) subject to the boundary condition of Eq. (7) is self-adjoint, i.e., $\phi_n^*(z)=\phi_n(z)$, $\psi_0^*(z)=\psi_0(z)$, $\psi_n^*(z)=\psi_n(z)$, $n=1,\dots,\infty$), the following system of infinite ODEs is obtained:

$$\begin{aligned}\dot{\alpha}_n&=(-\nu n^4+n^2)\alpha_n-f_{n\alpha}+\sum_{i=1}^{m}b_{i\alpha}^n u_i(t),\quad n=1,\dots,\infty,\\ \dot{\beta}_n&=(-\nu n^4+n^2)\beta_n-f_{n\beta}+\sum_{i=1}^{m}b_{i\beta}^n u_i(t),\quad n=0,\dots,\infty,\end{aligned} \tag{10}$$

where

$$\begin{aligned}f_{n\alpha}&=\int_{-\pi}^{\pi}\phi_n(z)\left(\sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z)+\sum_{n=0}^{\infty}\beta_n(t)\psi_n(z)\right)\left(\sum_{n=1}^{\infty}\alpha_n(t)\frac{\mathrm{d}\phi_n}{\mathrm{d}z}(z)+\sum_{n=0}^{\infty}\beta_n(t)\frac{\mathrm{d}\psi_n}{\mathrm{d}z}(z)\right)\mathrm{d}z,\\ f_{n\beta}&=\int_{-\pi}^{\pi}\psi_n(z)\left(\sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z)+\sum_{n=0}^{\infty}\beta_n(t)\psi_n(z)\right)\left(\sum_{n=1}^{\infty}\alpha_n(t)\frac{\mathrm{d}\phi_n}{\mathrm{d}z}(z)+\sum_{n=0}^{\infty}\beta_n(t)\frac{\mathrm{d}\psi_n}{\mathrm{d}z}(z)\right)\mathrm{d}z,\\ b_{i\alpha}^n&=\int_{-\pi}^{\pi}\phi_n(z)b_i(z)\mathrm{d}z,\quad n=1,\dots,\infty,\quad b_{i\beta}^n=\int_{-\pi}^{\pi}\psi_n(z)b_i(z)\mathrm{d}z,\quad n=0,\dots,\infty,\end{aligned} \tag{11}$$

and $\frac{\mathrm{d}\phi_n}{\mathrm{d}z}(z)=\frac{n}{\sqrt{\pi}}\cos(nz)$, $\frac{\mathrm{d}\psi_n}{\mathrm{d}z}(z)=-\frac{n}{\sqrt{\pi}}\sin(nz)$.

Owing to its infinite-dimensional nature, the system of Eq. (10) cannot be directly used for the design of controllers that can be implemented in practice (i.e., the practical implementation of controllers which are designed on the basis of this system will require the computation of infinite sums which cannot be done by a computer). Instead, we will base the controller design on linear finite-dimensional approximations of this system that capture the dynamics of the unstable modes. Specifically, in order to illustrate the proposed controller design method, we initially assume that $0.25<\nu<1$ (which, according to the stability analysis of the previous section, implies that two eigenvalues of the linearized system of Eq. (5) are unstable) and rewrite the system of Eq. (10) in the following form:

$$\begin{aligned}&\dot{\beta}_0=\sum_{i=1}^{m}b_{i\beta}u_i(t),\quad \dot{\alpha}_1=(-\nu+1)\alpha_1-f_{1\alpha}+\sum_{i=1}^{m}b_{i\alpha}^1u_i(t),\qquad \dot{\beta}_1=(-\nu+1)\beta_1-f_{1\beta}+\sum_{i=1}^{m}b_{i\beta}^n u_i(t),\\ &\dot{\alpha}_n=(-\nu n^4+n^2)\alpha_n-f_{n\alpha}+\sum_{i=1}^{m}b_{i\alpha}^n u_i(t),\quad n=2,\dots,\infty,\\ &\dot{\beta}_n=(-\nu n^4+n^2)\beta_n-f_{n\beta}+\sum_{i=1}^{m}b_{i\beta}^n u_i(t),\quad n=2,\dots,\infty.\end{aligned} \tag{12}$$

Performing a linearization of the above system around the origin and neglecting the differential equations corresponding to stable modes, we obtain

$$\dot{\beta}_0 = \sum_{i=1}^{m} b_{i\beta}^0 u_i(t), \quad \dot{\alpha}_1 = (-\nu+1)\alpha_1 + \sum_{i=1}^{m} b_{i\alpha}^1 u_i(t), \quad \dot{\beta}_1 = (-\nu+1)\beta_1 + \sum_{i=1}^{m} b_{i\beta}^1 u_i(t). \tag{13}$$

To stabilize the above system, we need to use three control actuators. This is because two control actuators are needed to stabilize the two identical unstable eigenvalues of the open-loop system, while a third control actuator is needed to ensure that conservation of mass is satisfied, i.e., $\int_{-\pi}^{\pi} x(z,t)\mathrm{d}z \equiv 0$ (or equivalently $\dot{\beta}_0 \equiv 0$ at steady state). Using the following formulas for the computation of the control actions:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} b_{1\beta}^0 & b_{2\beta}^0 & b_{3\beta}^0 \\ b_{1\alpha}^1 & b_{2\alpha}^1 & b_{3\alpha}^1 \\ b_{1\beta}^1 & b_{2\beta}^1 & b_{3\beta}^1 \end{bmatrix}^{-1} \begin{bmatrix} -\gamma_1 & 0 & 0 \\ 0 & -\gamma_2+\nu-1 & 0 \\ 0 & 0 & -\gamma_3+\nu-1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \alpha_1 \\ \beta_1 \end{bmatrix}, \tag{14}$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$ and substituting Eq. (14) into the system of Eq. (12), we obtain the following system:

$$\begin{aligned} &\dot{\beta}_0 = -\gamma_1\beta_0, \quad \dot{\alpha}_1 = -\gamma_2\alpha_1, \quad \dot{\beta}_1 = -\gamma_3\beta_1, \\ &\dot{\alpha}_n = (-\nu n^4 + n^2)\alpha_n + \begin{bmatrix} b_{1\alpha}^n & b_{2\alpha}^n & b_{3\alpha}^n \end{bmatrix} \\ &\quad \times \begin{bmatrix} b_{1\beta}^0 & b_{2\beta}^0 & b_{3\beta}^0 \\ b_{1\alpha}^1 & b_{2\alpha}^1 & b_{3\alpha}^1 \\ b_{1\beta}^1 & b_{2\beta}^1 & b_{3\beta}^1 \end{bmatrix}^{-1} \begin{bmatrix} (-\gamma_1)\beta_0 \\ (-\gamma_2+\nu-1)\alpha_1 \\ (-\gamma_3+\nu-1)\beta_1 \end{bmatrix}, \quad n = 2, \dots, \infty, \\ &\dot{\beta}_n = (-\nu n^4 + n^2)\beta_n + \begin{bmatrix} b_{1\beta}^n & b_{2\beta}^n & b_{3\beta}^n \end{bmatrix} \begin{bmatrix} b_{1\beta}^0 & b_{2\beta}^0 & b_{3\beta}^0 \\ b_{1\alpha}^1 & b_{2\alpha}^1 & b_{3\alpha}^1 \\ b_{1\beta}^1 & b_{2\beta}^1 & b_{3\beta}^1 \end{bmatrix}^{-1} \begin{bmatrix} (-\gamma_1)\beta_0 \\ (-\gamma_2+\nu-1)\alpha_1 \\ (-\gamma_3+\nu-1)\beta_1 \end{bmatrix}, \quad n = 2, \dots, \infty \end{aligned} \tag{15}$$

which is exponentially stable. This implies that the system of Eq. (12) under the control law of Eq. (14) is locally (for sufficiently small initial conditions $\beta_0(0), \alpha_1(0), \beta_1(0), \alpha_2(0), \beta_2(0), \dots$) exponentially stable, because its linearization around $x(z,t) = 0$ (i.e., system of Eq. (15)) is exponentially stable (see [17], Theorem 5.1.1). The fact that the controller of Eq. (14) can *globally* (i.e., for any initial condition in $L^2([-\pi,\pi])$) stabilize the zero solution of the Kuramoto–Sivashinsky equation with $0.25 < \nu < 1$ will be established in Theorem 1 below.

Now, we assume that $\frac{1}{9} < \nu < \frac{1}{4}$ and rewrite the system of Eq. (10) as follows:

$$\begin{aligned} &\dot{\beta}_0 = \sum_{i=1}^{m} b_{i\beta}^0 u_i(t), \quad \dot{\alpha}_1 = (-\nu+1)\alpha_1 - f_{1\alpha} + \sum_{i=1}^{m} b_{i\alpha}^1 u_i(t), \quad \dot{\beta}_1 = (-\nu+1)\beta_1 - f_{1\beta} + \sum_{i=1}^{m} b_{i\beta}^1 u_i(t), \\ &\dot{\alpha}_2 = (-16\nu+4)\alpha_2 - f_{2\alpha} + \sum_{i=1}^{m} b_{i\alpha}^2 u_i(t), \quad \dot{\beta}_2 = (-16\nu+4)\beta_2 - f_{2\beta} + \sum_{i=1}^{m} b_{i\beta}^2 u_i(t), \\ &\dot{\alpha}_n = (-\nu n^4 + n^2)\alpha_n - f_{n\alpha} + \sum_{i=1}^{m} b_{i\alpha}^n u_i(t), \quad n = 3, \dots, \infty, \\ &\dot{\beta}_n = (-\nu n^4 + n^2)\beta_n - f_{n\beta} + \sum_{i=1}^{m} b_{i\beta}^n u_i(t), \quad n = 3, \dots, \infty. \end{aligned} \tag{16}$$

One can easily check that, when $\nu = 0.2$, the above system is not controllable (see Appendix A for definition) at the origin for any selection of $b_1(z), b_2(z), b_3(z)$ and any control $u_1(t), u_2(t), u_3(t)$. This is because the first two unstable eigenvalues (which have multiplicity 2) become identical, i.e., $\mu_1 = \mu_2 = 0.8$, and thus, the system of Eq. (5) possesses four identical unstable eigenvalues, which implies that the use of three control actuators does not suffice to stabilize the system for all $\nu > \frac{1}{9}$. Studying further the structure of the eigenspectrum of $\mathcal{A}$, it can be shown that the maximum number of eigenvalues of $\mathcal{A}$ which are identical, for any ν, is four, and thus, the use of five control actuators is necessary and sufficient for stabilizing any controllable finite-dimensional approximation of the system of Eq. (10) with linear state feedback (four control actuators are needed due to the presence of four identical unstable eigenvalues and one control actuator is needed to ensure that the conservation of mass condition, $\int_{-\pi}^{\pi} x(z,t)\,\mathrm{d}z = 0$, is satisfied at steady state).

Theorem 1 that follows provides a linear state feedback control algorithm that uses five control actuators to globally exponentially stabilize the zero solution of the Kuramoto–Sivashinsky equation, for any value of ν.

Theorem 1. *Let the number of unstable eigenvalues of the system of Eq.* (5) *be l (i.e.,* $\nu > 1/(l+1)^2$), *and assume that the* $(2l+1)-$ *dimensional system*:

$$\dot{\alpha}_u = A_u \alpha_u + B_u u \tag{17}$$

where $\alpha_{\mathrm{u}} = [\beta_0 \;\; \alpha_1 \;\; \beta_1 \;\; \alpha_2 \;\; \beta_2 \;\; \dots \;\; \alpha_l \;\; \beta_l]^{\mathrm{T}}$ *and*

$$A_u = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\nu+1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\nu+1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -l^4\nu+l^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -l^4\nu+l^2 \end{bmatrix}, \quad B_u = \begin{bmatrix} b_{1\beta}^0 & b_{2\beta}^0 & \cdots & b_{5\beta}^0 \\ b_{1\alpha}^1 & b_{2\alpha}^1 & \cdots & b_{5\alpha}^1 \\ b_{1\beta}^1 & b_{2\beta}^1 & \cdots & b_{5\beta}^1 \\ \vdots & \vdots & \cdots & \vdots \\ b_{1\alpha}^l & b_{2\alpha}^l & \cdots & b_{5\alpha}^l \\ b_{1\beta}^l & b_{2\beta}^l & \cdots & b_{5\beta}^l \end{bmatrix} \tag{18}$$

is controllable in the sense that there exists a matrix K so that the matrix $A_u + B_u K$ *is Hurwitz (see Appendix A for definition). Then, the state feedback controller*:

$$u = K\alpha_u \tag{19}$$

ensures that the $x(z,t) = 0$ *solution of the closed-loop system* (*Eqs.* (2)–(19)) *is exponentially stable, for any initial condition in* $L^2([-\pi,\pi])$ *and any* $\nu > 1/(l+1)^2$.

Remark 1. *To show that the maximum number of eigenvalues of* $\mathcal{A}$ *which are identical, for any* ν, *is four, consider the expression for the nth eigenvalue of* $\mathcal{A}$, $\mu_n = -\nu n^4 + n^2$ *and recall that the multiplicity of each* μ_n *is two. This implies that we need to show that:* (a) *there exists at least one value of* ν *for which* $\mu_n = \mu_{n+l}$ *where l is a positive integer, and* (b) *there exists no* ν *for which there are* $\kappa \geq 3$ *eigenvalues* μ_n *that are identical. To show the first part, consider* $\nu = 0.2$ *in which case* $\mu_1 = \mu_2 = 0.8$, *and thus, four eigenvalues are identical for* $\nu = 0.2$. *For the second part, we need to show that* $\mu_n = \mu_{n+l_1} = \mu_{n+l_2} = \cdots = \mu_{n+l_\kappa}$ *where* $l_1, l_2, \dots, l_\kappa$ *are distinct positive integers. However, we have that* $\mu_n = \mu_{n+l_1}$ *if and only if* $\nu = 1/[(n+l_1)^2 + n^2]$, $\mu_n = \mu_{n+l_2}$ *if and only if* $\nu = 1/[(n+l_2)^2 + n^2]$, *and, in general,* $\mu_n = \mu_{n+l_\kappa}$ *if and only if* $\nu = 1/[(n+l_\kappa)^2 + n^2]$. *This means that, for a given* n, $\mu_n = \mu_{n+l_1} = \mu_{n+l_2} = \cdots = \mu_{n+l_\kappa}$ *if and only if* $l_1 = l_2 = \cdots = l_\kappa = l$. *Therefore, the maximum number of identical eigenvalues of* $\mathcal{A}$, *for any* ν, *is four.*

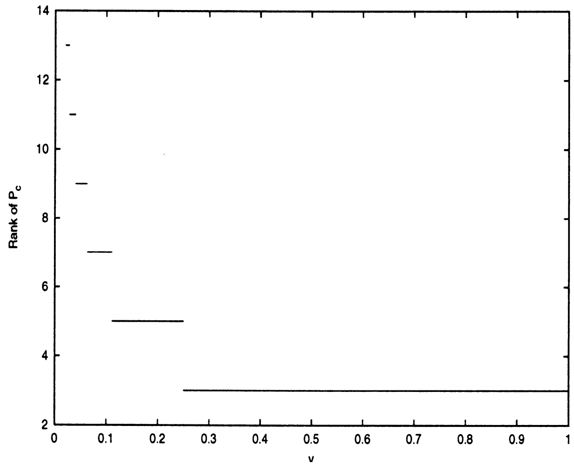

Fig. 1. Plot of rank of controllability matrix $P_c := [B_u | A_u B_u | A_u^2 B_u | \ldots | A_u^{n-1} B_u]$ versus ν — five control actuators are used.

Remark 2. *The requirement of Theorem* 1 *that the pair* $[A_u\, B_u]$ *is controllable depends on the structure of the matrix* B_u*, and therefore, on the shape of the five actuator distribution functions* $b_i(z)$*. Fig.* 1 *shows a plot of the rank of the controllability matrix* $P_c := [B_u | A_u B_u | A_u^2 B_u | \cdots | A_u^{n-1} B_u]$ *versus* ν*, for* $\nu \geq \frac{1}{49}$*, when five control actuators are used with distribution functions* $b_1(z) = 1.0/\sqrt{2\pi}$, $b_2(z) = \delta(z - \frac{\pi}{2})$, $b_3(z) = \delta(z - 0.0)$, $b_4(z) = \delta(z + \frac{\pi}{4})$ *and* $b_5(z) = \delta(z - \frac{\pi}{3})$*. It is clear that the rank of* P_c *is equal to the dimension of the square matrix* A_u *which implies that the pair* $[A_u \;\; B_u]$ *is controllable for* $\nu \geq \frac{1}{49}$ *(i.e., when at most 13 eigenvalues are in the closed right-half of the complex plane). This shows that it is possible to stabilize the zero solution of the KSE for a broad range of values of* ν *using five control actuators. For each* ν*, the computation of the gain K so that the matrix* $A_u + B_u K$ *is Hurwitz can be readily done by utilizing the subroutine "place" of the mathematical software* MATLAB. *However, when* $\nu < \frac{1}{49}$*, we have found that the pair* $[A_u B_u]$ *becomes uncontrollable for most of the values of* ν*. In this case, it is impossible to stabilize the zero solution of the Kuramoto–Sivashinsky equation with the above set of five control actuators. Potential remedies to such a problem are, either to change the location and shape of the five control actuators or to use more than five control actuators, to obtain a pair* $[A_u\; B_u]$ *which is controllable, and for which, there exists a matrix K of dimension* $J \times (2l + 1)$ *where* $J > 5$ *that makes the matrix* $A_u + B_u K$ *Hurwitz.*

Remark 3. *The exponential stability of the closed-loop system ensures a certain degree of robustness with respect to sufficiently small disturbances and uncertainty in process parameters (i.e.,* ν, $b_{i\alpha}^n$, $b_{i\beta}^n$*). The problem of designing robust controllers that explicitly account and compensate for the presence of uncertainty will not be addressed in this paper.*

Proof of Theorem 1. Under the controller of Eq. (19), the closed-loop system takes the form:

$$\frac{\partial x}{\partial t} = -\nu \frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z} + bK\alpha_u, \tag{20}$$

where $b = [b_1\, b_2\, b_3\, b_4\, b_5]$. Applying Galerkin's method to the above system, the following infinite set of ordinary differential equations can be obtained:

$$\begin{aligned}&\dot{\alpha}_u = A_u \alpha_u + B_u K \alpha_u - \tilde{f}, \quad \dot{\alpha}_n = (-\nu n^4 + n^2)\alpha_n - f_{n_\alpha} + b_\alpha^n K \alpha_u, \quad n = l+1, \ldots, \infty,\\ &\dot{\beta}_n = (-\nu n^4 + n^2)\beta_n - f_{n\beta} + b_\beta^n K \alpha_u, \quad n = l+1, \ldots, \infty,\end{aligned} \tag{21}$$

where $\tilde{f} = [0\ f_{1\alpha}\ f_{1\beta}\ f_{2\alpha}\ f_{2\beta}\ \dots\ f_{l\alpha}\ f_{l\beta}]^{\mathrm{T}}$, $b_{\alpha}^{n} = [b_{1\alpha}^{n}\ b_{2\alpha}^{n}\ \dots\ b_{5\alpha}^{n}]$ and $b_{\beta}^{n} = [b_{1\beta}^{n}\ b_{2\beta}^{n}\ \dots\ b_{5_\beta}^{n}]$. Computing the linearization of the above system around $x(z,t)=0$, we obtain the following linear system:

$$\dot{\alpha}_u = (A_u + B_u K)\alpha_u, \quad \dot{\alpha}_n = (-\nu n^4 + n^2)\alpha_n + b_{\alpha}^{n} K \alpha_u, \quad n = l+1, \dots, \infty,$$
$$\dot{\beta}_n = (-\nu n^4 + n^2)\beta_n + b_{\beta}^{n} K \alpha_u, \quad n = l+1, \dots, \infty, \tag{22}$$

which is exponentially stable for $\nu > 1/(l+1)^2$ (it follows from the lower triangular structure of the system and the assumption that the matrix $A_u + B_u K$ is Hurwitz). Owing to the exponential stability of the system of Eq. (22), there exists a positive constant c such that the linear closed-loop operator:

$$\tilde{\mathcal{A}}x = -\nu \frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} + bK \left[\int_{-\pi}^{\pi} \psi_0(z) x(z,t)\,\mathrm{d}z \int_{-\pi}^{\pi} \phi_1(z) x(z,t)\,\mathrm{d}z \int_{-\pi}^{\pi} \psi_1(z) x(z,t)\,\mathrm{d}z \dots \right.$$
$$\left. \int_{-\pi}^{\pi} \phi_l(z) x(z,t)\,\mathrm{d}z \int_{-\pi}^{\pi} \psi_l(z) x(z,t)\,\mathrm{d}z \right]^{\mathrm{T}} \tag{23}$$

satisfies $(x, \tilde{\mathcal{A}}x) \le -c\|x\|_2^2$. Now, multiplying the closed-loop system of Eq. (20) by x and integrating from $[-\pi, \pi]$, we obtain:

$$\int_{-\pi}^{\pi} x \frac{\partial x}{\partial t}\ \mathrm{d}z = \int_{-\pi}^{\pi} x \tilde{\mathcal{A}}\ x\,\mathrm{d}z - \int_{-\pi}^{\pi} x^2 \frac{\partial x}{\partial z}\ \mathrm{d}z. \tag{24}$$

Using the boundary conditions of Eq. (3), one can show that

$$\int_{-\pi}^{\pi} x^2 \frac{\partial x}{\partial z}\ \mathrm{d}z = 0 \tag{25}$$

which implies that Eq. (24) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||x||_2^2 = (x, \tilde{\mathcal{A}}x) \le -c||x||_2^2. \tag{26}$$

From the above inequality, the exponential stability of the solution, $x(z,t) = 0$, of the closed-loop system, for any initial condition in $L^2([-\pi, \pi])$, follows [31]. □

3.2. Dynamic output feedback control

In this subsection, we design a finite dimensional state observer that uses 5 measurements of the state $x(z,t)$, $y^m = [y_1^m\ y_2^m\ y_3^m\ y_4^m\ y_5^m]^{\mathrm{T}}$, where $y_j^m = \int_{-\pi}^{\pi} s_j(z) x(z,t)\,\mathrm{d}z$ ($s_j(z)$ is a function that depends on the shape of the measurement sensor (e.g., point/distributed sensing)), to produce estimates of α_u. We note that, similar to the case of control actuators, the largest number of measurements needed to obtain a stable observer for any observable (see Appendix A for definition) finite-dimensional approximation of the system of Eq. (10) is five. We then couple the state observer with the state feedback controller of Eq. (19), and establish that the resulting output feedback controller locally exponentially stabilizes the steady state, $x(z,t) = 0$, of the Kuramoto–Sivashinsky equation, for any ν (see Theorem 2 below).

Let l be the number of unstable eigenvalues and define $\bar{y}^m = [\bar{y}_1^m\ \bar{y}_2^m\ \bar{y}_2^m\ \bar{y}_3^m\ \bar{y}_4^m\ \bar{y}_5^m]^{\mathrm{T}} = S_u \bar{\alpha}_u$, where $\bar{\alpha}_u = [\bar{\beta}_0\ \bar{\alpha}_1\ \bar{\beta}_1\ \bar{\alpha}_2\ \bar{\beta}_2\ \dots\ \bar{\alpha}_l\ \bar{\beta}_l]^{\mathrm{T}}$, $\bar{y}_j^m = \int_{-\pi}^{\pi} s_j(z) \left(\sum_{n=1}^{l} \bar{\alpha}_n(t)\phi_n(z) + \sum_{n=0}^{l} \bar{\beta}_n(t)\psi_n(z) \right) \mathrm{d}z$, $j = 1, \dots, 5$, and S_u is a matrix whose explicit form is omitted for brevity. Suppose that the pair $[S_u A_u]$ is observable. Then, the following $(2l+1)$-dimensional linear system:

$$\dot{\bar{\alpha}}_u = A_u \bar{\alpha}_u + B_u u + L(y^m - \bar{y}^m) \tag{27}$$

is a local exponential observer for the system:

$$\dot{\alpha}_u = A_u\alpha_u + B_u u - \tilde{f} \tag{28}$$

provided that the eigenvalues of the matrix $A_u - LS_u$ are sufficiently far in the left-half of the complex plane (i.e., $A_u - LS_u = \frac{1}{\epsilon}\tilde{A}$, where ϵ is a sufficiently small positive parameter and $\tilde{A}$ is a Hurwitz matrix). The term local exponential observer for the system of Eq. (27) means that the exponential convergence of the estimates $\bar{\alpha}_u$ to the actual values α_u is guaranteed provided that the initial conditions of the systems of Eq. (27) and Eq. (28) are sufficiently small; see proof of Theorem 2 for details.

Combining the observer of Eq. (27) with the state feedback controller of Eq. (19), we obtain

$$\dot{\bar{\alpha}}_u = A_u\bar{\alpha}_u + B_u K\bar{\alpha}_u + L(y^m - \bar{y}^m), \quad u = K\bar{\alpha}_u. \tag{29}$$

Theorem 2 below establishes that the above output feedback controller locally exponentially stabilizes the $x(z,t) = 0$ solution of the Kuramoto–Sivashinsky equation.

Theorem 2. *Let the number of unstable eigenvalues of the system of Eq.* (5) *be* l (i.e., $\nu > l/(l+1)^2$), *and assume that the pairs* $[A_u\ \ B_u]$ *and* $[S_u\ \ A_u]$ *are controllable and observable, respectively. Suppose also that* $A_u - LS_u = (1/\epsilon)\tilde{A}$, *where* $\tilde{A}$ *is Hurwitz. Then, there exist* ϵ^*, δ *such that if* $\epsilon \in (0, \epsilon^*]$ *and* $\max\{|e_o(0)|, \|x_0\|_2\} \le \delta$, *where* $e_o = \bar{\alpha}_u - \alpha_u$, *the solution* $x(z,t) = 0$ *of Eq.* (2) *under the finite-dimensional dynamic output feedback controller of Eq. (29) is exponentially stable, for any* $\nu > 1/(l+1)^2$.

Proof. Under the control of Eq. (29), the closed-loop system takes the form:

$$\dot{\bar{\alpha}}_u = A_u\bar{\alpha}_u + B_u K\bar{\alpha}_u + L(y^m - \bar{y}^m), \quad \frac{\partial x}{\partial t} = -\nu\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z} + bK\bar{\alpha}_u. \tag{30}$$

Using the observer error vector $e_o = \bar{\alpha}_u - \alpha_u$, the above closed-loop system can be equivalently written as

$$\dot{e}_o = (A_u - LS_u)e_o + \tilde{f} + LD\alpha_s, \quad \frac{\partial x}{\partial t} = -\nu\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z} + bK(\alpha_u + e_o), \tag{31}$$

where $A_u - LS_u$ is a Hurwitz matrix, $\alpha_s = [\alpha_{l+1}\ \beta_{l+1}\ \ldots]^{\mathrm{T}}$ and D is a constant matrix. Performing a linearization of the above system, using that $A_u - LS_u = \frac{1}{\epsilon}\tilde{A}$, and multiplying by ϵ the $\dot{e}_o$ equation, we obtain:

$$\epsilon\dot{e}_o = \tilde{A}e_o + \epsilon LD\alpha_s \quad \frac{\partial x}{\partial t} = -\nu\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} + bK(\alpha_u + e_o) \tag{32}$$

The above system is in the standard singularly perturbed form [21], and possesses an exponentially stable ($\tilde{A}$ is Hurwitz) fast subsystem: $\mathrm{d}e_o/\mathrm{d}\tau = \tilde{A}e_o$, where $\tau = t/\epsilon$, and an exponentially stable (Theorem 1) slow subsystem: $\frac{\partial x}{\partial t} = -\nu\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} + bK\alpha_u$. This implies [21] that there exists an ϵ^* such that if $\epsilon \in (0, \epsilon^*]$, then the system of Eq. (32) is also exponentially stable. This, in turn, implies that there exists a δ such that if $\max\{|e_o(0)|, ||x_0||_2\} \le \delta$, then the system of Eq. (30) is exponentially stable ([17], Theorem 5.1.1). □

Remark 4. *From the proof of Theorem* 2, *it follows that the requirement* $\epsilon \in (0, \epsilon^*]$ *is due to the presence of the term* $LD\alpha_s$ *in the observer error dynamics* (*Eq.* (32)), *which appears because no restrictions are imposed on the shape of* $s_j(z)$, $j = 1, \ldots, 5$. *If* $s_j(z)$ *are chosen so that* $D = 0$, *then the matrix* $A_u - LS_u$ *only needs to be stable.*

Remark 5. *Following up on the results presented in Remark* 3, Fig. 2 *shows a plot of the rank of the observability matrix* $P_o := [S_u^{\mathrm{T}}|A_u^{\mathrm{T}}S_u^{\mathrm{T}}|A_u^{2\mathrm{T}}S_u^{\mathrm{T}}|\ldots|A_u^{n-1\mathrm{T}}S_u^{\mathrm{T}}]$ *versus* ν *when five measurement sensors are used with distribution functions* $s_1(z) = 1.0/\sqrt{2\pi}, s_2(z) = \delta(z - (\pi/2)), s_3(z) = \delta(z - 0.0), s_4(z) = \delta(z + (\pi/4))$ and $s_5(z) = \delta(z - (\pi/3))$. *It is clear that the pair* $[S_u A_u]$ *is observable for* $\nu \ge \frac{1}{49}$ *(i.e., when at most 13 eigenvalues are in the*

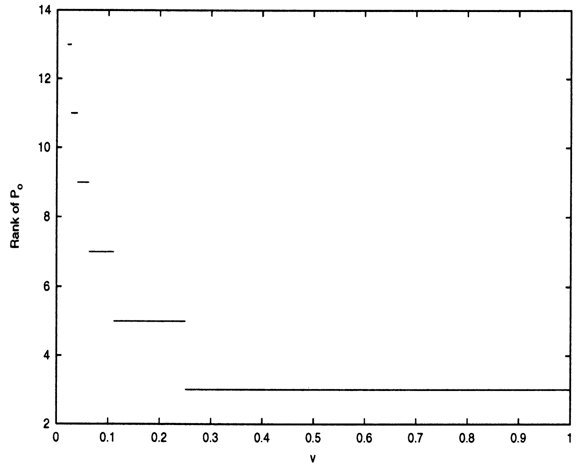

Fig. 2. Plot of rank of observability matrix $P_o := [S_u | A_u S_u | A_u^2 S_u | \cdots | A_u^{n-1} S_u]$ versus ν — five measurement sensors are used

closed right-half of the complex plane). *This shows that it is possible to design a local state observer of the form of Eq. (28) to estimate the unstable modes α_u for a broad range of values of ν using five measurement sensors; the computation of L so that the matrix $A_u - LS_u$ is Hurwitz can be done using* MATLAB. *However, when $\nu < \frac{1}{49}$, we have found that the pair $[S_u A_u]$ becomes unobservable for most of the values of ν. In this case, a stable observer can be designed either by changing the location and shape of the five measurement sensors or by using more than five measurement sensors, to obtain a pair $[S_u A_u]$ which is observable.*

4. Numerical results

In this section, we evaluate, through computer simulations, the ability of the proposed control algorithm to stabilize the system of Eqs. (2) and (3) at the steady state $x(z,t) = 0$. Two simulation runs were performed for $\nu = 0.4$ and 0.2. For the numerical simulation of the system, we used a 51-order nonlinear ordinary differential equation model obtained from the application of Galerkin's method to the system of Eq. (2) (the use of higher-order Galerkin approximations in simulating the system of Eqs. (2) and (3) led to identical numerical results, thereby implying that the following simulation runs are independent of the discretization). In all the simulation runs, the system is assumed to be at a spatially non-uniform initial condition, $x_0 = (2.5/\sqrt{\pi})\sum_{n=1}^{5}(\sin(nz) + \cos(nz)) \in L^2([-\pi, \pi])$.

In the first simulation run, we set $\nu = 0.4$ which implies that the linearized system of Eq. (5) possesses two eigenvalues in the right half of the complex plane. One distributed control actuator with $b_1(z) = 1.0/\sqrt{2\pi}$, two point control actuators placed at $z = -\pi/2$ and $z = \pi/6$ (i.e., $b_2(z) = \delta(z + \pi/2)$ and $b_3(z) = \delta(z - \pi/6)$), three distributed measurement sensors with shape functions $s_1(z) = 1/\sqrt{2\pi}$, $s_2(z) = 1/\sqrt{\pi}\sin(z)$, $s_3(z) = 1/\sqrt{\pi}\cos(z)$ were used to stabilize the system of Eqs. (2) and (3) at $x(z,t) = 0$. The control actions, $u_1(t), u_2(t), u_3(t)$, were computed from the formula of Eq. (29) with

$$K = \begin{bmatrix} -4.000 & -2.404 & 4.163 \\ 0.000 & 6.026 & -3.478 \\ 0.000 & 0.000 & -6.958 \end{bmatrix}, \quad L = \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ 0.00 & 2.00 & 0.00 \\ 0.00 & 0.00 & 3.00 \end{bmatrix}, \qquad (33)$$

and $\bar{\alpha}_u = [\bar{\beta}_0(t)\, \bar{\alpha}_1(t)\, \bar{\beta}_1(t)]^{\mathrm{T}}$. Fig 3 shows the closed-loop spatio-temporal profile of $x(z,t)$ and the profiles of the

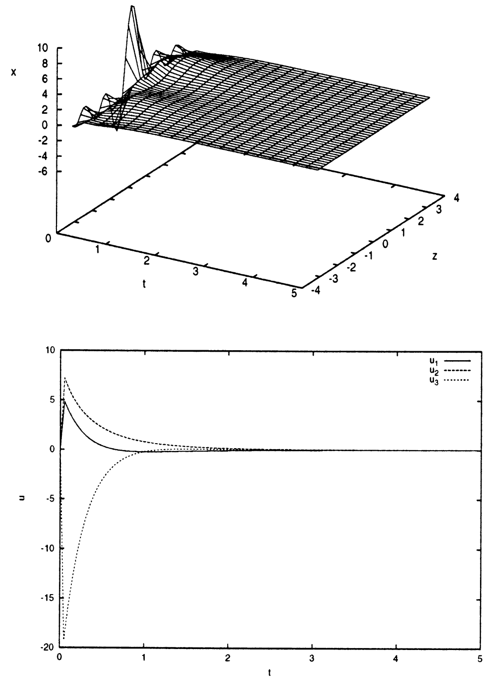

Fig. 3. Closed loop spatio-temporal profile of $x(z, t)$ (top figure) and manipulated input profiles (bottom figure), for $\nu = 0.4$.

three manipulated inputs. It is clear that the controller stabilizes the state of the system at $x(z, t) = 0$, indicating that the use of three control control actuator suffices to stabilize the system for $\nu = 0.4$.

In the second simulation run, we used $\nu = 0.2$ which implies that the linearized system of Eq. (5) possesses four identical unstable eigenvalues (i.e., $\mu_1 = \mu_2 = 0.8$; both μ_1, μ_2 are eigenvalues of multiplicity 2). Therefore, five control actuators and five measurement sensors of $x(z, t)$ are needed to stabilize the system at $x(z, t) = 0$. One distributed control actuator with $b_1(z) = 1.0/\sqrt{2\pi}$, four point control actuators placed at $z = \pi/2$, $z = 0.0$, $z = -\pi/4$ and $z = -\pi/2$, (i.e., $b_2(z) = \delta(z - (\pi/2))$, $b_3(z) = \delta(z - 0.0)$, $b_4(z) = \delta(z + (\pi/4))$ and $b_5(z) = \delta(z + (\pi/2))$), five spatially distributed sensors with shape functions $s_1(z) = 1/\sqrt{2\pi}$, $s_2(z) = (1/\sqrt{\pi})\sin(z)$, $s_3(z) = (1/\sqrt{\pi})\cos(z)$, $s_4(z) = (1/\sqrt{\pi})\sin(2z)$, $s_5(z) = (1/\sqrt{\pi})\cos(2z)$ were used to stabilize the system at $x(z, t) = 0$. The control actions, $u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)$, were computed from the formula of Eq. (29) with

$$K = \begin{bmatrix} -4.000 & 0.000 & 6.787 & 1.406 & -3.394 \\ 0.000 & -4.253 & -4.253 & 0.000 & 4.253 \\ 0.000 & 0.000 & -8.506 & -6.014 & 0.000 \\ 0.000 & 0.000 & 0.000 & 8.506 & 0.000 \\ 0.000 & 4.253 & -4.253 & -6.014 & 4.253 \end{bmatrix},$$

$$L = \begin{bmatrix} 1.50 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 2.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 2.50 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 3.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 3.50 \end{bmatrix}, \tag{34}$$

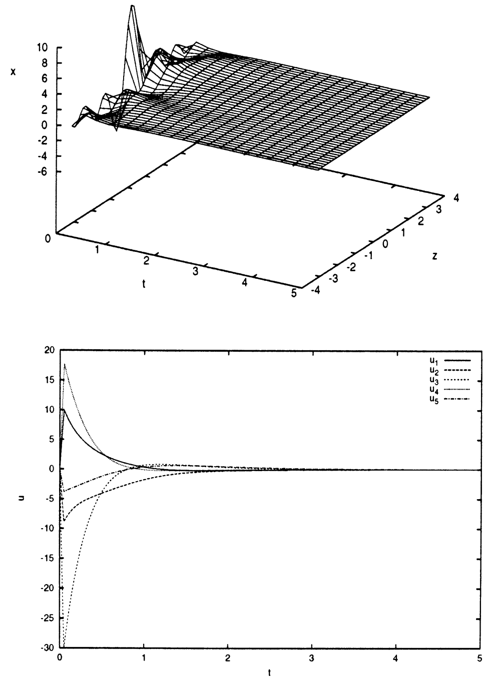

Fig. 4. Closed loop spatio-temporal profile of $x(z,t)$ (top figure) and manipulated input profiles (bottom figure), for $\nu = 0.2$.

and $\bar{\alpha}_u = [\bar{\beta}_0(t)\ \ \bar{\alpha}_1(t)\ \ \bar{\beta}_1(t)\ \ \bar{\alpha}_2(t)\ \ \bar{\beta}_2(t)]^{\mathrm{T}}$.The closed-loop profile of $x(z,t)$ and the profiles of the five manipulated inputs are displayed in Fig. 4. The stabilization of the state of the system at $x(z,t) = 0$, for $\nu = 0.2$, has been accomplished.

Remark 6. *We point out that in the case of point actuation (sensing) which influences (measures) the system at* z_0, *we approximate the function* $\delta(z - z_0)$ *by the finite value* $1/2\epsilon$ *in the interval* $[z_0 - \epsilon, z_0 + \epsilon]$ (*where* ϵ *is a small positive real number) and by zero elsewhere in the domain of definition of* z.

Appendix A. Definitions

- A square matrix A is said to be *Hurwitz* if all of its eigenvalues lie in the open left-half of the complex plane.
- Consider a linear dynamical system:

$$\dot{x} = Ax + Bu, \quad y = Sx, \tag{A.1}$$

  where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the vector of manipulated inputs, $y \in \mathbb{R}^p$ is the vector of measured outputs, A, B, S are matrices of appropriate dimensions and assume that all the eigenvalues of A are unstable. The system of Eq. (A.1) is said to be *controllable* if and only if the $n \times mn$ matrix $P_c := [B|AB|A^2B|\ldots|A^{n-1}B]$ has rank n. The system of Eq. (A.1) is said to be *observable* if and only if the $n \times pn$ matrix $P_o := [S^{\mathrm{T}}|A^{\mathrm{T}}S^{\mathrm{T}}|A^2\mathrm{T}S^{\mathrm{T}}|\ldots|A^{n-1\mathrm{T}}S^{\mathrm{T}}]$ has rank n. The reader may refer to [5] for more details on the concepts of controllability and observability for linear systems.

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