ELSEVIER

Computers and Chemical Engineering 24 (2000) 577–583

Computers & Chemical Engineering

www.elsevier.com/locate/compchemeng

Computation of optimal actuator locations for nonlinear controllers in transport-reaction processes

Charalambos Antoniades, Panagiotis D. Christofides *

Department of Chemical Engineering, University of California, Los Angeles, CA 90095-1592, USA

Abstract

This paper proposes a methodology for the computation of optimal locations of point control actuators for nonlinear feedback controllers in transport-reaction processes described by a broad class of quasi-linear parabolic partial differential equations (PDE). Initially, Galerkin's method is employed to derive finite-dimensional approximations of the PDE system which are used for the synthesis of stabilizing nonlinear state feedback controllers via geometric techniques. Then, the optimal location problem is formulated as the one of minimizing a meaningful cost functional that includes penalty on the response of the closed-loop system and the control action and is solved by using standard unconstrained optimization techniques. It is established that the solution to this problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal for the closed-loop infinite-dimensional system. The proposed methodology is successfully applied to a typical diffusion-reaction process. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Galerkin's method; Feedback linearization; Cost functionals; Diffusion–convection-reaction processes

1. Introduction

Transport-reaction processes with significant diffusive and convective phenomena are typically characterized by severe nonlinearities and spatial variations, and are naturally described by quasi-linear parabolic PDEs. Examples include tubular and packed-bed reactors, and chemical vapor deposition processes.

Parabolic PDE systems typically involve spatial differential operators whose eigenspectrum can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement (Friedman, 1976; Balas, 1979). This implies that the dynamic behavior of such systems can he approximately described by finite-dimensional systems. Therefore, the standard approach to the control of parabolic PDEs involves the application of Galerkin's method to the PDE system to derive ODE systems that describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for the synthesis of finite-dimensional controllers (Balas, 1979; Ray, 1981). A potential drawback of this approach is that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to complex controller design and high dimensionality of the resulting controllers. Motivated by this, recent efforts on control of parabolic PDE systems have focused on the problem of synthesizing low-order controllers on the basis of ODE models obtained through combination of Galerkin's method with approximate inertial manifolds (Christofides, 2000).

Even though the above methods allow to systematically design practically implementable nonlinear feedback controllers for transport-reaction processes, they do not deal with the problem of optimal actuator placement so that the desired control objectives are achieved with minimal energy use. The conventional approach to this problem is to select the actuator locations based on open-loop considerations to ensure that the necessary controllability requirements are satisfied. More recently, efforts have been made on the problem of integrating feedback control and optimal actuator placement for certain classes of linear distributed parameter systems including optimal placement of actuators for linear feedback control in parabolic PDEs (Xu, Warnitchai & Igusa, 1994; Demetriou, 1999) and in actively controlled structures

* Corresponding author.

0098-1354/00/$ - see front matter
PII: S0098-1354(00)00412-9

(Rao, Pan & Venkayya, 1991; Choe & Baruh, 1992). At this stage, there are no results available on optimal placement of actuators associated with nonlinear controllers for quasi-linear parabolic PDEs.

This paper proposes a methodology for the computation of optimal locations of point control actuators for nonlinear feedback controllers in transport-reaction processes described by a broad class of quasi-linear parabolic PDEs. Initially, Galerkin's method is employed to derive finite-dimensional approximations of the PDE system which are used for the synthesis of stabilizing nonlinear state feedback controllers via geometric techniques. Then, the optimal location problem is formulated as the one of minimizing a meaningful cost functional that includes penalty on the response of the closed-loop system and the control action and is solved by using standard unconstrained optimization techniques. It is established that the solution to this problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal for the closed-loop infinite-dimensional system. The proposed methodology is successfully applied to a typical diffusion-reaction process.

2. Preliminaries

We consider transport-reaction processes described by quasi-linear parabolic PDE systems of the form:

$$\frac{\partial \bar{x}}{\partial t} = A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2} + wb(z)u + f(\bar{x})$$

$$y_c^i = \int_\alpha^\beta c^i(z)k\bar{x}(z,t)\,dz, \quad i = 1, \ldots, l \tag{1}$$

subject to the boundary conditions:

$$C_1\bar{x}(\alpha,t) + D_1\frac{\partial \bar{x}}{\partial z}(\alpha,t) = R_1$$

$$C_2\bar{x}(\beta,t) + D_2\frac{\partial \bar{x}}{\partial z}(\beta,t) = R_2 \tag{2}$$

and the initial condition:

$$\bar{x}(z,0) = \bar{x}_0(z) \tag{3}$$

where $\bar{x}(z,t) = [\bar{x}_1(z,t)\cdots\bar{x}_n(z,t)]^T \in \mathbb{R}^n$ denotes the vector of state variables, $z\in[\alpha,\beta]\subset\mathbb{R}$ is the spatial coordinate, $t\in[0,\infty]$ is the time, $u = [u^1u^2\cdots u^l]\in\mathbb{R}^l$ denotes the vector of manipulated inputs, and $y_c^i\in\mathbb{R}$ denotes the ith controlled output. $\frac{\partial \bar{x}}{\partial z}, \frac{\partial^2 \bar{x}}{\partial z^2}$ denote the first- and second-order spatial derivatives of $\bar{x}$, $f(\bar{x})$ is a nonlinear vector function, w, k are constant vectors, A, B, C_1, D_1, C_2, D_2 are constant matrices, R_1, R_2 are column vectors, and $\bar{x}_0(z)$ is the initial condition. $b(z)$ is a known smooth vector function of z of the form $b(z) = [b^1(z)b^2(z)\cdots b^l(z)]$ where $b^i(z)$ describes how the control action $u^i(t)$ is distributed (point or distributing actuation) in the interval $[\alpha,\beta]$, and $c^i(z)$ is a known smooth function of z which is determined by the desired performance specifications. Whenever the control action enters the system at a single point z_0, with $z_0\in[\alpha,\beta]$ (i.e. point actuation), the function $b^i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_0-\epsilon, z_0+\epsilon]$, where ϵ is a small positive real number, and zero elsewhere in $[\alpha,\beta]$. Throughout the paper, we will use the order of magnitude notation $O(\epsilon)$. In particular, $\delta(\epsilon) = O(\epsilon)$ if there exist positive real numbers k_1 and k_2 such that: $|\delta(\varepsilon)| \le k_1|\varepsilon|, \forall|\varepsilon| < k_2$.

To precisely characterize the class of parabolic PDE systems considered in this work, we formulate the system of Eq. (1) as an infinite dimensional system in the Hilbert space $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$, with $\mathscr{H}$ being the space of n-dimensional vector functions defined on $[\alpha,\beta]$ that satisfy the boundary condition of Eq. (2), with inner product and norm:

$$(\omega_1,\omega_2) = \int_\alpha^\beta (\omega_1(z),\omega_2(z))_{\mathbb{R}^n}\,dz, \quad \|\omega_1\|_2 = (\omega_1,\omega_2)^{1/2} \tag{4}$$

where ω_1, ω_2 are two elements of $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function x on $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$ as:

$$x(t) = \bar{x}(z,t), \quad t>0, \quad z\in[\alpha,\beta] \tag{5}$$

the operator $\mathscr{A}$ in $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$ as:

$$\mathscr{A}x = A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2},$$

$$x\in D(\mathscr{A}) = \left\{x\in\mathscr{H}([\alpha,\beta];\mathbb{R}^n): C_1\bar{x}(\alpha,t) + D_1\frac{\partial \bar{x}}{\partial z}(\alpha,t) = R_1, \quad C_2\bar{x}(\beta,t) + D_2\frac{\partial \bar{x}}{\partial z}(\beta,t) = R_2\right\}$$

and the input and controlled output operators as:

$$\mathscr{B}u = wbu, \quad \mathscr{C}x = (c, kx) \tag{6}$$

the system of Eqs. (1)–(3) takes the form:

$$\dot{x} = \mathscr{A}x + \mathscr{B}u + f(x), \quad x(0) = x_0$$

$$y_c = \mathscr{C}x \tag{7}$$

where $f(x(t)) = f(\bar{x}(z,t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear term $f(x)$ is locally Lipschitz with respect to its arguments and satisfies $f(0) = 0$. For $\mathscr{A}$, the eigenvalue problem is defined as:

$$\mathscr{A}\phi_j = \lambda_j\phi_j, \quad j = 1, \ldots, \infty \tag{8}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction; the eigenspectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, is defined as the set of all eigenvalues of $\mathscr{A}$, i.e. $\sigma\mathscr{A} = \{\lambda_1, \lambda_2, \ldots, \}$. Assumption 1 that follows states that and a stable infinite-dimensional complement containing the remaining fast eigenvalues, and that the separation between the slow and fast eigenvalues of $\mathscr{A}$ is large. This assumption is satisfied by the majority of diffusion–convection-reaction processes (Christofides, 2000).

Assumption 1. *1.* $\text{Re}\{\lambda_1\} \geq \text{Re}\{\lambda_2\} \geq \cdots \geq \text{Re}\{\lambda_j\} \geq \cdots$, *where* $\text{Re}\{\lambda_j\}$ *denotes the real part of* λ_j.

2. $\sigma(\mathscr{A})$ *can be partitioned as* $\sigma(\mathscr{A}) = \sigma_1(\mathscr{A}) + \sigma_2(\mathscr{A})$, *where* $\sigma_1(\mathscr{A})$ *consists of the first m (with m finite) eigenvalues, i.e.* $\sigma_1(\mathscr{A}) = \{\lambda_1, \ldots, \lambda_m\}$, *and*

$$\frac{|\text{Re}\{\lambda_1\}|}{|\text{Re}\{\lambda_m\}|} = O(1).$$

3. $\text{Re}\,\lambda_{m+1} < 0$ *and* $\dfrac{|\text{Re}\{\lambda_m\}|}{|\text{Re}\{\lambda_{m+1}\}|} = O(\epsilon)$

where $\epsilon < 1$ *is a small positive number.*

3. Galerkin's method

In this section, we apply standard Galerkin's method to the system of Eq. (7) to derive an approximate finite-dimensional system. Let $\mathscr{H}_s$, $\mathscr{H}_f$ be modal subspaces of $\mathscr{A}$, defined as $\mathscr{H}_s = span\{\phi_1, \phi_2, \ldots, \}$ and $\mathscr{H}_f = span\{\phi_{m+1}, \phi_{m+2}, \ldots, \}$(the existence of $\mathscr{H}_s$, $\mathscr{H}_f$ follows from Assumption 1). Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq. (7) can be decomposed as:

$$x = x_s + x_f = P_s x + P_f x \tag{9}$$

Applying P_s and P_f to the system of Eq. (7) and using the above decomposition for x, the system of Eq. (7) can be equivalently written in the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathscr{A}_s x_s + \mathscr{B}_s u + f_s(x_s, x_f) \\ \frac{\partial x_f}{\partial t} &= \mathscr{A}_f x_f + \mathscr{B}_f u + f_f(x_s, x_f) \\ y_c &= \mathscr{C} x_s + \mathscr{C} x_f, \\ x_s(0) &= P_s x_0 = P_s x(0), \quad x_f(0) = P_f x(0) = P_f x_0 \end{aligned} \tag{10}$$

where $\mathscr{A}_s = P_s \mathscr{A} P_s$, $\mathscr{B} = P_s \mathscr{B}$, $f_s = P_s f$, $\mathscr{A}_f = P_f \mathscr{A} P_f$, $\mathscr{B} = P_f \mathscr{B}$, $f_f = P_f f$ and the notation $\dfrac{\partial x_f}{\partial t}$ is used to denote that the state x_f belongs in an infinite-dimensional space. In the above system, $\mathscr{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathscr{A}_s = diag\{\lambda_j\}$, $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are Lipschitz vector functions, and $\mathscr{A}_f$ is an unbounded differential operator which is exponentially stable (following from part 3 of Assumption 1 and the selection of $\mathscr{H}_s$, $\mathscr{H}_f$). Neglecting the fast and stable infinite-dimensional x_f-subsystem in the system of Eq. (10), the following m-dimensional slow system is obtained:

$$\begin{aligned} \frac{d\bar{x}_s}{dt} &= \mathscr{A}_s \bar{x}_s + \mathscr{B}_s u + f_s(\bar{x}_s, 0) \\ \bar{y}_c &= \mathscr{C} \bar{x}_s \end{aligned} \tag{11}$$

where the bar symbol in $\bar{x}_s$, $\bar{y}_c$ and $\bar{y}_m$ denotes that these variables are associated with a finite-dimensional system.

Remark 1. We note that the above model reduction procedure which led to the approximate ODE system of Eq. (10) can also be used, when empirical eigenfunctions of the system of Eq. (7) computed through Karhunen–Loéve expansion (Christofides, 2000) are used as basis functions in $\mathscr{H}_s$ and $\mathscr{H}_f$ instead of the eigenfunctions of $\mathscr{A}$.

4. Integrating nonlinear control and optimal actuator placement

4.1. Problem formulation

We address the problem of computing optimal locations of point control actuators associated with nonlinear state feedback control laws of the following general form:

$$u = \mathscr{F}(z_a, \bar{x}_s) \tag{12}$$

where $\mathscr{F}(z_a, \bar{x}_s)$ is a nonlinear vector function, and z_a denotes the vector of actuator locations so that the following properties are enforced in the closed-loop system: (a) exponential stability; and (b) near-optimality (in a sense made precise in theorem 1) with respect to a meaningful cost functional defined over the infinite time interval, that imposes penalty on the response of the closed-loop system and the control action.

4.2. Nonlinear controller synthesis

First, we address the problem of synthesizing nonlinear static state feedback control laws of the form of Eq. (12), that guarantee exponential stability of the closed-loop finite-dimensional system. To this end, we will need the following assumption (see Remark 3 below for a discussion on this assumption).

Assumption 2. *$l = m$ (i.e. the number of control actuators is equal to the number of slow modes), and the inverse of the matrix $\mathcal{B}_s$* exists.

Proposition 1 that follows provides the explicit formula for the state feedback controller that achieves the control objective.

Proposition 1. *Consider the finite-dimensional system of Eq.* (11) *for which Assumption* 2 *holds. Then, the state feedback controller*:

$$u = \mathcal{B}_s^{-1}((\Lambda_s - \mathcal{A}_s)\bar{x}_s - f_s(\bar{x}_s, 0)) \tag{13}$$

where Λ_s is a stable matrix, guarantees global exponential stability of the closed-loop finite-dimensional system.

Remark 2. The structure of the closed-loop finite-dimensional system under the controller of Eq. (13) has the following form:

$$\dot{x}_s = \Lambda_s \bar{x}_s \tag{14}$$

and thus, the response of this system depends only on the stable matrix Λ_s and the initial condition, $x_s(0)$, and is independent of the actuator locations.

Remark 3. The requirement $l = m$ is sufficient and not necessary, and it is made to simplify the solution of the controller synthesis problem. Full linearization of the closed-loop finite-dimensional system through coordinate change and nonlinear feedback can be achieved for any number of manipulated inputs (i.e. for any $l \in [1, m]$), provided that an appropriate set of involutivity conditions is satisfied by the corresponding vector fields of the system of Eq. (11) (Isidori, 1989).

Remark 4. We note that, owing to space limitations, we only focus here on the problem of computing optimal locations of control actuators for nonlinear state feedback controllers, and therefore, we assume throughout the paper that the state $\bar{x}$ of the PDE system of Eq. (1) is known; the solution to the problem of computing optimal locations of control actuators and measurement sensors for nonlinear output feedback controllers is presented in Antoniades and Christofides (2000).

4.3. Computation of optimal actuator locations

In this subsection, we compute the actuator locations so that the state feedback controller of Eq. (12) is near optimal for the full PDE system of Eq. (10) with respect to a meaningful cost functional which is defined over the infinite time interval and imposes a penalty on the response of the closed-loop system and the control action. To this end, we initially focus on the ODE system of Eq. (11) and consider the following cost functional:

$$J_s = \int_0^{\infty} (\bar{x}_s^T(x_s(0), t) Q_s \bar{x}_s(x_s(0), t) + u^T(\bar{x}_s(x_s(0), t), z_a) R u(\bar{x}_s(x_s(0), t), z_a))\, dt \tag{15}$$

where Q_s, and R are positive definite matrices. The cost of Eq. (15) is well-defined and meaningful since it imposes penalty on the response of the closed-loop finite-dimensional system and the control action. However, a potential problem of this cost is its dependence on the choice of a particular initial condition, $x_s(0)$, and thus, the solution to the optimal placement problem based on this cost may lead to actuator locations that perform very poorly for a large set of initial conditions. To eliminate this dependence and obtain optimality over a broad set of initial conditions, we follow Levine and Athans (1978) and consider an average cost over a set of m linearly independent initial conditions, $x_s^i(0)$, $i = 1, \ldots, m$ of the following form:

$$\hat{J}_{xs} = \frac{1}{m} \sum_{i=1}^{m} \int_0^{\infty} (\bar{x}_s^T(x_s^i(0), t) Q_s \bar{x}_s(x_s^i(0), t) + u^T(\bar{x}_s(x_s^i(0), t), z_a) R(\bar{x}_s(x_s^i(0), t), z_a))\, dt \tag{16}$$

Referring to the above cost, we first note that the penalty on the response of the closed-loop system:

$$\hat{J}_s = \frac{1}{m} \sum_{i=1}^{m} \int_0^{\infty} \bar{x}_s^T(x_s^i(0), t) Q_s \bar{x}_s(x_s^i(0), t)\, dt \tag{17}$$

is finite because the solution of the closed-loop system of Eq. (14) is exponentially stable by appropriate choice of Λ_s. Moreover, $\hat{J}_{xs}$ is independent of the actuator locations (Remark 2), and thus, the optimal actuator placement problem reduces to the one of minimizing the following cost which only includes penalty on the control action:

$$\hat{J}_{us} = \frac{1}{m} \sum_{i=1}^{m} \int_0^{\infty} u^T(\bar{x}_s(x_s^i(0), t), z_a) R u(\bar{x}_s(x_s^i(0), t), z_a)\, dt$$

$\hat{J}_{us}$ is a function of multiple variables, $z_a = [z_{a1} z_{a2}, \cdots z_{al}]$ and thus, it obtains its local minimum values when its gradient with respect to the actuator locations is equal to zero, i.e.

$$\frac{\partial \hat{J}_{us}}{\partial z_a} = \left[\frac{\partial \hat{J}_{us}}{\partial z_{a1}} \cdots \frac{\partial \hat{J}_{us}}{\partial z_{al}}\right]^T = [0 \cdots 0]^T \tag{18}$$

and $\nabla_{z_a z_a} \hat{J}_{us}(z_{am}) > 0$ where $\nabla_{z_a z_a} \hat{J}_{us}$ is the Hessian matrix of $\hat{J}_{us}$ and z_{am} is a solution of the system of nonlinear algebraic equations of Eq. (18) (which includes l equations with l unknowns). The solution z_{am} for which the above conditions are satisfied and $\hat{J}_{us}$ obtains its smallest value (global minimum) corresponds to the optimal actuator locations for the closed-loop finite-dimensional system. Theorem 1 that follows establishes that these locations are near-optimal for the closed-loop infinite-dimensional system (the proof is given in Antoniades and Christofides (2000).

Theorem 1. *Consider the infinite-dimensional system of Eq.* (10) *for which Assumption* 1 *holds, and the finite-dimensional system of Eq.* (11), *for which Assumption* 2 *holds. Then, there exists positive real numbers* μ_1, μ_2 and ϵ^* *such that if* $|x_s(0)| \leq \mu_1$, $\|x_f(0)\| \leq \mu_2$, *and* $\epsilon \in (0, \epsilon]$, *then the controller of Eq.* (13): (*a*) *guarantees exponential stability of the closed-loop infinite-dimensional system, and* (*b*) *the optimal locations of the point actuators obtained for the closed-loop finite-dimensional system are near-optimal for the closed-loop infinite-dimensional system in the sense that*:

$$\hat{J} = \frac{1}{m} \sum_{i=1}^{m} \int_0^{\infty} (x_s^T(x_s^i(0), t) Q_s x_s(x_s^i(0), t) + x_f^T(x_f^i(0), t) Q_f x_f(x_f^i(0), t) + u^T(x_s(x_s^i(0), t), z_a) R u(x_s(x_s^i(0), t), z_a))\, dt \rightarrow \hat{J}_s \quad \text{as} \quad \epsilon \rightarrow 0 \tag{19}$$

where Q_f *is an unbounded positive definite operator and* $\hat{J}$ *is the cost function associated with the controller of Eq.* (13) *and the infinite-dimensional system of Eq.* (10).

Remark 5. Note that even though the response of the closed-loop finite-dimensional system of Eq. (14) is independent of the actuator locations, the response of the closed-loop infinite-dimensional system does depend on the actuators locations. However, this dependence is scaled by ϵ, and therefore, it decreases as we increase the number of modes included in the finite-dimensional system used for controller design.

Remark 6. In general, the solution to the system of Eq. (18) can be computed through a combination of numerical integration techniques and multivariable Newton's method.

5. Application to a diffusion-reaction process

Consider a long, thin rod in a reactor. The reactor is fed with pure species A and a zeroth order exothermic catalytic reaction of the form A → B takes place on the rod. Since the reaction is exothermic, a cooling medium which is in contact with the rod is used for cooling. Under standard assumptions, the spatiotemporal evolution of the dimensionless rod temperature is described by the following parabolic PDE:

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T \, e^{-\frac{\gamma}{1+\bar{x}}} + \beta_U (b(z) u(t) - \bar{x}) - \beta_T \, e^{-\gamma}$$

subject to the boundary and initial conditions:

$$\bar{x}(0, t) = 0, \quad \bar{x}(\pi, t) = 0, \quad \bar{x}(z, 0) = \bar{x}_0(z) \tag{20}$$

where $\bar{x}$ denotes the dimensionless temperature in the reactor, β_T denotes a dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, $u(t)$ denotes the vector of manipulated inputs and $b(z)$ the vector of the corresponding actuator locations. The following typical values were given to the process parameters:

$$\beta_T = 50.0, \quad \beta_U = 2.0, \quad \gamma = 4.0 \tag{21}$$

For the above values, the operating steady state $\bar{x}(z, t) = 0$ is an unstable one (Fig. 1 shows the profile of evolution of open-loop rod temperature starting from initial conditions close to the steady state $\bar{x}(z, t) = 0$; the process moves to another stable steady state characterized by a maximum in the temperature profile in the center of the rod). Therefore, the control objective is to stabilize the rod temperature profile at the unstable steady state $\bar{x}(z, t) = 0$. The eigenvalue problem for the spatial differential operator of the process:

$$\mathcal{A}x = \frac{\partial^2 \bar{x}}{\partial z^2}, \quad x \in D(\mathcal{A}) = \{x \in \mathcal{H}([0, \pi]; \mathbb{R}): \bar{x}(0, t) = 0, \; \bar{x}(\pi, t) = 0\} \tag{22}$$

can he solved analytically and its solution is:

$$\lambda_j = -j^2, \quad \phi_j(z) = \bar{\phi}_j(z) = \sqrt{\frac{2}{\pi}} \sin(jz) \tag{23}$$

where λ_j, ϕ_j, $\bar{\phi}_j$ denote the eigenvalues, eigenfunctions and adjoined eigenfunctions of $\mathcal{A}$, respectively, and $j = 1, \ldots, \infty$.

In the remainder of this section, we use the proposed method to compute the optimal locations in the case of using two control actuators. Following Assumption 2, we use Galerkin's method to derive a second order ODE approximation of the PDE model which will be used for controller design and optimal actuator placement. The form of the approximate ODE system is given below:

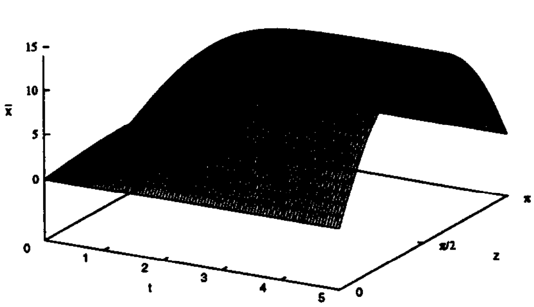

Fig. 1. Profile of evolution of open-loop rod temperature.

Table 1
Results for two control actuators

Case	Actuator locations	$\hat{J}_u$	$\hat{J}_x$	$\hat{J}$
Optimal	0.39π, 0.66π	0.8075	0.5332	1.3407
Linear	0.32π, 0.68π	0.8980	0.5414	1.4416
3	0.20π, 0.80π	5.2109	0.17957	7.0066
4	0.30π, 0.70π	0.9065	0.5608	1.5473

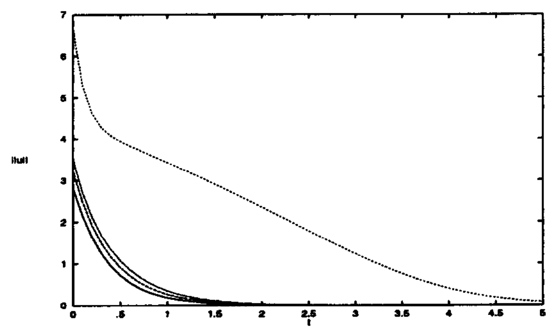

Fig. 2. Profile of the norm of the control action $\|u\|$, for the optimal case (solid line), the linearized case (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line) for $x(0) = \phi_1$.

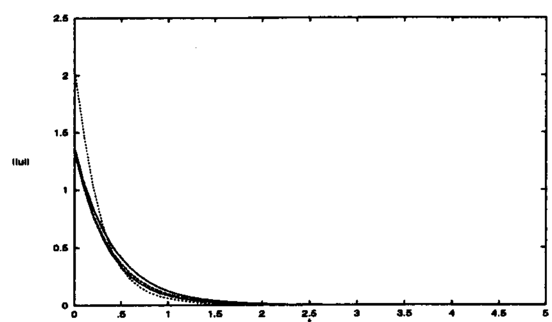

Fig. 3. Profile of the norm of the control action, $\|u\|$, for the optimal case (solid line), the linearized case (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line) for $x(0) = \phi_2$.

$$\begin{bmatrix} \dot{\bar{x}}_{s1} \\ \dot{\bar{x}}_{s2} \end{bmatrix} = \begin{bmatrix} \lambda_1 - \beta_U & 0 \\ 0 & \lambda_2 - \beta_U \end{bmatrix} \begin{bmatrix} \bar{x}_{s1} \\ \bar{x}_{s2} \end{bmatrix} + \beta_U \begin{bmatrix} \bar{\phi}_1(z_{a1}) & \bar{\phi}_1(z_{a2}) \\ \bar{\phi}_2(z_{a1}) & \bar{\phi}_2(z_{a2}) \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \begin{bmatrix} f_1(\bar{x}_s, 0) \\ f_2(\bar{x}_s, 0) \end{bmatrix} \tag{24}$$

where z_{a1} and z_{a2} are the locations of the two point actuators and the explicit forms of the terms $f_1(\bar{x}_s, 0)$ and $f_2(\bar{x}_s, 0)$ are omitted for brevity. For the system of Eq. (24), the nonlinear state feedback controller of Eq. (13) takes the form:

$$u = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = -\frac{1}{\beta_U} \begin{bmatrix} \bar{\phi}_1(z_{a1}) & \bar{\phi}_1(z_{a2}) \\ \bar{\phi}_2(z_{a1}) & \bar{\phi}_2(z_{a2}) \end{bmatrix}^{-1} * \left(\begin{bmatrix} \lambda_1 - \beta_U + \alpha & 0 \\ 0 & \lambda_2 - \beta_U + \beta \end{bmatrix} \begin{bmatrix} (\bar{x}_{s1}, \phi_1) \\ (\bar{x}_{s2}, \phi_2) \end{bmatrix} + \begin{bmatrix} (f_1(\bar{x}_s, 0), \phi_1) \\ (f_2(\bar{x}_s, 0), \phi_2) \end{bmatrix} \right) \tag{25}$$

Substituting the above controller into the system of Eq. (24), we obtain the following closed-loop ODE system:

$$\begin{bmatrix} \dot{\bar{x}}_{s1} \\ \dot{\bar{x}}_{s2} \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} \bar{x}_{s1} \\ \bar{x}_{s2} \end{bmatrix} \tag{26}$$

where α and β are positive real numbers. Since the response of the above system depends only on the parameters α, β and the initial condition $x_s(0)$, and is independent of the actuator locations, we compute the optimal actuator locations by minimizing the following cost functional, which only includes penalty on the control action:

$$\hat{J}_{us} = \frac{1}{2} \sum_{i=1}^{2} \int_0^{\infty} u^T(\bar{x}_s(x_s^i(0), t), z_a) R u(\bar{x}_s(x_s^i(0), t), z_a)\, dt$$

Using the following values for the parameters $\alpha = \beta = 1$, $x_s^1(0) = \phi_1$, and $x_s^2(0) = \phi_2$, and taking R, Q_s, Q_f to be unit matrices of appropriate dimensions, the optimal actuator locations were found to be: $z_{a1} = 0.39\pi$ and $z_{a2} = 0.66\pi$. Table 1 shows the values of the costs $\hat{J}_u$, $\hat{J}_x$, and $\hat{J}$ of the full-order closed-loop system under the state feedback controller of Eq. (25) in the case of optimal actuator placement, and for the sake of comparison, the values of these costs in the case of alternative actuator placements including optimal actuator placement based on the linearized system (second line).

The cost of the control action used to stabilize the system at $\bar{x}(z, t) = 0$ when the actuators are located at 0.39π and 0.66π is clearly smaller than in the other cases.

Figs. 2 and 3 show the norm of the control action, $\|u\|$, for $x(0) = \phi_1$ (Fig. 2) and $x(0) = \phi_2$ (Fig. 3), for the optimal case (solid line), the linearized case (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line). Clearly, the control action used for stabilization in the case of optimal actuator placement is smaller than all the other cases. Finally, Figs. 4 and 5 display the profiles of the evolution of the temperature of the rod, under state feedback control, for the opti-

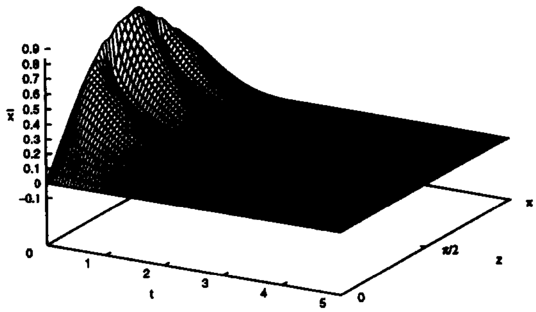

Fig. 4. Profile of evolution of rod temperature under state feedback control and optimal actuator locations for $x(0) = \phi_1$.

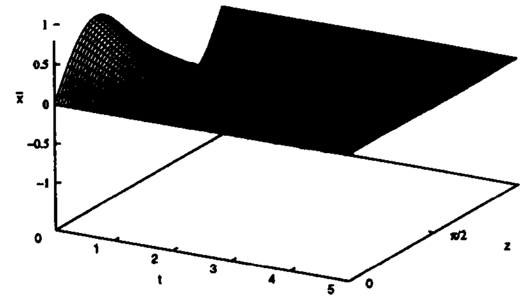

Fig. 5. Profile of evolution of rod temperature under state feedback control and optimal actuator locations for $x(0) = \phi_2$.

mal actuator locations, for $x(0) = \phi_1$ (Fig. 4) and for $x(0) = \phi_2$ (Fig. 5). In both cases, the nonlinear controllers achieve stabilization of the PDE system at $\bar{x}(z, t) = 0$.

References

Antoniades, C., & Christofides, P. D. (in press). Integrating nonlinear output feedback control and optimal actuator/sensor placement for transport-reaction processes. *Chemical Engineering Science* (submitted).

Balas, M. J. (1979). Feedback control of linear diffusion processes. *International Journal of Control, 29*, 523–533.

Choe, K., & Baruh, H. (1992). Actuator placement in structural control. *Journal of Guidance, 15*, 40–48.

Christofides, P. D. (in press). *Nonlinear and robust control of PDE systems: methods and applications to transport-reaction processes*. Boston: Birkhauser (in press).

Demetriou, M. A. (1999). Numerical investigation on optimal actuator/sensor location of parabolic PDEs. In: *Proceedings of the American control conference* (pp. 1722–1726). San Diego, CA.

Friedman, A. (1976). *Partial differential equations*. New York: Holt, Rinehart & Winston.

Isidori, A. (1989). *Non-linear control systems: an introduction* (2nd edition). New York: Springer-Verlag.

Levine, W. S., & Athans, M. (1978). On the determination of the optimal constant output feedback gains for linear multivariable systems. *IEEE Transactions on Automatic Control, 15*, 44–48.

Rao, S. S., Pan, T. S., & Venksyya, V. B. (1991). Optimal placement of actuators in actively controlled structures using genetic algorithms. *AIAA Journal, 29*, 942–943.

Ray, W. H. (1981). *Advanced process control*. New York: McGraw-Hill.

Xu, K., Warnitchal, P., & Igusa, T. (1994). Optimal placement and gains of sensors and actuators for feedback control. *Journal of Guidance & Control Dynamics, 17*, 929–934.