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Integrated optimal actuator/sensor placement and robust control of uncertain transport-reaction processes

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Abstract

This paper focuses on transport-reaction processes with unknown time-varying parameters and disturbances described by quasi-linear parabolic PDE systems, and addresses the problem of computing optimal actuator/sensor locations for robust nonlinear controllers. Initially, Galerkin's method is employed to derive finite-dimensional approximations of the PDE system which are used for the synthesis of robust nonlinear state feedback controllers via geometric and Lyapunov techniques and the computation of optimal actuator locations. The controllers enforce boundedness and uncertainty attenuation in the closed-loop system. The optimal actuator location problem is subsequently formulated as the one of minimizing a meaningful cost functional that includes penalty on the response of the closed-loop system and the control action. Owing to the boundedness of the state, the cost is defined over a finite-time interval (the final time is defined as the time needed for the process state to become smaller than the desired uncertainty attenuation limit), while the optimization is performed over a broad set of initial conditions and time-varying disturbance profiles. Subsequently, under the assumption that the number of measurement sensors is equal to the number of slow modes, we employ a standard procedure for obtaining estimates for the states of the approximate finite-dimensional model from the measurements. The optimal location of the measurement sensors is computed by minimizing a cost function of the estimation error in the closed-loop infinite-dimensional system. We show that the use of these estimates in the robust state feedback controller leads to a robust output feedback controller, which guarantees boundedness of the state and uncertainty attenuation in the infinite-dimensional closed-loop system, provided that the separation between the slow and the fast eigenvalues is sufficiently large. We also establish that the solution to the optimal actuator/sensor problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal in the sense that it approaches the optimal solution for the infinite-dimensional system as the separation between the slow and fast eigenvalues increases. The theoretical results are successfully applied to a typical diffusion-reaction process with nonlinearites and uncertainty to design a robust nonlinear output feedback controller and compute the optimal actuator/sensor locations for robust stabilization of an unstable steady state.

Keywords: Robust nonlinear control; Static output feedback control; Optimal actuator/sensor placement; Diffusion–convection-reaction processes

1. Introduction

Transport-reaction processes with significant diffusive and dispersive mechanisms are typically characterized by severe nonlinearities and spatial variations, and are naturally described by quasi-linear parabolic partial differential equations (PDEs). Examples include tubular and packed-bed reactors, as well as chemical vapor deposition and crystal growth processes. In addition to being nonlinear, parabolic PDE systems which model diffusion–convection-reaction processes are uncertain due to the presence of unknown or partially known process parameters and disturbances. Time-varying uncertain variables, if they are not appropriately accounted for in the controller design, may cause deterioration of the nominal closed-loop performance, and even lead to closed-loop instability.

Parabolic PDE systems typically involve spatial differential operators whose eigenspectrum can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement (Friedman, 1976; Balas, 1979; Ray, 1981; Christofides, 2001). This

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implies that the dynamic behavior of such systems can be approximately described by finite-dimensional systems. Therefore, the standard approach to the control of parabolic PDEs involves the application of Galerkin's method to the PDE system to derive ODE systems that describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for the synthesis of finite-dimensional controllers (Balas, 1979; Ray, 1981). A potential drawback of this approach is that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to complex controller design and high dimensionality of the resulting controllers. Motivated by this, research efforts on control of parabolic PDE systems over the last decade have focused on the problem of synthesizing low-order controllers on the basis of ODE models obtained through combination of Galerkin's method with approximate inertial manifolds (see the recent book of Christofides, 2001 for details and references). More recently, Lyapunov-based control methods for quasi-linear parabolic PDE systems with time-varying uncertain variables have been developed which lead to the synthesis of state (Christofides, 1998) and output (Christofides & Baker, 1999) feedback controllers that achieve closed-loop stability and uncertainty attenuation.

Even though the developed methods allow to systematically design practically-implementable nonlinear feedback controllers for transport-reaction processes using parabolic PDE systems, there is no work on the integration of nonlinear controller design with optimal placement of control actuators and measurement sensors for transport-reaction processes so that the desired control objectives are achieved with minimal energy use. Regarding the problem of optimal placement of control actuators, the conventional approach is to select the actuator locations based on open-loop considerations to ensure that the necessary controllability requirements for stabilization of the closed-loop system are satisfied. More recently, efforts have been made on the problem of integrating feedback control and optimal actuator placement for certain classes of *linear* distributed parameter systems including investigation of controllability measures and actuator placement in oscillatory systems (Arbel, 1981), optimal placement of actuators for linear feedback control of parabolic PDEs (Xu, Warnitchai, & Igusa, 1994; Demetriou, 1999) and of actively controlled structures (Rao, Pan, & Venkayya, 1991; Choe & Baruh, 1992).

On the other hand, the problem of selecting optimal locations for measurement sensors in linear distributed parameter systems has received very significant attention over the last 30 years. The essence of this problem is to use a finite number of measurements to compute the best estimate of the entire distributed state for all positions and times employing a state observer in the presence of measurement noise. Early efforts for the solution to this problem focused on linear PDE systems (Yu & Seinfeld, 1973; Chen & Seinfeld, 1975; Kumar & Seinfeld, 1978a; Omatu, Koide, & Soeda, 1978; Morari & O'Dowd, 1980) and the application of the results to optimal state estimation in tubular reactors (Colantuoni & Padmanabhan, 1977; Kumar & Seinfeld, 1978b; Harris, MacGregor, & Wright, 1980; Alvarez, Romagnoli, & Stefanopoulos, 1981; Waldraff, Dochain, Bourrel, & Magnus, 1998). The central idea to the solution involves the use of a spatial discretization scheme to obtain a lumped approximation of the original distributed parameter system followed by the formulation and solution of an optimal state estimation problem which involves computing sensor locations so that an appropriate functional that includes penalty on the estimation error and the measurement noise is minimized. Significant research efforts have also been made on the problem of optimal placement of controllers and sensors for various class of linear distributed parameter systems (see Amouroux, Di Pillo, & Grippo, 1976; Malandrakis, 1979; Omatu & Seinfeld, 1983 and the review paper Kubrusly & Malebranche, 1985).

Despite the progress on optimal sensor placement and the availability of results on the integration of linear feedback control with actuator placement for linear parabolic PDEs, there are no results on the integration of nonlinear control with optimal placement of control actuators and measurement sensors for transport-reaction processes described by nonlinear parabolic PDEs. Motivated by this, we have initiated a line of work on the computation of optimal actuator/sensor locations for nonlinear controllers in transport-reaction processes. In previous work (Antoniades & Christofides, 2000, 2001), we proposed a general and practical methodology for the integration of nonlinear output feedback control with optimal actuator/sensor placement for transport-reaction processes described by a broad class of quasi-linear parabolic PDEs. Under the hypothesis that the process model is perfectly known, our approach leads to actuator/sensor locations which allow enforcing the desired performance objectives in the closed-loop system with minimal control energy use.

In this paper, we consider transport-reaction processes with unknown time-varying parameters and disturbances modeled by quasi-linear parabolic PDE systems, and address the problem of computing optimal actuator/sensor locations for robust nonlinear controllers. Initially, Galerkin's method is employed to derive finite-dimensional approximations of the PDE system which are used for the synthesis of robust nonlinear state feedback controllers via geometric and Lyapunov techniques and the computation of optimal actuator locations. The controllers enforce boundedness and uncertainty attenuation in the closed-loop system. The optimal actuator location problem is subsequently formulated as the one of minimizing a meaningful cost functional that includes penalty on the response of the closed-loop system and the control action. Owing to the boundedness of the state, the cost is defined over a finite-time interval (the final time is defined as the time needed for the process state to become smaller than the desired uncertainty attenuation limit), while the optimization is performed over a broad set of initial conditions and time-varying disturbance profiles. Subsequently, under the assumption that the number of measurement sensors is equal to the number of slow modes, we employ a standard procedure for obtaining estimates for the states of the approximate finite-dimensional model from the measurements. The optimal location of the measurement sensors is computed by minimizing a cost function of the estimation error in the closed-loop infinite-dimensional system. We show that the use of these estimates in the robust state feedback controller leads to a robust output feedback controller, which guarantees boundedness of the state and uncertainty attenuation in the infinite-dimensional closed-loop system, provided that the separation between the slow and the fast eigenvalues is sufficiently large. We also establish that the solution to the optimal actuator/sensor problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal in the sense that it approaches the optimal solution for the infinite-dimensional system as the separation between the slow and fast eigenvalues increases. The theoretical results are successfully applied to a typical diffusion-reaction process with nonlinearities and uncertainty to design a robust nonlinear output feedback controller and compute the optimal actuator/sensor locations for robust stabilization of an unstable steady state.

2. Preliminaries

2.1. Description of parabolic PDE systems with uncertainty

We consider quasi-linear parabolic PDE systems with uncertain variables of the form:

$$\frac{\partial \bar{x}}{\partial t} = \mathbf{A}\frac{\partial \bar{x}}{\partial z} + \mathbf{B}\frac{\partial^2 \bar{x}}{\partial z^2} + \boldsymbol{w}b(z)u + f(\bar{x}) + W(\bar{x}, r(z)\theta(t)) \tag{1}$$

$$y_c^i = \int_\alpha^\beta c^i(z)\boldsymbol{k}\bar{x}\,\mathrm{d}z, \quad i = 1, \ldots, l$$

$$y_m^\kappa = \int_\alpha^\beta s^\kappa(z)\boldsymbol{\omega}\bar{x}\,\mathrm{d}z, \quad \kappa = 1, \ldots, p$$

subject to the boundary conditions:

$$\boldsymbol{C}_1\bar{x}(\alpha, t) + \boldsymbol{D}_1\frac{\partial \bar{x}}{\partial z}(\alpha, t) = \boldsymbol{R}_1 \tag{2}$$

$$\boldsymbol{C}_2\bar{x}(\beta, t) + \boldsymbol{D}_2\frac{\partial \bar{x}}{\partial z}(\beta, t) = \boldsymbol{R}_2$$

and the initial condition:

$$\bar{x}(z, 0) = \bar{x}_0(z) \tag{3}$$

where $\bar{x}(z, t) = [\bar{x}_1(z, t)\ \bar{x}_2(z, t) \ldots \bar{x}_n(z, t)]^T$ denotes the vector of state variables, $[\alpha, \beta] \subset \mathbb{R}$ is the domain of definition of the process, $z \in [\alpha, \beta]$ is the spatial coordinate, $t \in [0, \infty)$ is the time, $u(t) = [u^1(t)\ u^2(t) \ldots u^l(t)]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, $\theta(t) = [\theta_1(t)\ \theta_2(t) \ldots \theta_q(t)] \in \mathbb{R}^q$ denotes the vector of uncertain variables, which may include uncertain process parameters or exogenous disturbances, $y_c^i \in \mathbb{R}$ denotes a controlled output, $y_m^\kappa \in \mathbb{R}$ denotes a measured output, $\partial\bar{x}/\partial z$, $\partial^2\bar{x}/\partial z^2$ denote the first- and second-order spatial derivatives of $\bar{x}$, $f(\bar{x})$, $W(\bar{x}, r(z)\theta(t))$ are vector functions, $\boldsymbol{w}, \boldsymbol{k}, \boldsymbol{\omega}$ are vectors, $\mathbf{A}, \mathbf{B}, \mathbf{C}_1, \mathbf{D}_1, \mathbf{C}_2, \mathbf{D}_2$ are matrices, $\boldsymbol{R}_1, \boldsymbol{R}_2$ are column vectors, and $\bar{x}_0(z)$ is the initial condition.

The properties and role of the functions $b(z), r(z), c^i(z), s^i(z)$ are as follows: $b(z)$ is a known smooth vector function of z of the form $b(z) = [b^1(z)\, b^2(z) \cdots b^l(z)]$, where $b^i(z)$ describes how the control action $u^i(t)$ is distributed in the interval $[\alpha, \beta]$ (e.g. point/distributed actuation), $r(z) = [r_1(z) \cdots r_q(z)]$, where $r_k(z)$ is a known smooth function of z which specifies the position of action of the uncertain variable θ_k on $[\alpha, \beta]$, $c^i(z)$ is a known smooth function of z which is determined by the specification of the i-th controlled output in the interval $[\alpha, \beta]$, and $s^i(z)$ is a known smooth function of z which is determined by the location and type of the measurement sensors (e.g. point/distributed sensing). Throughout the paper, $O(\varepsilon)$ denotes the order of magnitude notation (i.e. $\delta(\varepsilon) = O(\varepsilon)$ if there exist positive real numbers k_1 and k_2 such that: $|\delta(\varepsilon)| \leq k_1|\varepsilon|$, $\forall|\varepsilon| < k_2$), $|\cdot|$ denotes the standard Euclidean norm, and $\|\theta\|$ denotes ess.sup.$|\theta(t)|$, $t \geq 0$ for any measurable function $\theta: \mathbb{R}_{\geq 0} \to \mathbb{R}^q$. Moreover, we will use the Lie derivative notation: $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f, $L_f^k h$ denotes the k-th order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative. Finally a function $\delta^{(\cdot)}$ is said to be of class.

Defining the state x in the infinite dimensional Hilbert space $\mathcal{H}([\alpha, \beta], \mathbb{R}^n)$ (with $\mathcal{H}$ being the space of n-dimensional vector functions defined on $[\alpha, \beta]$ that satisfy the boundary condition of Eq. (2) with inner product and norm:

$$(\omega_1, \omega_2) = \int_\alpha^\beta (\omega_1(z), \omega_2(z))_{\mathbb{R}^n}\mathrm{d}z, \quad \|\omega_1\|_2 = (\omega_1, \omega_2)^{\frac{1}{2}}$$

where ω_1, ω_2 are two elements of $\mathcal{H}$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$), as:

$$x(t) = \bar{x}(z, t), \quad t > 0, \quad z \in [\alpha, \beta], \tag{4}$$

and the operators as:

$$\mathscr{A}x = \mathbf{A}\frac{\partial \bar{x}}{\partial z} + \mathbf{B}\frac{\partial^2 \bar{x}}{\partial z^2}, \tag{5}$$

$$x \in D(\mathscr{A}) = \left\{ x \in \mathscr{H};\ \mathbf{C}_1\bar{x}(\alpha, t) + \mathbf{D}_1\frac{\partial \bar{x}}{\partial z}(\alpha, t) = \boldsymbol{R}_1,\ \mathbf{C}_2\bar{x}(\beta, t) + \mathbf{D}_2\frac{\partial \bar{x}}{\partial z}(\beta, t) = \boldsymbol{R}_2 \right\}$$

$$\mathscr{B}u = wbu, \quad Cx = (c, kx), \quad Sx = (s, wx)$$

where $c = [c^1\ c^2 \ldots c^l]$, the system of Eqs. (1)–(3)takes the form:

$$\dot{x} = \mathscr{A}x + \mathscr{B}u + f(x) + \mathscr{W}(x, \theta), \quad x(0) = x_0 \tag{6}$$

$$y_c = \mathscr{C}x,\ y_m = \mathscr{S}x$$

where $f(x(t)) = f(\bar{x}(z, t))$, $\mathscr{W}(x(t), \theta) = W(\bar{x}, r\theta)$ and $x_0 = \bar{x}_0(z)$.

For $\mathscr{A}$, the eigenvalue problem is defined as:

$$\mathscr{A}\phi_j = \lambda_j\phi_j, \quad j = 1, \ldots, \infty \tag{7}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction; the eigenspectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, is defined as the set of all eigenvalues of $\mathscr{A}$, i.e. $\sigma(\mathscr{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$. We will assume that $\sigma(\mathscr{A})$ satisfies the following properties.

Assumption 1. (Christofides, 2001)

1. $\mathrm{Re}\{\lambda_1\} \geq \mathrm{Re}\{\lambda_2\} \geq \cdots \geq \mathrm{Re}\{\lambda_j\} \geq \cdots$, where $\mathrm{Re}\{\lambda_j\}$ denotes the real part of λ_j.
2. $\sigma(\mathscr{A})$ can be partitioned as $\sigma(\mathscr{A}) = \sigma_1(\mathscr{A}) + \sigma_2(\mathscr{A})$, where $\sigma_1(\mathscr{A})$ consists of the first m (with m finite) eigenvalues, i.e. $\sigma_1(\mathscr{A}) = \{\lambda_1, \ldots, \lambda_m\}$, and $|\mathrm{Re}\{\lambda_1\}|/|\mathrm{Re}\{\lambda_m\}| = O(1)$.
3. $\mathrm{Re}\{\lambda_{m+1}\} < 0$ and $|\mathrm{Re}\{\lambda_m\}|/|\mathrm{Re}\{\lambda_{m+1}\}| = O(\varepsilon)$, where $\varepsilon = |\mathrm{Re}\{\lambda_1\}|/|\mathrm{Re}\{\lambda_{m+1}\}| < 1$ is a small positive number.

Remark 1. Referring to the system of Eq. (1), several remarks are in order: (a) the spatial differential operator is linear; this assumption is valid for diffusion–convection-reaction processes where the diffusion coefficient and the conductivity can be taken independent of temperature and concentrations; (b) the manipulated input enters the system in a linear and affine fashion; this is typically the case in many practical applications where, for example, the wall temperature is chosen as the manipulated input; and (c) the nonlinearities appear in an additive fashion (e.g. complex reaction rates, Arrhenius dependence of reaction rates on temperature).

Remark 2. Referring to Assumption 1, the hypothesis of finite number of unstable eigenvalues and discrete eigenspectrum are always satisfied for parabolic PDE systems defined in finite spatial domains, while the assumption of existence of only a few dominant modes that describe the dynamics of the parabolic PDE system is usually satisfied by the majority of diffusion–convection-reaction processes (see the catalytic rod example of Section 6).

2.2. Galerkin's method

We derive an m-dimensional approximation of the system of Eq. (6) using Galerkin's method. Letting $\mathscr{H}_s$, $\mathscr{H}_f$ be two subspaces of $\mathscr{H}$, defined as $\mathscr{H}_s = \mathrm{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathscr{H}_f = \mathrm{span}\{\phi_{m+1}, \phi_{m+2}, \ldots\}$, and defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq. (6) can be decomposed as:

$$x = x_s + x_f = P_s x + P_f x \tag{8}$$

Applying P_s and P_f to the system of Eq. (6) and using the above decomposition for x, the system of Eq. (6) can be equivalently written in the following form:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \mathscr{A}_s x_s + \mathscr{B}_s u + f_s(x_s, x_f) + \mathscr{W}_s(x_s, x_f, \theta) \tag{9}$$

$$\frac{\mathrm{d}x_f}{\mathrm{d}t} = \mathscr{A}_f x_f + \mathscr{B}_f u + f_f(x_s, x_f) + \mathscr{W}_f(x_s, x_f, \theta)$$

$$y_c = \mathscr{C}x_s + \mathscr{C}x_f, \quad y_m = \mathscr{S}x_s + \mathscr{S}x_f$$

$$x_s(0) = P_s x(0) = P_s x_0, \quad x_f(0) = P_f x(0) = P_f x_0$$

where $\mathscr{A}_s = P_s\mathscr{A}$, $\mathscr{B}_s = P_s\mathscr{B}$, $f_s = P_s f$, $\mathscr{W}_s = P_s\mathscr{W}$, $\mathscr{A}_f = P_f\mathscr{A}$, $\mathscr{B}_f = P_f\mathscr{B}$, $f_f = P_f f$, $\mathscr{W}_f = P_f\mathscr{W}$. In the above system, $\mathscr{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathscr{A}_s = \mathrm{diag}\{\lambda_j\}$, $f_s(x_s, x_f)$, $f_f(x_s, x_f)$, are Lipschitz vector functions, $\mathscr{W}_s(x_s, x_f, \theta)$, $\mathscr{W}_f(x_s, x_f, \theta)$ are Lipschitz matrix functions, and $\mathscr{A}_f$ is a stable unbounded differential operator. Using that $\varepsilon = |\mathrm{Re}\{\lambda_1\}|/|\mathrm{Re}\{\lambda_{m+1}\}|$, the system of Eq. (9) can be written in the following form:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \mathscr{A}_s x_s + \mathscr{B}_s u + f_s(x_s, x_f) + \mathscr{W}_s(x_s, x_f, \theta) \tag{10}$$

$$\varepsilon\frac{\partial x_f}{\partial t} = \mathscr{A}_{f\varepsilon} x_f + \varepsilon\mathscr{B}_f u + \varepsilon f_f(x_s, x_f) + \varepsilon\mathscr{W}_f(x_s, x_f, \theta)$$

$$y_c = \mathscr{C}x_s + \mathscr{C}x_f, \quad y_m = \mathscr{S}x_s + \mathscr{S}x_f$$

where $\mathscr{A}_{f\varepsilon}$ is an unbounded differential operator defined as $\mathscr{A}_{f\varepsilon} = \varepsilon\mathscr{A}_f$. Since $\varepsilon < 1$ and the operators $\mathscr{A}_s$, $\mathscr{A}_{f\varepsilon}$ generate semigroups with growth rates which are of the same order of magnitude, the system of Eq. (10) is in the standard singularly perturbed form (see Kokotovic, Khalil, & O'Reilly, 1986 for a precise definition of standard form), with x_s being the slow states and x_f being the fast states. Introducing the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq. (10):

$$\frac{\partial x_f}{\partial \tau} = \mathscr{A}_{f\varepsilon} x_f \tag{11}$$

From the fact that $\mathrm{Re}\{\lambda_{m+1}\} < 0$ and the definition of ε, we have that the above system is globally exponentially stable. Setting $\varepsilon = 0$ in the system of Eq. (10), we have that $x_f = 0$ and thus, the finite-dimensional slow system takes the form:

$$\frac{\mathrm{d}\tilde{x}_s}{\mathrm{d}t} = \mathscr{A}_s \tilde{x}_s + \mathscr{B}_s u + f_s(\tilde{x}_s, 0) + \mathscr{W}_s(\tilde{x}_s, 0, \theta) \tag{12}$$

$$y_c = \mathscr{C}\tilde{x}_s, \quad y_m = \mathscr{S}\tilde{x}_s$$

where the tilde notation in $\tilde{x}_s$ denotes that this is the state of a finite-dimensional system.

Remark 3. We note that the above model reduction procedure which led to the approximate ODE system of Eq. (12) can be also used, when empirical eigenfunctions of the system of Eq. (6) computed through Karhunen–Loéve expansion (see Christofides, 2001; Atwell & King, 2001 for applications of this method in the design of low-order controllers) are used as basis functions in $\mathscr{H}_s$ and $\mathscr{H}_f$, instead of the eigenfunctions of $\mathscr{A}$.

Remark 4. Referring to the model reduction of the PDE system of Eq. (1) (diffusion–convection–reaction process) using Galerkin's method, it is important to note the following: while the solution of the parabolic PDE system is expanded in terms of the eigenfunctions of the spatial differential operator, both the linear spatial differential operator (diffusion and convection terms) and the nonlinear term $f(\bar{x})$ and $W(\bar{w}, r(z)\theta(t))$ (reaction terms) are projected and used for the construction of the finite-dimensional approximation. To clearly see this point, consider the terms $f_s(\tilde{x}_s, 0) + \mathscr{W}_s(\tilde{x}_s, 0, \theta)$ in the finite-dimensional system of Eq. (12); these terms are the projections of the nonlinear terms $f(\bar{x}) + W(\bar{x}, r(z)\theta(t))$ that appear in the parabolic PDE of Eq. (1). Therefore, the finite-dimensional model not only approximates the diffusion and convection phenomena but also it approximates the reaction phenomena (modelled by the nonlinear terms $f(\bar{x}) + W(\bar{x}, r(z)\theta(t))$). Moreover, the separation of the infinite set of ODEs into fast and slow according to the fast and slow eigenvalues of the spatial differential operator is meaningful because the kinetics terms $(f(\bar{x}) + W(\bar{w}, r(z)\theta(t)))$ do not contribute to the two-time-scale behavior of the spatial modes of the parabolic PDE systems since they do not include terms of the type $1/\varepsilon$. Note that if the kinetics include terms of the type $1/\varepsilon$ the model reduction procedure can be extended to account for this type of behavior (see also Kumar, Christofides, & Daoutidis, 1998 for results on this problem). Finally, regarding the accuracy of the finite-dimensional model, it can be shown (Christofides, 2001) that when $\theta(t) \equiv 0$, the discrepancy between the solution of the finite-dimensional model of Eq. (12) and the solution of the parabolic PDE of Eq. (1) is of $O(\varepsilon)$. This result accounts for both the transport and reaction terms and was shown using singular perturbation arguments and results for stability of infinite-dimensional systems.

3. Problem statement and solution framework

In this paper, we address the problem of computing optimal locations of point control actuators and point measurement sensors associated with robust nonlinear output feedback control laws of the following general form:

$$u = \mathscr{F}(\boldsymbol{y}_m) \tag{13}$$

where $\mathscr{F}(\boldsymbol{y}_m)$ is a nonlinear vector function and $\boldsymbol{y}_m$ denotes the vector of measured outputs, so that the following properties are enforced in the closed-loop system: (a) boundedness of the state; (b) arbitrary degree of attenuation of the uncertain variables on the state of the slow subsystem; and (c) the optimal actuator/sensor locations, which are obtained on the basis of the closed-loop finite-dimensional system, are near-optimal in the sense that it approaches the optimal solution for the infinite-dimensional system as the separation of the slow and fast eigenmodes increases. To address this problem, we will initially synthesize, under the assumption of existence of bounding functions that capture the size of the uncertain terms, stabilizing robust nonlinear state feedback controllers via Lyapunov techniques on the basis of finite-dimensional approximations of the PDE system. The optimal actuator location problem will be subsequently formulated as the one of minimizing a meaningful cost functional that includes penalty on the response of the closed-loop system and the control action. Then, under the assumption that the number of measurement sensors is equal to the number of slow modes, we will employ a procedure proposed in Christofides and Baker (1999) for obtaining estimates for the states of the approximate finite-dimensional model from the measurements. The estimates will be combined with the robust state feedback controllers to derive robust output feedback controllers. The optimal location of the measurement sensors will be computed by minimizing a cost function of the estimation error in the closed-loop infinite-dimensional system. It will be established by using singular perturbation techniques that: (a) the robust output feedback controllers enforce boundedness of the state and uncertainty attenuation in the infinite-dimensional closed-loop system provided that the separation between the slow and the fast eigenvalues is sufficiently large; and (b) the solution to the optimal actuator/sensor problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal in the sense that it approaches the optimal solution for the infinite-dimensional system as the separation between the slow and fast eigenvalues increases.

4. Integrating robust control and optimal actuator placement

4.1. Robust state feedback controller synthesis

In this section, we assume that measurements of the states of the PDE system of Eq. (12) are available and address the problem of synthesizing robust nonlinear static state feedback control laws of the general form:

$$u = \mathscr{F}(z_a, \tilde{x}_s) \tag{14}$$

where $\mathscr{F}(z_a, \tilde{x}_s)$ is a nonlinear vector function and z_a denotes the vector of the actuator locations, that guarantee boundedness of the state of the closed-loop finite-dimensional system and make the effect of the uncertain variables on the state arbitrarily small by appropriate selection of the controller parameters. We note that owing to the fact that we allow the uncertain variables, $\theta(t)$, to change the equilibrium of the finite-dimensional system of Eq. (12), it is not possible to synthesize a continuous controller that enforces asymptotic stability in the closed-loop system. To proceed with the synthesis of the controller, we will need the following assumptions (see Remark 6 below for a discussion on this assumption).

Assumption 2. $l = m$ (i.e. the number of control actuators is equal to the number of slow modes), and the inverse of the matrix $\mathscr{B}_s$ exists.

Assumption 3 that follows requires the existence of a nonlinear time-varying bounding function that captures the size of the uncertain terms in the system of Eq. (12) and the existence of known bounds for the uncertain variables $\theta_k(t)$. Such bounding functions are typically obtained from physical considerations, preliminary simulations or experimental data. The requirement of existence of bounding functions is standard in all Lyapunov-based robust control methods (see Corless & Leitmann, 1981; Christofides, Teel, & Daoutidis, 1996).

Assumption 3. There exists a known function $c(x_s, t)$ and a known set of positive constants θ_{bk} such that the following conditions holds:

$$|\mathscr{W}_s(\tilde{x}_s, 0, \theta)| \leq c(x_s, t), \quad |\theta_k(t)| \leq \theta_{bk}, \quad k = 1, 2, \ldots, q \tag{15}$$

for all $x_s \in \mathscr{H}_s$, $\theta \in \mathbb{R}^q$, $t \geq 0$.

Proposition 1 that follows provides the explicit formula for the state feedback controller that achieves the aforementioned control objectives. The proof of the proposition is given in the Appendix A.

Proposition 1. *Consider the finite-dimensional system of* Eq. (12) *for which* Assumption 2 and Assumption 3 *hold. Then, the state feedback controller:*

$$u = \mathscr{B}_s^{-1}\left((\Lambda_s - \mathscr{A}_s)\tilde{x}_s - f_s(\tilde{x}_s, 0) - \chi c(\tilde{x}_s, t)\frac{\tilde{x}_s}{|\tilde{x}_s| + \phi}\right) \tag{16}$$

where Λ_s *is a stable matrix,* $\chi > 1$ *and* $\phi > 0$ *are positive parameters, guarantees boundedness of the state of the closed-loop finite-dimensional system and ensures that* $\lim_{t\to\infty} |\tilde{x}_s(t)| \leq \gamma(\phi)$ *where* $\gamma(\cdot)$ *is a class K function.*

Remark 5. The structure of the closed-loop finite-dimensional system of Eq. (12) under the controller of Eq. (16) has the following form:

$$\dot{\tilde{x}}_s = \Lambda_s\tilde{x}_s - \chi c(\tilde{x}_s, t)\frac{\tilde{x}_s}{|\tilde{x}_s| + \phi} + \mathscr{W}_s(\tilde{x}_s, 0, \theta) \tag{17}$$

From the above equation, it follows directly that the response of this system depends on the stable matrix Λ_s, the initial condition, $x_s(0)$, the uncertain variable $\theta(t)$, and the matrix $\mathscr{B}_s$ which accounts for the actuator locations. This is completely different from the optimal actuator placement for PDE systems without uncertain variables where the structure of the closed-loop finite-dimensional system was of the following form (Antoniades and Christofides, 2001):

$$\dot{\tilde{x}}_s = \Lambda_s\tilde{x}_s \tag{18}$$

and thus, the response was independent of the actuator locations.

Remark 6. The requirement $l = m$ is sufficient and not necessary, and it is made to simplify the solution of the controller synthesis problem. Full linearization (when $\theta(t) \in 0$) of the closed-loop finite-dimensional system through coordinate change and nonlinear feedback can be achieved for any number of manipulated inputs (i.e. for any $l \in [1, m]$), provided that an appropriate set of involutivity conditions is satisfied by the corresponding vector fields of the system of Eq. (12) (see Isidori, 1989 for details).

Remark 7. Referring to the role and selection of the parameters χ and ϕ of the controller of Eq. (16), we note that χ determines the gain of the controller term that compensates for the effect of the uncertainty and should be chosen greater than unity to achieve global boundedness of the state of the closed-loop finite-dimensional system (note that the choice $\chi = 1$ would provide local boundedness of the state). On the other hand, ϕ determines the degree of asymptotic attenuation of the effect of the uncertainty on the state and should be chosen to ensure uncertainty attenuation with smooth control action (note that as $\phi \to 0$ the controller of Eq. (16) becomes discontinuous and the control action non-smooth). Finally, it is important to point out that the robust controller of Eq. (16) compensates for the effect of uncertain variables on the state of the closed-loop finite-dimensional system without using high-gain feedback; this is a consequence of our objective to design a practically-implementable controller which would not destabilize the fast modes of the closed-loop infinite-dimensional system (note that there is guarantee that such a destabilization will be avoided if high-gain feedback of the type $1/\varepsilon$ is employed to compensate for the effect of the disturbances).

4.2. Computation of optimal location of control actuators

In this subsection, we compute the actuator locations so that the robust state feedback controller of Eq. (16) is near-optimal for the full PDE system of Eq. (9) with respect to a meaningful cost functional which imposes penalty on the response of the closed-loop system and the control action. Owing to the boundedness of the state, we consider a cost functional which is defined over a finite-time interval (the final time is defined as the needed for the process state to become smaller than the desired attenuation limit), T_f. From the stability bound $|\tilde{x}_s(t)| \le K|\tilde{x}_s(0)|e^{-\alpha_2 t} + \gamma(\phi)$, it follows directly that given the desired attenuation limit, d, so that $|\tilde{x}_s(t \ge T_f)| \le d$ and the value of ϕ used in the controller, the values of T_f can be computed (note that T_f is the time in which the state $\tilde{x}_s(t)$ satisfies $|\tilde{x}_s(t \ge T_f)| \le d$ for the first time). Specifically, we focus on the ODE system of Eq. (12) and consider the following cost functional:

$$J_s = \int_0^{T_f} ((\tilde{x}_s^T(x_s(0), t),\ \mathbf{Q}_s \tilde{x}_s(x_s(0), t)) + u^T(\tilde{x}_s(x_s(0), t), z_a)\ \mathbf{R}u(\tilde{x}_s(x_s(0), t), z_a))\, dt \quad (19)$$

where $\mathbf{Q}_s$ and $\mathbf{R}$ are positive definite matrices. The cost of Eq. (19) is well-defined and meaningful since it imposes penalty on the response of the closed-loop finite-dimensional system and the control action. However, a potential problem of this cost is its dependence on the choice of a particular initial condition, $x_s(0)$, and thus, the solution to the optimal placement problem based on this cost may lead to actuator locations that perform very poorly for a large set of initial conditions. To eliminate this dependence and obtain optimality over a broad set of initial conditions, we follow (Levine and Athans, 1978; Antoniades and Christofides, 2000) and consider an average cost over a set of m linearly independent initial conditions, $x_s^i(0)$, $i = 1, \ldots, m$, of the following form:

$$\bar{J}_s = \frac{1}{m} \sum_{i=1}^{m} \int_0^{T_{f_i}} ((\tilde{x}_s^T(x_s^i(0), t),\ \mathbf{Q}_s \tilde{x}_s(x_s^i(0), t)) + u^T(\tilde{x}_s(x_s^i(0), t), z_a)\ \mathbf{R}u(\tilde{x}_s(x_s^i(0), t), z_a))\, dt \quad (20)$$

Note that different initial conditions correspond to different times T_{f_i}. An additional issue that should be accounted for in the cost functional used to compute the optimal actuator locations is the fact that the evolution of the state, $\tilde{x}_s$, of the closed-loop finite-dimensional system depends on the profile of the vector of unknown variables $\theta(t)$. To account for this dependence, we modify the cost of Eq. (20) to average over a large set of disturbance profiles as follows:

$$\hat{J}_s = \frac{1}{m} \frac{1}{K} \sum_{\kappa=1}^{K} \sum_{i=1}^{m} \int_0^{T_{f_{i\kappa}}} ((\tilde{x}_s^T(x_s^i(0), \theta^\kappa, t),\ \mathbf{Q}_s \tilde{x}_s(x_s^i(0), \theta^\kappa, t)) + u^T(\tilde{x}_s(x_s^i(0), \theta^\kappa, t), z_a)\ \mathbf{R}u(\tilde{x}_s(x_s^i(0), \theta^\kappa, t), z_a))\, dt \quad (21)$$

Note again that different profiles of the vector of unknown variables, $\theta(t)$, might also correspond to different times $T_{f_{i\kappa}}$. In contrast to the case of PDE systems without uncertainty, the solution to optimal placement problem, z_{am}, for which the above cost is minimum can be only found numerically since it requires multiple integrations of the closed-loop finite-dimensional system of Eq. (17) for appropriate finite times, linearly independent initial conditions and different disturbance profiles (Remark 8 below discusses a numerical approach for computing the optimal actuator locations).

Theorem 1 that follows establishes that the robust controller of Eq. (16) guarantees the desired boundedness and uncertainty attenuation properties in the closed-loop finite-dimensional system and the locations for the control actuators that minimize the cost of Eq. (21) for the finite-dimensional closed-loop system are also near optimal for the closed-loop infinite-dimensional system, provided that the degree of separation between the fast and slow modes of the spatial differential operator of the PDE system is sufficiently large (the proof of this theorem can be found in the Appendix A).

Theorem 1. *Consider the infinite-dimensional system of* Eq. (9) *for which* Assumption 1 *holds, and the finite-dimensional system of* Eq. (12), *for which* Assumption 2 and Assumption 3 *hold. Then, there exist positive real numbers* μ_1, μ_2 *and* ε^* *such that if* $|x_s(0)| \le \mu_1$, $\|x_f(0)\|_2 \le \mu_2$, *and* $\varepsilon \in (0, \varepsilon^*]$, *then the controller of* Eq. (14)*:*

1. *guarantees boundedness of the state of the closed-loop infinite-dimensional system, and*

2. *the optimal locations of the point actuators obtained for the closed-loop finite-dimensional system are near-optimal for the closed-loop infinite-dimensional system in the sense that:*

$$\hat{J} = \frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{T_{f_{i\kappa}}}((x_s^T(x_s^i(0),\theta^\kappa,t),\ \mathbf{Q}_s x_s(x_s^i(0),\theta^\kappa,t)) + (x_f^T(x_s^i(0),\theta^\kappa,t),\ Q_f x_f(x_f^i(0),\theta^\kappa,t)) + u^T(x_s(x_s^i(0),\theta^\kappa,t),z_a)\ \mathbf{R}u(x_s(x_s^i(0),\theta^\kappa,t),z_a))\,dt \rightarrow \hat{J}_s \tag{22}$$

as $\varepsilon \rightarrow 0$

where Q_f is an unbounded positive definite operator and $\hat{J}$ is the average cost function associated with the controller of Eq. (16) *and the infinite-dimensional system of* Eq. (9).

Remark 8. Owing to the numerical complexity involved in computing the actuator locations that exactly minimize the cost of Eq. (21) (it involves search over an infinite number of locations), we initially assume a large number, say L, of equispaced locations along the length of the spatial domain in which the control actuators are possible to be placed. In determining this set of locations, we use standard controllability arguments for parabolic PDEs (see Ray, 1981) to exclude actuator locations for which the resulting system is uncontrollable. Since the disturbance vector $\theta(t)$ is unknown, we then use a stochastic approach (which is inspired by the methodology proposed in Gazi, Ungar, Seider, and Kuipers (1996), Gazi, Seider, and Ungar (1997) for robust controller verification) to compute a large number of randomly-generated disturbance profiles that satisfy $|\theta_k(t)| \leq \theta_{bk}$ for all times (Assumption 2). We finally compute the value of the cost of Eq. (21) for all possible combinations of the actuator locations to calculate the optimal locations. While the computational complexity of the above procedure is not very large for a small number of control actuators, its computational complexity grows very fast as the number of control actuators increases, making the practical implementation of this procedure very difficult. In this case, the following algorithm which employs a search procedure that uses the optimal actuator locations for the nominal system (see Remark 9 below for how this can be done) as initial guess can be followed. The procedure is as follows: we initially fix $l-1$ control actuators at the optimal locations computed on the basis of the nominal system and perform a search for the optimal location of the remaining control actuator that minimizes the cost for the uncertain system. This process is then performed $l-1$ more times for each one of the other control actuators to compute a new set of actuator locations for the uncertain system. This set of actuator locations is subsequently used as a new initial guess to carry out the above procedure for a second time. The algorithm is terminated when the change in the optimal actuator locations between two consecutive steps is smaller than a desired accuracy. This approach was successfully used in the diffusion-reaction process example discussed in Section 6.

Remark 9. It is important to point out that in the case of PDE systems without uncertainty, the closed-loop finite-dimensional system takes the form $\dot{\tilde{x}}_s = \Lambda_s \tilde{x}_s$ and thus the penalty on the response of this system:

$$\hat{J}_{xs} = \frac{1}{m}\sum_{i=1}^{m}\int_0^{\infty}(\tilde{x}_s^T(x_s^i(0),t),\ \mathbf{Q}_s\tilde{x}_s(x_s^i(0),t))\,dt \tag{23}$$

is finite and independent of the actuator locations. Therefore, the optimal actuator placement problem reduces to the one of minimizing the following cost which only includes penalty on the control action:

$$\hat{J}_{us} = \frac{1}{m}\sum_{i=1}^{m}\int_0^{\infty}u^T(\tilde{x}_s(x_s^i(0),t),z_a)\ \mathbf{R}u(\tilde{x}_s(x_s^i(0),t),z_a)\,dt$$

$\hat{J}_{us}$ is a function of multiple variables, $z_a = [z_{a1}\ z_{a2}\ \ldots\ z_{al}]$, and thus, it obtains its local minimum values when its gradient with respect to the actuator locations is equal to zero, i.e.:

$$\frac{\partial \hat{J}_{us}}{\partial z_a} = \left[\frac{\partial \hat{J}_{us}}{\partial z_{a1}} \cdots \frac{\partial \hat{J}_{us}}{\partial z_{al}}\right]^T = [0 \cdots 0]^T \tag{24}$$

and $\nabla_{z_a z_a}\hat{J}_{us}(z_{am}) > 0$, where $\nabla_{z_a z_a}\hat{J}_{us}$ is the Hessian matrix of $\hat{J}_{us}$ and z_{am} is a solution of the system of nonlinear algebraic equations of Eq. (24) (which includes l equations with l unknowns and can be obtained by standard methods). The solution z_{am} for which the above conditions are satisfied and $\hat{J}_{us}$ obtains its smallest value (global minimum) corresponds to the optimal actuator locations for the closed-loop finite-dimensional system.

5. Robust output feedback control with optimal sensor placement

In this section, we proceed with the synthesis of robust *output* feedback controllers that enforce stability and uncertainty attenuation in the closed-loop infinite-dimensional system, and compute the optimal locations for the measurement sensors. Specifically, we consider output feedback control laws of the general form:

$$u(t) = \mathcal{F}(y_m) \tag{25}$$

where $\mathcal{F}(y_m)$ is a nonlinear vector function and y_m is the vector of measured outputs. The synthesis of the controller of Eq. (25) will be achieved by combining the state feedback controller of Eq. (16) with a procedure proposed in Christofides & Baker (1999) for obtaining estimates for the states of the approximate ODE model of Eq. (12) from the measurements. To this end, we need to impose the following requirement on the number of measured outputs in order to obtain estimates of the states x_s of the finite-dimensional system of Eq. (12), from the measurements $\mathbf{y}_m^{\kappa}$, $\kappa = 1, 2, \ldots, p$.

Assumption 4. $p = m$ (i.e. the number of measurements is equal to the number of slow modes), and the inverse of the operator $\mathscr{S}$ exist, so that $\hat{x}_s = \mathscr{S}^{-1}\mathbf{y}_m$.

We note that the requirement that the inverse of the operator $\mathscr{S}$ exists can be achieved by appropriate choice of the location of the measurement sensors (i.e. functions $s^{\kappa}(z)$). The optimal locations for the measurement sensors can be computed by minimizing an average cost function of the estimation error of the closed-loop infinite-dimensional system of the form:

$$\hat{J}(e) = \frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{T_{f_{i\kappa}}} (\|x_s(x_s^i(0), \theta^{\kappa}, t) - \hat{x}_s(x_s^i(0), \theta^{\kappa}, t)\|_2)\, \mathrm{d}t \tag{26}$$

where x_s is the slow state of the closed-loop infinite-dimensional system of Eq. (9), $\hat{x}_s = \mathscr{S}^{-1}\mathbf{y}_m$, and $e(t) = \|x_s - \hat{x}_s\|_2$ is the estimation error. As in the case of the optimal location problem for the control actuators, the solution to the sensor placement problem requires the numerical integration of the closed-loop system (both finite- and infinite-dimensional) in order to compute x_s, and $\hat{x}_s$ (from the measurements $\mathbf{y}_m^{\kappa}$, $\kappa = 1, 2, \ldots, p$), and thus, it is computationally demanding (see Remark 11 for how compute the optimal sensor locations).

Theorem 2 that follows establishes that the proposed robust output feedback controller enforces stability in the closed-loop infinite-dimensional system and that the solution to the optimal actuator/sensor problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal in the sense that it approaches the optimal solution for the infinite-dimensional system as the separation of the slow and fast eigenmodes increases. The proof of this theorem can be found in the Appendix A.

Theorem 2. *Consider the system of* Eq. (9) *for which* Assumption 1 *hold, and the finite-dimensional system of* Eq. (12), *for which* Assumption 2, Assumption 3 and Assumption 4 *hold, under the robust output feedback controller:*

$$u = \mathscr{B}_s^{-1}\left((\Lambda_s - \mathscr{A}_s)\mathscr{S}^{-1}\mathbf{y}_m - f_s(\mathscr{S}^{-1}\mathbf{y}_m, 0) - \chi c(\mathscr{S}^{-1}\mathbf{y}_m, t)\frac{\mathscr{S}^{-1}\mathbf{y}_m}{|\mathscr{S}^{-1}\mathbf{y}_m| + \phi}\right) \tag{27}$$

Then, there exist positive real numbers μ_1, μ_2 and ε^* such that if $|x_s(0)| \le \mu_1$, $\|x_f(0)\|_2 \le \mu_2$, and $\varepsilon \in (0, \varepsilon^*]$, then the controller of Eq. (27)*:*

(a) *guarantees boundedness of the state of the closed-loop system, and*

(b) *the locations of the point actuators and measurement sensors are near-optimal in the sense that the cost function associated with the controller of* Eq. (27) *and the system of* Eq. (9) *satisfies:*

$$\begin{aligned}\hat{J} &= \frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{T_{f_{i\kappa}}} ((x_s^T(x_s^i(0), \theta^{\kappa}, t),\ \mathbf{Q}_s x_s(x_s^i(0), \theta^{\kappa}, t)) \\ &+ (x_f^T(x_s^i(0), \theta^{\kappa}, t),\ Q_f x_f(x_f^i(0), \theta^{\kappa}, t)) \\ &+ u^T(x_s(x_s^i(0), \theta^{\kappa}, t), \mathbf{z}_a)\ \mathbf{R}u(x_s(x_s^i(0), \theta^{\kappa}, t), \mathbf{z}_a))\, \mathrm{d}t \to \hat{J}_s\end{aligned} \tag{28}$$

as $\varepsilon \to 0$

where $\hat{J}$ and $\hat{J}_s$ are the average cost functions of the infinite-dimensional system of Eq. (9) *and the finite-dimensional system of* Eq. (12), *respectively, under the robust output feedback controller of* Eq. (27).

Remark 10. Note that in contrast to the case of state feedback control where the controller depends only the slow state x_s, the output feedback controller of Eq. (27) uses static feedback of the measured outputs y_m^{κ}, $\kappa = 1, 2, \ldots, p$, and thus, it feeds back both x_s and x_f. While this type of feedback of the state x_f has been shown to lead to closed-loop instability for certain classes of singularly perturbed systems (Kokotovic et al., 1986), in the present case, the large separation of the slow and fast modes of the spatial differential operator (i.e. assumption that ε is sufficiently small), together with the special structure of the fast subsystem of Eq. (10) (where ε multiplies the control input) and the fact that the controller does not include terms of the form $O(1/\varepsilon)$, do not allow such a destabilization to occur.

Remark 11. Similar to the case of optimal actuator placement, we perform the search for the optimal sensor locations so that the cost of Eq. (26) is minimized over a finite set of equispaced locations along the length of the spatial domain of the PDE.

6. Application to a diffusion-reaction process with uncertainty

Consider a long, thin rod in a reactor (Fig. 1). The reactor is fed with pure species A and a zeroth order exothermic catalytic reaction of the form $A \to B$ takes place on the rod. Since the reaction is exothermic, a cooling medium which is in contact with the rod is used for cooling. Under the assumptions of constant density and heat capacity of the rod, constant conductivity of the rod, and constant temperature at both ends of the rod, the mathematical model which describes the spatiotemporal evolution of the dimensionless rod temperature consists of the following parabolic PDE:

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T e^{-\gamma/(1+\bar{x})} + \beta_U (b(z)u(t) - \bar{x}) - \beta_{T,n} e^{-\gamma} \tag{29}$$

subject to the Dirichlet boundary conditions:

$$\bar{x}(0, t) = 0, \quad \bar{x}(\pi, t) = 0 \tag{30}$$

and the initial condition:

$$\bar{x}(z, 0) = \bar{x}_0(z) \tag{31}$$

where $\bar{x}$ denotes the dimensionless temperature of the rod, β_T denotes a dimensionless heat of reaction (which is assumed to be unknown and time-varying; uncertain variable), $\beta_{T,n}$ denotes a *nominal* dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, and u denotes the manipulated input (temperature of the cooling medium). The following typical values were given to the process parameters:

$$\beta_{T,n} = 50.0, \quad \beta_U = 2.0, \quad \gamma = 4.0 \tag{32}$$

The eigenvalue problem for the spatial differential operator of the process:

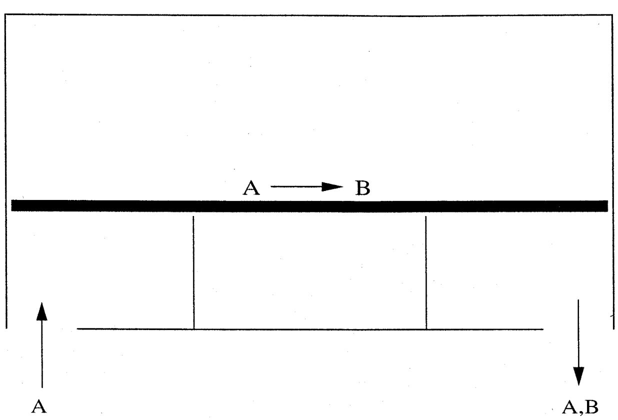

Fig. 1. Catalytic rod.

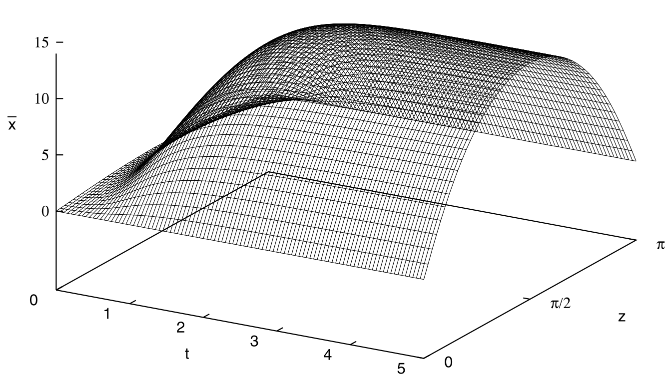

Fig. 2. Profile of evolution of rod temperature in the open-loop system.

$$\mathscr{A}x = \frac{\partial^2 \bar{x}}{\partial z^2}, \quad x \in D(\mathscr{A}) = \{x \in \mathscr{H}([0, \pi]; \mathbb{R});$$
$$\bar{x}(0, t) = 0, \quad \bar{x}(\pi, t) = 0\} \tag{33}$$

can be solved analytically and its solution is of the form:

$$\lambda_j = -j^2, \quad \phi_j(z) = \bar{\phi}_j(z) = \sqrt{\frac{2}{\pi}} \sin(j z),$$
$$j = 1, \ldots, \infty \tag{34}$$

where λ_j, ϕ_j, $\bar{\phi}_j$, denote the eigenvalues, eigenfunctions and adjoint eigenfunctions of $\mathscr{A}_i$, respectively.

Even though the eigenvalues of $\mathscr{A}$ are all stable, the spatially uniform operating steady-state $\bar{x}(z, t) = 0$ of the system of Eq. (29) is an unstable one (i.e. the linearization of the system of Eq. (29) around $\bar{x}(z, t) = 0$ possesses one unstable eigenvalue due to the exothermic nature of the reaction). Fig. 2 shows the spatiotemporal evolution of the open-loop dimensional rod temperature when no uncertainty is present, starting from initial conditions close to the steady state $\bar{x}(z, t) = 0$; the process moves to another stable steady state characterized by a maximum in the temperature profile, *hot-spot*, in the middle of the rod. We note that in all the simulation runs, a 20-th order Galerkin truncation of the system of Eq. (29) was used in our simulations in order to accurately describe the process (further increase on the order of the Galerkin truncation was found to give identical numerical results). Therefore, the control objective is to stabilize the dimensionless rod temperature at the unstable steady-state $\bar{x}(z, t) = 0$, in the presence of time-varying uncertainty in the dimensionless heat of reaction β_T.

We consider the first two Galerkin modes of the PDE system as the slow modes and use Galerkin's method to construct a second-order ODE system which is used for controller design and optimal actuator/sensor placement. This approximate system has the following form:

$$\begin{bmatrix} \dot{\tilde{x}}_{s1} \\ \dot{\tilde{x}}_{s2} \end{bmatrix} = \begin{bmatrix} \lambda_1 - \beta_U & 0 \\ 0 & \lambda_2 - \beta_U \end{bmatrix} \begin{bmatrix} \tilde{x}_{s1} \\ \tilde{x}_{s2} \end{bmatrix} \tag{35}$$
$$+ \beta_U \begin{bmatrix} \bar{\phi}_1(z_{a1}) & \bar{\phi}_1(z_{a2}) \\ \bar{\phi}_2(z_{a1}) & \bar{\phi}_2(z_{a2}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + (\beta_{T,n} + \theta(t)) \begin{bmatrix} f_1(\tilde{x}_s, 0) \\ f_2(\tilde{x}_s, 0) \end{bmatrix}$$

where z_{a1} and z_{a2} are the locations of the two point actuators, $\theta(t)$ is the uncertain term in the dimensionless heat of reaction and the explicit form of the terms $f_1(\tilde{x}_s, 0)$ and $f_2(\tilde{x}_s, 0)$ are omitted for brevity. We used the second-order ODE system of Eq. (35) for the synthesis of the nonlinear robust output feedback controller through an application of the formula of Eq. (16) (the explicit form is omitted for brevity). This controller was employed in the simulations with $\chi = 1.1$, $\theta_b = 12.5$ and $\phi = 0.005$.

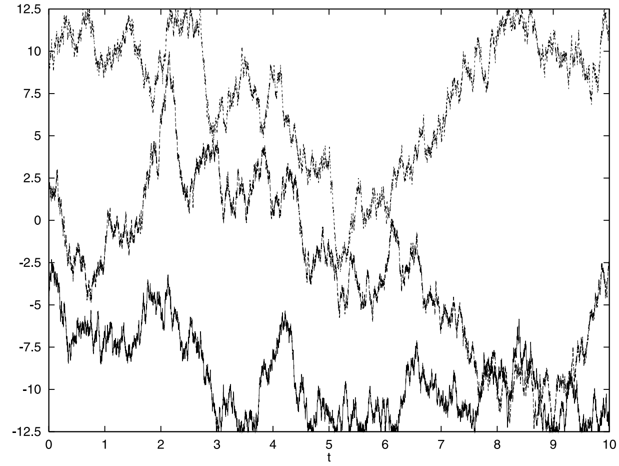

Fig. 3. Profile of three randomly-generated disturbances.

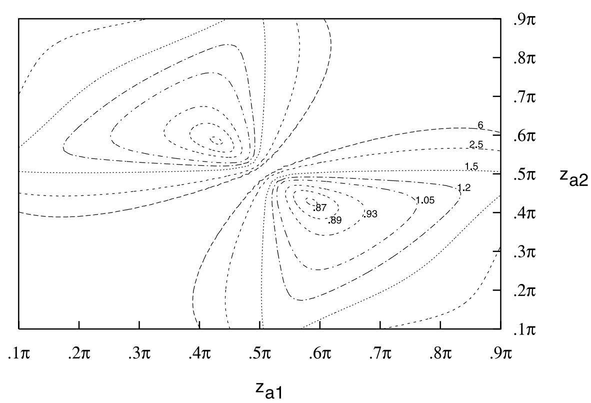

Fig. 4. Contour plot of constant $\hat{J}_s$ (0.87, 0.89, 0.93, 1.05, 1.2, 1.5, 2.5 and 6) for the closed-loop system under the robust state feedback controller for different locations of the two control actuators.

Table 1
Results for the actuator locations

Case	Actuator locations	$\hat{J}_s$
Optimal	0.42π, 0.59π	0.8678
2	0.39π, 0.66π	0.9198
3	0.32π, 0.68π	1.0380
4	0.25π, 0.75π	1.5060

Using the results of Theorem 1, the optimal locations of the control actuators were computed by minimizing a cost that includes penalty on the control action and averages over the set of two linearly independent initial conditions and a large set of randomly generated disturbance profiles $\theta(t)$ which satisfy $|\theta(t)| \leq \theta_b$ for all times. The specific cost used in our calculations has the following form:

$$\hat{J}_u = \frac{1}{40}\sum_{\kappa=1}^{20}\sum_{i=1}^{2}\int_0^{\bar{T}_f}((\tilde{x}_s^T(x_s^i(0), \theta^\kappa, t),\ Q_s\tilde{x}_s(x_s^i(0), \theta^\kappa, t)) \tag{36}$$

$$+ u^T(\tilde{x}_s(x_s^i(0), \theta^\kappa, t), z_a)\ \mathbf{R}u(x_s(\tilde{x}_s^i(0), \theta^\kappa, t), z_a))\, dt$$

where

$$\mathbf{Q}_s = \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\bar{T}_f = \max\{T_{f_{i\kappa}}\} = 10$ for $i = 1, 2$, $\kappa = 1, 2, \ldots, 20$, $x_s^1(0) = \phi_1$ and $x_s^2(0) = \phi_2$, are the two linearly independent initial conditions, and θ^κ, $\kappa = 1, 2, \ldots, 20$, are the randomly generated disturbance profiles that satisfy $\theta(t) \in [-0.25\,\beta_{T,n}, 0.25\,\beta_{T,n}]$ and $\theta(t+0.001) = \theta(t) + 0.04\,r$ where r is a random number $r \in [-1, 1]$. In Fig. 3, we can see three of these randomly generated profiles of the uncertainty. The optimal location of control actuators were computed to be: $z_{a1} = 0.42\pi$ and $z_{a2} = 0.59\pi$. In Fig. 4 we see the contour plot of constant $\hat{J}_s$ (0.87, 0.89, 0.93, 1.05, 1.2, 1.5, 2.5 and 6) for the closed-loop system under the robust state feedback controller for different locations of the two control actuators. The plot is symmetric about the 45° line as the two actuator locations are equivalent. We can see that the minimum $\hat{J}_s = 0.87$ is at the optimal locations that we have computed.

Using the result of Theorem 2, the vector of measured outputs $\bar{y}_m(t) \in \mathbb{R}^2$ is defined as:

$$\begin{bmatrix} \bar{y}_{m1} \\ \bar{y}_{m2} \end{bmatrix} = \begin{bmatrix} x(z_{s1}, t) \\ x(z_{s2}, t) \end{bmatrix} \tag{37}$$

where z_{s1} and z_{s2} are the locations of the two point sensors. The optimal sensor locations were computed by minimizing a cost functional of the estimation error which has the following form:

$$\hat{J}(e) = \frac{1}{40}\sum_{\kappa=1}^{20}\sum_{i=1}^{2}\int_0^{\bar{T}_f}(\|x_s(x_s^i(0), \theta^\kappa, t) - \hat{x}_s(x_s^i(0), \theta^\kappa, t)\|_2)\, dt \tag{38}$$

where x_s is obtained from the simulation of the high-order discretization of the PDE, and $\hat{x}_s$ is calculated from the following equations:

$$\begin{bmatrix} \hat{x}_{s1} \\ \hat{x}_{s2} \end{bmatrix} = \begin{bmatrix} \bar{\phi}_1(z_{s1}) & \bar{\phi}_1(z_{s2}) \\ \bar{\phi}_2(z_{s1}) & \bar{\phi}_2(z_{s2}) \end{bmatrix}^{-1} \begin{bmatrix} y_{m1}(z_{s1}, t) \\ y_{m2}(z_{s2}, t) \end{bmatrix}, \tag{39}$$

The optimal location of the measurement sensors were computed to be: $z_{s1} = 0.35\pi$ and $z_{s2} = 0.66\pi$.

In the remainder, we will present some of our simulation results. We initially focus on the state feedback control problem and compute, for the sake of comparison, the optimal locations of the control actuators, $\hat{J}_s$ for the case where no uncertainty in the dimensionless heat of reaction is considered ($\beta_T = \beta_{T,nom}$). We also compute $\hat{J}_s$ for two arbitrary actuator locations. The computed results for these cases are shown in Table 1. From the values in Table 1, one can see that a robust controller which uses the optimal actuator locations at $z_{a1} = 0.42\pi$ and $z_{a2} = 0.59\pi$, utilizes less control effort in order to stabilize the system at the spatially-uniform steady state $\tilde{x}(z, t) = 0$, compared to the other three actuator placements.

Fig. 5 shows the norm of the control effort, $\|u\|$, for the four different actuator placements reported in Table 1, optimal case (solid line), case 2 (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line), for $x_s(0) = \phi_1$ and for two different, randomly generated, disturbance profiles. We can see that the control effort used for the optimal case is less than the other cases. Fig. 6 presents the same results as Fig. 5 but for the initial condition $x_s(0) = \phi_2$. One observe that there is no significant difference in the control effort needed for stabilization and uncertainty attenuation among the different actuator placements.

We now turn to the problem of optimal placement of the measurement sensors. For the sake of comparison, Table 2 shows $\hat{J}(e)$, $\hat{J}_s$ for the optimal sensor placement and two other placements. We can see from Table 2 that the computed optimal sensor locations at $z_{s1} = 0.35\pi$ and $z_{s2} = 0.66\pi$, have much smaller estimation error than the other two sensor locations. Fig. 7 shows the norm of the closed-loop estimation error versus time, for the optimal actuator/sensor locations, for $x_s(0) = \phi_1$ (top plot) and $x_s(0) = \phi_2$ (bottom plot) for five different, randomly generated, disturbances. Clearly, the estimation error is very small for both initial conditions.

Fig. 8 displays the profiles of the evolution of the temperature of the rod, under robust output feedback

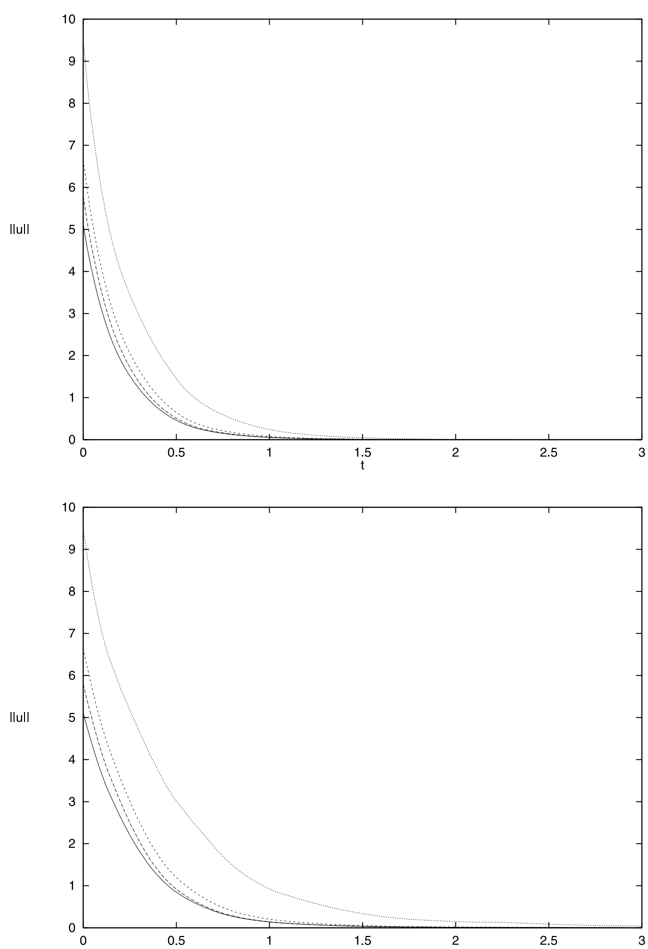

Fig. 5. Norm of the control effort, $\|u\|$ for two different disturbance profiles (top and bottom plots) for the optimal case (solid line), case 2 (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line)—$x_s(0) = \phi_1$.

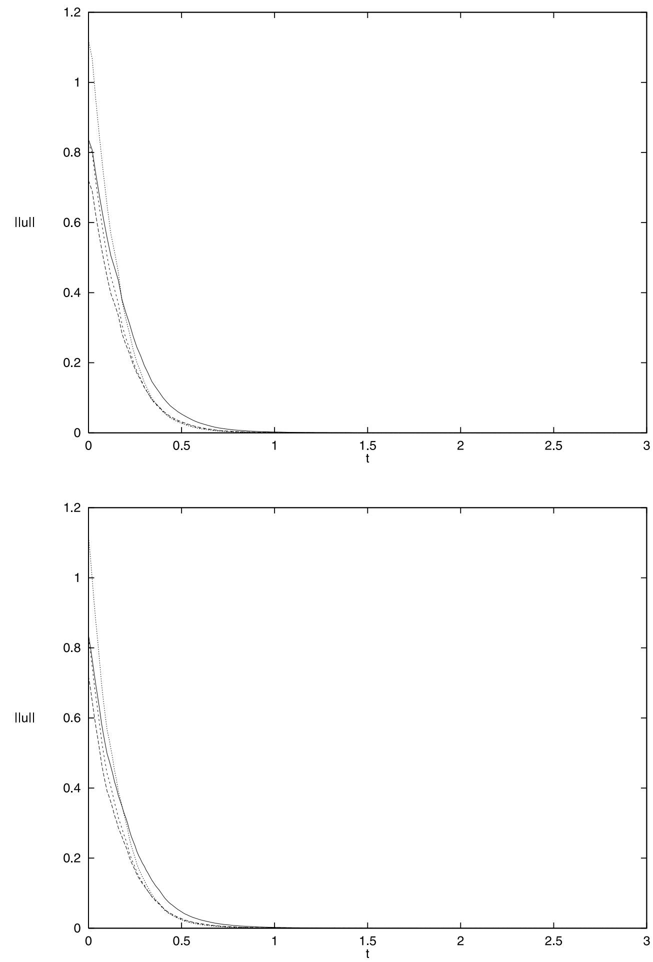

Fig. 6. Norm of the control effort, $\|u\|$ for two different disturbance profiles (top and bottom plots) for the optimal case (solid line), case 2 (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line)—$x_s(0) = \phi_2$.

Table 2
Results for the sensor locations

Case	Sensor locations	$\hat{J}(e)$	$\hat{J}_s$
Optimal	0.35π, 0.66π	8.862e−4	0.8992
2	0.30π, 0.64π	2.131e−2	0.9523
3	0.38π, 0.68π	5.435e−1	4.3531

control, for the optimal actuator/sensor locations, for $x_s(0)=\phi_1$ (top plot), and $x_s(0)=\phi_2$ (bottom plot). In both cases, the controller stabilizes the closed-loop PDE system at the spatially uniform operating steady-state very quickly despite the presence of time-varying uncertainty.

Finally, for the sake of comparison, we show in Figs. 9 and 10 the profiles of the evolution of the temperature of the rod under an output feedback controller that does not account for the presence of uncertainty for $x_s(0)=\phi_1$. In Fig. 9 we have used the optimal actuator/sensor locations ($z_{a1}=0.42\pi$, $z_{a2}=0.59\pi$ and $z_{s1}=0.35\pi$, $z_{s2}=0.66\pi$) and in Fig. 10 we have used the optimal actuator/sensor locations for the case where no uncertainty is considered ($z_{a1}=0.39\pi$, $z_{a2}=0.66\pi$ and $z_{s1}=0.31\pi$, $z_{s2}=0.72\pi$). We have used the same randomly generated disturbance profiles employed in the simulations shown in Fig. 8 (top plot), Figs. 9 and 10 in order for our comparison to be meaningful. We clearly observe that the closed-loop performance is significantly superior (in terms of uncertainty attenuation and speed of convergence to the steady-state) when the robust output feedback controller with the optimal actuator/sensor locations is applied to the system.

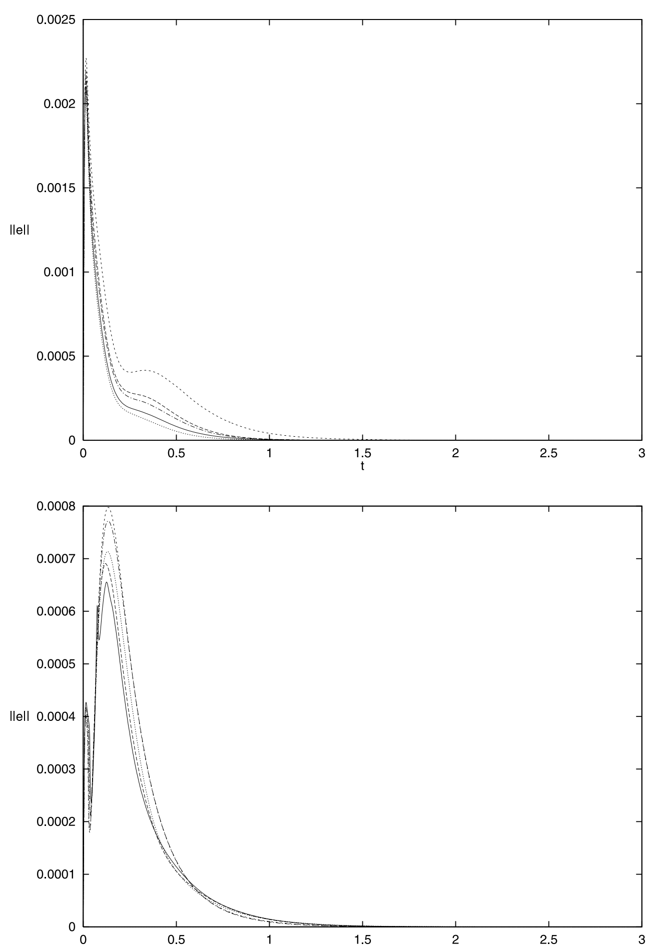

Fig. 7. Norm of the closed-loop estimation error $\|e\|$ versus time, for the optimal actuator/sensor locations, for five different disturbance profiles. Top plot: $x_s(0)=\phi_1$. Bottom plot: $x_s(0)=\phi_2$.

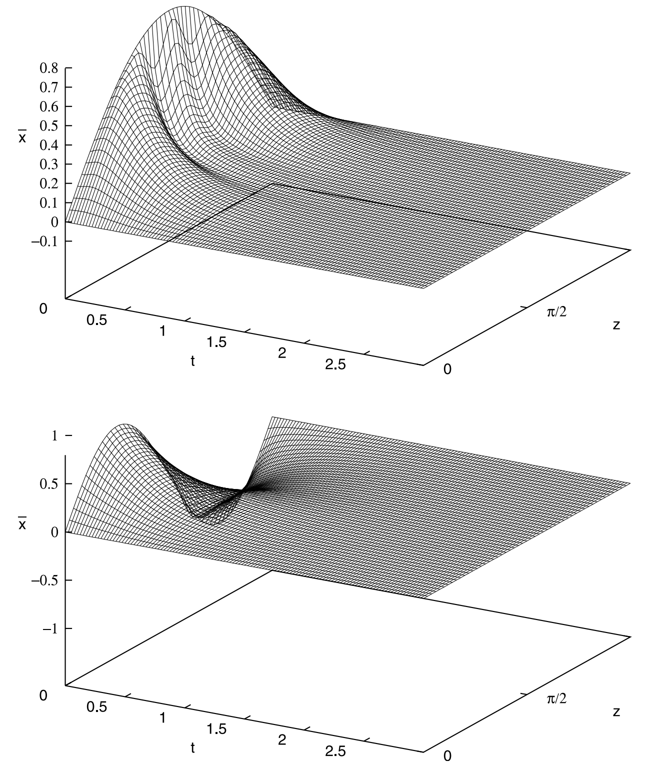

Fig. 8. Profile of evolution of the rod temperature under robust output feedback control with the optimal actuator/sensor locations. Top plot: $x_s(0)=\phi_1$. Bottom plot: $x_s(0)=\phi_2$.

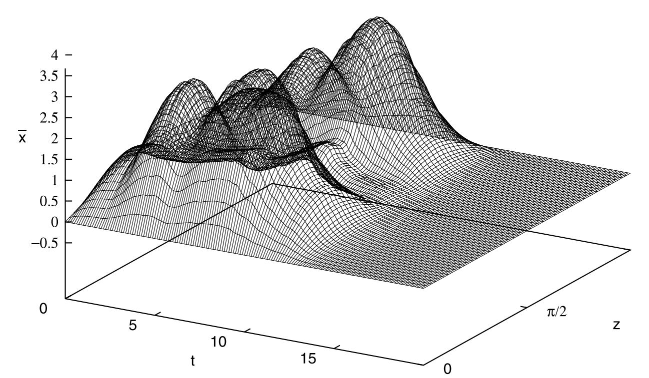

Fig. 9. Profile of evolution of the rod temperature under output feedback control without uncertainty attenuation using the optimal actuator/sensor locations—$x_s(0)=\phi_1$.

7. Conclusions

This paper focused on transport-reaction processes with unknown time-varying parameters and disturbances described by quasi-linear parabolic PDE systems, and addressed the problem of computing optimal actuator/sensor locations for robust nonlinear controllers. Galerkin's method was initially employed to derive finite-dimensional approximations of the PDE system which were used for the synthesis of robust nonlinear state feedback controllers via geometric and Lyapunov techniques and the computation of optimal actuator locations. The controllers enforce boundedness and uncertainty attenuation in the closed-loop system. The optimal actuator location problem was subsequently formulated as the one of minimizing a meaningful cost functional that includes penalty on the response of the closed-loop system and the control action. Owing to the boundedness of the state, the cost was defined over a finite-time interval (the final time is defined as the time needed for the process state to become smaller than the desired uncertainty attenuation limit), while the optimization was performed over a broad set of initial conditions and time-varying disturbance profiles. Subsequently, under the assumption that the number of measurement sensors is equal to the number of slow modes, we employed a standard procedure for obtaining estimates for the states of the approximate finite-dimensional model from the measurements. The optimal location of the measurement sensors was computed by minimizing a cost function of the estimation error in the closed-loop infinite-dimensional system. We showed that the use of these estimates in the robust state feedback controller leads to a robust output feedback controller, which guarantees boundedness of the state and uncertainty attenuation in the infinite-dimensional closed-loop system, provided that the separation between the slow and the fast eigenvalues is sufficiently large. We also established that the solution to the optimal actuator/sensor problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal in the sense that it approaches the optimal solution for the infinite-dimensional system as the separation between the slow and fast eigenvalues increases. The theoretical results were successfully applied to a typical diffusion-reaction process with nonlinearites and uncertainty to design a robust nonlinear output feedback controller and compute the optimal actuator/sensor locations for robust stabilization of an unstable steady state.

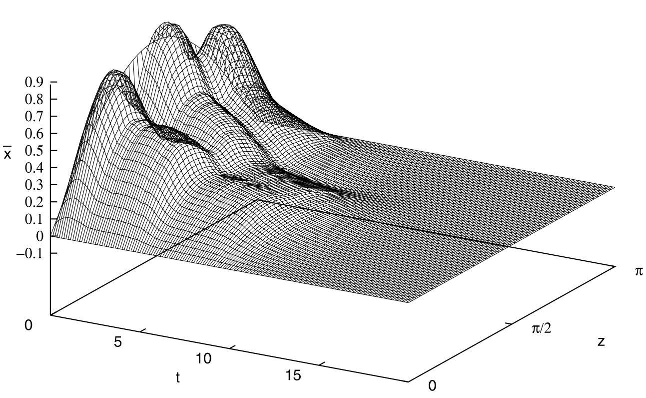

Fig. 10. Profile of evolution of the rod temperature, under output feedback control and optimal actuator/sensor placement which do not account for the disturbances—$x_s(0) = \phi_1$.

Acknowledgements

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Appendix A

Proof of Proposition 1. Substituting the control law of Eq. (16) into the finite-dimensional system of Eq. (12), we obtain:

$$\dot{\tilde{x}}_s = \Lambda_s \tilde{x}_s - \chi c(\tilde{x}_s, t) \frac{\tilde{x}_s}{|\tilde{x}_s| + \phi} + \mathcal{W}_s(\tilde{x}_s, 0, \theta) \tag{40}$$

To prove boundedness of the state of the above system and uncertainty attenuation, we use the following Lyapunov function candidate:

$$V = \frac{1}{2} \tilde{x}_s^2 \tag{41}$$

Computing the time-derivative of this function along the trajectories of the closed-loop system of Eq. (40), we obtain:

$$\dot{V} = \tilde{x}_s^T \dot{\tilde{x}}_s = \tilde{x}_s^T [\Lambda_s \tilde{x}_s - \chi c(\tilde{x}_s, t) \frac{\tilde{x}_s}{|\tilde{x}_s| + \phi} + \mathcal{W}_s(\tilde{x}_s, 0, \theta)] \tag{42}$$

Using the stability property of Λ_s and the bound of Assumption 3, the following bound for $\dot{V}$ can be computed:

$$\begin{aligned} \dot{V} &\leq -\alpha_1 \tilde{x}_s^2 - \chi c(\tilde{x}_s, t) \frac{\tilde{x}_s^2}{|\tilde{x}_s| + \phi} + |\tilde{x}_s| c(\tilde{x}_s, t) \\ &\leq -\alpha_1 \tilde{x}_s^2 + \frac{-\chi \tilde{x}_s^2 c(\tilde{x}_s, t) + \tilde{x}_s^2 c(\tilde{x}_s, t) + \phi |\tilde{x}_s| c(\tilde{x}_s, t)}{|\tilde{x}_s| + \phi} \\ &\leq -\alpha_1 \tilde{x}_s^2 + \frac{-(\chi - 1) \tilde{x}_s^2 c(\tilde{x}_s, t) + \phi |\tilde{x}_s| c(\tilde{x}_s, t)}{|\tilde{x}_s| + \phi} \end{aligned} \tag{43}$$

where α_1 is a strictly positive real number which is smaller or equal to the smallest eigenvalue of the matrix Λ_s. Since $|\tilde{x}_s|/|\tilde{x}_s| + \phi < 1$, we finally have:

$$\dot{V} \leq -\alpha_1 \tilde{x}_s^2 + (-(\chi - 1)|\tilde{x}_s| + \phi) c(\tilde{x}_s, t) \tag{44}$$

From the above inequality, it follows that when $|\tilde{x}_s| \geq \phi/(\chi - 1)$ (i.e. the state, $\tilde{x}_s$, is outside of a ball which is centered at the origin), then $-(\chi - 1)|\tilde{x}_s| + \phi \leq 0$, and thus, $\dot{V}$ satisfies the following bound:

$$\dot{V} \leq -\alpha_1 \tilde{x}_s^2 \tag{45}$$

This implies (Khalil, 1992) that there exist positive real numbers K, α_2 and a class K function $\gamma(\cdot)$ such that the state of the system of Eq. (40) satisfies:

$$|\tilde{x}_s(t)| \leq K|\tilde{x}_s(0)|e^{-\alpha_2 t} + \gamma(\phi) \tag{46}$$

From the above inequality, it follows directly that the state of the system of Eq. (40) remains bounded and asymptotically approaches a ball whose size is scaled by the controller parameter ϕ, $\lim_{t\to\infty} |\tilde{x}_s(t)| \leq \gamma(\phi)$.□

Proof of Theorem 1. The proof of this theorem will be obtained in two steps. In the first step, we will show boundedness and closeness of solutions for the closed-loop system of Eq. (9), provided that the initial conditions and ε are sufficiently small. In the second step, we will exploit the closeness of solutions result to show that the cost associated with the closed-loop PDE system approaches the optimal cost associated with the closed-loop finite-dimensional system under state feedback control, when the initial conditions and ε are sufficiently small, thereby establishing that the location of the control actuators obtained by using the finite-dimensional system is near-optimal.

Boundedness–closeness of solutions: Using that $\varepsilon = |\mathrm{Re}\{\lambda_1\}|/|\mathrm{Re}\{\lambda_{m+1}\}|$ and under the controller of Eq. (16), the closed-loop system of Eq. (9) takes the form:

$$\begin{aligned}\frac{dx_s}{dt} &= \Lambda_s x_s - \chi c(x_s, t)\frac{x_s}{|x_s| + \phi} + \mathscr{W}_s(x_s, x_f, \theta) + f_s(x_s, x_f)\\ &\quad - f_s(x_s, 0)\end{aligned} \tag{47}$$

$$\varepsilon\frac{\partial x_f}{\partial t} = \mathscr{A}_{f\varepsilon} x_f + \varepsilon\bar{f}_f(x_s, x_f) + \varepsilon\mathscr{W}_f(x_s, x_f, \theta)$$

where $\mathscr{A}_{f\varepsilon}$ is an unbounded differential operator defined as $\mathscr{A}_{f\varepsilon} = \varepsilon\mathscr{A}_f$, and $-\bar{f}_f(x_s, x_f) = \mathscr{B}_f\mathscr{B}_s^{-1}((\mathscr{A}_s - \Lambda_s)x_s + f_s(x_s, 0) - \chi c(x_s, t)(x_s/|x_s| + \phi)) + f_f(x_s, x_f)$. Since ε is a small positive number less than unity (Assumption 1, part 3), the system of Eq. (47) is in the standard singularly perturbed form, with x_s being the slow states and x_f being the fast states. Introducing the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq. (47):

$$\frac{\partial \tilde{x}_f}{\partial \tau} = \mathscr{A}_{f\varepsilon}\tilde{x}_f \tag{48}$$

where the tilde symbol in $\tilde{x}_f$, denotes that the state $\tilde{x}_f$ is associated with the approximation of the fast x_f-subsystem. From the fact that $\mathrm{Re}\{\lambda_{m+1}\} < 0$ and the definition of ε, we have that the above system is globally exponentially stable. Setting $\varepsilon = 0$ in the system of Eq. (47) and using that the operator $\mathscr{A}_{f\varepsilon}$ is invertible, we have that:

$$\tilde{x}_f = 0 \tag{49}$$

and thus the closed-loop of the finite-dimensional slow system takes the form:

$$\frac{d\tilde{x}_s}{dt} = \Lambda_s\tilde{x}_s - \chi c(\tilde{x}_s, t)\frac{\tilde{x}_s}{|\tilde{x}_s| + \phi} + \mathscr{W}_s(\tilde{x}_s, 0, \theta) \tag{50}$$

In Proposition 1, we proved that the state of the above slow subsystem is globally bounded. Since the state of the slow subsystem of Eq. (50) is bounded and the fast subsystem of Eq. (48) is globally exponentially stable, there exist positive real numbers μ_1, μ_2, and ε^* such that if $|x_s(t)| \leq \mu_1$, $\|x_f(t)\|_2 \leq \mu_2$, and $\varepsilon \in (0, \varepsilon^*]$, then the system of Eq. (47) is bounded and the solution $x_s(t)$, $x_f(t)$ of the system of Eq. (47) satisfies for all $t \in [t_b, T_f)$:

$$\begin{aligned}x_s(t) &= \tilde{x}_s(t) + O(\varepsilon)\\ x_f(t) &= \tilde{x}_f(t) + O(\varepsilon)\end{aligned} \tag{51}$$

where $t_b = O(\varepsilon)$ is the time required for $x_f(t)$ to approach $\tilde{x}_f(t)$. $\tilde{x}_s(t)$ and $\tilde{x}_f(t)$ are the solutions of the slow and fast subsystems of Eqs. (48) and (50), respectively (Christofides, 2001) and $T_f = O(1)$ is a positive constant.

Near-optimality of the actuator locations: The cost for the closed-loop infinite-dimensional system can be written as follows:

$$\begin{aligned}\hat{J} &= \frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{t_b}((x_s^T(x_s^i(0), \theta^\kappa, t),\ \mathbf{Q}_s x_s(x_s^i(0), \theta^\kappa, t))\\ &\quad + (x_f^T(x_s^i(0), \theta^\kappa, t),\ Q_f x_f(x_f^i(0), \theta^\kappa, t))\\ &\quad + u^T(x_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a)\ \mathbf{R}u(x_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a))\,dt\\ &\quad + \frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_{t_b}^{T_{f_{i\kappa}}}((x_s^T(x_s^i(0), \theta^\kappa, t),\ \mathbf{Q}_s x_s(x_s^i(0), \theta^\kappa, t))\\ &\quad + (x_f^T(x_s^i(0), \theta^\kappa, t),\ Q_f x_f(x_f^i(0), \theta^\kappa, t))\\ &\quad + u^T(x_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a)\ \mathbf{R}u(x_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a))\,dt\end{aligned} \tag{52}$$

From the closeness of solutions results of Eq. (51) and picking $T_{f_{i\kappa}} \leq T_f$ (this is always possible by appropriate choice of the controller parameters), it follows that for $t_b \leq t \leq T_{f_{i\kappa}}$ (outside of the boundary layer):

$$\begin{aligned}&x_s(x_s^i(0), \theta^\kappa, t) \to \tilde{x}_s(x_s^i(0), \theta^\kappa, t), \quad x_f(x_f^i(0), \theta^\kappa, t) \to 0,\\ &u(x_s(x_s^i(0), t), \mathbf{z}_a) \to u(\tilde{x}_s(x_s^i(0), t), \mathbf{z}_a) \qquad \text{as } \varepsilon \to 0\end{aligned} \tag{53}$$

and hence:

$$\begin{aligned}&\frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_{t_b}^{T_{f_{i\kappa}}}(x_s^T(x_s^i(0), \theta^\kappa, t)\ \mathbf{Q}_s x_s(x_s^i(0), \theta^\kappa, t)\\ &\quad + (x_f^T(x_f^i(0), \theta^\kappa, t)\ Q_f x_f(x_f^i(0), \theta^\kappa, t)\\ &\quad + u^T(x_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a)\ \mathbf{R}u(x_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a))\,dt\\ &\to \frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_{t_b}^{T_{f_{i\kappa}}}(\tilde{x}_s^T(x_s^i(0), \theta^\kappa, t)\ \mathbf{Q}_s\tilde{x}_s(x_s^i(0), \theta^\kappa, t)\\ &\quad + u^T(\tilde{x}_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a)\ \mathbf{R}u(\tilde{x}_s(x_s^i(0), \theta^\kappa, t), \mathbf{z}_a))\,dt\end{aligned} \tag{54}$$

as $\varepsilon \to 0$

From the boundedness of the state of the closed-loop system, we have that there exists a positive real number M that bounds the absolute values of the integrand of Eq. (52) for $t \in [0, T_f]$. Using the fact that $t_b = O(\varepsilon)$, we then have:

$$\frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{t_b}(x_s^T(x_s^i(0),\theta^\kappa,t)\ \mathbf{Q}_s x_s(x_s^i(0),\theta^\kappa,t) + (x_f^T(x_f^i(0),\theta^\kappa,t)\ Q_f x_f(x_f^i(0),\theta^\kappa,t) + u^T(x_s(x_s^i(0),\theta^\kappa,t),z_a)\ \mathbf{R}u(x_s(x_s^i(0),\theta^\kappa,t),z_a))\,dt \leq \int_0^{t_b} M\,dt \leq M\varepsilon = O(\varepsilon) \tag{55}$$

Similarly, from the stability of the closed-loop finite-dimensional system of Eq. (12) and the fact that $t_b = O(\varepsilon)$, we have that there exists a positive real number M' such that:

$$\frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{t_b}(\tilde{x}_s^T(x_s^i(0),\theta^\kappa,t)\ \mathbf{Q}_s\tilde{x}_s(x_s^i(0),\theta^\kappa,t) + u^T(\tilde{x}_s(x_s^i(0),\theta^\kappa,t),z_a)\ \mathbf{R}u(\tilde{x}_s(x_s^i(0),\theta^\kappa,t),z_a))\,dt \leq \int_0^{t_b} M'\,dt \leq M'\varepsilon = O(\varepsilon) \tag{56}$$

Combining Eqs. (54)–(56), we obtain:

$$\frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{T_{f_{i\kappa}}}(x_s^T(x_s^i(0),\theta^\kappa,t)\ \mathbf{Q}_s x_s(x_s^i(0),\theta^\kappa,t) + (x_f^T(x_f^i(0),\theta^\kappa,t)\ Q_f x_f(x_f^i(0),\theta^\kappa,t) + u^T(x_s(x_s^i(0),\theta^\kappa,t),z_a)\ \mathbf{R}u(x_s(x_s^i(0),\theta^\kappa,t),z_a))\,dt \to \frac{1}{K}\frac{1}{m}\sum_{\kappa=1}^{K}\sum_{i=1}^{m}\int_0^{T_{f_{i\kappa}}}(\tilde{x}_s^T(x_s^i(0),\theta^\kappa,t)\ \mathbf{Q}_s\tilde{x}_s(x_s^i(0),\theta^\kappa,t) + u^T(\tilde{x}_s(x_s^i(0),\theta^\kappa,t),z_a)\ \mathbf{R}u(\tilde{x}_s(x_s^i(0),\theta^\kappa,t),z_a))\,dt \tag{57}$$

as $\varepsilon \to 0$

This completes the proof of the theorem.

Proof of Theorem 2. Under the output feedback controller of Eq. (27), the closed-loop system takes the form:

$$\begin{aligned}\frac{dx_s}{dt} &= \Lambda_s x_s - \chi c(x_s+x_f,t)\frac{x_s+x_f}{|x_s+x_f|+\phi} + (\mathcal{A}_s-\Lambda_s)x_f + f_s(x_s,x_f) - f_s(x_s+x_f,0) + \mathcal{W}_s(x_s,x_f,\theta)\\ \varepsilon\frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\varepsilon}x_f + \varepsilon\mathcal{B}_f\mathcal{B}_s^{-1}((\mathcal{A}_s-\Lambda_s)(x_s+x_f)+f_s(x_s+x_f,0) - \chi c(x_s+x_f,t)\frac{x_s+x_f}{|x_s+x_f|+\phi}) + \varepsilon f_f(x_s,x_f) + \varepsilon\mathcal{W}_f(x_s,x_f,\theta)\end{aligned} \tag{58}$$

Using that ε is a small positive number less than unity (Assumption 1, part 3), and introducing the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, we obtain the following infinite-dimensional fast subsystem which describes the fast dynamics of the system of Eq. (58):

$$\frac{\partial\tilde{x}_f}{\partial\tau} = \mathcal{A}_{f\varepsilon}\tilde{x}_f \tag{59}$$

which is globally exponentially stable. Setting $\varepsilon = 0$ in the system of Eq. (58) and using that the operator $\mathcal{A}_{f\varepsilon}$ is invertible, we have that:

$$\tilde{x}_f = 0 \tag{60}$$

and thus the closed-loop of the finite-dimensional slow system takes the form:

$$\frac{d\tilde{x}_s}{dt} = \Lambda_s x_s - \chi c(x_s,t)\frac{x_s}{|x_s|+\phi} + \mathcal{W}_s(x_s,0,\theta) \tag{61}$$

which is globally bounded. Therefore, there exist positive real numbers μ_1, μ_1, and ε^* such that if $|x_s(t)| \leq \mu_1$, $\|x_f(t)\|_2 \leq \mu_2$, and $\varepsilon \in (0, \varepsilon^*]$, then the system of Eq. (10) is globally bounded and the solution $x_s(t)$, $x_f(t)$ of the system of Eq. (58) satisfies the estimates of Eq. (58).

Given the stability and closeness of solutions results for the closed-loop system, the near-optimality of the control actuators and measurement sensors in the sense described in Eq. (28) can be established by using similar calculations to the ones in part 2 of the proof of Theorem 1.

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