PERGAMON

Chemical Engineering Science 54 (1999) 5677–5709

Chemical Engineering Science

Feedback control of nonlinear differential difference equation systems

Charalambos Antoniades, Panagiotis D. Christofides*

Department of Chemical Engineering, University of California, Los Angeles, CA 90095-1592, USA

Received 20 March 1998; accepted 30 November 1998

Abstract

This paper proposes a methodology for the synthesis of nonlinear output feedback controllers for single-input single-output nonlinear differential difference equation (DDE) systems which include time delays in the states, the control actuator and the measurement sensor. Initially, DDE systems which only include state delays are considered and a novel combination of geometric and Lyapunov-based techniques is employed for the synthesis of nonlinear state feedback controllers that guarantee stability and enforce output tracking in the closed-loop system, independently of the size of the state delays. Then, the problem of designing nonlinear distributed state observers, which reconstruct the state of the DDE system while guaranteeing that the discrepancy between the actual and the estimated state tends exponentially to zero, is addressed and solved by using spectral decomposition techniques for DDE systems. The state feedback controllers and the distributed state observers are combined to yield distributed output feedback controllers that enforce stability and output tracking in the closed-loop system, independently of the size of the state delays. For DDE systems with state, control actuator and measurement delays, distributed output feedback controllers are synthesized on the basis of an auxiliary output constructed within a Smith-predictor framework. The proposed control method is successfully applied to a reactor-separator process with recycle and a fluidized catalytic cracker and is shown to outperform nonlinear controller designs that do not account for the presence of dead time associated with the recycle loop and the pipes transferring material from the reactor to the regenerator and vice versa, respectively. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Nonlinear control; Nonlinear state estimation; Time-delay processes

1. Introduction

The dynamic models of many chemical engineering processes involve severe nonlinearities and significant time delays and are naturally described by nonlinear differential difference equation (DDE) systems. Nonlinearities usually arise from complex reaction mechanisms and Arrhenius dependence of reaction rates on temperature, while time delays often occur due to transportation lag such as in flow through pipes, dead times associated with measurement sensors (measurement delays) and control actuators (manipulated input delays), and approximation of high-order dynamics. Typical examples of processes which involve nonlinearities and time delays include chemical reactors with recycle loops, fluidized catalytic cracking units, distillation columns, chemical vapor deposition processes to name a few.

The conventional approach to control linear/nonlinear DDE systems is to neglect the presence of time delays and address the controller design problem on the basis of the resulting linear/nonlinear ordinary differential equation (ODE) systems, employing standard control methods for ODE systems. However, it is well-known (see, for example, Stephanopoulos, 1984) that such an approach may pose unacceptable limitations on the achievable control quality and cause serious problems in the behavior of the closed-loop system including poor performance (e.g., sluggish response, oscillations) and instability.

Motivated by the above, significant research efforts have focused on the development of control methods for linear DDE systems that compensate for the effect of time delays. Research initially focused on linear systems with a single manipulated input delay which are described by transfer function models, in which the presence of the time delay prevents the use of large controller gains (i.e., the proportional gain of a proportional-integral controller should be sufficiently small in order to avoid destabilization of the closed-loop system), thereby leading to

* Corresponding author. Tel.: + 1-310-794-1015; fax: + 1-310-206-4107.

E-mail address: pdc@seas.ucla.edu (P.D. Christofides)

PII: S0009-2509(99)00148-7

sluggish closed-loop response. To overcome this problem, (Smith proposed in 1957) proposed a control structure, known as Smith predictor, which completely eliminates the time delay from the characteristic polynomial of the closed-loop system, allowing the use of larger controller gains. Since then many researchers have proposed alternatives or modifications of the Smith predictor structure (e.g., Vit, 1979), extensions to control structures for linear systems including inferential control (Brosilow, 1976) and internal model control (Garcia and Morari, 1982) other predictor structures such as the analytical predictor (Moore et al., 1970; Wellons and Edgar, 1987) and established connections of the Smith predictor with other predictors (Wong and Seborg, 1986). The Smith predictor structure has also been extended to linear multivariable systems with multiple input and output delays which are described by transfer function models, leading to multi-delay compensators (Jerome and Ray, 1986; Ogunnaike and Ray, 1979). Excellent reviews of results on Smith and other predictor structures can be found in Jerome and Ray (1986) and Wong and Seborg (1986).

Even though the above works provided powerful methods for dealing with control actuator and measurement dead time in linear systems, they do not explicitly account for the effect of time delays in the process state variables. This motivates research on the design of controllers for DDE systems with state delays. In this direction, the application of classical optimal control approaches to DDE system in order to design optimal distributed parameter (infinite-dimensional) controllers was initially studied (e.g., Ray, 1981; Soliman and Ray, 1972). Then, the distributed nature of the developed controllers motivated research on the problem of model reduction of linear DDE systems. This problem is the one of finding a linear low-dimensional ODE system that accurately reproduces the solutions of a linear DDE system. Approaches to address this problem include balanced approximation based on controllability and observability gramians (Lu et al., 1987), frequency response analysis (Gu et al., 1989) and approximations using Fourier–Laguerre models (Partington, 1991; Wahlberg, 1991) to name a few. An alternative approach for the synthesis of controllers for linear DDE systems with state delays that stabilize the closed-loop system independently of the size of the delays is based on the method of Lyapunov functionals (Hale and Verduyn Lunel, 1993). The central idea of this approach is to synthesize a linear controller so that the time derivative of an appropriate Lyapunov functional calculated along the trajectories of the closed-loop DDE system is negative definite, independently of the size of the delays. The method of Lyapunov functionals has been used in the design of linear stabilizing controllers for linear DDE systems in Nazaroff (1973) and Ross and Flugge-Lotz (1969).

Despite the abundance of results on control of linear DDE systems, most of the research on nonlinear DDE systems has focused on the derivation of conditions for existence and uniqueness of solutions, the understanding of qualitative and geometric properties of the solutions (see the book by Hale and Verduyn Lunel, 1993 for results and reference lists), and stability analysis through Razumikhin-type theorems and nonlinear small-gain theorem techniques (e.g., Teel, 1998; Xu and Liu, 1994). Very few results are available on control of nonlinear DDE systems, with the exception of optimal control methods (Koivo and Koivo, 1978; Ray, 1981). For nonlinear systems represented by input–output models, extensions of the Smith predictor have been proposed in Bartee et al. (1989) and Wong and Seborg (1988). Within a state-space framework, the only available results on controller synthesis are for single-input single-output nonlinear systems with a single manipulated input delay, for which a nonlinear Smith predictor structure was proposed in Kravaris and Wright (1989) under the assumption of open-loop stability, and further extended to open-loop unstable systems in Henson and Seborg, (1994). At this stage, there is no rigorous, yet practical, method for the design of nonlinear controllers for *nonlinear* DDE systems with state, manipulated input and measured output delays.

This paper proposes a methodology for the synthesis of nonlinear output feedback controllers for single-input single-output nonlinear differential difference equation (DDE) systems which include time delays in the states, the control actuator and the measurement sensor. Initially, DDE systems which only include state delays are considered and a novel combination of geometric and Lyapunov-based techniques is employed for the synthesis of nonlinear state feedback controllers that guarantee stability and enforce output tracking in the closed-loop system, independently of the size of the state delays. Then, the problem of designing nonlinear distributed state observers, which reconstruct the state of the DDE system while guaranteeing that the discrepancy between the actual and the estimated state tends exponentially to zero, is addressed and solved by using spectral decomposition techniques for DDE systems. The state feedback controllers and the distributed state observers are combined to yield distributed output feedback controllers that enforce stability and output tracking in the closed-loop system, independently of the size of the state delays. For DDE systems with state, control actuator and measurement delays, distributed output feedback controllers are synthesized on the basis of an auxiliary output constructed within a Smith-predictor framework. The proposed control method is successfully applied to an exothermic reactor-separator process with recycle and a fluidized catalytic cracker and is shown to outperform nonlinear controller designs that do not account for the presence of dead time associated with the recycle loop and the pipes transferring material from the reactor to the regenerator and vice versa, respectively.

2. Differential difference equation systems

2.1. Description of nonlinear DDE systems

We consider single-input single-output systems of nonlinear differential difference equations with the following state-space description:

$$\begin{aligned}\dot{\bar{x}} &= A\bar{x}(\bar{t}) + \sum_{k=1}^{q} B_k\,\bar{x}(\bar{t}-\bar{\alpha}_k)\\ &\quad + f(\bar{x}(\bar{t}), \bar{x}(\bar{t}-\bar{\alpha}_1), \ldots, \bar{x}(\bar{t}-\bar{\alpha}_q))\\ &\quad + g(\bar{x}(\bar{t}), \bar{x}(\bar{t}-\bar{\alpha}_1), \ldots, \bar{x}(\bar{t}-\bar{\alpha}_q))u(\bar{t}-\hat{\alpha}_1),\\ &\quad \bar{x}(\xi) = \bar{\eta}(\xi),\ \xi \in [-\bar{\alpha}, 0),\ \bar{x}(0) = \bar{\eta}_0,\\ y &= h(\bar{x}(\bar{t}-\hat{\alpha}_2)),\end{aligned} \tag{1}$$

where $\bar{x} \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}$ denotes the manipulated input, and $y \in \mathbb{R}$ denotes the controlled output (whose on-line measurements are assumed to be available). $\bar{\alpha}_k$, $k = 1, \ldots, q$, denotes the kth state delay, $\bar{\alpha} = \max\{\bar{\alpha}_1, \ldots, \bar{\alpha}_q\}$, $\hat{\alpha}_1$ denotes the manipulated input delay (control actuator dead time) and $\hat{\alpha}_2$ denotes the measured output delay (measurement sensor dead time). $A, B_1, \ldots, B_q$ are constant matrices of dimension $n \times n$, $f(\bar{x}(\bar{t}), \bar{x}(\bar{t}-\bar{\alpha}_1), \ldots, \bar{x}(\bar{t}-\bar{\alpha}_q))$, $g(\bar{x}(\bar{t}), \bar{x}(\bar{t}-\bar{\alpha}_1), \ldots, \bar{x}(\bar{t}-\bar{\alpha}_q))$ are locally Lipschitz nonlinear vector functions, $h(\bar{x}(\bar{t}-\hat{\alpha}_2))$ is a locally Lipschitz nonlinear scalar function, $\bar{\eta}(\xi)$ is a smooth vector function defined in the interval $[-\bar{\alpha}, 0)$ and $\bar{\eta}_0$ is a constant vector. We will assume that the vector function $f(\bar{x}(\bar{t}), \bar{x}(\bar{t}-\bar{\alpha}_1), \ldots, \bar{x}(\bar{t}-\bar{\alpha}_q))$, includes only nonlinear terms and satisfies $f(0, 0, \ldots, 0) = 0$ which implies that $x(t) \equiv 0$ is an equilibrium solution for the open-loop (i.e., $u(\bar{t}-\hat{\alpha}_1) \equiv 0$) system of Eq. (1).

There are many chemical engineering processes whose dynamic models involve time delays in the state variables and are naturally described by nonlinear DDE systems of the form of Eq. (1). Example of such processes include chemical reactors with recycle loops (where the state delays occur due to transportation lag in the recycle loops), fluidized catalytic cracking reactors (where the state delays occur due to dead time in pipes transferring material from the regenerator to the reactor and vice versa) and distillation columns (where the state delays occur due to dead time in reboiler and condenser recycle loops). Furthermore, control actuator and measurement sensor dead times are also very common sources of time delays in chemical process control, and they are explicitly accounted for in the DDE system of Eq. (1). The linear appearance of the manipulated input u in the system of Eq. (1) is also typical in most practical applications, where inlet flow rates, inlet temperatures and concentrations are typically chosen as manipulated inputs. Finally, the assumption that the controlled output is identical to the measured output is done in order to simplify the notation of the paper and can be readily relaxed (i.e., the extension of the proposed theory to systems in which the controlled output is different from the measured output is conceptually straightforward).

To simplify the presentation of the results of the paper, we will transform the DDE system of Eq. (1) into an equivalent DDE system which includes state and measurement delays and does not include manipulated input delay. We will also focus on DDE systems with a single state delay (the generalization of the results of the paper to the case of DDE systems with multiple state delays is conceptually straightforward and will not be presented here for reasons of brevity). To this end, we set $\tilde{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2$, $t = \bar{t} - \hat{\alpha}_1$, $q = 1$, $B_1 = B$, $\bar{\alpha} = \bar{\alpha}_1$, $\alpha = \hat{\alpha}_1 + \bar{\alpha}$, $x(t) = \bar{x}(t + \hat{\alpha}_1)$ and obtain the following system (which will be used in our development):

$$\begin{aligned}\dot{x} &= Ax(t) + Bx(t-\alpha) + f(x(t), x(t-\alpha))\\ &\quad + g(x(t), x(t-\alpha))u(t),\\ x(\xi) &= \bar{\eta}(\xi),\\ &\quad \xi \in [-\alpha, 0),\ x(0) = \bar{\eta}_0,\\ y &= h(x(t-\tilde{\alpha})).\end{aligned} \tag{2}$$

Remark 1. In order to compare the proposed approach for control of nonlinear DDE systems with existing approaches for control of linear DDE systems, we set $f(x(t), x(t-\alpha)) = 0$, $g(x(t), x(t-\alpha)) = c$ and $h(x(t-\tilde{\alpha})) = wx(t-\tilde{\alpha})$, where c, w are constant vectors, in the system of Eq. (2) to derive the following linear DDE system:

$$\begin{aligned}\dot{x} &= Ax(t) + Bx(t-\alpha) + cu(t),\\ x(\xi) &= \bar{\eta}(\xi),\ \xi \in [-\alpha, 0),\ x(0) = \bar{\eta}_0,\\ y &= wx(t-\tilde{\alpha}),\end{aligned} \tag{3}$$

which will be used to synthesize linear state feedback controllers and state observers.

In the next subsection, a typical chemical process example (Ogunnaike and Ray, 1979) is given in order to illustrate modeling of a chemical process in the form of Eq. (2).

2.2. Example of a chemical process modeled by a nonlinear DDE system

Consider the cascade of two perfectly mixed chemical reactors with recycle loop, which is shown in Fig. 1. A first-order irreversible reaction of the form $A \to B$ takes place in the reactors. The process possesses an inherent state delay due to the transportation lag in the recycle loop. Under standard modeling assumptions, the

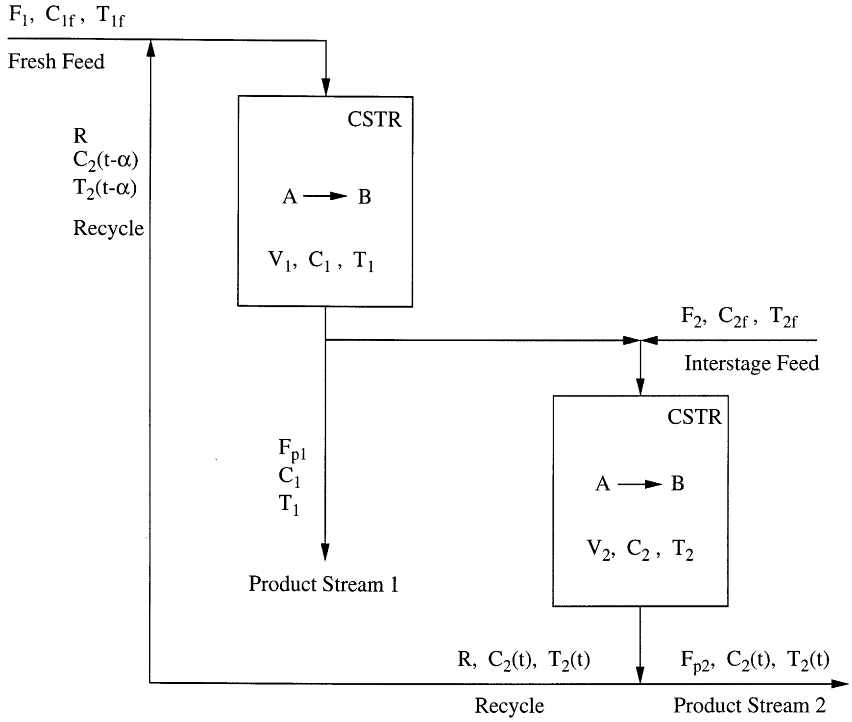

Fig. 1. Two chemical reactors with recycle loop.

dynamic model of the process can be derived from mass and energy balances and consists of the following system of four nonlinear differential difference equations:

$$\frac{dC_1}{dt} = \frac{F_1}{V_1} C_{1f} - \frac{(F_1 + R)}{V_1} C_1 + \frac{R}{V_1} C_2(t-\alpha) - k_0 e^{-E/RT_1} C_1,$$

$$\frac{dT_1}{dt} = \frac{F_1}{V_1} T_{1f} - \frac{(F_1 + R)}{V_1} T_1 + \frac{R}{V_1} T_2(t-\alpha) - \frac{(-\Delta H)}{\rho c_p} k_0 e^{-E/RT_1} C_1,$$

$$\frac{dC_2}{dt} = \frac{F_2}{V_2} C_{2f} - \frac{(F_{p2} + R)}{V_2} C_2 + \frac{(F_1 + R - F_{p1})}{V_2} C_1 - k_0 e^{-E/RT_2} C_2,$$

$$\frac{dT_2}{dt} = \frac{F_2}{V_2} T_{2f} - \frac{(F_{p2} + R)}{V_2} T_2 + \frac{(F_1 + R - F_{p1})}{V_2} T_1 - \frac{(-\Delta H)}{\rho c_p} k_0 e^{-E/RT_2} C_2, \quad (4)$$

where F_1, F_2 denote the flow rates of the inlet streams to the two reactors, V_1, V_2 denote the volumes of the two reactors, R denotes the recycle (from the second to the first reactor) flow rate, F_{p1}, F_{p2} denote the flow rate of the product streams from the two reactors, C_{1f}, C_{2f} denote the concentration of species A in the inlet streams to the reactors, C_1, C_2 denote the concentration of species A in the reactors, T_{1f}, T_{2f} denote the temperature of the inlet streams to the two reactors, T_1, T_2 denote the temperature in the two reactors, $k_0, E, \Delta H$ denotes the pre-exponential constant, the activation energy and the enthalpy of the reaction, c_p, ρ denote the heat capacity and the density of the reacting liquid, and x denotes the recycle loop dead time.

A typical control problem for this process can be formulated as the one of regulating the concentration of species A in the first reactor, C_1, by manipulating the feed concentration of A in the first reactor, C_{1f}. Setting

$$x_1 = C_1, x_2 = C_2, x_3 = T_1, x_4 = T_2, u = C_{1f}, y = C_1$$

the original set of equations can be put in the form of Eq. (2).

3. Mathematical properties of DDE systems

The objective of this section is to present the basic mathematical properties of DDE systems that will be used in our development. We will begin with the spectral properties of DDE systems, and we will continue with stability concepts and results.

3.1. Spectral properties

In this subsection, we formulate the system of Eq. (2) as an infinite-dimensional system in an appropriate Banach space and provide the statement and solution of the eigenvalue problem for the linear delay operator (see Eq. (7) below). The solution of the eigenvalue problem will be utilized in the design of nonlinear distributed state observers in Section 7. We formulate the system of Eq. (2) in the Banach space $\mathscr{C}([-\alpha, 0], \mathbb{R}^n)$ of continuous n-vector valued functions defined in the interval $[-\alpha, 0]$ with inner product and norm:

$$(\tilde{\omega}_1, \tilde{\omega}_2) = \tilde{\omega}_1(0)\tilde{\omega}_2(0) + \int_{-\alpha}^{0} \tilde{\omega}_1(z+\alpha)B\tilde{\omega}_2(z)\,dz,$$

$$\|\tilde{\omega}_1\|_2 = (\tilde{\omega}_1, \tilde{\omega}_1)^{1/2}, \tag{5}$$

where $\tilde{\omega}_1$ is an element of $\mathscr{C}^*([-\alpha, 0], \mathbb{R}^{n*})$, $\mathbb{R}^{n*}$ is the n-dimensional vector space of row vectors, and $\tilde{\omega}_2 \in \mathscr{C}$. On $\mathscr{C}$, the state function x of the system of Eq. (2) is defined as

$$x_t(\xi) = x(t+\xi),\ t \geq 0,\ \xi \in [-\alpha, 0], \tag{6}$$

the operator $\mathscr{A}$ as

$$\mathscr{A}\phi(\xi) = \begin{cases} \dfrac{d\phi(\xi)}{d\xi}, & \xi \in [-\alpha, 0), \\ A\phi(0) + B\phi(-\alpha), & \xi = 0, \end{cases} \tag{7}$$

$$\phi(\xi) \in D(\mathscr{A}) = \{\phi \in \mathscr{C}^*([-\alpha, 0], \mathbb{R}^{n*}):$$
$$\dot{\phi} \in \mathscr{C}, \dot{\phi}(0) = A\phi(0) + B\phi(-\alpha)\}, \tag{8}$$

and the output as

$$y(t) = h(Px_t), \tag{9}$$

where $P: \mathscr{C} \to \mathbb{R}^n$ is defined by $Px_t = x_t(0)$. $D(\mathscr{A})$ is a dense subset of $\mathscr{C}$. If $\eta \in D(\mathscr{A})$, then the system of Eq. (2) can be equivalently written as

$$\frac{dx_t}{dt} = Ax_t + f(Px_t, Qx_t) + g(Px_t, Qx_t)u,$$

$$x_0(\xi) = \bar{\eta},\ \xi \in (-x, 0),\ x_0(0) = \bar{\eta}_0,$$

$$y(t) = h(Px_t), \tag{10}$$

where $Qx_t = x_t(-\alpha)$.

The eigenvalue problem for the operator $\mathscr{A}$ is defined as

$$\mathscr{A}\phi_j = \lambda_j\phi_j, \quad j = 1, \ldots, \infty, \tag{11}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction (note that ϕ_j is a vector of dimension n); the eigenspectrum of $\mathscr{A}, \sigma(\mathscr{A})$, is defined as the set of all eigenvalues of $\mathscr{A}$, i.e., $\sigma(\mathscr{A}) = \{\lambda_1, \lambda_2, \ldots,\}$ and is given by (Hale and Verduyn Lunel, 1993)

$$\sigma(\mathscr{A}) = \{\lambda: \det(\lambda I - A - Be^{-\lambda\alpha}) = 0\}. \tag{12}$$

The eigenfunctions can be directly computed from the formula $\phi_\lambda = e^{\lambda\xi}\phi_\lambda(0)$, where $\phi_\lambda(0)$ satisfies the equation $(\lambda I - A - Be^{-\lambda\alpha})\phi_\lambda(0) = 0$. The adjoint operator $\bar{\mathscr{A}}$ is defined from the relation $(\mathscr{A}\phi, \psi) = (\phi, \bar{\mathscr{A}}\psi)$, where $(,)$ denotes the inner product of Eq. (5), and its eigenspectrum, $\sigma(\bar{\mathscr{A}})$, satisfies $\sigma(\bar{\mathscr{A}}) = \sigma(\mathscr{A})$.

Remark 2. Regarding the properties of the eigenspectrum of $\mathscr{A}$, several comments are in order (Hale and Verduyn Lunel, 1993; Manitius et al., 1987) (a) the eigenspectrum $\sigma(\mathscr{A})$ is a point spectrum consisting of eigenvalues, $\lambda \in \sigma(\mathscr{A})$, of finite multiplicity, $\kappa(\lambda)$, (b) the number of eigenvalues of $\sigma(\mathscr{A})$ which have positive real part (i.e., they are located in the right-half of the complex plane) is always finite, (c) the real parts of all the eigenvalues are bounded from above (i.e., there exists a positive real number β such that $|\mathrm{Re}\lambda_i| \leq \beta$ for all $i = 1, \ldots, \infty$), and (d) the eigenvalues are asymptotically distributed along nearly vertical asymptotes in the complex plane.

Remark 3. To illustrate the formulation of a DDE system in an infinite-dimensional Banach space, and the formulation and solution of the corresponding eigenvalue problem for the delay operator, we consider the following numerical example:

$$\dot{x} = \begin{bmatrix} -2.0 & 3.5 \\ 3.0 & -3.0 \end{bmatrix} x(t) + \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} x(t-3) + \begin{bmatrix} 4x_1x_2 - 3x_1^2 - x_2^2 \\ 0 \end{bmatrix} \tag{13}$$

For the above system, the solution $(x_1, x_2) = (0, 0)$ is the unique equilibrium solution, the Banach space is $\mathscr{C}([-3, 0], \mathbb{R}^2)$ and the operator $\mathscr{A}$ takes the form

$$\mathscr{A}\phi(\xi) = \begin{cases} \dfrac{d\phi(\xi)}{d\xi}, & \xi \in [-3, 0), \\ \begin{bmatrix} -2.0 & 3.5 \\ 3.0 & -3.0 \end{bmatrix} \phi(0) + \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \phi(-3), & \xi = 0, \end{cases} \tag{14}$$

$$\phi(\xi) \in D(\mathscr{A}) = \Bigg\{ \phi \in \mathscr{C}^*([-3, 0], \mathbb{R}^{2*}): \dot{\phi} \in \mathscr{C},$$
$$\dot{\phi}(0) = \begin{bmatrix} -2.0 & 3.5 \\ 3.0 & -3.0 \end{bmatrix} \phi(0) + \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \phi(-3) \Bigg\}. \tag{15}$$

The eigenvalue problem for $\mathscr{A}$ was solved numerically by using the mathematical software MAPLE and was found that $\sigma(\mathscr{A})$ includes two unstable eigenvalues $\lambda_1 = 0.58$, $\lambda_2 = 0.21$ and infinitely many stable eigenvalues; this implies that the equilibrium solution (0,0) of the system of Eq. (13) is unstable.

We finally note that even though the solution $(x_1, x_2) = (0,0)$ of the DDE system of Eq. (13) is unstable, the origin of the *undelayed* system:

$$\dot{x} = \begin{bmatrix} -4.0 & 3.5 \\ 3.0 & -3.0 \end{bmatrix} x(t) + \begin{bmatrix} 4x_1x_2 - 3x_1^2 - x_2^2 \\ 0 \end{bmatrix} \quad (16)$$

is globally asymptotically stable (the linearization of the above system around the origin possesses two stable eigenvalues: $\lambda_1 = -6.78, \lambda_2 = -0.22$ and $(x_1, x_2) = (0,0)$ is the unique equilibrium point).

3.2. Stability concepts and results

From the analysis of the previous subsection, it is evident that DDE systems of the form of Eq. (2) possess fundamentally different properties from ODE systems. The main difference is that the state space of a DDE system is infinite-dimensional, while the state space of an ODE system is finite-dimensional. Therefore, the rigorous analysis of the stability properties of the system of Eq. (2) requires the use of stability concepts and results for DDE systems. In what follows, we review the definitions of asymptotic stability and input-to-state stability for DDE systems as well as a basic theorem that provides sufficient conditions for assessing asymptotic stability. The reader may refer to the classic book (Hale and Verduyn Lunel, 1993) for a complete treatment of stability issues for nonlinear differential difference equations.

Consider the following system of nonlinear differential difference equations:

$$\dot{x}(t) = f(x(t), x(t-\alpha), \theta(t), \theta(t-\alpha)),$$

$$x(\xi) = \bar{\eta}(\xi),$$

$$\xi \in [-\alpha, 0), x(0) = \bar{\eta}_0, \quad (17)$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$, and suppose that $f(0,0,0,0) = 0$. Now, given a function $\theta: [-\alpha, \infty) \to \mathbb{R}^m$ and $t \in [0, \infty)$, $\theta_t(\xi)$ represents a function from $[-\alpha, 0]$ to $\mathbb{R}^m$ defined by $\theta_t(\xi) := \theta(t+\xi)$. We also define $|\theta_t(\xi)| := \max_{t-\alpha \leqslant \xi \leqslant t} |\theta_t(\xi)|_{\mathbb{R}^m}$, $\|\theta_t\| := \sup_{s \geqslant 0} |\theta_s(\xi)|$ and $\|\theta_t\|_T := \sup_{0 \leqslant s \leqslant T} |\theta_s(\xi)|$. Finally, $|\cdot|_{\mathbb{R}^m}$ denotes the standard Euclidean norm in $\mathbb{R}^m$. Definition 1 that follows provides a rigorous statement of the concept of input-to-state stability for the system of Eq. (17).

Definition 1. (Teel, 1998). Let γ be a function of class-Q (see definition of class-Q function in the appendix) and δ_x, δ_θ be positive real numbers. The zero solution of Eq. (17) is said to be input-to-state stable if $|x_0(\xi)| \leqslant \delta_x$ and $\|\theta_t\| \leqslant \delta_\theta$ imply that the solution of the system of Eq. (17) is defined for all times and satisfies

$$|x_t(\xi)| \leqslant \beta(|x_0(\xi)|, t) + \gamma(\|\theta_t\|), \forall t \geqslant 0. \quad (18)$$

The above definition, when $\theta(t) \equiv 0, \forall t \geqslant 0$ reduces to the definition of asymptotic stability for the zero solution of the DDE system of Eq. (17). Furthermore, when $\alpha = 0$, Definition 1 reduces to the standard definition of input-to-state stable for nonlinear ODE systems with external inputs (see, for example, Khalil, 1992). Finally, we note that from the definition of $|x_t(\xi)|$ and Eq. (18), it follows that $|x(t)|_{\mathbb{R}^n} \leqslant |x_t(\xi)| \leqslant \beta(|x_0(\xi)|, t) + \gamma(\|\theta_t\|), \forall t \geqslant 0$.

The following theorem provides sufficient conditions for the stability of the zero solution of the system of Eq. (17), expressed in terms of a suitable functional, and consists a natural generalization of the direct method of Lyapunov for ordinary differential equations. The result of this theorem will be directly used in the solution of the state feedback control problem in Section 5 below.

Theorem 1. (Hale and Verduyn Lunel 1993). *Consider the system of Eq.* (17) *with* $\theta(t) \equiv 0$ *and let* $\gamma_1, \gamma_2, \gamma_3$ be *functions of class Q. If* $\gamma_1(s), \gamma_2(s) > 0$ *for* $s > 0$ *and there is a continuous functional* $V: \mathscr{C} \to \mathbb{R}_{\geqslant 0}$ *such that*:

$$\gamma_1(|x(t)|_{\mathbb{R}^n}) \leqslant V(x_t(\xi)) \leqslant \gamma_2(|x_t(\xi)|),$$

$$\dot{V}(x_t(\xi)) \leqslant -\gamma_3(|x(t)|_{\mathbb{R}^n}) \quad (19)$$

then the solution $x = 0$ *is stable. If* $\gamma_3(s) > 0$ *for* $s > 0$, *then the solution* $x = 0$ *is locally asymptotically stable. If* $\gamma_1(s) \to \infty$ *as* $s \to \infty$ *and* $\gamma_3(s) > 0$ *for* $s > 0$, *then the solution* $x = 0$ *is globally asymptotically stable.*

Remark 4. Even though, at this stage, there is no systematic way for selecting the form of the functional $V(x_t(\xi))$ which is suitable for a particular application, a choice for $V(x_t(\xi))$, which is frequently used to show local exponential stability of a DDE system of the form of Eq. (17) via Theorem 1, is

$$V(x_t(\xi)) = x(t)^{\mathrm{T}} C x(t) + a^2 \int_{t-\alpha}^{t} x(s)^{\mathrm{T}} E x(s)\, \mathrm{d}s, \quad (20)$$

where E, C are symmetric positive-definite matrices and a is a positive real number. Clearly, the functional of Eq. (20) satisfies $K_1|x(t)|^2_{\mathbb{R}^n} \leqslant V(x_t(\xi)) \leqslant K_2|x_t(\xi)|^2$ for some positive K_1, K_2.

4. Nonlinear control of DDE systems with small time delays

In order to motivate the need for accounting for the presence of time delays in the controller design, we will now establish that a direct application of any nonlinear control method to DDE systems without accounting for the presence of time delays, will lead to the design of controllers that enforce stability and output tracking in the closed-loop system, provided that the time delays are sufficiently small.

We assume that there exists a nonlinear output feedback controller of the form

$$\dot{\omega} = \mathscr{F}(\omega, y, v),$$

$$u = \mathscr{P}(\omega, y, t), \tag{21}$$

where $\mathscr{F}(\omega, y, v)$ is a vector function, $\mathscr{P}(\omega, y, t)$ is a scalar function, and v is the set point, which has been designed on the basis of the system

$$\dot{x} = Ax(t) + Bx(t) + f(x(t), x(t)) + g(x(t), x(t))u(t),$$

$$x(0) = x_0,$$

$$y = h(x(t)) \tag{22}$$

so that the closed-loop system:

$$\dot{\omega} = \mathscr{F}(\omega, y, v), \omega(0) = \omega_0$$

$$\dot{x} = Ax(t) + Bx(t) + f(x(t), x(t))$$

$$+ g(x(t), \mathrm{G}x(t))\mathscr{P}(\omega, y, t), x(0) = x_0,$$

$$y = h(x(t)). \tag{23}$$

is locally exponentially stable (see Khalil (1992) for stability definitions for ODE systems) and the discrepancy between y and v is asymptotically zero (i.e., $\lim_{t\to\infty}|y - v|_{\mathbb{R}} = 0$).

Theorem 2 that follows establishes that if the controller of Eq. (21) enforces local exponential stability and asymptotic output tracking in the closed-loop system of Eq. (23), then it also enforces these properties in the closed-loop system of Eqs. (2)–(21), provided that the state and measurement delays are sufficiently small (the proof of the theorem can be found in the appendix).

Theorem 2. *If the ODE system of Eq.* (23) *is locally exponentially stable, then the nonlinear DDE system of Eq.* (2) *under the nonlinear output feedback controller of Eq.* (21) *is locally exponentially stable and the discrepancy between* y and v *tends asymptotically to zero (i.e.,* $\lim_{t\to\infty}|y - v|_{\mathbb{R}} = 0$*), provided that* α *and* $\tilde{\alpha}$ *are sufficiently small.*

Remark 5. Even though the result of the theorem clearly indicates the need for accounting for the presence of time delays in the controller design, it is worth mentioning that Theorem 2 establishes a very important robustness property of any nonlinear controller with respect to small time delays. This property may also be useful in many practical applications, because it could lead to simplifications in the controller design task, which, for DDE systems with small time delays, could be addressed on the basis of an ODE system instead of a DDE one.

5. Nonlinear control of DDE systems with large time delays

5.1. Problem statement

Since the application of controller design methods which do not account for the presence of time delays is limited to DDE systems with small time delays, we address, in this paper, the problem of synthesizing nonlinear output feedback controllers of the general form

$$\dot{\omega} = A\omega(t) + B\omega(t-\alpha) + f(\omega(t), \omega(t-\alpha))$$

$$+ g(\omega(t), \omega(t-\alpha))\mathscr{A}(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))$$

$$+ \mathscr{I}(y(t) - h(\omega(t))),$$

$$u = \mathscr{A}(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha)). \tag{24}$$

where $\mathscr{I}(\cdot)$ is a nonlinear integral operator, $\mathscr{A}(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))$ is a nonlinear scalar function, $\bar{v}(s) = [v(s)\ v^{(1)}(s)\ \cdots\ v^{(r-1)}(s)]^{\mathrm{T}}$, $s \in [t-\alpha, t]$ and $v^{(k)}$ denotes the kth time derivative of the reference input $v \in \mathbb{R}$, that enforce exponential stability, asymptotic output tracking and time-delay compensation in the closed-loop system, *independently of the size of the state and measurement delays.*

5.2. Methodological framework

To develop a comprehensive method for the synthesis of controllers of the form of Eq. (24) that enforce the requested properties in the closed-loop system, we will employ a methodology which involves the following steps:

1. Synthesis of nonlinear state feedback controllers that enforce stability and output tracking in the closed-loop system, independently of the size of the state delay.
2. Design of nonlinear distributed state observers (i.e., the observer itself is a system of nonlinear integro-differential equations) that produce estimates of the unknown state variables of the process with guaranteed asymptotic convergence of the error between the actual and estimated states to zero.
3. Synthesis of distributed nonlinear output feedback controllers through combination of the developed state feedback controllers with the distributed state observers.

Initially, nonlinear state feedback controllers will be synthesized for DDE systems which only include state delays by employing a novel combination of geometric control concepts with the method of Lyapunov functionals (i.e., the controllers will be synthesized in such a way so that the time derivative of an appropriate Lyapunov functional calculated along the trajectories of the closed-loop system is negative definite). Then, nonlinear distributed state observers will be constructed for DDE systems which only include state delays by using spectral decomposition techniques. Finally, for DDE systems with state and measurement delays, the output feedback controller will be synthesized on the basis of an auxiliary output constructed within a Smith-predictor framework.

6. Nonlinear state feedback control for DDE systems with state delays

In this section, we consider systems of the form of Eq. (2) without measurement delay, i.e., $\tilde{\alpha} = 0$, (this assumption will be removed below) and assume that measurements of the states are available (i.e., $x(s)$ for $s \in [t-\alpha, t]$, $t \in [0, \infty)$ is known). For these systems, we address the problem of synthesizing nonlinear static state feedback control laws of the general form

$$u = \mathscr{A}(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha)). \tag{25}$$

where $\mathscr{A}(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha))$ is a nonlinear scalar function, that (a) guarantees local exponential stability, (b) forces the output to asymptotically follow the reference input (i.e., ensure that $\lim_{t\to\infty}|y - v|_{\mathbb{R}} = 0$), and (c) compensate for the effect of the time delay on the output, in the closed-loop system. The structure of the control law of Eq. 25 is motivated by available results on stabilization of linear DDE systems (e.g., Nazaroff, 1973; Ross and Flugge-Lotz, 1969) and the requirement of output tracking.

In order to proceed with the explicit synthesis of the control law of Eq. (25), we will need to make certain assumptions on the structure and stability properties of the system of Eq. (2). To simplify the statement of these assumptions, we introduce the notation $f(x(t), x(t-\alpha)) = f_1(x(t)) + f_2(x(t), x(t-\alpha))$, $\tilde{f}(x(t)) = Ax(t) + f_1(x(t))$, and $\bar{p}(x(t), x(t-\alpha)) = Bx(t-\alpha) + f_2(x(t), x(t-\alpha))$, which allows us to rewrite the system of Eq. (2) in the following form:

$$\dot{x} = \tilde{f}(x(t)) + g(x(t), x(t-\alpha))u + \bar{p}(x(t), x(t-\alpha)),$$

$$y = h(x). \tag{26}$$

The first assumption is motivated by the requirement of output tracking and will play a crucial role in the synthesis of the controller.

Assumption 1. *Referring to the system of Eq.* (26), *there exists an integer r and a change of variables:*

$$\begin{bmatrix} \zeta(s) \\ \eta(s) \end{bmatrix} = \begin{bmatrix} \zeta_1(s) \\ \zeta_2(s) \\ \vdots \\ \zeta_r(s) \\ \eta_1(s) \\ \vdots \\ \eta_{n-r}(s) \end{bmatrix} = \mathscr{X}(x(s)) = \begin{bmatrix} h(x) \\ L_{\tilde{f}}h(x(s)) \\ \vdots \\ L_{\tilde{f}}^{r-1}h(x(s)) \\ \chi_1(x(s)) \\ \vdots \\ \chi_{n-r}(x(s)) \end{bmatrix}, \tag{27}$$

where $s \in [t-\alpha, t]$ and $\chi_1(x(s)), \ldots, \chi_{n-r}(x(s))$ *are scalar functions such that the system of Eq.* (26) *takes the form*

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2(t) + p_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\
&\vdots\\
\dot{\zeta}_{r-1} &= \zeta_r(t) + p_{r-1}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\
\dot{\zeta}_r &= L_{\tilde{f}}^r h(\mathscr{X}^{-1}(\zeta(t), \eta(t)))\\
&\quad + L_g L_{\tilde{f}}^{r-1} h(\mathscr{X}^{-1}(\zeta(t), \eta(t)))u\\
&\quad + p_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\
\dot{\eta}_1 &= \Psi_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\
&\vdots\\
\dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\
y &= \zeta_1,
\end{aligned} \tag{28}$$

where $p_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \ldots, p_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \Psi_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \ldots, \Psi_{n-r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))$ *are nonlinear Lipschitz functions and* $L_g L_{\tilde{f}}^{r-1} h(x) \neq 0$ *for all* $x(s) \in \mathbb{R}^n$ *and* $s \in [t-\alpha, t]$.

Assumption 1 provides the explicit form of a coordinate change (which is independent of the state delay present in the system of Eq. (2)) that transforms the nonlinear DDE system of Eq. (2) into an interconnection of two subsystems, the ζ-subsystem which describes the input/output dynamics of the system of Eq. (2) and the η-subsystem which includes the dynamics of the system of Eq. (2) which are unobservable from the output. Specifically, the interconnection of Eq. (28) is obtained by considering the change of variables of Eq. (27) with $s = t$, differentiating it with respect to time, and using that $x(t) = \mathscr{X}^{-1}(\zeta(t), \eta(t))$ and $x(t-\alpha) = \mathscr{X}^{-1}(\zeta(t-\alpha), \eta(t-\alpha))$ (note that this is possible because the coordinate change of Eq. (27) is assumed to be valid for $s \in [t-\alpha, t]$). The coordinate transformation of Eq. (27) is not restrictive from an application point of view (one can easily verify that Assumption 1 holds for the two chemical reactors with recycle of Section 2.2 and the two applications studied in Section 10). The assumption that $L_g L_{\tilde{f}}^{r-1} h(x) \neq 0$ for all $x(s) \in \mathbb{R}^n$ and $s \in [t-\alpha, t]$ is necessary in order to guarantee that the controller which will be synthesized is well-posed in the sense that it does not generate infinite control action for any values of the states of the process (compare with the structure of the controller given in Theorem 3).

To proceed with the controller design, we need to impose the following stability requirement on the η-subsystem of the system of Eq. (28) which will allow addressing the controller synthesis task on the basis of the low-order ζ-subsystem.

Assumption 2. *The dynamical system*

$$\begin{aligned}\dot{\eta}_1 &= \Psi_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))\end{aligned} \quad (29)$$

is input-to-state stable (see Definition 1 for a precise statement of this concept) with respect to the input $\zeta_t(\xi)$.

Loosely speaking, the above assumption states that if the state of the ζ-subsystem is bounded, then the state of the η-subsystem will also remain bounded (see Remark 8 for an interpretation of the η-subsystem). In practice, Assumption 2 can be verified by linearizing the system of Eq. (29) with $\zeta_t(\xi) = 0$ around the operating steady state and computing the eigenvalues of the resulting linear system. If all of these eigenvalues are in the left-half of the complex plane, then (Teel, 1998) Assumption 2 is satisfied locally (i.e., for sufficiently small initial conditions and $\zeta_t(\xi)$). An application of this approach for checking Assumption 2 is discussed in Section 10.2.2.

Using Assumption 2, the controller synthesis problem can be now addressed on the basis of the ζ-subsystem. Specifically, applying the following preliminary feedback law,

$$\begin{aligned}u = {}&\frac{1}{L_g L_{\tilde{f}}^{r-1} h(\mathscr{X}^{-1}(\zeta(t), \eta(t)))}(\tilde{u} - L_{\tilde{f}}^{r} h(\mathscr{X}^{-1}(\zeta(t), \eta(t)))\\ &- p_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))),\end{aligned} \quad (30)$$

where $\tilde{u}$ is an auxiliary input, to the system of Eq. (28) in order to cancel all the nonlinear terms that can be cancelled by using a feedback which utilizes measurements of $x(s)$ for $s \in [t-\alpha, t]$, we obtain the following modified system:

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 + p_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\ &\vdots\\ \dot{\zeta}_{r-1} &= \zeta_r + p_{r-1}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))),\\ \dot{\zeta}_r &= \tilde{u},\\ \dot{\eta}_1 &= \Psi_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\ y &= \zeta_1.\end{aligned} \quad (31)$$

Introducing the notation

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}, \quad b = \begin{bmatrix}0\\0\\0\\ \vdots\\ 1\end{bmatrix},$$

$$p(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) = \begin{bmatrix} p_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))\\ p_2(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))\\ \vdots\\ p_{r-1}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))\\ 0\end{bmatrix}, \quad (32)$$

the ζ-subsystem of the system of Eq. (31) can be written in the following compact form:

$$\begin{aligned}\dot{\zeta} &= \tilde{A}\zeta + b\tilde{u} + p(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),\\ y &= \zeta_1.\end{aligned} \quad (33)$$

The controller synthesis task has been now reduced to the one of synthesized $\tilde{u}$ to stabilize the ζ-subsystem and force the output y to asymptotically follow the reference input, v. To develop a solution to this problem, we will need to make the following assumption on the growth of the vector $p(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))$.

Assumption 3. *Let* $\bar{e}(s) = [(h(x(s)) - v(s))\ \ (L_{\tilde{f}} h(x(s)) - v^{(1)}(s)) \ldots (L_{\tilde{f}}^{r-1} h(x(s)) - v^{(r-1)}(s))]^{\mathrm{T}}, s \in [t-\alpha, t]$ *where* $v^{(k)}$ *denotes the kth time derivative of the reference input v. There exist positive real numbers* a_1, a_2 *such that the following bound can be written*:

$$\begin{aligned}&|p(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha))|^2_{\mathbb{R}^n}\\ &\quad \leqslant a_1\bar{e}^2(t) + a_2\bar{e}^2(t-\alpha).\end{aligned} \quad (34)$$

The above assumption on the growth of the vector $p(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha))$ does not need to hold globally (i.e., for any $\bar{e}(t), \eta(t)$), and thus, it is satisfied by most practical problems (see, for example, the applications studied in Section 10). Furthermore, Assumption 3 will allow us to synthesize a linear auxiliary feedback law of the form $\tilde{u} = K\bar{e}$ to stabilize the ζ-subsystem and enforce output tracking. The synthesis of such a $\tilde{u}$ will be performed by using the method of Lyapunov functionals. Specifically, $\tilde{u}$ will be designed so that the time derivative of the following Lyapunov functional:

$$V(\bar{e}_t(\xi)) = \bar{e}^{\mathrm{T}} P\bar{e} + a^2 \int_{t-\alpha}^{t} \bar{e}^{\mathrm{T}}(s)\bar{e}(s)\,\mathrm{d}s, \quad (35)$$

where a is a positive real number, calculated along the state of the closed-loop ζ-subsystem is negative definite. The incorporation of the integral term $\int_{t-\alpha}^{t} \bar{e}^{\mathrm{T}}(s)\bar{e}(s)\,\mathrm{d}s$ in the functional of Eq. (35) allows accounting for the distributed parameter (delayed) nature of the system of Eq. (2) in the controller design stage and synthesizing a controller that enforces the requested properties in the closed-loop system independently of the size of the state delay.

We are now in a position to state the main controller synthesis result of this section. Theorem 3 that follows provides the formula of the controller and conditions under which stability and output tracking is guaranteed in the closed-loop system (the proof of the theorem can be found in the appendix).

Theorem 3. *Consider the system of nonlinear differential difference equations of Eq.* (2) *with* $\tilde{\alpha} = 0$, *for which assumptions 1, 2, and 3 hold. Then, if the matrix equation*

$$\tilde{A}^{\mathrm{T}} P + P\tilde{A} - 2P^{\mathrm{T}} bR_2^{-1}b^{\mathrm{T}} P + (a^2 + a_1)I + P^2 = -R_1, \tag{36}$$

where $a^2 > a_2$, *and* R_1, R_2 *are positive-definite matrices, has a unique positive-definite solution for* P, *the nonlinear state feedback controller*:

$$\begin{aligned} u &= \mathscr{A}(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha)) \\ &:= \frac{1}{L_g L_{\tilde{f}}^{r-1} h(x)} (- R_2^{-1} b^{\mathrm{T}} P\bar{e}(t) + v^{(r)}(t) - L_{\tilde{f}}^{r} h(x) \\ &\quad - p_r(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha))) \end{aligned} \tag{37}$$

enforces (i) *local exponential stability*, *and* (ii) *asymptotic output tracking in the closed-loop system*, *independently of the size of the state delay.*

Remark 6. Regarding the structure, implementation and closed-loop properties of the nonlinear state feedback controller of Eq. (37), several remarks are in order: (a) it uses measurements of the states of the process at t and $t-\alpha$ (i.e., $x(t)$ and $x(t-\alpha)$), and thus, it belongs to the class of the requested control laws of Eq. (25), (b) its practical implementation requires the use of memory lines to store the values of x in the time interval $[t-\alpha, t]$, and (c) it enforces stability and asymptotic output tracking in the closed-loop system *independently of the size of the state delay.*

Remark 7. In order to apply the result of Theorem 3 to a chemical process application, one has to initially verify Assumptions 1–3 of the theorem on the basis of the process model and compute the parameters a_1 and a_2. Then, a, R_1, R_2 should be chosen so that $a^2 > a_2$ and the matrices R_1, R_2 are positive-definite to ensure that Eq. (36) has a unique positive-definite solution for P. Regarding the role of R_1, R_2 on closed-loop properties, we note that R_1 determines the speed of the closed-loop output response (namely, "larger" (in terms of the smallest eigenvalue) R_1 means faster response), while R_2 determines the penalty that should be imposed on the manipulated input in achieving stabilization and output tracking ("larger" R_2 means larger penalty on the control action). If these assumptions are satisfied, the synthesis formula of Eq. (37) can be directly used to derive the explicit form of the controller (see Section 10 for the application of this procedure to two chemical process examples).

Remark 8. In analogy to the case of nonlinear ODE systems (see, for example, Isidori, 1989; Kravaris and Arkun, 1991), one can show that the η-subsystem of Eq. (29) represents the inverse dynamics of the DDE system of Eq. (2). Moreover, the η-subsystem of Eq. (29) with $\zeta_t(\xi) = 0$, i.e.

$$\begin{aligned} \dot{\eta}_1 &= \Psi_1(0, \eta(t), 0, \eta(t-\alpha)), \\ &\vdots \\ \dot{\eta}_{n-r} &= \Psi_{n-r}(0, \eta(t), 0, \eta(t-\alpha)) \end{aligned} \tag{38}$$

represents the zero dynamics of the DDE system of Eq. (2) (i.e., the dynamics of Eq. (2) when the output is set identically equal to zero). Linearizing the zero dynamics of Eq. (38) around the zero solution, one can show that the eigenvalues of the resulting linear system are identical to the zeros (which are infinite) of the linear DDE system of Eq. (3).

Remark 9. Applying the proposed method for the synthesis of state feedback controllers for DDE systems to linear systems of the form of Eq. (3) with $\tilde{\alpha} = 0$, we end up with the following controller synthesis formula:

$$\begin{aligned} u &= [wA^{r-1}c]^{-1} (- R_2^{-1} b^{\mathrm{T}} P\bar{e}(t) + v^{(r)}(t) - wA^r x(t) \\ &\quad - wA^{r-1}Bx(t-\alpha)), \end{aligned} \tag{39}$$

where P is the solution of Eq. (36). The linear controller of Eq. (39) can be thought of as an extension of linear control laws of the form

$$u = F_1\, x(t) + F_2\, x(t-\alpha), \tag{40}$$

where F_1, F_2 are constant vectors of appropriate dimensions, which were considered in the context of stabilization of linear DDE systems (e.g., Nazaroff, 1973; Ross and Flugge-Lotz, 1969), to the problems of stabilization with output tracking. We note that the usual approach followed in the literature for the design of the vectors F_1, F_2 is based on the method of Lyapunov functionals.

Remark 10. A control problem which has attracted significant attention in the area of nonlinear process control is the one of specifying the explicit formula of a controller that enforces a linear response between the controlled output and the reference input in the closed-loop system. This problem was solved in Kravaris and Chung (1987) for systems of nonlinear ODEs, and more recently, in Christofides and Daoutidis (1996) for systems of nonlinear hyperbolic PDEs. In this remark, we formally pose this problem for DDE systems of the form of Eq. (2) with $\tilde{\alpha} = 0$ and show that it leads to the synthesis of a nonlinear feedback controller which is non-realizable, and thus, it cannot be implemented in practice. Specifically, we seek to design a nonlinear feedback controller that enforces the following linear input/output response:

$$\gamma_r \frac{d^r y}{dt^r} + \cdots + \gamma_1 \frac{dy}{dt} + y = v, \tag{41}$$

where r is the 'relative order' of the system of Eq. (2) (i.e., the smallest derivative of y which depends explicitly on the manipulated input u) and $\gamma_1, \gamma_2, \ldots, \gamma_r$ are adjustable parameters, in the closed-loop system. Calculating the time-derivatives of the output y, up to order r, in the system of Eq. (2), we obtain the following expressions:

$$\begin{aligned} y &= h(x), \\ \frac{dy}{dt} &= \psi_1(x(t), x(t-\alpha)), \\ \frac{d^2y}{dt^2} &= \psi_2(x(t), x(t-\alpha), x(t-2\alpha)), \\ &\vdots \\ \frac{d^r y}{dt^r} &= \psi_r(x(t), x(t-\alpha), \ldots, x(t-r\alpha)) \\ &\quad + \phi(x(t), x(t-\alpha), \ldots, x(t-r\alpha))u, \end{aligned} \tag{42}$$

where $\psi_1(x(t), x(t-\alpha)), \ldots, \psi_r(x(t), x(t-\alpha), \ldots, x(t-r\alpha))$, $\phi(x(t), x(t-\alpha), \ldots, x(t-r\alpha))$ are smooth nonlinear functions whose specific form is omitted for brevity. Substituting the expressions for the time derivatives of y in Eq. (41), assuming that the term $\phi(x(t), x(t-\alpha), \ldots, x(t-r\alpha)) \neq 0$ and solving for u, we obtain the following expression for the controller that enforces the linear response of Eq. (41) in the closed-loop system:

$$\begin{aligned} u(t) &= \frac{1}{\phi(x(t), x(t-\alpha), \ldots, x(t-r\alpha))} (v - h(x(t)) \\ &\quad - \sum_{v=1}^{r} \gamma_v \psi_v(x(t), x(t-\alpha), \ldots, x(t-v\alpha)). \end{aligned} \tag{43}$$

The above controller clearly uses measurements of $x(t-v\alpha)$ with $2 \leqslant v \leqslant r$ which may not be available, and thus, it cannot be implemented in practice. To illustrate this point, we consider the representation of the controller of Eq. (43) for $t = 0$ i.e.

$$\begin{aligned} u(0) &= \frac{1}{\phi(x(0), x(-\alpha), \ldots, x(-r\alpha))} (v - h(x(0)) \\ &\quad - \sum_{v=1}^{r} \gamma_v \psi_v(x(0), x(-\alpha), \ldots, x(-v\alpha)). \end{aligned} \tag{44}$$

Clearly, the calculation of the initial control action, $u(0)$, requires values of the state, such as $x(-2\alpha), \ldots, x(-r\alpha)$, which are not included in the initial data of the system of Eq. (2), and thus, $u(0)$, cannot be realized. We finally note that an extension of the initial conditions in the past as a remedy to this problem is not meaningful, since, from a mathematical point of view, it leads to an ill-defined DDE system, while, from a practical point of view, it may require past knowledge of the state at an arbitrarily large time interval which, in general, there is no guarantee that exists.

7. Nonlinear state observer design for DDE systems with state delay

In this section, we consider nonlinear DDE systems of the form of Eq. (2) with $\tilde{\alpha} = 0$ and focus on the design of nonlinear state observers that use measurements of process output, $y(s)$, for $s \in [t-\alpha, t]$ to produce estimates of the state variables $x(t)$, with guaranteed exponential convergence of the error between the actual and the estimated values of $x(t)$ to zero. Specifically, we consider the design of state observers with the following general state-space description:

$$\begin{aligned} \dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega(t), \omega(t-\alpha)) \\ &\quad + g(\omega(t), \omega(t-\alpha))u + \mathscr{I}(y(t) - h(\omega(t))), \\ \omega(\xi) &= \bar{\omega}(\xi),\ \xi \in [-\alpha, 0),\ \omega(0) = \bar{\omega}_0, \end{aligned} \tag{45}$$

where $\omega \in \mathbb{R}^n$ is the observer state, $\bar{\omega}(\xi)$ is a smooth vector function defined in $\xi \in [-\alpha, 0)$, $\bar{\omega}_0$ is a constant vector and $\mathscr{I}(\cdot)$ is a bounded nonlinear integral operator, mapping $\mathbb{R}$ into $\mathscr{C}$. The system of Eq. (45) consists of a replica of the system of Eq. (2) and the term $\mathscr{I}(y(t) - h(\omega(t)))$ which will be designed so that the system of Eq. (45) is locally exponentially stable and the discrepancy between $x(t)$ and $\omega(t)$ tends exponentially to zero. The derivation of the explicit form of the integral operator $\mathscr{I}(\cdot)$ will be performed by working with the infinite-dimensional formulation, Eq. (10), of the nonlinear DDE system of Eq. (2) presented in Section 3.1., and using a spectral decomposition of the DDE system and nonlinear observer design results for finite-dimensional systems.

We initially transform the system of Eq. (10) into an interconnection of a finite-dimensional system that describes the dynamics of the eigenmodes of $\mathscr{A}$ corresponding to the unstable eigenvalues of $\sigma(\mathscr{A})$, and an infinite-dimensional system that describes the dynamics of the remaining eigenmodes of $\mathscr{A}$. Let $H = \{\lambda: \lambda \in \sigma(\mathscr{A}) \text{ and } \mathrm{Re}\,\lambda \geqslant 0\}$ and assume, in order to simplify the development, that $\kappa(\lambda) = 1$ for each $\lambda \in \sigma(\mathscr{A})$ (i.e., the multiplicity of all the eigenvalues is assumed to be one; the usual case in most practical applications). Let m be the number of eigenvalues included in H (note that m is always finite). Also, let the elements of H be ordered as $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, where $\mathrm{Re}\,\lambda_1 \geqslant \mathrm{Re}\,\lambda_2 \geqslant \ldots \geqslant \mathrm{Re}\,\lambda_m$. Clearly, the eigenfunctions (column vectors) in the $n \times m$ matrix $\Phi_H = [\phi_1, \phi_2, \ldots, \phi_m]$ form a basis in $\mathscr{C}_{\lambda \geqslant 0}$. Also, let $\Psi_H = [\psi_1, \psi_2, \ldots, \psi_m]^T$ be the basis in $\mathscr{C}^*_{\lambda \geqslant 0}$ chosen so that $(\Psi_H, \Phi_H) = I$, where $(\Psi_H, \Phi_H) = [(\psi_i, \phi_j)]$, ψ_i and ϕ_j being the ith element and jth element of Ψ_H and Φ_H, respectively. Defining the orthogonal projection operators P_p and P_n such that $x_t^p = P_p x_t$, $x_t^n = P_n x_t$ (note that $x_t^p = \Phi_H(\Psi_H, x_t)$), the state x_t of the system of Eq. (10) can be decomposed as

$$x_t = x_t^p + x_t^n = P_p x_t + P_n x_t. \tag{46}$$

Applying P_p and P_n to the system of Eq. (10) and using the above decomposition for x_t, the system of Eq. (10) can be equivalently written in the following form:

$$\begin{aligned}
\frac{dx_t^p}{dt} &= \mathscr{A}_p x_t^n + f_p(P(x_t^p + x_t^n), Q(x_t^p + x_t^n)) \\
&\quad + g_p(P(x_t^p + x_t^n), Q(x_t^p + x_t^n))u, \\
\frac{\partial x_t^n}{\partial t} &= \mathscr{A}_n x_t^n + f_n(P(x_t^p + x_t^n), Q(x_t^p + x_t^n)) \\
&\quad + g_n(P(x_t^p + x_t^n), Q(x_t^p + x_t^n))u, \\
y &= h(P(x_t^p + x_t^n)), \\
x_t^p(0) &= P_p x(0) = P_p\bar{\eta},\ x_t^n(0) = P_n x(0) = P_n\bar{\eta},
\end{aligned} \tag{47}$$

where $\mathscr{A}_p = P_p \mathscr{A} P_p$, $g_p = P_p g$, $f_p = P_p f$, $\mathscr{A}_n = P_n \mathscr{A} P_n$, $g_n = P_n g$ and $f_n = P_n f$ and the notation $\partial x_t^n/\partial t$ is used to denote that the state x_t^n belongs in an infinite-dimensional space. In the above system, $\mathscr{A}_p = \Phi_H A_p$, where A_p is a matrix of dimension $m \times m$ whose eigenvalues are the ones included in H, f_p and f_n are Lipschitz vector functions, and $\mathscr{A}_n$ is an infinite range matrix which is stable (this follows from the fact that H includes all the eigenvalues of, $\sigma(\mathscr{A})$, which are in the closed right half of the complex plane). Neglecting the x_n-subsystem, the following finite-dimensional system is obtained:

$$\begin{aligned}
\frac{dx_t^p}{dt} &= \mathscr{A} P x_t^p + f_p(P x_t^p, Q x_t^p) + g_p(P x_t^p, Q x_t^p)u, \\
y_p &= h(P x_t^p),
\end{aligned} \tag{48}$$

where the subscript p in y_p denotes that the output is associated with an approximate finite dimensional system.

Assumption 4 that follows states that the system of of Eq. (48) is observable and is needed in order to design a nonlinear state observer for the system of Eq. (2) (the reader may refer to Soroush (1997) for a precise definition of the concept of observability for nonlinear finite-dimensional systems; see also Bhat and Koivo (1976) for definitions and characterizations of observability for linear DDE systems).

Assumption 4. *The pair* $[h(x_t^p)\mathscr{A}_p x_t^p + f_p(x^p, 0)]$ *is locally observable in the sense that there exists a nonlinear gain column vector* $L(x_t^p)$ of *dimension* m (*where* m *is the number of unstable eigenvalues of* $\mathscr{A}$) *so that the finite-dimensional dynamical system*:

$$\frac{d\omega_t^p}{dt} = \mathscr{A}_p \omega_t^p + f_p(P\omega_t^p, Q\omega_t^p) + L(\omega_t^p)(y_p - h(P\omega_t^p)) \tag{49}$$

is locally exponentially stable.

Theorem 4 that follows provides a nonlinear distributed state observer (the proof of the theorem can be found in the appendix).

Theorem 4. *Referring to the system of Eq.* (10) *with* $u(t) \equiv 0$ *and suppose that Assumption 4 holds. Then, if there exists a positive real number* a_1 *such that* $\|\bar{\omega} - \bar{\eta}\|_2 \leqslant a_1$, *the nonlinear infinite-dimensional dynamical system*

$$\begin{aligned}
\frac{d\omega_t}{dt} &= \mathscr{A}\omega_t + f(P\omega_t, Q\omega_t) \\
&\quad + \Phi_H L((\Psi_H, \omega_t))(y(t) - h(P\omega_t)), \\
\omega_0(\xi) &= \bar{\omega},\ \omega_0(0) = \bar{\omega}_0
\end{aligned} \tag{50}$$

is a local exponential observer for the system of Eq. (10) *in* $\xi \in (-a, 0)$, the sense that the estimation error, $e_t = \omega_t - x_t$, *tends exponentially to zero.*

The abstract dynamical system of Eq. (50) can be simplified by utilizing a procedure based on the method of characteristics for first-order hyperbolic PDE systems (this procedure is detailed in the appendix of the paper) to obtain the following nonlinear integro-differential equation system representation for the state observer:

$$\begin{aligned}
\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega(t), \omega(t-\alpha)) \\
&\quad + g(\omega(t), \omega(t-\alpha))u + \Phi_H(0)L((\Psi_H, \tilde{\omega}(\xi, t))) \\
&\quad \times (y(t) - h(\omega(t))) + B\int_0^\alpha \Phi_H(\xi - \alpha)L((\Psi_H, \tilde{\omega}(\xi, t))) \\
&\quad \times [y(t-\xi) - h(\omega(t-\xi))]\,d\xi, \\
\omega(\xi) &= \bar{\omega}(\xi),\ \xi \in [-\alpha, 0), \quad \omega(0) = \bar{\omega}_0.
\end{aligned} \tag{51}$$

Remark 11. Referring to the state observer of Eq. (51), it is worth noting that: (a) it includes an integral of the observer error, which is expected because the proposed observer design method accounts explicitly for the distributed parameter nature of the time-delay system of Eq. (2), and (b) it consists of a replica of the process model and a nonlinear integral gain acting on the discrepancy between the actual and the estimated output, and thus, it can be thought of as the analogue of nonlinear Luenberger-type observers (e.g., Kazantzis and Kravaris, 1995; Soroush, 1997) developed for nonlinear ODE systems in the case of nonlinear DDE systems.

Remark 12. For open-loop stable systems, the nonlinear gain $L((\Psi_H, \tilde{\omega}(\xi, t)))$ can be set identically equal to zero and the distributed state observer of Eq. (51) reduces to an open-loop DDE observer of the form

$$\begin{aligned}
\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega(t), \omega(t-\alpha)) \\
&\quad + g(\omega(t), \omega(t-\alpha))u.
\end{aligned} \tag{52}$$

Remark 13. For open-loop unstable systems, the practical implementation of the state observer of Eq. (51)

involves the design of a nonlinear gain, $L((\Psi_H, \tilde{\omega}(\xi, t)))$, on the basis of a nonlinear finite-dimensional system (Assumption 4). However, in most practical applications, the construction of a nonlinear gain requires performing extensive computations (for example, series solutions of nonlinear partial differential equations; Kazantzis and Kravaris, 1995), and thus, it cannot be easily performed. A computationally efficient way to address this problem is to design a constant gain, L, on the basis of a linear DDE system resulting from the linearization of the nonlinear DDE system around an operating steady-state and evaluating its validity through computer simulations (see the implementation of the observer of Eq. (51) in the reactor-separator system studied in Section 10). We note that the only computations needed to design a constant observer gain are: (a) the computation of the eigenvalues of the characteristic equation $\det(\lambda I - A - Be^{-\lambda\alpha}) = 0$ which are in the right-half of the complex plane (this can be done by using standard algorithms, for example, Manitius and Tran, 1985; Manitius et al., 1987) and (b) the computation of the eigenfunctions from the formula $\phi_\lambda = e^{\lambda\xi}\phi_\lambda(0)$, where $\phi_\lambda(0)$ satisfies the equation $(\lambda I - A - Be^{-\lambda\alpha})\phi_\lambda(0) = 0$.

Remark 14. To illustrate the application of the result of theorem 4, consider the numerical DDE example of remark 3 (Eq. (13)) with $y = x_2$ as the output. The eigenvectors $\phi_j(\xi)$ corresponding to the two unstable eigenvalues of the delay operator of Eq. (14) are

$$\phi_1(\xi) = \begin{bmatrix} -0.78 \\ 0.62 \end{bmatrix} e^{0.58\xi}, \quad \phi_2(\xi) = \begin{bmatrix} -0.68 \\ -0.73 \end{bmatrix} e^{0.21\xi}. \quad (53)$$

The following constant observer gain,

$$L = \begin{bmatrix} -1.0 \\ -16.0 \end{bmatrix}, \quad (54)$$

was found to satisfy Assumption 4 and yields the following nonlinear state observer:

$$\begin{aligned} \dot{\omega} = {} & \begin{bmatrix} -2.0 & 3.5 \\ 3.0 & -3.0 \end{bmatrix}\omega(t) + \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}\omega(t-3) \\ & + \begin{bmatrix} 4\omega_1\omega_2 - 3\omega_1^2 - \omega_2^2 \\ 0 \end{bmatrix} \\ & + \begin{bmatrix} -0.78 & -0.68 \\ 0.62 & -0.73 \end{bmatrix}\begin{bmatrix} -1.0 \\ -16.0 \end{bmatrix}(x_2 - \omega_2) \\ & + \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \\ & \times \int_0^3 \begin{bmatrix} -0.78e^{0.58(\xi-3.0)} & -0.68e^{0.21(\xi-3.0)} \\ 0.62e^{0.58(\xi-3.0)} & -0.73e^{0.21(\xi-3.0)} \end{bmatrix} \\ & \times \begin{bmatrix} -1.0 \\ -16.0 \end{bmatrix}(x_2(t-\xi) - \omega_2(t-\xi))\,d\xi. \end{aligned} \quad (55)$$

Remark 15. Applying the proposed observer design method to linear DDE systems of the form of Eq. (3) with $\tilde{\alpha} = 0$, we obtain the following linear state observer:

$$\begin{aligned} \dot{\omega} = {} & A\omega(t) + B\omega(t-\alpha) + cu \\ & + \Phi_H(0)L(wx(t) - w\omega(t)) \\ & + B\int_0^\alpha \Phi_H(\xi-\alpha)L[wx(t-\xi) - w\omega(t-\xi)]\,d\xi, \\ \omega(\xi) = {} & \bar{\omega}(\xi),\ \xi \in [-\alpha, 0),\ \omega(0) = \bar{\omega}_0. \end{aligned} \quad (56)$$

8. Nonlinear output feedback control of DDE systems with state delay

In this section, we consider DDE systems of the form of Eq. (2) with $\tilde{\alpha} = 0$ and address the problem of synthesizing distributed output feedback controllers that enforce local exponential stability and asymptotic output tracking in the closed-loop system, independently of the size of the state delay. The requisite output feedback controllers will be synthesized employing combination of the developed distributed state feedback controllers with distributed state observers.

Theorem 5 that follows provides a state-space realization of the distributed output feedback controller and the properties that it enforces in the closed-loop system (the proof of the theorem can be found in the appendix).

Theorem 5. *Consider the system of nonlinear differential difference equations of Eq.* (2) *with* $\tilde{\alpha} = 0$, *for which the Assumptions 1–4 hold. Then, if there exists a positive real number* a_0 *such that* $\|\bar{\omega} - \bar{\eta}\|_2 \leqslant a_0$ and the *matrix equation of Eq.* (36) *has a unique positive-definite solution for* P, *the distributed output feedback controller*:

$$\begin{aligned} \dot{\omega} = {} & A\omega(t) + B\omega(t-\alpha) + f(\omega(t), \omega(t-\alpha)) \\ & + \Phi_H(0)L((\Psi_H, \tilde{\omega}(\xi,t)))(y(t) - h(\omega(t))) \\ & + B\int_0^\alpha \Phi_H(\xi-\alpha)L((\Psi_H, \tilde{\omega}(\xi,t)))[y(t-\xi) \\ & - h(\omega(t-\xi))]d\xi + g(\omega(t), \omega(t-\alpha)) \\ & \times \frac{1}{L_g L_{\bar{f}}^{r-1}h(\omega)}(-R_2^{-1}b^T P\bar{e}(t) + v^{(r)}(t) \\ & - L_{\bar{f}}^r h(\omega) - p_r(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))), \\ \omega(\xi) = {} & \bar{\omega}(\xi),\ \xi \in [-\alpha, 0),\ \omega(0) = \bar{\omega}_0, \\ u = {} & \frac{1}{L_g L_{\bar{f}}^{r-1}h(\omega)}(-R_2^{-1}b^T P\bar{e}(t) + v^{(r)}(t) - L_{\bar{f}}^r h(\omega) \\ & - p_r(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))), \end{aligned} \quad (57)$$

where $\bar{e} = [(h(\omega) - v)\ (L_{\bar{f}}h(\omega) - v^{(1)}) \cdots (L_{\bar{f}}^{r-1}h(\omega) - v^{(r-1)})]^T$, *(a) guarantees local exponential stability of the* *closed-loop system, and (b) enforces asymptotic output tracking, independently of the size of the state delay.*

Remark 16. For open-loop stable systems, $L((\Psi_H, \tilde{\omega}(\xi, t)))$ can be set equal to zero and the distributed output feedback controller of Eq. (57) can be simplified to

$$\begin{aligned}\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega, \omega(t-\alpha))\\ &\quad + g(\omega(t), \omega(t-\alpha))\frac{1}{L_gL_{\bar{f}}^{r-1}h(\omega)}(-R_2^{-1}b^{\mathrm{T}}P\bar{e}(t)\\ &\quad + v^{(r)}(t) - L_{\bar{f}}^{r}h(\omega) - p_r(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))),\\ \bar{u} &= \frac{1}{L_gL_{\bar{f}}^{r-1}h(\omega)}(-R_2^{-1}b^{\mathrm{T}}P\bar{e}(t) + v^{(r)}(t) - L_{\bar{f}}^{r}h(\omega)\\ &\quad - p_r(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))). \end{aligned} \tag{58}$$

Remark 17. Referring to the result of Theorem 5, we note that no consistent initialization requirement has been imposed on the observer states in order to prove local exponential stability and asymptotic output tracking in the closed-loop system (i.e., it is not necessary that $\bar{\omega}(\xi) = \bar{\eta}(\xi)$, $\xi \in [-\alpha, 0)$ and $\bar{\omega}_0 = \bar{\eta}_0$).

Remark 18. The exponential stability of the closed-loop system guarantees that in the presence of small modeling errors (i.e., unknown model parameters and external disturbances) and initialization errors of the observer states, the states of the closed-loop system will remain bounded. Furthermore, it is possible to implement a linear error feedback controller around the $(y - v)$ loop to ensure asymptotic offsetless output tracking in the closed-loop system, in the presence of *constant* parametric uncertainty, external disturbances and initialization errors. In this case, one can use calculations similar to the ones in Daoutidis and Christofides (1995) to derive a mixed-error and output feedback controller, which possesses integral action, (i.e., a controller of the form of Eq. (57) with $\bar{e} = [(y(t) - v)\,(L_{\bar{f}}h(\omega) - v^{(1)} \cdots (L_{\bar{f}}^{r-1}h(\omega) - v^{(r-1)})]^{\mathrm{T}})$, that enforces exponential stability and asymptotic offsetless output tracking in the closed-loop system in the presence of constant modeling errors and disturbances.

9. Nonlinear output feedback control for DDE systems with state and measurement delay

In this section, we consider DDE systems of the form of Eq. (2) with $\tilde{\alpha} > 0$ and address the problem of synthesizing distributed output feedback controllers that enforce local exponential stability and asymptotic output tracking in the closed-loop system, independently of the size of the state delay. In order to account for the presence of the measurement delay in the controller design, the requisite output feedback controller will be obtained by working within a Smith Predictor framework (Henson and Seborg, 1994; Kravaris and Wright, 1989). Within this framework, the state feedback controller is synthesized on the basis of an auxiliary output $\bar{y}$, which represents the prediction of the output if there were no deadtime on the output, and can be obtained by adding a corrective signal δy to the on-line measurement of the actual output y:

$$\bar{y} = y + \delta y.$$

Assuming that the open-loop DDE system is observable, the corrective signal, δy, is obtained through a closed-loop Smith-type predictor; which for the problem in question is a nonlinear DDE system driven by the manipulated input, that simulates the difference in the responses between the process model without output deadtime and the process model with output deadtime:

$$\begin{aligned}\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega(t), \omega(t-\alpha))\\ &\quad + g(\omega(t), \omega(t-\alpha))u + \Phi_H(0)L((\Psi_H, \tilde{\omega}(\xi, t)))\\ &\quad \times (y(t-\tilde{\alpha}) - h(\omega(t-\tilde{\alpha}))) + B\int_0^{\alpha-\tilde{\alpha}} \Phi_H(\xi-\alpha)\\ &\quad \times L((\Psi_H, \tilde{\omega}(\xi, t)))[y(t-\xi-\tilde{\alpha}) - h(\omega(t-\xi-\tilde{\alpha}))]\,d\xi,\\ \omega(\xi) &= \bar{\omega}(\xi),\ \xi \in [-\alpha, 0),\ \omega(0) = \bar{\omega}_0. \end{aligned} \tag{59}$$

The resulting output feedback controller and the properties that it enforces in the closed-loop system are given in Theorem 6 below (the proof of the theorem is similar to the one of Theorem 5 and will be omitted for brevity).

Theorem 6. *Consider the nonlinear DDE system of* (2) *with* $\tilde{\alpha} > 0$, *for which Assumptions 1–3 hold, and an observability property similar to the one of Assumption 4 holds for* $0 \leqslant \tilde{\alpha} \leqslant \alpha$. *Then, if there exists a positive real number* a_0 *such that* $\|\bar{\omega} - \bar{\eta}\|_2 \leqslant a_0$ *and the matrix equation of Eq.* (36) *has a unique positive-definite solution for P, the distributed output feedback controller:*

$$\begin{aligned}\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega(t), \omega(t-\alpha))\\ &\quad + \Phi_H(0)L((\Psi_H, \tilde{\omega}(\xi, t)))(y(t-\tilde{\alpha}) - h(\omega(t-\tilde{\alpha})))\\ &\quad + B\int_0^{\alpha-\tilde{\alpha}} \Phi_H(\xi-\alpha)L((\Psi_H, \tilde{\omega}(\xi, t)))[y(t-\xi-\tilde{\alpha})\\ &\quad - h(\omega(t-\xi-\tilde{\alpha}))]d\xi + g(\omega(t), \omega(t-\alpha))\\ &\quad \times \frac{1}{L_gL_{\bar{f}}^{r-1}h(\omega)}(-R_2^{-1}b^{\mathrm{T}}P\bar{e} + v^{(r)}(t) - L_{\bar{f}}^{r}h(\omega)\\ &\quad - p_r(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))) \end{aligned} \tag{60}$$

$$\omega(\xi) = \bar{\omega}(\xi),\ \xi \in [-\alpha, 0),\ \omega(0) = \bar{\omega}_0,$$

$$\begin{aligned}\bar{u} &= \frac{1}{L_gL_{\bar{f}}^{r-1}h(\omega)}(-R_2^{-1}b^{\mathrm{T}}P\bar{e} + v^{(r)}(t) - L_{\bar{f}}^{r}h(\omega)\\ &\quad - p_r(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))),\end{aligned}$$

where $\bar{e} = [(y(t-\tilde{\alpha}) - v + h(\omega(t)) - h(\omega(t-\tilde{\alpha})))\ (L_{\bar{f}}h(\omega) - v^{(1)}) \cdots (L_{\bar{f}}^{r-1}h(\omega) - v^{(r-1)})]^{\mathrm{T}}$ *enforces local exponential* *stability, and asymptotic output tracking in the closed-loop system, independently of the size of the state delay.*

Remark 19. We note that no restrictions have been imposed on the stability properties of the open-loop DDE system (e.g., open-loop stable system) in order to derive the result of Theorem 6. This is because the predictor of Eq. (59) which is used to produce values of the corrective signal δy is a closed-loop one (we remark that the use of an open-loop predictor would require to assume that the open-loop DDE system is stable).

Remark 20. For open-loop stable DDE systems with measurement delay but no state delay (i.e., $\tilde{\alpha} > 0$ and $\alpha = 0$), the output feedback controller of Eq. (60) simplifies to

$$\dot{\omega} = A\omega(t) + B\omega(t) + f(\omega(t), \omega(t)) + g(\omega(t), \omega(t)) \times \frac{1}{L_g L_{\bar{f}}^{r-1} h(\omega)} (- R_2^{-1} b^T P \bar{e} + v^{(r)}(t) - L_{\bar{f}}^r h(\omega) - p_r(\omega(t), \bar{v}(t), \omega(t), \bar{v}(t))), \quad (61)$$

$$\bar{u} = \frac{1}{L_g L_{\bar{f}}^{r-1} h(\omega)} (- R_2^{-1} b^T P \bar{e} + v^{(r)}(t) - L_{\bar{f}}^r h(\omega) - p_r(\omega(t), \bar{v}(t), \omega(t), \bar{v}(t))).$$

where $\bar{e} = [(y(t-\tilde{\alpha}) - v + h(\omega(t)) - h(\omega(t-\tilde{\alpha})))\,(L_{\bar{f}} h(\omega) - v^{(1)}) \cdots (L^{r-1}{}_{\bar{f}} h(\omega) - v^{(r-1)})]^T$. The above controller is a Smith-predictor based open-loop output feedback controller similar to the one developed in Kravaris and Wright (1989).

Remark 21. The nonlinear distributed output feedback of Eqs. (57)–(60) are infinite-dimensional ones, due to the infinite-dimensional nature of the observers of Eqs. (51)–(59) respectively. Therefore, finite-dimensional approximation of these controllers have to be derived for on-line implementation. This task can be performed utilizing standard discretization techniques such as finite differences. We note that it is well-established (e.g., Soliman and Ray, 1972) that as the number of discretization points increases, the closed-loop system resulting from the DDE model plus an approximate finite-dimensional controller converges to the closed-loop system resulting from the DDE model plus the infinite-dimensional controller, guaranteeing the well-posedness of the approximate finite-dimensional controller.

10. Simulation studies

10.1. Application to a reactor–separator system with recycle

10.1.1. Process description – control problem formulation

Consider the process, shown in Fig. 2, which consists of a reactor and a separator (Lehman et al., 1995). An irreversible reaction of the form $A \rightarrow B$, where A is the reactant species and B is the product species, takes place in the reactor. The reaction is exothermic and a cooling jacket is used to remove heat from the reactor. The reaction rate is assumed to be of first-order and is given by

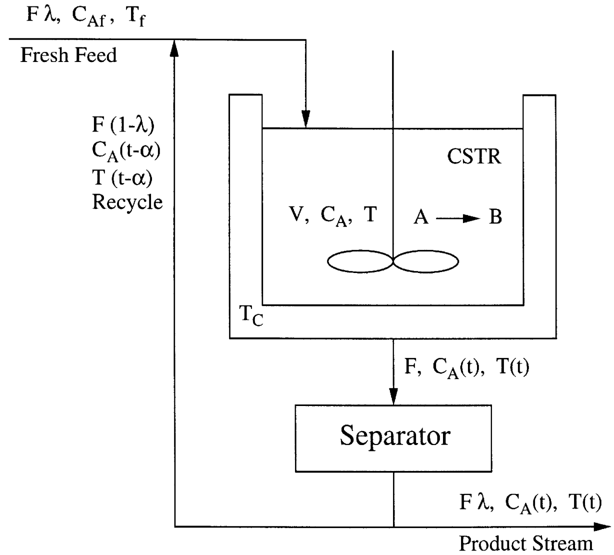

Fig. 2. A reactor–separator process with recycle.

$$r = k_0 \exp\left(-\frac{E}{RT} \right) C_A,$$

where k_0 and E denote the pre-exponential constant and activation energy of the reaction, and T and C_A denote the temperature and concentration of species A in the reactor. The outlet of the reactor is fed to a separator where the unreacted species A is separated from the product B. The unreacted amount of species A is fed back to the reactor through a recycle loop; this allows increasing the overall conversion of the reaction and minimizing reactant wastes. The inlet stream to the reactor consists of a fresh feed of pure A, at flow-rate λF, concentration C_{Af} and temperature T_f, and of the recycle stream at flow-rate $(1-\lambda)F$, concentration $C_A(t-\alpha)$ and temperature $T(t-\alpha)$, where F is the total reactor flow rate, λ is the recirculation coefficient (it varies from zero to one, with zero corresponding to total recycle and zero fresh feed and one corresponding to no recycle) and α is the recycle loop dead time. Under the assumptions of constant volume of the reacting liquid, V, negligible heat losses, constant density, ρ, heat capacity, c_p, of the reacting liquid, and constant jacket temperature, T_c, a process dynamic model can be derived from mass and energy balances and consists of the following two nonlinear differential difference equations:

$$\frac{dC_A}{dt} = \frac{\lambda F}{V} C_{Af} - \frac{F}{V} C_A + \frac{(1-\lambda)F}{V} C_A(t-\alpha) - k_0 \exp\left(-\frac{E}{RT} \right) C_A,$$

$$\frac{dT}{dt} = \frac{\lambda F}{V} T_f - \frac{F}{V} T + \frac{(1-\lambda)F}{V} T(t-\alpha) + \frac{(-\Delta H)}{\rho c_p} k_0 \exp\left(-\frac{E}{RT}\right) C_A - \frac{UA}{V\rho c_p}(T - T_c), \tag{62}$$

where ΔH denotes the enthalpy of the reaction, U denotes the heat transfer coefficient, and A denotes the heat transfer area. The values of the process parameters are given in Table 1. For these values the corresponding steady state is

$$C_{As} = 0.5000\ \text{mol/l},\ T_s = 296.16\ \text{K},\ C_{Afs} = 0.7090\ \text{mol/l} \tag{63}$$

where the subscript s denotes the steady state value. It was verified, through computation of the eigenvalues of the open-loop system, that this steady state is a stable one.

The control objective for the process is formulated as the one of regulating the temperature of the reactor, T, by manipulating the inlet concentration of the fresh feed C_{Af}. Setting $x_1 = C_A, x_2 = T, y = x_2$ and $u = C_{Af} - C_{Afs}$, the process model of Eq. (62) can be written in the form of Eq. (26) with

$$\tilde{f}(x(t)) = \begin{bmatrix} \frac{\lambda F}{V} C_{Afs} - \frac{F}{V} x_1 - k_0 \exp\left(-\frac{E}{Rx_2}\right) x_1 \\ \frac{\lambda F}{V} T_f - \frac{F}{V} x_2 + \frac{(-\Delta H)}{\rho c_p} k_0 \exp\left(-\frac{E}{Rx_2}\right) x_1 - \frac{UA}{V\rho c_p}(x_2 - T_c) \end{bmatrix}, \tag{64}$$

$$g(x(t), x(t-\alpha)) = \begin{bmatrix} \frac{\lambda F}{V} \\ 0 \end{bmatrix}, \bar{p}(x(t), x(t-\alpha)) = \begin{bmatrix} \frac{(1-\lambda)F}{V} x_1(t-\alpha) \\ \frac{(1-\lambda)F}{V} x_2(t-\alpha) \end{bmatrix} \tag{65}$$

10.1.2. State feedback controller design

For the system of Eq. (62), Assumption 1 is satisfied with $r = 2$ and the coordinate transformation of Eq. (27) takes the form

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \mathscr{X}(x) = \begin{bmatrix} h(x) \\ L_{\tilde{f}} h(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{\lambda F}{V} T_f - \frac{F}{V} x_2 + \frac{(-\Delta H)}{\rho c_p} k_0 \exp\left(-\frac{E}{Rx_2}\right) x_1 - \frac{UA}{V\rho c_p}(x_2 - T_c) \end{bmatrix}. \tag{66}$$

Using the above coordinate change, the process dynamic model can be equivalently written as

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + \frac{(1-\lambda)F}{V} \zeta_1(t-\alpha), \\ \dot{\zeta}_2 &= L_{\tilde{f}}^2 h(x) + L_g L_{\tilde{f}} h(x) u \\ &\quad + p_2(\zeta_1(t), \zeta_2(t), \zeta_1(t-\alpha), \zeta_2(t-\alpha)), \\ y &= \zeta_1, \end{aligned} \tag{67}$$

where the explicit form of the term $p_2(\zeta_1(t), \zeta_2(t), \zeta_1(t-\alpha), \zeta_2(t-\alpha))$ is omitted for brevity. For the above system Assumption 2 is trivially satisfied, while Assumption 3 is satisfied with $p(\bar{e}(t), + \bar{v}(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha)) = [0.1 e_1(t-\alpha)\ 0]^T$, $a_1 = 0$ and $a_2 = 0.01$. Utilizing the result of Theorem 3, the following matrix equation can be formed:

$$\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + a^2 I + P^2 = -R_1 \tag{68}$$

with $R_2 = 1.0$, $a = 0.101 (a^2 > a_2)$, and

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}. \tag{69}$$

Eq. (68) has a unique positive-definite solution for P of the form

$$P = \begin{bmatrix} 0.055 & -0.119 \\ -0.119 & 0.513 \end{bmatrix}, \tag{70}$$

which leads to the following nonlinear state feedback controller:

$$\begin{aligned} u = \frac{1}{L_g L_{\tilde{f}} h(x)} \big(&-0.119(x_2 - v) - 0.513(L_{\tilde{f}} h(x) \\ &+ 0.1 v(t-\alpha)) - L_{\tilde{f}}^2 h(x) \\ &- p_2(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha))\big). \end{aligned} \tag{71}$$

10.1.3. State observer and output feedback controller design

In order to avoid the lengthy computations involved in the design of a nonlinear observer gain, we will design a nonlinear state observer of the form of Eq. (51) with a constant observer gain, L, for the system of Eq. (62) (see also discussion in Remark 13). Computing the linearization of the system of Eq. (62) around the unstable steady state, $C_{As} = 2.36$ mol/l, $T_s = 320.0$ K ($v = 320.0$ K will be the new value of the reference input in the simulations discussed in the next subsection), we obtain the following linear

Table 1
Process parameters of the reactor–separator system

$V = 0.10$	m^3
$F = 13.3330 \times 10^{-3}$	$m^3\ s^{-1}$
$C_{Af} = 0.7090$	$kmol\ m^{-3}$
$\lambda = 0.25$	
$E = 1.20 \times 10^4$	$kcal\ kmol^{-1}$
$k_0 = 1.0 \times 10^7$	s^{-1}
$R = 1.987$	$kcal\ kmol^{-1}\ K^{-1}$
$\Delta H_0 = 5.0 \times 10^4$	$kcal\ kmol^{-1}$
$c_p = 0.001$	$kcal\ kg^{-1}\ K^{-1}$
$\rho = 1000.0$	$kg\ m^{-3}$
$T_f = 300.0$	K
$T_c = 295.0$	K
$U = 30.0$	$kcal\ m^{-2}\ s^{-1}\ K^{-1}$
$A = 1.0$	m^2

DDE system:

$$\dot{x} = Ax(t) + Bx(t-\alpha), \tag{72}$$

with

$$A = \begin{bmatrix} -0.1970 & -0.00885 \\ 3181.8 & 142.23 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \tag{73}$$

The system of Eq. (72) was found to possess two real unstable eigenvalues, $\lambda_1 = 142.036$ and $\lambda_2 = 0.031$ and infinitely many stable eigenvalues. The eigenfunctions corresponding to the unstable eigenvalues were found to be

$$\phi_1(\xi) = \begin{bmatrix} 0.00006 \\ -1.0 \end{bmatrix} e^{\lambda_1 \xi}, \quad \phi_2(\xi) = \begin{bmatrix} -0.0447 \\ 1.0 \end{bmatrix} e^{\lambda_2 \xi}. \tag{74}$$

The following constant observer gain:

$$L = \begin{bmatrix} -361.0 \\ -50.0 \end{bmatrix} \tag{75}$$

was found to satisfy Assumption 4 and yields the following nonlinear output feedback controller:

$$\begin{aligned} \dot{\omega} &= \tilde{f}(\omega(t), \omega(t-\alpha)) + g(\omega, \omega(t-\alpha))u + \bar{p}(\omega(t), \omega(t-\alpha)) \\ &\quad + \Phi_H(0)L(y(t) - h(\omega(t))) \\ &\quad + B\int_0^{\alpha} \Phi_H(\xi-\alpha)L[y(t-\xi) - h(\omega(t-\xi))]\,d\xi \\ u &= \frac{1}{L_g L_{\tilde{f}} h(\omega)}(-0.119(y(t) - v) - 0.513(L_{\tilde{f}} h(\omega) \\ &\quad + 0.1v(t-\alpha)) - L_{\tilde{f}}^2 h(\omega) \\ &\quad - p_2(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))) \end{aligned} \tag{76}$$

where $\Phi_H(\xi) = [\phi_1(\xi)\ \phi_2(\xi)]$. When both state and measurement delays are included in the system of Eq. (62), the following nonlinear output feedback controller was employed in the simulations described in the next subsection:

$$\begin{aligned} \dot{\omega} &= \tilde{f}(\omega(t), \omega(t-\alpha)) + g(\omega, \omega(t-\alpha))u \\ &\quad + \bar{p}(\omega(t), \omega(t-\alpha)) + \Phi_H(0)L(y(t) - h(\omega(t))) \\ &\quad + B\int_0^{\alpha-\tilde{\alpha}} \Phi_H(\xi-\alpha)L[y(t-\xi-\tilde{\alpha}) \\ &\quad - h(\omega(t-\xi-\tilde{\alpha}))]\,d\xi \\ u &= \frac{1}{L_g L_{\tilde{f}} h(\omega)}(-0.119(y(t-\tilde{\alpha}) - v + \omega_2(t) \\ &\quad - \omega_2(t-\tilde{\alpha}))) - 0.513(L_{\tilde{f}} h(\omega) + 0.1v(t-\alpha)) \\ &\quad - L_{\tilde{f}}^2 h(\omega) - p_2(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))) \end{aligned} \tag{77}$$

Note that according to the discussion of Remark 18, the controllers of Eqs. (76) and (77) possess integral action.

10.1.4. Closed-loop system simulations

We performed several sets of simulation runs to evaluate the stabilization and output tracking capabilities of the output feedback controllers of Eqs. (76) and (77) and compare their performance with nonlinear controllers that do not account for the presence of recycle loop dead time in the model of Eq. (62). In all the simulation runs, the process was initially assumed to be at the steady state of Eq. (63) and the user-friendly software package SIMULINK was used to simulate the closed-loop DDE system (SIMULINK is a toolbox of the mathematical software MATLAB that includes a delay function which can be readily used to simulate differential equations with time delays). The computation of the integrals in the controllers of Eqs. (76) and (77) was performed by discretizing the interval $[-\alpha, 0)$ into 10 equispaced intervals using finite differences (further increase on the number of discretization intervals was found to lead to negligible differences on the results).

In the first two sets of simulation runs, a 8.84 K increase in the reference input value (i.e., $v = 305.0$ K) was imposed at time $t = 0$ s. The new reference input value corresponds to a stable steady state. In the first set of simulation runs, we initially considered the process of Eq. (62) with $\alpha = 40$ s and $\tilde{\alpha} = 0$ s under the output feedback controller of Eq. (76) with $L = 0$ (due to operation at a stable region). Fig. 3 shows the closed-loop output and manipulated input profiles (solid lines). It is clear that the proposed controller drives quickly the output to the new reference input value, achieving an excellent transient response. For the sake of comparison, we also implemented on the process the same output feedback

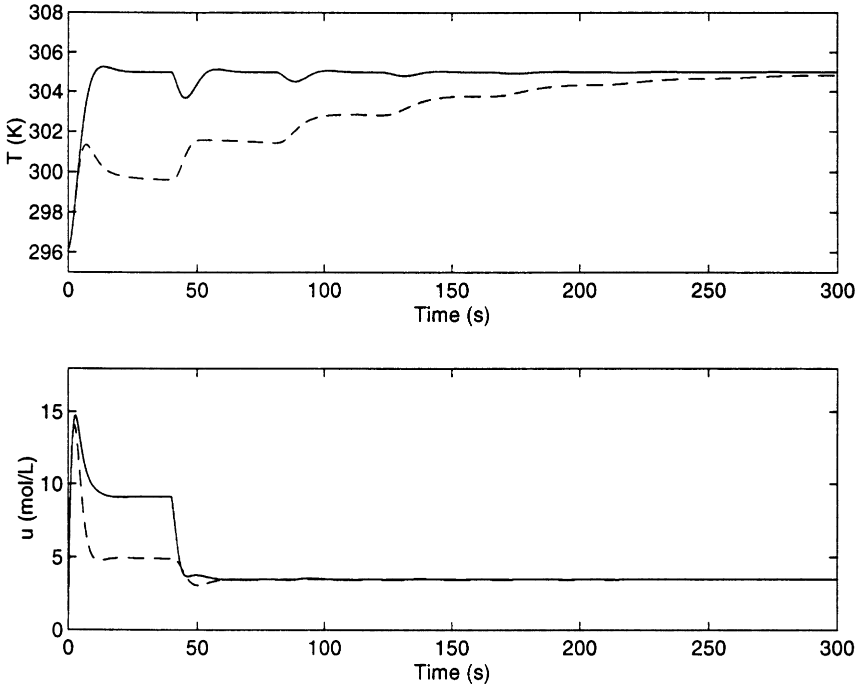

Fig. 3. Closed-loop output and manipulated input profiles with $\alpha = 40$ s and $\tilde{x} = 0$ s under the controller of Eq. (76) with $L = 0$ (solid lines) and the controller of Eq. (76) with $L = 0$ and $\alpha = 0$ s (dashed lines) – operation in stable region.

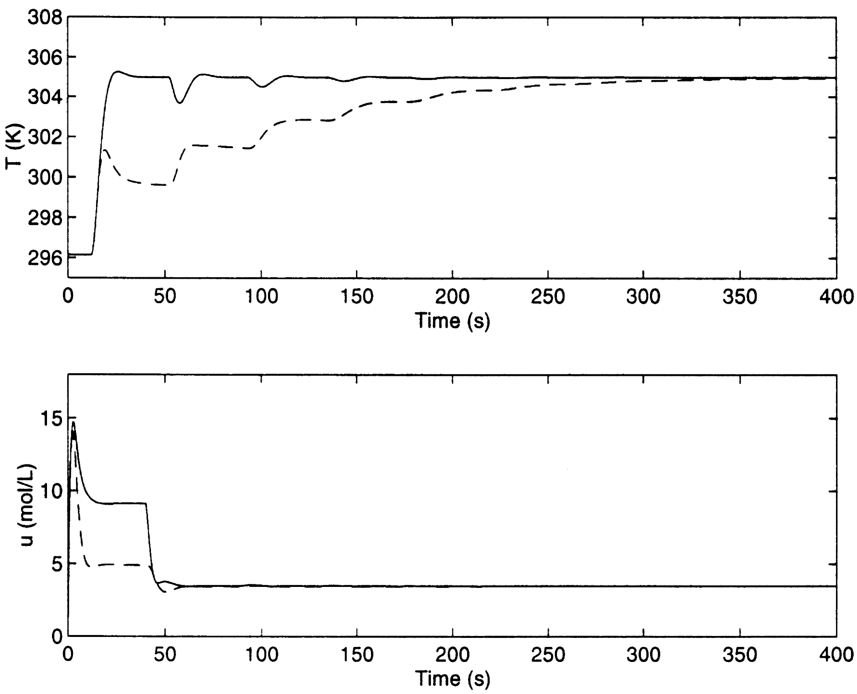

Fig. 4. Closed-loop output and manipulated input profiles with $\alpha = 40$ s and $\tilde{x} = 12$ s under the controller of Eq. (77) with $L = 0$ (solid lines) and the controller of Eq. (77) with $L = 0, \alpha = 0$ s and $\tilde{\alpha} = 12$ s (dashed lines) – operation in stable region.

controller with $\alpha = 0$ s. The closed-loop output and manipulated input profiles under this controller are also displayed in Fig. 3 (dashed lines). This controller yields a very poor transient response driving the output (dashed line) to the new reference input value very slowly. In the second set of simulation runs, we initially considered the process of Eq. (62) with $\alpha = 40$ s and $\tilde{\alpha} = 12$ s under the output feedback controller of Eq. (77) with $L = 0$.

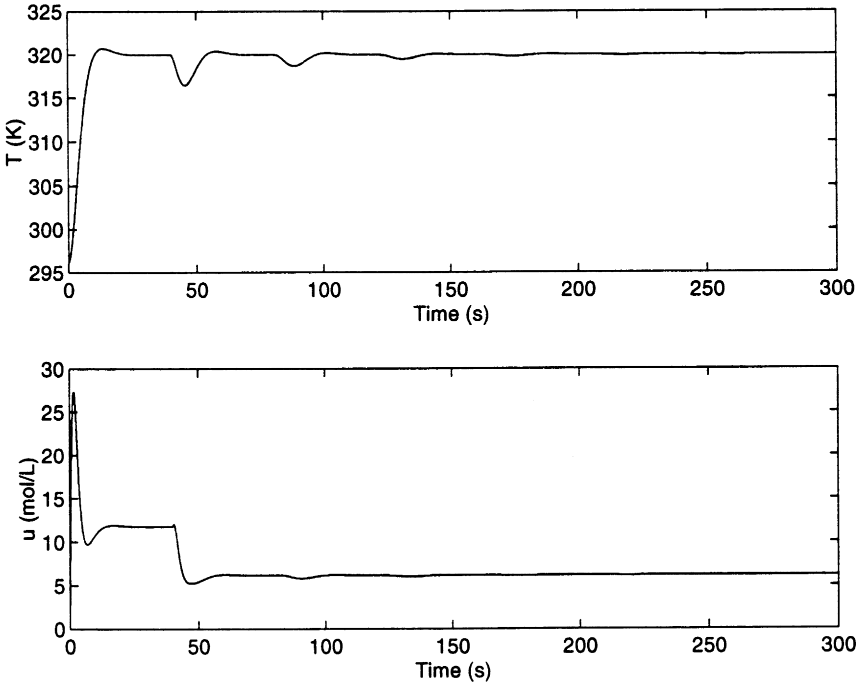

Fig. 5. Closed-loop output and manipulated input profiles with $\alpha = 40$ s and $\tilde{\alpha} = 0$ s under the controller of Eq. (76) – operation in unstable region.

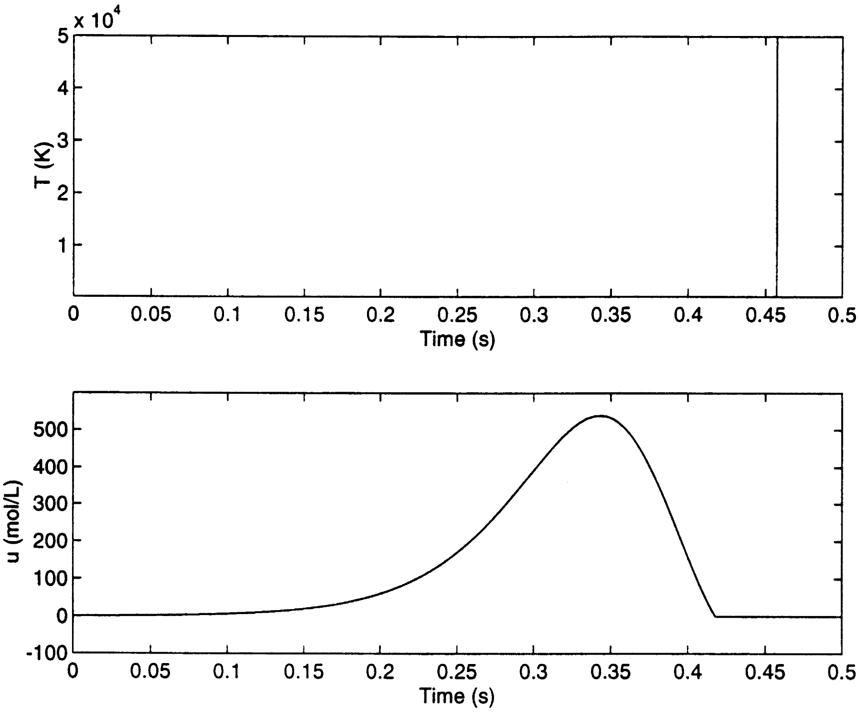

Fig. 6. Closed-loop output and manipulated input profiles with $\alpha = 40$ s and $\tilde{\alpha} = 0$ s under the controller of Eq. (76) with $\alpha = 0$ s – operation in unstable region.

The resulting closed-loop output and manipulated input profiles are presented in Fig. 4 (solid lines). The proposed controller, after the initial delay in the output of $t = 12$ s, which is caused by the presence of measurement delay, regulates successfully the output to the new reference input value. We also implemented on the process the same controller with $\alpha = 0$ s. The closed-loop output and manipulated input profiles are also shown in Fig. 4

(dashed lines). The transient performance of the closed-loop system is clearly inferior to the one obtained by the proposed controller.

In the next three sets of simulation runs, a 23.84 K increase in the reference input value (i.e., $v = 320.0$ K) was imposed at time $t = 0$ s. The new reference input value corresponds to an unstable steady state. We initially considered the process of Eq. (62) with $\alpha = 40$ s and $\tilde{\alpha} = 0$ s under the controller of Eq. (76). Fig. 5 shows the closed-loop output and manipulated input profiles. It is clear that the proposed controller drives quickly the output to the new reference input value. For the sake of comparison, we also implemented on the process the same output feedback controller with $\alpha = 0$ s. This controller led to an unstable closed-loop system (see the closed-loop output and manipulated input profiles in Fig. 6). Then, the process of Eq. (62) with $\alpha = 40$ s and $\tilde{\alpha} = 12$ s was considered under the output feedback controller of Eq. (77) and Fig. 7 shows the resulting closed-loop output and manipulated input profiles. The proposed controller, after the initial delay in the output of $t = 12$ s, which is caused by the measurement delay, regulates successfully the output to the new reference input value. The same controller with $\alpha = 0$ s was also implemented on the process. Again, this controller led to an unstable closed-loop system. Finally, we considered the process of Eq. (62) with $\alpha = 40$ s and $\tilde{\alpha} = 12$ s and studied the robustness properties of the controller of Eq. (76) in the presence of a 5 K increase in the value of the temperature of the fresh feed. Fig. 8 shows the closed-loop output and manipulated input profiles. Despite the presence of significant modeling errors and operation in unstable region, the controller drives the output of the closed-loop system close to the reference input value, exhibiting very good robustness properties.

10.2. Application to a fluidized catalytic cracker

10.2.1. Process modeling – control problem formulation

In this section, we illustrate the implementation of the developed control methodology on another important chemical engineering process, the fluidized catalytic cracker (FCC) shown in Fig. 9. The FCC unit consists of a cracking reactor, where the cracking of high boiling gas oil fractions into lighter hydrocarbons (e.g., gasoline) and the carbon formation reactions (undesired reactions) take place and a regenerator, where the carbon removal reactions take place. The reader may refer to: (a) Arbel et al. (1995), Denn (1986) and McFarlane et al. (1993) for a detailed discussion of the features of the FCC unit, (b) Arbel et al. (1996) for an analysis of the issue of the control structure, and (c) Huq et al. (1995), Monge and Georgakis (1987) and Christofides and Daoutidis (1997) for application of

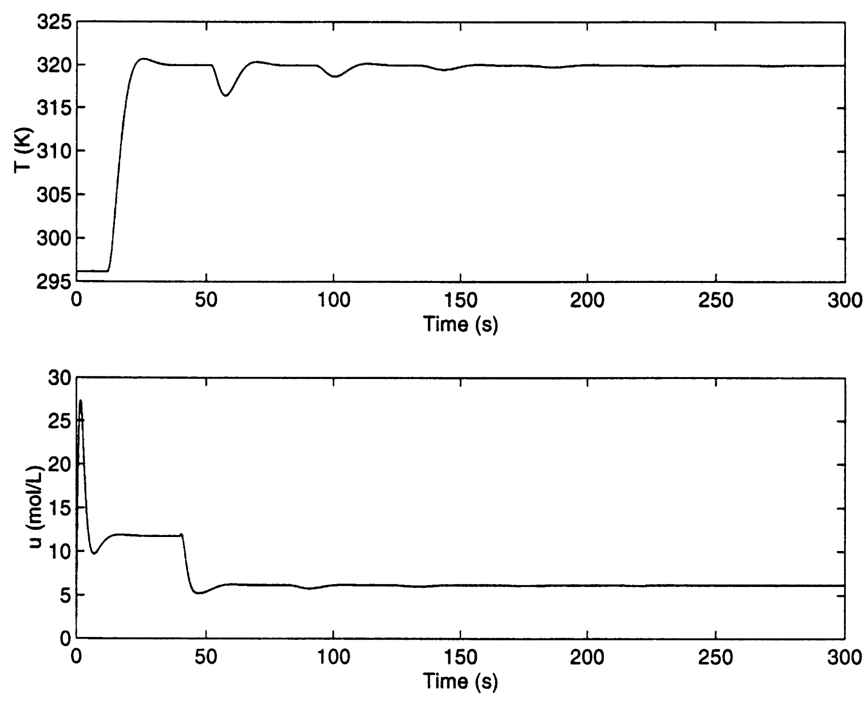

Fig. 7. Closed-loop output and manipulated input profiles with $\alpha = 40$ s and $\tilde{\alpha} = 12$ s under the the controller of Eq. (77) – operation in unstable region.

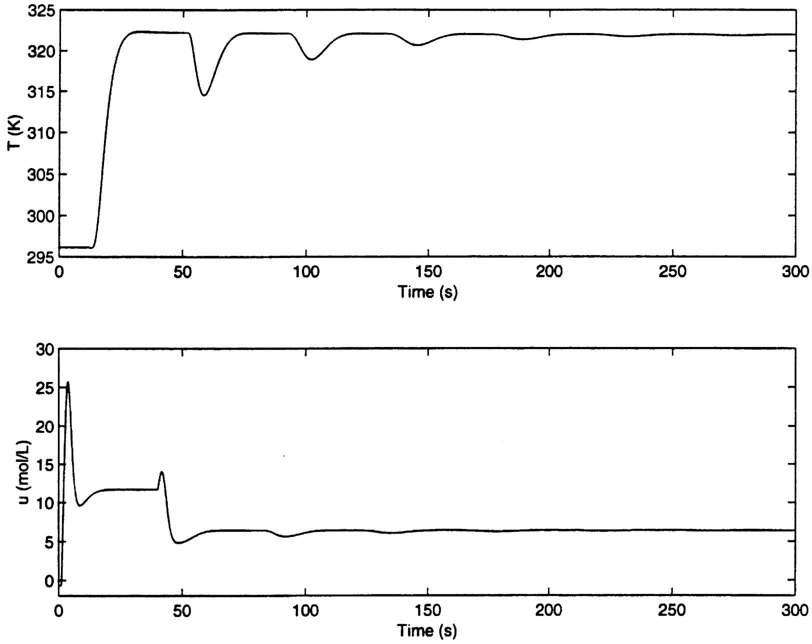

Fig. 8. Closed-loop output and manipulated input profiles with $\alpha = 40$ s and $\tilde{\alpha} = 12$ s under the controller of Eq. (77) in the presence of modeling error – operation in unstable region.

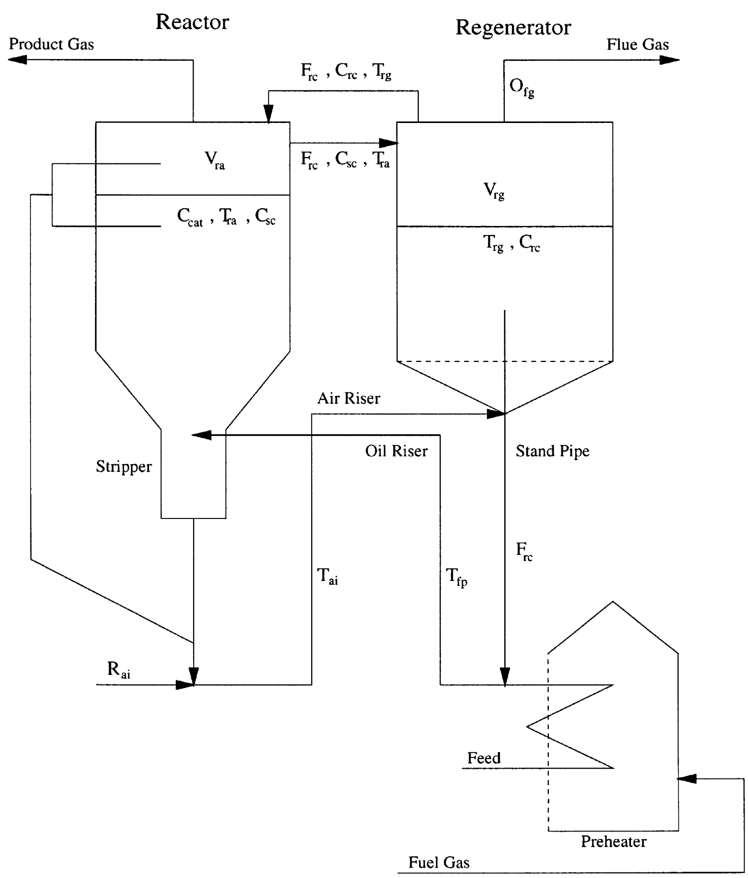

Fig. 9. A fluidized catalytic cracker.

linear and nonlinear control methods to the FCC unit, respectivly. Unfortunately, in all of these studies, the dead-time associated with the pipes transferring material from the reactor to the regenerator and vice versa were not accounted for both in modeling and controller design.

Under the standard modeling assumptions, of well-mixed reactive catalyst in the reactor, small-size catalyst particles, constant solid holdup in reactor and regenerator, uniform and constant pressure in reactor and regenerator, the process dynamic model takes the form (Denn, 1986):

$$V_{ra}\frac{\mathrm{d}C_{\mathrm{cat}}}{\mathrm{d}t} = -60F_{rc}C_{\mathrm{cat}}(t) + 50R_{cf}(C_{\mathrm{cat}}(t), C_{rc}(t-\alpha_1), T_{ra}),$$

$$V_{ra}\frac{\mathrm{d}C_{sc}}{\mathrm{d}t} = 60F_{rc}[C_{rc}(t-\alpha_1) - C_{sc}(t)] + 50R_{cf}(C_{\mathrm{cat}}(t), C_{rc}(t-\alpha_1), T_{ra}),$$

$$V_{ra}\frac{\mathrm{d}T_{ra}}{\mathrm{d}t} = 60F_{rc}[T_{rg}(t-\alpha_1) - T_{ra}(t)] + 0.875\frac{S_f}{S_c}D_{tf}R_{tf}[T_{fp} - T_{ra}(t)] + 0.875\frac{(-\Delta \mathrm{H}_{fv})}{S_c}D_{tf}R_{tf} + 0.5\frac{(-\Delta H_{cr})}{S_c}R_{oc}(C_{\mathrm{cat}}(t), C_{rc}(t-\alpha_1), T_{ra}),$$

$$V_{rg}\frac{\mathrm{d}C_{rc}}{\mathrm{d}t} = 60F_{rc}[C_{sc}(t-\alpha_2) - C_{rc}(t)] - 50R_{cb}(C_{rc}(t), T_{rg}(t)),$$

$$V_{rg}\frac{\mathrm{d}T_{rg}}{\mathrm{d}t} = 60F_{rc}[T_{ra}(t-\alpha_2) - T_{rg}(t)] + 0.5\frac{S_a}{S_c}R_{ai}[T_{ai} - T_{rg}(t)] - 0.5\frac{(-\Delta H_{rg})}{S_c}R_{cb}(C_{rc}(t), T_{rg}(t)), \quad (78)$$

where $C_{\mathrm{cat}}, C_{sc}, C_{rc}$ denote the concentrations of catalytic carbon on spent catalyst, the total carbon on spent catalyst, and carbon on regenerated catalyst, T_{ra}, T_{rg} denote the temperatures in the reactor and the regenerator, R_{ai} is the air flow rate in the regenerator, R_{tf} is the total feed flow rate, D_{tf} is the density of total feed, V_{ra}, V_{rg} denote the catalyst holdup of the reactor and the regenerator, $\Delta H_{rg}, \Delta H_{cr}$ are the heat of regeneration and cracking, ΔH_{fv} is the heat of feed vaporization, F_{rc} denotes the circulation flow rate of catalyst from reactor to regenerator and vice versa, S_a, S_c, S_f denote specific heats of the air, the catalyst, and the feed, T_{fp}, T_{ai} denote the inlet temperatures of the feed in the reactor and of the air in the regenerator, and R_{cf}, R_{oc}, R_{cb} denote the reaction rates of total carbon forming, of gas-oil cracking, and of coke burning. The analytic expressions for the reaction rates R_{cf}, R_{oc}, R_{cb} can be found in Denn (1986).

Table 2
Process parameters of the fluidized catalytic cracker

$E_{cc} = 18000.0$	Btu lb^{-1} mole^{-1}
$E_{cr} = 27000.0$	Btu lb^{-1} mole^{-1}
$E_{or} = 63000.0$	Btu lb^{-1} mole^{-1}
$k_{cc} = 8.59$	Mlb h^{-1} psia^{-1} t^{-1} $(\mathrm{wt}\%)^{-1.06}$
$k_{cr} = 11600$	Mbbl day^{-1} psia^{-1} t^{-1} $(\mathrm{wt}\%)^{-1.15}$
$k_{or} = 3{,}5 \times 10^{10}$	Mlb h^{-1} psia^{-1} t^{-1}
$V_{rg} = 200.0$	t
$V_{ra} = 60.0$	t
$F_{rc} = 40.0$	t h^{-1}
$T_{fps} = 744.0$	F
$T_{ai} = 175.0$	F
$P_{rg} = 25.0$	psia
$P_{ra} = 40.0$	psia
$\Delta H_{fv} = 60.0$	Btu lb^{-1}
$\Delta H_{cr} = 77.3$	Btu lb^{-1}
$\Delta H_{rg} = 10561.0$	Btu lb^{-1}
$S_a = 0.3$	Btu lb^{-1} F^{-1}
$S_c = 0.3$	Btu lb^{-1} F^{-1}
$S_f = 0.7$	Btu lb^{-1} F^{-1}
$R_{tf} = 100.0$	Mbbl/day
$D_{tf} = 7.0$	lb gal^{-1}
$\alpha = 0.3$	h
$R_{ai} = 400.0$	M lb min^{-1}
$(C_{\mathrm{cat}})_{s1} = 0.8723$	wt%
$(C_{sc})_{s1} = 1.5696$	wt%
$(C_{rc})_{s1} = 0.6973$	wt%
$(T_{ra})_{s1} = 930.62$	F
$(T_{rg})_{s1} = 1155.96$	F

The presence of time delay in the terms $C_{rc}(t-\alpha_1)$, $T_{rg}(t-\alpha_1)$ is due to deadtime in the pipes transferring regenerated catalyst from the regenerator to the reactor, and the time delay in the terms $C_{sc}(t-\alpha_2)$, $T_{ra}(t-\alpha_2)$ is due to deadtime in pipes transferring spent catalyst from the reactor to the regenerator. Even though the proposed method can be readily applied to the case where $\alpha_1 \neq \alpha_2$, we pick in order to simplify our development, $\alpha_1 = \alpha_2 = \alpha = 0.3$ h. The values of the remaining process parameters and the corresponding steady-state values are given in Table 2.

The control objective is formulated as the one of regulating the temperature in the regenerator, T_{rg}, by manipulating the temperature of the inlet air in the regenerator, T_{ai}. By setting $x = [x_1\, x_2\, x_3\, x_4\, x_5]^{\mathrm{T}} = [T_{rg}\, C_{rc}\, T_{ra}\, C_{sc}\, C_{\mathrm{cat}}]^{\mathrm{T}}$ $u = T_{ai} - T_{ai_s}$, $y_{,} = T_{rg}$, the process model of Eq. (78) can be written in the form of

Eq. (26) with

$$\tilde{f}(x(t)) = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \\ \tilde{f}_5 \end{bmatrix} = \begin{bmatrix} \frac{0.5 S_a R_{ai}}{S_c V_{rg}} T_{ai_s} - \left(\frac{60F_{rc}}{V_{rg}} + 0.5\frac{S_a R_{ai}}{S_c V_{rg}}\right) T_{rg}(t) - 0.5\frac{(-\Delta H_{rg,n})}{S_c V_{rg}} R_{cb}(C_{rc}(t), T_{rg}(t)) \\ -\frac{60F_{rc}}{V_{rg}} C_{rc}(t) - \frac{50}{V_{rg}} R_{cb}(C_{rc}(t), T_{rg}(t)) \\ -\frac{60F_{rc}}{V_{ra}} T_{ra}(t) + 0.875\frac{S_f}{S_c V_{ra}} D_{tf} R_{tf}[T_{fp} - T_{ra}(t)] + 0.875\frac{(-\Delta H_{fv})}{S_c V_{ra}} D_{tf} R_{tf} \\ -\frac{60F_{rc}}{V_{ra}} C_{sc}(t) \\ -\frac{60F_{rc}}{V_{ra}} C_{\mathrm{cat}}(t) \end{bmatrix}, \tag{79}$$

$$\bar{p}(x(t), x(t-\alpha)) = \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \\ \bar{p}_4 \\ \bar{p}_5 \end{bmatrix} = \begin{bmatrix} \frac{60F_{rc}}{V_{rg}} T_{ra}(t-\alpha) \\ \frac{60F_{rc}}{V_{rg}} C_{sc}(t-\alpha) \\ \frac{60F_{rc}}{V_{ra}} T_{rg}(t-\alpha) + 0.5\frac{(-\Delta H_{cr})}{S_c V_{ra}} R_{oc}(C_{\mathrm{cat}}(t), C_{rc}(t-\alpha), T_{ra}(t)) \\ \frac{60F_{rc}}{V_{ra}} C_{rc}(t-\alpha) + \frac{50}{V_{ra}} R_{cf}(C_{\mathrm{cat}}(t), C_{rc}(t-\alpha), T_{ra}(t)) \\ \frac{50}{V_{ra}} R_{cf}(C_{\mathrm{cat}}(t), C_{rc}(t-\alpha), T_{ra}(t)) \end{bmatrix}, \tag{80}$$

$$g(x(t), (x(t-\alpha)) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix} = \begin{bmatrix} \frac{0.5 S_a R_{ai}}{S_c V_{rg}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{81}$$

10.2.2. State feedback controller design

For the system of Eq. (78), Assumption 1 is satisfied with $r = 1$, and the coordinate transformation of Eq. (27) akes the form $[\zeta\ \eta_1\ \eta_2\ \eta_3\ \eta_4]^{\mathrm{T}} = [T_{rg}\ C_{rc}\ T_{ra}\ C_{sc}\ C_{\mathrm{cat}}]$ and yields the following system:

$$\dot{\zeta} = L_{\tilde{f}} h(\mathscr{X}^{-1}(\zeta, \eta)) + L_g h(\mathscr{X}^{-1}(\zeta, \eta)) u(t) + p_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)),$$
$$\dot{\eta} = \Psi(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \tag{82}$$

where the explicit form of $\Psi(\zeta(t), \eta(t), (t-\alpha), \eta(t-\alpha))$ is omitted for brevity. To verify Assumption 2, we consider the η-subsystem of Eq. (82) with $\zeta(t) = (t-\alpha) = \zeta_s = 1155.96$F, i.e., the system:

$$\dot{\eta} = \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\eta}_4 \end{bmatrix} = \begin{bmatrix} -\frac{60F_{rc}}{V_{rg}} \eta_1 - \frac{50}{V_{rg}} R_{cb}(\eta_1, \zeta_s) + \frac{60F_{rc}}{V_{rg}} \eta_3(t-\alpha) \\ -\frac{60F_{rc}}{V_{ra}} \eta_2 + 0.875\frac{S_f}{S_c V_{ra}} D_{tf} R_{tf}[T_{fp} - \eta_2] + 0.875\frac{(-\Delta H_{fv})}{S_c V_{ra}} D_{tf} R_{tf} + 0.5\frac{(-\Delta H_{cr})}{S_c V_{ra}} R_{oc}(\eta_4, \eta_1(t-\alpha), \eta_2) \\ -\frac{60F_{rc}}{V_{ra}} \eta_3 + \frac{60F_{rc}}{V_{ra}} \eta_1(t-\alpha) + \frac{50}{V_{ra}} R_{cf}(\eta_4, \eta_1(t-\alpha), \eta_2) \\ -\frac{60F_{rc}}{V_{ra}} \eta_4 + \frac{50}{V_{ra}} R_{cf}(\eta_4, \eta_1(t-\alpha), \eta_2) \end{bmatrix}. \tag{83}$$

The linearization of the above system around the steady state

$$C_{rc} = 0.6973\ \text{wt\%},\ T_{ra} = 930.62\ \text{F},\ C_{sc} = 1.5696\ \text{wt\%},$$
$$C_{cat} = 0.8723\ \text{wt\%} \tag{84}$$

was found to be exponentially stable, which implies that the -subsystem of Eq. (82) possesses a local input-to-state stability property with respect to $\zeta_t(\xi)$. Therefore, Assumption 2 holds and the controller synthesis problem can be addressed on the basis of the ζ-subsystem which is given below:

$$\begin{aligned}\dot{\zeta} = &-\left(\frac{60F_{rc}}{V_{rg}} + \frac{0.5S_aR_{ai}}{S_cV_{rg}}\right)\zeta(t)\\ &- 0.5\frac{(-\Delta H_{g,n})}{S_cV_{rg}}R_{cb}(\eta_1(t),\zeta(t)) + \left(\frac{0.5S_aR_{ai}}{S_cV_{rg}}\right)u(t)\\ &+\left(\frac{60F_{rc}}{V_{rg}}\right)\eta_2(t-\alpha).\end{aligned} \tag{85}$$

Setting $e = \zeta - v$ where v is the desired set point and using a preliminary control law of the form of Eq. (30), the above system becomes

$$\dot{e}(t) = -R_2^{-1}b^{\mathrm{T}}Pe(t). \tag{86}$$

For the above system, Assumption 4 is trivially satisfied since $p(x(t), x(t-\alpha)) = 0$. Utilizing the results of Theorem 3, the following equation can be formed:

$$\tilde{A}^{\mathrm{T}}P + P\tilde{A} - 2P^{\mathrm{T}}bR_2^{-1}b^{\mathrm{T}}P + a^2 + P^2 = -R_1 \tag{87}$$

with $R_2 = 1.0$, $a^2 = 0.5$ ($a^2 > a_2 = 0$), and

$$\tilde{A} = 0, \quad b = 1, \quad R_1 = 0.5. \tag{88}$$

Eq. (87) has a unique positive-definite solution for P of the form

$$P = 1.0 \tag{89}$$

which leads to the following nonlinear state feedback controller:

$$u = \frac{1}{L_gh(x(s))}(-(x_1 - v) - L_{\tilde{f}}h(x(s)) - p_1(x(t), x(t-\alpha))). \tag{90}$$

10.2.3. State observer and output feedback controller design

Since the open-loop process is stable, we set the observer gain L equal to zero and derive the following nonlinear output feedback controller using the result of Theorem 5:

$$\begin{aligned}\dot{\omega} &= \tilde{f}(\omega(t)) + g(\omega, \omega(t-\alpha))u + \bar{p}(\omega(t), \omega(t-\alpha)),\\ u &= \frac{1}{L_gh(\omega)}(-(y(t) - v) - L_{\tilde{f}}h(\omega)\\ &\quad - p_1(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))).\end{aligned} \tag{91}$$

When both state and measurement delays are included in the system of Eq. (78), the following nonlinear output feedback controller was derived by using the result of Theorem 6 and employed in the simulation described in the next subsection:

$$\begin{aligned}\dot{\omega} &= \tilde{f}(\omega(t)) + g(\omega, \omega(t-\alpha))u + \bar{p}(\omega(t), \omega(t-\alpha)),\\ u &= \frac{1}{L_gh(\omega)}(-(y(t) - \tilde{\alpha}) - v + \omega_1(t) - \omega_1(t-\tilde{\alpha}))\\ &\quad - L_{\tilde{f}}h(\omega) - p_1(\omega(t), \bar{v}(t), \omega(t-\alpha), \bar{v}(t-\alpha))).\end{aligned} \tag{92}$$

Note that the controllers of Eqs. (91) and (92) possess integral action.

10.2.4. Closed-loop system simulations

We performed several sets of simulation runs to evaluate the performance of the output feedback controllers of Eqs. (91) and (92) and compare their performance with nonlinear controllers that do not account for the presence of time delays in the model of Eq. (78). In all the simulation runs, the process was initially ($t = 0.0$ h) assumed to be at the steady state shown in Table 2 and the MATLAB toolbox SIMULINK was used to simulate the closed-loop DDE system.

In the first simulation run, we considered the process of Eq. (78) with $\alpha = 0.3$ h and $\tilde{\alpha} = 0$ h (i.e., no measurement delay is present) under the output feedback controller of Eq. (90). Fig. 10 shows the output and manipulated input profiles, for a 44 F increase in the reference input. It is clear that the proposed controller drives quickly the output to the new reference input value, compensating for the effect of the dead times associated with the pipes transferring material from the reactor to the regenerator and vice versa. For the sake of comparison, we also implemented on the process the same output feedback controller with $\alpha = 0$ h and $\tilde{\alpha} = 0$ h (i.e., we did not account for the presence of time delays in the design of the controller). Fig. 11 shows the output and manipulated input profiles; this controller leads to an unstable closed-loop system because it does not compensate for the detrimental effect of the time delays.

In the next simulation run, we considered the process of Eq. (78) with $\alpha = 0.3$ h and $\tilde{\alpha} = 0.05$ h (i.e., significant measurement delay is present) under the output feedback controller of Eq. (92). Fig. 12 shows the output and manipulated input profiles, for a 44°F increase in the reference input. Clearly, the controller of Eq. (92) drives the output of the closed-loop system, after an initial delay caused by the measurement dead time, to the new reference input value.

Finally, we considered the process of Eq. (78) with $\alpha = 0.3$ h and $\tilde{\alpha} = 0.05$ hr and studied the robustness properties of the controller of Eq. (90) in the presence

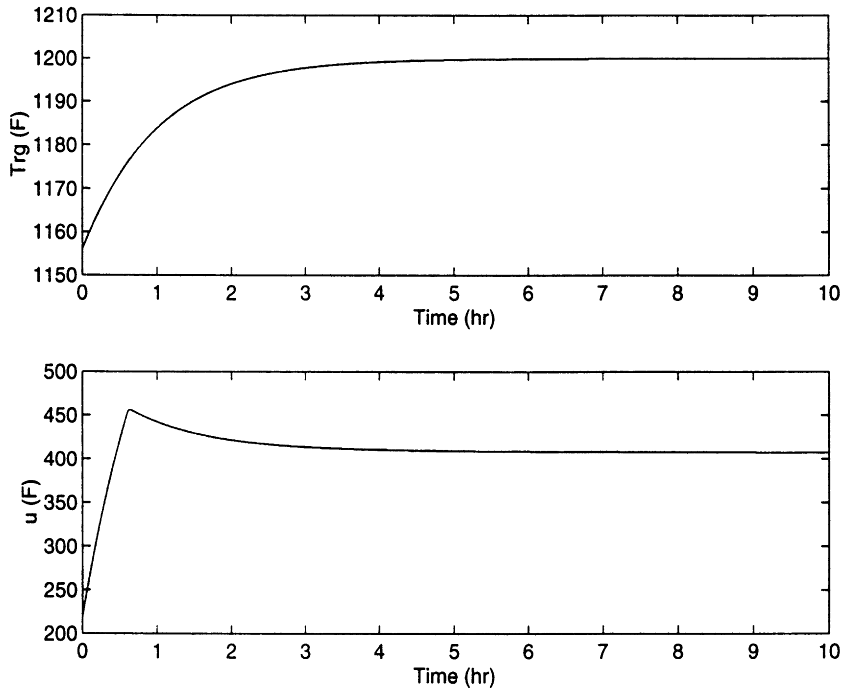

Fig. 10. Closed-loop output and manipulated input profiles with $\alpha = 0.3$ h and $\tilde{\alpha} = 0$ h under the controller of Eq. (91).

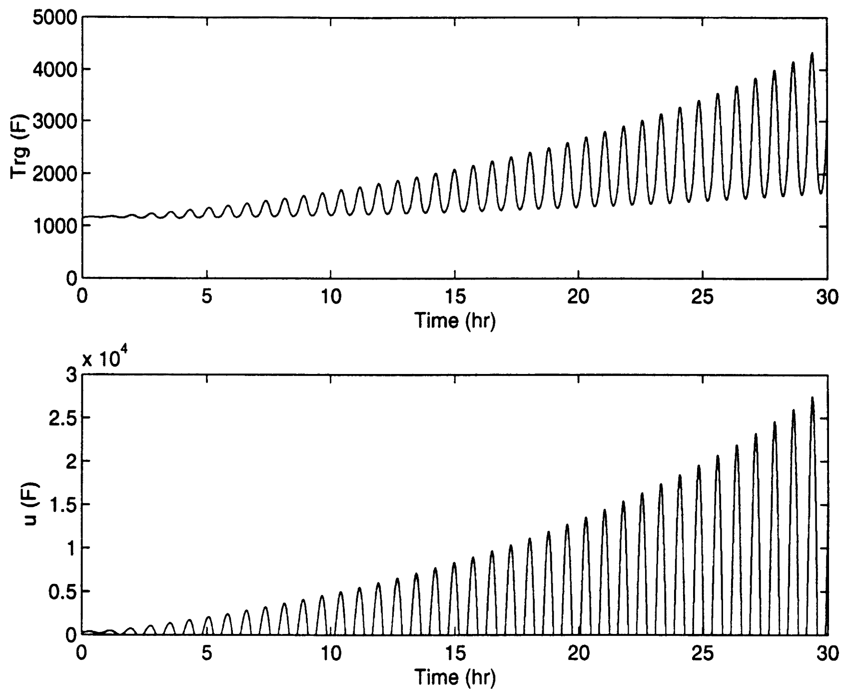

Fig. 11. Closed-loop output and manipulated input profiles with $\alpha = 0.3$ h and $\tilde{\alpha} = 0$ h under the controller of Eq. (91) with $\alpha = 0.0$ h.

of modeling errors. In particular, we simultaneously considered 5% error in the values of: (a) the catalyst circulation rate, F_{rc}, (b) the temperature of the feed in the reactor, T_{fp}, and (c) the inflow air rate in the regenerator, R_{ai}. Fig. 13 shows the output and manipulated input profiles, for a 44 F increase in the reference input. Despite the presence of a significant modeling errors the proposed controller drives the output of the closed-loop system to the new reference input value, exhibiting very good robustness properties.

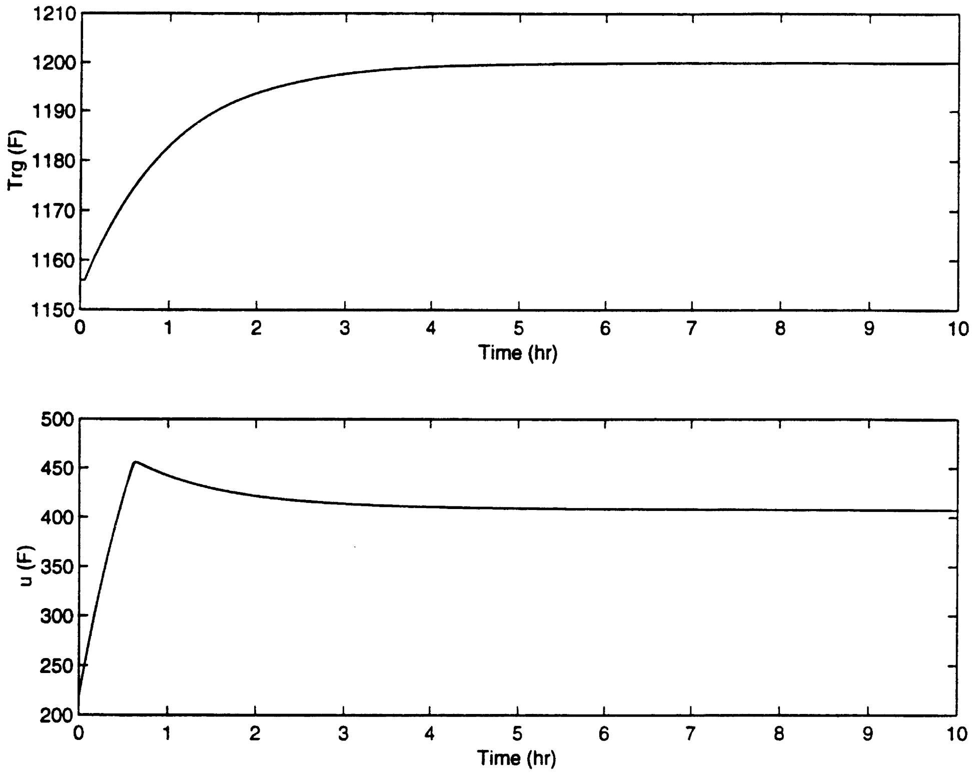

Fig. 12. Closed-loop output and manipulated input profiles with $\alpha = 0.3$ h and $\tilde{\alpha} = 0.05$ h under the controller of Eq. (92).

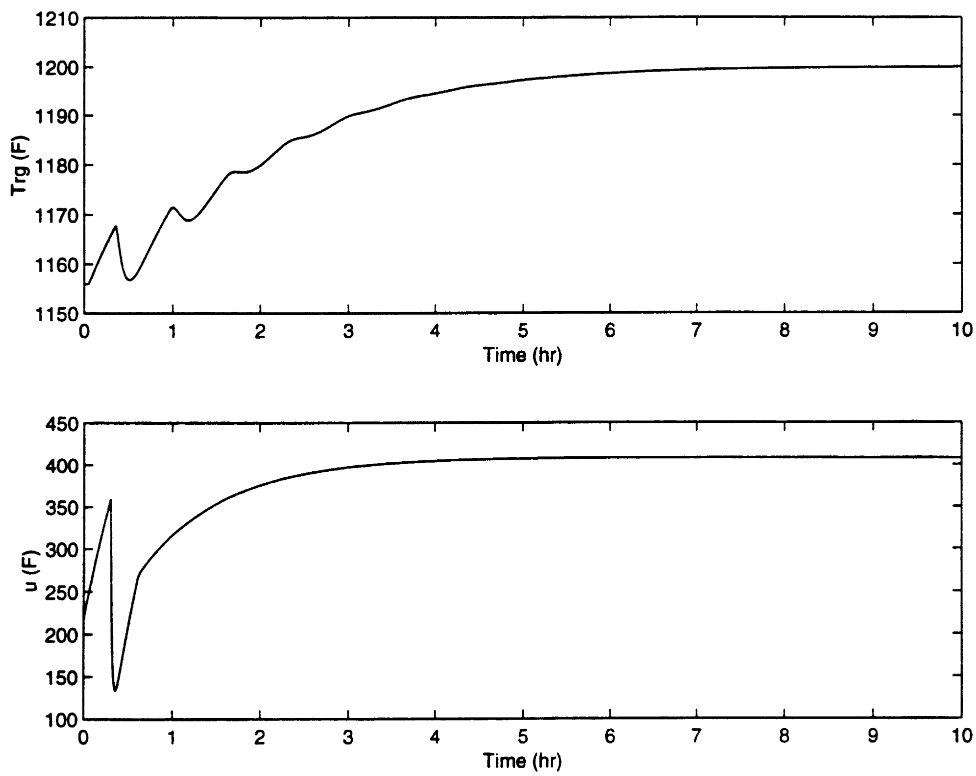

Fig. 13. Closed-loop output and manipulated input profiles with $\alpha = 0.3$ h and $\tilde{\alpha} = 0.05$ h under the controller of Eq. (92) in the presence of modeling errors.

Summarizing, the results of both simulation studies clearly show that it is necessary to compensate for the effect of dead-time associated with pipes transferring material from one unit to an other, as well as that the proposed control methodology is a very efficient tool for this purpose.

Remark 22. We finally note that even though the FCC unit exhibits two-time-scale behavior (Christofides and Daoutidis, 1997; Denn, 1986) owing to the significantly different holdups of the regenerator and reactor (i.e., the reactor dynamics are significantly faster than the regenerator dynamics), which, in general, may cause controller ill-conditioning (see, for example, Christofides and Daoutidis (1997) for a discussion on this issue), in the present study, the presence of time-scale multiplicity in the FCC process model was not taken into account in the controller design problem addressed in Sections 10.2.2 and 10.2.3 because the control problem that we formulated does not lead to the synthesis of an ill-conditioned controller.

11. Conclusions

In this article, we introduced a methodology for the synthesis of nonlinear output feedback controllers for nonlinear DDE systems which include time delays in the states, the control actuator and the measurement sensor. Initially, DDE systems with state delays were considered and a novel combination of geometric and Lyapunov-based techniques was employed for the synthesis of nonlinear state feedback controllers that guarantee stability and enforce output tracking in the closed-loop system, independently of the size of the state delay. Then, the problem of designing nonlinear distributed state observers, which reconstruct the state of the DDE system while guaranteeing that the discrepancy between the actual and the estimated state tends exponentially to zero, was addressed and solved by using spectral decomposition techniques for DDE systems. The state feedback controllers and the distributed state observers were combined to yield distributed output feedback controllers that enforce stability and asymptotic output tracking in the closed-loop system, independently of the size of the time delay. For DDE systems with state, control actuator and measurement delays, the output feedback controller was synthesized on the basis of an auxiliary output constructed within a Smith-predictor framework. The proposed control method was successfully applied to an exothermic reactor–separator process with recycle and a fluidized catalytic cracker and was shown to outperform nonlinear controller designs that do not account for the presence of deadtime associated with the recycle loop and the pipes transferring material from the reactor to the regenerator and vice versa, respectively.

Acknowledgements

The donors of The Petroleum Research Fund, administered by the ACS, for partial support of this research are acknowledged. C.A. would also like to acknowledge the Cyprus Fullbright Commission for their financial support through the Cyprus-America Scholarship Program.

Appendix A. Definition

- $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f. $L_f^k h$ denotes the kth order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.
- A function γ: $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class Q if it is continuous, zero at zero and nondecreasing. It is of class K if it is continuous, strictly increasing and is zero at zero.
- A function β: $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.

Proof of Theorem 2. Under the controller of Eq. (21), the closed-loop system takes the form.

$$\begin{aligned} \dot{\omega} &= \mathscr{F}(\omega, h(x(t-\alpha)), v), \\ \dot{x} &= Ax(t) + Bx(t-\alpha) + f(x(t), x(t-\alpha)) \\ &\quad + g(x(t), x(t-\alpha))\mathscr{P}(\omega, h(x(t-\alpha)), t), \\ x(\xi) &= \bar{\eta}(\xi),\ \xi \in [-\alpha, 0),\ x(0) = \bar{\eta}_0. \end{aligned} \tag{A.1}$$

Introducing the extended state vector $\hat{\omega} = [\omega^{\mathrm{T}}\ x^{\mathrm{T}}]^{\mathrm{T}}$, the above system can be written in the following compact form:

$$\dot{\hat{\omega}} = \bar{\mathscr{F}}(\hat{\omega}(t), \hat{\omega}(t-\alpha)), \tag{A.2}$$

where the specific form of the vector $\bar{\mathscr{F}}(\hat{\omega}(t), \hat{\omega}(t-\alpha))$ can be easily obtained by comparing Eqs. (A.1) and (A.2) and it is omitted here for brevity. From the fact that the controller of Eq. (21) enforces local exponential stability and asymptotic output tracking in the closed-loop system when $\alpha = 0$, we have that the system

$$\dot{\hat{\omega}} = \bar{\mathscr{F}}(\hat{\omega}(t), \hat{\omega}(t)) \tag{A.3}$$

is locally exponentially stable. For $t \geqslant 2\alpha$, the system of Eq. (A.2) can be rewritten as

$$\begin{aligned} \dot{\hat{\omega}} &= \bar{\mathscr{F}}(\hat{\omega}(t), \hat{\omega}(t)) + [\bar{\mathscr{F}}(\hat{\omega}(t), \hat{\omega}(t-\alpha)) - \bar{\mathscr{F}}(\hat{\omega}(t), \hat{\omega}(t))] \\ &= \bar{\mathscr{F}}(\hat{\omega}(t), \hat{\omega}(t)) - \int_{t-\alpha}^{t} \frac{\partial \bar{\mathscr{F}}}{\partial \hat{\omega}} \hat{\omega}(\theta), t)\bar{\mathscr{F}}(\hat{\omega}(\theta-\alpha), \hat{\omega}(\theta))\, \mathrm{d}\theta. \end{aligned} \tag{A.4}$$

We consider the function $V = \hat{\omega}^{T}(t)\hat{\omega}(t)$ as a Lyapunov function candidate for the system of Eq. (A.4). Computing the time derivative of V along the trajectory of the system of Eq. (A.4), we obtain

$$\dot{V} = 2\hat{\omega}(t)\bar{\bar{\mathscr{F}}}(t), \hat{\omega}(t)) -2\int_{t-\alpha}^{t} \hat{\omega}(t)\frac{\partial \bar{\bar{\mathscr{F}}}}{\partial \hat{w}}(\hat{\omega}F(\hat{\omega}(\theta), \hat{\omega}(\theta), t)\bar{\bar{\mathscr{F}}}(\hat{\omega}(\theta-\alpha), \times \hat{\omega}(\theta))\,\mathrm{d}\theta. \tag{A.5}$$

From the smoothness of the function $\bar{\bar{\mathscr{F}}}$, we have that there exists an L such that $|(\partial\bar{\bar{\mathscr{F}}}/\partial\hat{\omega}(\hat{\omega}(\theta), \hat{\omega}(\theta), t)|_{\mathbb{R}^n} \leqslant L$, and the bound on $\dot{V}$ of Eq. (A.5), takes the form

$$\begin{aligned}\dot{V} &\leqslant 2\hat{\omega}(t)\bar{\bar{\mathscr{F}}}(\hat{\omega}(t), \hat{\omega}(t)) + 2L^2\int_{t-\alpha}^{t}|\hat{\omega}(t)\hat{\omega}(\theta-\alpha)|_{\mathbb{R}^n}\,\mathrm{d}\theta \\ &\leqslant 2\hat{\omega}(t)\bar{\bar{\mathscr{F}}}(\hat{\omega}(t), \hat{\omega}(t)) + 2L^2|\hat{\omega}(t)|_{\mathbb{R}^n}\int_{t-\alpha}^{t}|\hat{\omega}(\theta-\alpha)|_{\mathbb{R}^n}\,\mathrm{d}\theta.\end{aligned} \tag{A.6}$$

Now, given a positive real number $\bar{q} > 1$, we consider the set of all $\hat{\omega}(t)$ that satisfy

$$\hat{\omega}^2(t-\xi) \leqslant \bar{q}^2\hat{\omega}^2(t), \quad 0 \leqslant \xi \leqslant 2\alpha \tag{A.7}$$

for which, we have that

$$\dot{V} \leqslant 2\hat{\omega}(t)\bar{\bar{\mathscr{F}}}(\hat{\omega}(t), \hat{\omega}(t)) + 2L^2\alpha\bar{q}|\hat{\omega}(t)|^2_{\mathbb{R}^n} \tag{A.8}$$

From the fact that the controller of Eq. (21) enforces local exponential stability and asymptotic output tracking in the closed-loop system when $\alpha = 0$, we have that the system

$$\dot{\hat{\omega}} = \bar{\bar{\mathscr{F}}}(\hat{\omega}(t), \hat{\omega}(t)) \tag{A.9}$$

is locally exponentially stable, which implies that

$$2\hat{\omega}(t)\bar{\bar{\mathscr{F}}}(\hat{\omega}(t), \hat{\omega}(t)) \leqslant -a|\hat{\omega}(t)|^2_{\mathbb{R}^n}, \tag{A.10}$$

where a is a positive real number. Substituting the above bound into Eq. (A.8), we obtain

$$\dot{V} \leqslant -a|\hat{\omega}(t)|^2_{\mathbb{R}^n} + 2L^2\alpha\bar{q}|\hat{\omega}(t)|^2_{\mathbb{R}^n} \leqslant -(a - 2L^2\alpha\bar{q})|\hat{\omega}(t)|^2_{\mathbb{R}^n}. \tag{A.11}$$

For $\alpha < a/2L^2\bar{q}$, we have that $\dot{V} \leqslant 0$, and using Theorem 4.2 from Hale and Verduyn Lunel (1993), we directly obtain that the system of Eq. (A.1) is exponentially stable. The proof that $\lim_{t\to\infty}|y - v|_{\mathbb{R}^n} = 0$ is conceptually similar and will be omitted for brevity.

Proof of Theorem 3. Substitution of the controller of Eq. (37) into the system of Eq. (2) with $\bar{\alpha} = 0$, yields the following system:

$$\begin{aligned}\dot{x} &= Ax + Bx(t-\alpha) + f(x, x(t-\alpha)) \\ &\quad + g(x, x(t-\alpha))\mathscr{A}(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha)), \\ y &= h(x).\end{aligned} \tag{A.12}$$

Using that $f(x(t), x(t-\alpha)) = f_1(x(t)) + f_2(x(t), x(t-\alpha))$, $\tilde{f}(x(t)) = Ax(t) + f_1(x(t))$, and $\bar{p}(x(t), x(t-\alpha)) = Bx(t-\alpha) + f_2(x(t), x(t-\alpha))$, the above system can be written as

$$\begin{aligned}\dot{x} &= \tilde{f}(x(t)) \\ &\quad + g(x(t), x(t-\alpha))\mathscr{A}x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha)) \\ &\quad + \bar{p}(x(t), x(t-\alpha)), \\ y &= h(x).\end{aligned} \tag{A.13}$$

Applying the coordinate transformation of Eq. (27) to the above system and setting, for ease of notation, $x(s) = \chi^{-1}(\zeta(s), \eta(s)), s \in [t-\alpha, t]$, we obtain

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 + p_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r + p_{r-1}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \\ \dot{\zeta}_{r-1} &= L_{\tilde{f}}^{T}h(x) \\ &\quad + L_gL_{\tilde{f}}^{r-1}h(x)\mathscr{A}(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha)) \\ &\quad + p_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \\ \dot{\eta}_1 &= \Psi_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \\ &\vdots \\ \eta_{n-r} &= \Psi_{n-r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \\ y &= \zeta_1.\end{aligned} \tag{A.14}$$

Introducing, the variables $e_i = \zeta_i - v^{(i-1)}$, $i = 1, \ldots ,r$, the system of Eq. (A.14) takes the form

$$\begin{aligned}\dot{e}_1 &= e_2 + p_1(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha)), \\ &\vdots \\ \dot{e}_{r-1} &= e_r + p_{r-1}(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \\ \dot{e}_r &= L_{\tilde{f}}^{r}h(x) - v^{(r)} \\ &\quad + L_gL_{\tilde{f}}^{r-1}h(x)\mathscr{A}(x(t), \bar{v}(t), \\ &\quad \times x(t-\alpha), \bar{v}(t-\alpha)), \eta(t-\alpha)), \\ &\quad + p_r(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha)), \\ \dot{\eta}_1 &= \Psi_1(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha)), \\ &\vdots \\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha)),\end{aligned} \tag{A.15}$$

where $\bar{e} = [e_1\, e_2 \cdots e_r]^{T}$. For the above system, we assume, without loss of generality, that when $\eta = 0$, $\bar{e} = 0$ is an equilibrium solution.

We now consider the $\bar{e}$-subsystem of the system of Eq. (A.15):

$$\begin{aligned}
\dot{e}_1 &= e_2 + p_1(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha)),\\
&\vdots\\
\dot{e}_{r-1} &= e_r\\
&\quad + p_{r-1}(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha)),\\
\dot{e}_r &= L_{\tilde{f}}^r h(x) - v^{(r)}\\
&\quad + L_g L_{\tilde{f}}^{r-1} h(x) \mathscr{A}(x(t), \tilde{v}(t), \dot{x}(t-\alpha), \bar{v}(t-\alpha))\\
&\quad + p_r(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha))
\end{aligned} \tag{A.16}$$

Using the explicit form of the controller formula of Eq. (37) and the definition for the matrix $\tilde{A}$ and the vectors b, p of Eq. (32), the above system can be written in the following compact form:

$$\begin{aligned}
\dot{\bar{e}} &= \tilde{A}\bar{e} - bR_2^{-1}b^T P\bar{e}\\
&\quad + p(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha)).
\end{aligned} \tag{A.17}$$

We will now show that if Eq. (36) has a unique positive definite solution for P and the state of the η-subsystem of Eq. (A.15) is bounded, then the system of Eq. (A.17) is exponentially stable, which implies that $\lim_{t\to\infty}|y - v|_{\mathbb{R}} = 0$. To establish this result, we consider the following smooth functional $V: \mathscr{C} \to \mathbb{R}_{\geq 0}$:

$$V(\bar{e}_t(\xi)) = \bar{e}^T P\bar{e} + a^2 \int_{t-\alpha}^{t} \bar{e}^T(s)\bar{e}(s)\,ds \tag{A.18}$$

which clearly satisfies, $K_1|\bar{e}(t)|^2_{\mathbb{R}^n} \leq V(\bar{e}_t(\xi)) \leq K_2|\bar{e}_t(\xi)|^2$, for some positive constants K_1 and K_2. Computing the time derivative of V along the trajectories of the system of Eq. (A.17), we obtain

$$\begin{aligned}
\dot{V} &= \dot{\bar{e}}^T P\bar{e} + \bar{e}^T P\dot{\bar{e}} + a^2(\bar{e}^T(t)\bar{e}(t) - \bar{e}^T(t-\alpha)\bar{e}(t-\alpha))\\
&\leq (\bar{e}^T\tilde{A}^T - \bar{e}^T P^T b R_2^{-1} b^T\\
&\quad + p^T(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha)\\
&\quad + \bar{v}(t-\alpha), \eta(t-\alpha)))P\bar{e}\\
&\quad + \bar{e}^T P(\tilde{A}\bar{e} - bR_2^{-1}b^T P\bar{e}\\
&\quad + p(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha)\\
&\quad + \bar{v}(t-\alpha), \eta(t-\alpha)))\\
&\quad + a^2(\bar{e}^T(t)\bar{e}(t) - \bar{e}^T(t-\alpha)\bar{e} + (t-\alpha))\\
&\leq \bar{e}^T(\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + a^2 I_{n\times n})\bar{e}\\
&\quad + 2p^T(\bar{e}(t)\\
&\quad + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha))P\bar{e}\\
&\quad - a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha)
\end{aligned} \tag{A.19}$$

where $I_{n\times n}$ denotes the identity matrix of dimension $n \times n$. Using the inequality $2x^T y \leq x^T x + y^T y$ where x and y are column vectors, we obtain

$$\begin{aligned}
\dot{V} &\leq \bar{e}^T(\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + a^2 I_{n\times n})\bar{e}\\
&\quad + p^2(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha))\\
&\quad + \bar{e}^T P^2 \bar{e} - a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha),\\
&\leq \bar{e}^T(\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + a^2 I_{n\times n} + P^2)\bar{e}\\
&\quad + p^2(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha))\\
&\quad - a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha).
\end{aligned} \tag{A.20}$$

Since the state of the η-subsystem of Eq. (A.15) is supposed to be bounded and assuming quadratic growth for the term $p^2(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha))$ (Assumption 3), we have that there exist positive real numbers a_1, a_2 such that the following bound can be written:

$$\begin{aligned}
&|p(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha),\\
&\quad \eta(t-\alpha))|_{\mathbb{R}^r} \leq a_1\bar{e}^2(t) + a_2\bar{e}^2(t-\alpha).
\end{aligned} \tag{A.21}$$

Substituting the above inequality on the bound for $\dot{V}$ in Eq. (A.20), we obtain

$$\begin{aligned}
\dot{V} &\leq \bar{e}^T(\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + a^2 I_{n\times n} + P^2)\bar{e}\\
&\quad + a_1\bar{e}^2(t) + a_2\bar{e}^2(t-\alpha) - a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha)\\
&\leq \bar{e}^T(\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + (a^2 + a_1) I_{n\times n}\\
&\quad + P^2)\bar{e} - (a^2 - a_2)\bar{e}^2(t-\alpha).
\end{aligned} \tag{A.22}$$

Now, if $a^2 > a_2$ and there exists a positive-definite symmetric matrix P which solves the following matrix equation:

$$\begin{aligned}
&\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + (a^2 + a_1) I_{n\times n}\\
&\quad + P^2 = -R_1,
\end{aligned} \tag{A.23}$$

where R_1 is a positive-definite matrix, then

$$\begin{aligned}
\dot{V} &\leq -\bar{e}^T \mathbb{R}_1 \bar{e} - a_3\bar{e}^2(t-\alpha)\\
&\leq -\lambda_{\min}(R_1)\bar{e}^2 - a_3\bar{e}^2(t-\alpha).
\end{aligned} \tag{A.24}$$

where a_3 is a positive real number and $\lambda_{\min}(R_1)$ denotes the smallest eigenvalue of the matrix R_1. Since V and $\dot{V}$ satisfy the assumptions of Theorem 1, we have that there exist positive real numbers K, a such that the state of the $\bar{e}$-subsystem of Eq. (A.16) is exponentially stable, i.e., it satisfies

$$|\bar{e}(t)|_{\mathbb{R}^r} \leq K e^{-at}|\bar{e}(0)|_{\mathbb{R}^r} \tag{A.25}$$

for every value of the delay, α, and thus, $\lim_{t\to\infty}|y - v|_{\mathbb{R}} = 0$.

To complete the proof of the theorem, we need to show that there exists a positive real number δ such that if $\max\{|x_0(\xi)|, \|\bar{v}\|\} \leq \delta$, then the state of the closed-loop system is exponentially stable and the output of the closed-loop system satisfies $\lim_{t\to\infty}|y - v|_{\mathbb{R}} = 0$. To establish this result, we will analyze the behavior of the DDE system of Eq. (A.15) using a two-step small-gain theorem type argument (Teel, 1998). In the first step, we will use a contradiction argument to show that the evolution of the states $\bar{e}, \eta$, starting from sufficiently small initial conditions, i.e., $|\bar{e}_0(\xi)| \leqslant \delta_{\bar{e}}$ and $|\eta_0(\xi)| \leqslant \delta_\eta$ (where $(\delta_{\bar{e}}, \delta_\eta)$ are positive real numbers which can be explicitly computed as functions of δ from the coordinate change of Eq. (27)), satisfies the following inequalities:

$$|\bar{e}(t)|_{\mathbb{R}^r} \leqslant \bar{\delta}_{\bar{e}}, \quad |\eta(t)|_{\mathbb{R}^{n-r}} \leqslant \bar{\delta}_\eta, \quad \forall t \in [0, \infty), \tag{A.26}$$

where $\bar{\delta}_{\bar{e}} > K\delta_{\bar{e}}$ and $\bar{\delta}_\eta > \delta_\eta$ are positive real number which will be specified below. In the second step, we will use the boundedness result obtained from the first step and the exponentially decaying bound of Eq. (A.25) to prove that the state of the η-subsystem of Eq. (A.15) decays exponentially to zero.

From Assumption 3, we have that the η-subsystem of the system of Eq. (A.15) is input-to-state stable, which implies that there exist a function $\bar{\gamma}_{\bar{e}}$ of class Q, a function β_η of class KL and positive real numbers $\bar{\delta}_{\bar{e}}, \delta_\eta$ such that if $|\eta_0(\xi)| < \delta_\eta$ and $\|\bar{e}_t\| < \bar{\delta}_{\bar{e}}$, then

$$|\eta(t)|_{\mathbb{R}^{n-r}} \leqslant |\eta_t(\xi)| \leqslant \beta(|\eta_0(\xi)|, t) + \bar{\gamma}_{\bar{e}}(\|\bar{e}_t\|), \quad \forall t \geqslant 0. \tag{A.27}$$

We will proceed by contradiction. Set $\bar{\delta}_\eta > \beta_\eta(\delta_\eta, 0) + \bar{\gamma}_{\bar{e}}(K\delta_{\bar{e}})$ and let $\bar{T}$ be the smallest time such that there is a $\hat{\delta}$ so that $t \in (\bar{T}, \bar{T} + \hat{\delta})$ implies either $|\bar{e}(t)|_{\mathbb{R}^r} > \bar{\delta}_{\bar{e}}$ or $|\bar{\eta}(t)|_{\mathbb{R}^{n-r}} > \bar{\delta}_\eta$. Then, for each $t \in [0, \bar{T}]$ the conditions of Eq. (A.26) hold.

Consider the functions $\bar{e}^T(t), \eta^T(t)$ defined as follows:

$$\bar{e}^{\bar{T}}(t) = \begin{cases} \bar{e}(t), & t \in [0, \bar{T}] \\ 0 & t \in (\bar{T}, \infty) \end{cases}, \quad \eta^{\bar{T}}(t) = \begin{cases} \eta(t), & t \in [0, \bar{T}], \\ 0 & t \in (\bar{T}, \infty). \end{cases} \tag{A.28}$$

Now, using Eq. (A.27) we have that

$$\sup_{0 \leqslant t \leqslant \bar{T}} (\beta_\eta(|\eta_0(\xi)|, t) + \bar{\gamma}_{\bar{e}}(\|\bar{e}_t\|)) \leqslant \beta_\eta(\delta_\eta, 0) + \bar{\gamma}_{\bar{e}}(K\delta_{\bar{e}}),$$
$$\sup_{0 \leqslant t \leqslant \bar{T}} (K|\bar{e}(0)|_{\mathbb{R}^r} e^{-at}) \leqslant K|\bar{e}(0)|_{\mathbb{R}^r} \leqslant K\delta_{\bar{e}}. \tag{A.29}$$

and that

$$\|\bar{e}^{\mathrm{T}}\| \leqslant K\delta_{\bar{e}} < \bar{\delta}_{\bar{e}},$$
$$\|\eta^{\bar{T}}\| \leqslant \beta_\eta(\delta_\eta, 0) + \bar{\gamma}_{\bar{e}}(K\delta_{\bar{e}}) < \bar{\delta}_\eta. \tag{A.30}$$

By continuity, we have that there exist some positive real number $\bar{k}$ such that $\|\bar{e}^{\bar{T}+\bar{k}}(t)\| \leqslant \bar{\delta}_{\bar{e}}$ and $\|\eta^{\bar{T}+\bar{k}}(t)\| \leqslant \bar{\delta}_\eta, \forall t \in [0, \bar{T} + \bar{k}]$. This contradicts the definition of $\bar{T}$. Hence, Eq. (A.26) holds $\forall t \geqslant 0$.

Since the states, $(\bar{e}, \eta)$, of the closed-loop system of Eq. (A.15) are bounded and $\bar{e}(t)$ decays exponentially to zero, a direct application of the result of theorem 2 in Teel (1998) implies that the η-subsystem of Eq. (A.15) is also exponentially stable, and thus, the system of Eq. (A.15) is exponentially stable and its output satisfies $\lim_{t \to \infty} |y - v|_{\mathbb{R}} = 0$.

Proof of Theorem 4. We initially construct that dynamical system which describes the dynamics of the estimation error, $e_t = \omega_t - x_t$. Differentiating e_t and using the systems of Eqs. (10) and (50), we obtain

$$\frac{de_t}{dt} = \mathscr{A} e_t + f(P(e_t + x_t), Q(e_t + x_t)) - f(Px_t, Qx_t) + \Phi_H L((\Psi_H, e_t + x_t))(y(t) - h(P(e_t + x_t))). \tag{A.31}$$

Computing the linearization of the above system and applying the spectral decomposition procedure, we obtain

$$\frac{de_t^p}{dt} = \mathscr{A}_p e_t^p - \Phi_H L(w(Pe_t^p + Pe_t^n)),$$
$$\frac{\partial e_t^n}{\partial t} = \mathscr{A}_n e_t^n,$$
$$e_t^p(0) = P_p e(0) = P_p(\bar{\omega} - \bar{\eta}), e_t^n(0) = P_n e(0) = P_n(\bar{\omega} - \bar{\eta}) \tag{A.32}$$

because $P_n \Phi_H L((\Psi_H, \omega_t))(y(t) - h(P\omega_t)) \equiv 0$ by construction. Since all the eigenvalues of the operator A_n lie in the left-half of the complex plane, this implies that the subsystem

$$\frac{\partial e_t^n}{\partial t} = \mathscr{A}_n e_t^n \tag{A.33}$$

is exponentially stable. Therefore, the infinite-dimensional system of Eq. (A.32) possesses identical stability properties with the finite-dimensional system:

$$\frac{de_t^p}{dt} = \mathscr{A}_p e_t^p + \Phi_H L w P e_t^p \tag{A.34}$$

From the observability Assumption 4, however, we have that the above system is exponentially stable, which implies that the error system is exponentially to zero, and thus, the estimation error, $e_t = \omega_t - x_t$, tends locally exponentially to zero.

Nonlinear state observer simplification: The abstract dynamical system of Eq. (50) can be simplified by utilizing the fact that

$$\mathscr{A}\omega_t(\xi) + \Phi_H L((\Psi_H, \phi))(y(t) - h(P\omega_t(\xi))) = \begin{cases} \dfrac{d\phi(\xi)}{d\xi} + \Phi_H(\xi) L((\Psi_H, \phi))(y(t) - h(P\omega_t(\xi))), & \xi \in [-\alpha, 0) \\ A\omega_t(0) + B\phi(-\alpha) - \Phi_H(0) L((\Psi_H, \phi)) h(P\omega_t(\xi)), & \xi = 0 \end{cases} \tag{A.35}$$

which implies that is equivalent to the following system of partial differential equations:

$$\frac{\partial \tilde{\omega}}{\partial t}(\xi, t) = \frac{\partial \tilde{\omega}}{\partial \xi}(\xi, t) + \Phi_H(\xi) L((\Psi_H, \tilde{\omega}(\xi, t)))(y(t) - h(\tilde{\omega}(0, t))), \tag{A.36}$$

$$\frac{\partial \tilde{\omega}}{\partial t}(0,t) = A\tilde{\omega}(0,t) + B\tilde{\omega}(-\alpha,t) + \Phi_H(0)L((\Psi_H,\tilde{\omega}(\xi,t)))(y(t) - h(\tilde{\omega}(0,t))), \quad \text{(A.37)}$$

where $\tilde{\omega}(\xi,t) = \omega_t(\xi)$. A further simplification can be performed by integrating the hyperbolic PDE of Eq. (A.36) along its characteristics to obtain a nonlinear integro-differential equation system for the observer. The characteristics of Eq. (A.36) are the family of lines with slope $dt/d\xi = -1$, i.e., the set of lines satisfying $t + \xi = c$, where c is a constant. Integrating the Eq. (A.36) along its characteristics, we obtain

$$\frac{d\tilde{\omega}}{d\xi}(c-\xi,\xi) = -\Phi(\xi)L((\Psi_H,\tilde{\omega}(\xi,t)))h(\tilde{\omega}(0,c-\xi)) + \Phi(\xi)L((\Psi_H,\tilde{\omega}(\xi,t)))y(c-\xi) \quad \text{(A.38)}$$

so that

$$\tilde{\omega}(c-\theta,\theta) = \tilde{\omega}(c,0) - \int_0^{\theta}\Phi(\xi)L((\Psi_H,\tilde{\omega}(\xi,t))) \times (h(\tilde{\omega}(0,c-\xi)) - y(c-\xi))\,d\xi,\ \theta\in[-\alpha,0], \quad \text{(A.39)}$$

or

$$\tilde{\omega}(t,\theta) = \tilde{\omega}(t+\theta,0) - \int_0^{\theta}\Phi(\xi)L((\Psi_H,\tilde{\omega}(\xi,t))) \times (h(\tilde{\omega}(0,t+\theta-\xi)) - y(t+\theta-\xi))\,d\xi. \quad \text{(A.40)}$$

Finally,

$$\begin{aligned}\tilde{\omega}(t-\alpha) &= \tilde{\omega}(t-\alpha,0) - \int_0^{-\alpha}\Phi(\xi)L((\Psi_H,\tilde{\omega}(\xi,t))) \times (h(\tilde{\omega}(0,t-\alpha-\xi)) - y(t-\alpha-\xi))\,d\xi \\ &= \tilde{\omega}(t-\alpha,0) + \int_0^{\alpha}\Phi(\xi-\alpha)L((\Psi_H,\tilde{\omega}(\xi,t))) \times (h(\tilde{\omega}(0,t-\xi)) - y(t-\xi))\,d\xi. \end{aligned} \quad \text{(A.41)}$$

Substituting $\tilde{\omega}(t+\theta,0) = \omega(t+\theta)$ into Eq. (A.37), we obtain the following system:

$$\begin{aligned}\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega(t),\omega(t-\alpha)) + g(\omega(t),\omega(t-\alpha))u \\ &\quad + \Phi_H(0)L((\Psi_H,\tilde{\omega}(\xi,t)))(y(t) - h(\omega(t))) \\ &\quad + B\int_0^{\alpha}\Phi_H(\xi-\alpha)L((\Psi_H,\tilde{\omega}(\xi,t)))[y(t-\xi) - h(\omega(t-\xi))]\,d\xi. \end{aligned} \quad \text{(A.42)}$$

which is identical to the one of Eq. (51).

Proof of Theorem 5. Under the output feedback controller of Eq. (57), the closed-loop system takes the form

$$\begin{aligned}\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + f(\omega(t),\omega(t-\alpha)) + \Phi_H(0)L((\Psi_H,\tilde{\omega}(\xi,t)))(y(t) - h(\omega(t))) \\ &\quad + B\int_0^{\alpha}\Phi_H(\xi-\alpha)L((\Phi_H,\tilde{\omega}(\xi,t)))[y(t-\xi) - h(\omega(t-\xi))]\,d\xi + g(w(t),\omega(t-\alpha)) \\ &\quad \times\frac{1}{L_gL_{\tilde{f}}^{r-1}h(\omega)}(-R_2^{-1}b^{T}P\bar{e}(t) + v^{(r)}(t) - L_{\tilde{f}}^{r}h(\omega) - p_r(\omega(t),\bar{v}(t),\omega(t-\alpha),\bar{v}(t-\alpha))), \quad \text{(A.43)} \\ \dot{x} &= Ax(t) + Bx(t-\alpha) + f(x(t),x(t-\alpha)) + g(x(t),x(t-\alpha))\frac{1}{L_gL_{\tilde{f}}^{r-1}h(\omega)}(-R_2^{-1}b^{T}P\bar{e}(t) \\ &\quad + v^{(r)}(t) - L_{\tilde{f}}^{r}h(\omega) - p_r(\omega(t),\bar{v}(t),\omega(t-\alpha)),\bar{v}(t-\alpha))).\end{aligned}$$

Since we are interested in proving local exponential stability of the above system, we compute its linearization around the zero solution and use that $f(0,0)=0$ and $f(x(t),x(t-\alpha))$ includes higher-order terms, and $g(0,0) = c$, where c is a constant vector, to obtain the following linear system:

$$\begin{aligned}\dot{\omega} &= A\omega(t) + B\omega(t-\alpha) + \Phi_H(0)L(wx(t) - w\omega(t)) + B\int_0^{\alpha}\Phi_H(\xi-\alpha)L[wx(t-\xi) - w\omega(t-\xi)]\,d\xi \\ &\quad + cu_{\text{lin}}(\omega(t),\bar{v}(t),\omega(t-\alpha),\bar{v}(t-\alpha)), \\ \dot{x} &= Ax(t) + Bx(t-\alpha) + cu_{\text{lin}}(\omega(t),\bar{v}(t),\omega(t-\alpha),\bar{v}(t-\alpha)),\end{aligned} \quad \text{(A.44)}$$

where L is the linearization of the nonlinear vector $L((\Psi_H,\tilde{\omega}(\xi,t)))$ around the zero solution, $w = (\partial h/\partial x)(0)$, and u_{lin} is the linearization of the term:

$$\frac{1}{L_gL_{\tilde{f}}^{r-1}h(\omega)}(-R_2^{-1}b^{T}P\bar{e}(t) + v^{(r)}(t) - L_{\tilde{f}}^{r}(\omega)) - p_r(\omega(t),\bar{v}(t),\omega(t-\alpha),\bar{v}(t-\alpha))) \quad \text{(A.45)}$$

around the zero solution.

Introducing the error vector $e_r = x - \omega$, the closed-loop system of Eq. (A.44) can be written as

$$\begin{aligned}\dot{e}_r &= Ae_r(t) + Be_r(t-\alpha) + \Phi_H(0)Lwe_r(t) + B\int_0^{\alpha}\Phi_H(\xi-\alpha)Lwe_r(t-\xi)\,d\xi, \\ \dot{x} &= Ax(t) + Bx(t-\alpha) + cu_{\text{lin}}(x(t) + e_r(t),\bar{v}(t),x(t-\alpha) + e_r(t-\alpha),\bar{v}(t-\alpha)).\end{aligned} \quad \text{(A.46)}$$

From the observability Assumption 3, we have that the error system

$$\dot{e}_r = Ae_r(t) + Be_r(t-\alpha) + \Phi_H(0)Lwe_r(t) + B\int_0^{\alpha} \Phi_H(\xi-\alpha)Lwe_r(t-\xi)\,\mathrm{d}\xi \tag{A.47}$$

is exponentially stable. Moreover, from the construction of the state feedback controller, we have that the system

$$\dot{x} = Ax(t) + Bx(t-\alpha) + cu_{\text{lin}}(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha)),$$
$$y = wx(t) \tag{A.48}$$

is exponentially stable and the output asymptotically follows the reference input (proof of Theorem 2). Therefore, we have that the linear system of Eq. (A.46) is an interconnection of two exponentially stable subsystems, which implies that it is also exponentially stable. Using the result of Theorem 1.1 in Hale and Verduyn Lunel (1993) Chapter 10 (which allows inferring the local stability properties of a nonlinear system based on its linearization) we obtain that the nonlinear closed-loop system of Eq. (A.43) is locally exponentially stable and the discrepancy between the output and the reference input asymptotically tends to zero.

References

Arbel, A., Rinard, I. H., & Shinnar, R. (1995). Dynamics and control of fluidized catalytic crackers. 1. Modeling of the current generation of the FCC's.. *Industrial and Engineering Chemistry Research*, *34*, 1228–1243.

Arbel, A., Rinard, I. H., & Shinnar, R. (1996). Dynamics and control of fluidized catalytic crackers. 3. Designing the control system: choice of manipulated and measured variables for partial control. *Industrial and Engineering Chemistry Research*, *35*, 2215–2233.

Bartee, J. F., Bloss, K. F., & Georgakis, C. (1989). Design of nonlinear reference system control structures. *A.I.Ch.E. annual meeting*, San Francisco, CA.

Bhat, K. P. M., & Koivo, H. N. (1976). Modal characterizations of controllability and observability in time delay systems. *IEEE Transactions on Automatic Control*, *21*, 292–293.

Brosilow, C. (1976). Structure and design of smith predictors from the viewpoint of inferential control, *Proceedings of joint American control conference*.

Christofides, P. D., & Daoutidis, P. (1996). Feedback control of hyperbolic PDE systems. *A.I.Ch.E. Journal*, *42*, 3063–3086.

Christofides, P. D., & Daoutidis, P. (1997). Robust control of multivariable two-time-scale nonlinear systems. *Journal of Process Control*, *7*, 313–328.

Daoutidis, P., & Christofides, P. D. (1995). Dynamic feedforward/output feedback control of nonlinear processes. *Chemical Engineering Science*, *50*, 1889–2007.

Denn, M. M. (1986). *Process modeling*. New York: Longam Inc.

Garcia, C. E., & Morari, M. (1982). Internal model control. 1: a unifying review and some new results. *IEC Proceedings Designs and Development*, *21*, 308–322.

Gu, G., Khargonekar, P. P., & Lee, E. B. (1989). Approximation of infinite dimensional systems. *IEEE Transactions on Automatic Control*, *34*, 610–618.

Hale, J. K., & Verduyn Lunel, S. M. (1993). Introduction to functional differential equations. . New York: Springer.

Henson, M. A., & Seborg, D. E. (1994). Time delay compensation for nonlinear processes. *Industrial and Engineering Chemistry Research*, *33*, 1493–1500.

Huq, I., Morari, M., & Sorensen, R. C. (1995). Modifications to model IV fluid catalytic cracking units to improve dynamic performance. *A.I.Ch.E. Journal*, *41*, 1481–1499.

Isidori, A. (1989). *Nonlinear control systems: an introduction*. Berlin: Springer.

Jerome, N. F., & Ray, W. H. (1986). High performance multivariable control strategies for systems having time delays. *A.I.Ch.E. Journal*, *32*, 914–931.

Kazantzis, N., & Kravaris, C. (1995). A nonlinear luenberger-type observer with application to catalytic activity estimation. *Proceedings of American control conference*, Seattle, WA, (pp. 1756–1760).

Khalil, H. K. (1992). Nonlinear systems. New York: Macmillan.

Koivo, H. N., & Koivo, A. J. (1978). Control and estimation of systems with time-delays. In Ray, W. H., & Lainiotis, D. G., *Distributed parameter systems*, New York.

Kravaris, C., & Arkun, Y. (1991). Geometric nonlinear control – an overview. In Arkun, Y., & Ray, W. H., *Proceedings of fourth international conference on chemical process control*, Padre Island, TX, (pp. 477–515).

Kravaris, C., & Chung, C. B. (1987). Nonlinear state feedback synthesis by global input/output linearization. *A.I.Ch.E. Journal*, *33*, 592–604.

Kravaris, C., & Wright, R. A. (1989). Deadtime compensation for nonlinear processes. *A.I.Ch.E. Journal*, *35*, 1535–1542.

Lehman, B., Widjaya, I., & Shujaee, K. (1995). Vibrational control of chemical reactions in a CSTR with delayed recycle stream. *Journal of Mathematical Analysis and Applications*, *193*, 28–59.

Lu, W., Lee, E. B., & Zhang, Q. T. (1987). Balanced approximation of two-dimensional and delay-differential equations. *International Journal of Control*, *46*, 2199–2218.

Manitius, A., & Tran, H. (1985). Computation of closed-loop eigenvalues associated with the optimal regulator problem for functional differential equations. *IEEE Transactions on Automatic Control*, *30*, 1245–1248.

Manitius, A., Tran, H., Payre, G., & Roy, R. (1987). Computation of eigenvalues associated with functional differential equations. *SIAM Journal of Scientific and Statistical Computing*, *8*, 222–247.

McFarlane, R. C., Reineman, R. C., Bartee, J. F., & Georgakis, C. (1993). Dynamic simulator for a model IV fluid catalytic cracking unit. *Computers and Chemical Engineering*, *17*, 275–293.

Monge, J. J., & Georgakis, C. (1987). The effect of operating variables on the dynamics of catalytic cracking processes. *Chemical Engineering Communications*, *60*, 1–15.

Moore, C. F., Smith, C. L., & Murril, P. W. (1970). Improved algorithm for direct digital control. *Instruments and Control Systems*, *43*, 70–79.

Nazaroff, G. J. (1973). Stability and stabilization of linear differential delay systems. *IEEE Transactions on Automatic Control*, *18*, 317–318.

Ogunnaike, B. A., & Ray, W. H. (1979). Multivariable controller design for linear systems having multiple time delays. *A.I.Ch.E. Journal*, *25*, 1042–1057.

Partington, J. R. (1991). Approximation of delay system by fourier-laguerre series. *Automatica*, *27*, 569–572.

Ray, W. H. (1981). *Advanced process control*. New York: McGraw-Hill.

Ross, D. W., & Flugge-Lotz, I. (1969). An optimal control problem for systems described by differential-difference equations. *SIAM Journal of Optimization and Control*, *7*, 609–623.

Smith (1957). Closer control of loops with dead time. *Chemical Engineering Progress*, *53*, 217–225.

Soliman, M. A., & Ray, W. H. (1972). Optimal feedback control for linear quadratic systems having time delays. *International Journal of Control*, *15*, 609–627.

Soroush, M. (1997). Nonlinear state observer design with application to reactors. *Chemical Engineering Science*, *52*, 387–404.

Stephanopoulos, G. (1984). *Chemical process control: an introduction to theory and practice*. Englewood Cliffs, New J.: Prentice Hall.

Teel, A. R. (1998). Connections between Razumikhin-type theorems and the ISS nonlinear small-gain theorem. *IEEE Transactions on Automatic Control*, *43*, 960–965.

Vit, K. (1979). Smithlike predictor for control of parameter-distributed processes. *International Journal of Control*, *30*, 179–191.

Wahlberg, B. (1991). System identification using laguerre models. *IEEE Transactions on Automatic Control*, *36*, 551–562.

Wellons, M. C., & Edgar, T. F. (1987). The generalized analytical predictor. *Industrial and Engineering Chemistry Research*, *26*, 1523–1532.

Wong, S. K. P., & Seborg, D. E. (1986). A theoretical analysis of smith and analytical predictors. *A.I.Ch.E. Journal*, *32*, 1597–1605.

Wong, S. K. P., & Seborg, D. E. (1988). Control strategy for single-input single-output nonlinear systems with time delays. *International Journal of Control*, *48*, 2303–2327.

Xu, B., & Liu, Y. (1994). An improved Razumikhin-type theorem and its applications. *IEEE Transactions on Automatic Control*, *39*, 839–841.