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ROBUST CONTROL OF NONLINEAR TIME-DELAY SYSTEMS

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This paper focuses on single-input single-output nonlinear differential difference equation (DDE) systems with uncertain variables. For such systems, a general methodology is developed for the synthesis of robust nonlinear state feedback controllers that guarantee boundedness of the states and ensure that the ultimate discrepancy between the output and the external reference input in the closed-loop system can be made arbitrarily small by an appropriate choice of controller parameters. The controllers are synthesized by using a novel combination of geometric and Lyapunov-based techniques and enforce the above properties in the closed-loop system independently of the size of the state delay. The proposed control method is successfully applied to a fluidized catalytic cracking unit with a time-varying uncertain variable and is shown to outperform a proportional integral (PI) controller, a nonlinear controller that does not account for the uncertainty, and a nonlinear controller that does not account for the state delays.

Keywords: model uncertainty, time-delays, nonlinear robust control, fluidized catalytic cracker

1. Introduction

The dynamic models of many chemical engineering processes involve nonlinearities, time delays and uncertain variables and are naturally described by nonlinear differential difference equation (DDE) systems. Time delays often occur due to transportation lag such as in flow through pipes, dead times associated with measurement sensors (measurement delays) and control actuators (manipulated input delays), and approximation of high-order dynamics. On the other hand, uncertain variables usually arise from unknown or partially known process parameters (e.g., heat transfer coefficients, kinetic constants, etc.) and external disturbances (e.g., concentration, temperature, and flow of inlet streams). Representative examples of processes which involve nonlinearities and time delays include chemical reactors with recycle loops, fluidized catalytic cracking units, distillation columns, chemical vapor deposition processes, etc. The presence of time-delays and uncertain variables, if it is not appropriately accounted for in the design of the controller, may pose unacceptable limitations

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on the achievable control quality and may cause serious problems in the behavior of the closed-loop system including poor performance (e.g., sluggish response) and instability (e.g., oscillations).

The problem of synthesizing linear and nonlinear controllers for uncertain systems that enforce output tracking with attenuation of the effect of the uncertain variables on the output has received considerable attention in the past. Specifically, research initially focused on the development of robust control methods to address this problem for linear systems in both frequency-domain and state-space domain including H_∞, μ-synthesis, etc. (Doyle *et al.*, 1989). In the context of nonlinear systems, adaptive control techniques were employed to solve this problem locally (i.e., for sufficiently small initial conditions and uncertain variables) (Kanellakopoulos *et al.*, 1991; Sastry and Isidori, 1989; Teel *et al.*, 1991) and globally (Krstic *et al.*, 1995; Marino and Tomei, 1993) for certain classes of nonlinear systems with constant uncertain variables. For nonlinear systems with time-varying uncertain variables that satisfy the so-called matching condition, robust state feedback controllers have been designed via Lyapunov's direct method to solve this problem locally (see the papers (Corless, 1993; Leitmann, 1993) for a review of results in this area). More recently, robust state (Christofides *et al.*, 1996) and output (Khalil, 1994) feedback controllers were designed that solve this problem for arbitrarily large initial conditions and uncertainty (semi-global result). Although these works provided powerful methods for adaptive and robust controller design for several classes of nonlinear systems, a direct application of these methods to nonlinear systems with uncertainty and time-delays may lead to poor performance and/or closed-loop instability.

In the area of control of systems with time-delays, research initially focused on linear systems with a manipulated input delay which are described by transfer function models, in which the presence of the time delay prevents the use of large controller gains (i.e., the proportional gain of a proportional-integral controller should be sufficiently small in order to avoid destabilization of the closed-loop system), thereby leading to sluggish closed-loop response. To overcome this problem, many researchers proposed various predictor schemes which completely eliminate the time delay from the characteristic polynomial of the closed-loop system, allowing the use of larger controller gains (Brosilow, 1976; Jerome and Ray, 1986; Smith, 1957). These results were extended to certain classes of nonlinear systems in (Henson and Seborg, 1994; Kravaris and Wright, 1989; Yanakiev and Kanellakopoulos, 1997). For DDE systems with state, measurement and manipulated input delays, a general method based on a novel integration of geometric concepts, Lyapunov functionals and predictor schemes was proposed in (Antoniades and Christofides, 1999) for the synthesis of nonlinear output feedback controllers. Research on control of uncertain DDE systems has mainly focused on linear systems where important contributions include robust control using the concept of quadratic stabilization (Cao *et al.*, 1997; Li *et al.*, 1997; Phoojaruenchanachai and Furuta, 1992) and linear matrix inequalities (Trofino *et al.*, 1997). In the context of nonlinear uncertain DDE systems, results include stabilization via variable structure control (Shyu and Yan, 1993; Thowsen, 1983) and dynamic disturbance decoupling (Moog *et al.*, 1996).

This paper focuses on single-input single-output nonlinear differential difference equation (DDE) systems with uncertain variables. For such systems, we propose a general methodology for the synthesis of robust nonlinear state feedback controllers that guarantee boundedness of the states and ensure that the ultimate discrepancy between the output and the external reference input in the closed-loop system can be made arbitrarily small by an appropriate choice of controller parameters. The controllers are synthesized by using a novel combination of geometric and Lyapunov-based techniques and enforce the aforementioned properties in the closed-loop system independently of the size of the state delay. The proposed control method is successfully applied to a fluidized catalytic cracking unit with a time-varying uncertain variable.

2. Notation

- $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f. $L_f^k h$ denotes the k-th order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.
- A function W: $\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be positive definite if $W(x)$ is positive for all nonzero x and is zero at zero.
- A function W: $\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be proper if $W(x)$ tends to $+\infty$ as $\|x\|_{\mathbb{R}^n}$ tends to $+\infty$, where $\|\cdot\|_{\mathbb{R}^n}$ denotes the standard Euclidean norm in $\mathbb{R}^n$.
- A function γ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{Q}$ if it is continuous, nondecreasing and zero at zero. It is of class $\mathcal{K}$ if it is continuous, strictly increasing and is zero at zero. It is of class $\mathcal{K}_\infty$ if, in addition, it is proper.
- A function β: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non increasing and tends to zero at infinity.

3. Preliminaries

We consider single-input single-output nonlinear DDE systems with uncertain variables with the following state-space description:

$$
\begin{aligned}
\dot{x} &= f\big(x(t), x(t-\alpha), \theta(t-\alpha_\theta)\big) + g\big(x(t), x(t-\alpha)\big)u(t),\\
x(\xi) &= \bar{\eta}(\xi), \quad \xi \in [-\alpha, 0],\\
y(t) &= h\big(x(t)\big),
\end{aligned}
\tag{1}
$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of the state variables, $u(t) \in \mathbb{R}$ denotes the manipulated input, $\theta(t-\alpha_\theta) = [\theta_1(t-\alpha_{\theta 1}) \ \cdots \ \theta_q(t-\alpha_{\theta q})] \in \mathbb{R}^q$ denotes the vector of the time-varying uncertain variables, $y(t) \in \mathbb{R}$ denotes the controlled output,

α denotes the state delay, $f(x(t), x(t-\alpha), \theta(t-\alpha_\theta))$, $g(x(t), x(t-\alpha))$ are locally Lipschitz nonlinear vector functions, $h(x(t))$ is a locally Lipschitz nonlinear scalar function and $\bar{\eta}(\xi)$ is a smooth vector function defined in the interval $[-\alpha, 0]$. We will assume that the vector function $f(x(t), x(t-\alpha), \theta(t-\alpha_\theta))$ satisfies $f(0,0,0) = 0$ which implies that $x(t) \equiv 0$ is an equilibrium solution for the open-loop system of eqn. (1) (i.e., system of eqn. (1) with $u(t) \equiv 0$).

There are many chemical engineering processes whose dynamic models involve uncertain variables and time delays in the state variables, and are naturally described by nonlinear DDE systems of the form (1). Uncertain variables typically arise from unknown or partially known process parameters (e.g., activation energies, heat of reaction, heat transfer coefficients) and external disturbances (e.g., upsets in inlet concentrations, temperatures). On the other hand, state delays usually occur due to dead-time in pipes used to transfer material among process units including fluidized catalytic cracking units (where the state delays occur due to dead time in pipes transferring material from the regenerator to the reactor and vice-versa) and distillation columns (where the state delays occur due to dead time in reboiler and condenser recycle loops). The linear appearance of the manipulated input u in the system of eqn. (1) is also typical in most practical applications, where inlet flow rates, inlet temperatures and concentrations are typically chosen as manipulated inputs.

In the remainder of this section, we review several stability concepts and theorems for DDE systems which will be used in our development (the reader may refer to (Hale and Verduyn Lunel, 1993) for more details).

We begin with some notation. Given a function $\theta : [-\alpha, \infty) \to \mathbb{R}^m$ and $t \in [0, \infty)$, $\theta_t(\xi)$ represents a function from $[-\alpha, 0]$ to $\mathbb{R}^m$ defined by $\theta_t(\xi) := \theta(t+\xi)$. We also define $|\theta_t(\xi)| := \max_{-\alpha \le \xi \le 0} \|\theta_t(\xi)\|_{\mathbb{R}^m}$, $\|\theta_t\| := \sup_{s \ge 0} |\theta_s(\xi)|$ and $\|\theta_t\|_T := \sup_{0 \le s \le T} |\theta_s(\xi)|$. Definition 1 that follows provides a rigorous statement of the concept of input-to-state stability for (1).

Definition 1. (Teel, 1998) Let β be a function of class $\mathcal{KL}$, γ be a function of class $\mathcal{Q}$ and δ_x, δ_θ be positive real numbers. The zero solution to (1) with $u(t) = 0$ is said to be ***input-to-state stable*** if $|x_0(\xi)| \le \delta_x$ and $\|\theta_t\| \le \delta_\theta$ imply that the solution to (1) is defined for all times and satisfies:

$$|x_t(\xi)| \le \beta(|x_0(\xi)|, t) + \gamma(\|\theta_t\|), \quad \forall t \ge 0. \tag{2}$$

The above definition, when $\theta(t) \equiv 0$, $\forall t \ge 0$ reduces to the definition of asymptotic stability for the zero solution to the DDE system (1). Furthermore, when $\alpha = 0$, Definition 1 reduces to the standard definition of input-to-state stability for nonlinear ODE systems with external inputs (Khalil, 1996). Finally, we note that from the definition of $|x_t(\xi)|$ and (2), it follows that $\|x(t)\|_{\mathbb{R}^n} \le |x_t(\xi)| \le \beta(|x_0(\xi)|, t) + \gamma(\|\theta_t\|)$, $\forall t \ge 0$.

The following theorem provides sufficient conditions for uniform ultimate boundedness of the system (1), expressed in terms of a suitable functional. The result of this theorem will be directly used in the solution to the robust control problem in Section 4 below.

Theorem 1. (Hale and Verduyn Lunel, 1993) *Consider (1) with $u(t) \equiv 0$ and let γ_1 be a function of class $\mathcal{K}_\infty$ and γ_2, γ_3, γ_4 be functions of class $\mathcal{Q}$. If there is a continuous functional $V : \mathcal{C} \to \mathbb{R}_{\geq 0}$ such that*

$$\gamma_1\big(\|x(t)\|_{\mathbb{R}^n}\big) \leq V\big(x_t(\xi)\big) \leq \gamma_2\big(|x_t(\xi)|\big),$$

$$\dot{V}\big(x_t(\xi)\big) \leq -\gamma_3\big(\|x(t)\|_{\mathbb{R}^n}\big),$$

$$\|x(t)\|_{\mathbb{R}^n} \geq \gamma_4\big(\|\theta_t\|\big), \tag{3}$$

then the solutions to (1) satisfy (2).

Remark 1. Even though, at this stage, there is no systematic way for selecting the form of the functional $V(x_t(\xi))$ which is suitable for a particular application, a choice for $V(x_t(\xi))$, which is frequently used to show stability of a DDE system of the form (1) via Theorem 1, is

$$V\big(x_t(\xi)\big) = x^T(t)Cx(t) + a^2\int_{t-\alpha}^{t} x^T(s)Ex(s)\,\mathrm{d}s, \tag{4}$$

where E, C are symmetric positive definite matrices and a is a positive real number. Clearly, V satisfies $K_1\|x(t)\|^2_{\mathbb{R}^n} \leq V(x_t(\xi)) \leq K_2|x_t(\xi)|^2$ for some positive K_1, K_2.

4. Robust Nonlinear Controller Synthesis

We consider systems of the form (1) and address the problem of synthesizing nonlinear static state feedback control laws of the general form:

$$u(t) = \mathcal{R}\big(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha)\big), \tag{5}$$

where $\mathcal{R}(x(t), \bar{v}(t), x(t-\alpha), \bar{v}(t-\alpha))$ is a nonlinear scalar function, $\bar{v}(s) = [v(s)\ v^{(1)}(s)\ \cdots\ v^{(r-1)}(s)]^T$, $s \in [t-\alpha, t]$ and $v^{(k)}$ denotes the k-th time-derivative of the reference input $v \in \mathbb{R}$. The controllers enforce the following properties in the closed-loop system: (a) boundedness of the trajectories, (b) robust output tracking for changes in the reference input with arbitrary degree of asymptotic attenuation of the effect of the uncertainty on the output, and (c) compensate for the effect of the time delay on the output, *independently of the size of the state delays*. The structure of the control law (5) is motivated by available results on stabilization of linear DDE systems (Nazaroff, 1973; Ross and Flugge-Lotz, 1969) and the requirement of output tracking with attenuation of the effect of the uncertainty on the output. The controllers will be synthesized by employing a novel combination of geometric control concepts with the method of Lyapunov functionals.

In order to proceed with the explicit synthesis of the control law (5), we will need to make certain assumptions on the structure and stability properties of (1). To simplify the statement of these assumptions, we introduce the notation:

$$f\big(x(t), x(t-\alpha), \theta(t-\alpha_\theta)\big) = f_{nom}\big(x(t), x(t-\alpha)\big) + \delta\big(x(t), x(t-\alpha), \theta(t-\alpha_\theta)\big),$$

$$f_{nom}\big(x(t), x(t-\alpha)\big) = \tilde{f}_{nom}\big(x(t)\big) + p\big(x(t), x(t-\alpha)\big),$$

$$\tilde{p}\big(x(t), x(t-\alpha), \theta(t-\alpha_\theta)\big) = \delta\big(x(t), x(t-\alpha), \theta(t-\alpha_\theta)\big) + p\big(x(t), x(t-\alpha)\big), \tag{6}$$

which allows us to rewrite (1) in the following form:

$$\dot{x} = \tilde{f}_{nom}(x(t)) + g(x(t), x(t-\alpha))u + \tilde{p}(x(t), x(t-\alpha), \theta(t-\alpha_\theta)),$$
$$y = h(x). \tag{7}$$

The first assumption is motivated by the requirement of output tracking with the attenuation of the effect of the uncertainty on the output and will play a crucial role in the synthesis of the controller.

Assumption 1. Referring to (7), there exists an integer r and a change of variables:

$$\begin{bmatrix} \zeta(s) \\ \eta(s) \end{bmatrix} = \begin{bmatrix} \zeta_1(s) \\ \zeta_2(s) \\ \vdots \\ \zeta_r(s) \\ \eta_1(s) \\ \vdots \\ \eta_{n-r}(s) \end{bmatrix} = \mathcal{X}(x(s)) = \begin{bmatrix} h(x(s)) \\ L_{\tilde{f}_{nom}} h(x(s)) \\ \vdots \\ L_{\tilde{f}_{nom}}^{r-1} h(x(s)) \\ \chi_1(x(s)) \\ \vdots \\ \chi_{n-r}(x(s)) \end{bmatrix}, \tag{8}$$

where $s \in [t-\alpha, t]$, $t \in [0, \infty)$ and $\chi_1(x(s)), \ldots, \chi_{n-r}(x(s))$ are nonlinear scalar functions such that (7) takes the form:

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2(t) + \tilde{p}_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)), \\
&\vdots \\
\dot{\zeta}_{r-1} &= \zeta_r(t) + \tilde{p}_{r-1}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)), \\
\dot{\zeta}_r &= L_{\tilde{f}_{nom}}^{r} h\left(\mathcal{X}^{-1}(\zeta(t), \eta(t))\right) + L_g L_{\tilde{f}_{nom}}^{r-1} h\left(\mathcal{X}^{-1}(\zeta(t), \eta(t)) u(t)\right) \\
&\quad + p_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\
&\quad + \delta_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)), \\
\dot{\eta}_1 &= \Psi_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)), \\
&\vdots \\
\dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)), \\
y &= \zeta_1(t),
\end{aligned} \tag{9}$$

where $p_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) + \delta_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)) = \tilde{p}_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta))$ and $L_g L_{\tilde{f}_{nom}}^{r-1} h(x) \neq 0$ for all $x(s) \in \mathbb{R}^n$, $s \in [t-\alpha, t]$. Moreover, for each $\theta(s) \in \mathbb{R}^q$, $s \in [t-\alpha_\theta, t]$, the states $\zeta(s)$ and $\eta(s)$, $s \in [t-\alpha, t]$, are bounded if and only if the state $x(s) \in \mathbb{R}^n$, $s \in [t-\alpha, t]$ is bounded.

Assumption 1 provides the explicit form of a coordinate change (which is independent of the state delay present in (1)) that transforms the nonlinear DDE system (1) into an interconnection of two subsystems, the ζ-subsystem which describes the input/output dynamics of (1) and the η-subsystem which describes the dynamics of (1) which are unobservable from the output. Specifically, the interconnection of (9) is obtained by considering the change of variables (8) with $s = t$, differentiating it with respect to time and using $x(t) = \mathcal{X}^{-1}(\zeta(t), \eta(t))$, $x(t-\alpha) = \mathcal{X}^{-1}(\zeta(t-\alpha), \eta(t-\alpha))$ (note that this is possible because the change of variables (8) is assumed to be valid for $s \in [t-\alpha, t]$). The assumption that $L_g L_{\tilde{f}_{nom}}^{r-1} h(x) \neq 0$ is necessary in order to guarantee that the resulting controller is well-posed in the sense that the controller does not generate an infinite control action for any values of the states of the process (compare with the structure of the controller given in Theorem 2).

The coordinate transformation (8) is not restrictive from an application point of view (one can easily verify that Assumption 1 holds for the fluidized catalytic cracking unit of Section 5), and it significantly facilitates the controller synthesis task which will be now addressed on the basis of the low-order partially-linear ζ-subsystem.

To proceed with the controller design, we need to impose the following stability requirement on the η-subsystem of the system of (9).

Assumption 2. The dynamical system:

$$\begin{aligned}\dot{\eta}_1 &= \Psi_1\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\end{aligned} \tag{10}$$

is input-to-state stable with respect to the inputs $\zeta_t(\xi)$ and $\theta_t(\xi)$.

The above assumption states that if the states of the ζ-subsystem, $\zeta_t(\xi)$, and the uncertainty, $\theta_t(\xi)$, are bounded, then the states of the η-subsystem will also remain bounded. In practice, Assumption 2 can be verified by linearizing (10) with $\zeta_t(\xi) = \theta_t(\xi) = 0$ around the operating steady state and computing the eigenvalues of the resulting linear DDE system (this can be done by using standard algorithms (for example Manitius and Tran, 1985; 1987)). If all of these eigenvalues are in the left-half of the complex plane, then (Teel, 1998) Assumption 2 is satisifed locally (i.e., for sufficiently small initial conditions, $||\zeta_t(\xi)||$, and $||\theta_t(\xi)||$).

Assumption 2 allows for addressing the controller synthesis problem on the basis of the ζ-subsystem. Specifically, applying the following preliminary feedback law:

$$\begin{aligned}u(t) = \frac{1}{L_g L_{\tilde{f}_{nom}}^{r-1} h(\mathcal{X}^{-1}(\zeta, \eta))}\Big(&\tilde{u}(t) - L_{\tilde{f}_{nom}}^{r} h(\mathcal{X}^{-1}(\zeta, \eta))\\ &- p_r\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)\big)\Big),\end{aligned} \tag{11}$$

where $\tilde{u}(t)$ is an auxiliary input, to (9) in order to cancel all the nonlinear terms that are known and can be cancelled by using a feedback which utilizes measurements of

$x(s)$ for $s \in [t-\alpha, t]$, we obtain the following modified system:

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2(t) + \tilde{p}_1\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\\
&\vdots\\
\dot{\zeta}_{r-1} &= \zeta_r(t) + \tilde{p}_{r-1}\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\\
\dot{\zeta}_r &= \tilde{u} + \delta_r\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\\
\dot{\eta}_1 &= \Psi_1\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\\
&\vdots\\
\dot{\eta}_{n-r} &= \Psi_{n-r}\big(\zeta(t), \eta(t), x(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\\
y &= \zeta_1(t),
\end{aligned} \tag{12}$$

where $\zeta_k(t) = L^{k-1}_{\tilde{f}_{nom}} h(x)$, $k = 1, \ldots, r$.

Introduce the notation:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

$$\begin{aligned}
&\bar{p}\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big)\\
&\quad = \begin{bmatrix} \tilde{p}_1\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big) \\ \tilde{p}_2\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big) \\ \vdots \\ \tilde{p}_{r-1}\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big) \\ 0 \end{bmatrix}.
\end{aligned} \tag{13}$$

Let $\bar{e}(s) = [(h(x(s)) - v(s)) \;\; (L_{\tilde{f}_{nom}} h(x(s)) - v^{(1)}(s)) \;\; \ldots \;\; (L^{r-1}_{\tilde{f}_{nom}} h(x(s)) - v^{(r-1)}(s))]^T$ and $\bar{v}(s) = [v(s) \;\; v^{(1)}(s) \;\; \ldots \;\; v^{(r-1)}(s)]^T$, $s \in [t-\alpha, t]$, where $v^{(k)}$ denotes the k-th time-derivative of the reference input v. The ζ-subsystem of (12) can be written in the following compact form:

$$\begin{aligned}
\dot{\bar{e}} &= \tilde{A}\bar{e}(t) + b\tilde{u}(t) + \bar{p}\big(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big)\\
&\quad + b\delta_r\big(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big),\\
y &= \bar{e}_1(t).
\end{aligned} \tag{14}$$

The controller synthesis task has been now reduced to that of synthesizing the auxiliary input $\tilde{u}(t)$ to achieve the asymptotic attenuation of the effect of the uncertainties of the $\bar{e}$-subsystem in (14), on the output. Specifically, the synthesis of $\tilde{u}(t)$ will be performed by using the method of Lyapunov functionals, so that the time derivative of the Lyapunov functional:

$$V = \bar{e}^T(t)P\bar{e}(t) + a^2 \int_{t-\alpha}^{t} \bar{e}^T(s)\bar{e}(s)\,\mathrm{d}s \tag{15}$$

where a is a positive real number, calculated along the state of the closed-loop $\bar{e}$-subsystem is negative definite. The incorporation of the integral term $\int_{t-\alpha}^{t} \bar{e}^T(s)\bar{e}(s)\,\mathrm{d}s$ in the functional (15) allows accounting for the distributed parameter (delayed) nature of (1) in the controller design stage and synthesizing a controller that enforces the requested properties in the closed-loop system independently of the size of the state delay.

To explicitly synthesize $\tilde{u}(t)$, we need to assume the existence of known state-dependent, possibly time-varying, upper bound that capture the size of the uncertainty for all times (such bounds are usually obtained from physical considerations, preliminary simulations, experimental data, etc.). Assumptions 3 and 4 formalize our requirements.

Assumption 3. There exists a known function $\tilde{c}_0(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), t)$ such that the following condition holds:

$$\begin{aligned} &\big|\delta_r\big(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big)\big|_{\mathbb{R}} \\ &\qquad \leq \tilde{c}_0\big(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), t\big) \end{aligned} \tag{16}$$

where $|\cdot|_{\mathbb{R}}$ denotes the absolute value of a scalar, for all $\bar{e}(s) \in \mathbb{R}^r$, $s \in [t-\alpha, t]$, $\theta(s) \in \mathbb{R}^q$, $s \in [t-\alpha_\theta, t]$, $\bar{v}(s) \in \mathbb{R}^r$, $s \in [t-\alpha, t]$, $\eta(s) \in \mathbb{R}^{n-r}$, $s \in [t-\alpha, t]$.

Assumption 4. There exist positive real numbers a_1, a_2 such that the following bound can be written:

$$\begin{aligned} &\big\|\bar{p}\big(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)\big)\big\|_{\mathbb{R}^r}^2 \\ &\qquad \leq a_1\bar{e}^2(t) + a_2\bar{e}^2(t-\alpha) \end{aligned} \tag{17}$$

for all $\bar{e}(s) \in \mathbb{R}^r$, $s \in [t-\alpha, t]$, $\theta(s) \in \mathbb{R}^q$, $s \in [t-\alpha_\theta, t]$, $\bar{v}(s) \in \mathbb{R}^r$, $s \in [t-\alpha, t]$, $\eta(s) \in \mathbb{R}^{n-r}$, $s \in [t-\alpha, t]$.

The above assumption on the growth of the vector $\bar{p}(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta))$ does not need to hold globally (i.e., for any $\bar{e}(t), \eta(t)$), and thus, it is satisfied by most practical problems (see e.g. the fluidized catalytic cracking reactor of Section 5).

We are now in a position to state the main controller synthesis result of this paper in the form of a theorem (the proof of this theorem can be found in the Appendix).

Theorem 2. *Consider the nonlinear DDE system (1) for which Assumptions 1–4 hold. Suppose that the matrix equation:*

$$\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + (a^2 + a_1) I_{n\times n} + P^2 = -R_1 \tag{18}$$

where $a^2 > a_2$, *and* R_1 *is a positive definite matrix, has a unique positive definite solution for* P, *and consider the closed-loop system under the robust nonlinear state feedback controller:*

$$\begin{aligned} u &= \mathcal{R}\big(x(t), x(t-\alpha), t\big) \\ &= \frac{1}{L_g L_{\bar{f}}^{r-1} h\big(x(t)\big)} \Big(-R_2^{-1} b^T P\bar{e}(t) + v^r(t) - \bar{\chi}\tilde{c}_0\big(x(t), x(t-\alpha), t\big) w\big(\bar{e}(t), \phi\big) \\ &\quad - L_{\bar{f}_{nom}}^r h\big(x(t)\big) - p_r\big(x(t), x(t-\alpha)\big)\Big) \end{aligned} \tag{19}$$

where $w(\bar{e}(t), \phi)$ *is a scalar function given by:*

$$w\big(\bar{e}(t), \phi\big) = \frac{b^T P\bar{e}(t)}{|b^T P\bar{e}(t)|_{\mathbb{R}} + \phi} \tag{20}$$

and $\bar{\chi}, \phi$ *are adjustable parameters with* $\bar{\chi} > 1$ *and* $\phi > 0$. *Then there exist positive real numbers* δ, d, ϕ^* *such that if* $\max\{|x_0(\xi)|, \|\bar{v}_t\|, \|\theta_t\|\} \leq \delta$ *and* $\phi \in (0, \phi^*]$, *the output of the closed-loop system satisfies a relation of the form:*

$$\lim_{t\to\infty} \sup \big|y(t) - v(t)\big|_{\mathbb{R}} \leq d \tag{21}$$

independently of the size of the state delays.

Remark 2. Regarding the structure, implementation and closed-loop properties of the nonlinear state feedback controller of (19), several remarks are in order: (a) it uses measurements of the states of the process evaluated at t and $t-\alpha$ (i.e., $x(t)$ and $x(t-\alpha)$), and thus, it belongs to the class of the requested control laws (5), (b) its practical implementation requires the use of memory lines to store the values of x in the time interval $[t-\alpha, t]$, and (c) it enforces stability and asymptotic output tracking in the closed-loop system *independently of the size of the state delay.*

Remark 3. In order to apply the result of Theorem 2 to a chemical process application, one has to initially verify Assumptions 1–3 of the theorem on the basis of the process model and compute the parameters a_1 and a_2 and the function $\tilde{c}_0(x(t), x(t-\alpha), t)$. Then, a, R_1, R_2 should be chosen so that $a^2 > a_2$ and the matrices R_1, R_2 are positive definite to ensure that the matrix equation (18) has a unique positive definite solution for P. Note that R_1 determines the speed of the closed-loop output response (namely, 'larger' (in terms of the smallest eigenvalue) R_1 means a faster response), while R_2 determines the penalty that should be imposed on the manipulated input in achieving stabilization and output tracking ('larger' R_2 means a larger penalty on the control action). If these assumptions are satisfied, the synthesis formula (19) can be directly used to derive the explicit form of the controller (see Section 5 for an application of this procedure to a chemical process example).

Remark 4. The robust nonlinear controller (19) possesses a robustness property with respect to fast and exponentially stable unmodeled dynamics (i.e., the controller enforces boundedness of the state and output tracking with uncertainty attenuation in the closed-loop system despite the presence of additional dynamics in the process, as long as they are exponentially stable and sufficiently fast). This property of the controller (19) can be rigorously established by analyzing the closed-loop system with unmodeled dynamics using singular perturbations and is of particular importance for many practical applications where unmodeled dynamics often occur due to actuator and sensor dynamics, fast process dynamics, etc.

5. Application to a Fluidized Catalytic Cracker

In this section, we illustrate the implementation of the developed control methodology on the fluidized catalytic cracker (FCC) shown in Fig. 1. The FCC unit consists of a cracking reactor, where the cracking of high boiling gas oil fractions into lighter hydrocarbons (e.g., gasoline) and the carbon formation reactions (undesired reactions) take place and a regenerator, where the carbon removal reactions take place. The reader may refer to (Arbel *et al.*, 1995; Denn, 1986; McFarlane *et al.*, 1993) for a detailed discussion of the features of the FCC unit, (Arbel *et al.*, 1996) for an analysis of the issue of control structure selection and (Antoniades and Christofides, 1999; Christofides and Daoutidis, 1997; Huq *et al.*, 1995; Monge and Georgakis, 1987) for application of linear and nonlinear control methods to the FCC unit, respectivly.

Under the assumptions of well-mixed reactive catalyst in the reactor, small-size catalyst particles, constant solid holdup in reactor and regenerator, uniform and constant pressure in reactor and regenerator, a dynamic model for the FCC unit can be derived which takes the form (Antoniades and Christofides, 1999):

$$\begin{aligned}
V_{ra}\frac{\mathrm{d}C_{cat}}{\mathrm{d}t} &= -60F_{rc}C_{cat}(t) + 50R_{cf}\big(C_{cat}(t), C_{rc}(t-\alpha_1), T_{ra}\big),\\
V_{ra}\frac{\mathrm{d}C_{sc}}{\mathrm{d}t} &= 60F_{rc}\big[C_{rc}(t-\alpha_1) - C_{sc}(t)\big] + 50R_{cf}\big(C_{cat}(t), C_{rc}(t-\alpha_1), T_{ra}\big),\\
V_{ra}\frac{\mathrm{d}T_{ra}}{\mathrm{d}t} &= 60F_{rc}\big[T_{rg}(t-\alpha_1) - T_{ra}(t)\big] + 0.875\frac{S_f}{S_c}D_{tf}R_{tf}\big[T_{fp} - T_{ra}(t)\big]\\
&\quad +0.875\frac{(-\Delta H_{fv})}{S_c}D_{tf}R_{tf} + 0.5\frac{(-\Delta H_{cr})}{S_c}R_{oc}\big(C_{cat}(t), C_{rc}(t-\alpha_1), T_{ra}\big),\\
V_{rg}\frac{\mathrm{d}C_{rc}}{\mathrm{d}t} &= 60F_{rc}\big[C_{sc}(t-\alpha_2) - C_{rc}(t)\big] - 50R_{cb}\big(C_{rc}(t), T_{rg}(t)\big),\\
V_{rg}\frac{\mathrm{d}T_{rg}}{\mathrm{d}t} &= 60F_{rc}\big[T_{ra}(t-\alpha_2) - T_{rg}(t)\big] + 0.5\frac{S_a}{S_c}R_{ai}\big[T_{ai} - T_{rg}(t)\big]\\
&\quad - 0.5\frac{(-\Delta H_{rg})}{S_c}R_{cb}\big(C_{rc}(t), T_{rg}(t)\big),
\end{aligned} \tag{22}$$

where C_{cat}, C_{sc}, C_{rc} denote respectively the concentrations of catalytic carbon on spent catalyst, the total carbon on spent catalyst, and carbon on regenerated catalyst, T_{ra}, T_{rg} denote the temperatures in the reactor and the regenerator, R_{ai} is the air flow rate in the regenerator, R_{tf} is the total feed flow rate, D_{tf} is the density of total feed, V_{ra}, V_{rg} denote the catalyst holdup of the reactor and the regenerator, ΔH_{rg}, ΔH_{cr} are the heat of regeneration and cracking, ΔH_{fv} is the heat of feed vaporization, F_{rc} denotes the circulation flow rate of catalyst from reactor to regenerator and vice-versa, S_a, S_c, S_f denote specific heats of the air, the catalyst, and the feed, T_{fp}, T_{ai} denote the inlet temperatures of the feed in the reactor and of the air in the regenerator, and R_{cf}, R_{oc}, R_{cb} denote the reaction rates of total carbon forming, of gas-oil cracking, and of coke burning. Analytic expressions for the reaction rates R_{cf}, R_{oc}, R_{cb} can be found in (Denn, 1986).

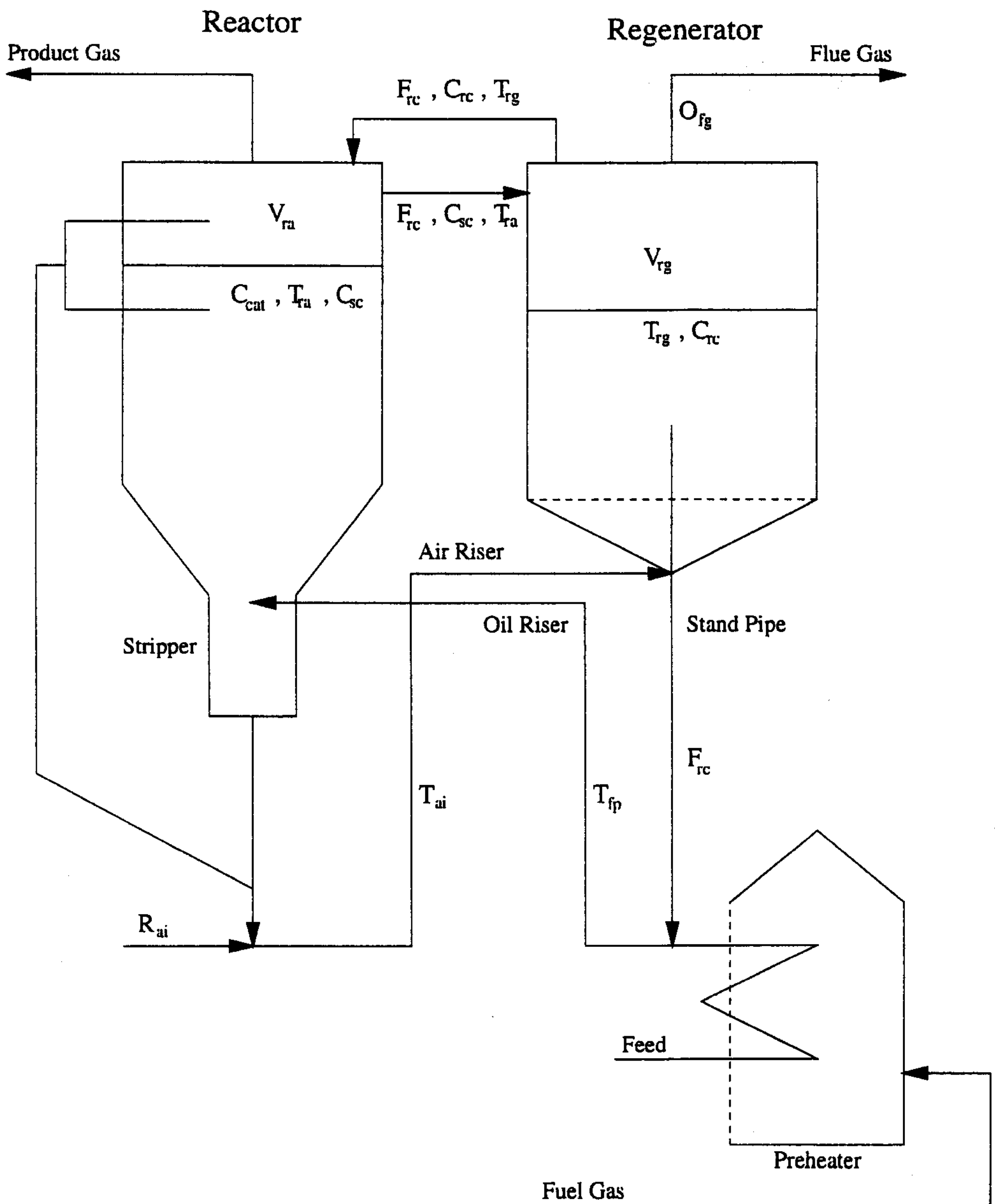

Fig. 1. Fluidized catalytic cracker.

The presence of the time delay in the terms $C_{rc}(t-\alpha_1)$ and $T_{rg}(t-\alpha_1)$ is due to dead-time in the pipes transferring regenerated catalyst from the regenerator to the reactor, and the time delay in the terms $C_{sc}(t-\alpha_2)$, $T_{ra}(t-\alpha_2)$ is due to dead time in the pipes transferring spent catalyst from the reactor to the regenerator. Even though the proposed method can be readily applied to the case where $\alpha_1 \neq \alpha_2$, we pick in order to simplify our development, $\alpha_1 = \alpha_2 = \alpha = 0.3\,\text{hr}$. The values of the remaining process parameters and the corresponding steady-state values are given in Table 1.

The control objective is the regulation of the temperature in the regenerator, T_{rg}, by manipulating the temperature of the inlet air in the regenerator, T_{ai}. The heat of combustion in the regeneration is assumed to be the uncertain variable, $\Delta H_{rg} = \Delta H_{rg,n} + \theta(t)$, where $\Delta H_{rg,n}$ is the nominal value of the heat of combustion and $\theta(t)$ is the time-varying uncertainty. Setting $x = [x_1\ x_2\ x_3\ x_4\ x_5]^T = [T_{rg}\ C_{rc}\ T_{ra}\ C_{sc}\ C_{cat}]$, $u = T_{ai}$, $y = T_{rg}$, the process model (22) can be written in the form (7) with:

$$\tilde{f}_{nom}(x(t)) = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \\ \tilde{f}_5 \end{bmatrix} = \begin{bmatrix} \frac{0.5 S_a R_{ai}}{S_c V_{rg}} T_{ais} - \left(\frac{60F_{rc}}{V_{rg}} + 0.5\frac{S_a R_{ai}}{S_c V_{rg}}\right) T_{rg}(t) \\ \quad -0.5\frac{(-\Delta H_{rg,n})}{S_c V_{rg}} R_{cb}(C_{rc}(t), T_{rg}(t)) \\ -\frac{60F_{rc}}{V_{rg}} C_{rc}(t) - \frac{50}{V_{rg}} R_{cb}(C_{rc}(t), T_{rg}(t)) \\ -\frac{60F_{rc}}{V_{ra}} T_{ra}(t) + 0.875\frac{S_f}{S_c V_{ra}} D_{tf} R_{tf}\left[T_{fp} - T_{ra}(t)\right] \\ \quad +0.875\frac{(-\Delta H_{fv})}{S_c V_{ra}} D_{tf} R_{tf} \\ -\frac{60F_{rc}}{V_{ra}} C_{sc}(t) \\ -\frac{60F_{rc}}{V_{ra}} C_{cat}(t) \end{bmatrix}, \tag{23}$$

$$\tilde{p}(x(t), x(t-\alpha), \theta(t)) = \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \\ \tilde{p}_5 \end{bmatrix} = \begin{bmatrix} \frac{60F_{rc}}{V_{rg}} T_{ra}(t-\alpha) \\ \quad -0.5\frac{(-\theta(t))}{S_c V_{rg}} R_{cb}(C_{rc}(t), T_{rg}(t)) \\ \frac{60F_{rc}}{V_{rg}} C_{sc}(t-\alpha) \\ \frac{60F_{rc}}{V_{ra}} T_{rg}(t-\alpha) + 0.5\frac{(-\Delta H_{cr})}{S_c V_{ra}} \\ \quad \times R_{oc}(C_{cat}(t), C_{rc}(t-\alpha), T_{ra}(t)) \\ \frac{60F_{rc}}{V_{ra}} C_{rc}(t-\alpha) + \frac{50}{V_{ra}} \\ \quad \times R_{cf}(C_{cat}(t), C_{rc}(t-\alpha), T_{ra}(t)) \\ \frac{50}{V_{ra}} R_{cf}(C_{cat}(t), C_{rc}(t-\alpha), T_{ra}(t)) \end{bmatrix}, \tag{24}$$

$$g\big(x(t), x(t-\alpha)\big) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix} = \begin{bmatrix} \dfrac{0.5 S_a R_{ai}}{S_c V_{rg}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

For the system (22), Assumption 1 is satisfied with $r = 1$, the coordinate transformation (8) takes the form $[\zeta_1 \ \eta_1 \ \eta_2 \ \eta_3 \ \eta_4]^T = [T_{rg} \ C_{rc} \ T_{ra} \ C_{sc} \ C_{cat}]$ and yields the following system:

$$\begin{aligned} \dot{\zeta} &= L_{\tilde{f}_{nom}} h\big(\mathcal{X}^{-1}(\zeta, \eta)\big) + L_g h\big(\mathcal{X}^{-1}(\zeta, \eta)\big) u(t) \\ &\quad + \tilde{p}_1\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t)\big), \\ \dot{\eta} &= \Psi\big(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)\big), \end{aligned} \tag{26}$$

whose explicit form is given below in (27) and (29). To verify Assumption 2, we consider the η-subsystem of (26) with $\zeta(t) = \zeta(t-\alpha) = \zeta_s = 1155.96°\text{F}$, i.e., the system:

$$\dot{\eta} = \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\eta}_4 \end{bmatrix} = \begin{bmatrix} -\dfrac{60F_{rc}}{V_{rg}}\eta_1 - \dfrac{50}{V_{rg}} R_{cb}(\eta_1, \zeta_s) + \dfrac{60F_{rc}}{V_{rg}}\eta_3(t-\alpha) \\ -\dfrac{60F_{rc}}{V_{ra}}\eta_2 + 0.875\dfrac{S_f}{S_c V_{ra}} D_{tf} R_{tf}[T_{fp} - \eta_2] + 0.875\dfrac{(-\Delta H_{fv})}{S_c V_{ra}} \\ \times D_{tf} R_{tf} + 0.5\dfrac{(-\Delta H_{cr})}{S_c V_{ra}} R_{oc}\big(\eta_4, \eta_1(t-\alpha), \eta_2\big) \\ -\dfrac{60F_{rc}}{V_{ra}}\eta_3 + \dfrac{60F_{rc}}{V_{ra}}\eta_1(t-\alpha) + \dfrac{50}{V_{ra}} R_{cf}\big(\eta_4, \eta_1(t-\alpha), \eta_2\big) \\ -\dfrac{60F_{rc}}{V_{ra}}\eta_4 + \dfrac{50}{V_{ra}} R_{cf}\big(\eta_4, \eta_1(t-\alpha), \eta_2\big) \end{bmatrix}. \tag{27}$$

Its linearization around the steadystate

$$\begin{aligned} C_{rc} &= 0.6973\,\text{wt\%}, \quad T_{ra} = 930.62°\text{F}, \\ C_{sc} &= 1.5696\,\text{wt\%}, \quad C_{cat} = 0.8723\,\text{wt\%}, \end{aligned} \tag{28}$$

was found, through simulations, to be exponentially stable, which implies that the η-subsystem of (26) possesses a local input-to-state stability property with respect to $\zeta_t(\xi)$. Therefore, Assumption 2 holds and the controller synthesis problem can be addressed on the basis of the ζ-subsystem which is given as:

$$\begin{aligned}\dot{\zeta} = &-\left(\frac{60F_{rc}}{V_{rg}} + \frac{0.5S_aR_{ai}}{S_cV_{rg}}\right)\zeta(t) \\ &- 0.5\frac{(-\Delta H_{rg,n})}{S_cV_{rg}}R_{cb}(\eta_1(t),\zeta(t)) + \left(\frac{0.5S_aR_{ai}}{S_cV_{rg}}\right)u(t) \\ &+ \left(\frac{60F_{rc}}{V_{rg}}\right)\eta_2(t-\alpha_2) + 0.5\frac{\theta(t)}{S_cV_{rg}}R_{cb}(\eta_1(t),\zeta(t)).\end{aligned} \tag{29}$$

Setting $e(t) = \zeta(t) - v$, where v is the desired set point, and implementing a preliminary control law of the form (11), the above system becomes:

$$\begin{aligned}\dot{e} = &-R_2^{-1}b^TPe(t) - \bar{\chi}\bar{c}_0(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), t) \\ &\times w(e(t),\phi) + 0.5\frac{\theta(t)}{S_cV_{rg}}R_{cb}(\eta_1(t), e(t) + \bar{v}(t)).\end{aligned} \tag{30}$$

Assumption 4 is trivially satisfied since $\bar{p}(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha)) = 0$ and the functions $\bar{c}_0(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), t)$ and $w(e(t),\phi)$ take the form:

$$\begin{aligned}&\bar{c}_0(\bar{e}(t) + \bar{v}(t), \eta(t), \bar{e}(t-\alpha) + \bar{v}(t-\alpha), \eta(t-\alpha), t) \\ &\qquad = 0.5\frac{\theta_b}{S_cV_{rg}}R_{cb}(\eta_1(t), e(t) + v),\end{aligned} \tag{31}$$

$$w(e(t),\phi) = \frac{Pe(t)}{|Pe(t)|_{\mathbb{R}} + \phi},$$

where θ_b denotes the upper bound on the size of the uncertain variable and ϕ is an adjustable parameter. Utilizing the result of Theorem 2, the following matrix equation can be formed:

$$\tilde{A}^TP + P\tilde{A} - 2P^TbR_2^{-1}b^TP + a^2I_{n\times n} + P^2 = -R_1, \tag{32}$$

with $R_2 = 1/3$, $a^2 = 2.5$ $(a^2 > a_2 = 0)$, and

$$\tilde{A} = 0, \quad b = 1, \quad R_1 = 2.5. \tag{33}$$

Equation (32) has a unique positive definite solution for P of the form:

$$P = 1, \tag{34}$$

which leads to the following nonlinear state feedback controller:

$$\begin{aligned}u = \frac{1}{L_gh(x(s))}\Bigg[&-3(x_1(t) - v) - \bar{\chi}0.5\frac{\theta_b}{S_cV_{rg}}R_{cb}(x_2(t), x_1(t)) \\ &\times\left(\frac{x_1(t) - v}{|x_1(t) - v|_{\mathbb{R}} + \phi}\right) - L_{\tilde{f}_{nom}}h(x(s)) - p_1(x(t), x(t-\alpha))\Bigg].\end{aligned} \tag{35}$$

We performed several sets of simulation runs to evaluate the performance and the robustness properties of the controller (35) and compare them with the ones of: (a) a nonlinear controller that does not account for the time-varying uncertain variables (i.e., controller (35) with $\tilde{c}_0(x(t), x(t-\alpha), t) = 0$), (b) a PI controller, and (c) a robust nonlinear controller that does not account for the presence of dead-times associated with the pipes transferring material from the regenerator to reactor and vice versa (i.e., controller (35) with $\alpha = 0$). The user-friendly software package SIMULINK was used to carry out all the simulation runs (SIMULINK is a toolbox of the mathematical software MATLAB that includes a delay function which can be readily used to simulate differential equations with time delays). In all the simulation runs, the process was initially ($t = 0.0\,\mathrm{hr}$) assumed to be at the steady-state shown in Table 1 and a time-varying uncertain variable of the form:

$$\theta(t) = 300\cos(0.5t), \tag{36}$$

was considered. The upper bound on the uncertainty was taken to be $\theta_b = 300\,\mathrm{Btu\,lb}^{-1}$. The controller of (35) was implemented with $\phi = 0.05$ and $\bar{\chi} = 1.1$ to guarantee that the output of the closed-loop system satisfies a relation of the form:

$$\lim_{t\to\infty} \sup |y - v|_{\mathbf{R}} \leq 1^{\circ}\mathrm{F}. \tag{37}$$

In the first set of simulation runs, we tested the ability of the controller to maintain the output at the steadystate despite the presence of uncertainty. The output profile and the corresponding manipulated input profile are given in Fig. 2. Clearly, the controller regulates the output at the operating steady state compensating for the effect of the uncertainty and satisfying the requirement of (37). For the sake of comparison, we also implemented the nonlinear controller (35) with $\tilde{c}_0(x(t), x(t-\alpha), t) = 0$ and a PI controller with $K_c = 10$ and $\tau_I = 0.1$. Figure 3 shows the output profiles and the corresponding manipulated inputs of all the three controllers; the performance of the proposed controller is clearly superior to the ones of the other two controllers.

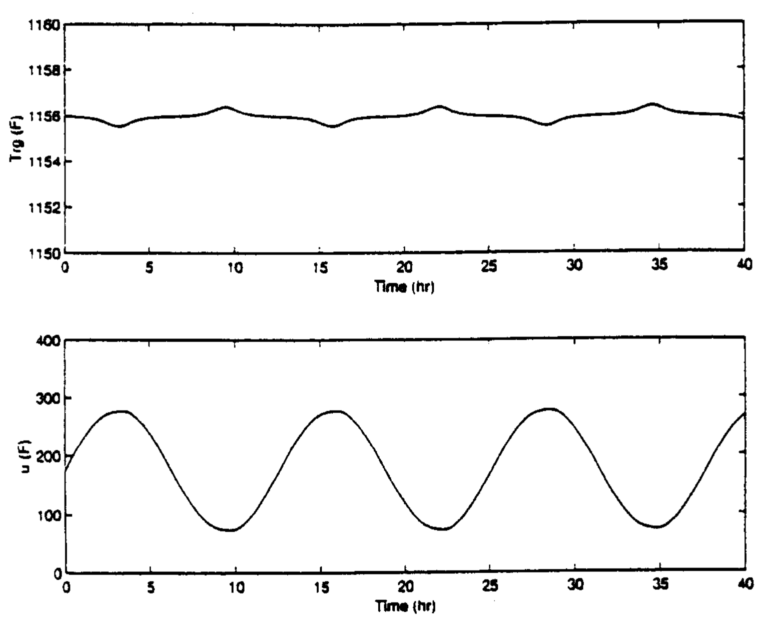

Fig. 2. Closed-loop output and manipulated input profiles under the controller (35).

In the second set of simulation runs, we tested the output tracking capabilities of the nonlinear controller (35) for a 44° F increase in the reference input. Figure 4 shows the output and manipulated input profiles. The controller drives the output to the new reference input value compensating for the effect of the uncertainty and state delays, and satisfying the requirement of (37). We also tested the controller (35) with $\tilde{c}_0(x(t), x(t-\alpha), t) = 0$ and a PI controller with $K_c = 10$ and $\tau_I = 0.1$. Figure 5 displays the output and manipulated input profiles of all the three controllers. Again, the superiority of the proposed controller is evident.

Finally, we also implemented a nonlinear controller which does not account for the presence of the state delays (no uncertainty was considered in this simulation run). Figure 6 shows the closed-loop output profile and manipulated input profile; this controller leads to an unstable closed-loop system because it does not account for the presence of significant state delays.

Table 1. Process parameters and steady-state values.

$E_{cc} = 18000.0$	Btu $lb^{-1}mole^{-1}$
$E_{cr} = 27000.0$	Btu $lb^{-1}mole^{-1}$
$E_{or} = 63000.0$	Btu $lb^{-1}mole^{-1}$
$k_{cc} = 8.59$	Mlb $hr^{-1}psia^{-1}ton^{-1}(wt\%)^{-1.06}$
$k_{cr} = 11600$	Mbbl $day^{-1}psia^{-1}ton^{-1}(wt\%)^{-1.15}$
$k_{or} = 3{,}5 \times 10^{10}$	Mlb $hr^{-1}psia^{-1}ton^{-1}$
$V_{rg} = 200.0$	ton
$V_{ra} = 60.0$	ton
$T_{fps} = 744.0$	F
$T_{ai} = 175.0$	F
$P_{rg} = 25.0$	psia
$P_{ra} = 40.0$	psia
$\Delta H_{fv} = 60.0$	Btu lb^{-1}
$\Delta H_{cr} = 77.3$	Btu lb^{-1}
$\Delta H_{rg} = 10561.0$	Btu lb^{-1}
$S_a = 0.3$	Btu $lb^{-1}F^{-1}$
$S_c = 0.3$	Btu $lb^{-1}F^{-1}$
$S_f = 0.7$	Btu $lb^{-1}F^{-1}$
$F_{rc} = 40.0$	ton hr^{-1}
$R_{tf} = 100.0$	Mbbl/day
$D_{tf} = 7.0$	lb gal^{-1}
$\alpha_1 = 0.3$	hr
$\alpha_2 = 0.3$	hr
$R_{ai} = 400.0$	Mlb min^{-1}
$(C_{cat})_s = 0.8723$	wt%
$(C_{sc})_s = 1.5696$	wt%
$(C_{rc})_s = 0.6973$	wt%
$(T_{ra})_s = 930.62$	F
$(T_{rg})_s = 1155.96$	F

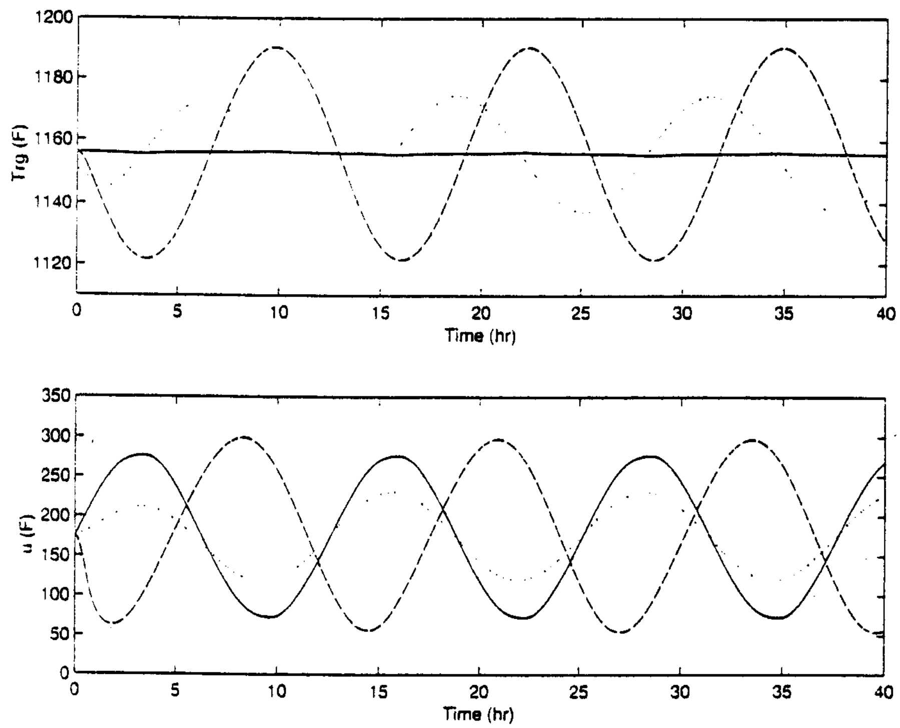

Fig. 3. Closed-loop output and manipulated input profiles under the controller (35) (solid line), the controller (35) without accounting for the uncertainty (dashed line), and a PI controller (dotted line).

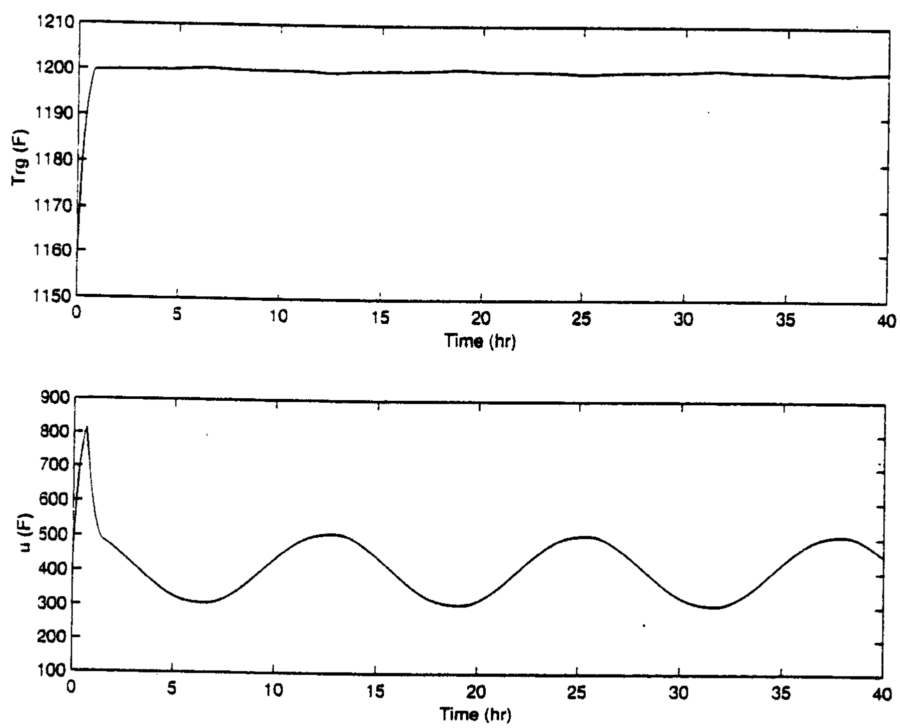

Fig. 4. Closed-loop output and manipulated input profiles under the controller (35) for a 44° F increase in the reference input.

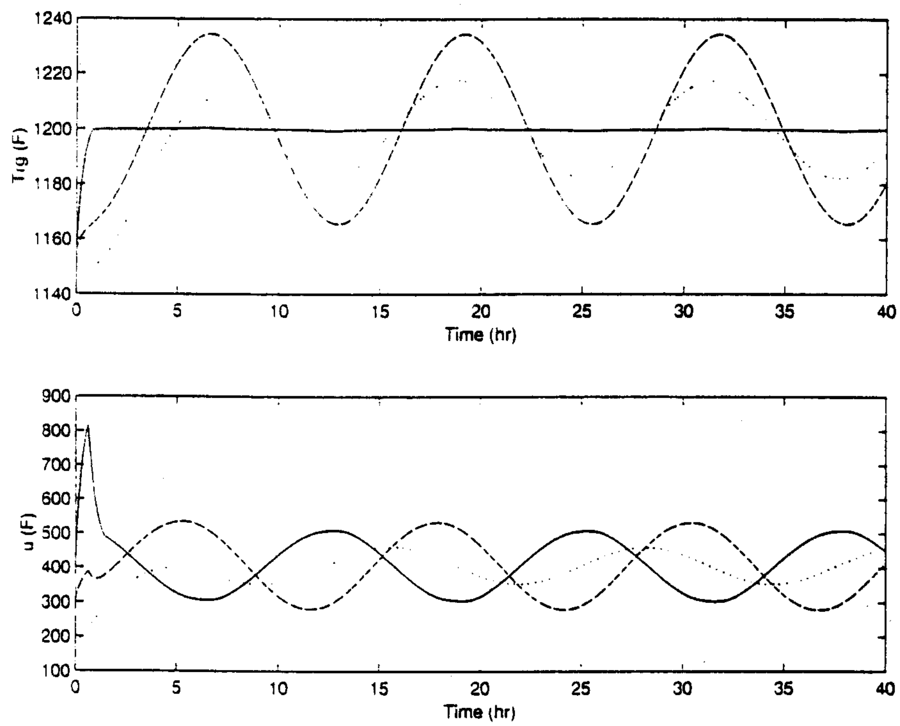

Fig. 5. Closed-loop output and manipulated input profiles under the controller (35) (solid line), the controller (35) without accounting for the uncertainty (dashed line), and a PI controller (dotted line) for a 44° F increase in the reference input.

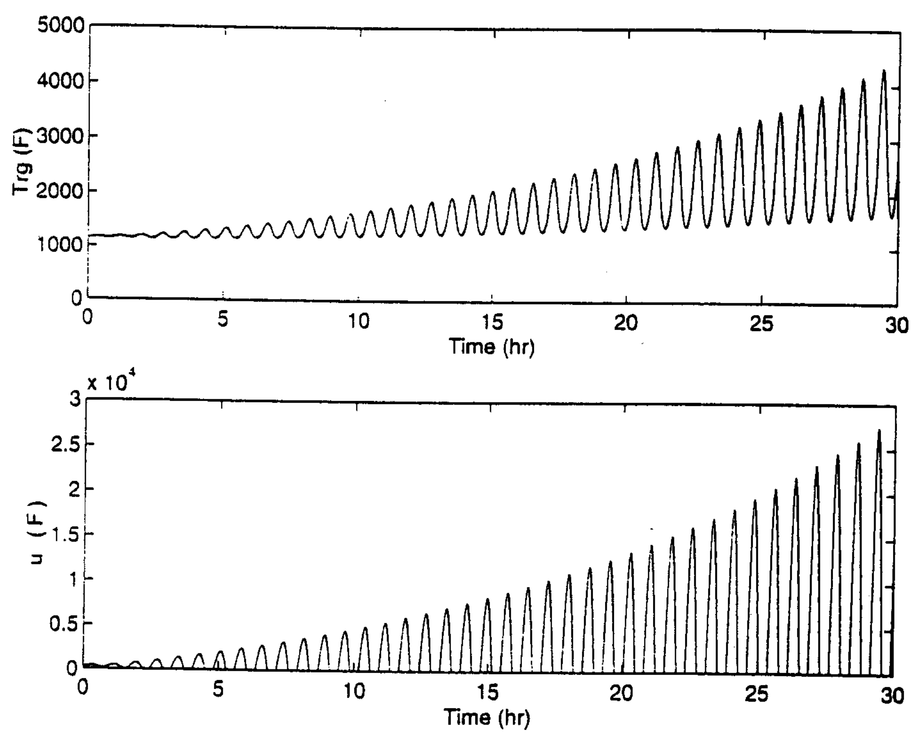

Fig. 6. Closed-loop output and manipulated input profiles under the controller (35) with $\alpha = 0$; no model uncertainty is present.

6. Conclusions

We have considered single-input single-output nonlinear DDE systems with uncertain variables and proposed a general methodology for the synthesis of robust nonlinear state feedback controllers that guarantee boundedness of the states and ensure that the ultimate discrepancy between the output and the external reference input in the closed-loop system can be made arbitrarily small by an appropriate choice of controller parameters. The controllers were synthesized by using a novel combination of geometric and Lyapunov-based techniques. The proposed control method was successfully applied to a fluidized catalytic cracking unit with a time-varying uncertain variable and was shown to outperform a PI controller, a nonlinear controller that does not account for the uncertainty, and a nonlinear controller that does not account for the state delays. We currently work on the extension of these results to systems of partial differential equations with time-delays that arise in the modeling of transport-reaction processes (Christofides and Daoutidis, 1996; Skliar and Ramirez, 1998).

Acknowledgement

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Appendix

Proof of Theorem 2. Consider the DDE system (7) under the robust controller (19):

$$\begin{aligned}\dot{x} &= \tilde{f}_{nom}(x(t)) + g(x(t), x(t-\alpha))\mathcal{R}(x(t), x(t-\alpha), t) \\ &\quad + \tilde{p}(x(t), x(t-\alpha), \theta(t-\alpha_\theta)), \\ y &= h(x).\end{aligned} \tag{A1}$$

Applying the coordinate transformation (8) to the above system and setting, for ease of notation, $x(s) = \mathcal{X}^{-1}(\zeta(s), \eta(s), \theta(s))$, $s \in [t-\alpha, t]$, $t \in [0, \infty)$, we obtain:

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2(t) + \tilde{p}_1(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)), \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r(t) + \tilde{p}_{r-1}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)), \\ \dot{\zeta}_r &= L_{\tilde{f}_{nom}}^r h(x(s)) + L_g L_{\tilde{f}_{nom}}^{r-1} h(x(s))\mathcal{R}(x(t), x(t-\alpha), t) \\ &\quad + \tilde{p}_r(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha), \theta(t-\alpha_\theta)),\end{aligned}$$

$$\begin{aligned}\dot{\eta}_1 &= \Psi_1\big(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}\big(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ y &= \zeta_1(t).\end{aligned} \tag{A2}$$

Introducing the variables $e_i(t) = \zeta_i(t) - v^{(i-1)}$, $i = 1,\dots,r$ and the notation $\bar{v}(s) = [v(s)\ \ v^{(1)}(s)\ \ \cdots\ \ v^{(r-1)}(s)]^T$, $s \in [t-\alpha, t]$, where $v^{(k)}$ denotes the k-th time derivative of the reference input v, which is assumed to be a sufficiently smooth function of time, and $\tilde{p}_r(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)) = p_r(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha)) + \delta_r(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta))$, the system (A2) takes the form:

$$\begin{aligned}\dot{e}_1 &= e_2(t) + \tilde{p}_1\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ &\vdots\\ \dot{e}_{r-1} &= e_r(t) + \tilde{p}_{r-1}\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ \dot{e}_r &= L^r_{\tilde{f}_{nom}} h\big(x(s)\big) - v^{(r)} + L_g L^{r-1}_{\tilde{f}_{nom}} h\big(x(s)\big)\mathcal{R}\big(x(t),x(t-\alpha),t\big)\\ &\quad + p_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha)\big)\\ &\quad + \delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ \dot{\eta}_1 &= \Psi_1\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\end{aligned} \tag{A3}$$

where $\bar{e}(t) = [e_1(t)\ \ e_2(t)\ \ \cdots\ \ e_r(t)]^T$ and $\eta(t) = [\eta_1(t)\ \ \eta_2(t)\ \ \cdots\ \ \eta_{n-r}(t)]^T$. For the above system, we assume, without loss of generality, that $[\bar{e}\ \ \eta] = [0\ \ 0]$ is an equilibrium solution.

We now consider the $\bar{e}$-subsystem of (A3):

$$\begin{aligned}\dot{e}_1 &= e_2(t) + \tilde{p}_1\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ &\vdots\\ \dot{e}_{r-1} &= e_r(t) + \tilde{p}_{r-1}\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big),\\ \dot{e}_r &= L^r_{\tilde{f}_{nom}} h\big(x(s)\big) - v^{(r)} + L_g L^{r-1}_{\tilde{f}_{nom}} h\big(x(s)\big)\mathcal{R}\big(x(t),x(t-\alpha),t\big)\\ &\quad + p_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha)\big)\\ &\quad + \delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big).\end{aligned} \tag{A4}$$

Using the explicit form of the controller formula (19) and the definition for the matrix $\tilde{A}$ and the vectors b, $\bar{p}(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta))$, the above system can be written in the following compact form:

$$\begin{aligned}\dot{\bar{e}} = {} & \tilde{A}\bar{e}(t)+b\Big(-R_2^{-1}b^TP\bar{e}\\ & -\bar{\chi}\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)w\big(\bar{e}(t),\phi\big)\Big)\\ & +\bar{p}\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\\ & +b\delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t)\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big).\end{aligned} \tag{A5}$$

We will now show that if (18) holds and the state of the η-subsystem of (A3) is bounded, then (A5) ensures that the ultimate discrepancy between the output and the external reference inputs in the closed-loop system can be made arbitrarily small by an appropriate choice of controller parameters, which implies that $\lim_{t\to\infty}\sup|y(t)-v(t)|_{\mathbb{R}}\leq d$. To establish this result, we consider the following smooth functional $V:\ C\to\mathbb{R}_{\geq 0}$:

$$V\big(\bar{e}_t(\xi)\big)=\bar{e}^T(t)P\bar{e}(t)+a^2\int_{t-\alpha}^{t}\bar{e}^T(s)\bar{e}(s)\,\mathrm{d}s, \tag{A6}$$

where P is a positive definite symmetric matrix, a is a positive scalar, and V clearly satisfies, $K_1\|\bar{e}(t)\|^2_{\mathbb{R}^r}\leq V(\bar{e}_t(\xi))\leq K_2|\bar{e}_t(\xi)|^2$, for some positive constants K_1 and K_2.

From Assumption 3, we have $|\delta_r(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta))|_{\mathbb{R}}<\tilde{c}_0(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t)$. Computing the time-derivative of V along the trajectories of (A5), we obtain:

$$\begin{aligned}\dot{V} = {} & \dot{\bar{e}}(t)^TP\bar{e}(t)+\bar{e}^T(t)P\dot{\bar{e}}(t)+a^2\big(\bar{e}^T(t)\bar{e}(t)-\bar{e}^T(t-\alpha)\bar{e}(t-\alpha)\big)\\ \leq {} & \Big(\bar{e}^T(t)\tilde{A}^T-\bar{e}^T(t)PbR_2^{-1}b^T\\ & -\bar{\chi}w\big(\bar{e}(t),\phi\big)^T\tilde{c}_0^T\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)b^T\\ & +\bar{p}^T\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\\ & +\delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)b^T\Big)P\bar{e}(t)\\ & +\bar{e}^T(t)P\Big(\tilde{A}\bar{e}(t)-bR_2^{-1}b^TP\bar{e}(t)\\ & -b\bar{\chi}\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)w\big(\bar{e}(t),\phi\big)\end{aligned}$$

$$
\begin{aligned}
&+\bar{p}\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\\
&+b\delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\Big)\\
&+a^2\big(\bar{e}^T(t)\bar{e}(t)-\bar{e}^T(t-\alpha)\bar{e}(t-\alpha)\big)\\
\le\ &\bar{e}^T(t)\big(\tilde{A}^TP+P\tilde{A}-2P^TbR_2^{-1}b^TP+a^2I_{n\times n}\big)\bar{e}(t)\\
&+2\bar{p}^T\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)P\bar{e}(t)\\
&+2\Big(-\bar{\chi}\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)w\big(\bar{e}(t),\phi\big)\\
&+\delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\Big)b^TP\bar{e}(t)\\
&-a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha),
\end{aligned}
\tag{A7}
$$

where $I_{n\times n}$ denotes the $n\times n$ identity matrix. Using the inequality $2x^Ty\le x^Tx+y^Ty$, where x and y are column vectors, we obtain:

$$
\begin{aligned}
\dot{V}\le\ &\bar{e}^T(t)\big(\tilde{A}^TP+P\tilde{A}-2P^TbR_2^{-1}b^TP+a^2I_{n\times n}+P^2\big)\bar{e}(t)\\
&+\bar{p}^2\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\\
&-2\bar{\chi}\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)\frac{|b^TP\bar{e}|_{\mathbb{R}}^2}{|b^TP\bar{e}|_{\mathbb{R}}+\phi}\\
&+2\delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)b^TP\bar{e}\\
&-a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha)\\
\le\ &\bar{e}^T(t)\big(\tilde{A}^TP+P\tilde{A}-2P^TbR_2^{-1}b^TP+a^2I_{n\times n}+P^2\big)\bar{e}(t)\\
&+\bar{p}^2\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\\
&-\frac{2\bar{\chi}\tilde{c}_0(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t)|b^TP\bar{e}|_{\mathbb{R}}^2}{|b^TP\bar{e}|_{\mathbb{R}}+\phi}\\
&+\frac{2\delta_r(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta))|b^TP\bar{e}|_{\mathbb{R}}^2}{|b^TP\bar{e}|_{\mathbb{R}}+\phi}\\
&+\frac{2\phi\delta_r(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta))|b^TP\bar{e}|_{\mathbb{R}}}{|b^TP\bar{e}|_{\mathbb{R}}+\phi}\\
&-a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \bar{e}^T(t)\big(\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + a^2 I_{n\times n} + P^2\big)\bar{e}(t) \\
&\quad + \bar{p}^2\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big) \\
&\quad - \frac{2(\bar{\chi}-1)\tilde{c}_0(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t)|b^T P\bar{e}|_{\mathbb{R}}^2}{|b^T P\bar{e}|_{\mathbb{R}}+\phi} \\
&\quad + \frac{2\phi\delta_r(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta))|b^T P\bar{e}|_{\mathbb{R}}}{|b^T P\bar{e}|_{\mathbb{R}}+\phi} \\
&\quad - a^2\bar{e}^T(t-\alpha)\bar{e}(t-\alpha). \qquad \text{(A8)}
\end{aligned}
$$

Assumption 4 implies that there exist positive real numbers a_1, a_2 such that the following bound holds:

$$
\begin{aligned}
&\big\|\bar{p}\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big)\big\|_{\mathbb{R}^r}^2 \\
&\qquad \leq a_1\bar{e}^2(t) + a_2\bar{e}^2(t-\alpha). \qquad \text{(A9)}
\end{aligned}
$$

Substituting the above inequality on the bound for $\dot{V}$ in (A8), we obtain:

$$
\begin{aligned}
\dot{V} &\leq \bar{e}^T(t)\big(\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + (a^2+a_1) I_{n\times n} + P^2\big)\bar{e}(t) \\
&\quad - (a^2 - a_2)\bar{e}^2(t-\alpha) \\
&\quad - 2(\bar{\chi}-1)\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)|b^T P\bar{e}|_{\mathbb{R}} \\
&\quad + 2\phi\delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big). \qquad \text{(A10)}
\end{aligned}
$$

Now, if $a^2 - a_2 = a_3$, where a_3 is a positive real number, and there exists a positive definite symmetric matrix P which solves the matrix equation

$$
\tilde{A}^T P + P\tilde{A} - 2P^T b R_2^{-1} b^T P + (a^2+a_1) I_{n\times n} + P^2 = -R_1, \qquad \text{(A11)}
$$

where R_1 is a positive definite matrix, then:

$$
\begin{aligned}
\dot{V} &\leq -\bar{e}^T(t)R_1\bar{e}(t) - a_3\bar{e}^2(t-\alpha) \\
&\quad - 2(\bar{\chi}-1)\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)|b^T P\bar{e}|_{\mathbb{R}} \\
&\quad + 2\phi\delta_r\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)\big). \qquad \text{(A12)}
\end{aligned}
$$

Using the inequalities $-\bar{e}^T(t)R_1\bar{e}(t) \leq -\lambda_{\min}(R_1)\bar{e}^2(t)$ and $\delta_r(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),\theta(t-\alpha_\theta)) \leq \tilde{c}_0(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t)$, we obtain:

$$
\begin{aligned}
\dot{V} &\leq -\lambda_{\min}(R_1)\bar{e}^2(t) - a_3\bar{e}^2(t-\alpha) \\
&\quad - 2(\bar{\chi}-1)\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big)|b^T P\bar{e}|_{\mathbb{R}} \\
&\quad + 2\phi\tilde{c}_0\big(\bar{e}(t)+\bar{v}(t),\eta(t),\bar{e}(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha),t\big). \qquad \text{(A13)}
\end{aligned}
$$

Therefore, $\dot{V}$ satisfies the following property:

$$\dot{V} \leq -\lambda_{\min}(R_1)\bar{e}^2(t) - a_3\bar{e}^2(t-\alpha), \quad a_4\big\|\bar{e}(t)\big\|_{\mathbb{R}^r} \geq \big|b^T P\bar{e}(t)\big|_{\mathbb{R}} \geq \frac{\phi}{\bar{\chi}-1}, \qquad \text{(A14)}$$

where a_4 is a positive real number. The above property implies that the ultimate bound on the state $\bar{e}(t)$ of (A3) depends only on the parameter ϕ and is independent of the state η and the vector of uncertain variables θ. Specifically, from Theorem 1 we have that the above inequality implies that there exist positive real numbers K, β, and a class $\mathcal{K}$ function γ_ϕ, such that the state of the $\bar{e}$-subsystem of (A4) satisfies:

$$\big|\bar{e}_t(\xi)\big| \leq Ke^{-\beta t}\big|\bar{e}_0(\xi)\big| + \gamma_\phi(\phi), \qquad \text{(A15)}$$

for every value of the delay, α. Now, since $|\bar{e}_1(t)|_{\mathbb{R}} \leq \|\bar{e}(t)\|_{\mathbb{R}^r} \leq |\bar{e}_t(\xi)|$, for any positive number d, there exists a ϕ^* such that if $\phi \in (0, \phi^*]$, it follows that $\lim\limits_{t\to\infty} \sup|y(t) - v(t)|_{\mathbb{R}} \leq d$.

To complete the proof of the theorem, we need to show that there exists a positive real number δ such that if $\max\{|x_0(\xi)|, \|\bar{v}_t\|, \|\theta_t\|\} \leq \delta$, then the state of the closed-loop system is ultimately bounded and the output of the closed-loop system satisfies $\lim\limits_{t\to\infty} \sup|y(t) - v(t)|_{\mathbb{R}} \leq d$. This result can be established by analyzing the behaviour of the DDE system (A3) using a small gain theorem type argument similar to the one used in the proof of Theorem 1 in (Christofides *et al.*, 1996); the details of this part of the proof will be omitted for brevity.

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