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Non-linear feedback control of parabolic partial differential difference equation systems

CHARALAMBOS ANTONIADES† and PANAGIOTIS D. CHRISTOFIDES†*

This paper proposes a general method for the synthesis of non-linear output feedback controllers for single-input single-output quasi-linear parabolic partial differential difference equation (PDDE) systems, for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement. Initially, a non-linear model reduction scheme which is based on combination of Galerkin's method with the concept of approximate inertial manifold is employed for the derivation of differential difference equation (DDE) systems that describe the dominant dynamics of the PDDE system. Then, these DDE systems are used as the basis for the explicit construction of non-linear output feedback controllers through combination of geometric and Lyapunov techniques. The controllers guarantee stability and enforce output tracking in the closed-loop parabolic PDDE system independently of the size of the state delay, provided that the separation of the slow and fast eigenvalues of the spatial differential operator is sufficiently large and an appropriate matrix is positive definite. The methodology is successfully employed to stabilize the temperature profile of a tubular reactor with recycle at a spatially non-uniform unstable steady-state.

1. Introduction

Quasi-linear parabolic partial differential difference equation (PDDE) systems arise naturally in the modelling of diffusion–convection–reaction processes with time-delays. A typical example of such processes is the tubular reactor with recycle loop, where the non-linearities arise from the complex reaction mechanisms and the Arrhenius dependence of the reaction rates on temperature, and the time delays occur due to transportation lag in the recycle loop and dead times associated with the measurement sensor and the control actuator. The presence of time-delays, if it is not appropriately accounted for in the controller design, may lead to serious problems in the behaviour of the closed-loop system including poor performance (e.g. sluggish response, oscillations) and instability, while it may pose unacceptable limitations on the achievable control quality.

In the past, most of the research on control of diffusion–convection–reaction processes has been carried out in the context of linear/quasi-linear parabolic partial differential equations (PDEs) (see, for example, Balas 1982, Lasieckea 1995, Christofides 2000 and the references therein). Motivated by the fact that the eigenspectrum of the parabolic spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement, the standard approach to synthesize feedback controllers for parabolic PDEs (e.g. Balas 1979, Chen and Chang 1992) involves initially the application of standard Galerkin's method to the PDE system to derive ODE systems that accurately describe its dominant dynamics. These ODE systems are subsequently used as the basis for the synthesis of finite-dimensional controllers. A potential drawback of this approach is that the number of modes that should be retained to derive an ODE system which yields the desired degree of approximation may be very large, thereby leading to complex controller synthesis and high dimensionality of the resulting controllers.

To overcome these controller synthesis and implementation problems, recent research efforts have focused on the synthesis of low-order controllers for parabolic PDE systems by taking advantage of the concept of inertial manifold (IM) (Temam 1988). If it exists, an IM is a positively invariant, exponentially attracting, finite-dimensional Lipschitz manifold. When the trajectories of the PDE system are on the IM, then this system is *exactly* described by a low-order ODE system (called inertial form), which can be used for the synthesis of low-order controllers. The main obstacle in implementing this approach is the computation of the closed-form expression of the IM, which, in most practical applications, is an almost impossible task. Fortunately, the development of efficient procedures for the construction of accurate approximations of the function that describes the inertial manifold (called approximate inertial manifolds (AIMs)) (see, for example, Foias *et al.* 1989a,b, Christofides and Daoutidis 1997) has allowed the construction of approximations of the inertial form of desired accuracy. These developments led to the synthesis of low-order linear output feedback controllers for boundary stabilization of parabolic PDEs (Kunimatsu and Sano 1994, Sano and Kunimatsu 1994, 1995), as well as the synthesis of non-linear low-order output feedback controllers that enforce closed-loop stability and output tracking in quasi-linear

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* Author for correspondence. e-mail: pdc@seas.ucla.edu
† Department of Chemical Engineering, University of California, Los Angeles, CA 90095-1592, USA.

parabolic PDE systems with distributed control action (Christofides and Daoutidis 1997).

The conventional approach to control parabolic PDDE systems is to neglect the presence of time delays and address the controller design problem on the basis of the resulting parabolic PDE system, employing one of the aforementioned methods. However, such an approach may pose unacceptable limitations on the achievable control quality and cause serious problems in the behaviour of the closed-loop system including poor performance and instability. These problems have motivated significant research on control of time-delay systems. Specifically, most of the research has focused on control of linear/non-linear ODE systems with manipulated input delay, in which the presence of the time delay prevents the use of large controller gains, thereby leading to sluggish closed-loop response. To overcome this problem, many researchers proposed various predictor schemes which completely eliminate the time delay from the characteristic polynomial of the input/output dynamics of the closed-loop system, allowing the use of larger controller gains (see, for example, Smith 1957, Brosilow 1976, Garcia and Morari 1985, Jerome and Ray 1986, Kravaris and Wright 1989). For differential difference equation (DDE) systems with state, measurement and manipulated input delays, a general method which integrates geometric concepts, Lyapunov functionals and predictor schemes was proposed in Antoniades and Christofides (1999) for the synthesis of non-linear output feedback controllers. In the area of control of parabolic distributed parameter systems with time-delays, most of the research has focused on the synthesis of optimal controllers (e.g. Wang 1975, Kowalewski 1998).

This paper focuses on single-input single-output quasi-linear parabolic partial differential difference equation (PDDE) systems, for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement. For such systems, we propose a general methodology for the synthesis of non-linear output feedback controllers that guarantee stability and enforce output tracking in the closed-loop system. Initially, a non-linear model reduction scheme which is based on combination of Galerkin's method with the concept of approximate inertial manifold is employed for the derivation of DDE systems that describe the dominant dynamics of the PDDE system. Then, these DDE systems are used as the basis for the explicit construction of non-linear output feedback controllers through combination of geometric and Lyapunov techniques. The controllers enforce the desired properties in the closed-loop parabolic PDDE system independently of the size of the state delays, provided that the separation of the slow and fast eigenvalues of the spatial differential operator is sufficiently large and an appropriate matrix is positive definite. The methodology is successfully employed to stabilize the temperature profile of a tubular reactor with recycle at a spatially non-uniform unstable steady-state.

2. Preliminaries

2.1. *Description of parabolic PDDE systems*

We consider parabolic partial differential difference equations systems with a state-space representation of the form

$$\left.\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2} + wb(z)u(t) \\ &\quad + f(\bar{x}(z,t),\bar{x}(z,t-\alpha)) \\ y_c &= \int_{\bar{\alpha}}^{\bar{\beta}} c(z)k\bar{x}(z,t)\,\mathrm{d}z \\ y_m &= \int_{\bar{\alpha}}^{\bar{\beta}} s(z)\omega\bar{x}(z,t-\tilde{\alpha})\,\mathrm{d}z \end{aligned}\right\} \tag{1}$$

subject to the boundary conditions

$$\left.\begin{aligned} C_1\bar{x}(\bar{\alpha},t) + D_1\frac{\partial \bar{x}}{\partial z}(\bar{\alpha},t) &= R_1 \\ C_2\bar{x}(\bar{\beta},t) + D_2\frac{\partial \bar{x}}{\partial z}(\bar{\beta},t) &= R_2 \end{aligned}\right\} \tag{2}$$

and the initial condition

$$\bar{x}(z,\xi) = \bar{\eta}(z,\xi), \quad \xi \in [-\alpha,0] \tag{3}$$

where $\bar{x}(z,t) = [\bar{x}_1(z,t) \cdots \bar{x}_n(z,t)]^{\mathrm{T}}$ denotes the vector of state variables, $z \in [\bar{\alpha},\bar{\beta}] \subset \mathbb{R}$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $u \in \mathbb{R}$ denotes the manipulated input, $y_c \in \mathbb{R}$ denotes the controlled output, and $y_m \in \mathbb{R}$ denotes the measured output. $\partial\bar{x}/\partial z, \partial^2\bar{x}/\partial z^2$ denote the first- and second-order spatial derivatives of $\bar{x}$, $f(\bar{x})$ is a non-linear vector function, w,k,ω are constant vectors, A,B,C_1,D_1,C_2,D_2 are constant matrices, R_1,R_2 are column vectors, α denotes the state delay, $\tilde{\alpha}$ denotes the measured output delay (measurement sensor dead time), and $\bar{\eta}(z,\xi)$ is the initial condition.

The vector function $b(z)$ is assumed to be known and smooth, and describes how the control action $u(t)$ is distributed in the interval $[\bar{\alpha},\bar{\beta}]$, $c(z)$ is a known smooth function of z which is determined by the desired performance specifications in the interval $[\bar{\alpha},\bar{\beta}]$, and $s(z)$ is a known smooth function of z which describes the 'shape' of the measurement sensor (e.g. point/distributed sensing). Whenever the control action enters the system at a single point z_0, with $z_0 \in [\bar{\alpha},\bar{\beta}]$ (i.e. point actuation), the function $b(z)$ is taken to be non-zero in a finite spatial interval of the form $[z_0-\epsilon, z_0+\epsilon]$, where ϵ is a small positive real number, and zero elsewhere in

$[\bar{\alpha},\bar{\beta}]$. Throughout the paper, we will use the order of magnitude notation $O(\epsilon)$. In particular, $\delta(\epsilon) = O(\epsilon)$ if there exist positive real numbers k_1 and k_2 such that $|\delta(\epsilon)| \le k_1|\epsilon|$ and $\forall\ |\epsilon| < k_2$. Finally, we will use the Lie derivative notation: $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f, $L_f^k h$ denotes the kth order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.

Referring to the system of equation (1), we note: (a) the spatial differential operator is linear; this assumption is valid for diffusion–convection–reaction processes for which the diffusion coefficient and the thermal conductivity can be taken to be independent of concentrations and temperature (see, for example, the tubular reactor example of § 5), (b) the manipulated input enters the system in a linear and affine fashion; this is typically the case in many practical applications where, for example, the wall temperature is chosen as the manipulated input, (c) the controlled output is different than the measured output; this is done to allow more flexibility in the formulation of the control problem for diffusion–convection–reaction processes, and (d) there is no manipulated input delay taken into account; this is done to simplify the notation of the theoretical development and the case of input delay is addressed in Remark 9.

2.2. *Formulation of parabolic PDDE system in Hilbert space*

We initially consider an infinite dimensional Hilbert space $\mathcal{H}([\bar{\alpha},\bar{\beta}],\mathbb{R}^n)$ of n-dimensional vector functions defined on $[\bar{\alpha},\bar{\beta}]$ that satisfy the boundary condition of equation (2), with inner product and norm:

$$\left.\begin{aligned} (\omega_1,\omega_2) &= \int_{\bar{\alpha}}^{\bar{\beta}} (\omega_1(z),\omega_2(z))_{\mathbb{R}^n}\,\mathrm{d}z \\ ||\omega_1||_2 &= (\omega_1,\omega_1)^{1/2} \end{aligned}\right\} \tag{4}$$

where ω_1,ω_2 are two elements of $\mathcal{H}([\bar{\alpha},\bar{\beta}];\mathbb{R}^n)$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. We formulate the parabolic PDDE system of equation (1) in $\mathcal{H}([\bar{\alpha},\bar{\beta}],\mathbb{R}^n)$ by defining the state function x as

$$x(t) = \bar{x}(z,t), \quad t > 0, \quad z \in [\bar{\alpha},\bar{\beta}] \tag{5}$$

the operator $\mathcal{A}$ in $\mathcal{H}([\bar{\alpha},\bar{\beta}],\mathbb{R}^n)$ as

$$\mathcal{A}x = A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2}$$

$$x \in D(\mathcal{A}) = \left\{ \begin{aligned} & x \in \mathcal{H}([\bar{\alpha},\bar{\beta}],\mathbb{R}^n) \\ & C_1\bar{x}(\bar{\alpha},t) + D_1\frac{\partial \bar{x}}{\partial z}(\bar{\alpha},t) = R_1 \\ & C_2\bar{x}(\bar{\beta},t) + D_2\frac{\partial \bar{x}}{\partial z}(\bar{\beta},t) = R_2 \end{aligned} \right\} \tag{6}$$

and the input and output operators as

$$\mathcal{B}u = wbu, \quad \mathcal{C}x = (c,kx), \quad \mathcal{S}x = (s,\omega x) \tag{7}$$

The system of equations (1)–(3) then takes the form

$$\left.\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}u + f(x(t),x(t-\alpha)) \\ y_c &= \mathcal{C}x(t), \quad y_m = \mathcal{S}x(t-\tilde{\alpha}) \\ x(\xi) &= \bar{\eta}(\xi), \quad \xi \in [-\alpha,0] \end{aligned}\right\} \tag{8}$$

where $f(x(t),x(t-\alpha)) = f(\bar{x}(z,t),\bar{x}(z,t-\alpha))$ and $\bar{\eta}(\xi) = \bar{\eta}(z,\xi)$. We assume that the non-linear term $f(x(t),x(t-\alpha))$ is locally Lipschitz with respect to its arguments and satisfies $f(0,0) = 0$.

In the remainder of this subsection, we precisely characterize the class of parabolic PDDE systems of the form of equation (1) considered in this work in terms of the properties of the eigenspectrum of $\mathcal{A}$. To this end, we consider the eigenvalue problem

$$\mathcal{A}\phi_j = \lambda_j\phi_j, \quad j = 1,\ldots,\infty \tag{9}$$

where λ_j denotes an eigenvalue and ϕ_j denotes the corresponding eigenfunction. The eigenspectrum of $\mathcal{A}$ is defined as $\sigma(\mathcal{A}) = \{\lambda_1,\lambda_2,\ldots\}$. Assumption 1 that follows was introduced in Christofides and Daoutidis (1997) and states that $\sigma(\mathcal{A})$ can be partitioned into a finite set containing m eigenvalues which are close to the imaginary axis and an infinite-dimensional complement containing eigenvalues which are far in the left-half of the complex plane. This assumption is satisfied by the majority of non-linear parabolic PDDE systems arising in the modelling of diffusion–convection–reaction processes (see, for example, the tubular reactor example of § 5).

Assumption 1:

(1) $\mathrm{Re}\,\{\lambda_1\} \ge \mathrm{Re}\,\{\lambda_2\} \ge \cdots \ge \mathrm{Re}\,\{\lambda_j\} \ge \cdots$, *where* $\mathrm{Re}\,\{\lambda_j\}$ *denotes the real part of* λ_j.

(2) $\sigma(\mathcal{A})$ *can be partitioned as* $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) + \sigma_2(\mathcal{A})$, *where* $\sigma_1(\mathcal{A})$ *consists of the first* m (*with* m *finite*) *eigenvalues, i.e.* $\sigma_1(\mathcal{A}) = \{\lambda_1,\ldots,\lambda_m\}$, *and* $|\mathrm{Re}\,\{\lambda_1\}|/|\mathrm{Re}\,\{\lambda_m\}| = O(1)$.

(3) $\mathrm{Re}\,\lambda_{m+1} < 0$ *and* $|\mathrm{Re}\,\{\lambda_m\}|/|\mathrm{Re}\,\{\lambda_{m+1}\}| = O(\epsilon)$ *where* $\epsilon < 1$ *is a small positive number.*

2.3. *Galerkin's method*

In this subsection, we review standard Galerkin's method in the context of the PDDE system of equation (8). Let $\mathcal{H}_s$, $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as

$$\mathcal{H}_s = \mathrm{span}\,\{\phi_1,\phi_2,\ldots,\phi_m\}$$

and

$$\mathcal{H}_f = \mathrm{span}\,\{\phi_{m+1},\phi_{m+2},\ldots\}$$

(the existence of $\mathcal{H}_s$, $\mathcal{H}_f$ follows from Assumption 1). Clearly, dim $\mathcal{H}_s$+ dim $\mathcal{H}_f$= dim $\mathcal{H}$. Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of equation (8) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x \tag{10}$$

Applying P_s and P_f to the system of equation (8) and using the above decomposition for x, the system of equation (8) can be equivalently written in the form

$$\left.\begin{aligned}
\frac{dx_s}{dt} &= \mathcal{A}_s x_s(t) + \mathcal{B}_s u \\
&\quad + f_s(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\
\frac{dx_f}{dt} &= \mathcal{A}_f x_f(t) + \mathcal{B}_f u \\
&\quad + f_f(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\
y_c &= \mathcal{C}x_s(t) + \mathcal{C}x_f(t) \\
y_m &= \mathcal{S}x_s(t-\tilde{\alpha}) + \mathcal{S}x_f(t-\tilde{\alpha}) \\
x_s(\xi) &= P_s x(\xi) = P_s \bar{\eta}(\xi) = \bar{\eta}_s(\xi) \\
x_f(\xi) &= P_f x(\xi) = P_f \bar{\eta}(\xi) = \bar{\eta}_f(\xi), \quad \xi \in [-\alpha, 0]
\end{aligned}\right\} \tag{11}$$

where $\mathcal{A}_s = P_s \mathcal{A} P_s$, $\mathcal{B}_s = P_s \mathcal{B}$, $f_s = P_s f$, $\mathcal{A}_f = P_f \mathcal{A} P_f$, $\mathcal{B}_f = P_f \mathcal{B}$ and $f_f = P_f f$. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s =$ diag $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, $f_s(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha))$ and $f_f(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha))$ are Lipschitz vector functions, and $\mathcal{A}_f$ is an unbounded differential operator which is exponentially stable (this follows from part (3) of Assumption 1 and the selection of $\mathcal{H}_s, \mathcal{H}_f$). Neglecting the fast modes, the DDE system, which describes the slow dynamics of the system of equation (11), is derived as

$$\left.\begin{aligned}
\frac{dx_s}{dt} &= \mathcal{A}_s x_s(t) + \mathcal{B}_s u + f_s(x_s(t), x_s(t-\alpha), 0, 0) \\
y_{cs} &= \mathcal{C}x_s(t), \quad y_{ms} = \mathcal{S}x_s(t-\tilde{\alpha}) \\
x_s(\xi) &= \bar{\eta}_s(\xi), \quad \xi \in [-\alpha, 0]
\end{aligned}\right\} \tag{12}$$

where the subscript s in y_{cs} and y_{ms} denotes that these outputs are associated with a slow system.

Remark 1: Whenever the eigenfunctions ϕ_j of the operator $\mathcal{A}$ cannot be calculated analytically, one can still use Galerkin's method to perform model reduction by using 'empirical eigenfunctions' of the PDDE system as basis functions in $\mathcal{H}_s$ and $\mathcal{H}_f$ (such 'empirical eigenfunctions' can be extracted from detailed numerical simulations of the PDDE system using the Karhunen–Loève expansion; see Holmes *et al.* (1996) for details on the Karhunen–Loève expansion).

2.4. *Spectral and stability properties of DDE systems*

In this subsection, we initially formulate the DDE system of equation (12) as an infinite dimensional system in an appropriate Banach space and provide the statement and solution of the eigenvalue problem for the linear delay operator (see equation (16) below). The solution of the eigenvalue problem will be utilized in the design of non-linear distributed state observers in §4. To this end, we rewrite the system of equation (12) as

$$\left.\begin{aligned}
\frac{dx_s}{dt} &= \bar{\mathcal{A}}_s x_s(t) + \tilde{\mathcal{A}}_s x_s(t-\alpha) + \mathcal{B}_s u \\
&\quad + f_{ns}(x_s(t), x_s(t-\alpha), 0, 0) \\
y_{cs} &= \mathcal{C}x_s(t), \quad y_{ms} = \mathcal{S}x_s(t-\tilde{\alpha}) \\
x_s(\xi) &= \bar{\eta}_s(\xi), \quad \xi \in [-\alpha, 0]
\end{aligned}\right\} \tag{13}$$

where $\bar{\mathcal{A}}_s, \tilde{\mathcal{A}}_s$ are constant matrices and the term $f_{ns}(x_s(t), x_s(t-\alpha), 0, 0)$ includes only higher-order terms.

We formulate the system of equation (13) in the Banach space $\mathcal{C}_s([-\alpha, 0], \mathcal{H}_s)$ of continuous n-vector valued functions defined in the interval $[-\alpha, 0]$ with inner product and norm

$$\left.\begin{aligned}
(\tilde{\omega}_1, \tilde{\omega}_2) &= \tilde{\omega}_1(0)\tilde{\omega}_2(0) + \int_{-\alpha}^{0} \tilde{\omega}_1(z+\alpha)\tilde{\mathcal{A}}_s \tilde{\omega}_2(z)\, dz \\
\|\tilde{\omega}_1\|_2 &= (\tilde{\omega}_1, \tilde{\omega}_1)^{1/2}
\end{aligned}\right\} \tag{14}$$

where $\tilde{\omega}_1$ is an element of $\mathcal{C}_s^*([-\alpha, 0], \mathcal{H}_s^*)$, $\mathcal{H}_s^*$ is the n-dimensional vector space of row vectors, and $\tilde{\omega}_2 \in \mathcal{C}_s$. On $\mathcal{C}_s$, the state function x_s of the system of equation (13) is defined as

$$x_{st}(\xi) = x_s(t+\xi), \quad t \geq 0, \quad \xi \in [-\alpha, 0] \tag{15}$$

and the operator $\mathcal{D}$ as

$$\mathcal{D}\phi(\xi) = \left\{\begin{aligned}
&\frac{d\phi(\xi)}{d\xi}, \quad \xi \in [-\alpha, 0) \\
&\bar{\mathcal{A}}_s \phi(0) + \tilde{\mathcal{A}}_s \phi(-\alpha), \quad \xi = 0
\end{aligned}\right\} \tag{16}$$

$$\begin{aligned}
\phi(\xi) \in D(\mathcal{D}) = \{\phi &\in \mathcal{C}_s^*([-\alpha, 0], \mathbb{R}^{n^*}) : \dot{\phi} \in \mathcal{C}_s, \\
&\dot{\phi}(0) = \bar{\mathcal{A}}_s \phi(0) + \tilde{\mathcal{A}}_s \phi(-\alpha)\}
\end{aligned} \tag{17}$$

$D(\mathcal{D})$ is a dense subset of $\mathcal{C}_s$. If $\bar{\eta}_s \in D(\mathcal{D})$, then the system of equation (13) can be equivalently written as

$$\left.\begin{aligned}\frac{\mathrm{d}x_{st}}{\mathrm{d}t} &= \mathcal{D}x_{st} + \mathcal{B}_s u + f_{ns}(Px_{st}, Qx_{st}, 0, 0)\\ y(t) &= \mathcal{C}Px_{st}, \quad y_{ms} = \mathcal{S}Rx_{st}\\ x_s(\xi) &= \bar{\eta}_s(\xi), \quad \xi \in [-\alpha, 0]\end{aligned}\right\} \tag{18}$$

where $Px_{st} = x_{st}(0)$, $Qx_{st} = x_{st}(-\alpha)$ and $Rx_{st} = x_{st}(-\tilde{\alpha})$.

The eigenvalue problem for the operator $\mathcal{D}$ is defined as

$$\mathcal{D}\tilde{\phi}_j = \mu_j \tilde{\phi}_j, \quad j = 1, \ldots, \infty \tag{19}$$

where μ_j denotes an eigenvalue and $\tilde{\phi}_j$ denotes the corresponding eigenfunction (note that $\tilde{\phi}_j$ is a vector function of dimension m); the eigenspectrum of $\mathcal{D}$, $\sigma(\mathcal{D})$, is defined as the set of all eigenvalues of $\mathcal{D}$, i.e. $\sigma(\mathcal{D}) = \{\mu_1, \mu_2, \ldots\}$ and is given by Hale and Lunel (1993)

$$\sigma(\mathcal{D}) = \{\mu\text{: } \det(\mu I - \bar{\mathcal{A}}_s - \tilde{\mathcal{A}}_s \,\mathrm{e}^{-\mu\alpha}) = 0\} \tag{20}$$

The eigenfunctions can be directly computed from the formula $\tilde{\phi}_j(\xi) = \mathrm{e}^{\mu_j \xi}\tilde{\phi}_j(0)$, where $\tilde{\phi}_j(0)$ satisfies the equation $(\mu_j I - \bar{\mathcal{A}}_s - \tilde{\mathcal{A}}_s \,\mathrm{e}^{-\mu_j \alpha})\tilde{\phi}_j(0) = 0$, where $\xi \in [-\alpha, 0]$ and $j = 1, \ldots, \infty$. The adjoint operator $\bar{\mathcal{D}}$ of $\mathcal{D}$ is defined from the relation $(\mathcal{D}\tilde{\phi}, \tilde{\psi}) = (\tilde{\phi}, \bar{\mathcal{D}}\tilde{\psi})$, where (,) denotes the inner product of equation (14), and its eigenspectrum, $\sigma(\bar{\mathcal{D}})$, satisfies $\sigma(\bar{\mathcal{D}}) = \sigma(\mathcal{D})$.

In the remainder of this subsection, we review the definitions of asymptotic stability and input-to-state stability for DDE systems as well as a basic theorem that provides sufficient conditions for assessing asymptotic stability. To this end, we consider the following system of non-linear differential difference equations

$$\left.\begin{aligned}\dot{x}_s(t) &= f(x_s(t), x_s(t-\alpha), \theta(t), \theta(t-\alpha))\\ x_s(\xi) &= \bar{\eta}_s(\xi), \quad \xi \in [-\alpha, 0]\end{aligned}\right\} \tag{21}$$

where $x_s \in \mathcal{H}_s$, $\theta \in \mathcal{H}_s$, and suppose that $f(0,0,0,0) = 0$. Moreover, given a function $\theta : [-\alpha, \infty) \to \mathcal{H}_s$, $\theta_t(\xi)$ represents a function from $[-\alpha, 0]$ to $\mathcal{H}_s$ defined by $\theta_t(\xi) := \theta(t + \xi)$. We also define $|\theta_t(\xi)| := \max_{t-\alpha \le \xi \le t} ||\theta_t(\xi)||_2$, $||\theta_t|| := \sup_{s \ge 0} |\theta_s(\xi)|$ and $||\theta_t||_T := \sup_{0 \le s \le T} |\theta_s(\xi)|$. Finally, a function γ: $\mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{Q}$ if it is continuous, non-decreasing and is zero at zero. It is of class K if it is continuous, strictly increasing and is zero at zero. A function β: $\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.

Definition 1: Let γ be a function of class-$\mathcal{Q}$ and δ_x, δ_θ be positive real numbers. The system of equation (21) is said to be input-to-state stable if $|x_{s0}(\xi)| \le \delta_x$ and $||\theta_t|| \le \delta_\theta$ imply that the solution of the system of equation (21) is defined for all times and satisfies

$$|x_{st}(\xi)| \le \beta(|x_{s0}(\xi)|, t) + \gamma(||\theta_t||), \ \forall\ t \ge 0 \tag{22}$$

The above definition, when $\theta(t) \equiv 0$, $\forall\ t \ge 0$ reduces to the definition of asymptotic stability for the zero solution of the DDE system of equation (21).

The following theorem provides sufficient conditions for the stability of the zero solution of the system of equation (21), expressed in terms of a suitable functional.

Theorem 1 (Hale and Lunel 1993)**:** *Consider the system of equation* (21) *with* $\theta(t) \equiv 0$ *and let* $\gamma_1, \gamma_2, \gamma_3$ *be functions of class* $\mathcal{Q}$. *If* $\gamma_1(s), \gamma_2(s) > 0$ *for* $s > 0$ *and there is a continuous functional* $V : \mathcal{C}_s \to R_{\ge 0}$ *such that*

$$\begin{aligned}\gamma_1(||x_s(t)||_2) &\le V(x_{st}(\xi)) \le \gamma_2(|x_{st}(\xi)|)\\ \dot{V}(x_{st}(\xi)) &\le -\gamma_3(||x_s(t)||_2)\end{aligned} \tag{23}$$

then the solution $x_s = 0$ *is stable. If* $\gamma_3(s) > 0$ *for* $s > 0$, *then the solution* $x_s = 0$ *is locally asymptotically stable. Moreover if* $\gamma_1(s) \to \infty$ *as* $s \to \infty$ *and* $\gamma_3(s) > 0$ *for* $s > 0$, *then the solution* $x_s = 0$ *is globally asymptotically stable.*

Remark 2: Regarding the properties of the eigenspectrums of the operators $\mathcal{A}$ and $\mathcal{D}$, we note the following similarities: (a) the eigenspectrums $\sigma(\mathcal{A})$ and $\sigma(\mathcal{D})$ are both point spectrums containing eigenvalues of finite multiplicity, (b) the number of eigenvalues of both $\sigma(\mathcal{A})$ and $\sigma(\mathcal{D})$ which have positive real part (i.e. they are located in the right-half of the complex plane) is always finite, and (c) the real parts of all the eigenvalues of $\sigma(\mathcal{A})$ and $\sigma(\mathcal{D})$ are bounded from above (i.e. there exists a positive real number β such that $\max\{|\mathrm{Re}\,\lambda_i|, |\mathrm{Re}\,\mu_i|\} \le \beta$ for all $i = 1, \ldots, \infty$). On the other hand, $\sigma(\mathcal{A})$ and $\sigma(\mathcal{D})$ exhibit significant structural differences: specifically, $\sigma(\mathcal{A})$ can be partitioned into a finite set containing eigenvalues which are close to the imaginary axis and an infinite-dimensional complement containing eigenvalues which are far in the left-half of the complex plane (Assumption 1), while the eigenvalues of $\sigma(\mathcal{D})$ are asymptotically distributed along nearly vertical asymptotes in the complex plane (see Manitius *et al.* 1987, Hale and Lunel 1993 for more details). These structural differences of $\sigma(\mathcal{A})$ and $\sigma(\mathcal{D})$ directly affect the approach that is used for controller design for parabolic PDE and DDE systems, respectively. In particular, the control of parabolic PDE systems is addressed on the basis of finite-dimensional approximations obtained through linear/non-linear Galerkin's method, while the control of DDE systems is addressed by using combination of geometric concepts with the method of Lyapunov functionals on the basis of the DDE system (see §4.2 for more details).

Remark 3: Even though, at this stage, there is no systematic way for selecting the form of the functional

$V(x_{st}(\xi))$ which is suitable for a particular application, a choice for $V(x_{st}(\xi))$, which is frequently used to show local exponential stability of a DDE system of the form of equation (21) via Theorem 1, is

$$V(x_{st}(\xi)) = x_s(t)^{\mathrm{T}} C x_s(t) + a^2 \int_{t-\alpha}^{t} x_s(s)^{\mathrm{T}} E x_s(s)\,\mathrm{d}s \quad (24)$$

where E, C are symmetric positive definite matrices and a is a positive real number. Clearly, the functional of equation (24) satisfies $K_1 \|x_s(t)\|_2^2 \le V(x_{st}(\xi)) \le K_2 |x_{st}(\xi)|^2$ for some positive K_1, K_2.

3. Non-linear model reduction

In this section, we construct non-linear DDE systems that accurately reproduce the dynamics of the parabolic PDDE system of equation (8). The construction of the DDE systems is achieved by generalizing a non-linear model reduction procedure introduced in Christofides and Daoutidis (1997) for parabolic PDE systems, to the class of systems of equation (8). The non-linear model reduction procedure is based on a combination of standard Galerkin's method with the concept of approximate inertial manifold.

We initially formulate the system of equation (12) within a singular perturbation framework. Using that $\epsilon = |\mathrm{Re}\,\lambda_1|/|\mathrm{Re}\,\lambda_{m+1}|$, the system of equation (11) can be written in the form

$$\left.\begin{aligned}
\frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{A}_s x_s(t) + \mathcal{B}_s u \\
&\quad + f_s(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\
\epsilon \frac{\mathrm{d}x_f}{\mathrm{d}t} &= \mathcal{A}_{f\epsilon} x_f(t) + \epsilon \mathcal{B}_f u \\
&\quad + \epsilon f_f(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\
x_s(\xi) &= \bar{\eta}_s(\xi), \quad x_f(\xi) = \bar{\eta}_f(\xi), \quad \xi \in [-\alpha, 0]
\end{aligned}\right\} \quad (25)$$

where $\mathcal{A}_{f\epsilon}$ is an unbounded differential operator defined as $\mathcal{A}_{f\epsilon} = \epsilon \mathcal{A}_f$. Since ϵ is a small positive number less than unity (Assumption 1, part (3)), the system of equation (25) is in the standard singularly perturbed form (Kokotovic *et al.* 1986), with x_s being the slow states and x_f being the fast states.

Introducing the fast time-scale $\tau = t/\epsilon$ in equation (25) and setting $\epsilon = 0$, the infinite-dimensional fast subsystem is obtained as

$$\frac{\mathrm{d}x_f}{\mathrm{d}\tau} = \mathcal{A}_{f\epsilon} x_f \quad (26)$$

which is globally exponentially stable since the operator $\mathcal{A}_{f\epsilon}$ generates an exponentially stable semigroup.

Now, we introduce the concepts of inertial manifold and approximate inertial manifold. Our definition of the concept of inertial manifold is a direct generalization of the one used in Christofides and Daoutidis (1997) (see also Temam 1988) for parabolic PDE systems. An inertial manifold $\mathcal{M}$ for the system of equation (8) is a subset of $\mathcal{H}$, which satisfies the properties:

(i) $\mathcal{M}$ is a Lipschitz manifold;

(ii) $\mathcal{M}$ is a graph of a Lipschitz function $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ mapping $[0,\infty) \times \mathcal{H}_s \times \mathbb{R}^l \times (0, \epsilon^*]$ into $\mathcal{H}_f$ and for every solution $x_s(t), x_f(t)$ of equation (25), with $x_f(\xi) = \bar{\eta}_f(\xi)$, $\xi \in [-\alpha, 0]$, then

$$\left.\begin{aligned}
x_f(t) &= \Sigma(x_s(t), \bar{\eta}_s(t-\alpha), u(t), \epsilon), \quad \forall\, t \in (0, \alpha] \\
x_f(t) &= \Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon), \quad \forall\, t > \alpha
\end{aligned}\right\} \quad (27)$$

(iii) $\mathcal{M}$ attracts every trajectory exponentially.

The evolution of the state x_f on $\mathcal{M}$ is given by equation (27), while the evolution of the state x_s is governed by the DDE system (called inertial form)

$$\left.\begin{aligned}
\frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s(t), \bar{\eta}_s(t-\alpha), \Sigma(x_s(t), \\
&\qquad \bar{\eta}_s(t-\alpha), u(t), \epsilon), \bar{\eta}_f(t-\alpha)), \quad \forall\, t \in (0, \alpha] \\
\frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s(t), x_s(t-\alpha), \Sigma(x_s(t), \\
&\qquad x_s(t-\alpha), u(t), \epsilon), \Sigma(x_s(t-\alpha), \\
&\qquad x_s(t-2\alpha), u(t-\alpha), \epsilon)), \quad \forall\, t > \alpha
\end{aligned}\right\} \quad (28)$$

Assuming that $u(t)$ is smooth, differentiating equation (27) and utilizing equation (25), $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ can be computed as the solution of the partial differential difference equation

$$\left.\begin{aligned}
&\epsilon\left(\frac{\partial \Sigma}{\partial x_s}\dot{x}_s + \frac{\partial \Sigma}{\partial u}\dot{u}\right) \\
&\quad = \mathcal{A}_{f\epsilon}\Sigma(x_s(t), \bar{\eta}_s(t-\alpha), u(t), \epsilon) \\
&\qquad + \epsilon \mathcal{B}_f u + \epsilon f_f(x_s(t), \bar{\eta}_s(t-\alpha), \\
&\qquad \Sigma(x_s(t), \bar{\eta}_s(t-\alpha), u(t), \epsilon), \bar{\eta}_f(t-\alpha)), \\
&\qquad \forall\, t \in (0, \alpha] \\
&\epsilon\left(\frac{\partial \Sigma}{\partial t}\dot{x}_s + \frac{\partial \Sigma}{\partial x_s(t-\alpha)}\dot{x}_s(t-\alpha) + \frac{\partial \Sigma}{\partial u}\dot{u}\right) \\
&\quad = \mathcal{A}_{f\epsilon}\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon) \\
&\qquad + \epsilon \mathcal{B}_f u + \epsilon f_f(x_s(t), x_s(t-\alpha), \\
&\qquad \Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon), \Sigma(x_s(t-\alpha), \\
&\qquad x_s(t-2\alpha), u(t-\alpha), \epsilon)), \quad \forall\, t > \alpha
\end{aligned}\right\} \quad (29)$$

which $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ has to satisfy for all $x_s \in \mathcal{H}_s$, $u \in \mathbb{R}^l$, $\epsilon \in (0, \epsilon^*]$.

From the complex structure of equation (29), it is obvious that the computation of the explicit form of $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ is impossible in most practical applications. To circumvent this problem, a procedure based on singular perturbations (introduced in Christofides and Daoutidis 1997) is used to compute approximations of $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ (approximate inertial manifolds) and approximations of the inertial form, of desired accuracy. More specifically, the vectors $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ and $u(t)$ are expanded in a power series in ϵ as

$$\left.\begin{aligned} u &= \bar{u}_0 + \epsilon\bar{u}_1 + \epsilon^2\bar{u}_2 + \cdots \\ &\quad + \epsilon^k\bar{u}_k + O(\epsilon^{k+1}) \\ \Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon) &= \Sigma_0(x_s(t), x_s(t-\alpha), u(t)) \\ &\quad + \epsilon\Sigma_1(x_s(t), x_s(t-\alpha), u(t)) \\ &\quad + \epsilon^2\Sigma_2(x_s(t), x_s(t-\alpha), u(t)) \\ &\quad + \cdots \\ &\quad + \epsilon^k\Sigma_k(x_s(t), x_s(t-\alpha), u(t)) \\ &\quad + O(\epsilon^{k+1}) \end{aligned}\right\} \tag{30}$$

where $\bar{u}_\nu$ and Σ_ν, $\nu = 0, \ldots, k$ are smooth vector functions. Substituting the expressions of equation (30) into equation (29), and equating terms of the same power in ϵ, one can obtain approximations of $\Sigma(t, x_s(t), x_s(t-\alpha), u, \epsilon)$ up to a desired order. Substituting the $O(\epsilon^{k+1})$ approximation of $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ and u into equation (28), the following approximation of the inertial form is obtained as

$$\left.\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s(\bar{u}_0 + \epsilon\bar{u}_1 + \cdots + \epsilon^k\bar{u}_k) \\ &\quad + f_s(x_s(t), \bar{\eta}_s(t-\alpha), \Sigma_0(x_s(t), \bar{\eta}_s(t-\alpha), u(t)) \\ &\quad + \cdots + \epsilon^k\Sigma_k(x_s(t), \bar{\eta}_s(t-\alpha), u(t)), \bar{\eta}_f(t-\alpha)), \\ &\quad \forall\ t \in (0, \alpha] \\ \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s(\bar{u}_0 + \epsilon\bar{u}_1 + \cdots + \epsilon^k\bar{u}_k) \\ &\quad + f_s(x_s(t), x_s(t-\alpha), \Sigma_0(x_s(t), x_s(t-\alpha), u(t)) \\ &\quad + \cdots + \epsilon^k\Sigma_k(x_s(t), x_s(t-\alpha), u(t)), \\ &\quad \Sigma_0(x_s(t-\alpha), x_s(t-2\alpha), u(t-\alpha)) \\ &\quad + \cdots + \epsilon^k\Sigma_k(x_s(t-\alpha), x_s(t-2\alpha), u(t-\alpha))), \\ &\quad \forall\ t > \alpha \end{aligned}\right\} \tag{31}$$

In order to characterize the stability of the system of equation (25) from the stability properties of the slow subsystem of equation (31) and the fast subsystem of equation (26), we need to impose the following exponential stability requirement on the system of equation (31) with $u(t) \equiv 0$ and $\epsilon = 0$.

Assumption 2: *The DDE system of equation* (31) *with* $u(t) \equiv 0$ *and* $\epsilon = 0$ *is exponentially stable, in the sense that there exists a smooth Lyapunov functional* $V: \mathcal{C}_s \to \mathbb{R}_{\geq 0}$ *and a set of positive real numbers* $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$, *such that for all* $x_s \in \mathcal{H}_s$ *that satisfy* $||x_s||_2 \leq a_7$, *the following conditions hold*

$$\left.\begin{aligned} a_1||x_s||_2^2 \leq V(x_{st}) &\leq a_2|x_{st}|^2 \\ \dot{V}(x_{st}) &\leq -a_3||x_s(t)||_2^2 - a_4||x_s(t-\alpha)||_2^2 \\ \left|\left|\frac{\partial V}{\partial x_s}\right|\right|_2 &\leq a_5||x_s(t)||_2 + a_6||x_s(t-\alpha)||_2 \end{aligned}\right\} \tag{32}$$

We are now in a position to state the main stability result of this section. The proof is given in the appendix.

Theorem 2: *Consider the system of equation* (8) *with* $u(t) \equiv 0$, *for which Assumption* 1 *holds. Suppose also that Assumption* 2 *holds. Then, there exist positive real numbers* μ_1, μ_2, ϵ^* *such that if* $|x_{s0}(\xi)| \leq \mu_1$, $|x_{f0}(\xi)| \leq \mu_2$ *and* $\epsilon \in (0, \epsilon^*]$, *and the matrix*

$$\begin{bmatrix} S_1 & -S_2^{\mathrm{T}} \\ -S_2 & S_3 \end{bmatrix} \tag{33}$$

where S_1, S_2, S_3 *are matrices defined in equation* (65), *is positive definite, then there exist positive real numbers* $K \geq 1, \sigma$ *such that*

$$\begin{bmatrix} ||x_s||_2 \\ ||x_f||_2 \end{bmatrix} \leq K\,\mathrm{e}^{-\sigma t}\begin{bmatrix} |x_{s0}(\xi)| \\ |x_{f0}(\xi)| \end{bmatrix} \tag{34}$$

independently of the size of α.

Remark 4: The requirement that the matrix of equation (33) is positive definite arises from our desire to derive a general result for assessing the stability properties of the PDDE system of equation (8) in terms of the stability properties of the DDE system of equation (31) independently of the size of the state delay. Of course, such a requirement is not needed when time delays are not present in the system of equation (8) (see Christofides and Daoutides 1997). Finally, even though the stability requirement of Assumption 2 is imposed on the system of equation (31) with $\epsilon = 0$ and $u(t) \equiv 0$, the result of Theorem 2 can be also shown to hold when the stability requirement of Assumption 2 is imposed on the system of equation (31) with $\epsilon \neq 0$ and $u(t) \equiv 0$.

Remark 5: The expansion of $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon)$ in a power series in ϵ (equation (30)) is well-posed with respect to ϵ because as $\epsilon \to 0$, $\Sigma(x_s(t),\ x_s(t-\alpha),\ u(t), \epsilon) \to \Sigma_0(x_s(t),\ x_s(t-\alpha),\ u(t)) = 0$, and the corresponding approximate inertial form is

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + \mathcal{B}_s \bar{u}_0 + f_s(x_s(t), x_s(t-\alpha), 0, 0) \tag{35}$$

which is the DDE system obtained from the standard Galerkin's method. Furthermore, for $k = 1$, the expansion of equation (30) yields $\Sigma(x_s(t), x_s(t-\alpha), u(t), \epsilon) = \epsilon\Sigma_1(x_s(t), x_s(t-\alpha), \bar{u}_0(t)) = -\epsilon(\mathcal{A}_{f\epsilon})^{-1}[\mathcal{B}_f \bar{u}_0(t) + f_f(x_s(t), x_s(t-\alpha), 0, 0)]$, and the corresponding approximate inertial form is

$$\left.\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s(\bar{u}_0 + \epsilon\bar{u}_1) + f_s(x_s(t), \eta_s(t-\alpha), \\ &\quad \epsilon\Sigma_1(x_s(t), \eta_s(t-\alpha), \bar{u}_0(t)), \eta_f(t-\alpha)), \\ &\quad \forall\ t \in (0, \alpha] \\ \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s(\bar{u}_0 + \epsilon\bar{u}_1) + f_s(x_s(t), x_s(t-\alpha), \\ &\quad \epsilon\Sigma_1(x_s(t), x_s(t-\alpha), \bar{u}_0(t)), \\ &\quad \epsilon\Sigma_1(x_s(t-\alpha), x_s(t-2\alpha), \bar{u}_0(t-\alpha)), \\ &\quad \forall\ t > \alpha \end{aligned}\right\} \tag{36}$$

In contrast to the system of equation (35), the above system utilizes information about the structure of the fast subsystem, thereby yielding solutions which are asymptotically closer to the solutions of the open-loop system of equation (8) (i.e. $\lim_{t\to\infty} \|x(t) - x_s(t)\|_2 = O(\epsilon^2)$) compared to the solutions of the system of equation (35) for which $\lim_{t\to\infty} \|x(t) - x_s(t)\|_2 = O(\epsilon)$. Finally, the expansion of u in a power series in ϵ in equation (30) is motivated by our intention to appropriately modify the synthesis of the controller such that the controlled output of the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form, y_{cs}, satisfies $\lim_{t\to\infty} |y_{cs} - v| = 0$, where v is the reference input.

4. Non-linear output feedback control

In this section, we synthesize non-linear output feedback controllers that guarantee local exponential stability and force the output of the closed-loop PDDE system to asymptotically follow the reference input independently of the size of the state delay, provided that ϵ is sufficiently small.

4.1. *A general result*

We initially use the result of Theorem 1 to establish that any non-linear output feedback controller that guarantees stability and enforces asymptotic output tracking in the DDE system of equation (31), exponentially stabilizes the closed-loop PDDE system and ensures that the discrepancy between the output of the closed-loop DDE system and the output of the closed-loop PDDE system is asymptotically of $O(\epsilon^{k+1})$, provided that ϵ is sufficiently small. The non-linear output feedback controller which enforces the desired control objectives in the system of equation (31) is constructed through a standard combination of a state feedback controller with a state observer.

More specifically, we consider non-linear state feedback control laws of the general form

$$\left.\begin{aligned} u &= \bar{u}_0 + \epsilon\bar{u}_1 + \epsilon^2\bar{u}_2 + \cdots + \epsilon^k\bar{u}_k \\ &= \mathcal{A}_0(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha)) + \cdots \\ &\quad + \epsilon^k[\mathcal{A}_k(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha))] \\ &=: \mathcal{A}_\epsilon(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha)) \end{aligned}\right\} \tag{37}$$

where $\mathcal{A}_0(x_s(t),\ \bar{v}(t),\ x_s(t-\alpha),\ \bar{v}(t-\alpha)), \ldots, \mathcal{A}_k(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha))$ are smooth scalar functions, $\bar{v}(s) = [v(s)\ v^{(1)}(s)\ \cdots\ v^{(r-1)}(s)]^T$, $s \in [t-\alpha, t]$ and $v^{(k)}$ denotes the kth time-derivative of the reference input $v \in \mathbb{R}$.

The non-linear controllers are constructed by following a sequential procedure. Specifically, the component $\bar{u}_0 = \mathcal{A}_0(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha))$ is initially synthesized on the basis of the $O(\epsilon)$ approximation of the inertial form; then the component $\bar{u}_1 = \mathcal{A}_1(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha))$ is synthesized on the basis of the $O(\epsilon^2)$ approximation of the inertial form. In general, at the kth step, the component $\bar{u}_k = \mathcal{A}_k(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha))$ is synthesized on the basis of the $O(\epsilon^{k+1})$ approximation of the inertial form (equation (31)). The explicit synthesis of $\mathcal{A}_\nu(x_s(t), \bar{v}(t), x_s(t-\alpha), \bar{v}(t-\alpha))$, $\nu = 0, \ldots, k$ to guarantee stability and force the controlled output of the $O(\epsilon^{\nu+1})$ approximation of the inertial form to asymptotically follow the reference input will be addressed in §4.2 below.

Since measurements of the states $\bar{x}$ (and thus x_s) are usually not available in practice, we will use the non-linear state observer for the implementation of the state feedback law of equation (37) as

$$\left.\begin{aligned} \dot{\omega}_s &= \mathcal{A}_s\omega_s + \mathcal{B}_s(\bar{u}_0 + \epsilon\bar{u}_1 + \cdots + \epsilon^k\bar{u}_k) \\ &\quad + f_s(\omega_s(t), \bar{\eta}_s(t-\alpha), \Sigma_0(\omega_s(t), \bar{\eta}_s(t-\alpha), u(t)) \\ &\quad + \cdots + \epsilon^k\Sigma_k(\omega_s(t), \bar{\eta}_s(t-\alpha), u(t)), \bar{\eta}_f(t-\alpha)) \\ &\quad + \Phi_{C_s}(0)L(y_m(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha})) \\ &\quad + \tilde{\mathcal{A}}_s\int_0^{\alpha-\tilde{\alpha}} \Phi_{C_s}(\xi-\alpha)L(y_m(t-\xi-\tilde{\alpha}) \\ &\quad - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))\,d\xi, \quad \forall\ t \in (0, \alpha] \end{aligned}\right\} \tag{38}$$

$$
\left.
\begin{aligned}
\dot{\omega}_s = {} & \mathcal{A}_s\omega_s + \mathcal{B}_s\Big(\bar{u}_0 + \epsilon\bar{u}_1 + \cdots + \epsilon^k\bar{u}_k\Big) \\
& + f_s(\omega_s(t), \omega_s(t-\alpha), \Sigma_0(\omega_s(t), \omega_s(t-\alpha), u(t)) \\
& + \cdots + \epsilon^k\Sigma_k(\omega_s(t), \omega_s(t-\alpha), u(t)), \\
& \Sigma_0(\omega_s(t-\alpha), \omega_s(t-2\alpha), u(t-\alpha)) \\
& + \cdots + \epsilon^k\Sigma_k(\omega_s(t-\alpha), \omega_s(t-2\alpha), u(t-\alpha))) \\
& + \Phi_{\mathcal{C}_s}(0)L(y_m(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha})) \\
& + \tilde{\mathcal{A}}_s\int_0^{\alpha-\tilde{\alpha}} \Phi_{\mathcal{C}_s}(\xi-\alpha)L(y_m(t-\xi-\tilde{\alpha}) \\
& - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))\,\mathrm{d}\xi, \quad \forall\ t > \alpha \\
\omega_s(\xi) = {} & \bar{\omega}_s(\xi), \quad \xi \in [-\alpha, 0]
\end{aligned}
\right\}
\tag{38}
$$

where $\omega_s \in \mathbb{R}^m$ is the observer state, $\bar{\omega}_s(\xi)$ is a smooth vector function defined in $\xi \in [-\alpha, 0]$, and $\Phi_{\mathcal{C}_s} = [\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_m]$. L is a constant column vector chosen so that all the eigenvalues of the operator

$$
\begin{aligned}
\bar{\mathcal{A}}_s\omega_s(t) + \tilde{\mathcal{A}}_s\omega_s(t-\alpha) - \Phi_{\mathcal{C}_s}(0)L\mathcal{S}\omega_s(t-\tilde{\alpha}) \\
- \tilde{\mathcal{A}}_s\int_0^{\alpha-\tilde{\alpha}} \Phi_{\mathcal{C}_s}(\xi-\alpha)L\mathcal{S}\omega_s(t-\xi-\tilde{\alpha})\,\mathrm{d}\xi
\end{aligned}
$$

are in the left-half of the complex plane.

Assumption 3 states the desired control objectives that the output feedback controller of equations (37) and (38) enforces in the closed-loop PDDE system.

Assumption 3: *The non-linear output feedback controller of the form of equations* (37) *and* (38) *exponentially stabilizes the* $O(\epsilon^{k+1})$ *approximation of the closed-loop inertial form and ensures that its output satisfy* $\lim_{t\to\infty}|y_{cs}(t) - v| = 0$, *independently of the size of the state delay.*

Theorem 3 provides precise conditions under which the non-linear controller of equations (37) and (38) enforces exponential stability and asymptotic output tracking in the closed-loop PDDE system (the proof is given in the appendix).

Theorem 3: *Consider the PDDE system of equation* (8), *for which Assumptions* 1 *and* 3 *hold. Then, there exist positive real numbers* $\bar{\mu}_1, \bar{\mu}_2, \bar{\epsilon}^*, \bar{\sigma}$ *such that if* $|x_{s0}(\xi)| \le \bar{\mu}_1$, $|x_{f0}(\xi)| \le \bar{\mu}_2$ *and* $\bar{\epsilon} \in (0, \bar{\epsilon}^*]$, *and the matrix*

$$
\begin{bmatrix} \bar{S}_1 & -\bar{S}_2^{\mathrm{T}} \\ -\bar{S}_2 & \bar{S}_3 \end{bmatrix}
\tag{39}
$$

where $\bar{S}_1, \bar{S}_2, \bar{S}_3$ *are matrices defined in the proof* (*see equation* (65)), *is positive definite, then the controller of equations* (37) *and* (38) *enforces:* (*a*) *local exponential stability, and* (*b*) *asymptotic output tracking* ($\lim_{t\to\infty}|y_{cs}(t) - v| = O(\epsilon^{k+1})$) *in the closed-loop PDDE system, independently of the size of the state delay.*

Remark 6: Owing to the infinite-dimensional range of $\Sigma_k(\omega_s(t), \omega_s(t-\alpha), u(t))$, the practical implementation of the output feedback controller of equations (37) and (38) involves approximating $\Sigma_k(\omega_s(t), \omega_s(t-\alpha), u(t))$ by a finite-dimensional vector function $\Sigma_k^t(\omega_s(t), \omega_s(t-\alpha), u(t))$, which can be derived (Christofides and Daoutides 1997) by keeping the first $\bar{m}$ elements of $\Sigma_k(\omega_s(t), \omega_s(t-\alpha), u(t))$ and neglecting the remaining infinite ones. Clearly, as $\bar{m} \to \infty$, $\Sigma_k^t(\omega_s(t), \omega_s(t-\alpha), u(t))$ approaches $\Sigma_k(\omega_s(t), \omega_s(t-\alpha), u(t))$. This implies that by picking $\bar{m}$ to be sufficiently large, the controller of equations (37) and (38) with $\Sigma_k^t(\omega_s(t), \omega_s(t-\alpha), u(t))$ instead of $\Sigma_k(\omega_s(t), \omega_s(t-\alpha), u(t))$ guarantees stability and enforces output tracking in the closed-loop parabolic PDDE system.

4.2. *Controller synthesis*

In this section, we will synthesize a non-linear output feedback controller for the system of equation (8) on the basis of the system of equation (31), using geometric control methods and Lyapunov techniques. To this end, we need to make certain assumptions on the structure and stability properties of the system of equation (13). The reader may refer to Antoniades and Christofides (1999) for details on the nature of these assumptions. To simplify the statement of these assumptions, we introduce the notation $f_{ns}(x_s(t), x_s(t-\alpha), 0, 0) = f_1(x_s(t)) + f_2(x_s(t), x_s(t-\alpha))$, $\tilde{f}_0(x_s(t)) = \bar{\mathcal{A}}_s x_s(t) + f_1(x_s(t))$, $\mathcal{B}_s = g(x_s(t), x(t-\alpha))$, $\bar{p}_0(x_s(t), x_s(t-\alpha)) = \tilde{\mathcal{A}}_s x_s(t-\alpha) + f_2(x_s(t), x_s(t-\alpha))$ and $\mathcal{C}x_s(t) = h_0(x_s(t))$ which allows us to rewrite the system of equation (13) in the form

$$
\left.
\begin{aligned}
\dot{x}_s &= \tilde{f}_0(x_s(t)) + g(x_s(t), x_s(t-\alpha))\bar{u}_0 \\
&\quad + \bar{p}_0(x_s(t), x_s(t-\alpha)) \\
y_{cs} &= h_0(x_s(t))
\end{aligned}
\right\}
\tag{40}
$$

The first assumption is motivated by the requirement of output tracking and will play a crucial role in the synthesis of the controller.

Assumption 4 (Antoniades and Christofides 1999)**:** *Referring to the system of equation* (40), *there exists an integer r and a change of variables*

$$
\begin{bmatrix} \zeta(s) \\ \eta(s) \end{bmatrix} = \begin{bmatrix} \zeta_1(s) \\ \zeta_2(s) \\ \vdots \\ \zeta_r(s) \\ \eta_1(s) \\ \vdots \\ \eta_{n-r}(s) \end{bmatrix} = \mathcal{X}(x_s(s)) = \begin{bmatrix} h_0(x_s) \\ L_{\tilde{f}_0}h_0(x_s(s)) \\ \vdots \\ L_{\tilde{f}_0}^{r-1}h_0(x_s(s)) \\ \chi_1(x_s(s)) \\ \vdots \\ \chi_{n-r}(x_s(s)) \end{bmatrix}
\tag{41}
$$

where $s \in [t-\alpha, t]$ *and* $\chi_1(x_s(s)), \ldots, \chi_{m-r}(x_s(s))$ *are scalar functions such that the system of equation* (41) *takes the form*

$$\left.\begin{aligned} \dot{\zeta}_1 &= \zeta_2(t) + p_{01}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r(t) + p_{0(r-1)}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ \dot{\zeta}_r &= L_{\tilde{f}_0}^r h_0(\mathcal{X}^{-1}(\zeta(t), \eta(t))) \\ &\quad + L_g L_{\tilde{f}_0}^{r-1} h_0(\mathcal{X}^{-1}(\zeta(t), \eta(t)))\bar{u}_0 \\ &\quad + p_{0r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ \dot{\eta}_1 &= \Psi_{01}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ &\vdots \\ \dot{\eta}_{m-r} &= \Psi_{0(m-r)}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ y_{cs} &= \zeta_1 \end{aligned}\right\} \quad (42)$$

where $p_{01}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \ldots, p_{0r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))$, $\Psi_{01}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)), \ldots, \Psi_{0(m-r)}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))$ *are non-linear Lipschitz functions and* $L_g L_{\tilde{f}_0}^{r-1} h_0(x_s) \neq 0$ *for all* $x_s(s) \in \mathbb{R}^m$ *and* $s \in [t-\alpha, t]$.

Assumption 4 provides the explicit form of a coordinate change (which is independent of the state delay present in the system of equation (31)) that transforms the non-linear DDE system of equation (31) into an interconnection of two subsystems, the ζ-subsystem which describes the input/output dynamics of the system of equation (31) and the η-subsystem which includes the dynamics of the system of equation (31) which are unobservable from the output. Specifically, the interconnection of equation (42) is obtained by considering the change of variables of equation (41) with $s = t$, differentiating it with respect to time, and using that $x(t) = \mathcal{X}^{-1}(\zeta(t), \eta(t))$ and $x(t-\alpha) = \mathcal{X}^{-1}(\zeta(t-\alpha), \eta(t-\alpha))$ (note that this is possible because the coordinate change of equation (41) is assumed to be valid for $s \in [t-\alpha, t]$). The coordinate transformation of equation (41) is not restrictive from an application point of view (one can easily verify that Assumption 4 holds, see for example the two applications studied in Section 10 of Antoniades and Christofides (1999)). The assumption that $L_g L_{\tilde{f}_0}^{r-1} h_0(x_s) \neq 0$ for all $x_s(s) \in \mathbb{R}^m$ and $s \in [t-\alpha, t]$ is necessary in order to guarantee that the controller which will be synthesized is well-posed in the sense that it does not generate infinite control action for any values of the states of the process (compare with the structure of the controller given in Theorem 4).

Assumption 5 that follows imposes a standard stability requirement on the η-subsystem of the system of equation (42) which will allow addressing the controller synthesis task on the basis of the low-order ζ-subsystem.

Assumption 5 (Antoniades and Christofides 1999)**:** *The dynamical system*

$$\left.\begin{aligned} \dot{\eta}_1 &= \Psi_{01}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ &\vdots \\ \dot{\eta}_{m-r} &= \Psi_{0(m-r)}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \end{aligned}\right\} \quad (43)$$

is input-to-state stable with respect to the input $\zeta_t(\xi)$.

Loosely speaking, the above assumption states that if the state of the ζ-subsystem is bounded, then the state of the η-subsystem will also remain bounded. In practice, Assumption 5 can be verified by linearizing the system of equation (43) with $\zeta_t(\xi) = 0$ around the operating steady-state and computing the eigenvalues of the resulting linear system. If all of these eigenvalues are in the left-half of the complex plane, then (Teel 1998) Assumption 5 is satisfied locally (i.e. for sufficiently small initial conditions and $\zeta_t(\xi)$). An application of this approach for checking Assumption 5 is discussed in subsection 10.2.2 of Antoniades and Christofides (1999).

Using Assumption 5, the controller synthesis problem can now be addressed on the basis of the ζ-subsystem. Specifically, applying the following preliminary feedback law

$$\begin{aligned} \bar{u}_0 = &\frac{1}{L_g L_{\tilde{f}_0}^{r-1} h_0(\mathcal{X}^{-1}(\zeta, \eta))} (\tilde{u}_0 - L_{\tilde{f}_0}^r h_0(\mathcal{X}^{-1}(\zeta, \eta)) \\ &- p_{0r}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))) \end{aligned} \quad (44)$$

where $\tilde{u}_0$ is an auxiliary input, to the system of equation (42) in order to cancel all the non-linear terms that can be cancelled by using a feedback which utilizes measurements of $x_s(s)$ for $s \in [t-\alpha, t]$, we obtain the modified system

$$\left.\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + p_{01}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r + p_{0(r-1)}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha))) \\ \dot{\zeta}_r &= \tilde{u}_0 \\ \dot{\eta}_1 &= \Psi_{01}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ &\vdots \\ \dot{\eta}_{m-r} &= \Psi_{0(m-r)}(\zeta(t), \eta(t), \zeta(t-\alpha), \eta(t-\alpha)) \\ y_{cs} &= \zeta_1 \end{aligned}\right\} \quad (45)$$

Introducing the notation

$$\left.\begin{aligned}\tilde{A}_0 &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \\ b_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ p_0(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha)) &= \begin{bmatrix} p_{01}(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha)) \\ p_{02}(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha)) \\ \vdots \\ p_{0(r-1)}(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha)) \\ 0 \end{bmatrix}\end{aligned}\right\} \tag{46}$$

the ζ-subsystem of the system of equation (45) can be written in the compact form

$$\left.\begin{aligned}\dot{\zeta} &= \tilde{A}_0\zeta + b_0\tilde{u}_0 + p_0(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha)) \\ y_{cs} &= \zeta_1\end{aligned}\right\} \tag{47}$$

The controller synthesis task has been now reduced to the one of synthesized $\tilde{u}_0$ to stabilize the ζ-subsystem and force the output y_{cs} to asymptotically follow the reference input, v. To develop a solution to this problem, we will need to make the following assumption on the growth of the vector $p_0(\zeta(t),\eta(t),\zeta(t-\alpha),\eta(t-\alpha))$.

Assumption 6 (Antoniades and Christofides 1999)**:** *Let*

$$\bar{e}_0(s) = [(h(x_s(s)) - v(s))\ (L_{\tilde{f}_0}h_0(x_s(s)) - v^{(1)}(s))\ \cdots\ (L_{\tilde{f}_0}^{r-1}h_0(x_s(s)) - v^{(r-1)}(s))]^{\mathrm{T}}, s \in [t-\alpha, t]$$

where $v^{(k)}$ denotes the kth time-derivative of the reference input v. There exist positive real numbers a_1, a_2 such that the following bound can be written

$$|p_0(\bar{e}_0(t)+\bar{v}(t),\eta(t),\bar{e}_0(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha))|^2_{\mathbb{R}^m} \leq a_{01}\bar{e}_0^2(t) + a_{02}\bar{e}_0^2(t-\alpha) \tag{48}$$

where $|\cdot|_{\mathbb{R}^m}$ denotes the standard Euclidean norm in $\mathbb{R}^m$.

The above assumption on the growth of the vector $p_0(\bar{e}_0(t)+\bar{v}(t),\eta(t),\bar{e}_0(t-\alpha)+\bar{v}(t-\alpha),\eta(t-\alpha))$ does not need to hold globally (i.e. for any $\bar{e}_0(t),\eta(t)$), and thus, it is satisfied by most practical problems (see, for example, the applications studied in § 10 of Antoniades and Christofides (1999)).

Theorem 4 provides the synthesis formula of the output feedback controller and conditions that guarantee closed-loop stability in the case of considering an $O(\epsilon^2)$ approximation of the exact slow system for the synthesis of the controller. The derivation of synthesis formulas for higher-order approximations of the output feedback controller is notationally complicated, although conceptually straightforward, and thus will be omitted for reasons of brevity (the proof of the theorem is similar to the one of Theorem 3 and will be omitted for brevity).

Theorem 4: *Consider the non-linear parabolic PDDE system of equation* (13) *with $\alpha > 0$ and $\tilde{\alpha} > 0$, for which Assumption 1 holds. Suppose also that the DDE system of equation* (31) *satisfies Assumptions* 4, 5 *and* 6, *L is chosen so that the observer of equation* (38) *is locally exponentially stable, and the matrix equations*

$$\tilde{A}_\nu^{\mathrm{T}}P_\nu + P_\nu\tilde{A}_\nu - 2P_\nu^{\mathrm{T}}b_\nu R_{\nu 2}^{-1}b_\nu^{\mathrm{T}}P_\nu + (a_\nu^2 + a_{\nu 1})I + P_\nu^2 = -R_{\nu 1} \tag{49}$$

where $a_\nu^2 > a_{\nu 2}$, and $R_{\nu 1}, R_{\nu 2}$ are positive definite matrices, have unique positive definite solutions for P_ν for $\nu = 0,1$. Then, there exist positive real numbers $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\epsilon}^*, \tilde{\sigma}$ such that if $|x_{s0}(\xi)| \leq \tilde{\mu}_1$, $|x_{f_0}(\xi)| \leq \tilde{\mu}_2$ and $\tilde{\epsilon} \in (0, \tilde{\epsilon}^*]$, and a matrix of the form

$$\begin{bmatrix} \tilde{S}_1 & -\tilde{S}_2^{\mathrm{T}} \\ -\tilde{S}_2 & \tilde{S}_3 \end{bmatrix} \tag{50}$$

is positive definite (the explicit form of $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3$, is omitted for brevity), the distributed output feedback controller

$$\left.\begin{aligned}\dot{\omega}_s &= \mathcal{A}_s\omega_s + \mathcal{B}_s u + f_s(\omega_s(t),\bar{\eta}_s(t-\alpha), \\ &\quad \Sigma_0(\omega_s(t),\bar{\eta}_s(t-\alpha),u(t)) \\ &\quad + \epsilon\Sigma_1(\omega_s(t),\bar{\eta}_s(t-\alpha),u(t)),\bar{\eta}_f(t-\alpha)) \\ &\quad + \Phi_{\mathcal{C}_s}(0)L(y_m(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha})) \\ &\quad + \tilde{A}_s\int_0^{\alpha-\tilde{\alpha}} \Phi_{\mathcal{C}_s}(\xi-\alpha)L(y_m(t-\xi-\tilde{\alpha}) \\ &\quad - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))\,\mathrm{d}\xi, \quad \forall\ t \in (0,\alpha] \\ \dot{\omega}_s &= \mathcal{A}_s\omega_s + \mathcal{B}_s u + f_s(\omega_s(t),\omega_s(t-\alpha), \\ &\quad \Sigma_0(\omega_s(t),\omega_s(t-\alpha),u(t)) \\ &\quad + \epsilon\Sigma_1(\omega_s(t),\omega_s(t-\alpha),u(t)), \\ &\quad \Sigma_0(\omega_s(t-\alpha),\omega_s(t-2\alpha),u(t-\alpha)) \\ &\quad + \epsilon\Sigma_1(\omega_s(t-\alpha),\omega_s(t-2\alpha),u(t-\alpha))) \\ &\quad + \Phi_{\mathcal{C}_s}(0)L(y_m(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha})) \\ &\quad + \tilde{A}_s\int_0^{\alpha-\tilde{\alpha}} \Phi_{\mathcal{C}_s}(\xi-\alpha)L(y_m(t-\xi-\tilde{\alpha}) \\ &\quad - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))\,\mathrm{d}\xi, \quad \forall\ t > \alpha\end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned} &\omega_s(\xi) = \bar{\omega}_s(\xi), \quad \xi \in [-\alpha, 0), \quad \omega_s(0) = \bar{\omega}_{s0} \\ &\quad u = \frac{1}{L_g L_{\tilde{f}_0}^{r-1} h_0(\omega_s)} (-R_{02}^{-1} b_0^{\mathrm{T}} P_0 \bar{e}_0 + v^{(r)}(t) \\ &\qquad - L_{\tilde{f}_0}^r h_0(\omega_s) - p_{0r}(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha))) \\ &\qquad + \epsilon \frac{1}{L_g L_{\tilde{f}_1}^{r-1} h_1(\omega_s)} (-R_{12}^{-1} b_1^{\mathrm{T}} P_1 \bar{e}_1 + v^{(r)}(t) \\ &\qquad - L_{\tilde{f}_1}^r h_1(\omega_s) - p_{1r}(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha))) \end{aligned}\right\} \tag{51}$$

where

$$\bar{e}_0 = [(h_0(\omega_s(t)) - v) \;\; (L_{\tilde{f}_0} h_0(\omega_s(t)) - v^{(1)}) \;\; \cdots \;\; (L_{\tilde{f}_0}^{r-1} h_0(\omega_s(t)) - v^{(r-1)})]^{\mathrm{T}}$$

and

$$\bar{e}_1 = [(h_1(\omega_s(t)) - v) \;\; (L_{\tilde{f}_1} h_1(\omega_s(t)) - v^{(1)}) \;\; \cdots \;\; (L_{\tilde{f}_1}^{r-1} h_1(\omega_s(t)) - v^{(r-1)})]^{\mathrm{T}}$$

enforces: (*a*) *local exponential stability, and* (*b*) *asymptotic output tracking* ($\lim_{t\to\infty} |y_{cs}(t) - v| = O(\epsilon^2)$) *in the closed-loop PDDE system, independently of the size of the state delay.*

Remark 7: Note that the exponential stability of the closed-loop system under the controller of equation (51) guarantees robustness with respect to sufficiently small initialization errors on the observer states, uncertainty in the model parameters and external disturbances.

Remark 8: The non-linear distributed output feedback controller of equation (51) is an infinite dimensional one owing to the infinite dimensional nature of the observer of equation (38) (it includes time delays). Therefore, finite-dimensional approximation of these controllers have to be derived for on-line implementation. This task can be performed by utilizing standard discretization techniques such as finite differences. We note that it is well-established (e.g. Soliman and Ray 1972) that as the number of discretization points increases, the closed-loop system resulting from the DDE model plus an approximate finite-dimensional controller converges to the closed-loop system resulting from the DDE model plus the infinite-dimensional controller, guaranteeing the well-posedness of the approximate finite-dimensional controller.

Remark 9: When manipulated input delay is present in the system of equation (13), the proposed approach can still be employed for controller design. The only difference is that the output feedback controller of equation (51) will be designed on the basis of an auxiliary output constructed within a state-space Smith predictor framework; the reader may refer to Antoniades and Christofides (1999) for details on how such a controller design can be performed.

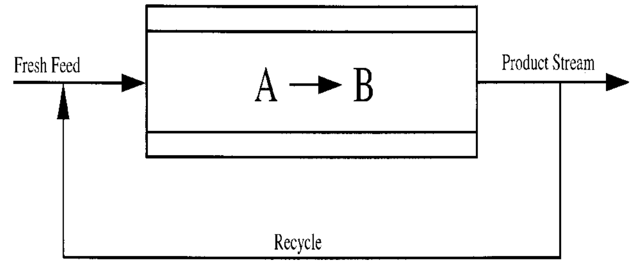

Figure 1. A tubular reactor with recycle.

5. Application to a tubular reactor with recycle

We consider a non-isothermal tubular reactor shown in figure 1, where an irreversible first-order reaction of the form $A \to B$ takes place. The reaction is exothermic and a cooling jacket is used to remove heat from the reactor. The outlet of the reactor is fed to a separator where the unreacted species A is separated from the product B. The unreacted amount of species A is then fed back to the reactor through a recycle loop. Under standard modelling assumptions, the dynamic model of the process can be derived from mass and energy balances and takes the dimensionless form

$$\left.\begin{aligned} \frac{\partial \bar{x}_1}{\partial t} &= -\frac{\partial \bar{x}_1}{\partial z} + \frac{1}{Pe_T}\frac{\partial^2 \bar{x}_1}{\partial z^2} \\ &\quad + B_T B_C \exp\left(\frac{\gamma \bar{x}_1}{1+\bar{x}_1}\right)(1+\bar{x}_2) \\ &\quad + \beta_T(b(z)u(t) - \bar{x}_1) \\ \frac{\partial \bar{x}_2}{\partial t} &= -\frac{\partial \bar{x}_2}{\partial z} + \frac{1}{Pe_C}\frac{\partial^2 \bar{x}_2}{\partial z^2} - B_C \exp\left(\frac{\gamma \bar{x}_1}{1+\bar{x}_1}\right)(1+\bar{x}_2) \end{aligned}\right\} \tag{52}$$

subject to the boundary conditions

$$\left.\begin{aligned} z = 0, \quad & \frac{\partial \bar{x}_1}{\partial z} = Pe_T(\bar{x}_1 - (1-r)x_{1f} - r\bar{x}_1(1, t-\alpha)) \\ & \frac{\partial \bar{x}_2}{\partial z} = Pe_C(\bar{x}_2 - (1-r)x_{2f} - r\bar{x}_2(1, t-\alpha)) \\ z = 1, \quad & \frac{\partial \bar{x}_1}{\partial z} = 0, \quad \frac{\partial \bar{x}_2}{\partial z} = 0 \end{aligned}\right\} \tag{53}$$

where $\bar{x}_1$ and $\bar{x}_2$ denote dimensionless temperature and concentration of species A in the reactor, respectively, Pe_T and Pe_C are the heat and thermal Peclet numbers, respectively, B_T and B_C denote a dimensionless heat of reaction and a dimensionless pre-exponential factor, respectively, r is the recirculation coefficient (it varies from zero to one, with one corresponding to total recycle and zero fresh feed and zero corresponding to no recycle), α is the recycle loop dead time, γ is a dimensionless activation energy, β_T is a dimensionless heat transfer coefficient, u is a dimensionless jacket temperature (chosen to be the manipulated input), and $b(z)$ is the actuator distribution function.

In order to transform the boundary condition of equation (53) into a homogeneous one, we insert the non-homogeneous part of the boundary condition into the differential equation and obtain the PDDE representation of the processes

$$\left.\begin{aligned}\frac{\partial \bar{x}_1}{\partial t} &= -\frac{\partial \bar{x}_1}{\partial z} + \frac{1}{Pe_T}\frac{\partial^2 \bar{x}_1}{\partial z^2} \\ &\quad + B_T B_C \exp\left(\frac{\gamma \bar{x}_1}{1+\bar{x}_1}\right)(1+\bar{x}_2) \\ &\quad + \beta_T (b(z)u(t) - \bar{x}_1) \\ &\quad + \delta(z-0)((1-r)x_{1f} + r\bar{x}_1(1,t-\alpha)) \\ \frac{\partial \bar{x}_2}{\partial t} &= -\frac{\partial \bar{x}_2}{\partial z} + \frac{1}{Pe_C}\frac{\partial^2 \bar{x}_2}{\partial z^2} - B_C \exp\left(\frac{\gamma \bar{x}_1}{1+\bar{x}_1}\right)(1+\bar{x}_2) \\ &\quad + \delta(z-0)((1-r)x_{2f} + r\bar{x}_2(1,t-\alpha))\end{aligned}\right\} \tag{54}$$

where $\delta(\cdot)$ is the standard Dirac function, subject to the homogeneous boundary conditions

$$\left.\begin{aligned} z=0, \quad \frac{\partial \bar{x}_1}{\partial z} &= Pe_T \bar{x}_1, \quad \frac{\partial \bar{x}_2}{\partial z} = Pe_C \bar{x}_2 \\ z=1, \quad \frac{\partial \bar{x}_1}{\partial z} &= 0, \quad \frac{\partial \bar{x}_2}{\partial z} = 0 \end{aligned}\right\} \tag{55}$$

The following values for the process parameters were used in our calculations:

$$\left.\begin{aligned} Pe_T = 7.0, \quad Pe_C = 7.0, \quad B_C = 0.1, \quad B_T = 2.5 \\ \beta_T = 2.0, \quad \gamma = 10.0, \quad r = 0.5, \quad \alpha = 0.1 \end{aligned}\right\} \tag{56}$$

For the above values, the operating steady-state of the open-loop system is unstable (the linearization around the steady-state possesses one real unstable eigenvalue, $\mu = 0.0328$, and infinitely many stable eigenvalues), thereby implying the need to operate the process under feedback control. We note that in the absence of recycle-loop (i.e. $r = 0$), the above process parameters correspond to a unique stable steady-state for the open-loop system.

The spatial differential operator of the system of equation (54) is of the form

$$\left.\begin{aligned}\mathcal{A}x &= \begin{bmatrix} \mathcal{A}_1 x_1 & 0 \\ 0 & \mathcal{A}_2 x_2 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1}{Pe_T}\dfrac{\partial^2 \bar{x}_1}{\partial z^2} - \dfrac{\partial \bar{x}_1}{\partial z} & 0 \\ 0 & \dfrac{1}{Pe_C}\dfrac{\partial^2 \bar{x}_2}{\partial z^2} - \dfrac{\partial \bar{x}_2}{\partial z} \end{bmatrix}\end{aligned}\right\} \tag{57}$$

The solution of the eigenvalue problem for $\mathcal{A}_i$ can be obtained by utilizing standard techniques from linear operator theory (see, for example, Ray 1981) and is of the form

$$\left.\begin{aligned} \lambda_{ij} &= \frac{\bar{a}_{ij}^2}{Pe} + \frac{Pe}{4}, \quad i=1,2, \ j=1,\ldots,\infty \\ \phi_{ij}(z) &= B_{ij}\, e^{Pe(z/2)}\left(\cos(\bar{a}_{ij}z) + \frac{Pe}{2\bar{a}_{ij}}\sin(\bar{a}_{ij}z)\right), \\ &\qquad i=1,2, \ j=1,\ldots,\infty \\ \bar{\phi}_{ij}(z) &= e^{-Pez}\phi_{ij}(z), \quad i=1,2, \ j=1,\ldots,\infty \end{aligned}\right\} \tag{58}$$

where $Pe = Pe_T = Pe_C$, and $\lambda_{ij}, \phi_{ij}, \bar{\phi}_{ij}$, denote the eigenvalues, eigenfunctions and adjoint eigenfunctions of $\mathcal{A}_i$, respectively. $\bar{a}_{ij}, B_{ij}$ can be calculated from the formulas

$$\left.\begin{aligned} \tan(\bar{a}_{ij}) &= \frac{Pe\bar{a}_{ij}}{\bar{a}_{ij}^2 - (Pe/2)^2}, \quad i=1,2, \ j=1,\ldots,\infty \\ B_{ij} &= \left\{\int_0^1 \left(\cos(\bar{a}_{ij}z) + \frac{Pe}{2\bar{a}_{ij}}\sin(\bar{a}_{ij}z)\right)^2 dz\right\}^{-1/2}, \\ &\qquad i=1,2, \ j=1,\ldots,\infty \end{aligned}\right\} \tag{59}$$

A direct computation of the first nine eigenvalues of $\mathcal{A}$ yields: $\lambda_{11} = \lambda_{21} = -2.36$, $\lambda_{12} = \lambda_{22} = -4.60$, $\lambda_{13} = \lambda_{23} = -9.13$, $\lambda_{14} = \lambda_{24} = -16.30$, $\lambda_{15} = \lambda_{25} = -26.22$, $\lambda_{16} = \lambda_{26} = -38.94$, $\lambda_{17} = \lambda_{27} = -54.47$, $\lambda_{18} = \lambda_{28} = -72.81$, and $\lambda_{19} = \lambda_{29} = -93.96$.

A 400th order Galerkin truncation of the system of equations (54)–(56) was used in our simulations in order to accurately describe the process (further increase on the order of the Galerkin truncation was found to give negligible improvement on the accuracy of the results). Figure 2 shows the open-loop profile of $\bar{x}_1$ along the length of the reactor which corresponds to the operating unstable steady-state. Therefore, the control problem is to manipulate the wall temperature, $u(t)$, in order to stabilize the reactor at the open-loop unstable steady-state. The controlled output was defined as $y_c(t) = \int_0^1 e^{-Pez}\phi_{11}(z)\bar{x}_1\, dz$, the actuator distribution function was taken to be $b(z) = 1$ (uniform wall temperature in space), and the measurement sensor shape function was assumed to be of the form $s(z) = e^{-Pez}\phi_{11}(z)$ (distributed sensing).

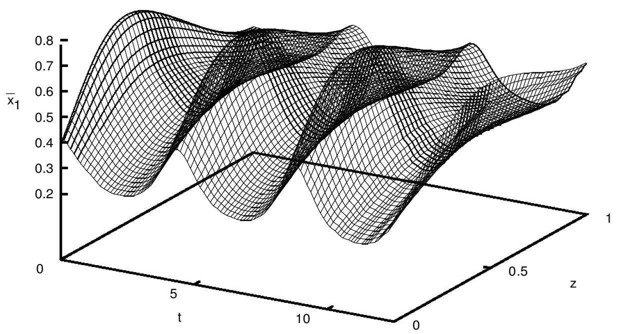

Figure 2. Spatiotemporal evolution of $\bar{x}_1$ in the open-loop system.

Two sets of simulation runs were performed to: (a) evaluate the improvement on the performance of the controller which is achieved when the controller is synthesized on the basis of DDE models derived from combination of Galerkin's method with approximate inertial manifolds (first set of simulations), and (b) compare the performance of the proposed non-linear controllers with non-linear controllers which are designed on the basis of approximate ODE models which do not account for the time delay in the recycle loop (second set of simulations). In all the simulations runs, the process was initially ($t = 0$) assumed to be at the unstable steady state, and the reference input value was set at $v = 0.12$.

In the first set of simulations, we assumed the time-delay of the recycle loop to be $\alpha = 0.1$. We initially employed the standard Galerkin's method to derive an approximate DDE system which was used for the synthesis of a non-linear output feedback controller. It was found that the *lowest-order* DDE model obtained from standard Galerkin's method which leads to the synthesis of a controller that stabilizes the closed-loop system is 10 (i.e. $m = 10$, $\tilde{m} = 0$). Figures 3 and 4 (short-dashed lines) show the evolution of the output of the closed-loop system and the profile of the manipulated input under a non-linear output feedback controller of the form of equation (51) synthesized on the basis of the approximate model comprised of ten differential difference equations (the controller parameters are $\epsilon = 0$, $\Phi_{C_s}(0) = [1\ 1\ \ldots\ 1\ 1]^{\mathrm{T}}_{10\times 1}$ and $L = 2$). It is clear that this controller stabilizes the system far away from the desired reference input value. Figure 6 (short-dashed line) shows the corresponding closed-loop steady-state profile of $\bar{x}_1$, which is clearly not close to the desired one (dotted line). We subsequently used the proposed combination of Galerkin's method with approximate inertial manifolds to derive a DDE system which was used for the synthesis of a non-linear output feedback controller. We constructed a 10th order DDE model which uses a 12th order approximation for the state x_f (i.e. $m = 10$, $\tilde{m} = 12$) and used it for the design of a non-linear output

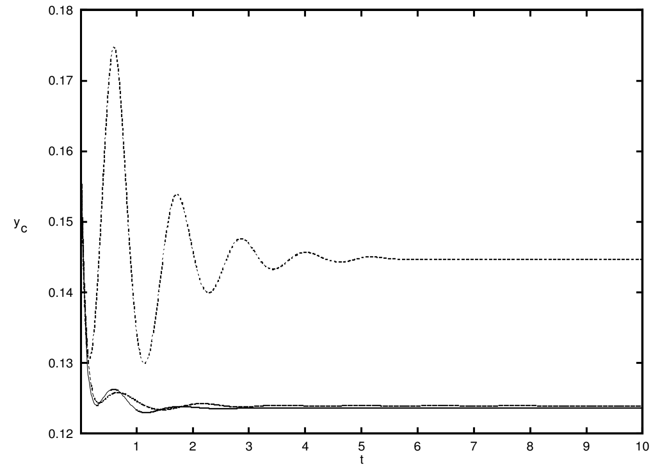

Figure 3. Closed-loop controlled output profiles under a controller of the form of equation (51) with $m = 10$, $\tilde{m} = 12$ (solid line), a controller of the form of equation (51) with $m = 10$, $\tilde{m} = 0$ (short-dashed line), and a controller of the form of equation (51) with $m = 22$, $\tilde{m} = 0$ (long-dashed line).

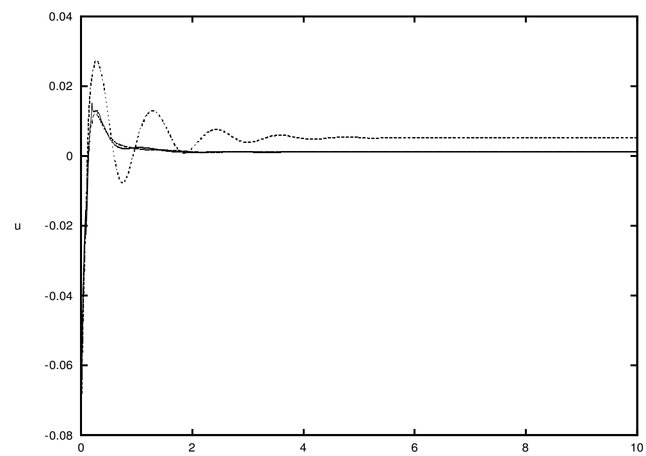

Figure 4. Manipulated input profiles under a controller of the form of equation (51) with $m = 10$, $\tilde{m} = 12$ (solid line), a controller of the form of equation (51) with $m = 10$, $\tilde{m} = 0$ (short-dashed line), and a controller of the form of equation (51) with $m = 22$, $\tilde{m} = 0$ (long-dashed line).

feedback controller of the form of equation (51) (the controller parameters are $\epsilon = 0.016$, $\Phi_{C_s}(0) = [1\ 1\ \ldots\ 1\ 1]^{\mathrm{T}}_{10\times 1}$ and $L = 2$). Figures 3 and 4 (solid lines) show the evolution of the output of the closed-loop system and the profile of the manipulated input, while figure 5 shows the spatiotemporal evolution of $\bar{x}_1$ and figure 6 (solid line) shows the final steady-state profile of $\bar{x}_1$. This controller clearly regulates the system very close to the reference input value. Finally, we also implemented on the process a non-linear output feedback controller of the form of equation (51) which was synthesized on the basis of a DDE model obtained with

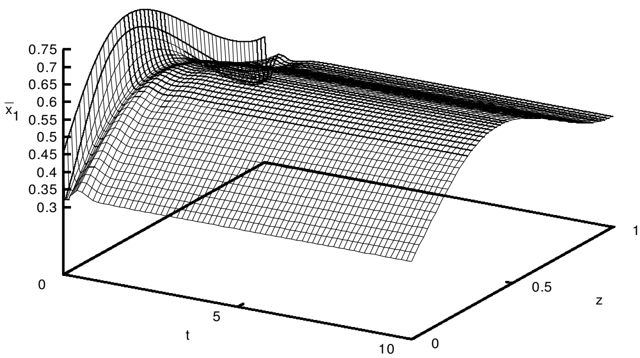

Figure 5. Spatiotemporal evolution of $\bar{x}_1$ under a controller of the form of equation (51) with $m = 10$, $\tilde{m} = 12$.

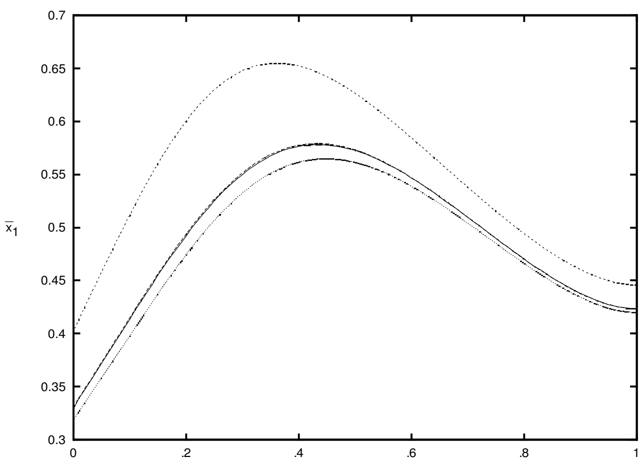

Figure 6. Final steady-state profile of $\bar{x}_1$ under a controller of the form of equation (51) with $m = 10$, $\tilde{m} = 12$ (solid line), a controller of the form of equation (51) with $m = 10$, $\tilde{m} = 0$ (short-dashed line), a controller of the form of equation (51) with $m = 22$, $\tilde{m} = 0$ (long-dashed line), and profile of the desired operating steady state (dotted line).

$m = 22$ and $\tilde{m} = 0$ (the controller parameters are $\epsilon = 0$, $\Phi_{\mathcal{C}_s}(0) = [1\ 1\ \dots\ 1\ 1]^{\mathrm{T}}_{22\times 1}$ and $L = 2$). Figures 2 and 3 (long-dashed lines) show the corresponding closed-loop output and manipulated input profiles, while figure 6 (long-dashed line) shows the profile of $\bar{x}_1$. The performance of this controller is very close to the one of the controller synthesized on the basis of the DDE model obtained with $m = 10$ and $\tilde{m} = 12$, indicating the effectiveness of the proposed approach for the synthesis of high-performance controllers based on low-order DDE systems.

In the second set of simulation runs, our objective is to demonstrate that the closed-loop performance of the proposed non-linear controllers is superior to the one of non-linear controllers which are designed on the basis of approximate ODE models which do not account for the time delay in the recycle loop. To this end, we assumed a larger time-delay for the recycle loop, $\alpha = 0.5$. We initially used the proposed combination of Galerkin's method with approximate inertial manifolds to construct a 22nd order DDE model which uses a 24th order approximation for the state x_f (i.e. $m = 20$, $\tilde{m} = 24$) and used it for the design of a non-linear output feedback controller of the form of equation (51) (the controller parameters are $\epsilon = 0.0606$, $\Phi_{\mathcal{C}_s}(0) = [1\ 1\ \dots\ 1\ 1]^{\mathrm{T}}_{22\times 1}$ and $L = 2$). Figures 7 and 8 (solid lines) show the evolution of the output of the closed-loop system and the profile of the manipulated input,

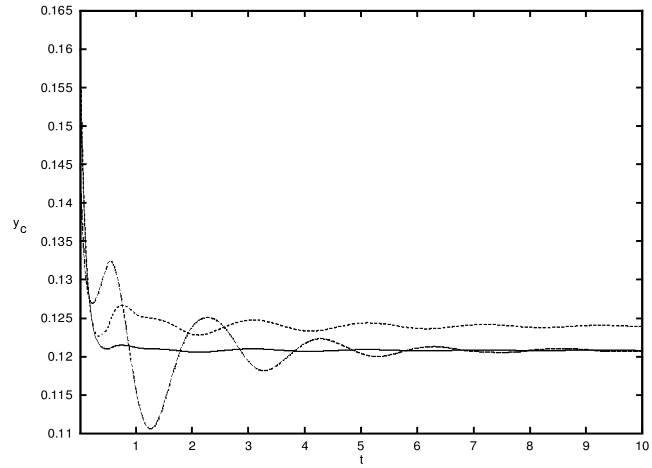

Figure 7. Closed-loop controlled output profiles under a controller of the form of equation (51), with $\alpha = 0.5$, and $m = 22$, $\tilde{m} = 24$ (solid line), a controller of the form of equation (51), with $\alpha = 0$, and $m = 22$, $\tilde{m} = 24$ (long-dashed line), and a controller of the form of equation (51), with $\alpha = 0.5$, and $m = 22$, $\tilde{m} = 0$ (short-dashed line).

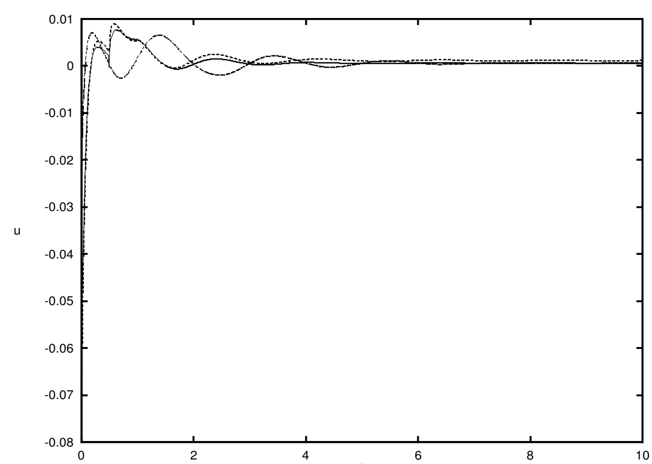

Figure 8. Manipulated input profiles under a controller of the form of equation (51), with $\alpha = 0.5$, and $m = 22$, $\tilde{m} = 24$ (solid line), a controller of the form of equation (51), with $\alpha = 0$, and $m = 22$, $\tilde{m} = 24$ (long-dashed line), and a controller of the form of equation (51), with $\alpha = 0.5$, and $m = 22$, $\tilde{m} = 0$ (short-dashed line).

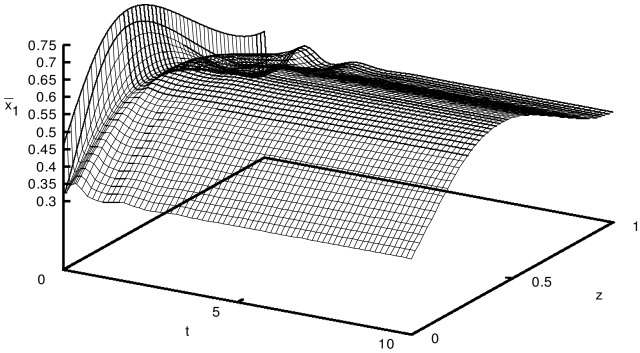

Figure 9. Spatiotemporal evolution of $\bar{x}_1$ under a controller of the form of equation (51), with $\alpha = 0.5$, and $m = 22$, $\tilde{m} = 24$ (solid line).

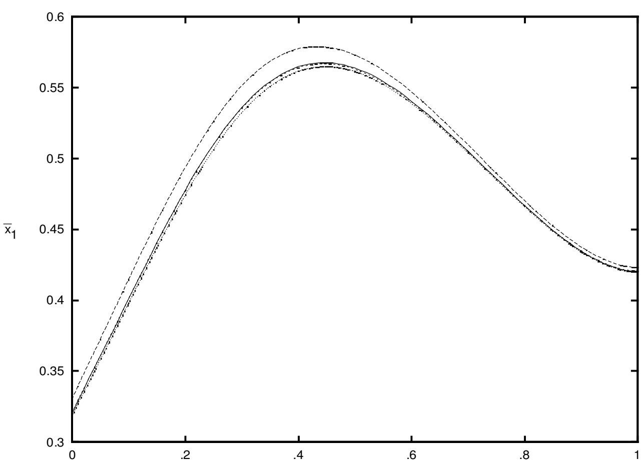

Figure 10. Final steady-state profile of $\bar{x}_1$ under a controller of the form of equation (51), with $\alpha = 0.5$, and $m = 22$, $\tilde{m} = 24$ (solid line), a controller of the form of equation (51), with $\alpha = 0$, and $m = 22$, $\tilde{m} = 24$ (short-dashed line), and a controller of the form of equation (51), with $\alpha = 0.5$, and $m = 22$, $\tilde{m} = 0$ (long-dashed line), and profile of the desired operating steady state (dotted line).

while figure 9 shows the spatiotemporal evolution of $\bar{x}_1$ and figure 10 (solid line) shows the final steady-state profile of $\bar{x}_1$. This controller clearly drives quickly the output very close to the reference input value, exhibiting a very good transient response. Note that owing to the larger time-delay (0.5 versus 0.1), the order of the DDE model needed to derive a stabilizing controller is higher (22 versus 10). To demonstrate the effect of the time delay on the closed-loop performance, we implemented on the process the non-linear output feedback controller used in the previous simulation run with $\alpha = 0$ (i.e. the time-delay is not accounted for in the controller; the other controller parameters are the same as in the previous run). Figures 7 and 8 (long-dashed lines) show the evolution of the output of the closed-loop system and the profile of the manipulated input, while figure 10 (short-dashed line) shows the final steady-state profile of $\bar{x}_1$. This controller yields a very poor transient response before driving the output close to the new reference input value. Finally, we used Galerkin's method to construct a 22nd order DDE model x_f (i.e. $m = 22$, $\tilde{m} = 0$) and used it for the design of a non-linear output feedback controller of the form of equation (51). Figures 7 and 8 (short-dashed lines) show the evolution of the output of the closed-loop system and the profile of the manipulated input, while figure 10 (long-dashed line) shows the corresponding closed-loop steady-state profile of $\bar{x}_1$, which is not close to the desired one (dotted line). It is clear that this controller stabilizes the system away from the desired reference input value.

The results of the above simulation study clearly indicate the effectiveness of the proposed approach for the synthesis of non-linear feedback controllers based on low-order DDE systems, as well as the importance of accounting for the effect of time-delays in the controller design.

6. Conclusions

In this article, we introduced a methodology for the synthesis of non-linear output feedback controllers for non-linear parabolic PDDE systems. Initially, a non-linear model reduction scheme which is based on combination of Galerkin's method with the concept of approximate inertial manifold was employed for the derivation of DDE systems that accurately capture the dominant dynamics of the PDDE system. These DDE systems were subsequently used as the basis for the explicit construction of non-linear output feedback controllers through combination of geometric and Lyapunov techniques. The controllers guarantee stability and enforce output tracking in the closed-loop parabolic PDDE system, independently of the size of the state delays, provided that the separation of the slow and fast eigenvalues of the spatial differential operator is sufficiently large and an appropriate matrix is positive definite. The methodology was successfully employed to stabilize the temperature profile of a tubular reactor with recycle at a spatially non-uniform unstable steady-state.

Acknowledgments

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Appendix

Proof of Theorem 2: The system of equation (25), with $u \equiv 0$, can be equivalently written as

$$\left.\begin{aligned} \frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{A}_s x_s + f_s(x_s(t), x_s(t-\alpha), 0, 0) \\ &\quad + [f_s(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\ &\quad - f_s(x_s, x_s(t-\alpha), 0, 0)] \\ \epsilon \frac{\mathrm{d}x_f}{\mathrm{d}t} &= \mathcal{A}_{f\epsilon} x_f + \epsilon f_f(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \end{aligned}\right\} \tag{60}$$

Let μ_1^*, μ_2^* with $\mu_1^* \geq a_5$ and $\mu_2^* \geq b_5$ be two positive real numbers such that if $||x_s||_2 \leq \mu_1^*$ and $\|x_f\|_2 \leq \mu_2^*$, then there exist positive real numbers $(k_1, k_2, k_3, k_4, k_5, k_6)$ such that

$$\left.\begin{aligned} &\| f_s(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\ &\qquad - f_s(x_s(t), x_s(t-\alpha), 0, 0)\|_2 \\ &\qquad \leq k_1 \|x_f(t)\|_2 + k_2 \|x_f(t-\alpha)\|_2 \\ &\| f_f(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha))\|_2 \\ &\qquad \leq k_3 \|x_s(t)\|_2 + k_4 \|x_s(t-\alpha)\|_2 \\ &\qquad + k_5 \|x_f(t)\|_2 + k_6 \|x_f(t-\alpha)\|_2 \end{aligned}\right\} \tag{61}$$

Furthermore, since the fast subsystem of equation (26) is exponentially stable, there exists a smooth Lyapunov functional $W\colon \mathcal{C}_f \rightarrow \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(b_1, b_2, b_3, b_4, b_5, b_6, b_7)$, such that for all $x_f \in \mathcal{H}_f$ that satisfy $\|x_f\|_2 \leq b_7$, the following conditions hold

$$\left.\begin{aligned} &b_1 ||x_f(t)||_2^2 \leq W(x_{f_t}) \leq b_2 |x_{f_t}|_2^2 \\ &\dot{W}(x_{ft}) = \frac{1}{\epsilon}\frac{\partial W}{\partial x_f} \mathcal{A}_{f\epsilon} x_f(t) \leq -\frac{b_3}{\epsilon} ||x_f(t)||_2^2 \\ &\qquad - b_4 ||x_f(t-\alpha)||_2^2 \\ &\left\|\frac{\partial W}{\partial x_f}\right\|_2 \leq b_5 ||x_f(t)||_2 + b_6 ||x_f(t-\alpha)||_2 \end{aligned}\right\} \tag{62}$$

Pick $\mu_1 < a_7 < \mu_1^*$ and $\mu_2 < b_7 < \mu_2^*$ and consider the smooth time-varying function $L\colon \mathcal{C}_s \times \mathcal{C}_f \rightarrow \mathbb{R}_{\geq 0}$:

$$L(x_{st}, x_{f_t}) = V(x_{st}) + W(x_{f_t}) \tag{63}$$

where $V(x_{st})$ was defined in Assumption 2, as a Lyapunov function candidate for the system of equation (60). From equations (32) and (62), we have that $L(x_{st}, x_{f_t})$ is positive definite, proper (tends to $+\infty$ as $|x_{st}| \rightarrow \infty$, or $|x_{f_t}| \rightarrow \infty$) and decrescent, with respect to its arguments. Computing the time-derivative of L along the trajectories of the system of equation (60), and using the bounds of equations (32) and (62) and the estimates of equation (61), the following expressions can be obtained

$$\left.\begin{aligned} \dot{L}(x_{st}, x_f) &= \frac{\partial V}{\partial x_s}\dot{x}_s + \frac{\partial W}{\partial x_f}\dot{x}_f \\ &= \frac{\partial V}{\partial x_s}[\mathcal{A}_s x_s + f_s(x_s(t), x_s(t-\alpha), 0, 0)] \\ &\quad + \frac{\partial V}{\partial x_s}[f_s(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\ &\quad - f_s(x_s(t), x_s(t-\alpha), 0, 0)] + \frac{1}{\epsilon}\frac{\partial W}{\partial x_f}\mathcal{A}_{f\epsilon} x_f \\ &\quad + \frac{\partial W}{\partial x_f} f_f(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha)) \\ &\leq -a_3 ||x_s(t)||_2^2 - a_4 ||x_s(t-\alpha)||_2^2 \\ &\quad + (a_5 ||x_s(t)||_2 + a_6 ||x_s(t-\alpha)||_2) \\ &\quad \times (k_1 \|x_f(t)\|_2 + k_2 \|x_f(t-\alpha)\|_2) \\ &\quad - \frac{b_3}{\epsilon} \|x_f\|_2^2 - b_4 ||x_f(t-\alpha)||_2^2 \\ &\quad + (b_5 ||x_f(t)||_2 + b_6 ||x_f(t-\alpha)||_2) \\ &\quad \times (k_3 ||x_s(t)||_2 + k_4 ||x_s(t-\alpha)||_2 \\ &\quad + k_5 \|x_f(t)\|_2 + k_6 \|x_f(t-\alpha)\|_2) \\ &\leq -\left[||x_s(t)||_2 \quad \|x_f(t)\|_2 \right] \\ &\quad \times \begin{bmatrix} a_3 & -\dfrac{a_5 k_1 + b_5 k_3}{2} \\ -\dfrac{a_5 k_1 + b_5 k_3}{2} & \dfrac{b_3}{\epsilon} - b_5 k_5 \end{bmatrix} \\ &\quad \times \begin{bmatrix} ||x_s(t)||_2 \\ \|x_f(t)\|_2 \end{bmatrix} \\ &\quad + 2\left[||x_s(t)||_2 \quad \|x_f(t)\|_2 \right] \\ &\quad \times \begin{bmatrix} \dfrac{a_5 k_2}{2} & \dfrac{b_6 k_3}{2} \\ \dfrac{a_6 k_1 + b_5 k_4}{2} & \dfrac{b_5 k_6 + b_6 k_5}{2} \end{bmatrix} \\ &\quad \times \begin{bmatrix} ||x_s(t-\alpha)||_2 \\ \|x_f(t-\alpha)\|_2 \end{bmatrix} \\ &\quad - \left[||x_s(t-\alpha)||_2 \quad \|x_f(t-\alpha)\|_2 \right] \\ &\quad \times \begin{bmatrix} a_4 & -\dfrac{a_6 k_2 + b_6 k_4}{2} \\ -\dfrac{a_6 k_2 + b_6 k_4}{2} & b_4 - b_6 k_6 \end{bmatrix} \end{aligned}\right\} \tag{64}$$

$$\left.\begin{aligned}
&\times\begin{bmatrix}||x_s(t-\alpha)||_2\\ \|x_f(t-\alpha)\|_2\end{bmatrix}\\
&\leq -\big[||x_s(t)||_2 \quad \|x_f(t)\|_2\big]S_1\begin{bmatrix}||x_s(t)||_2\\ \|x_f(t)\|_2\end{bmatrix}\\
&\quad +2\big[||x_s(t)||_2 \quad \|x_f(t)\|_2\big]S_2\\
&\quad\times\begin{bmatrix}||x_s(t-\alpha)||_2\\ \|x_f(t-\alpha)\|_2\end{bmatrix}\\
&\quad -\big[||x_s(t-\alpha)||_2 \quad \|x_f(t-\alpha)\|_2\big]S_3\\
&\quad\times\begin{bmatrix}|x_s(t-\alpha)|\\ \|x_f(t-\alpha)\|_2\end{bmatrix}\\
&\leq -\big[||x_s(t)||_2 \ \|x_f(t)\|_2 \ \|x_s(t-\alpha)\|_2 \ \|x_f(t-\alpha)\|_2\big]\\
&\quad\times\begin{bmatrix}S_1 & -S_2^{\mathrm{T}}\\ -S_2 & S_3\end{bmatrix}\begin{bmatrix}||x_s(t)||_2\\ \|x_f(t)\|_2\\ ||x_s(t-\alpha)||_2\\ \|x_f(t-\alpha)\|_2\end{bmatrix}
\end{aligned}\right\}\tag{64}$$

where S_1, S_2, S_3 are constant matrices of the form

$$\left.\begin{aligned}
S_1 &= \begin{bmatrix} a_3 & -\dfrac{a_5k_1+b_5k_3}{2}\\ -\dfrac{a_5k_1+b_5k_3}{2} & \dfrac{b_3}{\epsilon}-b_5k_5\end{bmatrix}\\
S_2 &= \begin{bmatrix} \dfrac{a_5k_2}{2} & \dfrac{b_6k_3}{2}\\ \dfrac{a_6k_1+b_5k_4}{2} & \dfrac{b_5k_6+b_6k_5}{2}\end{bmatrix}\\
S_3 &= \begin{bmatrix} a_4 & -\dfrac{a_6k_2+b_6k_4}{2}\\ -\dfrac{a_6k_2+b_6k_4}{2} & b_4-b_6k_6\end{bmatrix}
\end{aligned}\right\}\tag{65}$$

Picking

$$\epsilon^* = \frac{(a_3-\sqrt{\delta})b_3}{(a_3-\sqrt{\delta})(b_5k_5+\sqrt{\delta})+\left(\dfrac{a_5k_1+b_5k_3}{2}\right)^2}$$

where $\delta \in \mathbb{R}_{>0}$, we have that if $\epsilon \in (0,\epsilon^*]$ the matrix S_1 is positive definite. Then, under the assumption that the matrix

$$\begin{bmatrix}S_1 & -S_2^{\mathrm{T}}\\ -S_2 & S_3\end{bmatrix}\tag{66}$$

is positive definite, one can show (see Hale and Lunel 1993 for details) that there exists a $\delta_1 \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}\dot{L}(x_{s_t}, x_{f_t}) \leq -\delta_1(||x_s(t)||_2^2 + \|x_f(t)\|_2^2\\ + ||x_s(t-\alpha)||_2^2 + \|x_f(t-\alpha)\|_2^2)\end{aligned}\tag{67}$$

which from the properties of L directly implies that the state of the system of equation (60) is exponentially stable, i.e. there exists positive real numbers $K \geq 1, \sigma$ such that the relation of equation (34) holds. □

Proof of Theorem 3: Substituting the output feedback controller of equations (37) and (38) into the system of equation (25) and defining $\Sigma_0(\omega_s(t), \bar{\eta}_s(t-\alpha), u(t)) + \cdots + \epsilon^k\Sigma_k(\omega_s(t), \bar{\eta}_s(t-\alpha), u(t)), \bar{\eta}_f(t-\alpha) =: \Sigma^k(\omega_s(t), \bar{\eta}_s(t-\alpha), u(t))$, we get

$$\left.\begin{aligned}
\dot{\omega}_s &= \mathcal{A}_s\omega_s + \mathcal{B}_s\mathcal{A}_\epsilon(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha))\\
&\quad + f_s(\omega_s(t), \bar{\eta}_s(t-\alpha), \Sigma^k(\omega_s(t), \bar{\eta}_s(t-\alpha), u(t)),\\
&\quad \bar{\eta}_f(t-\alpha))\\
&\quad + \Phi_{\mathcal{C}_s}(0)L(y_m(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha}))\\
&\quad + \tilde{\mathcal{A}}_s\int_0^{\alpha-\tilde{\alpha}}\Phi_{\mathcal{C}_s}(\xi-\alpha)L(y_m(t-\xi-\tilde{\alpha})\\
&\quad - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))\,\mathrm{d}\xi, \quad \forall\ t \in (0,\alpha]\\
\dot{\omega}_s &= \mathcal{A}_s\omega_s + \mathcal{B}_s\mathcal{A}_\epsilon(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha))\\
&\quad + f_s(\omega_s(t), \omega_s(t-\alpha), \Sigma^k(\omega_s(t), \omega_s(t-\alpha), u(t)),\\
&\quad \Sigma^k(\omega_s(t-\alpha), \omega_s(t-2\alpha), u(t-\alpha)))\\
&\quad + \Phi_{\mathcal{C}_s}(0)L(y_m(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha}))\\
&\quad + \tilde{\mathcal{A}}_s\int_0^{\alpha}\Phi_{\mathcal{C}_s}(\xi-\alpha)L(y_m(t-\xi-\tilde{\alpha})\\
&\quad - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))d\xi, \quad \forall\ t > \alpha\\
\frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathcal{A}_s x_s(t) + \mathcal{B}_s\mathcal{A}_\epsilon(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha))\\
&\quad + f_s(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha))\\
\epsilon\frac{\mathrm{d}x_f}{\mathrm{d}t} &= \mathcal{A}_{f\epsilon}x_f(t) + \epsilon\mathcal{B}_f\mathcal{A}_\epsilon(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha))\\
&\quad + \epsilon f_f(x_s(t), x_s(t-\alpha), x_f(t), x_f(t-\alpha))\\
y_{cs} &= \mathcal{C}x_s(t) + \mathcal{C}x_f(t)
\end{aligned}\right\}\tag{68}$$

The fast dynamics of the above closed-loop system are described by the system

$$\frac{\mathrm{d}x_f}{\mathrm{d}\tau} = \mathcal{A}_{f\epsilon}x_f\tag{69}$$

which is exponentially stable (Assumption 2). The $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form is

$$
\left.\begin{aligned}
\dot{\omega}_s &= \mathcal{A}_s\omega_s + \mathcal{B}_s\mathcal{A}_\epsilon(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha)) \\
&\quad + f_s(\omega_s(t), \bar{\eta}_s(t-\alpha), \Sigma^k(\omega_s(t), \bar{\eta}_s(t-\alpha), u(t)), \\
&\quad \bar{\eta}_f(t-\alpha)) \\
&\quad + \Phi_{\mathcal{C}_s}(0)L(y_{ms}(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha})) \\
&\quad + \tilde{\mathcal{A}}_s\int_0^{\alpha-\tilde{\alpha}} \Phi_{\mathcal{C}_s}(\xi-\alpha)L(y_{ms}(t-\xi-\tilde{\alpha}) \\
&\quad - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))\,d\xi, \quad \forall\, t \in [0,\alpha] \\
\dot{\omega}_s &= \mathcal{A}_s\omega_s + \mathcal{B}_s\mathcal{A}_\epsilon(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha)) \\
&\quad + f_s(\omega_s(t), \omega_s(t-\alpha), \Sigma^k(\omega_s(t), \omega_s(t-\alpha), u(t)), \\
&\quad \Sigma^k(\omega_s(t-\alpha), \omega_s(t-2\alpha), u(t-\alpha))) \\
&\quad + \Phi_{\mathcal{C}_s}(0)L(y_{ms}(t-\tilde{\alpha}) - \mathcal{S}\omega_s(t-\tilde{\alpha})) \\
&\quad + \tilde{\mathcal{A}}_s\int_0^{\alpha-\tilde{\alpha}} \Phi_{\mathcal{C}_s}(\xi-\alpha)L(y_{ms}(t-\xi-\tilde{\alpha}) \\
&\quad - \mathcal{S}\omega_s(t-\xi-\tilde{\alpha}))\,d\xi, \quad \forall\, t > \alpha \\
\frac{dx_s}{dt} &= \mathcal{A}_s x_s(t) + \mathcal{B}_s\mathcal{A}_\epsilon(\omega_s(t), \bar{v}(t), \omega_s(t-\alpha), \bar{v}(t-\alpha)) \\
&\quad + f_s(x_s(t), x_s(t-\alpha), \Sigma^k(x_s(t), x_s(t-\alpha), u(t)), \\
&\quad \Sigma^k(x_s(t-\alpha), x_s(t-2\alpha), u(t-\alpha))) \\
y_{cs} &= \mathcal{C}x_s(t) + \epsilon\mathcal{C}(\Sigma_0(x_s(t), x_s(t-\alpha), u(t)) + \cdots \\
&\quad + \epsilon^k\Sigma_k(x_s(t), x_s(t-\alpha), u(t)))
\end{aligned}\right\} \tag{70}
$$

From Assumption 3, we have that the above system is exponentially stable and the output asymptotically follows the reference input ($\lim_{t\to\infty}|y_{cs}(t) - v| = 0$). To conclude the proof of the theorem, we need to utilize the result of Theorem 2. Specifically, the exponential stability of the systems of equations (69) and (70) implies that there exist positive real numbers $\bar{\mu}_1, \bar{\mu}_2, \bar{\epsilon}^*$ such that if $|x_{s0}(\xi)| \le \bar{\mu}_1$, $|x_{f0}(\xi)| \le \bar{\mu}_2$ and $\epsilon \in (0, \bar{\epsilon}^*]$, and the matrix

$$
\begin{bmatrix} \bar{S}_1 & -\bar{S}_2^{\mathrm{T}} \\ -\bar{S}_2 & \bar{S}_3 \end{bmatrix} \tag{71}
$$

is positive definite, then the closed-loop system of equation (68) is exponentially stable and the closed-loop output satisfies $\lim_{t\to\infty}|y_c(t) - v| = O(\epsilon^{k+1})$. We note that $\bar{S}_1, \bar{S}_2, \bar{S}_3$ are constant matrices whose structure is similar to the one shown in equation (65); the computation of their explicit forms requires defining the appropriate bounds for the non-linearities of the system of equation (68) and is omitted for brevity. □

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