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# Estimation-Based Networked Predictive Control of Nonlinear Systems

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**Abstract.** In this work we focus on estimation-based networked predictive control of nonlinear systems. We propose an output feedback controller based on the combination of a high-gain observer with a Lyapunov-based model predictive controller that takes data losses explicitly into account both in the controller design and in the implementation. We provide precise bounds on the data loss sequence such that stability of the closed-loop system is guaranteed. The theoretical results are demonstrated using a chemical process example.

**Keywords.** Networked control systems; Nonlinear systems; Output feedback control; High-gain observers; Lyapunov-based model predictive control; Uncertain systems.

## 1 Introduction

New developments in wireless technologies have generated a vast field of new applications for wireless networked control systems like wireless sensor networks [1, 3] and cooperative control of multi-agent systems [2, 21]. Wireless networks can improve the flexibility and fault-tolerance of chemical processes, see for example [12]. From a control theory point of view, networks introduce new dynamics in the closed-loop system which have to be taken into account at the controller design stage. For example, wireless sensors may be characterized by a significant data loss rate. The reader may refer to [22, 18] and the references therein for a survey on networked control applications.

The stability and robustness properties of nonlinear systems under state feedback control in the presence of data losses have been studied in [20, 17, 12]. However, these results are based on the assumption that full state measurements are available. In many applications, this assumption does not hold and an output feedback control scheme has to be used [8, 6, 4]. Recently, output feedback control of nonlinear systems subject to data losses has been studied in [16]. In [16], however, when data losses occur the control actuator output is fixed at the last available input.

In the present work, we adopt a different strategy based on using an estimate of the state computed via the nominal model of the plant to decide the control actuator ouput over the period of time in which feedback is lost between consecutively received measurements, see for example [13, 14]. In particular, we use Lyapunov-based model predictive control (LMPC), see [10, 11]. Lyapunov-based MPC is based on uniting receding horizon control with control Lyapunov functions and computes the manipulated input trajectory solving a finite-horizon constrained optimal control problem. When data losses occur, we update the input taking advantage of the predictions obtained from the solution of the optimization problem.

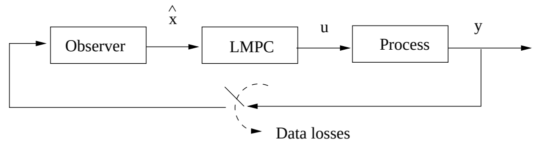


Figure 1: Closed-loop system with sensor data losses.

Figure 1 shows a schematic of the class of closed-loop systems under consideration. When sensor data losses occur, the observer does not receive new measurements to update the estimated state and the controller must operate in open-loop. In this paper, we study the stability and robustness properties of a combination of a Lyapunov-based model predictive controller with a high-gain observer in the presence of sensor data losses. We provide precise bounds on the data loss sequence such that stability of the closed-loop system is guaranteed.

## 2 Notation

Throughout the paper, the notation $|\cdot|$ will be used to denote the standard Euclidean norm of a vector. The notation $L_f^k h(\cdot)$ denotes the standard $k$-th order Lie derivative of a scalar function $h(\cdot)$ with respect to the vector function $f(\cdot)$. The notation $L_g L_f h(\cdot)$ denotes the mixed Lie derivative of a scalar function $h(\cdot)$ with respect to the vector functions $f(\cdot)$ and $g(\cdot)$. The notation $\Omega_r$ denotes the set $\Omega_r := \{x \in R^n | V(x) \leq r\}$ for a given positive definite scalar function $V(\cdot)$. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed $s$, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$, and for each fixed $r$, the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$.

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## 3 Preliminaries

We consider single-input single-output (SISO) nonlinear systems with the following state-space description:

$$\begin{aligned} \dot{x} &= f(x,w) + g(x,w)u \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in R^n$ is the state, $u \in R$ is the input, $w \in R^m$ is the disturbance variable and $y \in R$ is the measured output. The disturbance satisfies $|w| \leq \theta$. To simplify our notation, we focus on SISO systems but extensions of the results to multi-input multi-output systems are conceptually straightforward.

We assume that functions $f(\cdot), g(\cdot)$ and $h(\cdot)$ are sufficiently smooth in $x$, $f(0,0) = 0$ and $h(0) = 0$. This means that the origin is an equilibrium point for system (1) with $u = 0$ and $w = 0$. We also assume that there exists a state feedback controller that renders the origin globally asymptotically stable for the closed-loop nominal system; that is, system (1) with $w(t) \equiv 0$ for all times has a globally asymptotically stable equilibrium point at the origin $x = 0$ for a given feedback control $k_l : R^n \to R$ which satisfies $k_l(0) = 0$. Using converse Lyapunov theorems (see [7]), this assumption implies that there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a Lyapunov function $V$ for the closed-loop nominal system (system (1) with $u = k_l(x)$ and $w(t) \equiv 0$ for all times), which is continuous and bounded in $R^n$ that satisfies $V(x) > 0$, $V(0) = 0$ and

$$\frac{\partial V(x)}{\partial x}[f(x,0) + g(x,0)k_l(x)] \leq -\alpha(V(x)). \tag{2}$$

Note that stabilizing state feedback control laws for nonlinear systems have been developed using Lyapunov techniques; the reader may refer to [9, 5] for results on this area. These techniques can be used to obtain $k_l(x)$. The feedback controller $k_l(x)$ will be used to design the contractive constraints of the LMPC controller.

The closed-loop system is subject to sensor data losses. We model data losses with an increasing sequence of times $\{t_{i \geq 0}\}$ that determine when the output is available ($t \in [t_{2i}, t_{2i+1})$) or the sensor data is lost ($t \in [t_{2i+1}, t_{2i+2})$). This switching sequence is a random sequence and in this paper we provide sufficient conditions on the structure of this sequence under which the output-feedback LMPC controller proposed below in Section 4 achieves closed-loop stability.

### 3.1 Lyapunov-based MPC

In this section, we review previous results on state feedback LMPC for nonlinear systems subject to data losses, see [15]. These results are used to prove the main contribution of this work.

Lyapunov-based model predictive control is based on solving a finite horizon optimal control problem. In order to define a finite-dimensional optimization problem, the manipulated input trajectory (i.e., the free variable of the LMPC optimization problem) is constrained to belong to the family of piece-wise constant functions $S(\Delta)$ with sampling period $\Delta$ and length equal to the prediction horizon. The proposed LMPC controller is based on the following optimization problem

$$\min_{u(t) \in S(\Delta)} \int_0^{N\Delta} [\tilde{x}(t)^T Q_c \tilde{x}(t) + u(t)^T R_c u(t)]dt \tag{3a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t),0) + g(\tilde{x}(t),0)u(t) \tag{3b}$$

$$\tilde{x}(0) = x \tag{3c}$$

$$V(\tilde{x}(t)) \leq V(\hat{x}(t)), \ \forall t \in [0, N\Delta] \tag{3d}$$

where $S(\Delta)$ is the family of piece-wise constant functions with sampling period $\Delta$, $\tilde{x}(t)$ is the predicted trajectory of the nominal system for the input trajectory computed by the LMPC, $Q_c, R_c$ are positive definite weight matrices that define the cost, $N$ is the prediction horizon and $x$ is the initial state that is given as a parameter. The trajectory $\hat{x}(t)$ is the nominal sampled trajectory of system (1) under the feedback controller $k_l(x)$ starting from initial state $x$ and is obtained by solving recursively

$$\dot{\hat{x}}(t) = f(\hat{x}(t),0) + g(\hat{x}(t),0)k_l(\hat{x}(t_k)), \ t \in [t_k, t_{k+1})$$

where $t_k = k\Delta$, $k = 0, 1, \ldots, N$ and $\hat{x}(0) = x$. Note that $t_k$ defines the sample times along the time domain $[0, N\Delta]$ of the optimization problem. These sample times are independent of the actual sampling times $\tau_k$ defined below. Constraint (3d) is based on the the nominal sampled trajectory and guarantees that the value of the Lyapunov function of the predicted trajectory is smaller or equal to the value obtained if the Lyapunov-based controller $u = k_l(x)$ is implemented in the nominal closed-loop system along the prediction horizon. This is the contractive constraint that allows one to prove that the closed-loop system under state feedback LMPC is stable.

Measurements from the sensor are available in a continuous manner when no data losses are present. Sampling is introduced artificially by the LMPC scheme in order to solve a finite-dimensional optimization problem. If no data losses are present, the state is sampled at times $\tau_k = \tau_0 + k\Delta$. The optimization problem (3) is solved at each sampling time $\tau_k$ with initial state $\tilde{x}(0)$ equal to the measured state $x(\tau_k)$. The optimal manipulated input trajectory $u^*(t|\tilde{x}(0) = x(\tau_k))$ is applied to the closed-loop system for $t \in [\tau_k, \tau_{k+1})$. When data losses are present, the periodic sampling strategy is not appropriate because the data losses occur in an asynchronous manner and are independent of $\Delta$. If full state measurements are available, the sampling times are decided using the following strategy: a) if at sampling time $\tau_k$ the state is available (i.e., $\tau_k \in [t_{2i}, t_{2i+1})$), then the next sampling time is $\tau_{k+1} = \tau_k + \Delta$; b) if the state is not available (i.e., $\tau_k \in [t_{2i+1}, t_{2i+2})$), then the next sampling time takes place when the next measurement is available, i.e., $\tau_{k+1} = t_{2i+2}$. In this way, when a sample is lost, the next sample is taken as soon as a new measurement is available.

When data losses occur and a new sample is not available at sampling time $\tau_k$, the controller has to decide the input based on a function that depends on the available information, namely the last sampled state $x_s(\tau_k)$ and the corresponding sampling time $\tau_s(\tau_k)$. In this case the input is computed as follows: $u(t) = K(x_s(\tau_k), t - \tau_s(\tau_k))$. Note that $t - \tau_s(\tau_k)$ is the time in which the system has been operating in open-loop since the last measurement was received. Standard approaches set the control actuator output to a fixed value or to the "last available input". In this work, we update the manipulated input (and the control actuator output) based on the model of the process, see also [13, 14]. Using this approach the closed-loop performance is improved. This is done applying the optimal manipulated input trajectory obtained solving problem (3), i.e.,

$$K(x, t) = u^*(t|\tilde{x}(0) = x).$$

Recall that $u^*$ is defined in $[0, N\Delta]$ so $K(x, t)$ is not defined for all times. This limits the maximum time that the controller can operate in open-loop. The values of $x_s(k)$ and $\tau(k)$ change at each sampling time $\tau_k$ depending on whether the current state $x(\tau_k)$ is available or not.

The dynamics of the closed-loop system under the state feedback LMPC controller subject to data losses are defined by the following model that depends on the system and controller parameters, the data loss time sequence $\{t_{i\geq 0}\}$ and the sampling time $\Delta$:

$$\begin{aligned} \dot{x} &= f(x(t), w(t)) + g(x(t), w(t))u(t) \\ u(t) &= K(x_s(k), t - \tau_s(k)),\ \forall t \in [\tau_k, \tau_{k+1}) \end{aligned} \quad (4)$$

where the sequence of sampling times $\tau_k$ and corresponding auxiliary variables $x_s(k), \tau_s(k)$ are obtained using the following expressions

- If $\tau_k \in [t_{2i}, t_{2i+1})$, then $\tau_{k+1} = \tau_k + \Delta$, $x_s(k) = x(\tau_k)$ and $\tau_s(k) = \tau_k$.
- If $\tau_k \in [t_{2i+1}, t_{2i+2})$, then $\tau_{k+1} = t_{2i+2}$, $x_s(k) = x_s(k-1)$ and $\tau_s(k) = \tau_s(k-1)$.

Note that the current state is not available at every sampling time.

The state feedback LMPC controller allows for an explicit characterization of the stability region and guarantees that this region is an invariant set for the closed-loop system under data losses if the maximum time in which the loop is open is shorter than a given constant that depends on the parameters of the system and the Lyapunov-based controller that is used to formulate the optimization problem. These properties are stated in the form of an upper bound on the maximum time in which the system operates in open-loop

$$\delta_o = \max_i t_{2i+2} - t_{2i+1}.$$

The following proposition is based on the results presented in [15] and characterizes the closed-loop properties of system (1) under the proposed LMPC controller.

**Proposition 1** *Consider the closed-loop system under the state feedback LMPC controller subject to data losses described by (4). Then, given any positive real numbers $\rho, d$, there exist positive real numbers $\Delta^*$, $\theta^*$ such that if $\Delta \leq \Delta^*$, $\theta \leq \theta^*$, $\delta_o = 0$ (i.e., there are no data losses) and $x(t_0) \in \Omega_\rho$, then $x(t) \in \Omega_\rho$ for all times and $\limsup_{t\to\infty} V(x(t)) \leq d$. Also, there exist positive real numbers $\delta_o^*, N^*$ such that if $\delta_o \leq \delta_o^*$, $N \geq N^*$ and $x(t_0) \in \Omega_\rho$, then $x(t) \in \Omega_\rho$ for all times.*

## 3.2 High-gain observer

In this section, we present well-known results on output feedback control of nonlinear systems. In particular we present an output feedback controller based on a linear high-gain observer (see, for example, [8, 6, 4]). The results presented in this section will be used to prove the main result of this work in the next section.

We assume that the system (1) is fully input-output linearizable (i.e., the relative degree of the output with respect to the input is $n$ [7]) and that there exists a set of coordinates

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \quad (5)$$

such that system (1) takes the form:

$$\begin{aligned} \dot{z} &= Az + B[L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u] \\ y &= Cz \end{aligned}$$

where $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in R^n$ with

$$A = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}^T \end{bmatrix},\ B = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0_{n-1} \end{bmatrix}^T$$

where $I_{n-1}$ and $0_{n-1}$ are the identity matrix and a vector of zeros of dimension $n-1$ respectively.

We note that the change of variables is invertible, since for every $x$, the variable $z$ is uniquely determined by the transformation $z = T(x)$ which does not depend on the uncertainty vector $w$; that is, $L_f^i h(x)$ is independent of $w$ for all $i = 1, \ldots, n-1$. The linear high-gain observer (see for example [8, 6, 4]) provides estimates of the derivatives of the output up to order $n-1$, and thus estimates of the variable $z$. Using these estimates, the estimated state of the system $\hat{x}$ is obtained using $T^{-1}(\cdot)$. Proposition 2 below presents an output feedback controller based on the combination of a high-gain observer with the Lyapunov-based controller $k_l(x)$ and characterizes its stability properties. The proof of the proposition, which invokes singular perturbation arguments, is a special case of Theorem 1 in [4] and is omitted for brevity.

**Proposition 2** *Consider the nonlinear system (1) under the output feedback controller*

$$\dot{\hat{z}} = A\hat{z} + L(y - C\hat{z}),\ u = k_l(\hat{x}) \quad (6)$$

*with*

$$L = \begin{bmatrix} \frac{a_1}{\epsilon} & \frac{a_2}{\epsilon^2} & \ldots & \frac{a_n}{\epsilon^n} \end{bmatrix}^T,\ \hat{x} = T^{-1}(sat(\hat{z}))$$

*where the parameters $a_i$ are chosen such that the roots of*

$$s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0$$

*are in the open left-half of the complex plane.*

*Given positive real numbers $\rho, d$, there exist positive real numbers $\epsilon^*, \theta^*$ such that if $\epsilon \leq \epsilon^*$, $\theta \leq \theta^*$, $|z(t_0)| \leq z_m$, $x(t_0) \in \Omega_\rho$ and $sat(\cdot) = \min\{1, z_m/|(\cdot)|\}(\cdot)$ with $z_m$ being the maximum of the vector $z$ for $|z| \leq \beta_z(\delta_z, 0)$ where $\beta_z$ is a class $\mathcal{KL}$ function and $\delta_z = \max\{|T(x)|, x \in \Omega_\rho\}$; then $x(t) \in \Omega_\rho$ for all times and $\limsup_{t\to\infty} V(x(t)) \leq d$. This stability property implies that given $\epsilon \in (0, \epsilon^*]$ and some positive real number $e_m > 0$ there exists positive real number $t_b$ such that if $x(t_0) \in \Omega_\rho$ and $|z(t_0)| \leq z_m$, then $|x(t) - \hat{x}(t)| \leq e_m$ for all $t > t_0 + t_b$.*

To eliminate the peaking phenomenon associated with the high-gain observer, we use the saturation function, $sat(\cdot)$, to eliminate wrong estimates of the output derivatives for short times, see for example [8]. We consider that the estimated state $\hat{x}$ has converged to the actual state $x$, when the estimation error $|x - \hat{x}|$ is less or equal than a given bound $e_m$. The time needed to converge is given by $t_b$ which is an increasing function of the observer gain $1/\epsilon$ (recall that we have eliminated the peaking phenomenon).

# 4 Output feedback LMPC for systems subject to sensor data losses

In this section we present an output feedback LMPC controller that consists of a combination of the high-gain observer of Proposition 2 and of the state feedback LMPC controller of Proposition 1 that takes into account data losses in an explicit way. In this section, we provide explicit bounds on the data losses such that the closed-loop system is stable.

The high-gain observer estimates the state using sensor data in $t \in [t_{2i}, t_{2i+1})$ and maintains the last estimated state when no feedback is available, i.e., $\dot{z} = 0$, for all $t \in [t_{2i+1}, t_{2i+2})$. Each time feedback is recovered, the estimation error grows before converging close to zero because of the value of the high-gain observer gain. For this reason, the observer estimates are used in the controller after $t_b$ time, so it is guaranteed that the estimation error is less than $e_b$. The proposed output feedback LMPC scheme takes this issue into account and uses the last optimal manipulated input trajectory to decide the input when data losses occur using the following implementation procedure: a) when sensor data is available, the observer estimates the state as in (6), b) when sensor data are not available, the observer does not modify the estimated state, c) the LMPC controller only samples the estimated state when the estimated state is assured to have converged close to the actual state, and d) when new measurements are not available, the control actuator implements the last optimal input trajectory.

To model the evolution of the closed-loop system under the output feedback LMPC controller subject to data losses, we use the following model that depends on the system and controller parameters, the data loss time sequence $\{t_{i\geq 0}\}$ and the sampling time $\Delta$:

$$\begin{array}{l} \dot{x} = f(x(t), w(t)) + g(x(t), w(t))u(t) \\ u(t) = K(\hat{x}_s(k), t - t_s(k)),\ \forall t \in [\tau_k, \tau_{k+1}) \\ y = h(x) \\ \hat{x} = T^{-1}(sat(\hat{z})) \\ \dot{\hat{z}} = \begin{cases} A\hat{z} + L(y - C\hat{z}) & t \in [t_{2i}, t_{2i+1}) \\ 0 & t \in [t_{2i+1}, t_{2i+2}) \end{cases} \end{array} \quad (7)$$

where the sequence of sampling times and auxiliary variables are obtained using the following expressions

- If $\tau_k \in [t_{2i} + t_b, t_{2i+1})$ then $\tau_{k+1} = \tau_k + \Delta$, $\hat{x}_s(k) = \hat{x}(\tau_k)$, $x_s(k) = x(\tau_k)$ and $\tau_s(k) = \tau_k$.
- If $\tau_k \in [t_{2i+1}, t_{2i+2} + t_b)$ then $\tau_{k+1} = \min\{t_{2i+2} + t_b, t_{2i+3}\}$, $\hat{x}_s(k) = \hat{x}_s(k-1)$, $x_s(k) = x_s(k-1)$ and $\tau_s(k) = \tau_s(k-1)$.

Note that at each sample time the estimated state $\hat{x}_s(k)$ is different from the actual state $x_s(k)$.

System (10) is a nonlinear asynchronous system in which the switching between the different modes is governed by the external random sequence of times that characterizes the data losses. The main contribution of this paper is to prove using Proposition 2 that if certain conditions are satisfied, the closed-loop system under the proposed output feedback LMPC controller is equivalent in a neighborhood of the origin to a state feedback closed-loop system with a higher dimension uncertainty vector. Based on this equivalent system and Proposition 1, it is proved that the proposed output feedback controller allows for an explicit characterization of the stability region and guarantees that this region is an invariant set for the closed-loop system under data losses if $\hat{\delta}_o$ is small enough, where

$$\begin{array}{rl} \hat{\delta}_o = & \max_i \min_{j>i} t_{2j} - t_{2i+1} \\ & \text{s.t. } t_{2j+1} - t_{2j} > t_b \end{array}$$

The variable $\hat{\delta}_o$ takes into account that if in a given period of time we have measurements but the observer is not able to converge to the actual state, then the controller should operate in open-loop for the whole period of time. The main result of the present paper is presented in the following theorem:

**Theorem 1** *Consider the closed-loop system under the output feedback LMPC controller subject to data losses described by (7). Then, given any positive real numbers $d, \rho$, there exist positive real numbers $\epsilon^*$, $\Delta^*$, $\theta^*$ such that if $\epsilon \leq \epsilon^*$, $\Delta \leq \Delta^*$, $\theta \leq \theta^*$, $\hat{\delta}_o = 0$ (i.e., no data losses are present) and $x(t_0) \in \Omega_\rho$, then $x(t) \in \Omega_\rho$ for all times and $\limsup_{t\to\infty} V(x(t)) \leq d$. Also, there exist positive real numbers $\hat{\delta}_o^*, N^*$ such that if $\hat{\delta}_o \leq \hat{\delta}_o^*$, $N \geq N^*$ and $x(t_0) \in \Omega_\rho$, then $x(t) \in \Omega_\rho$ for all times.*

**Proof:** Due to space limitations we provide a sketch of the proof. Considering smoothness of functions $f(\cdot), g(\cdot)$ and continuity of the optimal solution of the optimization problem (3), given $\rho$ there exists positive real number $\Gamma$ such that the following inequality holds

$$|K(x,t) - K(x',t)| \leq \Gamma|x - x'| \quad (8)$$

for all $x, x' \in \Omega_\rho$, $|x - x'| \leq e_b$.

Following Proposition 2, given positive numbers $d, \rho, e_b$, there exist positive real numbers $\theta^*, \epsilon^*, t_b$ such that if $\epsilon \leq \epsilon^*$, $\theta \leq \theta^*$ and $x(0), \hat{x}(0) \in \Omega_\rho$, then $x(t) \in \Omega_\rho$ for all times and $|x(t) - \hat{x}(t)| \leq e_b$ for all $t > t_b$. The proposed output feedback LPM controller samples the estimated state after the observer has already operated for at least $t_b$ time in closed-loop. Following this implementation, it is guaranteed that each time a new estimated state is fed to the LMPC controller, the estimation error is smaller than $e_b$; that is

$$|x_s(k) - \hat{x}_s(k)| \leq e_b, \quad \forall k. \tag{9}$$

Taking into account (7), (8) and (9), system (1) in closed-loop with the proposed ouput feedback LMPC controller is equivalent to the following system for all $x(t) \in \Omega_\rho$

$$\begin{aligned} \dot{x} &= f(x(t), w(t)) + g(x(t), w(t))(u(t) + \Gamma w_e(t)) \\ u(t) &= K(x_s(k), t - t_s(k)), \ \forall t \in [\tau_k, \tau_{k+1}] \end{aligned} \tag{10}$$

with $w_e(t)$ an additional uncertainty vector that satisfies $|w_e(t)| \leq e_b$, and $x_s(k), \hat{x}_s(k), \tau_s(k)$ are obtained as in (7). This system is in the form of system (4) and Proposition 1 can be applied.

Applying Proposition 1, positive real numbers $\Delta^*$, $\delta_o^*$, $N^*$ are obtained such that if $\Delta \leq \Delta^*$, $N \geq N^*$ and the maximum time between consecutive samples is smaller than $\delta_o^*$, then the claims of Theorem 1 hold. Taking into account how the sampling times are defined in (7), in order to obtain a sample sequence that satisfies the conditions of Proposition 1, the data losses must satisfy $\hat{\delta}_o \leq \delta_o^* - t_b$. In this way it is guaranteed that the maximum time between two samples is smaller that $\delta_o^*$. It follows that $\hat{\delta}_o^* = \delta_o^* - t_b$. **QED**

# 5 Chemical reactor application

Consider a well mixed, non-isothermal continuous stirred tank reactor where three parallel irreversible elementary exothermic reactions take place of the form $A \rightarrow B$, $A \rightarrow C$ and $A \rightarrow D$. B is the desired product and C and D are byproducts. The feed to the reactor consists of pure A at flow rate $F$, molar concentration $C_{A0}$ and temperature $T_{A0} + \Delta T_{A0}$ where $\Delta T_{A0}$ is an unknown time-varying uncertainty. Due to the non-isothermal nature of the reactor, a jacket is used to remove/provide heat to the reactor. Using first principles and standard modeling assumptions, the following mathematical model of the process can be obtained

$$\begin{aligned} \frac{dT}{dt} &= \frac{F}{V_r}(T_{A0} + \Delta T_{A0} - T) - \sum_{i=1}^{3} \frac{\Delta H_i}{\sigma c_p} k_{i0} e^{\frac{-E_i}{RT}} C_A \\ &\quad + \frac{Q}{\sigma c_p V_r} \\ \frac{dC_A}{dt} &= \frac{F}{V_r}(C_{A0} - C_A) + \sum_{i=1}^{3} k_{i0} e^{\frac{-E_i}{RT}} C_A \end{aligned} \tag{11}$$

where $C_A$ denotes the concentration of the reactant $A$, $T$ denotes the temperature of the reactor, $Q$ denotes the rate of heat input/removal, $V_r$ denotes the volume of the reactor, $\Delta H_i, k_{i0}, E_i, i = 1, 2, 3$ denote the enthalpies, pre-exponential constants and activation energies of the three reactions, respectively, $c_p$ and $\sigma$ denote the heat capacity and the density of the fluid in the reactor. The values of the process parameters are shown in Table 1.

Table 1: Process parameters

| | | | |
|---|---|---|---|
| $F$ | 4.998 $[m^3/h]$ | $k_{10}$ | 3*$10^6$ $[h^{-1}]$ |
| $V_r$ | $1[m^3]$ | $k_{20}$ | 3*$10^5$ $[h^{-1}]$ |
| $R$ | 8.314 $[KJ/kmol \cdot K]$ | $k_{30}$ | 3*$10^5$ $[h^{-1}]$ |
| $T_{A0}$ | 300 $[K]$ | $E_1$ | 5*$10^4$ $[KJ/kmol]$ |
| $C_{A0}$ | 4 $[kmol/m^3]$ | $E_2$ | 7.53*$10^4$ $[KJ/kmol]$ |
| $\Delta H_1$ | -5.0*$10^4$ $[KJ/kmol]$ | $E_3$ | 7.53*$10^4$ $[KJ/kmol]$ |
| $\Delta H_2$ | -5.2*$10^4$ $[KJ/kmol]$ | $\sigma$ | 1000 $[kg/m^3]$ |
| $\Delta H_3$ | -5.4*$10^4$ $[KJ/kmol]$ | $c_p$ | 0.231 $[KJ/kg \cdot K]$ |

System (11) with $\Delta T_{A0} = 0$ has three steady-states (two locally asymptotically stable and one unstable). The control objective is to stabilize the system at the open-loop unstable steady state at $T_s = 388K$, $C_{As} = 3.59 mol/l$ assuming that only measurements of the concentration of $A$ are available and that data can be lost in the communication links. The manipulated input is the rate of heat input $Q$. We consider a time-varying uncertainty in the temperature of the inflow stream which satisfies $|\Delta T_{A0}| \leq 10K$.

To demonstrate the theoretical results, we first design a Lyapunov based feedback law using the method presented in [19]. System (11) belongs to class of nonlinear systems (1) where $x^T = [T \ C_A]$ is the state, $u = Q$ is the input and $w = \Delta T_{A0}$ is the time varying bounded disturbance. Consider the control Lyapunov function $V(x) = x^T P x$ with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 10^4 \end{bmatrix}.$$

The values of the weights have been chosen to account for the different range of numerical values for each state. The following feedback law [19] asymptotically stabilizes the open-loop unstable steady-state of the nominal system (i.e., $w(t) = 0$)

$$k_l(x) = \begin{cases} -\frac{L_f V + \sqrt{L_f V^2 + L_g V^4}}{L_g V} & \text{if } L_g V \neq 0 \\ 0 & \text{if } L_g V = 0 \end{cases} \tag{12}$$

where $L_f V$ and $L_g V$ are evaluated for $w(t) = 0$. This feedback law will be used to design the LMPC controller below.

This model satisfies the assumptions introduced in Section 3 when the concentration of the reactant $A$, $C_A$, is the output. The relative degree of the output $C_A$ with respect to the input $Q$ is 2 and the change of variables is invertible and independent of the uncertainty $w(t)$. The observer parameters are given by $\epsilon = 0.0005$, $a_1 = 2$, $a_2 = 1$, $\delta = 1000$ and $z_m = 7.07$. With these parameters the error goes below $e_m = 0.001$ in less than $t_b = 0.0025h$. Time $t_b$ is used in the controller implementation to decide when the estimated state has converged close enough to the actual state.

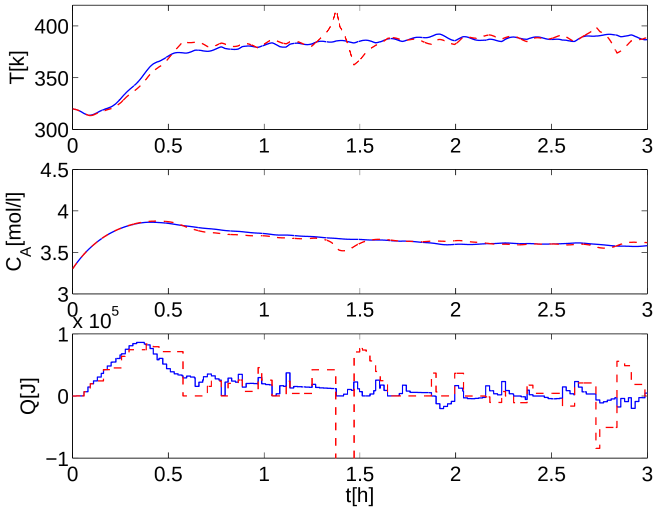


Figure 2: State and actuator ouput trajectories of the closed-loop system under the proposed LMPC scheme (solid-line) and "last available input" strategy (dashed-line) with data losses defined by $W = 50, p = 0.6$.

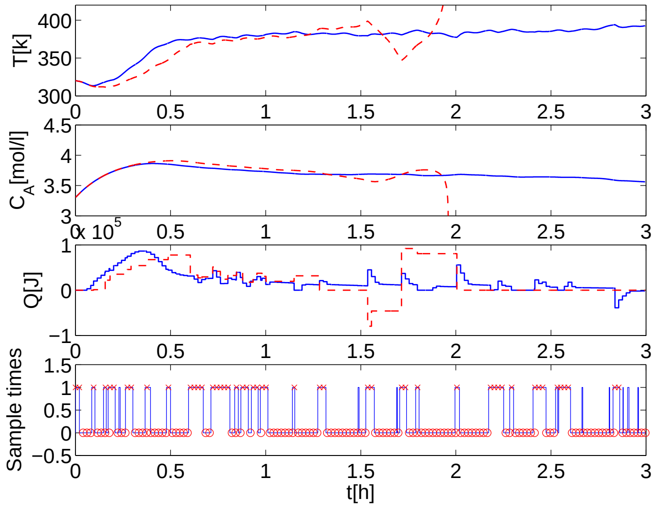


Figure 3: State and actuator ouput trajectories along with the sampling times of the closed-loop system under the proposed LMPC scheme (solid-line) and "last available input" strategy (dashed-line) with data losses defined by $W = 50, p = 0.75$.

To generate the increasing sequence of times that determines when the output is available $\{t_{i\geq 0}\}$, we use a random Poisson process as in [12]. The Poisson process is defined by the number of events per unit time $W$, and a probability $p$ of losing sensor data. At a given time $t$, an event takes place that determines whether the system is in the unstable or in the stable mode for the following period of time. This event is generated using a random variable $\zeta \in [0, 1]$ chosen from a uniform probability distribution. For a given probability $p$, if $\zeta \leq p$, then the controller is operating in open-loop, while if $\zeta > p$ the controller is operating in closed-loop. The length of the period of time, is generated randomly based on $W$, the number of events per unit time of the Poisson process. The time for which the system will remain in the chosen mode is given by $\Delta = \frac{-\ln \xi}{W}$, where $\xi \in [0, 1]$ is another random variable chosen from a uniform probability distribution. At $t+\Delta$, another event takes place. The time sequence obtained using this procedure is random. Below, we have carried out two different simulations with different values for the number of events $W$ and the probability of losing data.

A sampling time of $\Delta = 0.02h$ is used to implement the LMPC controller. The cost function is defined by the weight matrices $Q_c = P$ and $R_c = 10^{-6}$. The values of the weights have been tuned in such a way that the values of the manipulated inputs are comparable to the ones computed by the Lyapunov-based controller (i.e., same order of magnitude of the input signal and convergence time of the closed-loop system when no uncertainty or data losses are present). The constrained optimization problem is solved using MATLAB function `fmincon` and the set of ODEs is integrated using the Euler method with a fixed integration step of $0.0002min$. A new value for the uncertainty is obtained from a uniform distribution at each integration step.

In the following simulations, we compare the proposed output feedback LMPC scheme which uses the model to update the input when measurements are lost, with the strategy of keeping the control actuator output to the value of the input at the time in which the last estimated state was received (refer to as the "last available input" strategy). The "last available input" strategy follows the same scheme as the proposed LMPC controller regarding sampling times and observer implementation, but sets $K(x,t) = u^*(0)$ when data losses occur. This strategy has been used in previous works on nonlinear systems subject to data losses. It has been proved that the closed-loop system is practically stable for sufficiently small data losses and uncertainties. The following simulations, however, demonstrate that the proposed scheme is more robust with respect to data losses; that is, it is able to handle more data losses than the "last available input" strategy and also improves the closed-loop performance. Figure 2 shows a simulation of the closed-loop system under the proposed controller and the "last available input" strategy. The data losses have been obtained using $W = 50$ events per unit time, and probability $p = 0.6$ of losing data. For this particular realization, the maximum number of consecutive sampling times in which the controller operates in open-loop is 10. The initial state is $T(0) = 320K$ and $C_A(0) = 3.3Kmol/l$. For both controllers, $N = 12$. Figure 2 shows that when data losses occur, the proposed output feedback LMPC scheme keeps on updating the input, while the "last available input" strategy maintains the input fixed at the last value. It can be seen that although both controllers enforce practical stability in the closed-loop system, the performance of the proposed output feedback LMPC approach is better than the performance of the "last available input" strategy.

For the second simulation, a higher amount of data losses are introduced. In this case, the data losses are obtained using $W = 50$ and $p = 0.75$ and the maxi-

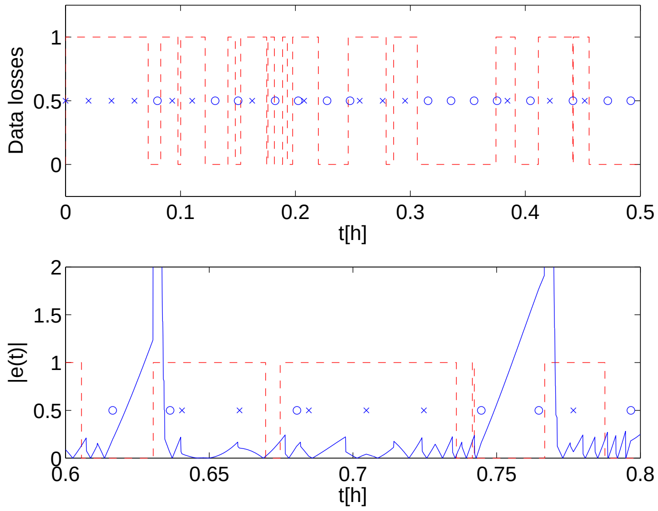


Figure 4: Close up of the sample times and norm of the estimation error of the closed-loop system for the simulation presented in Figure 2.

mum number of consecutive sampling times in which the controller operates in open-loop is 12. Figure 3 shows a simulation of the closed-loop system under the proposed output feedback LMPC controller and the "last available input" strategy. It can be seen that the state of the closed-loop system under the proposed LMPC controller remains in a neighborhood of the origin, while the state of the closed-loop system under the "last available input" strategy moves away from the equilibrium point. The bottom plot of Figure 3 shows the data loss time sequence and the corresponding sampling times. When the signal is equal to 1, the system is in closed-loop, and when the signal is equal to 0, the system is in open-loop. In Figure 3, crosses represent sampling times in which the state is available and the optimization problem is solved. Circles represent instants in which the input is updated following the previous optimal manipulated input trajectory. It can be seen that the state of the closed-loop system under the "last available input" trajectory starts moving away from the equilibrium point approximately between $t = 1.25h$ and $t = 2h$, a period of time with a high number of data losses.

Figure 4 shows two close ups of the sampling times and the norm of the estimation error corresponding to the simulation shown in Figure 2. Note the different time scales. The norm of the estimation error grows when the process operates in open-loop and for a short period of time when the loop closes (because of the high-gain of the observer). It can be seen that if the loop is closed only for a short period of time, in particular smaller than $t_b$, then the controller does not obtain a new estimate of the state to update the current optimal trajectory and keeps implementing the previous optimal trajectory. It can also be seen that the time between two samples depends on the data loss time sequence.

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