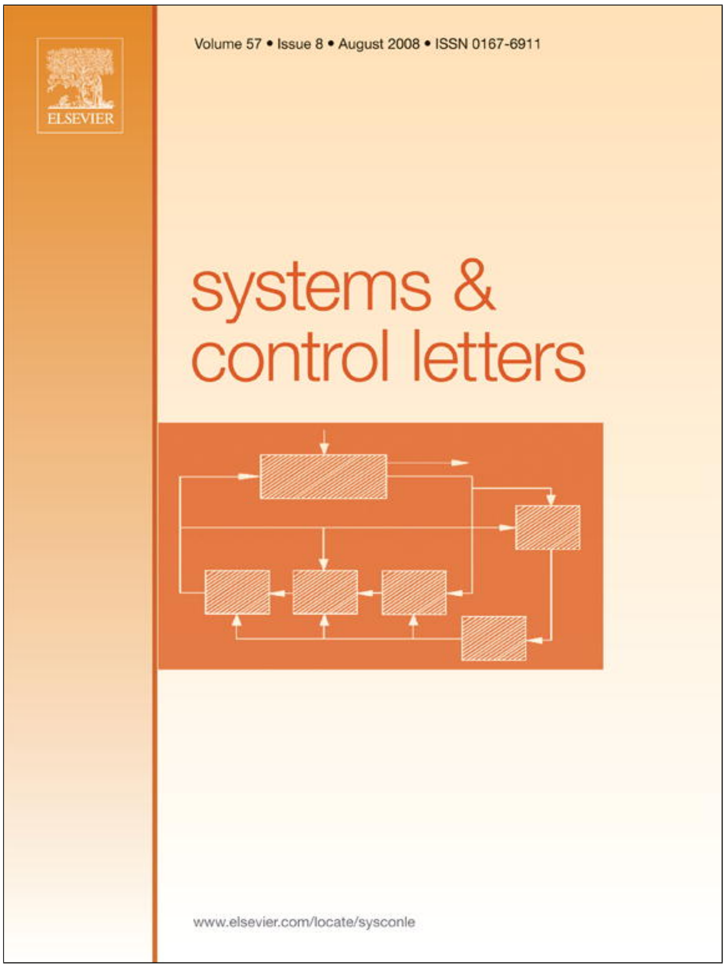
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Output feedback control of nonlinear systems subject to sensor data losses

David Muñoz de la Peña[1], Panagiotis D. Christofides*

Department of Chemical and Biomolecular Engineering, University of California, Los Angeles, CA 90095-1592, United States
Department of Electrical Engineering, University of California, Los Angeles, CA 90095-1592, United States

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Abstract

In this work, we focus on output feedback control of nonlinear systems subject to sensor data losses. We initially construct an output feedback controller based on a combination of a Lyapunov-based controller with a high-gain observer. We then study the stability and robustness properties of the closed-loop system in the presence of sensor data losses for both the continuous and sampled-data systems. We state a set of sufficient conditions under which the closed-loop system is guaranteed to be practically stable. The theoretical results are demonstrated using a chemical process example.

Keywords: Output feedback control; Nonlinear systems; Networked control systems; Process control

1. Introduction

Recently, an increasing number of control applications which have the control loops closed via a shared communication network have been discussed, see for example [44,41,31] and the references therein. These control systems are known as networked control systems (NCS) and differ from standard control systems (in which direct point-to-point links are used), in that the network introduces additional dynamics in the closed-loop system. There are different ways of modeling the dynamics introduced in the closed-loop system by the network, like time-varying delays, data losses or data quantization. In the present work, we focus on output feedback control of nonlinear systems subject to sensor data losses. This class of systems are of particular interest for wireless NCS [33], which play a prominent role in several areas of interest like sensor networks [1,5], multi-agent systems [4,40] and chemical processes [25,27]. New wireless, relatively low cost, sensors are available from vendors. These sensors can be used to implement new control loops, or to add redundancy in already working plants. These sensors are implemented in a wireless setting and are susceptible to communication network interference which would result in time intervals in which readings may not be provided to the control system; this set-up leads to the control problem formulation that is considered in this work. Several applications of networked control systems based on wireless communication links have been presented in the literature, see for example [32,43,42,15].

There are some recent works in the literature focusing on the analysis of the stability and robustness properties of nonlinear systems under state feedback control in the presence of data losses [38,39,30,29,25,35,10]. In these works it is proved that, if the maximum time in which the system operates in open-loop (i.e., without feedback) is small enough, practical stability is guaranteed. However, these results are based on the assumption that full-state measurements are available. In many systems, this assumption does not hold and an output feedback control scheme such as high-gain observers [20,17,37,24,6,11–13] or robust finite-time convergence observers [14,22] has to be used. However, output feedback control of nonlinear systems subject to sensor data losses has not been studied.

Motivated by the above considerations, we consider the problem of output feedback control of nonlinear systems subject to sensor data losses. Fig. 1 shows a schematic of the class of closed-loop systems under consideration. The process output is fed to the observer, which provides an estimate of

* Corresponding author at: Department of Chemical and Biomolecular Engineering, University of California, Los Angeles, CA 90095-1592, United States. Tel.: +1 310 794 1015; fax: +1 310 206 4107.
E-mail addresses: davidmps@cartuja.us.es (D. Muñoz de la Peña), pdc@seas.ucla.edu (P.D. Christofides).

[1] David Muñoz de la Peña is now with the Departmento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Sevilla, 41004, Spain. Financial support from NSF, CTS-0529295, and MEC, DPI2007-66718-C04-01, is gratefully acknowledged.

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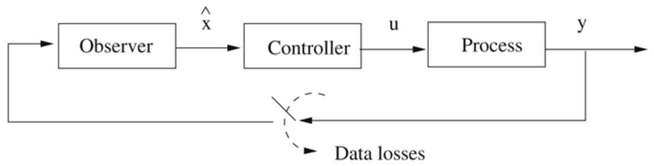

Fig. 1. Closed-loop system with sensor data losses.

the state to the controller. When sensor data losses occur, the observer does not receive new measurements to update the estimated state. In this paper, we study the stability and robustness properties of a combination of a Lyapunov-based controller with a high-gain observer in the presence of sensor data losses. Due to the nature of the fast dynamics of the observer, to obtain results that differ from output feedback control of nonlinear sampled-data systems [9,19] (which are a degenerate case of nonlinear systems subject to sensor data losses), it is necessary to approximately decouple the dynamics of the observer from the sensor data losses. To this end, the minimum time that the control system operates in closed-loop between consecutive periods without measurements, must be bounded from below. The main idea is that the estimated state must converge to the actual state before new sensor data losses occur. Once the dynamics of the high-gain observer are approximately decoupled from the sensor data losses, practical stability is guaranteed if the maximum time without measurements is smaller than an upper bound that practically depends on the properties of the closed-loop system under state feedback control.

The paper is organized as follows: In Section 2, preliminary notation and results on Lyapunov-based control and high-gain observers are introduced. In Section 3, the main contribution of the paper is presented. In Section 4 we explicitly consider the issue of measurement-sampling in the output feedback controller. In Section 5, the results are demonstrated using a chemical process example. In Section 6, we present some concluding remarks.

2. Preliminaries

The main objective of this paper is to study the stability and robustness properties of an output feedback controller based on a combination of a Lyapunov-based controller with a high-gain observer with respect to sensor data losses. We assume that this controller has been designed *a priori* under the assumption of flawless communication. This approach has been followed in previous works to study state feedback control of nonlinear systems subject to sensor data losses, see for example [38,39,30,29,25]. In this section, we present the class of nonlinear systems and output feedback controllers under consideration, along with some properties. We will use these properties in the analysis of the closed-loop system subject to sensor data losses.

Specifically, in this work we assume that the process in Fig. 1 is modeled by a single-input single-output (SISO) nonlinear system with the following state-space description:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in R^n$ is the state, $u \in R$ is the input and $y \in R$ is the measured output. To simplify our notation, we focus on SISO systems but extensions of these results to multi-input multi-output systems are conceptually straightforward.

Throughout the paper, the notation $|\cdot|$ will be used to denote the standard Euclidean norm of a vector. The notation $L_f^k h(\cdot)$ denotes the standard kth-order Lie derivative of a scalar function $h(\cdot)$ with respect to the vector function $f(\cdot)$. The notation $L_g L_f h(\cdot)$ denotes the mixed Lie derivative of a scalar function $h(\cdot)$, with respect to vector functions $f(\cdot)$ and $g(\cdot)$. The notation Ω_r denotes the set $\Omega_r := \{x \in R^n | V(x) \leq r\}$ for a given positive definite scalar function $V(\cdot)$. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$, and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

In order to proceed with the design of the output feedback controller, we need to impose the following assumptions on system (1):

Assumption 1. Functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are sufficiently smooth in x, $f(0) = 0$ and $h(0) = 0$.

This means that the origin is an equilibrium point for system (1) with $u = 0$. We also assume that there exists a state feedback controller that renders this equilibrium point globally asymptotically (and locally exponentially) stable. This assumption is stated below.

Assumption 2. System (1) has a globally asymptotically (and locally exponentially) stable equilibrium at the origin $x = 0$ for a given feedback control $k : R^n \to R$ which satisfies $k(0) = 0$.

Using converse Lyapunov theorems (see [18]), Assumption 2 implies that there exist a class $\mathcal{K}$ function $\rho(\cdot)$ and a Lyapunov function V for the closed-loop system (system (1) with $u = k(x)$), which is continuous and bounded in R^n that satisfies $V(x) > 0$, $V(0) = 0$ and

$$\dot{V}(x) \leq -\rho(V(x)). \tag{2}$$

Note that stabilizing state feedback control laws for nonlinear systems have been developed using Lyapunov techniques; the reader may refer to [21,7] for results on this area. These techniques can be used to obtain $k(x)$. In Section 4, a method such as the one presented in [34] is used.

Remark 1. The assumption of global asymptotic stability of the origin can be relaxed to asymptotic stability of the origin. In this case, functions $L_f V$ and $L_g V$ must be sufficiently smooth in the region of attraction of the stable equilibrium point of the closed-loop system. The size of this region depends on the class of nonlinear systems considered.

We assume next that the system (1) is fully input–output linearizable (i.e., the relative degree of the output with respect to the input is n [18]). This is not necessary but simplifies the notation of the paper. The results can be extended to a more general class of systems (for example systems with ISS inverse dynamics [6,11,13,12]). Assumption 3 states this requirement.

Assumption 3. There exists a set of coordinates

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \tag{3}$$

such that system (1) takes the form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u \\ y &= z_1 \end{aligned}$$

where $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in R^n$.

Using Assumption 3, the system of Eq. (1) can be rewritten in the form

$$\begin{aligned} \dot{z} &= Az + B[L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u] \\ y &= Cz \end{aligned}$$

with

$$A = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}^{\mathrm{T}} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0_{n-1} \end{bmatrix}^{\mathrm{T}}$$

where I_{n-1} and 0_{n-1} are the identity matrix and a vector of zeros of dimension $n-1$ respectively.

We note that the change of variables is invertible, since for every x, the variable z is uniquely determined by the transformation $z = T(x)$. The output feedback controller proposed is based on a linear high-gain observer (see for example [20,17,37,24,6,11–13]) which provides estimates of the derivatives of the output up to order $n-1$, and thus estimates of the variable z. Using these estimates, the estimated state of the system $\hat{x}$ is obtained using $T^{-1}(\cdot)$. Proposition 1 presents the output feedback controller used and characterizes its stability properties. The proof of the proposition, which invokes singular perturbation arguments, is a special case of Theorem 2 in [12], and is omitted for brevity.

Proposition 1. *Consider the nonlinear system (1) for which Assumptions 1–3 hold, under the output feedback controller*

$$\dot{z} = Az + L(y - Cz), u = k(\hat{x}) \tag{4}$$

with

$$L = \begin{bmatrix} \frac{a_1}{\epsilon} & \frac{a_2}{\epsilon^2} & \cdots & \frac{a_n}{\epsilon^n} \end{bmatrix}^{\mathrm{T}}, \qquad \hat{x} = T^{-1}(sat(z))$$

where the parameters a_i are chosen such that the roots of

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

are in the open left-half of the complex plane.

Then given δ, there exists ϵ^* such that if $\epsilon \in (0, \epsilon^*]$, $|z(t_0)| \leq z_m$, $x(t_0) \in \Omega_\delta$ and $sat(\cdot) = \min\{1, z_m/|(\cdot)|\}(\cdot)$ with z_m being the maximum of the vector z for $|z| \leq \beta_z(\delta_z, 0)$ where β_z is a class $\mathcal{KL}$ function and $\delta_z = \max\{|T(x)|, x \in \Omega_\delta\}$; the origin of the closed-loop system is asymptotically (and locally exponentially) stable. This stability property implies that given $\epsilon \in (0, \epsilon^*]$ and some positive constant $e_m > 0$ there exists positive real constant t_b such that if $x(t_0) \in \Omega_\delta$ and $|z(t_0)| \leq z_m$, then $|x(t) - \hat{x}(t)| \leq e_m$ for all $t > t_0 + t_b$.

Remark 2. To eliminate the peaking phenomenon associated with the high-gain observer, we use the saturation function, $sat(\cdot)$, to eliminate wrong estimates of the output derivatives for short times, see for example [20].

Remark 3. We consider that the estimated state $\hat{x}$ has converged to the actual state x, when the estimation error $|x - \hat{x}|$ is less than or equal to a given bound e_m. The time needed to converge, is given by t_b which is proportional to the observer gain $1/\epsilon$ (recall that we have eliminated the peaking phenomenon). During this transient, the value of the Lyapunov function $V(x)$ may increase.

Remark 4. We consider high-gain observers because they can be designed to provide guaranteed stability of the closed-loop system. In the results presented in this paper these stability properties are used to study the robustness of the closed-loop system with respect to data losses. Other observer schemes such as robust finite-time convergence observers [14,22] (see also [3,2] for other results on finite-time stability and control) can provide different closed-loop properties and can be the subject of further research.

3. Stabilization subject to sensor losses

In this section, we consider system (1) subject to sensor data losses in closed-loop with the output feedback controller (4) introduced in the previous section. When sensor data is lost in the sensor link, the observer no longer has access to the output to update the estimated state. There are different potential actions that the controller can take when sensor data is lost. One strategy is to set the input to zero (or any fixed value) [33]. Other approaches [26,27] use the model of the system to estimate in open-loop the actual state and update accordingly the input (note that in this case uncertainty has to be taken into account in an explicit way). In this paper, we consider that the estimated state is held at the value computed using the last available measurement as done in [38,39,30,29,25]. This means that the input is kept constant at the last value computed using measurements from the system, however, our results can be extended to other strategies as discussed in Remark 10 below. The closed-loop system subject to sensor data losses is described by:

$$\begin{aligned} \dot{x} &= f(x) + g(x)k(\hat{x}) \\ \hat{x} &= T^{-1}(sat(z)) \\ \dot{z} &= \begin{cases} Az + L(y - Cz) & t \in [t_{2i}, t_{2i+1}) \\ 0 & t \in [t_{2i+1}, t_{2i+2}) \end{cases} \\ y &= h(x) \end{aligned} \tag{5}$$

where the partition $\{t_{i\geq 0}\}$ is an increasing sequence of times that determine when the output is available ($t \in [t_{2i}, t_{2i+1})$) or the sensor data is lost ($t \in [t_{2i+1}, t_{2i+2})$).

The objective of this paper, is to establish a set of sufficient conditions on the partition $\{t_{i\geq 0}\}$, that guarantee convergence of the state of the closed-loop system to a desired neighborhood of the origin. These conditions are given in the form of constraints on the following properties of the partition $\{t_{i\geq 0}\}$

$$\delta_o = \max_i t_{2i+2} - t_{2i+1} \tag{6}$$

$$\delta_c = \min_i t_{2i+1} - t_{2i}. \tag{7}$$

Constant δ_o is the maximum time that the controller operates in open-loop in a consecutive manner, while δ_c is the minimum time that the controller operates in closed-loop in a consecutive manner, that is, the minimum time between two consecutive open-loop periods. These constants are used in the following theorem:

Theorem 1. *Consider system (5) for which Assumptions 1–3 hold. Then, given positive real numbers d and δ that satisfy $\gamma < d < \delta$ with*

$$\begin{aligned} \gamma = \max V(x(t_i + t_b)) - V(x(t_i)) \\ s.t.\ x(t_i) \in \Omega_\delta, u(t) \in k(\Omega_\delta) \end{aligned} \tag{8}$$

where $k(\Omega_\delta)$ is the set of all possible inputs $k(x)$ for any $x \in \Omega_\delta$; there exists a positive constant $\epsilon^*(\delta) = \epsilon^*$ such that if $\epsilon \in (0, \epsilon^*]$, there exist positive constants $\delta_o^*(\epsilon) = \delta_o^*$ and $\delta_c^*(\epsilon) = \delta_c^*$ such that if $\delta_o \leq \delta_o^*$, $\delta_c \geq \delta_c^*$, $|z(t_0)| \leq z_m$ and $x(t_0) \in \Omega_\delta$, then $\limsup_{t\to\infty} V(x(t)) \leq d$.

Proof. The proof consists of three parts. In the first part, we study the trajectories of system (5) when the output measurements are available (that is, $t \in [t_{2i}, t_{2i+1})$ for all i). In this part we provide a sufficient condition for approximately decoupling the observer dynamics from the sensor data losses. In the second part, we characterize the trajectories when sensor data is lost (that is, $t \in [t_{2i+1}, t_{2i+2})$ for all i) assuming that at time t_{2i+1} the estimated state is close to the actual state. If at time t_{2i} the conditions of Part 1 hold, then it is guaranteed that the conditions of Part 2 hold at time t_{2i+1}, and moreover, if at time t_{2i+1} the conditions of Part 2 hold, then it is guaranteed that the conditions of Part 1 hold at time t_{2i+2}. This allows us to use the results of Parts 1 and 2 recursively from t_0 to characterize the evolution of the closed-loop system for all times. This is done in Part 3 of the proof, where convergence of the state of the closed-loop system to a neighborhood of the origin is proved. The main idea is to prove that between two consecutive times in which the loop is closed or open (from t_j to t_{j+1}), it is guaranteed that the value of the Lyapunov function has decreased.

Part 1: In this part, we study the trajectories of the state of system (5) in $t \in [t_{2i}, t_{2i+1})$. At time t_{2i}, the loop is closed and measurements are available so the results of Proposition 1 can be applied. Proposition 1 fixes ϵ^* such that given δ, if $\epsilon \in (0, \epsilon^*]$, $|z(t_{2i})| \leq z_m$[2] and $x(t_{2i}) \in \Omega_\delta$, then the origin of the closed-loop system is asymptotically (and locally exponentially) stable. These conditions hold for $i = 0$. In part 2 of this proof a sufficient condition is given such that it can be proved recursively that the conditions hold for all i. The proof is presented in Part 3.

We are going to define δ_c^*, in a way such that if $\delta_c \geq \delta_c^*$, $|z(t_{2i})| \leq z_m$ and $x(t_{2i}) \in \Omega_\delta$ then

$$V(x(t)) \leq V(x(t_{2i})) + \gamma, \quad t \in [t_{2i}, t_{2i+1}) \tag{9}$$

$$V(x(t_{2i+1})) < V(x(t_{2i})) < \delta \tag{10}$$

$$|x(t_{2i+1}) - \hat{x}(t_{2i+1})| \leq e_m \tag{11}$$

$$|z(t_{2i+1})| \leq z_m. \tag{12}$$

The above inequalities imply that at time t_{2i+1}, when sensor data is lost and the controller starts operating in open-loop, the estimated state is close to the real state, i.e., the observer dynamics are approximately decoupled from the data losses.

We distinguish two time periods in $t \in [t_{2i}, t_{2i+1})$, before and after the estimated state value, $\hat{x}$, has converged close to the actual state, x. The stability property implies that given some positive constant $e_m > 0$ there exists positive real constant t_b such that $|x(t) - \hat{x}(t)| \leq e_m$ for all $t > t_0 + t_b$. Taking into account (8), we conclude that

$$V(x(t)) \leq V(x(t_{2i})) + \gamma, \quad t \in [t_{2i}, t_{2i} + t_b).$$

This means that the Lyapunov function remains bounded although it may achieve a value greater than δ. After the estimation error has decreased below e_m, because the origin is asymptotically stable under the state feedback controller, the state of the closed-loop system converges towards the equilibrium point, so by continuity of $V(x(t))$, there exists $t_c > 0$ such that

$$V(x(t)) \leq V(x(t_{2i})), \quad \forall t > t_{2i} + t_b + t_c.$$

For a choice of $\delta^* = t_b + t_c$, inequalities (9)–(12) hold.

Part 2: In this part of the proof, we study the evolution of system (5) in $t \in [t_{2i+1}, t_{2i+2})$. We assume that $|x(t_{2i+1}) - \hat{x}(t_{2i+1})| \leq e_m$, that is, when the sensor data is lost ($t = t_{2i+1}$), the estimated state has converged close to the actual state. This implies that the observer dynamics have been practically decoupled from the sensor data losses. In Part 1 of this proof a sufficient condition is given such that it can be proved recursively that the conditions hold for all i. The proof is presented in Part 3.

We are going to define δ_o^*, in a way such that if $\delta_o \leq \delta_o^*$, $x(t_{2i+1}) \in \Omega_\delta$ and $|z(t_{2i+1})| \leq z_m$ then the following inequalities hold

$$V(x(t_{2i+2})) \leq s \leq \delta, \quad \forall x(t_{2i+1}) \in \Omega_{\frac{s}{2}} \tag{13}$$

$$V(x(t_{2i+2})) \leq V(x(t_{2i+1})) \leq \delta, \quad \forall x(t_{2i+1}) \notin \Omega_{\frac{s}{2}} \tag{14}$$

$$|z(t_{2i+2})| \leq z_m \tag{15}$$

with $s = d - \gamma$. Note that in order to guarantee that the system is ultimately bounded in Ω_d, we must take into account (following

[2] z_m was defined in Proposition 1.

inequality (13) of Part 1) that during the time period in which the observer state converges to the actual state, the Lyapunov function can grow at most γ before starting to converge towards the equilibrium point.

To guarantee (13), δ_o^* must satisfy the following constraint

$$\begin{aligned} s \le \max\ & V(x(t_0+\delta_o^*)) \\ \text{s.t. } & x(t_{2i+1}) \in \Omega_{\frac{s}{2}},\ u(t) = k(\hat{x}(t_{2i+1})) \\ & |x(t_{2i+1}) - \hat{x}(t_{2i+1})| \le e_m. \end{aligned} \quad (16)$$

This inequality can be always satisfied because of the continuity of the trajectories $x(t)$ of the closed-loop system.

We now derive a constraint on δ_c^* that assures that (14) holds. The input is kept constant for the time period in which the controller is operating in open-loop, that is $u(t) = k(\hat{x}(t_{2i+1}))$ for $t \in [t_{2i+1}, t_{2i+2})$ (recall that $\hat{x}(t_{2i+1})$ is the last estimated state). It follows that

$$\dot{x} = f(x) + g(x)k(\hat{x}(t_{2i+1})), \quad t \in [t_{2i+1}, t_{2i+2}).$$

In this time period, the time derivative of the Lyapunov function $V(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= L_f V(x) + LgV(x)k(\hat{x}(t_{2i+1})) \\ &= L_f V(\hat{x}(t_{2i+1})) + LgV(\hat{x}(t_{2i+1}))k(\hat{x}(t_{2i+1})) \\ &\quad + L_f V(x) - LfV(\hat{x}(t_{2i+1})) \\ &\quad + (LgV(x) - LgV(\hat{x}(t_{2i+1})))k(\hat{x}(t_{2i+1})). \end{aligned}$$

Since $L_f V$ and $L_g V$ are continuous, there exist positive constants K_f, K_g such that

$$\begin{aligned} &|L_f V(x) - LfV(x')| \le K_f |x - x'| \\ &|(LgV(x) - LgV(x'))k(x')| \le K_g |x - x'| \end{aligned}$$

for all $x, x' \in \Omega_\delta$. Using these bounds and taking into account (2), we obtain the following bound on the time derivative of the Lyapunov function $V(x)$:

$$\dot{V}(x) \le -\rho(V(\hat{x}(t_{2i+1}))) + (K_f + K_g)|x - \hat{x}(t_{2i+1})|$$

for all $x, x(t_{2i+1}) \in \Omega_\delta$. Define the error vector as $e = x - \hat{x}$. Taking into account that $\dot{\hat{z}} = 0$, the error has the following dynamics

$$\dot{e} = f(x) + g(x)k(\hat{x}(t_{2i+1})), \quad t \in [t_{2i+1}, t_{2i+2}).$$

By Assumption 1, we know that there exists constant M such that

$$|f(x) + g(x)k(x')| \le M, \quad \forall x, x' \in \Omega_\delta.$$

As shown in Part 1, $\delta_c^* > t_b$, so $|e(t_{2i+1})| \le e_m$. It follows that the norm of the error is upper bounded by

$$|e(t)| \le e_m + M(t - t_{2i+1}).$$

Using this bound we obtain the following inequality for all $x(t_{2i+1}) \notin \Omega_{\frac{s}{2}}$

$$\dot{V}(x) \le -\rho\left(\frac{s}{2}\right) + (K_f + K_g)(e_m + M(t - t_{2i+1})).$$

To guarantee (14), a sufficient condition is that the Lyapunov function has a negative time derivative for all $t \in [t_{2i+1}, t_{2i+2})$.

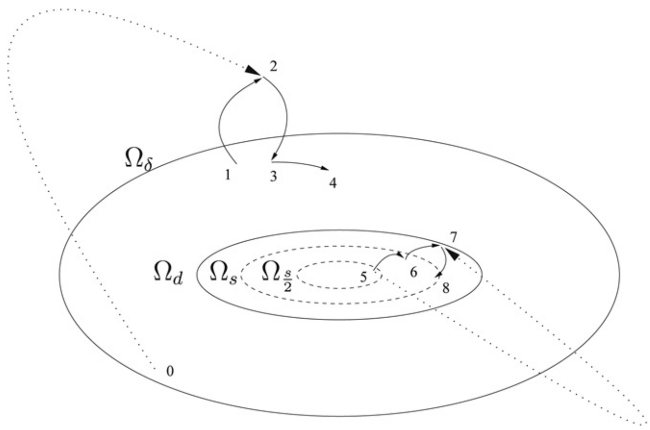

Fig. 2. Trajectories in the state space of the closed-loop system.

For a choice of δ_o^* such that

$$\delta_o^* < \frac{\rho(\frac{s}{2}) - (K_f + K_g)e_m}{(K_f + K_g)M}, \quad (17)$$

if $x(t_{2i+1}) \notin \Omega_{\frac{s}{2}}$, then $\dot{V} \le 0$ for all $t \in [t_{2i+1}, t_{2i+2})$. Note that e_m can be made arbitrarily small by decreasing t_b.

Because $\dot{z}(t) = 0$ for $t \in [t_{2i+1}, t_{2i+2})$, when sensor data losses occur, the observer does not modify the estimated state. This means that if $|z(t_{2i+1})| \le z_m$ then (15) is guaranteed.

Part 3: In this part, we prove that if $\epsilon < \epsilon^*$, $\delta_c^* = t_b + t_c$ and δ_o^* satisfies (16) and (17), then the trajectories of the closed-loop system (5) are ultimately bounded in a neighborhood of the origin if $\delta_c > \delta_c^*$, $\delta_o < \delta_o^*$, $x(t_0) \in \Omega_\delta$ and $|z(t_0)| \le z_m$.

At time t_0, $x(t_0) \in \Omega_\delta$ and $|z(t_0)| \le z_m$ so inequalities (9)–(12) hold. Applying recursively (9)–(12) and (13)–(15), it follows that (9)–(12) and (13)–(15) hold for all i. This means that the conditions (and the results) of Parts 1 and 2 hold for all i. We will now prove that the state of the system reaches Ω_s in finite time. Using (10) and (14), we obtain that for all j, if $x(t_j) \notin \Omega_s$, then $V(x(t_j)) < V(x(t_{j+1}))$. This means that there exists j^* such that $x(t_{j^*}) \in \Omega_s$. Once the state reaches Ω_s, the trajectories remain bounded in Ω_d for all future times. Following (9) and (10), if $x(t_{2i}) \in \Omega_s$, then $V(x(t)) \le d$ for all $t \in [t_{2i}, t_{2i+1})$ and $x(t_{2i+1}) \in \Omega_s$. Following (13) and (14), if $x(t_{2i+1}) \in \Omega_s$, then $V(x(t)) \le d$ for all $t \in [t_{2i+1}, t_{2i})$ and $x(t_{2i+2}) \in \Omega_s$. Repeating this argument recursively, it holds that

$$\limsup_{i \to \infty} V(x(t_i)) \le d$$

so the state x of the closed-loop system is ultimately bounded in Ω_d. □

Remark 5. The state x of the closed-loop system might leave the set Ω_δ while the estimated state $\hat{x}$ converges to x. In Fig. 2, the actual state starts from point 1, while the estimated state starts from point 0. Both points are inside δ but the initial estimation error is very high. As the estimated state converges, the actual state moves to point 2 leaving Ω_δ. Note that although it may leave Ω_δ, it is still bounded is some set (recall (9) and the definition of γ). The estimated state might peak before converging to point 2, but because the saturation function is used, it also remains bounded. Once it has converged, both the actual state and the estimated state move toward the origin because the origin is asymptotically stable. Because $\delta_c^* = t_b + t_c$, it is assured that before sensor data is lost, the actual state returns to Ω_δ (from point 2 to point 3). When feedback is lost at point 3, the estimated state remains fixed, while the actual state is assured to keep approaching the equilibrium point (from point 3 to point 4) because $\delta_o \leq \delta_o^*$.

Remark 6. If the actual and the estimated states, x and $\hat{x}$ respectively, are close and lie inside $\Omega_{\frac{s}{2}}$ when sensor data losses occur, then because of (13), the Lyapunov function can increase at most to s. In Fig. 2, this trajectory is represented from point 5 to point 6. In the meantime, the estimated state remains in point 5. When feedback is recovered, the state does not start immediately to move towards the origin, as it would do in the case of state feedback. While the estimated state converges (from point 5 to point 7), the actual state keeps moving away (from point 6 to point 7) leaving Ω_s, but it remains inside Ω_d. Once the estimated state has converged, the actual system state moves again inside Ω_s (from point 7 to point 8).

Remark 7. Constants δ_c^* and δ_o^* depend on ϵ which is upper bounded by Proposition 1. The observer gain $\frac{1}{\epsilon}$ can be increased to obtain smaller t_b and γ, improving this way the robustness of the output feedback controller with regard to sensor data losses. It should be noted, however, that it is well known that increasing the observer gain, can amplify measurement noise and induce poor closed-loop performance. This points to a fundamental trade-off that cannot be resolved by simply changing the estimation scheme, owing to the lack of separation principle in nonlinear systems.

Remark 8. The technique employed in Part 2 of the proof, that is, providing an upper bound on the maximum time that the system can operate in open-loop while assuring a negative time derivative of the Lyapunov function, is shared by the works on state feedback control of nonlinear systems subject to data losses [38,39,30,29,25] and sampled-data systems [8, 28]. In Part 1, sensor data losses are approximately decoupled from the observer dynamics providing the lower bound on δ_c. This sufficient condition is particular of output feedback control systems based on high-gain observers and is part of the main contribution of this work.

Remark 9. In this work we require that the controller sets $\dot{z} = 0$ when operating in open-loop. It is also possible to rely on open-loop predictions based on the model by setting

$$\dot{z} = Az + B[L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u]$$

for $t \in [t_{2i+1}, t_{2i+2})$ as done for example in model-based networked control [26]. In this case, the same stability and robustness result is obtained, although the estimated values of δ_o^* would have a different expression (possibly tighter). This change would not affect Part 1 of the proof, that is, the approximate decoupling of the observer dynamics from the sensor data losses. The actual performance of both strategies would be different (it is reasonable to expect that the model-based approach would perform better in most cases). Note that in this case, uncertainties would have to be introduced in the model to take into account that the predicted state would be different from the actual state. The predictions would be made with the nominal model.

Remark 10. The proposed implementation of the output feedback controller subject to sensor data losses of Eq. (5) always computes the input using the estimated state of the observer. For this reason, it is expected that the input trajectory exhibits abrupt changes each time the estimated state deviates from the actual state when the loop is open, and converges to the actual state again once the loop is closed. The abrupt changes on the estimated state cannot be avoided because they are due to the fast dynamics of the high-gain observer, however, the abrupt changes of the input can be avoided if the controller only uses the estimated state once convergence of the estimation error has been achieved, that is, after t_b time has passed from the moment the loop closed. This strategy is defined by the following changes in Eq. (5):

$$\dot{x} = f(x) + g(x)u$$
$$u(t) = \begin{cases} k(\hat{x}(t)) & t \in [t_{2i} + t_b, t_{2i+1}) \\ u(t_{2i+1}) & t \in [t_{2i+1}, t_{2i+2} + t_b). \end{cases} \tag{18}$$

This implementation strategy avoids an abrupt change in the input, because in $t \in [t_{2i}, t_{2i} + t_b)$, the controller does not modify the input. Within the time period of length t_b, the estimation error is converging close to zero, but $\hat{x}$ can take any value inside Ω_δ (recall that the saturation function is used). If $u = k(\hat{x})$ is implemented, the input can take any value inside $k(\Omega_\delta)$. The input jumps after t_b seconds, but it is reasonable to expect this jump to be smaller than the jumps that occur in the implementation of Eq. (5). Also note that if $t_{2i+1} - t_{2i} < t_b$ then the controller never uses the estimated state $\hat{x}$ in $t \in [t_{2i}, t_{2i+1})$. From a practical point of view, the loop remained open in this period of time (the input does not change). Following this idea and using the same techniques used in the proof of Theorem 1, this implementation technique guarantees practical stability of the closed-loop system if $\hat{\delta}_o \leq \delta_c^* - t_b$, where

$$\hat{\delta}_o = \max_i \min_{j>i} t_{2j} - t_{2i+1}$$
$$\text{s.t. } t_{2j+1} - t_{2j} > t_b.$$

This is a sufficient condition that takes into account that if in a given period of time we have measurements but the observer is not able to converge to the actual state, then the controller should operate in open-loop for the whole period of time. Note that this implementation technique is just a modification of the proposed output feedback scheme, and that it builds on the results presented in Theorem 1. See the example for an application of this controller implementation scheme.

Remark 11. Assumption 2 states that the origin must be an asymptotically and locally exponentially stable equilibrium point for the closed-loop system under the state feedback controller. This assumption can be relaxed to only asymptotic stability without the local exponential stability property. In this case, the origin of the closed-loop system under the output feedback controller without data losses can be shown to be semi-globally practically stable under the assumptions of Proposition 1. This modification does not alter the result obtained in Theorem 1 that indicates that the closed-loop system with data losses is practically stable if certain conditions on δ_c and δ_o are satisfied. In this case, the proof and the expressions have to be modified to take into account the changes in the stability properties of the system without data losses.

Remark 12. The robustness of the closed-loop system with respect to data losses depends on the nonlinear process model structure and the output feedback controller used in the implementation. This is an important question and no general statement can be made. It should be addressed on a case-by-case basis. In the chemical reactor example, we demonstrate that Lyapunov-based control can be used to obtain a reasonable robustness with respect to sensor data losses.

Remark 13. Although the proof of Theorem 1 is constructive, the constants obtained are conservative. This is the case with most of the results of the type presented in this paper, see for example [28,36] for further discussion on this issue. The inequalities are more useful as guidelines on the interaction between the different parameters that define the system and the controllers. The main point is that output feedback controllers are less robust to sensor data losses than the corresponding state feedback controller, because if sensor data losses are short, but frequent, and the observer is not able to converge to the actual state, stability might be lost. In practice, an estimate of the robustness properties of an output feedback controller would be better obtained off-line using extensive simulations.

4. Sampled-data output feedback controller

It is important to put into perspective the result of Theorem 1 with respect to the existing results of sampled-data high-gain observer-based output feedback control of nonlinear systems [9,19]. In this subsection we explicitly consider the issue of measurement-sampling in the output feedback controller and study the robustness to data losses of a sampled-data output feedback controller. We start by scaling the observer variables to avoid inherent ill-conditioning of the partial differential equation for small ϵ. Let

$$q = Dz$$

with $D = diag(1, \epsilon, \ldots, \epsilon^{n-1})$. The discrete version of the output feedback controller (4) takes the form

$$\begin{aligned}
u &= k(\hat{x}_k), \ \ t \in [t_k, t_{k+1}] \\
\hat{x}_k &= T^{-1}(sat(z_k)) \\
z_k &= C_q q_k + D_q y_k \\
q_{k+1} &= A_q q_k + B_q y_k
\end{aligned} \tag{19}$$

where $t_k = t_0 + k\Delta$, Δ is the sampling time and matrices A_q, B_q, C_q and D_q depend on the system parameters and the discretization method used. The subindex k denotes the value of a given variable at sampling time t_k; that is, $y_k = y(t_k)$. For a forward difference discretization method, the matrices take the following form [9]:

$$\begin{aligned}
A_q &= I + \Delta D(A - LC)D^{-1} \\
B_q &= \Delta DL \\
C_q &= D^{-1} \\
D_q &= 0.
\end{aligned}$$

In [9,19], it is proved that a high-gain observer-based output feedback controller that enforces global (asymptotic) stability in the closed-loop system under continuous implementation also enforces closed-loop semi-global practical stability under sample-and-hold implementation (with zero-order hold) if the hold time is of the order of ϵ. This result is formalized in the following proposition.

Proposition 2. *Consider the nonlinear system (1) for which Assumptions 1–3 hold, under the sampled-data output feedback controller (19) based on the output feedback controller (4). Then given constants $\delta > \chi > 0$ there exist $\epsilon^*(\delta)$ and $\Delta(\epsilon)$ such that if $\epsilon \in (0, \epsilon^*]$, $\Delta \in (0, \Delta^*]$, $|\hat{z}(t_0)| \leq z_m$ and $x(t_0) \in \Omega_\delta$; the closed-loop system is stable and ultimately bounded in Ω_χ. This stability property implies that given $\epsilon \in (0, \epsilon^*]$, $\Delta \in (0, \Delta^*]$ and some positive constant $e_m > 0$ there exists positive real constant t_b such that if $x(t_0) \in \Omega_\delta$ and $|z(t_0)| \leq z_m$, then $|x(t) - \hat{x}(t)| \leq e_m$ for all $t > t_0 + t_b$.*

This proposition stems from the results presented in [9,19] and can be proved following the same line of thought. The main idea, is that in order to guarantee performance recovery; that is, that the trajectories of the sampled-data system are close enough to the trajectories of the continuous time system, a sufficiently small sampling time of the order of ϵ has to be used. In [9] the sampling time $\Delta = \alpha\epsilon$, where α is a bounded, number that depends on the parameters of the system but is independent of ϵ. This expression implies that if ϵ tends to zero, the sampling time also tends to zero approaching a continuous time implementation; thus, a fast enough sampling is needed to maintain closed-loop stability in the case of sampled-data high-gain output feedback control. While from a sampled-data system point of view, this result is reasonable, from a networked control point of view constraining the maximum time that the loop can be open to be bounded by $k\epsilon$ is very conservative. To this end, in the present work we follow a different approach to prove closed-loop stability under high-gain output feedback control in the presence of sensor data losses by approximately decoupling the tasks of state estimation and feedback control. The sampled-data output feedback controller (19) subject to data losses takes the following form

$$\begin{aligned}
u &= k(\hat{x}_k), \ \ t \in [t_k, t_{k+1}] \\
\hat{x}_k &= T^{-1}(sat(z_k)) \\
z_k &= \begin{cases} C_q q_k + D_q y_k & t_k \in [t_{2i}, t_{2i+1}) \\ z_{k-1} & t_k \in [t_{2i+1}, t_{2i+2}) \end{cases} \\
q_{k+1} &= \begin{cases} C_q q_k + D_q y_k & t_k \in [t_{2i}, t_{2i+1}) \\ q_k & t_k \in [t_{2i+1}, t_{2i+2}) \end{cases}
\end{aligned} \tag{20}$$

where it has been taken into account that when data losses occur, the estimation is not updated so the controller keeps implementing the same manipulated input. The following theorem presents the stability properties of the closed-loop system under the sampled-data output feedback controller subject to data losses. As in the continuous case, the result is given in the form of bound on δ_c and δ_o.

Theorem 2. *Consider system (1) for which Assumptions 1–3 hold under the sampled-data output feedback controller (19). Then, given positive real numbers d and δ that satisfy* $\gamma < d < \delta$; *there exist positive constants* $\epsilon^*(\delta) = \epsilon^*$, $\Delta^*(\epsilon^*)$ *such that if* $\epsilon \in (0, \epsilon^*]$ *and* $\Delta \in (0, \Delta^*]$, *there exist positive constants* $\delta_o^*(\epsilon) = \delta_o^*$ *and* $\delta_c^*(\epsilon) = \delta_c^*$ *such that if* $\delta_o \le \delta_o^*$, $\delta_c \ge \delta_c^*$, $|z(t_0)| \le z_m$ *and* $x(t_0) \in \Omega_\delta$, *then* $\limsup_{t\to\infty} V(x(t)) \le d$.

Proof. The proof of the theorem follows the same lines of the proof of Theorem 1. In what follows we provide a sketch of the proof. First, an upper bound Δ^* on the sampling time is fixed following Proposition 2 in order to recover the performance of the continuous implementation. For $\Delta \in (0, \Delta^*]$, the stability properties of Proposition 2 states that the closed-loop sampled-data system satisfies the following property after the observer has converged close to the real state:

$$\dot{V}(x) \le -\hat{\rho}(V(x)), \quad \forall V(x) \ge c \tag{21}$$

where $\hat{\rho}(\cdot)$ is a positive definite function and c is a positive number that depend on the parameters of the system and on the sampling time Δ.

Next the tasks of state estimation and feedback control are approximately decoupled fixing a lower bound δ_c^* on the minimum time on which the system operated in close-loop after a period of time in which data has been lost. This bound guarantees that the value of the Lyapunov function is lower than or equal to the value of the Lyapunov function at the beginning of the time period, and that the estimation of the state is close enough.

At last, using (21) an upper bound δ_o^* on the maximum time in which the system operated in open-loop is provided. This bound is obtained to guarantee that the system has a negative definite derivative of the Lyapunov function along the whole period of time if the system is outside a region that contains the origin, and that if the system is close enough to the origin, it does not leave a given region. Note that in this case, the upper bound δ_o^* is given on behalf of $\hat{\rho}(\cdot)$ and the region must contain Ω_c. □

Note that the expressions obtained for the bounds on δ_c and δ_o are different from the continuous time case.

Remark 14. As in the continuous time case, the approximate decoupling is done by computing a lower bound on the time the loop shall remain closed to guarantee convergence of the state estimates to the true states and an upper bound on the time that the loop afterwards can remain open to guarantee that the time derivative of the Lyapunov function remains negative outside a neighborhood of the origin. The upper bound on the time interval in which the loop must stay closed depends on the observer properties and is of $O(\epsilon)$. The upper bound on the time interval in which the loop can stay open is independent of ϵ and depends on the closed-loop system properties under state feedback control. Therefore, the loop can remain open for a time interval whose size is larger than $O(\epsilon)$ if this is allowable by the closed-loop system under the state feedback controller.

Remark 15. Note that when a high-gain observer-based output feedback controller is considered in the closed-loop system, as it is done in our work, the packet loss sequence is very different (from the sampling rate-based type of sequence obtained when the sampled-data implementation of a generic dynamic controller is studied) since in this case the lower bound is a function of the rate of convergence of the estimation error only and the upper bound is a function of the state feedback control problem only. This separation is obscured when a generic output feedback control system is considered and its sampled-data implementation is studied and closed-loop stability is proved provided that the sampling rate is sufficiently small, as it is done in [23].

5. Application to a chemical reactor

Consider a well mixed, nonisothermal continuous stirred tank reactor where three parallel irreversible elementary exothermic reactions take place of the form $A \to B$, $A \to C$ and $A \to D$. B is the desired product and C and D are byproducts. The feed to the reactor consists of pure A at flow rate F, temperature T_{A0} and molar concentration C_{A0}. Due to the nonisothermal nature of the reactor, a jacket is used to remove/provide heat to the reactor. Using first principles and standard modeling assumptions, the following mathematical model of the process is obtained:

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{F}{V_r}(T_{A0} - T) - \sum_{i=1}^{3} \frac{\Delta H_i}{\rho_f c_p} k_{i0} \mathrm{e}^{\frac{-E_i}{RT}} C_A + \frac{Q}{\rho_f c_p V_r} \\ \frac{\mathrm{d}C_A}{\mathrm{d}t} &= \frac{F}{V_r}(C_{A0} - C_A) + \sum_{i=1}^{3} k_{i0} \mathrm{e}^{\frac{-E_i}{RT}} C_A\end{aligned} \tag{22}$$

where C_A denotes the concentration of the reactant A, T denotes the temperature of the reactor, Q denotes the rate of heat input/removal, V_r denotes the volume of the reactor, $\Delta H_i, k_{i0}, E_i, i = 1, 2, 3$ denote the enthalpies, pre-exponential constants and activation energies of the three reactions, respectively, and c_p and ρ_f denote the heat capacity and the density of the fluid in the reactor. The values of the process parameters are shown in Table 1. This model satisfies Assumption 1 when the rate of heat input/removal is Q, the input and the concentration of the reactant A, C_A, is the output.

The system of equation (22) has three steady states (two locally asymptotically stable and one unstable). The control objective is to stabilize the system at the open-loop unstable steady state at $T_s = 388\,\mathrm{K}$, $C_{As} = 3.59\,\mathrm{mol/l}$ assuming that only measurements of the concentration of A are available. Data can be lost in the sensor-controller communication link.

We first design a Lyapunov-based state feedback law using the results presented in [34]. We will use this feedback in

Table 1
Process parameters

F	4.998 (m^3/h)	k_{10}	$3*10^6$ (h^{-1})
V_r	1 (m^3)	k_{20}	$3*10^5$ (h^{-1})
R	8.314 (kJ/kmol K)	k_{30}	$3*10^5$ (h^{-1})
T_{A0}	300 (K)	E_1	$5*10^4$ (kJ/kmol)
C_{A0}	4 (kmol/m^3)	E_2	$7.53*10^4$ (kJ/kmol)
ΔH_1	$-5.0*10^4$ (kJ/kmol)	E_3	$7.53*10^4$ (kJ/kmol)
ΔH_2	$-5.2*10^4$ (kJ/kmol)	ρ_f	1000 (kg/m^3)
ΔH_3	$-5.4*10^4$ (kJ/kmol)	c_p	0.231 (kJ/kg K)

combination with a high-gain observer. We identify $x^{\mathrm{T}} = [T\ C_A]$ as the state and $u = Q$ as the input. Consider the control Lyapunov function $V(x) = x^{\mathrm{T}}Px$ with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 10^4 \end{bmatrix}.$$

The values of the weights have been chosen to compensate for the different range of numerical values for each process state. The following feedback law satisfies (2) and globally asymptotically stabilizes system (22) in the case of full-state measurements and no sensor data losses:

$$k(x) = \begin{cases} -\dfrac{L_fV + \sqrt{L_fV^2 + LgV^4}}{L_gV} & \text{if } L_gV \neq 0 \\ 0 & \text{if } L_gV = 0. \end{cases} \tag{23}$$

Functions L_fV and L_gV corresponding to the closed-loop system are sufficiently smooth in the region of interest of this simulation. This has been tested through simulation. Note that this controller does not guarantee local exponential stability of the origin for the closed-loop system under state feedback control, however, the results provided by Theorem 1 still hold, see Remark 11.

The relative degree of the output C_A with respect to the input Q is 2. So system (22) satisfies Assumption 3. We will use the output feedback controller of Proposition 1 with $k(x)$ from (23). The observer parameters are given by $\epsilon = 0.0005$, $a_1 = 2$, $a_2 = 1$, $\delta = 1000$ and $z_m = 7.07$. With these parameters, the closed-loop system is semi-globally asymptotically stable under output feedback control and the error goes below $e_m = 0.001$ in less than $t_b = 0.0025$ h. Fig. 3 shows a simulation of system (22) in closed-loop with the proposed output feedback controller with $k(x)$ from (23) without sensor data losses. The initial state is $T_s = 330\,\mathrm{K}$, $C_{As} = 3.2\,\mathrm{mol/l}$.

When sensor data losses occur, if the minimum time that the system remains in closed-loop after losing sensor data, δ_c, is too small, practical stability of the closed-loop system is not guaranteed. To demonstrate this point, we next show two different simulations with different sensor data losses starting from the same initial condition of the simulation of Fig. 3. To generate the increasing sequence of times that determine when the output is available $\{t_{i\geq 0}\}$, we first use a random Poisson process as in [25] to obtain an auxiliary random sequence $\{\hat{t}_{i\geq 0}\}$. The Poisson process is defined by the number of events per unit time W, and a probability p of losing sensor data. At a given time t, an event takes place that determines whether the

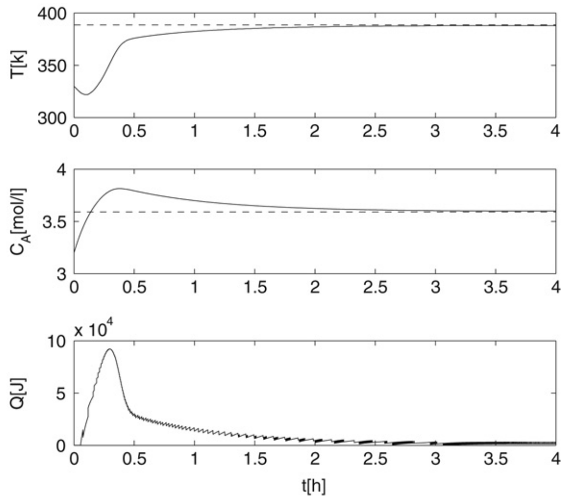

Fig. 3. Trajectories of system (22) in closed-loop with the proposed output feedback controller based on (23) without sensor data losses.

system is in the unstable or in the stable mode for the following period of time. This event is generated using a random variable $\mu \in [0, 1]$ chosen from a uniform probability distribution. For a given probability p, if $\mu \leq p$, then the controller is operating in open-loop, while if $\mu > p$ the controller is operating in closed-loop. The length of the period of time, is generated randomly based on W, the number of events per unit time of the Poisson process. The time for which the system will remain in the chosen mode is given by $\Delta = \frac{-\ln \xi}{W}$, where $\xi \in [0, 1]$ is another random variable chosen from a uniform probability distribution. At $t + \Delta$, another event takes place. This sequence is random and δ_c and δ_o can take any value. In order to impose limits on δ_c and δ_o (recall that the main result is stated in terms of these values), we generate the sequence that is used in the simulations $\{t_{i\geq 0}\}$ as follows:

(1) $t_0 = \hat{t}_0$
(2) for each $i \geq 0$
- if i is odd then $t_{i+1} = t_i + \max\{\hat{t}_{i+1} - \hat{t}_i, \delta_c\}$
- if i is even then $t_{i+1} = t_i + \min\{\hat{t}_{i+1} - \hat{t}_i, \delta_o\}$.

Note that 0 is considered an odd number, and that the system begins in closed-loop. This recursion only modifies those intervals that do not satisfy the constraints on δ_c and δ_o. In the following simulations, the values of the parameters used to generate the time sequence, that is, the number of events per unit time W and probability p of the Poisson process, and the values of δ_c and δ_o, are reported. Also, the rate of sensor data losses is provided, that is the fraction of time of the duration of the simulation in which the controller is operating in open-loop. This parameter is often used to characterize the quality of the communication link, see for example [16,25].

Note that in order to evaluate δ_c^* and δ_o^* the expressions of Theorem 1 are conservative. In practice these constants are better determined by extensive off-line simulations. For the example presented in this section, we have estimated the following values: $\delta_c^* \simeq 0.001$ and $\delta_o^* \simeq 0.05$. We next present two different simulations, one that does not satisfy the constraints on δ_c, and a second one that does satisfy the

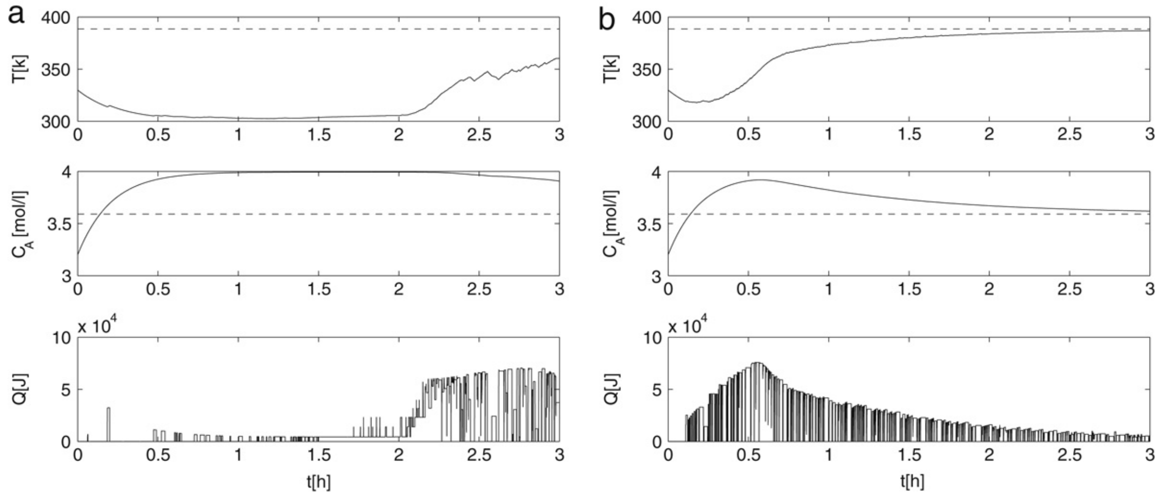

Fig. 4. Trajectories of system (1) in closed-loop with output feedback controller (4) with two different δ_c: (a) $\delta_c = 0.0001$ h and (b) $\delta_c = 0.005$ h.

constraint. Fig. 4(a) shows the trajectories of system (22) in closed-loop with the output feedback controller based on (23) subject to sensor data losses obtained with $W = 500$ events/h, $p = 0.75$, $\delta_o = 0.03$ h and $\delta_c = 0.0001$ h. The resulting time sequence has a sensor data loss rate equal to 74.19%. It can be seen that the state does not converge to the equilibrium. Moreover, it drifts towards one of the other steady states of system (22), namely, $T_s = 400$ K, $C_{As} = 3$ mol/l.

On the other hand, in Fig. 4(b), sensor losses have been obtained with $W = 500$ events/h, $p = 0.9$, $\delta_o = 0.03$ h and $\delta_c = 0.005$ h. The resulting time sequence has a sensor data loss rate equal to 75.16% (although the probability of a sensor data loss event used in the Poisson process was $p = 0.9$) which is similar to the sensor data loss rate of the simulation shown in Fig. 4(a). In this case, the closed-loop trajectory, is stable. This is due to the fact, that in the simulation shown in Fig. 4(a), $\delta_c = 0.0001$ was higher than $\delta_c^* = 0.001$, while in the second simulation δ_c satisfied the lower bound. It follows that, if δ_c is not large enough, the closed-loop system might not be stable.

In both simulations, the input profiles of the output feedback controller show high frequency changes. These abrupt changes take place in the period of time in which the estimated state is converging to the actual state after sensor data losses have occurred (this means that there is an initial estimation error). Because the transient of the estimation error scales with the observer gain, the transient can be very abrupt and the input can take any value in $k(\Omega_\delta)$. One solution to avoid these abrupt changes, is to maintain the input fixed at the last value, until the estimated state has converged to the real state, that is, when measurements are regained at time t_{2i}, the input is not updated as $u = k(\hat{x})$ until time $t_{2i} + t_b$. In this way, when the controller switches to the input computed by using the estimated state of the observer $\hat{x}$, the estimation error is already small. Roughly speaking, we assume that the system is in open-loop also during the time in which the observer is converging. This modification was discussed in Remark 10. In Fig. 5 a simulation with the same parameters as the one in Fig. 4(b) (that is, $W = 500$ events/h, $p = 0.9$, $\delta_o = 0.03$ h and $\delta_c = 0.005$ h) using the modified implementation with $t_b = 0.005$ h is shown. It can be seen that the input trajectory is smoother in comparison to the one in Fig. 4(b) while still preserving closed-loop stability.

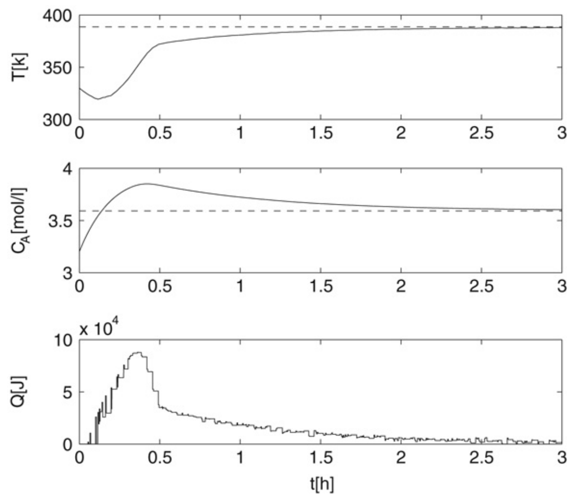

Fig. 5. Trajectories of system (1) in closed-loop with output feedback controller (4) with the modified implementation introduced in Remark 10 and $\delta_c = 0.005$ h.

In Fig. 6, the trajectories of the norm of the estimation error $|e(t)|$ are shown for different simulations. These simulations demonstrate the necessity of a lower bound on the minimum time that the loop should remain closed to achieve approximate decoupling between the estimation of the state, and the stabilization of the closed-loop system. The approximate decoupling is obtained, if when the controller starts operating in open-loop, the estimated state is close to the actual state. In this case the input which will be applied for the whole duration of the open-loop period has been computed on behalf of an estimate with a sufficiently small error. Trajectory $|e_1(t)|$ shows a simulation done with $t_{2i+1} - t_{2i} = 0.01$ h and $t_{2i+2} - t_{2i+1} = 0.03$ h for all i.[3] When sensor data losses occur and feedback is lost in $t \in [0.01, 0.04]$, the error grows slowly with time. In this

[3] Note that the sequence is not random.

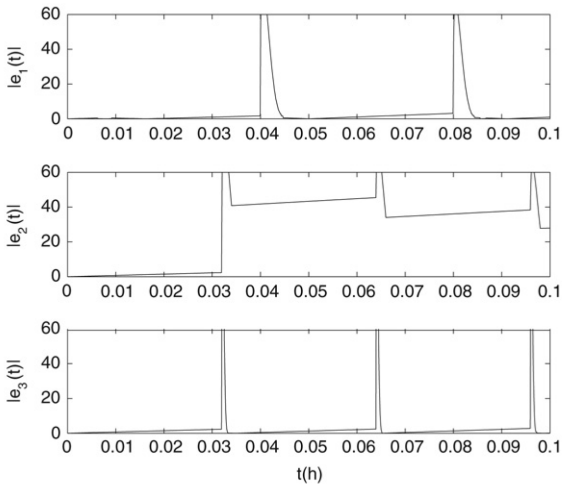

Fig. 6. Estimation errors for different values of δ_c^*, δ_o^* and ϵ.

period of time the estimated state and the input are fixed at the last computed value. When feedback is regained in interval $t \in [0.04, 0.05]$, the estimated state peaks because of the dynamics of the high-gain observer. In this simulation, the system remains in closed-loop long enough for the error to converge to zero, so when feedback is lost again, the estimation error is small. In the second simulation, $|e_2(t)|$ is obtained with a time sequence defined by $t_{2i+1} - t_{2i} = 0.002$ h and $t_{2i+2} - t_{2i+1} = 0.03$ h for all i. In this case, the estimation error does not converge to zero before new sensor data losses occur. In this case, when the controller fixes the input to the last evaluated value, this value has been computed by using an estimate with a very high error. This input does not guarantee that the derivative of the Lyapunov function is negative, furthermore, it may drive the system further away from the equilibrium point (recall that it may take any value in $k(\Omega_\delta)$). In the third simulation, ϵ is set to 0.0001 while the same sensor data losses trajectory as in the second simulation is used, i.e., $t_{2i+1} - t_{2i} = 0.002$ h and $t_{2i+2} - t_{2i+1} = 0.03$ h. In this case, as the observer gain is higher, it is large enough for the estimation error to converge in 0.002 h so that the error is small enough when sensor data losses occur and the loop is open.

Finally, in Fig. 7 the trajectories of the state, the input and the discrete state of the high-gain observer of system (1) in closed-loop with the sampled-data output feedback controller (20) are shown. For this simulation the data losses have been obtained with $W = 100$ events/h, $p = 0.5$, $\delta_o = 0.03$ h and $\delta_c = 0.0001$ h. The sampling time is $\Delta = 0.0005 = \epsilon$. For a higher sampling time, the closed-loop system is not stable. It can be seen, that the closed-loop system can tolerate losses for periods of time greater than ϵ, however, the sampling time of the sampled-data implementation is of the same order of magnitude of ϵ. Fig. 7(b) shows the discrete state of the observer, and it can be seen how the state peaks after feedback is regained. Recall, that the effect of this peaking in the closed-loop system is reduced by the use of a saturation function on the observer estimates (20). For bigger sampling times, or data losses, the observer is not able to converge fast enough to the actual state and the closed-loop system becomes unstable.

6. Conclusions

In this work, we considered the problem of output feedback control of nonlinear systems subject to sensor data losses. We have studied the stability and robustness properties of an output feedback controller resulting from a combination of a Lyapunov-based controller with a high-gain observer for both the continuous and the sampled-data cases. We have proved that in order to approximately decouple the estimation of the actual state from the feedback stabilization the minimum time between two consecutive open-loop periods should be bounded from below, so the estimated state is guaranteed to converge to the actual state value when the loop is closed. Once the approximate decoupling is achieved, practical stability is guaranteed if the maximum time between consecutive measurements is bounded as in state feedback control of nonlinear systems subject to sensor data losses. This result

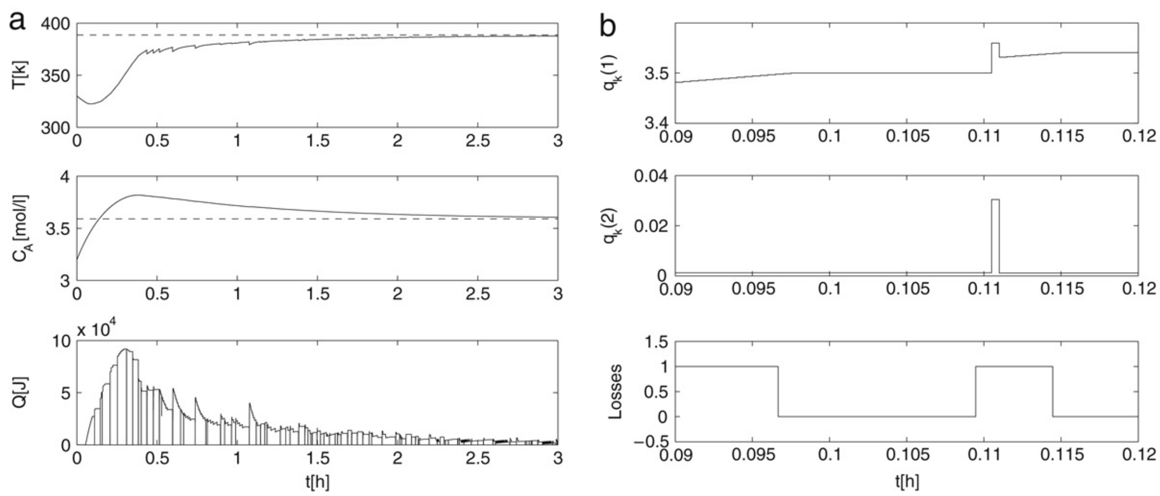

Fig. 7. Trajectories of system (1) in closed-loop with the sampled-data output feedback controller (20): (a) State and input trajectories and (b) observer discrete state and data loss trajectories.

states that in general, output feedback controllers are less robust to sensor data losses than state feedback controllers, and suggests that sensor data losses should be taken into account in the design of the controller and of the observer. Also, alternative output feedback controller implementations in the presence of sensor data losses have been suggested, in particular, using the model to update the input when the loop is open, and switching to the output feedback controller only when the estimation error is sufficiently small.

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