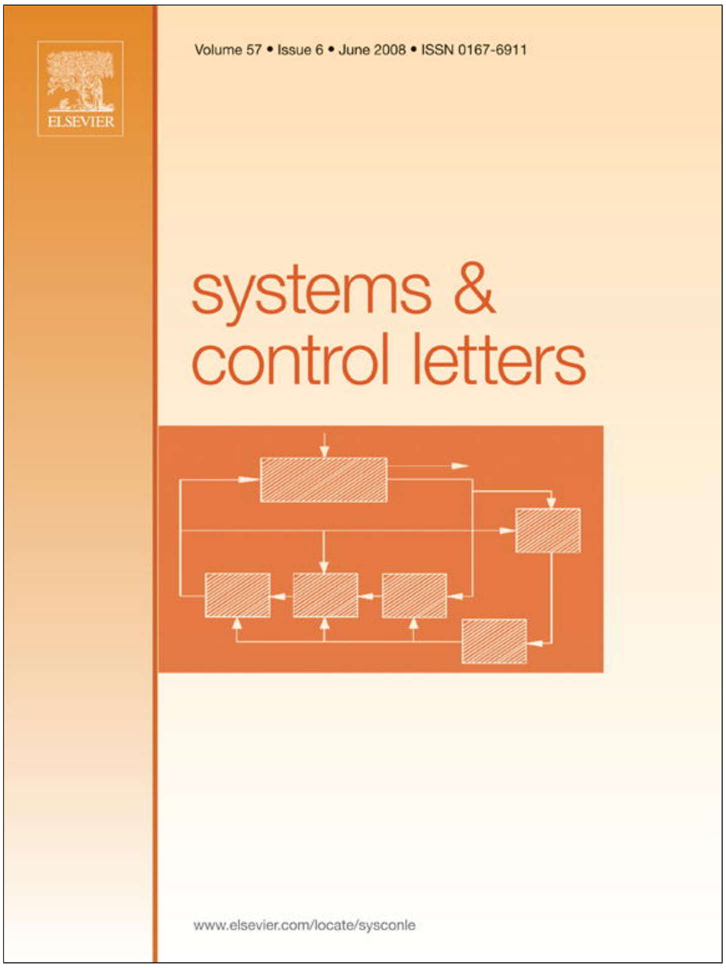
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Volume 57 • Issue 6 • June 2008 • ISSN 0167-6911
systems &
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Systems & Control Letters 57 (2008) 465–473
systems & control letters
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Stability of nonlinear asynchronous systems

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Received 30 January 2007; received in revised form 16 November 2007; accepted 18 November 2007
Available online 31 January 2008

Abstract

In this work, we focus on a class of nonlinear asynchronous systems defined by two different modes of operation, one stable and the other one unstable. The switching between the two modes of operation is driven by external asynchronous events. It is assumed that on any time interval of a given length, the maximum time in which the system evolves in the unstable mode is bounded. This property is given in the form of a rate constraint. Under this assumption, we study the behavior of this class of systems and provide existential results of conditions on this rate constraint under which various types of stability of the origin of the nonlinear asynchronous system can be assured.

Keywords: Asynchronous dynamical systems; Nonlinear systems; Lyapunov theory

1. Introduction

Asynchronous systems are systems with strongly coupled continuous and discrete variables in which the continuous variables obey continuous time ordinary differential equations and the evolution of the discrete variables is randomly determined. They are a class of switched nonlinear systems in which the switching between the different modes of operation is determined by external asynchronous random events. Applications that can be modeled by nonlinear asynchronous systems include control systems in which signals are transmitted over an asynchronous communication network; parallelized numerical algorithms in which the algorithm is separated into several local algorithms with cross-algorithm communication; and queuing networks in which jobs arrive and are serviced asynchronously at fixed rates [1].

Most of the works in systems theory literature that study the stability properties of this class of systems focus on asynchronous systems characterized by linear continuous dynamics, see for example [2] for results on jump linear systems, [3,4] for results on switched linear systems and [5–7] for results on linear asynchronous and stochastic hybrid systems. There are few results that deal with nonlinear dynamics, see for example [8] and the references therein for results on stochastic nonlinear differential equations.

Within control theory, asynchronous systems have been studied in the context of networked control systems (NCS). NCS are control systems which have the control loops closed via a network and can often be modeled as asynchronous dynamical systems [9,10]. The different modes of operation of the asynchronous model represent different modes of operation of the network. For example, an unreliable communication channel may be transmitting data or not. This is the case, for example, in control systems based on wireless communications [11]. Stability of NCS has been studied in [12–18]. These results are based on techniques similar to those used in the analysis of sampled-data systems (see [19,20] for example).

In this work, we focus on a class of nonlinear asynchronous systems defined by two different modes of operation, one stable and the other one unstable. This model captures most of the possible behaviors of more complex asynchronous systems defined by a higher number of modes. The switching between the two modes of operation is driven by external asynchronous events. It is assumed that the rate of occurrence of these events is constrained in such a way that on any time interval of a given

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doi:10.1016/j.sysconle.2007.11.006

length, the fraction of time in which the system evolves in the unstable mode is bounded. Using a Lyapunov-based approach as the one used in [1], we precisely state the conditions under which stability of the origin (in a sense to be made precise in Section 3) can be guaranteed. The results are demonstrated through numerical examples.

In the next section, we introduce the class of systems considered. In Section 3, we present the main results of the paper. In Section 4, we show some numerical examples. In Section 5, we present some concluding remarks.

2. Nonlinear asynchronous systems

We focus on nonlinear asynchronous systems with the following state-space description

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \tag{1}$$

where $x(t) \in R^n$ is the continuous state, and $\sigma(t) \in \{s, u\}$ is the discrete state. The discrete state σ determines at each time instant, which vector field f_σ governs the evolution of the continuous state. We assume that $x = 0$ is a globally asymptotically stable equilibrium point for $\dot{x} = f_s(x)$ (i.e., the stable mode) and a single unstable equilibrium point for $\dot{x} = f_u(x)$ (i.e., the unstable mode). This means that $f_s(0) = f_u(0) = 0$. We also assume that there exists a continuously differentiable Lyapunov-like function $V : R^n \to R$ such that

$$\begin{aligned} & V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{in } R^n - \{0\} \\ & |x| \to \infty \Rightarrow V(x) \to \infty \\ & L_{f_s} V(x) \le -\alpha_s V^{m_s}(x) \\ & L_{f_u} V(x) \le -\alpha_u V^{m_u}(x) \end{aligned} \tag{2}$$

with $\alpha_s > 0$, $\alpha_u < 0$ and $m_s, m_u > 0$. The notation $L_f V(\cdot)$ denotes the standard Lie derivative of the scalar function $V(\cdot)$, with respect to the vector function $f(\cdot)$. In the previous works on stability of asynchronous nonlinear systems, see for example [12–16], no assumption is made on the open-loop upper bound on the time derivative of the Lyapunov-like function. The knowledge of this bound, in particular the exponent, leads to sharper stability results.

The value of the discrete state of system (1), $\sigma(t)$, depends on external asynchronous events that trigger switching between the two modes of operation (and so between the two possible discrete states). This model is of particular interest in NCS when the controller and the system are connected by an unreliable communication link. In this case, the closed-loop system can be described by two different modes of operation, one in which data from the sensor to the controller is transmitted so the input applied depends on the actual state of the system (closed-loop operation), and another mode of operation, in which data is successfully transmitted, and the input is decided without any feedback, for example setting it to zero or to the last computed value (open-loop operation). In this case, if the open-loop system is unstable and the controller is carefully designed, one of the modes is stable (when data is transmitted) and the other may be potentially unstable (when there is no feedback).

We assume that the discrete state $\sigma(t)$ is piecewise constant, that is, for every bounded interval of time, the points where $\sigma(t)$ is discontinuous are finite (see [21]). The trajectories of the discrete state of (1), also satisfy the following rate constraint

$$\frac{1}{T}\int_t^{t+T} e(\sigma(\tau))\mathrm{d}\tau \le r, \quad \forall t > 0 \tag{3}$$

where $r, T > 0$ and $e(\cdot) : \{s, u\} \to \{0, 1\}$ is a function that indicates if system (1) is operating in the stable mode ($e(s) = 0$) or in the unstable mode ($e(u) = 1$). This constraint implies that over every finite-time interval of length T, system (1) is operating in the unstable mode for a total time equal to or less than $T \times r$. This constraint will allow us to consider deterministic stability notions; see Remark 3 below.

Remark 1. The trajectories of system (1) may fail to be defined on all times because of possible finite escape time. In the results presented in this work however, conditions are given such that the trajectories are well defined (i.e., $x(t)$ is continuous and bounded for all times).

Remark 2. The two-mode model captures the main stability properties of asynchronous dynamical systems because for systems with multiple modes, these modes can be divided into two groups, stable modes and unstable modes. Each of these two groups can then be characterized with a single mode using a worst case approach.

Remark 3. If the stochastic event generation allows for arbitrary switching trajectories (i.e. arbitrary evolution of $\sigma(t)$ with respect to t), deterministic stability cannot be assured because the system may evolve indefinitely in an unstable mode. For this reason it is necessary to include the rate constraint (3) defined in a finite-time interval of length T. Such a constraint is also meaningful in the context of networked control problems, where the performance of a network is often measured in terms of the fraction of successful transmissions during a given time interval.

Remark 4. If the rate constraint (3) is defined in the infinite-time interval (i.e., $T = \infty$), the system may remain in the unstable mode for arbitrarily long periods of time. In this case, no stability guarantees can be obtained. This is explained in detail in the following section.

Remark 5. The asynchronous systems under consideration are a generalization of the class of systems studied in [1]. In [1], although multiple nodes were taken into account, only exponential stability was considered, that is $m_s = m_u = 1$.

Remark 6. The class of systems considered in this paper is different from the class of systems studied in [22,23] and other related works. In those works, hybrid systems with stable continuous dynamics are studied. No stochastic properties are assumed in the switching and no unstable modes are considered.

Remark 7. The rate constraint (3) introduced in this section is not related with the definition of average dwell time used in switched systems, see for example [24].

Remark 8. It is possible to study the stability properties of this class of systems under alternative assumptions on the dynamics of the stable and unstable modes. In this direction, it would be interesting to investigate proving Lyapunov stability assuming, for example, the existence of a symmetric positive definite matrix $P \in R^{n\times x}$ that satisfies the so-called "relaxed one-sided Lipschitz condition".

3. Stability of nonlinear asynchronous systems

In this section, we present a Lyapunov-based stability theory for the class of systems presented in the previous section. We extend the results presented in [1], studying systems with asymptotically stable and unstable modes. We characterize a rate r^* and an interval of length T^* such that if $r < r^*$ and $T < T^*$, then it is guaranteed that the origin of system (1) is stable (in a sense to be made precise below). The main idea, is that r and T must assure that the Lyapunov-like function $V(x)$ decreases *on average*, and that the trajectories $x(t)$ are well defined for all times (i.e., $x(t)$ is continuous and bounded for all times).

In the following theorems we show that the stability properties of system (1) are mainly characterized by the relative values of m_s and m_u. There are three possible cases:

- $m_s = m_u$
- $m_s > m_u$
- $m_s < m_u$.

In what follows, three different theorems characterize the stability properties of system (1) on behalf of the relative values of m_s and m_u. We begin with the case $m_s = m_u = m$ and distinguish three cases, $m < 1$, $m = 1$ and $m > 1$. This theorem uses the definitions of asymptotic and finite-time stability introduced in [25,26].

Theorem 1. *Consider system (1) under a switching sequence that satisfies (3) and let $V(x)$ be a Lyapunov-like function that satisfies (2) with $m_s = m_u = m$. If $m < 1$, there exists a constant r^* such that if $r \in (0, r^*]$, then for any finite T, $x = 0$ is a globally finite-time stable equilibrium point for system (1). If $m = 1$, there exists a constant r^* such that if $r \in (0, r^*]$, then for any finite T, $x = 0$ is a globally asymptotically stable equilibrium point for system (1). If $m > 1$, given a positive real number $\rho > 0$, there exist constants r^* and T^* such that if $r \in (0, r^*]$ and $T \in (0, T^*]$ then $x = 0$ is an asymptotically stable equilibrium point for system (1) with region of attraction Ω_ρ.*[1]

Proof. The proof consists of three parts, one for each claim.

Part 1: In this part of the proof, we are going to consider the case $m < 1$. In order to prove finite-time stability of the origin we need to prove finite-time convergence of the state to the origin and Lyapunov stability (which implies that the trajectories are well defined for all times, see [25]). Because the system switches between a stable and an unstable mode of operation, the time derivative of the Lyapunov function is not negative for all times. This means that in order to apply a Lyapunov approach, we have to obtain a different convergence property, in particular we will prove that for a sufficiently small r, the Lyapunov function is decreasing on average on any interval of length T; that is $V(x(t_0)) > V(x(t_0+T))$ for all t_0. Then, using this property, finite-time convergence of the state to the origin and Lyapunov stability are proved.

Consider the time period $t \in [t_0, t_0+T]$. An increasing sequence of $M+1$ times $\{t_{i>0}\}$ can be found such that $t_0 = kT$, $t_M = t_0 + T$, and the discrete state of the system is a constant in each interval; that is, (abusing notation $\sigma(\cdot)$) $\sigma(t) = \sigma(i)$, $\forall t \in [t_i, t_{i+1}]$. Integrating the time derivative of $V(x)$ (recall (2)), the following inequality is obtained

$$\frac{1}{1-m}(V(x(t))^{1-m} - V(x(t_i))^{1-m}) \leq -\alpha_{\sigma(i)}(t-t_i) \tag{4}$$

for all $t \in [t_i, t_{i+1})$. Note that $V(x) \geq 0$. If for a given time t_f, $V(x(t))$ converges to zero, the state has converged to the equilibrium point and remains there; that is, $x(t) = 0, \forall t \geq t_f$. In this case, finite-time convergence to the origin has been achieved, see [26]. We are going to use this inequality to relate the value of the Lyapunov function at t_0 with the value at t_M where $t_M = t_0 + T$.

Using (4), the following inequality that relates the value of $V(x)$ between switching times in the time interval of length T for every solution of system (1) is obtained

$$V(x(t_{i+1}))^{1-m} \leq V(x(t_i))^{1-m} - (1-m)\alpha_{\sigma(i)}(t_{i+1}-t_i).$$

Applying recursively this inequality from t_0 to t_M and taking into account the definition of function $e(\cdot)$, the following inequality follows

$$\begin{aligned} V(x(t_M))^{1-m} \leq\ & V(x(t_0))^{1-m} \\ & - (1-m)\alpha_s\left(1 - \int_{t_0}^{t_M} e(\sigma(\tau))\mathrm{d}\tau\right) \\ & - (1-m)\alpha_u \int_{t_0}^{t_M} e(\sigma(\tau))\mathrm{d}\tau. \end{aligned} \tag{5}$$

Supposing that the system evolves for the maximum amount of time equal to rT in the unstable mode (note that if the system remains in the unstable mode for a shorter time, the value of $V(x)$ will be lower), and taking into account that $t_M - t_0 = T$, $\alpha_s > 0$ and $\alpha_u < 0$, the following upper bound in the value of $V(x(t_M))$ is obtained using the rate constraint (3)

$$\begin{aligned} V(x(t_M))^{1-m} \leq\ & V(x(t_0))^{1-m} \\ & - (1-m)(\alpha_s(1-r) + \alpha_u r)T. \end{aligned} \tag{6}$$

For a choice of

$$r^* \leq \frac{\alpha_s - \alpha^*}{\alpha_s - \alpha_u} \tag{7}$$

[1] The notation Ω_r denotes the set $\Omega_r := \{x \in R^n | V(x) \leq r\}$.

where $\alpha^* \in (0, \alpha_s)$, Eq. (6) yields

$$V(x(t_M))^{1-m} \leq V(x(t_0))^{1-m} - (1-m)\alpha^* T.$$

Note that this equality only holds for $V(x) \geq 0$. Using this inequality, we will prove finite-time convergence to the origin from any initial state $x(0)$. Applying this inequality recursively and taking into account that when $V(x) = 0$, the system has converged to the origin and remains there, the following bound on the value of the Lyapunov-like function at time $t = kT$ with $k = 0, 1, \ldots$ is obtained

$$V(x(kT))^{1-m} \leq \max\{V(x(0))^{1-m} - k(1-m)\alpha^* T, 0\}$$

For any initial state $x(0)$, there exists $k^*(x(0))$ such that $V(x(t)) = 0$ for all $t > k^*T$ which implies that $x(t) = 0$ for all $t > k^*T$. Finite-time convergence to the origin is proved for all finite T. Note that if $T = \infty$ as in [1], k^*T is not a finite quantity for finite k^*.

Lyapunov stability, i.e., for each $\epsilon > 0$, there is $\delta > 0$ such that $V(x(0)) < \delta$ implies that $V(x(t)) < \epsilon$ for all $t > t_f$ (see [25]), is guaranteed for all $r \in (0, r^*]$ and any finite T because finite-time stability implies that given any ϵ, $V(x(t)) < \epsilon$ for all $t > t_f$ where $t_f = k^*T$; that is, the time in which the system converges to the origin. Global finite-time stability of the origin is guaranteed because $V(x)$ is radially unbounded, see [25], Theorem 3.2.

Part 2: In this part we are going to consider the case $m = 1$. In order to prove asymptotic stability of the origin we need to prove asymptotic convergence of the state to the origin and Lyapunov stability. Following the same reasoning used in Part 1 to obtain (4), the following inequality is obtained for the value of the Lyapunov-like function between switching times t_i for every solution of system (1)

$$V(x(t)) \leq V(x(t_i))\mathrm{e}^{-\alpha_{\sigma(i)}(t-t_i)} \tag{8}$$

for all $t \in [t_i, t_{i+1})$. In this case finite-time convergence to the origin is not possible. Averaging over the time interval $[t_0, t_0 + T]$, and for a choice of $r^* > 0$ and $\alpha^* > 0$ that satisfy (7), we have that

$$V(x(t_0 + T)) \leq V(x(t_0))\mathrm{e}^{-\alpha^* T}. \tag{9}$$

Applying (9) recursively we obtain the following upper bound on the trajectories of the Lyapunov function along the trajectories from initial state $x(0)$

$$V(x(kT)) \leq V(x(0))\mathrm{e}^{-\alpha^* kT}. \tag{10}$$

This proves the asymptotic convergence of the state to the origin.

We now have to prove stability in the Lyapunov sense. Using recursively (13) it holds that if $V(x(0)) \leq \delta$ then $V(x(kT)) \leq \delta$. We need to introduce an auxiliary function $w(\tau, \delta)$ that bounds the value of the Lyapunov-like function when the system evolves in the unstable mode for τ time from an initial value of δ. Taking into account (8), for $m = m_u = 1$

$$w(\tau, \delta) = \delta\mathrm{e}^{-\alpha_u \tau}.$$

Note that this function takes a different expression depending on the value of m_u. The maximum value that the Lyapunov function can attain from an initial state in Ω_δ in the time period T is given by the trajectory in which the system operates at the beginning of the time period in the unstable mode of operation for the maximum time; that is, rT. It follows that for all t_0

$$V(x(t)) \leq w(rT, V(x(t_0))); \quad \forall t \in [t_0, t_0 + T].$$

Using this inequality, for any $\epsilon > 0$, for a choice of δ such that

$$\epsilon > w(rT, \delta)$$

it holds that if $V(x(0)) \leq \delta$, then $V(x(t)) \leq \epsilon$ for all times. It is always possible to find a δ that satisfies this inequality because

$$\lim_{\delta \to 0} w(\tau, \delta) = 0$$

for $m = 1$. Global asymptotic stability of the origin is guaranteed because $V(x)$ is radially unbounded, see [25], Theorem 3.2.

Part 3: In this part we are going to consider the case $m > 1$. In order to prove asymptotic stability of the origin we need to prove asymptotic convergence of the state to the origin and Lyapunov stability. Following the same reasoning used in Part 1 to obtain (4), the following upper bound on the value of $V(x(t))$ between switching times for every solution of system (1) is obtained

$$\frac{1}{V(x(t))^{m-1}} \geq \frac{1}{V(x(t_i))^{m-1}} + (m-1)\alpha_{\sigma(i)}(t - t_i). \tag{11}$$

Averaging over the time interval $[t_0, t_0 + T]$, and using (3), for a choice of $r^* > 0$ and $\alpha^* > 0$ that satisfy (7), we have that

$$\frac{1}{V(x(t_0 + T))^{m-1}} \geq \frac{1}{V(x(t_0))^{m-1}} + (m-1)\alpha^* T. \tag{12}$$

Applying (12) recursively we obtain the following upper bound on the trajectories of the Lyapunov function along the trajectories from initial state $x(0)$

$$\frac{1}{V(x(kT))^{m-1}} \geq \frac{1}{V(x(0))^{m-1}} + (m-1)\alpha^* kT. \tag{13}$$

This proves the asymptotic convergence of the state to the origin.

We now have to prove stability in the Lyapunov sense (which implies that the trajectories must be well defined for all times). In this case, because $m > 1$ and $\alpha_u < 0$, finite escape time might occur if the system evolves in the unstable mode long enough. For the unstable mode, as $\alpha_u < 0$, the upper bound on the value of the cost function goes to infinity in $t \in [t_i, t_{i+1})$ if

$$\frac{1}{V(x(t_i))^{m-1}} < -(m-1)\alpha_u(t_{i+1} - t_i). \tag{14}$$

In order to guarantee that when $x(0) \in \Omega_\rho$, the trajectory of $x(t)$ will be well defined for all times, an appropriate finite T must be chosen such that for any initial state inside Ω_ρ, any possible trajectory will not go to infinity in the interval of length T; that is, the upper bound remains finite in any time interval in which the systems evolves in the unstable mode. Now, for a given T, r and ρ, the worst case scenario involves an initial value $V(x(t_0)) = \rho$, and a time partition such that the system operates in the unstable mode for the maximum time possible; that is, for rT. To guarantee that the value of $V(x(t))$ remains

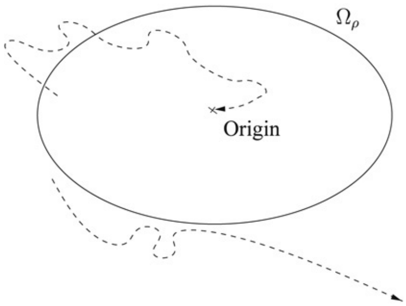

Fig. 1. Possible trajectories of the state when $m_s = m_u > 1$. Theorem 1 states that if $r \in (0, r^*]$, $T \in (0, T^*]$ and $x(t_0)$ is inside Ω_ρ, then it is guaranteed that the system converges to the origin even if it leaves Ω_ρ. On the other hand, if $x(t_0)$ is outside Ω_ρ, there is the possibility that the system state goes to infinity in finite time. Note that in this case the system state never enters Ω_ρ.

bounded in that time interval, (14) must hold for $t_{i+1} - t_i = rT$. It follows that for a choice of

$$T^* < \frac{-1}{r^* \rho^{m-1}(m-1)\alpha_u} \tag{15}$$

it is assured that $V(x(t))$ remains bounded for all $r \in (0, r^*]$ in $t \in [t_0, t_0 + T]$ for any initial value $V(x(t_0)) \leq \rho$. The $\epsilon - \delta$ stability condition is proved following the same ideas as in part 2 and taking into account that for $m > 1$

$$w(\tau, \delta) = \frac{\delta}{\sqrt[m-1]{1 + \delta(m-1)\alpha_u \tau}}$$

which implies that

$$\lim_{\delta \to 0} w(\tau, \delta) = 0.$$

In conclusion, picking $r \in (0, r^*]$ and $T \in (0, T^*]$ we can guarantee that for all $x(0) \in \Omega_\rho$ the trajectories converge asymptotically to the origin and that the origin is stable in the Lyapunov sense. In Fig. 1 possible trajectories of the state when $m_s = m_u > 1$ are shown. □

Remark 9. In the proof of Theorem 1 uniqueness of the solution has not been proved, however the stability results of Theorem 1 hold for every solution of system (1). Note that it cannot be assumed that the vector fields f_s and f_u are locally Lipschitz for $m < 1$, and additional technical assumptions would have to be used instead to prove uniqueness for this case.

Remark 10. For nonlinear asynchronous systems the constraint that the time derivative of the Lyapunov function must be negative is relaxed to be negative on the average in a time interval of length T. For this reason, the proof of Theorem 1 begins characterizing an upper bound on the rate r^* that guarantees this property. After this property is proved, the rest of the proof follows standard Lyapunov arguments.

Remark 11. Global exponential stability can be proved for the case $m_u = m_s = 1$ if additional assumptions are considered, in particular if there exist positive constants K_1, K_2 and p such that

$$K_1|x|^p \leq V(x) \leq K_2|x|^p$$

for all $x \in R^n$.

Remark 12. When $m_s = m_u = m$, the maximum rate r^* for a given mean decay rate α^*, is independent of T, m and ρ. It only depends on α_s and α_u, because stability is achieved assuring that $V(x)$ decreases *on average*. For $m > 1$, the maximum length of the interval T^* depends on m, ρ, r^* and α_u. It does not depend on α_s, because it is fixed by the minimum finite escape time of the unstable mode for an initial state inside Ω_ρ.

Remark 13. The upper limit on T^*, is not related to the maximum allowable transmission time studied in the NCS literature (see, for example [12–16]). In those works, the maximum time the system can remain in open-loop (i.e., in the stable mode) is evaluated in such a way that it is guaranteed that the time derivative of the Lyapunov-like function is negative for all times. This can only be assured outside a neighborhood of the origin, so only practical stability was obtained. In this work, however, we allow the Lyapunov-like function to grow, as long as it decreases between intervals of length T (i.e., it decreases on average). For this reason, in the results presented in this paper, asymptotic stability can be guaranteed. The upper limit on T^*, must be included to assure that the origin is stable in the Lyapunov sense. If the system evolves in an arbitrarily large interval of time in the unstable mode, it may move away from the origin, and the stability in the $\epsilon - \delta$ sense is not guaranteed.

Theorem 2. *Consider system (1) under a switching sequence that satisfies (3) and let $V(x)$ be a Lyapunov-like function that satisfies (2) with $m_s > m_u$. If $m_u \leq 1$, given a real number $d > 0$, there exist constants r^* and T^* such that if $r \in (0, r^*]$, $T \in (0, T^*]$, then for any initial state $x(0)$, $x(t)$ is well defined for all times and $\limsup_{t\to\infty} V(x(t)) \leq d$. If $m_u > 1$, given real numbers $\rho, d > 0$, there exist constants r^* and T^* such that if $r \in (0, r^*]$, $T \in (0, T^*]$ and $x(0) \in \Omega_\rho$, then $x(t)$ is well defined for all times and $\limsup_{t\to\infty} V(x(t)) \leq d$.*

Proof. For a choice of $\hat{\alpha}_u = \alpha_u(\frac{d}{2})^{m_u - m_s}$, we have that

$$L_{f_u} V(x) \leq -\alpha_u V^{m_u}(x) \leq -\hat{\alpha}_u V^{m_s}(x), \quad \forall x \notin \Omega_{\frac{d}{2}}. \tag{16}$$

Now, we will follow the same path as in the proof of Theorem 1, to prove that there exist constants r^* and T^* such that if $r \in (0, r^*]$, $T \in (0, T^*]$ and $x(0) \in \Omega_\rho$, there exists a finite time t_f such that $x(t_f) \in \Omega_{\frac{d}{2}}$.

Assume that $m = m_s < 1$. Taking into account that (16) holds only for $x \notin \Omega_{\frac{d}{2}}$, for a choice of r^* and α^* that satisfy (7) for $\hat{\alpha}_u$ and α_s, we obtain the following inequality for every solution of system (1) for which $V(x(t)) \geq \frac{d}{2}$ for $t \in [t_0, t_0+T]$

$$V(x(t_0 + T))^{1-m} \leq V(x(t_0))^{1-m} - (1-m)\alpha^* T.$$

By continuity of the trajectory of $V(x(t))$ (recall (4)), if $x(t_0) \in \Omega_\rho$ there exists a finite time t_f such that $V(x(t_f)) \leq \frac{d}{2}$ because $V(x(t_0 + kT))$ is a decreasing series of numbers.

Once inside $\Omega_{\frac{d}{2}}$, we cannot assure that the Lyapunov-like function will decrease on average because (16) does not hold. However, we can guarantee that for the worst possible case, if the state reaches $\Omega_{\frac{d}{2}}$, it will remain inside Ω_d for all times. For a given T and d, the worst possible case involves an initial value $V(t_0) = \frac{d}{2}$, and a time partition such that the system operates in

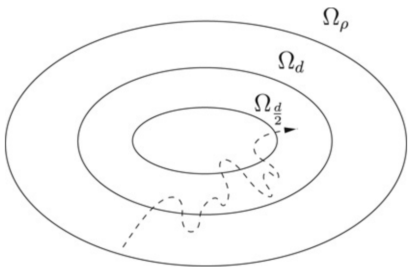

Fig. 2. Possible trajectory of the state when $m_s > m_u > 1$. Theorem 2 states that if $r \in (0, r^*]$, $T \in (0, T^*]$ and $x(t_0)$ is inside Ω_ρ, the system reaches $\Omega_{\frac{d}{2}}$. Once the system enters $\Omega_{\frac{d}{2}}$, it remains bounded in Ω_d for all times.

the unstable mode for the maximum time possible; that is, for rT. If the upper bound of this worst case trajectory is smaller than or equal to d, and r is chosen such that $V(x(t_0+T)) \le V(x(t_0)) \le \frac{d}{2}$, then it is assured that if $x(t_f) \in \Omega_{\frac{d}{2}}$, then $x(t) \in \Omega_d$ for all $t \in [t_0, t_0+T]$ and $x(t_0+T) \in \Omega_{\frac{d}{2}}$. Applying this result recursively, practical stability is proved. In order to characterize the rate constraint that guarantees this statement, we take into account that

$$\max\{V(x(\tau)),\ \text{s.t. } \dot{x} = f_u(x), x(0) \in \Omega_\delta\} \le w(\tau, \delta)$$

For a choice of r^* and T^* such that the following inequality is satisfied

$$w\left(r^*T^*, \frac{d}{2}\right) \le d. \tag{17}$$

it holds that if $r < r^*$ and $T < T^*$ then once the system reaches $\Omega_{\frac{d}{2}}$, it remains bounded in Ω_d for all times. If this inequality is satisfied, then practical stability of the equilibrium point is guaranteed for all initial states. The proof for $m_s \ge 1$ is similar.

For $m_u \le 1$, ρ can take any value (i.e., global practical stability). If $m_u > 1$ however, in addition to constraint (17), an additional constraint must be taken into account to guarantee that no finite escape time occurs during an interval; that is, T^* must also satisfy (15) for the given ρ. In this case, practical stability is guaranteed for all initial states inside Ω_ρ. In Fig. 2 possible trajectories of the state when $m_s > m_u > 1$ are shown. □

Remark 14. Using the same approach as in Theorem 2 it can be proved that if $m_s > m_u$, then the origin cannot be asymptotically stable. For any $r > 0$, because $m_s > m_u$, as the state x approaches the origin, the norm of the time derivative of the stable mode $|L_{f_s}V(x)|$ goes to zero faster than the norm of the time derivative of the unstable mode $|L_{f_u}V(x)|$; that is

$$\lim_{x\to 0} \frac{|L_{f_s}V(x)|}{|L_{f_u}V(x)|} = \infty.$$

This means that no matter how small r is, or how large α_s is, in a neighborhood of the origin, the Lyapunov-like function will grow on average, and so, only practical stability can be assured.

Theorem 3. *Consider system (1) under a switching sequence that satisfies (3) and let $V(x)$ be a Lyapunov-like function that satisfies (2) with $m_s < m_u$. Then given a real number $\rho > 0$, there exist constants r^* and T^* such that if $r \in (0, r^*]$ and $T \in (0, T^*]$, then $x = 0$ is an asymptotically stable equilibrium point for (1) with region of attraction Ω_ρ.*

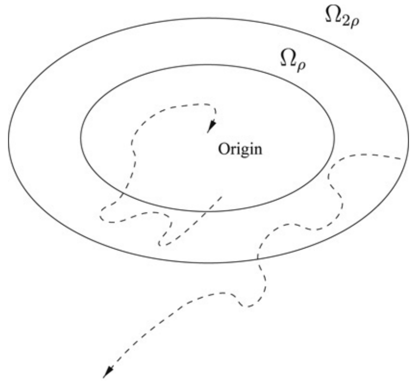

Fig. 3. Possible trajectories of the state when $m_s < m_u$. Theorem 3 states that if $r \in (0, r^*]$, $T \in (0, T^*]$ and $x(t_0)$ is inside Ω_ρ, then it is guaranteed that the system reaches the origin and that the trajectory remains bounded in $\Omega_{2\rho}$. On the other hand, if $x(t_0)$ is outside Ω_ρ, there is the possibility that the systems goes to infinity because it may leave $\Omega_{2\rho}$ and then the unstable mode is dominant. Note that in this case the system never enters Ω_ρ.

Proof. For a choice of $\hat{\alpha}_u = \alpha_u(2\rho)^{m_u-m_s}$, we have that

$$L_{f_u}V(x) \le -\alpha_u V^{m_u}(x) \le -\hat{\alpha}_u V^{m_s}(x), \quad \forall x \in \Omega_{2\rho}. \tag{18}$$

We consider first the case $m_s < 1$ and set $m_s = m$. Taking into account that (18) holds for $x \in \Omega_{2\rho}$, for a choice of r^* and α^* that satisfy (7) for α_s and $\hat{\alpha}_u$, we have that for every solution of system (1) for which $V(x(t)) \le 2\rho$ for all $t \in [t_i, t_i+T]$

$$V(x(t_i+T))^{1-m} \le \max\{V(x(t_i))^{1-m} - (1-m)\alpha^*T, 0\} \tag{19}$$

First we chose r^* and T^* such that if $x(t_i) \in \Omega_\rho$ then $x(t) \in \Omega_{2\rho}$ for $t \in (t_i, t_i+T]$. For a given ρ, the worst case scenario involves an initial state such that $V(x(t_i)) = \rho$, and a time partition such that the system operates in the unstable mode for the maximum time possible; that is, for r^*T^*. For this case, we have that if following inequality is satisfied

$$w(r^*T^*, \rho) \le 2\rho \tag{20}$$

it follows that $V(x(t)) \le 2\rho$ for $t \in (t_i, t_i+T]$. This implies that if $r \in (0, r^*]$, $T \in (0, T^*]$ and $x(t_i) \in \Omega_\rho$ then $x(t) \in \Omega_{2\rho}$ for $t \in (t_i, t_i+T]$. Thus, if (20) is satisfied and $m_u \le 1$, asymptotic stability of the origin for all $x(0) \in \Omega_\rho$ can be proved by applying (19) recursively to prove that $x(t) \in \Omega_{2\rho}$ for all times and $\lim_{t\to\infty} V(x(t)) = 0$. If $m_u > 1$, in addition to constraints (7) and (20), an additional constraint must be taken into account to guarantee that no finite escape time occurs during an interval of length T; that is, T^* must also satisfy (15) for the given ρ. The proof for $m_s \ge 1$ is similar. In Fig. 3 possible trajectories of the state when $m_s < m_u$ are shown. □

Remark 15. When $m_s < m_u$, only local results can be obtained. This is because, if the rate constraint is defined in a finite-time interval, for a closed set containing the origin, the stable mode is dominant and asymptotic stability of the origin is assured. However, for states far from the origin, the unstable mode dominates the stable one, and the system becomes unstable on average.

Remark 16. In Theorems 1–3, the upper bound on the rate r^* does not depend on the length of the interval of the rate constraint T. It only depends on the system parameters. For a given data loss rate, in order to guarantee stability, the maximum time in which the system operates in the unstable mode must be bounded. This is guaranteed by the upper bound T^* which does depend on r^*. This demonstrates that both the rate and the interval of definition of this rate are important to characterize the behavior of asynchronous nonlinear systems.

Remark 17. The stability analysis results presented in this paper can be applied in the context of networked control systems to obtain sharper results when the open-loop exponent of the upper bound of the Lyapunov-like function is known. In this way, for a nonlinear system, if the controller is chosen properly so that $m_s = m_u$, asymptotic stability of the closed-loop system with data losses can be obtained, where the previous results would only guarantee practical stability.

Remark 18. Although the proofs of the theorems of this section are constructive, the constants obtained are conservative. This is the case with most of the results of the type presented in this paper, see for example [20] for further discussion on this issue. The inequalities are more useful as guidelines on the interaction between the different parameters that define the system.

4. Numerical examples

4.1. Switching events generation

In this section we present some numerical examples to demonstrate the application of the theoretical results. First, we describe how we model the switching between the two modes of system (1). We are going to use a Poisson process as in [16]. At a given time t, an event takes place that determines whether the system is in the unstable or in the stable mode for the following period of time. This event is generated using a random variable $p \in [0, 1]$ chosen from a uniform probability distribution. For a given rate r, if $p \leq r$, then the system is in the unstable mode, while if $p > r$ the system is in the stable mode. The length of the period of time, is generated randomly based on W, the number of events per unit time of the Poisson process. The time for which the system will remain in the chosen mode is given by $\Delta = \frac{-\ln \xi}{W}$, where $\xi \in [0, 1]$ is another random variable chosen from a uniform probability distribution. At $t+\Delta$, another event takes place. In order to generate, in practice, a switching sequence that satisfies a given rate constraint r in an interval T, the number of events per unit time, W, must be chosen sufficiently large. In this way, we obtain a sufficiently large number of events over every finite-time interval T such that the rate constraint (3) is satisfied (see [16] for more details).

4.2. $m_u = m_s = 1$

Consider system (1) described by the following continuous vector fields

$$f_s(x) = -x, \qquad f_u(x) = x. \tag{21}$$

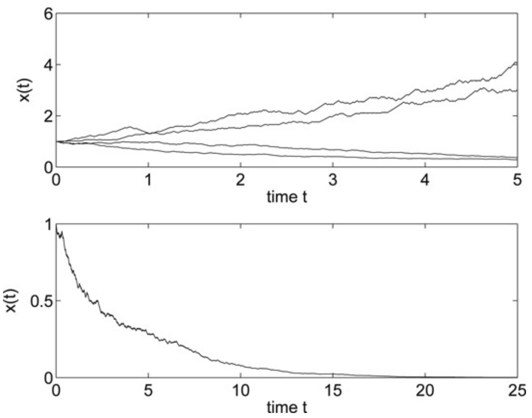

Fig. 4. Upper figure: Simulations of system (1) with continuous vector fields (21) for $r = 0.3, 0.4, 0.6, 0.7$. Lower figure: Simulation of system (1) with continuous vector fields (21) for $r = 0.4$.

For a Lyapunov-like function $V(x) = \frac{1}{2}x^2$, we obtain that

$$L_{f_s}V(x) = -2V(x), \qquad L_{f_u}V(x) = 2V(x).$$

Following Theorem 1, $r^* = 0.5$. In Fig. 4, simulations for different values of r are given. In the upper plot, simulations for $r = 0.3, 0.4, 0.6, 0.7$ are shown. It can be seen that the two trajectories are unstable, those with $r = 0.5, 0.7$, while the other two are stable ($r = 0.3, 0.4$). In the lower plot the trajectory with $r = 0.4$ is given. This trajectory converges to the origin asymptotically. The number of events per unit time used to generate the switching sequences is $W = 100$.

4.3. $m_u = m_s = 2$

Consider system (1) described by the following continuous vector fields

$$f_s(x) = -x^3, \qquad f_u(x) = x^3. \tag{22}$$

For a Lyapunov-like function $V(x) = \frac{1}{2}x^2$, we obtain that

$$L_{f_s}V(x) = -4V(x)^2, \qquad L_{f_u}V(x) = 4V(x)^2.$$

Following Theorem 1, for $\rho = 1$, $r^* = 0.5$ and $T^* = 1$. In Fig. 5, simulations for different values of the initial state are given. The rate is $r = 0.4$. The number of events per unit time used to generate the switching sequences is $W = 100$. It can be seen that with $x(0) = 2$, the state of the system converges to the origin, while with $x(0) = 5$, the state of the system is not well defined, because the state goes to infinity.

4.4. $m_u < m_s$

Consider system (1) described by the following continuous vector fields

$$f_s(x) = -x^3, \qquad f_u(x) = x. \tag{23}$$

For a Lyapunov-like function $V(x) = \frac{1}{2}x^2$, we obtain that

$$L_{f_s}V(x) = -4V(x)^2, \qquad L_{f_u}V(x) = 2V(x).$$

Following Theorem 2, the origin is globally practically stable. In Fig. 6, simulations for two different values of the rate are given, namely, for $r = 0.05$ and $r = 0.2$. It can be seen

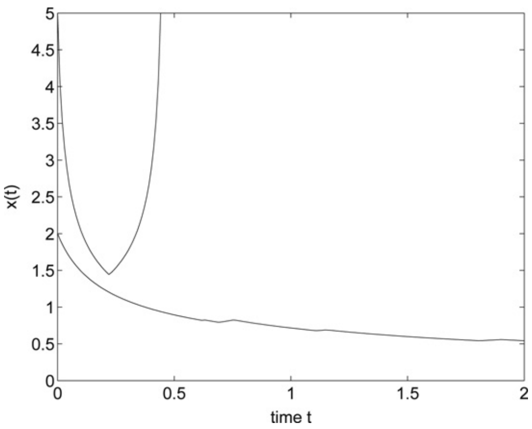

Fig. 5. Simulations of system (1) with continuous vector fields (22) for different initial states for $r = 0.4$.

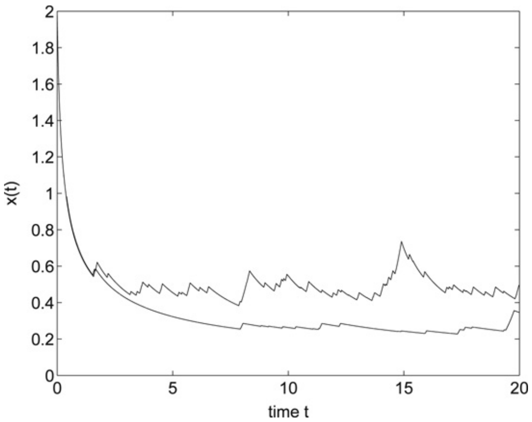

Fig. 6. Simulations of system (1) with continuous vector fields (23) for $r = 0.05, 0.2$.

that although the rate r that characterizes how much time the system is in the unstable mode, is very low, the system does not converge to the origin. With a rate $r = 0.2$, the system remains further away from the origin than with a rate $r = 0.05$. The number of events per unit time used to generate the switching sequences is $W = 100$.

4.5. $m_u > m_s$

Consider system (1) described by the following continuous vector fields

$$f_s(x) = -x, \qquad f_u(x) = x^3. \tag{24}$$

For a Lyapunov-like function $V(x) = \frac{1}{2}x^2$, we obtain that

$$L_{f_s}V(x) = -2V(x), \qquad L_{f_u}V(x) = 4V(x)^2.$$

Following Theorem 3, the origin is asymptotically stable. In Fig. 7, simulations for three different values of the initial state are given. The rate is $r = 0.3$. In order to obtain switching sequences that guarantee the rate constraint, the number of events per unit time used to generate the switching sequences is $W = 400$. It can be seen that with $x(0) = 0.5$, or $x(0) = 1$, the system converges to the origin, while with $x(0) = 1.5$, the origin is not stable, because far from the origin, the unstable mode dominates the stable one for this rate constraint.

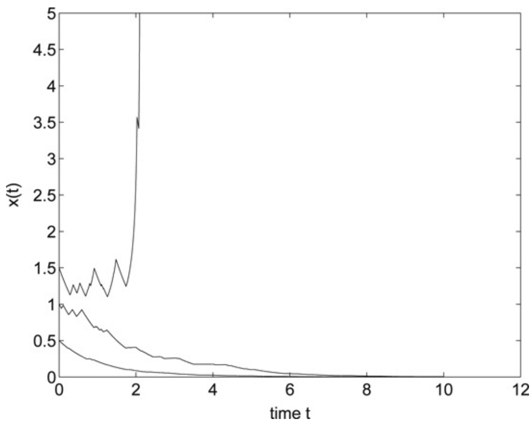

Fig. 7. Simulations of system (1) with continuous vector fields (24) for different initial states for $r = 0.3$.

5. Conclusions

In this paper, we considered the stability analysis of a class of nonlinear asynchronous systems defined by two different modes, one stable and the other one unstable. We assumed that the evolution of the discrete state of the system (i.e., the state of the system which determines the mode of operation) was driven by external asynchronous events whose rate of events was bounded on a given time period. For this class of systems, specific constraints on the rates of events have been given, in order to assure stability of the equilibrium point. It has been shown how depending on the stability properties of the stable and the unstable modes, and on the relative convergence rates; finite-time, asymptotic or practical stability of the equilibrium point can be guaranteed. Also, it has been proved that these results may be local or global depending on the nature of the unstable mode. The results presented provide a new insight into the dynamics of asynchronous nonlinear systems and demonstrate that the behavior of this class of systems might be very complex.

Acknowledgments

The authors would like to thank Professor Iasson Karafyllis for providing valuable comments on this manuscript. Financial support from NSF, CTS-0529295, is gratefully acknowledged.

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