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Economic model predictive control designs for input rate-of-change constraint handling and guaranteed economic performance

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ABSTRACT

Economic model predictive control (EMPC) has been a popular topic in the recent chemical process control literature due to its potential to improve process profit by operating a system in a time-varying manner. However, time-varying operation may cause excessive wear of the process components such as valves and pumps. To address this issue, input magnitude constraints and input rate-of-change constraints can be added to the EMPC optimization problem to prevent possible frequent and extreme changes in the requested inputs. Specifically, we develop input rate-of-change constraints that can be incorporated in Lyapunov-based EMPC (LEMPC) that ensure controller feasibility and closed-loop stability. Furthermore, we develop a terminal equality constraint for LEMPC that can ensure that the performance of LEMPC is at least as good as that of a Lyapunov-based controller in finite-time and in infinite-time. Chemical process examples demonstrate the incorporation of input rate-of-change constraints and terminal state constraints in EMPC.

1. Introduction

As environmental requirements tighten and chemical processing companies are increasingly interested in operating processes in the most economically efficient but safe manner, advanced process control is being exploited as a means to achieve these objectives. Real-time optimization (RTO), coupled with model predictive control (MPC) and a distributed control system (DCS) architecture, has been used in industry to improve production profits (Darby et al., 2011; Marlin and Hrymak, 1996). Typical industrial implementations of the RTO-MPC paradigm have structures that are considered to make the advanced control strategy safe to use, including logic steps at the RTO level to evaluate RTO solutions before implementing them (Marlin and Hrymak, 1996), a tracking MPC formulation with a quadratic objective, and penalties on changes in the manipulated inputs between two sampling periods of the prediction horizon to prevent aggressive movement of the actuation elements (Qin and Badgwell, 2003).

A fairly recent development in the MPC literature that is often viewed as an alternative to the RTO-MPC hierarchy is economic model predictive control (EMPC) which, like standard tracking MPC, optimizes an objective function subject to constraints to determine the optimal control actions that meet these constraints. Unlike tracking MPC, however, the objective function of EMPC is not required to have its minimum at a steady-state of the process because it is based on the concept that processes may operate more profitably off steady-state than at steady-state. To attain greater economic profitability than the steady-state operating strategy dictated by the RTO-MPC control architecture, EMPC may calculate widely varying or bang-bang type control actions (Mendoza-Serrano and Chmielewski, 2012; Ellis and Christofides, 2014; Bailey et al., 1971). Such time-varying control policies are consistent with the optimal process operation literature, in which it has been repeatedly shown, both theoretically and experimentally, that periodic operation of certain chemical processes may be more economically optimal than steady-state operation (Silveston, 1987; Shu et al., 1989; Sterman and Ydstie, 1991; Özgülşen et al., 1992). Though the economic results from time-varying operation in such cases may be highly favorable, the possible extreme movement required by the actuation elements brings up safety concerns with respect to whether such movement might cause actuators or other components that regulate the process flow rates, such as pumps, to wear out early and thus fail when they are used for safety-critical processes or are crucial to compliance with environmental regulations. If such an issue were to occur, the economic benefits from time-varying process operation under EMPC would no longer matter or be realized.

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The matter of reducing the aggressiveness of input changes has been a consideration in the tracking MPC literature since its inception, however most theoretical studies in EMPC to date have not focused on this issue, but the majority of the literature has instead focused on other concerns, both theoretical and practical, that arise from the use of EMPC. For example, convergence to a steady-state was established for systems under an EMPC for which the optimal control problem exhibited an exact turnpike property (Faulwasser and Bonvin, 2015). In addition, the closed-loop stability of processes under numerous EMPC formulations has been investigated by various methods (e.g., ensuring that a process with a cyclic steady-state can be driven to this periodic solution by an EMPC Huang et al., 2012). A common method for guaranteeing closed-loop stability is to impose additional constraints in the EMPC formulation, such as terminal steady-state constraints requiring the process state to be at the steady-state at the end of the horizon (Diehl et al., 2011), terminal region constraints (Amrit et al., 2011), and Lyapunov-based constraints (Heidarinejad et al., 2012) (see also the review paper Ellis et al., 2014a for further information and references on EMPC).

To extend the foundational results on EMPC to address the issue of input change aggressiveness, additional constraints may be added to EMPC. One type of constraint that has been used extensively for tracking MPC formulations to prevent rapid changes of the actuator output and consequently to prevent rapid changes of the process states is a rate of change constraint on the values of the inputs calculated by the MPC (see, e.g., Qin and Badgwell, 2000, 2003; Camacho and Bordons, 2007; De Souza et al., 2010 for both industrial and research work incorporating such a constraint). For example, in Muske and Rawlings (1993), feasibility and closed-loop stability of linear, discrete-time systems under MPC with input magnitude and rate of change constraints are proven for both open-loop stable and unstable systems. In Mhaskar and Kennedy (2008), an MPC formulation for input-affine nonlinear systems accounting for input magnitude constraints and input rate of change constraints using a penalty in the objective and hard constraints when possible is proven to be feasible and to ensure closed-loop stability for bounded process uncertainty. Input rate of change constraints have also been used in several works on EMPC. In particular, an MPC including input magnitude and rate of change constraints was used to improve the economic performance of a heat pump by incorporating electricity price and weather forecasts (Tahersima et al., 2012), and EMPC including magnitude and rate of change constraints on the inputs was applied for power production and use (Hovgaard et al., 2010). Though input rate of change constraints have been applied to several EMPC examples in the literature, no proof of general feasibility and closed-loop stability for a nonlinear system under an EMPC strategy incorporating both input magnitude and input rate of change constraints with Lyapunov-based constraints that ensure closed-loop stability in the presence of disturbances has yet been developed. The development of such an EMPC strategy will be one of the topics covered in this work.

Despite the benefits from an actuator durability perspective of incorporating constraints that prevent an EMPC from calculating aggressive control actions, it would be expected that limiting the control actions that the EMPC can calculate would reduce the economic profitability of the EMPC compared to the case that no input rate of change constraints are used. However, determining whether there is still an economic benefit of EMPC compared with the traditional steady-state paradigm when input rate of change constraints are used in EMPC requires the development of proofs regarding the economic performance of EMPC with input rate of change constraints. Previous proofs of the economic performance of EMPC have not explicitly addressed the case when input rate of change constraints are included in the EMPC formulation. The proofs for many of the methods use terminal constraints in the EMPC (Amrit et al., 2011; Ellis et al., 2016; Angeli et al., 2012) or an EMPC prediction horizon that is sufficiently long with some additional technical conditions (e.g., Grüne, 2013; Müller and Grüne, 2015). Studies to investigate the economic performance of EMPC have been carried out for EMPC with a stage cost and terminal constraints that change with time (Angeli et al., 2015), for EMPC with a generalized terminal region constraint and self-tuning terminal cost (Müller et al., 2014), for EMPC without terminal costs or constraints for discrete-time systems meeting certain assumptions including controllability and dissipativity assumptions (Grüne and Stieler, 2014), and a two-layer EMPC structure including performance constraints (Ellis and Christofides, 2014a).

Motivated by all of the above, in this work, we introduce a Lyapunov-based economic model predictive control (LEMPC) architecture that can incorporate input rate of change constraints with provable feasibility, stability, and closed-loop performance properties. First, we introduce input rate of change constraints in the context of LEMPC and show that when the constraints are formulated with reference to a Lyapunov-based controller, the LEMPC can be proven to be feasible and to maintain closed-loop stability for a sufficiently small sampling period. Through a chemical process example, we demonstrate that the incorporation of input magnitude and rate of change constraints in EMPC can prevent significant variations in the process inputs while improving the profit compared to steady-state operation. Subsequently, we develop an LEMPC design incorporating a terminal equality constraint based on an explicit stabilizing Lyapunov-based controller for which closed-loop economic performance improvement guarantees with respect to the Lyapunov-based controller (and with respect to steady-state operation when the Lyapunov-based controller is exponentially stabilizing) may be proven for nominal operation. A chemical process example demonstrates the use of this LEMPC strategy. We then show that LEMPC with the terminal equality constraint based on a Lyapunov-based controller, with input magnitude constraints, and with input rate of change constraints retains these provable performance guarantees for nominal operation.

2. Preliminaries

2.1. Notation

The symbol $|\cdot|$ signifies the Euclidean norm of a vector. A continuous, strictly increasing function α: $[0, a) \to [0, \infty)$ belongs to class $\mathcal{K}$ if $\alpha(0) = 0$. The notation Ω_ρ signifies a level set of a positive definite scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and is defined by $\Omega_\rho := \{x \in \mathbb{R}^n \ | \ V(x) \leq \rho, \rho > 0\}$. The notation $t_k = k\Delta$, $k = 0, 1, 2, \ldots$ signifies the time at the beginning of a sampling period of length Δ for synchronously sampled time intervals. Set subtraction is signified by '/' (e.g., $x \in A/B := \{x \in A \mid x \notin B\}$). The symbol $S(\Delta)$ denotes the family of piecewise constant vector-valued functions with period $\Delta > 0$. More specifically, $u(\cdot) \in S(\Delta)$ means that the function u can be described by a sequence $\{u^{(j)}\}_{j=k}^{k+N-1}$ where $u^{(j)} \in \mathbb{R}^m$, or

$$u(t) = u^{(j)}$$

for $t \in [t_j, t_{j+1})$, $j = k, \ldots, k + N - 1$.

2.2. Class of systems

The class of systems of nonlinear first-order ordinary differential equations considered in this work is that of the general form:

$$\dot{x} = f(x, u, w) \tag{1}$$

where $x \in \mathbb{R}^n$, $u = [u_1 \ \ u_2 \ \ \cdots \ \ u_m]^T \in \mathbb{R}^m$, and $w \in \mathbb{R}^l$ are the state, input, and disturbance vectors, respectively, and are related to the time-derivative of the state vector through the nonlinear vector function f. In addition, we assume that the states $x(t)$ are restricted to the set $\mathbb{X}$ ($x(t) \in \mathbb{X} \subset \mathbb{R}^n$), that $u_i(t)$, $i = 1, \ldots, m$, are bounded ($u_i(t) \in \mathbb{U}_i := \{u_{i,\min} \leq u_i(t) \leq u_{i,\max}\}$), and that the disturbance $w(t)$ is bounded within a set $\mathbb{W} \subset \mathbb{R}^l$ ($w(t) \in \mathbb{W} := \{w(t) \mid |w(t)| \leq \theta, \theta > 0\}$). For simplicity of presentation in the following, we will use the notation $u(t) \in \mathbb{U} \subset \mathbb{R}^m$ to denote that each component $u_i(t)$ of $u(t)$ is bounded within its respective set $\mathbb{U}_i$. The vector function $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W}$ is assumed to be locally Lipschitz with respect to its arguments.

It is assumed that the process economic cost for the system of Eq. (1) can be represented by an economic stage cost function $l_e : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ that is continuous on $\mathbb{X} \times \mathbb{U}$. In addition, it is assumed that there is a steady-state and steady-state input pair (x_s^*, u_s^*) for the nominal ($w(t) \equiv 0$) system (on $\mathbb{X} \times \mathbb{U}$) that minimizes the economic cost in the sense that the minimum of l_e is attained at the pair (x_s^*, u_s^*) when the time derivative of the nominal state in Eq. (1) is zero. For simplicity, the minimizing pair is assumed to be unique. With these assumptions, the minimizing steady-state pair is given by:

$$(x_s^*, u_s^*) = \underset{x \in \mathbb{X}, u \in \mathbb{U}}{\arg\min} \left\{ l_e(x, u) : f(x, u, 0) = 0 \right\}.$$

Without loss of generality, the minimizing pair will be taken to be the origin of the nominal system of Eq. (1).

2.3. Lyapunov-based controller stabilizability assumptions under continuous implementation

We consider two stabilizability assumptions for the system of Eq. (1) that a Lyapunov-based controller $h(x) = [h_1(x)\ h_2(x) \cdots h_m(x)]^T$ exists for the nominal system of Eq. (1) that renders the origin either locally asymptotically stable or locally exponentially stable in a sense to be made precise in the following two assumptions (Massera, 1956; Khalil, 2002), while also meeting the input constraints. The first assumption covers the weaker of the two cases, that of asymptotic stability, while the second covers exponential stability. With slight abuse of notation, the same notation is used in both assumptions.

Assumption 1. There exists a locally Lipschitz feedback controller $h : \mathbb{X} \to \mathbb{U}$ with $h(0) = 0$ for the nominal system of Eq. (1) that renders the origin of the closed-loop system $\dot{x} = f(x, h(x), 0)$ asymptotically stable when applied continuously in the sense that there exists a sufficiently smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following inequalities hold:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2a}$$

$$\frac{\partial V(x)}{\partial x} f(x, h(x), 0) \leq -\alpha_3(|x|) \tag{2b}$$

$$\left| \frac{\partial V(x)}{\partial x} \right| \leq \alpha_4(|x|) \tag{2c}$$

for all $x \in D$ where D is an open neighborhood of the origin and $\alpha_i \in \mathcal{K}$, $i = 1, 2, 3, 4$.

Assumption 2. There exists a locally Lipschitz feedback controller $h : \mathbb{X} \to \mathbb{U}$ with $h(0) = 0$ for the nominal system of Eq. (1) that renders the origin of the system $\dot{x} = f(x, h(x), 0)$ exponentially stable when applied continuously in the sense that there exists a sufficiently smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following inequalities hold:

$$c_1|x|^2 \leq V(x) \leq c_2|x|^2 \tag{3a}$$

$$\frac{\partial V(x)}{\partial x} f(x, h(x), 0) \leq -c_3|x|^2 \tag{3b}$$

$$\left| \frac{\partial V(x)}{\partial x} \right| \leq c_4|x| \tag{3c}$$

for all $x \in D$ where D is an open neighborhood of the origin and c_i, $i = 1, 2, 3, 4$ are positive constants.

We define the set $\Omega_\rho \subseteq \mathbb{X} \subset D$, which is an estimate of the region of attraction of the nominal closed-loop system under a feedback controller meeting either Assumption 1 or Assumption 2, as the stability region of the closed-loop system for that controller. Methods for designing Lyapunov-based feedback controllers can be found in works such as Lin and Sontag, 1991 and Christofides and El-Farra, 2005.

A consequence of our assumption of the Lipschitz continuity of $h(x)$ meeting either Assumption 1 or 2 is that its components are Lipschitz continuous in x, and thus $L_{h_L} > 0$ exists such that

$$|h_i(x) - h_i(\hat{x})| \leq L_{h_L}|x - \hat{x}| \tag{4}$$

for all $x, \hat{x} \in \Omega_\rho$. Here, L_{h_L} is chosen such that it satisfies the bound in Eq. (4) with the same value for each $h_i(x)$ (i.e., $L_{h_L} = \max\{L_{h_{L_1}}, \ldots, L_{h_{L_m}}\}$, where $L_{h_{L_i}}$, $i = 1, \ldots, m$, is the smallest positive constant such that $|h_i(x) - h_i(\hat{x})| \leq L_{h_{L_i}}|x - \hat{x}|$ for all $x, \hat{x} \in \Omega_\rho$). The requirement that $h(x)$ and its components are Lipschitz continuous in x does not pose significant practical restrictions.

We note that from the assumption of Lipschitz continuity of f and the bounds on w and u_i, $i = 1, \ldots, m$, there exist $M > 0$, $L_x > 0$, and $L_w > 0$ such that:

$$|f(x, u, w)| \leq M \tag{5}$$

$$|f(x, u, w) - f(\hat{x}, u, 0)| \leq L_x|x - \hat{x}| + L_w|w| \tag{6}$$

for all $x, \hat{x} \in \Omega_\rho$, $u_i \in \mathbb{U}_i$, $i = 1, \ldots, m$, and $|w| \leq \theta$. Furthermore, since V is sufficiently smooth, f is locally Lipschitz and Ω_ρ is compact, there exist $L'_x > 0$ and $L'_w > 0$ such that the following also holds:

$$\left| \frac{\partial V(x)}{\partial x} f(x, u, w) - \frac{\partial V(\hat{x})}{\partial x} f(\hat{x}, u, 0) \right| \leq L'_x|x - \hat{x}| + L'_w|w| \tag{7}$$

for all $x, \hat{x} \in \Omega_\rho$, $u_i \in \mathbb{U}_i$, $i = 1, \ldots, m$, and $|w| \leq \theta$.

2.4. Lyapunov-based controller stabilizability results for sample-and-hold implementation

Though the system of Eq. (1) is continuous time and it is assumed that a controller $h(x)$ can be designed that can stabilize the nominal closed-loop system as described in Assumptions 1 and 2 when implemented continuously, the Lyapunov-based controller will be used in this work to design stability constraints for an economic model predictive control method that is implemented in sample-and-hold. Thus, we develop in this section the stability properties of the nominal closed-loop system of Eq. (1) under $h(x)$ applied in sample-and-hold, where $h(x)$ meets either Assumption 1 or Assumption 2 and Eq. (4) when applied continuously. Specifically, we consider the following nonlinear sampled-data system:

$$\dot{x}(t) = f(x(t), h(x(t_k)), 0) \tag{8}$$

for $t \in [t_k, t_{k+1})$, where $k = 0, 1, \ldots$. We present two propositions that follow from standard results in the nonlinear sampled-data systems literature to state the stability results for the process under a sample-and-hold controller. The first proposition states that the origin of the sampled-data system of Eq. (8) using $h(x)$ that satisfies Assumption 1 is rendered practically stable (i.e., the closed-loop state trajectory will converge to a small neighborhood of the origin where it will be maintained thereafter). This result follows from standard results found in the literature (e.g., Teel et al., 1998; Muñoz de la Peña and Christofides, 2008). The second proposition states that the origin of the sampled-data system of Eq. (8) using $h(x)$ that satisfies Assumption 2 is rendered exponentially stable. This result is stronger than the result that can be obtained when $h(x)$ satisfies Assumption 1, and the proof can be found, for example, in Corollary 1 of Ellis et al. (2014b) as well as Laila et al. (2006) and the results contained therein.

Proposition 1. *Let Assumption 1 hold and V and Ω_ρ be the Lyapunov function that satisfies Eq. (2) and the resulting stability region, respectively. Given $\rho_{\min} \in (0, \rho)$, there exists $\Delta^* > 0$ such that for any $\Delta \in (0, \Delta^*)$ and $x(t_0) \in \Omega_\rho$, the closed-loop state trajectory of the sampled-data system of Eq. (8) is always bounded in Ω_ρ and is (uniformly) ultimately bounded in $\Omega_{\rho_{\min}}$ and*

$$\limsup_{t\to\infty} x(t) \in \Omega_{\rho_{\min}}. \tag{9}$$

Proposition 2. *Let Assumption 2 hold and V and Ω_ρ be the Lyapunov function that satisfies Eq. (3) and the resulting stability region, respectively. There exists $\Delta_e^* > 0$ such that for any $\Delta \in (0, \Delta_e^*)$ the closed-loop state trajectory of the sampled-data system of Eq. (8) is always bounded in Ω_ρ and the origin of the sampled-data system of Eq. (8) is exponentially stable for all initial states in Ω_ρ.*

We can also develop stability properties for the closed-loop system of Eq. (1) in the presence of disturbances under $h(x)$ applied in sample-and-hold, where $h(x)$ meets either Assumption 1 or Assumption 2 and Eq. (4) when applied continuously. Specifically, these results are derived for the following nonlinear sampled-data system:

$$\dot{x}(t) = f(x(t), h(x(t_k)), w(t)) \tag{10}$$

for $t \in [t_k, t_{k+1})$, where $k = 0, 1, \ldots$. The following two properties address this sample-and-hold system with disturbances. In both the case that $h(x)$ in Eq. (10) meets Assumption 1 and the case that it meets Assumption 2, only uniform ultimate boundedness of the closed-loop state can be proven. We note that the sampling times and regions within which uniform ultimate boundedness of the state are proven are different from those in Propositions 1 and 2. The proof of the results of the following two propositions can be found, for example, in Muñoz de la Peña and Christofides (2008).

Proposition 3. *Let Assumption 1 hold and V and Ω_ρ be the Lyapunov function that satisfies Eq. (2) and the resulting stability region, respectively. Given $\rho^*_{\min} \in (0, \rho)$, there exists $\Delta^*_w > 0$ such that for any $\Delta \in (0, \Delta^*_w)$ and $x(t_0) \in \Omega_\rho$, the closed-loop state trajectory of the sampled-data system of Eq. (10) is always bounded in Ω_ρ and is (uniformly) ultimately bounded in $\Omega_{\rho^*_{\min}}$ and*

$$\limsup_{t\to\infty} x(t) \in \Omega_{\rho^*_{\min}}. \tag{11}$$

Proposition 4. *Let Assumption 2 hold and V and Ω_ρ be the Lyapunov function that satisfies Eq. (3) and the resulting stability region, respectively. Given $\rho^*_{\min,e} \in (0, \rho)$, there exists $\Delta^*_{w,e} > 0$ such that for any $\Delta \in (0, \Delta^*_{w,e})$ and $x(t_0) \in \Omega_\rho$, the closed-loop state trajectory of the sampled-data system of Eq. (10) is always bounded in Ω_ρ and is (uniformly) ultimately bounded in $\Omega_{\rho^*_{\min,e}}$ and*

$$\limsup_{t\to\infty} x(t) \in \Omega_{\rho^*_{\min,e}}. \tag{12}$$

It is noted that Assumption 2 is stronger than Assumption 1 (i.e., whenever Assumption 2 is satisfied, Assumption 1 is also satisfied). Therefore, for clarity in the remainder of this manuscript regarding which results require the stronger conditions in Assumption 2 to hold, we will state that the results require Assumption 1 when only the conditions of that assumption are required (though the result is then also satisfied if Assumption 2 is met), and we will reserve mention of Assumption 2 only for those results that require the stronger conditions in that assumption to hold.

2.5. Economic model predictive control

This work develops a formulation for an optimization-based control strategy known as economic model predictive control (EMPC) with constraints guaranteeing closed-loop stability, satisfaction of bounds on the input rate of change, and an upper bound on the economic cost for the process under the controller. The general formulation of EMPC is given by:

$$\min_{u(\cdot)\in S(\Delta)} \int_{t_k}^{t_{k+N}} l_e(\tilde{x}(\tau), u(\tau))\, d\tau \tag{13a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0) \tag{13b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{13c}$$

$$u_i(t) \in \mathbb{U}_i, \quad i = 1, \ldots, m \tag{13d}$$

$$\tilde{x}(t) \in \mathbb{X}, \quad \forall \ t \in [t_k, t_{k+N}) \tag{13e}$$

Eq. (13) is a general nonlinear optimization problem that minimizes a stage cost $l_e(x(t), u(t))$ (Eq. (13a)) subject to a model of the nominal system (Eq. (13b)) and the initial condition in Eq. (13c) that comes from a measurement of the process state at time t_k. The calculated inputs u_i, $i = 1, \ldots, m$, and the predicted states $\tilde{x}(t)$, $t \in [t_k, t_{k+N})$, are restricted to their respective sets as shown in Eqs. (13d) and (13e). In general, additional equality or inequality constraints may be added to a general EMPC with the form in Eq. (13) as desired.

The optimization variable in Eq. (13) is the piecewise constant optimal control trajectory $u(t)$ over a prediction horizon with N sampling periods of length Δ. Thus, the input profile that is the solution to the EMPC optimization problem is a set of N vectors denoted by $u^*(t|t_k)$, $t = t_k, \ldots, t_{k+N-1}$, of which only the first, $u^*(t_k|t_k)$, is implemented on the process in a sample-and-hold fashion. At t_{k+1}, the EMPC optimization problem is re-solved.

2.6. Lyapunov-based economic model predictive control

The specific type of EMPC that will be the focus of this work is Lyapunov-based economic model predictive control (LEMPC) (Heidarinejad et al., 2012), which is an EMPC with the form of Eq. (13) but with the addition of Lyapunov-based stability constraints that define two modes of operation, as shown below:

$$\min_{u(\cdot)\in S(\Delta)} \int_{t_k}^{t_{k+N}} l_e(\tilde{x}(\tau), u(\tau))\, d\tau \tag{14a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0) \tag{14b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{14c}$$

$$u_i(t) \in \mathbb{U}_i, \quad i = 1, \ldots, m \tag{14d}$$

$$\tilde{x}(t) \in \mathbb{X}, \quad \forall \ t \in [t_k, t_{k+N}) \tag{14e}$$

$$V(\tilde{x}(t)) \le \rho_e, \quad \forall \ t \in [t_k, t_{k+N}) \quad \text{if} \quad t_k < t' \quad \text{and} \quad V(x(t_k)) \le \rho_e \tag{14f}$$

$$\frac{\partial V(x(t_k))}{\partial x} f(x(t_k), u(t_k), 0) \le \frac{\partial V(x(t_k))}{\partial x} f(x(t_k), h(x(t_k)), 0)$$
$$\text{if} \quad t_k \ge t' \quad \text{or} \quad V(x(t_k)) > \rho_e \tag{14g}$$

where the notation follows that in Eq. (13), with the added constraints in Eqs. (14f) and (14g) that define two modes of operation of the LEMPC (Modes 1 and 2), and t' is a pre-determined time at which it is desired to apply only Mode 2 of the LEMPC. Mode 1 is active when $t_k < t'$ and when the measured state is within Ω_{ρ_e}, which is a subset of Ω_ρ defined such that if the state at t_k is within Ω_{ρ_e}, then by t_{k+1}, it is still within Ω_ρ, even in the presence of bounded process disturbances or plant-model mismatch. Mode 2 is activated when the measured state is outside of Ω_{ρ_e} due to process disturbances or plant-model mismatch, or once the time t' has been reached. This dual-mode strategy guarantees that the closed-loop state trajectories of the process under LEMPC are maintained within Ω_ρ at all times.

3. LEMPC formulation with input magnitude constraints, input rate of change constraints, and an equality terminal constraint based on a Lyapunov-based controller

This work introduces input rate of change constraints that can be used in an LEMPC framework while guaranteeing closed-loop stability and feasibility of the controller, and it also addresses the performance guarantees that can be made for this LEMPC incorporating input rate of change constraints for nominal process operation (the performance guarantees are also shown to hold in the absence of the input rate of change constraints). The guarantees will be made for a general cost function, so that they will hold even if the objective function of the LEMPC is designed to reduce the input rate of change (for example, a penalty on the input rate of change may be added to the objective function). This work thus addresses the questions of not only how to add input rate of change constraints to LEMPC in a manner that does not affect the feasibility and closed-loop stability of the controller, but also of whether that reduces the economic benefits of using LEMPC for a given process.

To develop the answers to these questions, the contributions of this work are divided into three parts. In Part 1, we introduce a Lyapunov-based economic model predictive control (LEMPC) architecture that incorporates input magnitude and rate of change constraints with provable feasibility and stability properties, even in the presence of disturbances. In Part 2, we develop a terminal equality constraint based on a Lyapunov-based controller that, when used in an LEMPC for a process with no disturbances or plant-model mismatch (nominal process) ensures that the economic performance of the resulting LEMPC is at least as good as that of the Lyapunov-based controller implemented in sample-and-hold. In Part 3, the results of the first two sections will be combined to show that the nominal process of Eq. (1) under an LEMPC with input magnitude and rate of change constraints and a terminal equality constraint based on a Lyapunov-based controller performs at least as well as it does under the Lyapunov-based controller implemented in sample-and-hold.

3.1. Part 1: LEMPC with input magnitude and rate of change constraints

In this section, we develop LEMPC with input magnitude constraints that restrict the calculated control actions between an upper and a lower bound, as well as input rate of change constraints, which prevent the calculated inputs between two sampling periods from differing from each other by more than a pre-specified amount. Specifically, we add input rate of change constraints to the LEMPC of Eq. (14) (which has input magnitude constraints in Eq. (14d)). The input rate of change constraints developed are written with respect to a Lyapunov-based controller, but we demonstrate that for a sufficiently small sampling period and an appropriate value of a parameter of the constraints, the constraints developed ensure that the difference between the control actions calculated for two subsequent sampling periods can be bounded by any desired value. We prove that the LEMPC incorporating input rate of change constraints is feasible and furthermore that it ensures closed-loop stability of a process even in the presence of bounded disturbances. The results presented hold for the case that the Lyapunov-based controller meets Assumption 1, except where it is noted that Assumption 2 is required. Finally, we present a chemical process example to demonstrate the effect of incorporating input rate of change constraints in addition to input magnitude constraints in EMPC, which shows that the manner in which the input rate of change constraints are enforced in EMPC can significantly affect whether the closed-loop process is able to meet other hard constraints.

3.1.1. Part 1: Formulation of LEMPC with input magnitude and input rate of change constraints

As noted in Section 1 of this work, it may be desirable to add input rate of change constraints to LEMPC, especially since Mode 1 of LEMPC attempts to dynamically optimize process operation within the stability region and does not drive the process to a steady-state. The result of this is that an LEMPC may request input trajectories with sharp changes in the requested control actions (an example is shown in Section 3.1.4 of this work) to maximize profit subject to the constraints. Restricting the range of allowable control actions in such a case (e.g., increasing $u_{i,min}$ and/or decreasing $u_{i,max}$) may ameliorate this issue, but it may be necessary to drastically decrease this range to reduce the difference between two calculated control actions to a desired level, particularly if the LEMPC calculates bang-bang type control actions. Such a drastic reduction in the allowable range of control actions may significantly reduce the process profit; thus, input rate of change constraints may instead be considered as an alternative constraint that achieves the same goal but with potentially higher profit.

The desired form of the input rate of change constraints, assuming that the actuators bring the actuator outputs to the requested values $u_i^*(t_{k-1}|t_{k-1})$, $i=1, \ldots, m$, before t_k when the LEMPC is resolved, is as follows:

$$|u_i^*(t_k|t_k) - u_i^*(t_{k-1}|t_{k-1})| \le \epsilon_{desired}, \quad \forall \ \ i = 1, \ldots, m \tag{15}$$

where $\epsilon_{desired} > 0$ is a bound on the difference between the control action $u_i^*(t_k|t_k)$ implemented at t_k and the immediate past value of the actuator output that was implemented on the process. To make the predicted state trajectories within the LEMPC more consistent with the actual state trajectory, it may also be desirable that the other control actions in the prediction horizon that are not implemented meet the following constraints:

$$|u_i^*(t_j|t_k) - u_i^*(t_{j-1}|t_k)| \le \epsilon_{desired}, \quad \forall \ \ i = 1, \ldots, m, \quad j = k+1, \ldots, k+N-1 \tag{16}$$

The chemical process example in Section 3.1.4 will show that the decision to enforce both Eqs. (15) and (16) (imposing restrictions on both the implemented and not implemented control actions) or only Eq. (15) (imposing a constraint on the implemented control actions only) may significantly affect the results obtained for LEMPC with input rate of change constraints, and thus should be carefully considered.

If the input rate of change constraints are written as in Eqs. (15) and (16) and directly added into the LEMPC of Eq. (14), it is not possible to prove feasibility of the resulting LEMPC, as will be further discussed in Section 3.1.3 of this work. For this reason, modified constraints are added to the LEMPC of Eq. (14) that constrain the calculated control actions to differ by no more than a constant $\epsilon_r \ge 0$ from the value of the Lyapunov-based control law at $x(t_k)$. These modified constraints ensure, as will be demonstrated in Section 3.1.3, that the LEMPC is feasible, and they also ensure that the desired constraints of Eqs. (15) and (16) are met for any $\epsilon_{desired}$ when ϵ_r and Δ are suitably chosen.

Incorporating the above considerations, the proposed LEMPC with both input magnitude and rate of change constraints is as follows:

$$\min_{u(\cdot)\in S(\Delta)} \int_{t_k}^{t_{k+N}} l_e(\tilde{x}(\tau), u(\tau))\,d\tau \tag{17a}$$

$$\text{s.t.}\quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0) \tag{17b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{17c}$$

$$u_i(t) \in \mathbb{U}_i, \quad i = 1, \ldots, m \tag{17d}$$

$$\tilde{x}(t) \in \mathbb{X}, \quad \forall\ t \in [t_k, t_{k+N}) \tag{17e}$$

$$|u_i(t_k) - h_i(x(t_k))| \le \epsilon_r, \quad i = 1, \ldots, m \tag{17f}$$

$$|u_i(t_j) - h_i(\tilde{x}(t_j))| \le \epsilon_r, \quad i = 1, \ldots, m, \quad j = k+1, \ldots, k+N-1 \tag{17g}$$

$$V(\tilde{x}(t)) \le \rho_e, \quad \forall t \in [t_k, t_{k+N}) \quad \text{if} \quad t_k < t' \quad \text{and} \quad V(x(t_k)) \le \rho_e \tag{17h}$$

$$\frac{\partial V(x(t_k))}{\partial x} f(x(t_k), u(t_k), 0) \le \frac{\partial V(x(t_k))}{\partial x} f(x(t_k), h(x(t_k)), 0)$$
$$\text{if} \quad t_k \ge t' \quad \text{or} \quad V(x(t_k)) > \rho_e \tag{17i}$$

where the notation follows that in Eqs. (13) and (14). The rate of change constraints of Eqs. (15) and (16) are imposed through Eqs. (17f) and (17g), which require that the values of $u_i^*(t_j|t_k)$, $i = 1, \ldots, m$, $j = k, \ldots, k+N-1$, be within $\epsilon_r \ge 0$ of the values of $h_i(\tilde{x}(t_j))$. Note that ϵ_r in Eqs. (17f) and (17g) is not the same as $\epsilon_{desired}$ in Eqs. (15) and (16), which will be justified in the next section.

Remark 1. It is noted that in the LEMPC formulation of Eq. (17), as well as in the other LEMPC formulations developed throughout this work, the number of constraints considered is kept to a minimum, and the form of the objective function and the manner of developing such an objective function are not discussed. This is done so that the theoretical developments in this work in the proofs to be presented are kept as general as possible and are not obscured by the additional considerations that may arise when the optimization problem is augmented. The results presented in this work could be extended, however, to certain cases with additional constraints and may hold practically even when the formulation/assumptions of this work are not met, though such an extended study is outside the scope of the present work.

3.1.2. Part 1: Rate of change constraints analysis

In this section, we prove that given $\epsilon_{desired}$, we can ensure that the desired rate of change constraints of Eqs. (15) and (16) are met by enforcing the rate of change constraints with respect to $h_i(\tilde{x}(t_j))$, $i = 1, \ldots, m$, $j = k, \ldots, k+N-1$, in Eqs. (17f) and (17g) for a suitable ϵ_r value, with $h(x)$ meeting Assumption 1.

Theorem 1. *Consider the closed-loop input trajectories of the process of Eq. (1) operated under the LEMPC of Eq. (17), with $h(x)$ meeting Assumption 1 and Eq. (4). There exist ϵ_r and $0 < \Delta < \min\{\Delta_w^*, \Delta_1\}$ such that for any chosen $\epsilon_{desired} > 0$, if*

$$2\epsilon_r + L_{h_L} M\Delta \le \epsilon_{desired} \tag{18}$$

then Eqs. (15) and (16) are satisfied for all t_k with $k > 0$ and $u_i^*(t_0|t_0) = h_i(x(t_0))$, $i = 1, \ldots, m$.

Proof. From the bound on f in Eq. (5) and continuity of x, the following bound holds for all $x(t)$, $x(t_{k-1}) \in \Omega_\rho$ and $t \in [t_{k-1}, t_k]$, where $x(t)$ is the solution of Eq. (1) at time t:

$$|x(t) - x(t_{k-1})| \le M\Delta \tag{19}$$

for Δ sufficiently small (i.e., $\Delta < \Delta_1$, where Δ_1 is the largest value of Δ for which Eq. (19) holds; it is noted that Δ_1 always exists since Ω_ρ is forward invariant for a sufficiently small Δ from Proposition 3 and Eq. (5) holds in Ω_ρ). In addition, because the bound in Eq. (5) and continuity of x hold when $w(t) \equiv 0$ as well, the following inequality holds for the predicted state of the nominal closed-loop system for the LEMPC of Eq. (17) (Eq. (17b)):

$$|\tilde{x}(t) - \tilde{x}(t_{j-1})| \le M\Delta \tag{20}$$

for $\tilde{x}(t)$, $\tilde{x}(t_{j-1}) \in \Omega_\rho$, $t \in [t_{j-1}, t_j]$, $j = k+1, \ldots, k+N$, and $\Delta < \min\{\Delta_w^*, \Delta_1\}$. It is noted that because $\Omega_\rho \subseteq \mathbb{X}$, $x(t) \in \Omega_\rho$ implies that $x(t) \in \mathbb{X}$ as required by Eq. (17e); the fact that the LEMPC formulation of Eq. (17) maintains the state within Ω_ρ is proven in Section 3.1.3.

From Eqs. (19) and (20) and the Lipschitz continuity property of $h_i(x)$ in x (Eq. (4)), the following bounds hold for $x(t_k) \in \Omega_\rho$ and $x(t_{k-1}) \in \Omega_\rho$:

$$|h_i(x(t_k)) - h_i(x(t_{k-1}))| \le L_{h_L}|x(t_k) - x(t_{k-1})| \le L_{h_L} M\Delta \tag{21}$$

$$|h_i(\tilde{x}(t_j)) - h_i(\tilde{x}(t_{j-1}))| \le L_{h_L}|\tilde{x}(t_j) - \tilde{x}(t_{j-1})| \le L_{h_L} M\Delta \tag{22}$$

for $\Delta < \min\{\Delta_w^*, \Delta_1\}$, $j = k+1, \ldots, k+N-1$.

Under the assumption that a feasible solution to the LEMPC of Eq. (17) exists and thus that Eqs. (17f) and (17g) are satisfied (which will be proven in Section 3.1.3), we use the constraints of Eqs. (17f) and (17g) and Eqs. (21) and (22), in addition to the triangle inequality, to develop upper bounds on the value of the desired rate of change constraints in Eqs. (15) and (16) when the LEMPC of Eq. (17) is used to control the process as follows:

$$\begin{aligned}
&|u_i^*(t_k|t_k) - u_i^*(t_{k-1}|t_{k-1})| \\
&= |u_i^*(t_k|t_k) - u_i^*(t_{k-1}|t_{k-1}) - h_i(x(t_k)) + h_i(x(t_k)) - h_i(x(t_{k-1})) \\
&\quad + h_i(x(t_{k-1}))| \le |u_i^*(t_k|t_k) - h_i(x(t_k))| + |u_i^*(t_{k-1}|t_{k-1}) \\
&\quad - h_i(x(t_{k-1}))| + |h_i(x(t_k)) - h_i(x(t_{k-1}))| \le 2\epsilon_r + L_{h_L} M\Delta
\end{aligned} \tag{23}$$

$$\begin{aligned}
&|u_i^*(t_j|t_k) - u_i^*(t_{j-1}|t_k)| \\
&= |u_i^*(t_j|t_k) - u_i^*(t_{j-1}|t_k) - h_i(\tilde{x}(t_j)) + h_i(\tilde{x}(t_j)) - h_i(\tilde{x}(t_{j-1})) \\
&\quad + h_i(\tilde{x}(t_{j-1}))| \le |u_i^*(t_j|t_k) - h_i(\tilde{x}(t_j))| + |u_i^*(t_{j-1}|t_k) \\
&\quad - h_i(\tilde{x}(t_{j-1}))| + |h_i(\tilde{x}(t_j)) - h_i(\tilde{x}(t_{j-1}))| \le 2\epsilon_r + L_{h_L} M\Delta
\end{aligned} \tag{24}$$

for $\Delta < \min\{\Delta_w^*, \Delta_1\}$ and $j = k+1, \ldots, k+N-1$. It is noted that by assuming $u_i^*(t_0|t_0) = h_i(x(t_0))$, $i = 1, \ldots, m$, then $|u_i^*(t_{k-1}|t_{k-1}) - h_i(x(t_{k-1}))| = 0 \le \epsilon_r$ in Eq. (23) when $k = 1$, which allows the result of Eq. (23) to hold for all $k > 0$. For any $\epsilon_{desired}$, there always exist ϵ_r and Δ that are sufficiently small such that $2\epsilon_r + L_{h_L} M\Delta \le \epsilon_{desired}$. When these values of ϵ_r and Δ are chosen, the desired rate of change constraints in Eqs. (15) and (16) are met, which follows from Eqs. (23) and (24) with the bound in Eq. (18). □

Remark 2. The value of $\epsilon_{desired}$ would typically be chosen based on practical considerations. Because $\epsilon_{desired}$ depends on the sampling time in Eq. (18), one of these practical considerations may be the minimum sampling time possible with the controller software/hardware.

3.1.3. Part 1: Feasibility and stability analysis

In this section, we extend the proofs of feasibility and closed-loop stability from Heidarinejad et al. (2012) for nonlinear processes under LEMPC without input rate of change constraints to those under LEMPC including input rate of change constraints. The results are developed when $h(x)$ used in the design of LEMPC meets Assumption 1, and stronger closed-loop stability results are presented for that case that it meets Assumption 2. We first state several propositions to define functions used to state the theorems giving the conditions under which feasibility and closed-loop stability of nonlinear processes under LEMPC with input magnitude and rate of change constraints are guaranteed.

Proposition 5. *(c.f. Mhaskar et al., 2013; Heidarinejad et al., 2012) Consider the systems*

$$\dot{x}_a(t) = f(x_a(t), u(t), w(t))$$

$$\dot{x}_b(t) = f(x_b(t), u(t), 0)$$

with initial states $x_a(t_0) = x_b(t_0) \in \Omega_\rho$. *There exists a* $\mathcal{K}$ *function* $f_W(\cdot)$ *such that*

$$|x_a(t) - x_b(t)| \leq f_W(t - t_0),$$

for all $x_a(t), x_b(t) \in \Omega_\rho$ *and all* $w(t) \in W$ *with*

$$f_W(\tau) = \frac{L_w \theta}{L_x}(e^{L_x \tau} - 1).$$

Proposition 6. *(c.f. Mhaskar et al., 2013; Heidarinejad et al., 2012) Consider the Lyapunov function* $V(\cdot)$ *of the system of Eq. (1) under* $h(x)$ *meeting Assumption 1. There exists a quadratic function* $f_V(\cdot)$ *such that*

$$V(x) \leq V(\hat{x}) + f_V(|x - \hat{x}|)$$

for all $x, \hat{x} \in \Omega_\rho$ *with*

$$f_V(s) := \alpha_4(\alpha_1^{-1}(\rho))s + M_V s^2$$

where M_V *is a positive constant.*

Proposition 7. *(c.f. Muñoz de la Peña and Christofides, 2008; Alanqar et al., 2015) Consider the Lyapunov function* $V(\cdot)$ *of the system of Eq. (1) under* $h(x)$ *meeting Assumption 2. There exists a quadratic function* $f_V(\cdot)$ *such that*

$$V(x) \leq V(\hat{x}) + f_V(|x - \hat{x}|)$$

for all $x, \hat{x} \in \Omega_\rho$ *with*

$$f_V(s) := \frac{c_4\sqrt{\rho}}{\sqrt{c_1}}s + \beta s^2$$

where β *is a positive constant.*

In the following theorems, we use the notation developed in Propositions 5–7 and prove feasibility and closed-loop stability of a process under LEMPC with input magnitude and rate of change constraints in the presence of bounded process disturbances. The theorems extend the results of Heidarinejad et al. (2012) by requiring a modified bound on Δ based on the results of Section 3.1.2 to ensure that the LEMPC of Eq. (17) computes control actions that satisfy the desired constraints in Eqs. (15) and (16). We present proofs for LEMPC designed using both the asymptotically stabilizing $h(x)$ (Assumption 1) and the exponentially stabilizing $h(x)$ (Assumption 2).

Theorem 2. *Consider the system of Eq. (1) in closed-loop under the LEMPC design of Eq. (17) based on a controller* $h(x)$ *that satisfies the conditions of Eq. (4) and Assumption 1, and assume that* $u_i^*(t_0|t_0) = h_i(x(t_0))$, $i = 1, \ldots, m$. *Let* $\epsilon_w > 0$, $0 < \Delta < \min\{\Delta_w^*, \Delta_1\}$, $\theta > 0$, $\rho > \rho_e \geq \rho_{\min}^* \geq \rho_s > 0$ *satisfy*

$$\rho_e \leq \rho - f_V(f_W(\Delta)), \tag{25}$$

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + L'_x M\Delta + L'_w\theta \leq \frac{-\epsilon_w}{\Delta}, \tag{26}$$

and

$$2\epsilon_r + L_{h_L} M\Delta \leq \epsilon_{desired} \tag{27}$$

where f_V *and* f_W *are defined in Propositions 5 and 6. If* $x(t_0) \in \Omega_\rho$ *and* $N \geq 1$ *where*

$$\rho_{\min}^* = \max\{V(x(t+\Delta)) : V(x(t)) \leq \rho_s\} \tag{28}$$

then the state $x(t)$ *of the closed-loop system is always bounded in* Ω_ρ *and is (uniformly) ultimately bounded in* $\Omega_{\rho_{\min}^*}$.

Proof. The proof of the closed-loop stability of nonlinear processes under LEMPC with both input magnitude and rate of change constraints, where $h(x)$ satisfies Eq. (4) and Assumption 1, follows along the lines of that for the LEMPC without input rate of change constraints in Heidarinejad et al. (2012) because the stability proof depends only on the Lyapunov-based stability constraints in Eqs. (17h) and (17i) and is unaffected by the added rate of change constraints. Feasibility follows because $u_i(t_k) = h_i(x(t_k))$, $i = 1, \ldots, m$, and $u_i(t_j) = h_i(\tilde{x}(t_j))$, $i = 1, \ldots, m$, $j = k+1, \ldots, k+N-1$, is a solution that meets the Lyapunov-based constraints of Eqs. (17h) and (17i), the input constraints of Eq. (17d), and the rate of change constraints of Eqs. (17f) and (17g), as well as the assumption that the first value of u_i calculated, $u_i^*(t_0|t_0)$, $i = 1, \ldots, m$, is set to $h_i(x(t_0))$, $i = 1, \ldots, m$ (the assumption that $u_i^*(t_0|t_0) = h_i(x(t_0))$, $i = 1, \ldots, m$, is made so that every input calculated by the EMPC, which is all inputs calculated from time t_1 and after since the components of the input vector at t_0 are fixed to $h_i(x(t_0))$, meets the desired constraints of Eqs. (15) and (16)). The constraint of Eq. (17e) is satisfied when $x(t) \in \Omega_\rho$ due to the definition of Ω_ρ, and is thus always satisfied when the LEMPC optimization problem is feasible due to the closed-loop stability proof of the LEMPC which shows that $x(t) \in \Omega_\rho$ for $t \geq t_0$ if the LEMPC is feasible and $x(t_0) \in \Omega_\rho$. □

Theorem 3. *Consider the system of Eq. (1) in closed-loop under the LEMPC design of Eq. (17) based on a controller* $h(x)$ *that satisfies the conditions of Eq. (4) and Assumption 2, and assume that* $u_i^*(t_0|t_0) = h_i(x(t_0))$, $i = 1, \ldots, m$. *Let* $\bar{\epsilon}_w > 0$, $0 < \Delta < \min\{\Delta_{w,e}^*, \Delta_1\}$, $\theta > 0$, $\rho > \rho_e \geq \rho_{\min,e}^* \geq \bar{\rho}_s > 0$ *satisfy*

$$\rho_e \leq \rho - f_V(f_W(\Delta)), \tag{29}$$

$$-\frac{c_3}{c_2}\bar{\rho}_s + L'_x M\Delta + L'_w\theta \leq \frac{-\bar{\epsilon}_w}{\Delta}, \tag{30}$$

and

$$2\epsilon_r + L_{h_L} M\Delta \leq \epsilon_{desired} \tag{31}$$

where f_V *and* f_W *are defined in Propositions 5 and 7. If* $x(t_0) \in \Omega_\rho$ *and* $N \geq 1$ *where*

$$\rho_{\min,e}^* = \max\{V(x(t+\Delta)) : V(x(t)) \leq \bar{\rho}_s\} \tag{32}$$

then the state $x(t)$ *of the closed-loop system is always bounded in* Ω_ρ, *and is (uniformly) ultimately bounded in* $\Omega_{\rho_{\min,e}^*}$.

Proof. The proof of feasibility of the LEMPC with both input magnitude and rate of change constraints, where $h(x)$ satisfies Eq. (4) and Assumption 2, is the same as the proof of feasibility of the LEMPC where $h(x)$ satisfies Assumption 1 (the proof for feasibility for Theorem 2). The proof of closed-loop stability of a process under this LEMPC is an extension of the results in Heidarinejad et al. (2012), and the major steps of this proof will be presented here to outline how this extension proceeds.

We first examine the case when $x(t_k) \in \Omega_{\rho_e}$. In this case, the proof that $x(t_{k+1}) \in \Omega_\rho$ when $x(t_k) \in \Omega_{\rho_e}$ and Eq. (29) holds follows that in Heidarinejad et al. (2012). When instead $x(t_k) \in \Omega_\rho/\Omega_{\rho_e}$, the Mode 2 constraint of Eq. (17i) is activated which leads to the following bound:

$$\frac{\partial V(x(t_k))}{\partial x} f(x(t_k), u(t_k), 0) \leq \frac{\partial V(x(t_k))}{\partial x} f(x(t_k), h(x(t_k)), 0) \leq -c_3|x(t_k)|^2 \tag{33}$$

where the upper bound follows from Eq. (3b). The bound in Eq. (33) is then used to bound the time derivative of the Lyapunov function as follows:

$$\dot{V}(x(t)) \leq -c_3|x(t_k)|^2 + \left|\frac{\partial V(x(t))}{\partial x} f(x(t), u(t_k), w(t)) - \frac{\partial V(x(t_k))}{\partial x} f(x(t_k), u(t_k), 0)\right| \tag{34a}$$

$$\overset{\text{Eq. 7}, |w| \leq \theta}{\leq} -c_3|x(t_k)|^2 + L'_x|x(t) - x(t_k)| + L'_w\theta \tag{34b}$$

$$\overset{\text{Eqs. 19, 3a}}{\leq} -\frac{c_3}{c_2}\overline{\rho}_s + L'_x M\Delta + L'_w\theta \tag{34c}$$

for all $t \in [t_k, t_{k+1})$.

If Eq. (30) holds, then the time derivative of the Lyapunov function along the closed-loop state trajectories is negative and can be integrated as in Heidarinejad et al. (2012) to show that when $x(t_k) \in \Omega_\rho/\Omega_{\rho_e}$, the Lyapunov function decreases between two sampling times, bringing the state back into Ω_{ρ_e} in finite time. When the state re-enters Ω_{ρ_e}, either the Mode 1 constraint in Eq. (17h) is re-activated, which maintains the state within Ω_ρ between two sampling times as noted above in this proof, or if $t_k \geq t'$, the Mode 2 constraint continues to be enforced, which decreases the state by at least as much as it would decrease under the sample-and-hold implementation of $h(x)$ meeting Assumption 2 in the presence of disturbances. This decreases the state into the region $\Omega_{\rho^*_{\min,e}}$ as stated in Proposition 4 and maintains it there thereafter. This completes the proof of Theorem 3. □

Remark 3. In prior sections, it was mentioned that $\epsilon_r \geq 0$ and $\epsilon_{desired} > 0$, for practical implementation reasons related to controller feasibility. Specifically, the optimization problem of Eq. (17) is feasible if $\epsilon_r = 0$, but the only feasible solution is $u_i(t_k) = h_i(x(t_k))$, $i = 1, \ldots, m$, and $u_i(t_j) = h_i(\tilde{x}(t_j))$, $i = 1, \ldots, m$, $j = k+1, \ldots, k+N-1$, in which case the application of LEMPC to a process when $\epsilon_r = 0$ will be the same as applying the Lyapunov-based controller in sample-and-hold (Eq. (10)), for which the more complex LEMPC architecture is not necessary. If $\epsilon_{desired} = 0$, however, no $\Delta > 0$ can be chosen as seen from Eqs. (27) and (31), so it is never possible to require $\epsilon_{desired} = 0$ in this LEMPC.

Remark 4. The reason that the constraints of Eqs. (17f) and (17g) are enforced with respect to $h(x)$, rather than being enforced as the desired constraints of Eqs. (15) and (16), is because there is no guarantee that the constraints of Eqs. (15) and (16) are feasible within the LEMPC since they are unrelated to the controller $h(x)$ upon which the two constraints of Eqs. (17h) and (17i) that also must be satisfied are based.

3.1.4. Part 1: Application to a chemical process example

In this section, we use a chemical process example to demonstrate the effect on the computed control actions and process profit of incorporating input rate of change constraints in EMPC. We perform this demonstration by comparing the closed-loop results for a process under three EMPC's: one which does not incorporate input rate of change constraints, a second which imposes input rate of change constraints only on the implemented inputs, and a third which imposes input rate of change constraints on all control actions in the prediction horizon. This chemical process example shows that input rate of change constraints can be used to reduce wide variations in the control actions and thus process variables while still providing economic benefit compared to steady-state operation, and furthermore shows that the number of sampling periods of the prediction horizon over which the input rate of change constraints are enforced in an EMPC can have a significant impact on whether the EMPC can satisfy other process constraints.

Table 1
Ethylene oxide process parameters (Özgülşen et al., 1992).

Parameter	Value	Parameter	Value
γ_1	−8.13	B_1	7.32
γ_2	−7.12	B_2	10.39
γ_3	−11.07	B_3	2170.57
A_1	92.80	B_4	7.02
A_2	12.66	T_c	1.0
A_3	2412.71		

The chemical process considered is the oxidation of ethylene to ethylene oxide in a nonisothermal continuously stirred tank reactor (CSTR), which is assumed to occur according to the following three complex reactions:

$$C_2H_4 + \frac{1}{2}O_2 \rightarrow C_2H_4O \tag{35}$$

$$C_2H_4 + 3O_2 \rightarrow 2CO_2 + 2H_2O \tag{36}$$

$$C_2H_4O + \frac{5}{2}O_2 \rightarrow 2CO_2 + 2H_2O \tag{37}$$

In Özgülşen et al. (1992), the dimensionless material and energy balances for this reactor are developed, with the rate laws for the reactions in Eqs. (35)–(37) taken from Alfani and Carberry (1970). The resulting dimensionless equations defining the relationship $\dot{x} = f(x, u, 0)$ where $u = [u_1\, u_2]^T$ are as follows:

$$\dot{x}_1 = u_1(1 - x_1x_4) \tag{38a}$$

$$\dot{x}_2 = u_1(u_2 - x_2x_4) - A_1e^{\gamma_1/x_4}(x_2x_4)^{0.5} - A_2e^{\gamma_2/x_4}(x_2x_4)^{0.25} \tag{38b}$$

$$\dot{x}_3 = -u_1x_3x_4 + A_1e^{\gamma_1/x_4}(x_2x_4)^{0.5} - A_3e^{\gamma_3/x_4}(x_3x_4)^{0.5} \tag{38c}$$

$$\dot{x}_4 = \frac{u_1}{x_1}(1 - x_4) + \frac{B_1}{x_1}e^{\gamma_1/x_4}(x_2x_4)^{0.5} + \frac{B_2}{x_1}e^{\gamma_2/x_4}(x_2x_4)^{0.25} + \frac{B_3}{x_1}e^{\gamma_3/x_4}(x_3x_4)^{0.5} - \frac{B_4}{x_1}(x_4 - T_c) \tag{38d}$$

where the dimensionless state variables x_1, x_2, x_3, and x_4 represent the dimensionless gas density, ethylene concentration, ethylene oxide concentration, and temperature in the reactor, respectively, and u_1 and u_2 are inputs to the process, with u_1 being the feed volumetric flow rate and u_2 the concentration of ethylene in the feed. The parameters in Eqs. (38a)–(38d) are constants and have the values defined in Table 1, which are taken from Özgülşen et al. (1992).

The goal of the process operating strategy is to maximize the yield of ethylene oxide for a limited reactant feedstock, where the yield is defined by the following equation:

$$Y(t_f) = \frac{\int_0^{t_f} u_1(\tau)x_3(\tau)x_4(\tau)d\tau}{\int_0^{t_f} u_1(\tau)u_2(\tau)d\tau} \tag{39}$$

where t_f is the time at the end of operation. We assume that the available reactant material is fixed by the following integral material constraint:

$$\int_0^{t_f} u_1(\tau)u_2(\tau)d\tau = 0.175t_f \tag{40}$$

Thus, the EMPC's considered in this example will maximize the following stage cost:

$$l_e(x, u) = u_1(t)x_3(t)x_4(t) \tag{41}$$

In addition, due to actuator limitations, u_1 and u_2 are restricted to the following sets:

$$0.0704 \leq u_1 \leq 0.7042, \qquad 0.2465 \leq u_2 \leq 2.4648 \tag{42}$$

The reactor is initialized at $x_I = [x_{1I}\ x_{2I}\ x_{3I}\ x_{4I}]^T = [0.997\ 1.264\ 0.209\ 1.004]^T$, and a sampling period of $\Delta = 9.36$ is used. The Explicit Euler numerical integration method is used to integrate the ordinary differential equations in Eqs. (38a)–(38d) using an integration step size of $h_I = 10^{-4}$ within the EMPC and $h_p = 10^{-5}$ for the model used to simulate the process behavior (which is again Eqs. (38a)–(38d) since it is assumed that there are no disturbances/plant-model mismatch). The open-source interior point optimization software Ipopt (Wächter and Biegler, 2006) was used for all optimizations.

To accomplish the above control objectives, we develop an EMPC, referred to as $EMPC-1$, as follows:

$$\min_{u(\cdot)\in S(\Delta)} \int_{t_k}^{t_{k+N_k}} -u_1(\tau)\tilde{x}_3(\tau)\tilde{x}_4(\tau)d\tau \tag{43a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0) \tag{43b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{43c}$$

$$0.0704 \le u_1(t) \le 0.7042, \quad \forall \ t \in [t_k, t_{k+N_k}) \tag{43d}$$

$$0.2465 \le u_2(t) \le 2.4648, \quad \forall \ t \in [t_k, t_{k+N_k}) \tag{43e}$$

$$\frac{1}{t_p}\int_{jt_p}^{t_k} u_1^*(\tau)u_2^*(\tau)d\tau + \frac{1}{t_p}\int_{t_k}^{t_{k+N_k}} u_1(\tau)u_2(\tau)d\tau = 0.175 \tag{43f}$$

where the notation is as in Eq. (17) except that the problem is implemented with a shrinking horizon of length N_k and the material constraint is implemented over operating periods of length t_p to reduce the computation time. Ten operating periods, each of length $t_p = 46.8$, under this EMPC were simulated. The index j signifies the number of operating periods that have passed prior to the current one ($j = 0, \ldots, 9$), and $u_1^*(t)$ and $u_2^*(t)$ represent the previously computed and applied input trajectories ($u_1^*(t) = u_1^*(t_q|t_q)$ for $t \in [t_q, t_{q+1})$, and $u_2^*(t) = u_2^*(t_q|t_q)$ for $t \in [t_q, t_{q+1})$, where t_q varies between jt_p and t_{k-1} in Eq. (43f)). N_k is initialized to $t_p/\Delta = 5$ at the beginning of each operating period and is decremented by one at the beginning of each sampling period. The results of the simulations under $EMPC-1$ are shown as the solid lines in Figs. 1 and 2. As seen in Fig. 2, the EMPC determines that the optimal input trajectories are those for which the inputs make extreme jumps throughout each operating period, which in turn causes significant variation in the state variables, as shown in Fig. 1, especially in x_2 and x_3.

We now suppose that we do not want to have such rapid changes in the requested control actions. As a result, we impose input rate of change constraints in the EMPC to allow it to continue to optimize the process economics throughout the whole range of u_1 and u_2, but without taking extreme, sudden action to do so. We enforce that the difference between two control actions can be no more than 0.1. We formulate two EMPC's with input rate of change constraints, the first of which ($EMPC-2$) enforces the rate of change constraint only on the first control action that will be implemented, and the second of which ($EMPC-3$) enforces the rate of change constraint at each sampling period in the shrinking prediction horizon N_k. Thus, $EMPC-2$ solves the optimization problem of Eq. (43) with the added constraints:

$$|u_1(t_k) - u_1^*(t_{k-1}|t_{k-1})| \le 0.1 \tag{44a}$$

$$|u_2(t_k) - u_2^*(t_{k-1}|t_{k-1})| \le 0.1 \tag{44b}$$

and $EMPC-3$ solves the optimization problem of Eq. (43) with the added constraints in Eqs. (44a) and (44b) plus the additional constraints:

$$|u_1(t_j) - u_1(t_{j-1})| \le 0.1, \quad j = k+1, \ldots, k+N_k-1 \tag{45a}$$

$$|u_2(t_j) - u_2(t_{j-1})| \le 0.1, \quad j = k+1, \ldots, k+N_k-1 \tag{45b}$$

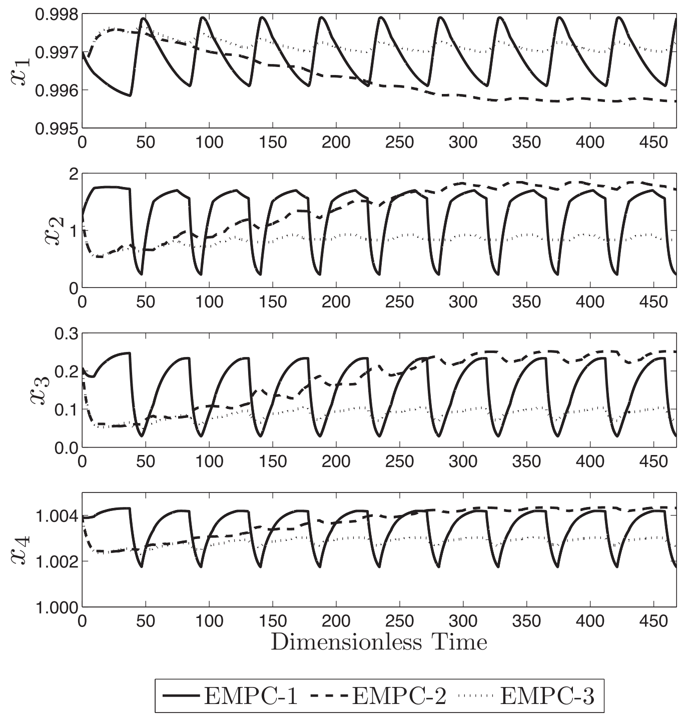

Fig. 1. State trajectories for the process of Eqs. (38a)–(38d) under $EMPC-1$, $EMPC-2$, and $EMPC-3$.

The Lyapunov-based constraints in Eqs. (17h) and (17i) were not considered for this example because the process is operated within a region around an open-loop asymptotically stable steady-state and showed no closed-loop stability issues during the simulations of the three EMPC's. Because no Lyapunov-based constraints were employed, the input rate of change constraints used were those in Eqs. (44a)–(45b), which are written in terms of the desired rate of change as in Eqs. (15) and (16), rather than based off of the

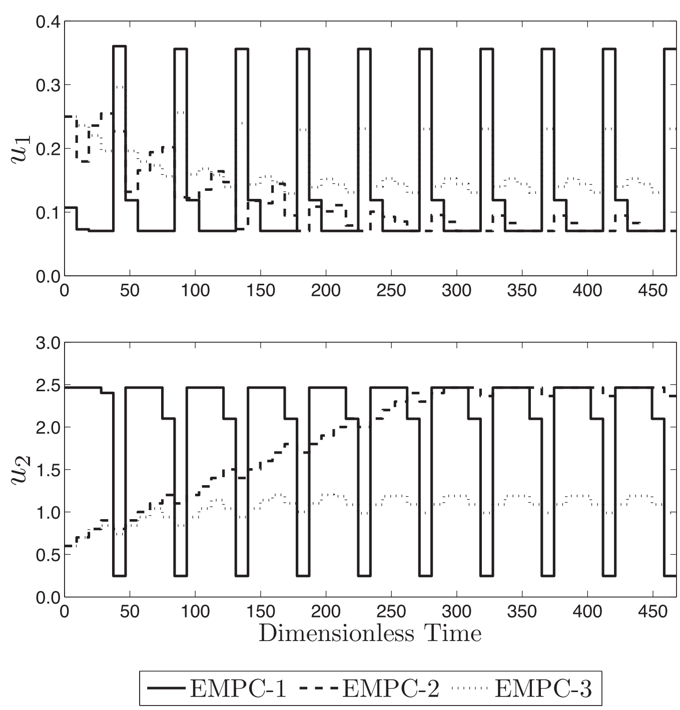

Fig. 2. Input trajectories for the process of Eqs. (38a)–(38d) under $EMPC-1$, $EMPC-2$, and $EMPC-3$.

Table 2
Process yield.

Process	Yield
EMPC − 1	9.61%
EMPC − 2	9.56%
EMPC − 3	8.23%
SS	6.63%

controller $h(x)$ as was done in the LEMPC of Eq. (17). The state and input trajectories for the simulations of the closed-loop system of Eqs. (38a)–(38d) under *EMPC* − 2 and *EMPC* − 3 are plotted against the state and input trajectories for the closed-loop system under *EMPC* − 1 in Figs. 1 and 2.

The yields (according to Eq. (39)) for the process under the three different EMPC's are shown in Table 2 and compared to the yield for the steady-state case (*SS* in the table) obtained by starting at x_I and using the constant input vector $u_s = [u_{1s}\ u_{2s}]^T = [0.35\ 0.5]^T$ to bring the process states to the open-loop asymptotically stable steady-state $[x_{1s}\ x_{2s}\ x_{3s}\ x_{4s}] = [0.998\ 0.424\ 0.032\ 1.002]$.

As shown in Figs. 1 and 2, the input rate of change constraints significantly reduce the variability in the state and input trajectories as desired, while still allowing optimization of the process economics, as shown by the periodic trajectories that still exist in the state and input trajectories for *EMPC* − 3, though with reduced amplitude compared to those under *EMPC* − 1. As expected, the addition of input rate of change constraints reduces the ability of the EMPC to maximize the yield of the process to its fullest extent (the yield under *EMPC* − 1 is 16.8% greater than that under *EMPC* − 3). However, even with the input rate of change constraints, *EMPC* − 3 outperforms steady-state operation (the yield under *EMPC* − 3 is 24.1% greater than that for steady-state operation).

Because the input rate of change constraints in *EMPC* − 2 are not enforced at every sampling period, *EMPC* − 2 becomes infeasible in the last sampling period of all operating periods after the third (when Ipopt determines a problem is locally infeasible, it returns a solution that locally minimizes the constraint violation using a separate optimization problem (Wächter, 2009)). In each of the operating periods in which *EMPC* − 2 is infeasible, the process' use of reactant material exceeded the value of the integral constraint, in some operating periods by as much as 8.7%. Thus, the value of the yield reported in Table 2 for *EMPC* − 2 cannot be compared with the yields of *EMPC* − 1 and *EMPC* − 3 because *EMPC* − 1 and *EMPC* − 3 met the process constraints, while *EMPC* − 2 did not. The violation of the integral constraint by *EMPC* − 2 occurs because *EMPC* − 2 predicts that there can be sharp changes in all inputs in the prediction horizon except those for the first sampling period, which are forced to stay within 0.1 of the previous input value. Thus, because of the lack of foresight of *EMPC* − 2, the implemented control actions for the first four sampling periods in most of the operating periods use too much of the reactant material, with the result that there is no way that the integral constraint can be met in the last sampling period of the operating periods if the rate of change constraints and hard bounds on the inputs are also to be met.

We note that though we did not formulate the constraints of *EMPC* − 3 with respect to a Lyapunov-based controller which, as was proven above in this work, ensures feasibility of the optimization problem, no issues with feasibility of the solution of *EMPC* − 3 were encountered. This can occur in practice, and emphasizes that the requirements for feasibility of an EMPC with input rate of change constraints as developed in this paper, such as formulating the constraints as in Eqs. (17f) and (17g) rather than as in Eqs. (15) and (16), are conservative. For example, in this problem, we set $u^*(t_{-1}|t_{-1}) = u_s$. The input vector u_s satisfies all constraints in Eqs.

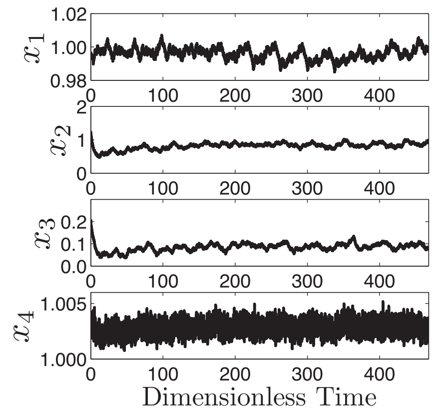

Fig. 3. State trajectories for the process of Eqs. (38a)–(38d) under *EMPC* − 3 in the presence of bounded disturbances.

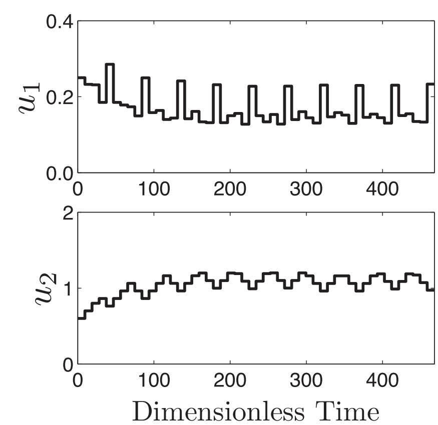

Fig. 4. Input trajectories for the process of Eqs. (38a)–(38d) under *EMPC* − 3 in the presence of bounded disturbances.

(43)–(45b), and thus is itself a feasible input trajectory. Because *EMPC* − 3 recognizes that all future inputs in the prediction horizon must meet the input rate of change constraint, when it finds a solution that outperforms steady-state operation but is able to satisfy the constraints, this solution is feasible both in the current operating period and also, in reverse, in the next (because for this problem, we assume that the plant follows the nominal process model and that all constraints depend only on u_1 and u_2, not on process states, so the input trajectory just implemented, in reverse, will be feasible for the next operating period). By progressing in this manner, the full input trajectory that *EMPC* − 3 takes is feasible in reverse, and it is able to settle to an off steady-state input trajectory without feasibility issues.

To evaluate the robustness of *EMPC* − 3 when there are process disturbances ($w(t) \not\equiv 0$), the process of Eqs. (38a)–(38d) was simulated under *EMPC* − 3, but with bounded Gaussian white noise added to the right-hand side of Eqs. (38a)–(38d) for the simulation of the process outside of the EMPC, with zero mean, standard deviation vector $[\sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4}]^T = [0.6\ 10\ 1.8\ 0.6]^T$, and bound vector $[\theta_{x_1}\theta_{x_2}\theta_{x_3}\theta_{x_4}]^T = [1.8\ 30\ 5.4\ 1.8]^T$. The standard deviations and bounds were chosen such that the noise had a significant effect on the process states. The simulation results are shown in Figs. 3 and 4, and demonstrate that *EMPC* − 3 incorporating input rate of change constraints maintained closed-loop stability of the process in the presence of bounded disturbances. In addition, it met the integral material constraint and was feasible in all sampling periods, demonstrating the robustness of the controller.

Remark 5. It is noted that the periodic state trajectories for $EMPC-1$ and $EMPC-3$ in Fig. 1 are the result of the periodic input policies in Fig. 2. The periodic input policies are chosen by the EMPC because the EMPC found that that was the most economically optimal input policy; this is consistent with prior work on the ethylene oxide production process example (e.g., Özgülşen et al., 1992; Ellis and Christofides, 2014), which showed that a periodic operating policy is more economically optimal than steady-state operation for this example. The integral constraint of Eq. (40) plays a role in the periodic trajectories observed because it requires that within each operating period, only a certain amount of material can be used. Despite the periodic nature of the trajectories, the closed-loop states remain in a bounded region in state-space around the asymptotically stable steady-state such that the closed-loop process is stable in the sense that the states remain within a bounded region.

3.2. Part 2: LEMPC with a terminal constraint design based on a Lyapunov-based controller

In this section, we further build toward the development of provable performance guarantees for LEMPC with input rate of change constraints by developing an LEMPC formulation (without input rate of change constraints) for which provable performance guarantees can be made for nominal operation. Specifically, an LEMPC incorporating a terminal equality constraint based on a Lyapunov-based controller will be developed, and performance guarantees will be made for this LEMPC for both finite-time and infinite-time. While a number of performance results have been developed for other EMPC formulations (such as EMPC with a terminal steady-state equality constraint (Diehl et al., 2011; Angeli et al., 2012) or a terminal region constraint (Amrit et al., 2011)), few performance results have appeared for LEMPC. Previous performance results for LEMPC have been developed utilizing solutions from an auxiliary tracking MPC (Ellis and Christofides, 2014a; Heidarinejad et al., 2013); the performance guarantees for LEMPC developed in this section compare the closed-loop performance under LEMPC not with that under MPC but with that under a Lyapunov-based controller implemented in sample-and-hold. Like many other EMPC performance guarantees, those made in this work rely on the use of a terminal constraint and thus hold only for nominal process operation; however, they have several advantages over performance guarantees developed for some other formulations of EMPC in that an *a priori* characterization of the feasible region is possible and because the terminal constraint is not necessarily the economically optimal steady-state (in the design the terminal constraint asymptotically converges to the economically optimal steady-state or a neighborhood of it, depending on the properties of $h(x)$), the resulting LEMPC may give a larger feasible region relative to an EMPC with a terminal equality constraint equal to the economically optimal steady-state.

To develop the LEMPC with a terminal equality constraint based on a Lyapunov-based controller and its provable performance guarantees, this section begins with a description of the LEMPC formulation for which provable performance guarantees can be made, which is an LEMPC of the form of Eq. (14) but without Eq. (14g) (since only nominal operation is considered) and with the addition of a terminal equality constraint based on the same Lyapunov-based controller as is used to develop the Lyapunov-based stability constraint of Eq. (14f) (we note that the LEMPC, like that with input magnitude and rate of change constraints developed in Part 1, has input magnitude constraints with the form in Eq. (14d), but since the input trajectories themselves are not the focus of Part 2, this will not be further highlighted in Part 2). Subsequently, it is shown that when there are no disturbances and when there is no plant-model mismatch, the LEMPC with a terminal equality constraint based on $h(x)$ is feasible and maintains closed-loop stability of the nominal process in the sense of boundedness of the closed-loop state. Following this development, the performance properties of the controller are proven on both the finite-time and infinite-time intervals for a Lyapunov-based controller satisfying Assumption 1 and for a Lyapunov-based controller satisfying Assumption 2 (for $h(x)$ meeting Assumption 2, the infinite-time performance result is equivalent to the statement that the nominal process under LEMPC with a terminal constraint based on a Lyapunov-based controller performs at least as well in infinite-time as it does under steady-state operation). Finally, a chemical process example is presented that demonstrates the use of the LEMPC incorporating a terminal equality constraint based on a Lyapunov-based controller and shows that for a short prediction horizon used in an EMPC, the use of terminal equality constraints in the EMPC may be crucial to improving process economic performance over steady-state operation.

3.2.1. Part 2: Formulation of LEMPC with a terminal equality constraint based on a Lyapunov-based controller

In the standard LEMPC design (Heidarinejad et al., 2012), an EMPC scheme was designed by taking advantage of a Lyapunov-based controller (meeting Assumption 1 or Assumption 2), a corresponding Lyapunov function, and the stability region. Though feasibility, closed-loop stability, and robustness to sufficiently small disturbances may be proven for this standard LEMPC design, guaranteed closed-loop performance under the resulting LEMPC cannot be proven without additional technical conditions and a sufficiently long prediction horizon because the standard LEMPC design does not incorporate terminal constraints (e.g., Grüne, 2013; Müller and Grüne, 2015). Nevertheless, owing to the availability of the Lyapunov-based controller of Assumption 1 or Assumption 2, the corresponding Lyapunov function, and the stability region used to design LEMPC, a terminal equality constraint may be readily designed for the LEMPC problem that allows performance guarantees to be made for nominal process operation while maintaining the unique recursive feasibility property of LEMPC for all initial states in Ω_ρ. In this work, the terminal constraint is computed from the solution of the sampled-data system of Eq. (8) (where h meets either Assumption 1 or Assumption 2).

Because the terminal constraint is derived from the solution of Eq. (8), it is necessary to define notation that distinguishes the solution of Eq. (8) from the solution of the LEMPC. To distinguish the state and input trajectories of the system under the Lyapunov-based controller implemented in sample-and-hold (Eq. (8)) from the state and input trajectories of the closed-loop system under LEMPC incorporating a terminal equality constraint derived from Eq. (8), z and v will be used for the former, and x and u^* will be used for the latter. Thus, for simplicity of notation, the sampled-data system consisting of the nominal system of Eq. (1) under the sample-and-hold implementation of the Lyapunov-based controller is given by:

$$\begin{aligned} \dot{z}(t) &= f(z(t), v(t), 0) \\ v(t) &= h(z(t_k)) \end{aligned} \tag{46}$$

for $t \in [t_k, t_{k+1})$, $k = 0, 1, \ldots$ with initial condition $z(0) = z_0 \in \Omega_\rho$. The sampled-data system consisting of the nominal system of Eq. (1) under the sample-and-hold inputs computed by the LEMPC with a terminal equality constraint based on the Lyapunov-based controller is given by:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u^*(t), 0) \\ u^*(t) &= u^*(t_k|t_k) \end{aligned} \tag{47}$$

for $t \in [t_k, t_{k+1})$, $k = 0, 1, \ldots$ with initial condition $x(0) = x_0 \in \Omega_\rho$, where $x_0 = z_0$. It is noted that the two sampled-data systems in Eqs. (46) and (47) are initiated from the same initial condition, but the system of Eq. (46) only incorporates feedback of $z(t_k)$ without any reference to the measured state of the sampled-data system of Eq. (47), and the sampled-data system of Eq. (47) only incorporates feedback of $x(t_k)$ (though it does require $z(t_{k+N})$ for the determination of the input $u^*(t)$ that is applied to the system, as will be shown subsequently).

The solution of the sampled-data system of Eq. (46) is used to design a terminal equality constraint for LEMPC that requires that the predicted state at the end of the prediction horizon ($\tilde{x}(t_{k+N})$) be equal to the solution of Eq. (46) at time t_{k+N} ($z(t_{k+N})$), where the function h in Eq. (46) is the same h used to design V and Ω_ρ in the LEMPC. The terminal condition $z(t_{k+N})$ is determined at each sampling time as follows:

Step 1. At the initial time $t_0 = 0$, $z(t_N)$ is computed by first initializing the system of Eq. (46) at $z_0 = x_0$ (a measurement of the state of Eq. (47) at the initial time) and recursively solving the system from the initial time to $t_N = N\Delta$. Then, the state $z(t_N)$ is used as a terminal equality constraint in an LEMPC problem solved at $t = 0$.

Step 2. For all sampling times after t_0, the terminal constraint that is imposed in the LEMPC problem at t_k is computed by recursively solving the system of Eq. (46) from $z(t_k)$ to $z(t_{k+N})$ (because only nominal operation is considered and $z(t_{k+N-1})$ was computed at the previous sampling time, it is only necessary to recursively solve the system of Eq. (46) from t_{k+N-1} to t_{k+N} to obtain the solution from $z(t_k)$ to $z(t_{k+N})$ if the solution from the previous sampling time was stored; for added robustness, especially to numerical and discretization errors, one may reinitialize the system of Eq. (46) with a state measurement $z(t_k)$ at each sampling time and numerically integrate forward from this measurement to compute $z(t_{k+N})$, but in the nominal operating setting considered here, numerical and discretization errors are not considered).

Using the terminal equality constraint described above, the formulation of the LEMPC with the terminal equality constraint formulated based on the state z obtained under the Lyapunov-based controller is given by the problem:

$$\min_{u(\cdot) \in S(\Delta)} \int_{t_k}^{t_{k+N}} l_e(\tilde{x}(\tau), u(\tau))d\tau \quad (48a)$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0) \quad (48b)$$

$$\tilde{x}(t_k) = x(t_k) \quad (48c)$$

$$\tilde{x}(t_{k+N}) = z(t_{k+N}) \quad (48d)$$

$$u(t) \in \mathbb{U}, \quad \forall \ t \in [t_k, t_{k+N}) \quad (48e)$$

$$\tilde{x}(t) \in \mathbb{X}, \quad \forall \ t \in [t_k, t_{k+N}) \quad (48f)$$

$$V(\tilde{x}(t)) \leq \rho, \quad \forall \ t \in [t_k, t_{k+N}) \quad (48g)$$

where the notation follows that in Eqs. (13) and (14) and, as noted in Section 2.2, $u(t) \in \mathbb{U}$ is equivalent to $u_i \in \mathbb{U}_i$, $i = 1, \ldots, m$. Because $\rho_e = \rho$ for nominal process operation, the Mode 1 constraint of Eq. (48g) enforces that the predicted state remain in Ω_ρ throughout the prediction horizon. The terminal constraint of Eq. (48d) forces the computed input trajectory to steer the predicted state trajectory to the state $z(t_{k+N})$ at the end of the prediction horizon. This terminal constraint differs from traditional terminal equality constraints in the sense that it is not necessarily a steady-state. However, the terminal constraint in the LEMPC of Eq. (48) converges to a neighborhood of the steady-state owing to the stability properties of the Lyapunov-based controller (if h used in the design of Eq. (48) meets Assumption 1, then $z(t)$ eventually enters $\Omega_{\rho_{\min}}$ from Proposition 1, and if h used in the design of Eq. (48) meets Assumption 2, then $z(t)$ reaches the steady-state in infinite-time from Proposition 2). A difference between the standard LEMPC design of Eq. (14) and the LEMPC incorporating a terminal equality constraint in Eq. (48) is that there is no contractive constraint in the LEMPC with a terminal equality constraint. The reason for this difference is that only nominal operation is considered for the LEMPC with a terminal equality constraint, so only the constraint in Eq. (48g) is required to ensure closed-loop stability in the sense that the state trajectory will be maintained within the stability region Ω_ρ for all times. The effect of including the contractive constraint will be discussed in Section 3.2.2, along with the stability and feasibility properties of the LEMPC incorporating a terminal equality constraint based on the Lyapunov-based controller.

The LEMPC of Eq. (48) is implemented according to a standard receding horizon implementation. At each sampling time t_k, a state measurement $x(t_k)$ is received and the terminal constraint $z(t_{k+N})$ is computed. The optimization problem of Eq. (48) is solved with the computed $z(t_{k+N})$ to obtain the input trajectory over the prediction horizon. However, only the control action computed for the first sampling period of the prediction horizon is implemented on the system. At the next sampling time, a new state measurement is obtained, a new terminal constraint is computed, and the optimization problem is re-solved with the updated parameters to obtain the control action for the next sampling period.

3.2.2. Part 2: Feasibility and stability analysis

In this section, we develop a theorem stating that the LEMPC of Eq. (48) is feasible and maintains closed-loop stability of the nominal system of Eq. (1) when the Lyapunov-based controller used in the design of the LEMPC meets Assumption 1 and a sufficiently small sampling period is utilized.

Theorem 4. *Consider the system of Eq. (1) with $w(t) \equiv 0$ in closed-loop under the LEMPC design of Eq. (48) based on a controller h that satisfies the conditions of Eq. (4) and Assumption 1. Let $\rho > 0$, and $0 < \Delta < \min\{\Delta^*, \Delta_1\}$. If $x(t_0) \in \Omega_\rho$ and $N \geq 1$, then the state $x(t)$ of the closed-loop system is always bounded in Ω_ρ.*

Proof. Recursive feasibility of the optimization problem of Eq. (48) is guaranteed when the conditions of Theorem 4 are met because the sample-and-hold input trajectory obtained from the Lyapunov-based controller is a feasible solution to the optimization problem at t_0 (i.e., the input trajectory $u(t) = v(t)$, $t \in [t_0, t_N)$ satisfies the input constraint of Eq. (48e) and the terminal constraint of Eq. (48d) by design, it satisfies Eq. (48g) because the Lyapunov-based controller implemented in sample-and-hold maintains the state within Ω_ρ from Proposition 1, and it satisfies Eq. (48f) since Eq. (48g) is satisfied and $\Omega_\rho \subseteq \mathbb{X}$). At the next sampling time (t_1), $u(t) = u^*(t_j|t_0)$ for $t \in [t_j, t_{j+1})$, $j = 1, \ldots, N-1$ (which drives $\tilde{x}(t_N)$ to $z(t_N)$ since $u^*(t_j|t_0)$, $j = 0, \ldots, N-1$, was feasible at the previous sampling time and thus Eq. (48d) is satisfied for this input trajectory), and $u(t) = h(z(t_N))$ for $t \in [t_N, t_{N+1})$ is a feasible solution to the optimization problem because nominal operation is considered. At subsequent sampling times (i.e., at t_k), a feasible solution to the LEMPC of Eq. (48) is, similarly, the part of the solution from the previous sampling time that was not implemented followed by $h(z(t_{k+N-1}))$ utilized for the last sampling period in the prediction horizon. This shows that for nominal operation, the LEMPC of Eq. (48) is always feasible. Closed-loop stability of the LEMPC of Eq. (48) in the sense that the closed-loop state trajectory is maintained within Ω_ρ at all times is guaranteed when the optimization problem is feasible owing to the fact that the Lyapunov-based constraint of Eq. (48g) is imposed in the optimization problem and nominal operation is considered. □

Though the terminal condition itself converges to the origin or a neighborhood of it, the input trajectory generated by applying the input calculated for the first sampling period of each prediction horizon may not drive the process state to the origin or a small neighborhood of it. The LEMPC of Eq. (48) may be extended to include the two-mode control strategy of Eq. (14), or the contractive constraint in Eq. (14g) may be added to the LEMPC of Eq. (48) to drive the process state to a neighborhood of the origin, even in the presence of disturbances, if the resulting LEMPC remains feasible. However, the performance results to be developed in Section 3.2.3 hold for the nominal case.

Remark 6. It has been previously noted in this paper that the feasible region of LEMPC with a terminal equality constraint based on a Lyapunov-based controller can be explicitly characterized *a priori*. Theorem 4 and its proof show that the feasible region is the stability region of the LEMPC.

3.2.3. Part 2: Closed-loop performance analysis

In this section, we prove that the economic performance of the LEMPC of Eq. (48) is at least as good as that of the Lyapunov-based controller used in its design in both finite-time and infinite-time. The analysis techniques used follow those of Angeli et al. (2012), which analyzes the closed-loop performance of EMPC formulated with an equality terminal constraint equal to x_s^*. In the following, $J_e^*(x(t_k))$ denotes the optimal value of the objective function of the optimization problem of Eq. (48) at time t_k given the state measurement $x(t_k)$.

The first performance result, presented in the following theorem, gives the finite-time performance of the process under the LEMPC of Eq. (48) designed with a Lyapunov-based controller that satisfies Assumption 1.

Theorem 5. *Consider the closed-loop system of Eq. (1) with $w(t)\equiv 0$ under the LEMPC of Eq. (48) based on a Lyapunov-based controller that satisfies Assumption 1. Let $\Delta\in(0,\Delta^*)$ where $\Delta^*>0$ is the conclusion of Proposition 1. For any strictly positive finite integer T, the closed-loop economic performance under the LEMPC of Eq. (48) is bounded by:*

$$\int_0^{T\Delta} l_e(x(t),u^*(t))dt \leq \int_0^{(T+N)\Delta} l_e(z(t),v(t))dt \tag{49}$$

where x and u^* denote the closed-loop state and input trajectories of the system of Eq. (47) and z and v denote the state and input trajectories of the system of Eq. (46) where $z(0)=x(0)\in\Omega_\rho$.

Proof. Let $u^*(t|t_k)$ for $t\in[t_k, t_{k+N})$ be the optimal input trajectory of Eq. (48) at t_k. The piecewise defined input trajectory consisting of $u(t)=u^*(t|t_k)$ for $t\in[t_{k+1}, t_{k+N})$ and $u(t)=h(z(t_{k+N}))$ for $t\in[t_{k+N}, t_{k+N+1})$ is a feasible solution to the optimization problem at t_{k+1}. Utilizing this feasible solution to the problem of Eq. (48) at t_{k+1}, the difference between the optimal values of Eq. (48) at any two successive sampling times t_k and t_{k+1} may be bounded as follows:

$$J_e^*(x(t_{k+1}))-J_e^*(x(t_k)) \leq \int_{t_{k+N}}^{t_{k+N+1}} l_e(z(t),h(z(t_{k+N})))dt - \int_{t_k}^{t_{k+1}} l_e(x(t),u^*(t_k|t_k))dt. \tag{50}$$

Let T be any positive finite integer. Summing the differences between the optimal values of Eq. (48) at two subsequent sampling times, the following upper bound is derived:

$$\sum_{k=0}^{T-1}[J_e^*(x(t_{k+1}))-J_e^*(x(t_k))] \leq \int_{t_N}^{(T+N)\Delta} l_e(z(t),v(t))dt - \int_0^{T\Delta} l_e(x(t),u^*(t))dt \tag{51}$$

Without loss of generality, take $l_e(x,u)\geq 0$ for all $x\in\Omega_\rho$ and $u\in\mathbb{U}$. Then the left-hand side of Eq. (51) is bounded below by:

$$\begin{aligned}\sum_{k=0}^{T-1}[J_e^*(x(t_{k+1}))-J_e^*(x(t_k))] &= J_e^*(x(T\Delta))-J_e^*(x(0))\\ &\geq -J_e^*(x(0))\end{aligned} \tag{52}$$

where the inequality follows from the fact that $l_e(x,u)\geq 0$ for all $x\in\Omega_\rho$ and $u\in\mathbb{U}$. Owing to optimality, the optimal value of Eq. (48) at the initial time may be bounded by the cost under a feasible solution; thus, it may be bounded by the cost under the Lyapunov-based controller implemented in sample-and-hold over the prediction horizon:

$$J_e^*(x(0)) \leq \int_0^{t_N} l_e(z(t),v(t))dt. \tag{53}$$

Combining Eqs. (51)–(53), the closed-loop economic performance from the initial time to $T\Delta$ is no worse than the closed-loop performance under the Lyapunov-based controller from the initial time to $(T+N)\Delta$, which proves the bound of Eq. (49). □

The upper limit of integration of the right-hand side of Eq. (49) ($(T+N)\Delta$) arises from the fact that a fixed prediction horizon is used in the LEMPC of Eq. (48). If, instead, $T\Delta$ represents the final operating time of a given system, one could employ a shrinking horizon from time $(T-N)\Delta$ to $T\Delta$ in the LEMPC and the upper limit of integration of the right-hand side of Eq. (49) would be $T\Delta$. Specifically, for $t_k\in[t_0, t_{T-N})$, we have from Eq. (51):

$$\sum_{k=0}^{T-N-1}[J_e^*(x(t_{k+1}))-J_e^*(x(t_k))] \leq \int_{N\Delta}^{T\Delta} l_e(z(t),v(t))dt - \int_0^{(T-N)\Delta} l_e(x(t),u^*(t))dt \tag{54}$$

and from Eq. (49) we have:

$$\int_0^{(T-N)\Delta} l_e(x(t),u^*(t))dt \leq \int_0^{T\Delta} l_e(z(t),v(t))dt. \tag{55}$$

For sampling times between t_{T-N} and t_T, we employ a shrinking horizon in the EMPC. That is, let $\bar{N}_k=N-j$ be the horizon used at sampling time t_k for $k\in\{T-N,\ldots,T-1\}$ where $j=k-T+N$. With slight abuse of notation, let

$$J_e^*(x(t_k)) = \int_{t_k}^{t_{k+\bar{N}_k}} l_e(x(t),u^*(t|t_k))dt \tag{56}$$

be the optimal value of the EMPC problem at sampling times t_k, $k\in\{T-N,\ldots,T-1\}$ where the EMPC is formulated with a prediction horizon of $\bar{N}_k$. By the principle of optimality, the difference between the optimal value of the EMPC problem at two subsequent sampling times is

$$J_e^*(x(t_{k+1}))-J_e^*(x(t_k)) = -\int_{t_k}^{t_{k+1}} l_e(x(t),u^*(t_k|t_k))dt \tag{57}$$

for $k \in \{T-N, \ldots, T-2\}$. Because $T\Delta$ for this shrinking horizon case represents the final time of operation, the EMPC is not solved at that time and thus there is no value of $J_e^*(x(t_T))$. For this reason, we consider the following summation of the terms in Eq. (57):

$$\sum_{k=T-N}^{T-2} [J_e^*(x(t_{k+1})) - J_e^*(x(t_k))] - J_e^*(x(t_{T-1}))$$

$$= -J_e^*(x(t_{T-N})) = -\int_{(T-N)\Delta}^{T\Delta} l_e(x(t), u^*(t))dt. \tag{58}$$

where the left-hand side is equivalent to the summation of the terms in Eq. (57) from $k = T-N$ to $k = T-1$ with $J_e^*(x(t_T)) := 0$. The sum of the differences between the optimal values of Eq. (48) at two subsequent sampling times between 0 and $T-2$ with $J_e^*(x(t_{T-1}))$ subtracted from this sum gives:

$$\begin{aligned}\sum_{k=0}^{T-2} [J_e^*(x(t_{k+1})) - J_e^*(x(t_k))] - J_e^*(x(t_{T-1})) &= \sum_{k=0}^{T-N-1} [J_e^*(x(t_{k+1})) - J_e^*(x(t_k))] + \sum_{k=T-N}^{T-2} [J_e^*(x(t_{k+1})) - J_e^*(x(t_k))] - J_e^*(x(t_{T-1})) \\ &\overset{(54),(58)}{\leq} \int_{N\Delta}^{T\Delta} l_e(z(t), v(t))dt - \int_0^{(T-N)\Delta} l_e(x(t), u^*(t))dt - \int_{(T-N)\Delta}^{T\Delta} l_e(x(t), u^*(t))dt\end{aligned} \tag{59}$$

Also,

$$\begin{aligned}\sum_{k=0}^{T-2} [J_e^*(x(t_{k+1})) - J_e^*(x(t_k))] - J_e^*(x(t_{T-1})) &= J_e^*(x(t_{T-1})) - J_e^*(x(0)) - J_e^*(x(t_{T-1})) = -J_e^*(x(0)) \\ &\geq -\int_0^{N\Delta} l_e(z(t), v(t))dt\end{aligned} \tag{60}$$

where the last inequality follows from the same arguments used to write Eq. (53) above. Combining Eqs. (59) and (60), the required performance bound is obtained for the shrinking horizon case as follows:

$$\int_0^{T\Delta} l_e(x(t), u^*(t))dt \leq \int_0^{T\Delta} l_e(z(t), v(t))dt \tag{61}$$

This completes the proof of the finite-time performance bound for the shrinking horizon case.

Again considering the case that no shrinking horizon is used, we note that as a consequence of the performance bound of Eq. (49), the average finite-time economic performance may be bounded as follows:

$$\frac{1}{T\Delta}\int_0^{T\Delta} l_e(x, u^*)dt \leq \frac{1}{T\Delta}\int_0^{T\Delta} l_e(z, v)dt + \frac{1}{T\Delta}\int_{T\Delta}^{(T+N)\Delta} l_e(z, v)dt \tag{62}$$

for any integer $T > 0$. From the right-hand side of Eq. (62), the significance of the second term on the right-hand side dissipates as T gets large. Thus, the results of Theorem 5 show that the average closed-loop economic performance over a finite-time operating interval under LEMPC with a terminal equality constraint based on h that meets Assumption 1 is at least as good as the average closed-loop economic performance under h implemented in sample-and-hold plus a term that dissipates as the length of operation increases.

In the above discussion, we developed economic performance guarantees for LEMPC with a terminal equality constraint based on a Lyapunov-based controller satisfying Assumption 1 on the finite-time interval. We now consider the infinite-time (asymptotic average) performance. The provable result on asymptotic average economic performance varies depending on whether the Lyapunov-based controller satisfies Assumption 1 or Assumption 2. We first present a theorem for the infinite-time performance for a controller satisfying Assumption 1 (the performance result when the Lyapunov-based controller satisfies Assumption 2 is stronger and will be presented subsequently).

Theorem 6. *Consider the closed-loop system of Eq. (1) with $w(t) \equiv 0$ under the LEMPC of Eq. (48) where the Lyapunov-based controller satisfies Assumption 1 and $z(0) = x(0) \in \Omega_\rho$. Let $\Delta \in (0, \Delta^*)$ where Δ^* is the conclusion of Proposition 1. The asymptotic average performance is bounded by:*

$$\limsup_{T\to\infty} \frac{1}{T\Delta}\int_0^{T\Delta} l_e(x(t), u^*(t))dt \leq \max_{x,y \in \Omega_{\rho_{\min}}} l_e(x, h(y)). \tag{63}$$

Proof. To develop the proof of Theorem 6, we first consider the asymptotic average economic performance of the nominal system of Eq. (1) under the Lyapunov-based controller that satisfies Assumption 1 implemented in sample-and-hold (i.e., the sampled-data system of Eq. (46)) for $\Delta \in (0, \Delta^*)$ where $\Delta^* > 0$ is the conclusion of Proposition 1. Owing to the fact that z and v are bounded in compact sets and l_e and h are continuous on $\Omega_\rho \times \mathbb{U}$ and Ω_ρ, respectively, the asymptotic average economic performance, which is given by the left-hand side of Eq. (64) below, is bounded. Moreover, z converges to $\Omega_{\rho_{\min}}$ from Proposition 1. Therefore, the following inequality, which represents the worst-case asymptotic average performance under the sample-and-hold Lyapunov-based controller, follows:

$$\limsup_{T\to\infty} \frac{1}{T\Delta}\int_0^{T\Delta} l_e(z(t), v(t))dt \leq \max_{x,y \in \Omega_{\rho_{\min}}} l_e(x, h(y)). \tag{64}$$

where the Lyapunov-based controller is evaluated at y instead of x since y does not necessarily equal x due to the sample-and-hold implementation of the controller. Given that for any finite-time interval, the bound of Eq. (49) holds, the inequality of Eq. (63) follows from the fact that x and u^* are bounded in compact sets, the fact that l_e is continuous on $\Omega_\rho \times \mathbb{U}$, and the bound of Eq. (64). □

As noted, Theorem 6 characterizes the worst-case infinite-time (asymptotic average) performance for the process under the LEMPC based on a Lyapunov-based controller that satisfies Assumption 1, and states that it is no worse than the worst-case asymptotic average performance under the Lyapunov-based controller. Though this is a weaker result than showing that the asymptotic average performance is at least as good as that for steady-state operation, the level set $\Omega_{\rho_{\min}}$ may be selected arbitrarily small, at the expense of requiring a faster sampling rate.

We now focus on the performance guarantees that can be made in infinite-time when h meets Assumption 2. We first present a lemma on the infinite-time performance of the nominal process of Eq. (1) under the Lyapunov-based controller meeting Assumption 2 implemented in sample-and-hold (Eq. (46)). We will then present a theorem relating this result to the infinite-time performance of the process under the LEMPC with a terminal constraint based on

h. The lemma that will now be presented states that the asymptotic average economic performance under a Lyapunov-based controller that satisfies Assumption 2 is no worse than the economic cost at the optimal steady-state pair (x_s^*, u_s^*).

Lemma 1. *The asymptotic average economic cost of the closed-loop system of Eq. (46) under a feedback controller that satisfies Assumption 2 for any initial condition* $z(0) \in \Omega_\rho$ *is*

$$\lim_{T\to\infty} \frac{1}{T\Delta} \int_0^{T\Delta} l_e(z(t), v(t))dt = l_e(x_s^*, u_s^*) \tag{65}$$

where $\Delta \in (0, \Delta_e^*)$ ($\Delta_e^* > 0$ *is the conclusion of Proposition 2) and* z *and* v *denote the state and input trajectories of the system of Eq. (46).*

Proof. Recall, the economic stage cost function l_e is continuous on the compact set $\Omega_\rho \times \mathbb{U}$ and $z(t) \in \Omega_\rho$ and $v(t) \in \mathbb{U}$ for all $t \geq 0$. Thus, the integral:

$$\frac{1}{T\Delta} \int_0^{T\Delta} l_e(z(t), v(t))dt < \infty \tag{66}$$

for any integer $T > 0$. Since $z(t)$ and $v(t)$ exponentially converge to the optimal steady-state pair (x_s^*, u_s^*) as $t \to \infty$, the limit of the integral of Eq. (66) as T tends to infinity exists and is equal to $l_e(x_s^*, u_s^*)$. To prove the limit, it is sufficient to show that for any $\epsilon > 0$, there exists a T^* such that for $T > T^*$, the following holds:

$$\left| \frac{1}{T\Delta} \int_0^{T\Delta} l_e(z(t), v(t))dt - l_e(x_s^*, u_s^*) \right| < \epsilon \tag{67}$$

To simplify the presentation, define $I(T_1, T_2)$ as the following integral:

$$I(T_1, T_2) := \int_{T_1\Delta}^{T_2\Delta} l_e(z(t), v(t))dt \tag{68}$$

where the arguments of I are integers representing the integers of the lower and upper limits of integration, respectively. Since $z(t)$ and $v(t)$ converge to x_s^* and u_s^* as t tends to infinity, respectively, $l_e(x(t), v(t)) \to l_e(x_s^*, u_s^*)$ as t tends to infinity. Furthermore, $z(t) \in \Omega_\rho$ and $v(t) \in \mathbb{U}$ for all $t \geq 0$, so for every $\epsilon > 0$, there exists an integer $\tilde{T} > 0$ such that

$$|l_e(z(t), v(t)) - l_e(x_s^*, u_s^*)| < \epsilon/2 \tag{69}$$

for $t \geq \tilde{T}\Delta$. For any $T > \tilde{T}$,

$$\begin{aligned} |I(0, T) - T\Delta l_e(x_s^*, u_s^*)| &= |I(0, \tilde{T}) + I(\tilde{T}, T) - T\Delta l_e(x_s^*, u_s^*)| \\ &\leq \int_0^{\tilde{T}\Delta} |l_e(z(t), v(t)) - l_e(x_s^*, u_s^*)|dt + \int_{\tilde{T}\Delta}^{T\Delta} |l_e(z(t), v(t)) - l_e(x_s^*, u_s^*)|dt \\ &< \tilde{T}\tilde{M}\Delta + (T - \tilde{T})\Delta\epsilon/2 \end{aligned} \tag{70}$$

where

$$\tilde{M} := \sup_{t\in[0,\tilde{T}\Delta]} \left\{ |l_e(z(t), v(t)) - l_e(x_s^*, u_s^*)| \right\}.$$

For any $T > T^* = 2\tilde{T}(\tilde{M} - \epsilon/2)/\epsilon$ (which implies $(\tilde{M} - \epsilon/2)\tilde{T}/T < \epsilon/2$), the following inequality is satisfied:

$$\begin{aligned} |I(0, T)/(T\Delta) - l_e(x_s^*, u_s^*)| &< \tilde{T}\tilde{M}/T + (1 - \tilde{T}/T)\epsilon/2 \\ &= (\tilde{M} - \epsilon/2)\tilde{T}/T + \epsilon/2 < \epsilon \end{aligned} \tag{71}$$

which proves the limit of Eq. (65). □

Utilizing Lemma 1, one may prove that the asymptotic average closed-loop economic performance under the LEMPC of Eq. (48) designed with a Lyapunov-based controller that satisfies Assumption 2 is no worse than the closed-loop performance at the economically optimal steady-state (from Lemma 1, this is the same as stating that the asymptotic average performance of the nominal process under the LEMPC of Eq. (48) designed with h that meets Assumption 2 is no worse than the asymptotic average performance under h implemented in sample-and-hold). This result is stated in the following theorem.

Theorem 7. *Consider the system of Eq. (1) with* $w(t) \equiv 0$ *under the LEMPC of Eq. (48) based on a Lyapunov-based controller that satisfies Assumption 2. Let* $\Delta \in (0, \Delta_e^*)$ *where* $\Delta_e^* > 0$ *is the conclusion of Proposition 2. The closed-loop asymptotic average economic performance is no worse than the economic cost at steady-state; that is, the following bound holds:*

$$\limsup_{T\to\infty} \frac{1}{T\Delta} \int_0^{T\Delta} l_e(x(t), u^*(t))dt \leq l_e(x_s^*, u_s^*). \tag{72}$$

Proof. From Theorem 5, for any integer $T > 0$:

$$\frac{1}{T\Delta} \int_0^{T\Delta} l_e(x(t), u^*(t))dt \leq \frac{1}{T\Delta} \int_0^{(T+N)\Delta} l_e(z(t), v(t))dt. \tag{73}$$

As T increases, both sides of the inequality of Eq. (73) remain finite owing to the fact that l_e is continuous and the state and input trajectories are bounded in compact sets. The limit of the right-hand side as $T \to \infty$ is equal to $l_e(x_s^*, u_s^*)$ (Lemma 1). Therefore, the result in Eq. (72) is obtained. □

Remark 7. For systems with average constraints, the average constraint design methodologies for asymptotic average constraints (Angeli et al., 2012) and for transient average constraints (Müller et al., 2014), which were presented for EMPC with a terminal equality constraint equal to x_s^*, may be extended to the LEMPC of Eq. (48) when the average constraint is satisfied under the Lyapunov-based controller.

Remark 8. The performance results of this section hold for any prediction horizon size even when $N = 1$. The use of a short horizon may be computationally advantageous for real-time application. Also, owing to the fact that the terminal equality constraint of Eq. (48d) may be a point in the state-space away from the steady-state, the feasible region of the LEMPC of Eq. (48) may be larger than the feasible region of EMPC with a terminal equality constraint equal to the steady-state especially when a short prediction horizon is used.

3.2.4. Part 2: Application to a chemical process example

In this section, we use a chemical process example to demonstrate that the nominal process under LEMPC with a terminal equality constraint based on a Lyapunov-based controller can show improved economic performance compared to the process under the sample-and-hold Lyapunov-based controller. The LEMPC of Eq. (48) is applied to a chemical process example consisting of a continuously stirred-tank reactor (CSTR) within which two parallel reactions occur (Bailey et al., 1971):

$$R \to P_1 \tag{74a}$$

$$R \to P_2 \tag{74b}$$

where P_1 is the desired product and P_2 is a by-product. The rates of the reactions are second-order and first-order in R, respectively.

To model the reactor, it is assumed that there is no significant heat of reaction or heat of mixing, that the temperature dependence of the reaction rates can be modeled through the Arrhenius equation, and that the reactor mixture density, heat capacity, and inlet and outlet volumetric flow rates are constant. Applying these assumptions, the dimensionless dynamic model of the reactor, obtained from the conservation equations, is

$$\dot{x}_1 = -a_1 e^{-1/x_3} x_1^2 - a_2 e^{-\delta/x_3} x_1 - x_1 + 1 \quad (75a)$$

$$\dot{x}_2 = a_1 e^{-1/x_3} x_1^2 - x_2 \quad (75b)$$

$$\dot{x}_3 = -x_3 + u \quad (75c)$$

where x_1 is the dimensionless R concentration, x_2 is the dimensionless P_1 concentration, x_3 is the dimensionless temperature, and the manipulated input, denoted by u, is a dimensionless quantity related to the heat flux provided to the reactor. The process parameters are $a_1 = 1.0 \times 10^4$, $a_2 = 400$, and $\delta = 0.55$, and the input is restricted to take values in the interval [0.049, 0.449].

The operating profit of the CSTR is assumed to scale with the flow of the desired product out of the reactor. Owing to the fact that the volumetric inlet and outlet flow rates are constant, the stage cost minimized in LEMPC to maximize the operating profit of the reactor is given by:

$$l_e(x, u) = -x_2. \quad (76)$$

The economically optimal steady-state that minimizes Eq. (76) is $x_s^* = [0.0832\ 0.0846\ 0.149]$ corresponding to the steady-state input $u_s^* = 0.149$. Regarding the implementation details of the LEMPC, the sampling period is $\Delta = 0.05$, the prediction horizon consists of sixty sampling periods ($N = 60$), and the Lyapunov-based controller is chosen to be a proportional controller with saturation to account for the bound on the input (i.e., $\bar{h}(x) = -K(x_2 - x_{2s}^*) + u_s^*$ where $K = 3.3$ and $h(x) = \bar{h}(x)$ if $\bar{h}(x) \in [0.049, 0.449]$; else if $\bar{h}(x) < 0.049$ then $h(x) = 0.049$; else $h(x) = 0.449$). The closed-loop simulations were written in Python. To integrate the ODEs and solve the corresponding sensitivity information required to solve the nonlinear optimization problem, CVODE (Hindmarsh et al., 2005) and automatic differentiation via CasADi (Andersson et al., 2012) were used, respectively. The resulting nonlinear program was solved using Ipopt (Wächter and Biegler, 2006).

To demonstrate that using the LEMPC with a terminal equality constraint based on h can indeed lead to better economic performance than using the Lyapunov-based controller (in this case, the proportional controller with saturation) in sample-and-hold, two closed-loop simulations were completed: the closed-loop system under the LEMPC and the closed-loop system under the Lyapunov-based feedback controller. Fig. 5 gives the closed-loop trajectories under LEMPC for a closed-loop simulation over a length of 10.0 dimensionless time units. From Fig. 5, the LEMPC of Eq. (48) dictates a periodic-like operating policy. On the other hand, the Lyapunov-based controller dictates a steady-state operating policy. The average closed-loop economic performance is given by the index:

$$\bar{J}_e = \frac{1}{10.0} \int_0^{10.0} x_2(t) dt. \quad (77)$$

The average performance under the LEMPC is 0.0919, while the average performance under the feedback controller is 0.0849; the performance under LEMPC is 8.3% better than that under the feedback controller. It is important to note that it has been demonstrated that time-varying operation of this example improves closed-loop performance relative to steady-state operation (Bailey et al., 1971).

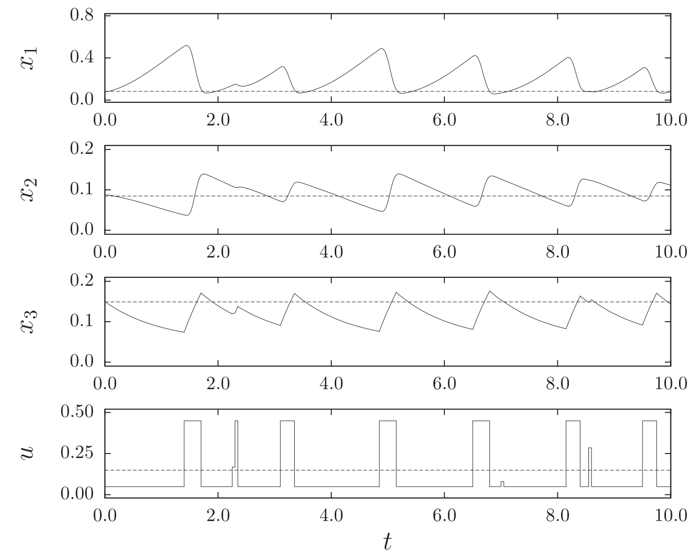

Fig. 5. Closed-loop trajectories for the system of Eq. (75) under the LEMPC of Eq. (48) (solid trajectories). The horizontal (dashed) trajectories indicate the steady-state value of each state and input.

Two potentially interesting issues to address are the closed-loop performance under EMPC with and without a terminal constraint and the closed-loop performance under EMPC with different terminal constraint formulations. While these issues may be difficult to address in general, we may explore these issues with simulation for this particular example. Fig. 6 displays the average closed-loop performance for several closed-loop simulations over 10.0 dimensionless time units for three different EMPC schemes and different horizon lengths. In particular, the three EMPC's considered are the LEMPC of Eq. (48), EMPC with a terminal constraint equal to the economically optimal steady-state, and EMPC without terminal constraints. Overall, the closed-loop performance for the two EMPC schemes with terminal constraints is relatively similar and for each horizon length the closed-loop performance realized was better than the profit at the economically optimal steady-state and also better than the closed-loop performance under the Lyapunov-based controller. On the other hand, there is a noticeable dependence of the average closed-loop performance on

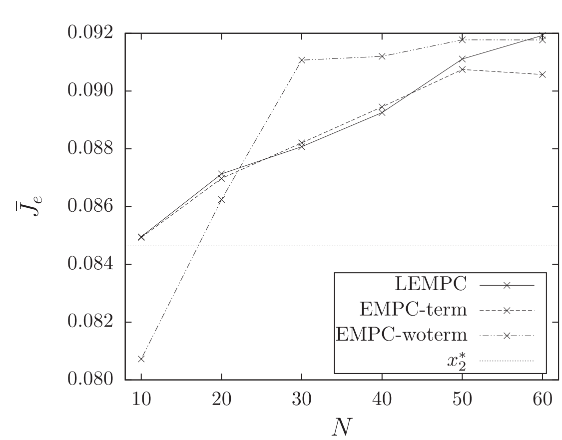

Fig. 6. Closed-loop economic performance with prediction horizon length for the process of Eq. (75) under the LEMPC of Eq. (48) (solid line, denoted as LEMPC), under an EMPC with a terminal equality constraint equal to the economically optimal steady-state (dashed line, denoted as EMPC-term), and under EMPC without a terminal constraint (dashed-dotted line, denoted as EMPC-woterm). For comparison, the closed-loop economic performance for operation at the economically optimal steady-state is also plotted (dotted line, denoted as x_2^*).

the prediction horizon length. For $N=10$, the closed-loop performance under the EMPC without terminal constraints was worse than that under the Lyapunov-based controller, illustrating that performance-based constraints imposed in EMPC may be needed to ensure acceptable closed-loop economic performance for shorter prediction horizons.

3.3. Part 3: LEMPC with input magnitude constraints, input rate of change constraints, and an equality terminal constraint based on a Lypaunov-based controller

In this section, we combine the results of Parts 1 and 2 on LEMPC with input magnitude and rate of change constraints and on LEMPC with a terminal state constraint based on a Lyapunov-based controller to show that the performance of LEMPC with input magnitude and rate of change constraints can be proven to be at least as good as it would be under a Lyapunov-based controller implemented in sample-and-hold for nominal process operation.

The formulation of LEMPC incorporating a terminal state constraint based on the Lyapunov-based controller, input magnitude constraints, and input rate of change constraints, assuming nominal process operation, is as follows:

$$\min_{u(\cdot)\in S(\Delta)} \int_{t_k}^{t_{k+N}} l_e(\tilde{x}(\tau), u(\tau))d\tau \tag{78a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0) \tag{78b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{78c}$$

$$\tilde{x}(t_{k+N}) = z(t_{k+N}) \tag{78d}$$

$$u(t) \in \mathbb{U}, \quad \forall \ t \in [t_k, t_{k+N}) \tag{78e}$$

$$\tilde{x}(t) \in \mathbb{X}, \quad \forall \ t \in [t_k, t_{k+N}) \tag{78f}$$

$$|u_i(t_j) - h_i(z(t_j))| \le \epsilon_r, \quad i = 1, \ldots, m, \quad j = k, \ldots, k+N-1 \tag{78g}$$

$$V(\tilde{x}(t)) \le \rho, \quad \forall \ t \in [t_k, t_{k+N}) \tag{78h}$$

where the notation follows that of Eqs. (17) and (48), and the implementation strategy is like that of Eq. (48) (at each t_k, a state measurement $x(t_k)$ is received and $z(t_{k+N})$ is updated before the LEMPC optimization problem is solved), except that each $h_i(z(t_j))$, $i=1, \ldots, m, j=k, \ldots, k+N-1$, is also determined and incorporated into the LEMPC of Eq. (78) at each sampling time.

We will now briefly address how the properties of the LEMPC in Eq. (78) compare with those of the LEMPC of Eq. (17) and of the LEMPC of Eq. (48). Specifically, we will address the bounds on ϵ_r and Δ required for the LEMPC of Eq. (78) to satisfy the desired input rate of change constraints in Eqs. (15) and (16) when $h(x)$ meets Assumption 1, the feasibility of the LEMPC optimization problem, the closed-loop stability properties of a process under the LEMPC, and the performance guarantees that can be made for the nominal process under LEMPC. We note that we will not address the robustness of the method, because only nominal operation is considered for the LEMPC of Eq. (78) due to the use of the terminal equality constraint.

Using arguments similar to those in Eqs. (19)–(24), it can be shown that the desired input rate of change constraints of Eqs. (15) and (16) are met when the LEMPC of Eq. (78) is feasible (Eq. (78g) is met by the calculated control actions) and $2\epsilon_r + L_{h_L} M\Delta \le \epsilon_{desired}$. The proof of feasibility of the LEMPC is similar to that noted in the proof of Theorem 4 in that, because nominal operation is considered, $u(t)=h(z(t_j))$ for $j=k, \ldots, k+N-1$ is a feasible solution to the LEMPC of Eq. (78) at t_0, with $u(t)=u^*(t|t_k)$ for $t \in [t_{k+1}, t_{k+N})$ and $u(t)=h(z(t_{k+N}))$ for $t \in [t_{k+N}, t_{k+N+1})$ being a feasible solution at time t_{k+1} when $u(t)=u^*(t|t_k)$ for $t \in [t_k, t_{k+N})$ is the solution at time t_k. Closed-loop stability of a process under the LEMPC in Eq. (78) is ensured for nominal operation in the sense that the state is always maintained within the compact set Ω_ρ due to the constraint in Eq. (78h).

Finally, we compare the performance of the nominal process of Eq. (1) under the LEMPC of Eq. (78) with the performance of the process under the Lyapunov-based controller implemented in sample-and-hold. Because this comparison is only fair if the process meets the same constraints under both controllers, we note that the process under the Lyapunov-based controller implemented in sample-and-hold meets all state and input constraints in Eq. (78) for the reasons mentioned in the proof that this control law is a feasible solution to the LEMPC at t_0; it also satisfies the desired rate of change constraints of Eqs. (15) and (16) if the terms $u^*(t_k|t_k)$ are replaced by $h_i(z(t_k))$, $i=1, \ldots, m$, since these are the implemented control actions under the sample-and-hold Lyapunov-based controller. With that replacement, Eqs. (15) and (16) become a requirement that $|h_i(z(t_k)) - h_i(z(t_{k-1}))| \le \epsilon_{desired}$, $i=1, \ldots, m$, which holds for all $\epsilon_{desired} > 0$ for $L_{h_L} M\Delta \le \epsilon_{desired}$ from Eq. (21). Thus, when the control actions calculated by the LEMPC meet Eqs. (15) and (16) (i.e., $2\epsilon_r + L_{h_L} M\Delta \le \epsilon_{desired}$), it is also true that $L_{h_L} M\Delta \le \epsilon_{desired}$ so that the control actions implemented by the sample-and-hold Lyapunov-based controller also satisfy the desired input rate of change constraints. This establishes that a fair comparison can be made between the performance of the process under the LEMPC of Eq. (78) and the sample-and-hold Lyapunov-based controller. The performance results of Theorem 5 hold for h meeting Assumption 1 on the finite-time interval, the performance results of Theorem 6 hold for h meeting Assumption 1 on the infinite-time interval, and the performance results of Theorem 7 hold for h meeting Assumption 2 on the infinite-time interval. It is noted that these performance results hold for the LEMPC of Eq. (78) regardless of the form of the cost function; this proves that for nominal operation, the performance of LEMPC with a terminal equality constraint and input rate of change constraints is no worse on both the finite-time and infinite-time intervals than that of an alternative controller that enforces steady-state operation, regardless of whether the cost function includes additional penalties on the input rate of change to reduce actuator wear. Like the LEMPC with a terminal equality constraint but without input rate of change constraints (Eq. (48)), the LEMPC of Eq. (78) has a number of advantages over other EMPC formulations for which performance guarantees have been made, particularly that the feasible region can be characterized *a priori*.

Remark 9. The motivation for adding input rate of change constraints to LEMPC (that the LEMPC may dictate a dynamic operating policy) is also motivation for the addition of input rate of change constraints to EMPC in general. Thus, it is noted that input rate of change constraints can be added to other EMPC formulations for which performance guarantees have been previously developed (such as the steady-state terminal equality constraint formulation) as well. However, as noted above, LEMPC has a number of advantages over some of the other EMPC formulations that make it more attractive for incorporating input rate of change constraints and making performance guarantees for the resulting formulation.

4. Conclusions

In this work, we developed a formulation of LEMPC incorporating input magnitude and rate of change constraints and a terminal equality constraint based on a Lyapunov-based controller that allows provable performance guarantees to be made for the LEMPC. The LEMPC formulation was developed in three parts. In Part 1 of this work, we demonstrated that input rate of change constraints written with respect to a Lyapunov-based controller can be added to LEMPC, and that the implemented inputs can then be ensured to differ by no more than a desired bound between two subsequent sampling periods. The formulation of LEMPC with input rate of change constraints developed was shown to be feasible and to maintain closed-loop stability of a process even in the presence of bounded disturbances. A chemical process example demonstrated that the number of sampling periods of the prediction horizon over which the input rate of change constraints are enforced may have a significant impact on whether other process constraints such as integral material constraints can be met.

In Part 2, we developed an LEMPC formulation with a terminal equality constraint based on the Lyapunov-based controller utilized in the formulation of the LEMPC. With this terminal equality constraint, the LEMPC formulation was proven to be not only feasible and stable in the sense of boundedness of the closed-loop state for nominal operation, but was also proven to have finite-time and infinite-time economic performance properties such that the process under LEMPC performs no worse than it does under an asymptotically stabilizing or exponentially stabilizing Lyapunov-based controller implemented in sample-and-hold. When the exponentially stabilizing controller is utilized to design the LEMPC, the asymptotic average performance of the process under the LEMPC was proven to be no worse than that under steady-state operation. The LEMPC formulation presented has advantages over other EMPC's for which performance guarantees have been made, such as that the feasible region can be explicitly characterized *a priori*. A chemical process example demonstrated the economic benefits of incorporating a terminal equality constraint in EMPC when a short prediction horizon is used.

In Part 3, the results of Parts 1 and 2 were combined to develop an LEMPC formulation incorporating a terminal equality constraint based on a Lyapunov-based controller that also had input magnitude and rate of change constraints. It was proven that the closed-loop performance under this LEMPC with input magnitude and rate of change constraints is no worse than that under an asymptotically or exponentially stabilizing Lyapunov-based controller for nominal operation on the finite-time and infinite-time operating intervals (this means that the infinite-time performance under the LEMPC based on an exponentially stabilizing controller is no worse than that under steady-state operation), regardless of the form of the cost function or any penalties on the input rate of change in the cost function. This is significant because it may be desirable from a safety perspective to reduce input variations under EMPC but without reducing the economic performance of the process below that obtainable with the traditional steady-state operating strategies.

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