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Economic model predictive control of nonlinear singularly perturbed systems

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ABSTRACT

We focus on the development of a Lyapunov-based economic model predictive control (LEMPC) method for nonlinear singularly perturbed systems in standard form arising naturally in the modeling of two-time-scale chemical processes. A composite control structure is proposed in which, a "fast" Lyapunov-based model predictive controller (LMPC) using a quadratic cost function which penalizes the deviation of the fast states from their equilibrium slow manifold and the corresponding manipulated inputs, is used to stabilize the fast dynamics while a two-mode "slow" LEMPC design is used on the slow subsystem that addresses economic considerations as well as desired closed-loop stability properties by utilizing an economic (typically non-quadratic) cost function in its formulation and possibly dictating a time-varying process operation. Through a multirate measurement sampling scheme, fast sampling of the fast state variables is used in the fast LMPC while slow-sampling of the slow state variables is used in the slow LEMPC. Appropriate stabilizability assumptions are made and suitable constraints are imposed on the proposed control scheme to guarantee the closed-loop stability and singular perturbation theory is used to analyze the closed-loop system. The proposed control method is demonstrated through a nonlinear chemical process example.

1. Introduction

Optimizing process economics while achieving desired closed-loop stability properties in the context of chemical processes is of paramount importance. Usually, economic optimization is addressed through a two-layer framework. The upper layer deals with real-time optimization to obtain an economically optimal process operating steady-state through taking advantage of the steady-state process model while the lower layer utilizes appropriate feedback control techniques to ensure steering the process state to the economically optimal steady-state computed by the upper layer [21]. While this methodology has a number of advantages, it restricts the optimal process operation in the neighborhood of a steady-state. To overcome this potential drawback, a possible time-varying operation may be considered to provide a wider range of optimal operation policies; however, closed-loop stability needs to be carefully addressed if an unsteady-state process operation is considered. Model predictive control (MPC) is widely used as a process control technique in the lower layer due to its ability to address state and control input constraints through an online optimization-based approach. In a conventional MPC, the deviation of the predicted state and input, using the nominal process model along the prediction horizon, from their corresponding steady-state values are penalized through a quadratic cost function [7]; however, to address economic considerations, economic MPC (EMPC) has been introduced which utilizes a general cost function (typically non-quadratic) in its formulation which may directly address economic considerations, thereby achieving a tighter integration of the process economics and process control layers.

Recently, EMPC has become a research subject of increasing interest. Specifically, closed-loop stability of EMPC through an appropriate terminal constraint [9], considering EMPC for energy reduction purposes [19] as well as closed-loop stability of EMPC of cyclic processes [13] have been studied. In a previous work, we proposed a two-mode Lyapunov-based economic MPC (LEMPC) method for a broad class of nonlinear systems which is capable of handling asynchronous and delayed measurements [10] and extended this method in the context of state estimation-based output feedback control [11] and distributed MPC [4]. In particular, the first mode of the two-mode LEMPC

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method deals with economic considerations by dictating a possible time-varying economically optimal operation while ensuring that the closed-loop system state is maintained in a predefined invariant set, while the second mode addresses convergence of the closed-loop state to an economically optimal steady-state.

In addition to accounting for economics and control in a single framework, the development of optimal process control, automation and management methodologies while addressing time-scale multiplicity due to the strong coupling of slow and fast phenomena occurring at different time-scales is an important issue in the context of chemical process control. In the context of multiple-time-scale systems, closed-loop stability as well as controller design are usually addressed through explicit separation of fast and slow states in a standard singular perturbation setting [15,5,16] or by taking advantage of change of coordinates for two-time-scale systems in nonstandard singularly perturbed form [1,2]. Recently, we have developed methods for slow time-scale MPC as well as composite fast-slow MPC for nonlinear singularly perturbed systems [3,4]; however, EMPC of nonlinear singularly perturbed systems has not been studied.

Motivated by the above, we present an economic MPC method for a broad class of nonlinear singularly perturbed systems. Specifically, a "fast" Lyapunov-based MPC (LMPC) using a conventional quadratic cost is employed to stabilize the fast closed-loop dynamics at their equilibrium slow manifold while a "slow" two-mode LEMPC which may dictate time-varying operation to address economic considerations and/or convergence to an economically optimal steady-state is utilized for the slow dynamics. Multirate sampling of the process states is considered involving fast-sampling of the fast states and slow-sampling of the slow states used in the fast and slow MPC systems, respectively. Closed-loop stability of the proposed control scheme is addressed through singular perturbation theory. The proposed control method is demonstrated through a chemical process example which exhibits two-time-scale behavior.

2. Preliminaries

2.1. Notation

The notation $|\cdot|$ is used to denote the Euclidean norm of a vector. For any measurable (with respect to the Lebesgue measure) function $w:[0,\infty)\to R^l$, $||w||$ denotes ess.sup.$|w(t)|$, $t\geq 0$ where R^l denotes an l dimensional space. A continuous function $\alpha:[0,a)\to[0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0)=0$. A continuous function $\beta:[0,a)\times[0,\infty)\to[0,\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$, and for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s)\to 0$ as $s\to\infty$. The symbol Ω_r is used to denote the set $\Omega_r:=\{x\in R^{n_x}:V(x)\leq r\}$ where V is a continuously differentiable, positive definite scalar function and $r>0$, and the operator '\' denotes set subtraction, that is, $A\setminus B:=\{x\in R^{n_x}:x\in A, x\notin B\}$. The symbol $diag(v)$ denotes a matrix whose diagonal elements are the elements of vector v and all the other elements are zeros.

2.2. Class of nonlinear singularly perturbed systems

We consider nonlinear singularly perturbed systems in standard form with the following state-space description:

$$\begin{aligned}\dot{x}(t)&=f(x(t),z(t),\epsilon,u_s(t),w(t)),\quad x(t_0)=x_0\\ \epsilon\dot{z}(t)&=g(x(t),z(t),\epsilon,u_f(t),w(t)),\quad z(t_0)=z_0\end{aligned}\tag{1}$$

where $x(t)\in R^n$ and $z(t)\in R^m$ denote the vectors of state variables, ϵ is a small positive parameter, $w(t)\in R^l$ denotes the vector of process disturbances and $u_s(t)\subset R^s$ and $u_f(t)\subset R^f$ are the control (manipulated) inputs. The manipulated inputs are restricted to be in nonempty convex sets $U_s\subseteq R^s$ and $U_f\subseteq R^f$, which are defined as $U_s:=\{u_s(t)\in R^s:|u_s(t)|\leq u_s^{\max}\}$ and $U_f:=\{u_f(t)\in R^f:|u_f(t)|\leq u_f^{\max}\}$ where $u_s^{\max}$ and $u_f^{\max}$ are positive real numbers, specifying the input constraints. The disturbance $w(t)$ is assumed to be absolutely continuous and bounded, i.e., $W:=\{w(t)\in R^l:|w(t)|\leq\theta\}$ where θ is a positive real number. Due to the multiplication of the small parameter ϵ with $\dot{z}(t)$ in Eq. (1), the fast and slow dynamics are explicitly separated and the model of Eq. (1) is said to be in the standard singularly perturbed form. Thus, through the rest of the paper, we will refer to $x(t)$ and $z(t)$ as the slow and fast states, respectively. Furthermore, we assume that the vector functions f and g are sufficiently smooth in $R^n\times R^m\times[0,\bar{\epsilon})\times R^s\times R^l$ and $R^n\times R^m\times[0,\bar{\epsilon})\times R^f\times R^l$, respectively, for some $\bar{\epsilon}>0$, and that the origin is an equilibrium point of the unforced nominal system (i.e., system of Eq. (1) with $u_s=0$, $u_f=0$ and $w=0$).

We assume that the fast states $z(t)$ are sampled synchronously and are available at time instants indicated by the time sequence $\{t_{k_f\geq 0}\}$ with $t_{k_f}=t_0+k\Delta_f$, $k=0,1,\ldots$ where t_0 is the initial time and Δ_f is the measurement sampling time of the fast states. Similarly, we assume that the slow states $x(t)$ are sampled synchronously and are available at time instants indicated by the time sequence $\{t_{k_s\geq 0}\}$ with $t_{k_s}=t_0+k\Delta_s$, $k=0,1,\ldots$ where t_0 is the initial time and Δ_s is the measurement sampling time of the slow states. With respect to the control problem formulation, it is assumed that the controls $u_f(t)$ and $u_s(t)$, which are responsible for the fast and slow dynamics, are computed every Δ_f and Δ_s, respectively. For the sake of simplicity, we assume that Δ_s/Δ_f is a positive integer.

2.3. Two-time-scale decomposition

The explicit separation of slow and fast variables in the system of Eq. (1) allows decomposing it into two separate reduced-order systems evolving in different time-scales. To proceed with such a two-time-scale decomposition and in order to simplify the notation of the subsequent development, we will first address the issue of controlling the fast dynamics. Similar to our previous work [3], it has been assumed that there exists a "fast" model predictive controller u_f that renders the fast dynamics asymptotically stable in a sense to be made precise in Assumption 2 below and moreover, $u_f(t)$ does not modify the open-loop equilibrium slow manifold for the fast dynamics. This implies that when we set $\epsilon=0$ in the system of Eq. (1) to derive the model that describes the slow dynamics, $u_f=0$, and the system of Eq. (1) takes the following form:

$$\frac{d\bar{x}}{dt}=f(\bar{x}(t),\bar{z}(t),0,u_s(t),w(t))\tag{2a}$$

$$0 = g(\bar{x}(t), \bar{z}(t), 0, 0, w(t)) \tag{2b}$$

where the bar in $\bar{x}$ and $\bar{z}$ denotes association with the slow dynamics.

Assumption 1. The equation $g(\bar{x}(t), \bar{z}(t), 0, 0, w(t)) = 0$ possesses a unique root

$$\bar{z}(t) = \tilde{g}(\bar{x}(t), w(t)) \tag{3}$$

where $\tilde{g} : R^n \times R^l \to R^m$ and its partial derivatives $\frac{\partial \tilde{g}}{\partial \bar{x}}, \frac{\partial \tilde{g}}{\partial w}$ are sufficiently smooth and $\left|\frac{\partial \tilde{g}}{\partial w}\right| \le L_{\tilde{g}}$.

Assumption 1 is a standard requirement in singularly perturbation theory (please see, for example [15]). It ensures that the system of Eq. (1) has an isolated equilibrium manifold for the fast dynamics while on this manifold, $\bar{z}(t)$ can be expressed in terms of $\bar{x}(t)$ and $w(t)$ using an algebraic expression. It should be emphasized that $g(\bar{x}(t), \bar{z}(t), 0, 0, w(t))$ is, in this present case, independent of the expression of the "fast" model predictive controller $u_f(t)$. We note that Assumption 1 does not pose any significant limitations in practical applications, but it is a necessary one in the singular perturbation framework to construct a well-defined slow subsystem.

Utilizing $\bar{z}(t) = \tilde{g}(\bar{x}(t), w(t))$, we can re-write Eq. (2) as follows:

$$\dot{\bar{x}} = f(\bar{x}(t), \tilde{g}(\bar{x}(t), w(t)), 0, u_s(t), w(t)) \equiv f_s(\bar{x}(t), u_s(t), w(t)) \tag{4}$$

We will refer to the subsystem of Eq. (4) as the slow subsystem.

Introducing the fast time scale $\tau = (t - t_0)/\epsilon$ and the deviation variable $y = z - \tilde{g}(x, w)$, we can rewrite the nonlinear singularly perturbed system of Eq. (1) as follows:

$$\begin{aligned}
\frac{dx}{d\tau} &= \epsilon f(x(\epsilon\tau + t_0), y(\epsilon\tau + t_0) + \tilde{g}(x(\epsilon\tau + t_0), w(\epsilon\tau + t_0)), \epsilon, u_s(\epsilon\tau + t_0), w(\epsilon\tau + t_0)) \\
\frac{dy}{d\tau} &= g(x(\epsilon\tau + t_0), y(\epsilon\tau + t_0) + \tilde{g}(x(\epsilon\tau + t_0), w(\epsilon\tau + t_0)), \epsilon, u_f(\epsilon\tau + t_0), w(\epsilon\tau + t_0)) \\
&\quad -\epsilon \frac{\partial \tilde{g}}{\partial x} f(x(\epsilon\tau + t_0), y(\epsilon\tau + t_0) + \tilde{g}(x(\epsilon\tau + t_0), w(\epsilon\tau + t_0)), \epsilon, u_s(\epsilon\tau + t_0), w(\epsilon\tau + t_0)) - \epsilon \frac{\partial \tilde{g}}{\partial w} \dot{w}(\epsilon\tau + t_0)
\end{aligned} \tag{5}$$

Setting $\epsilon = 0$, we obtain the following fast subsystem whose state is denoted by $\bar{y}$:

$$\frac{d\bar{y}}{d\tau} = g(x(t_0), \bar{y}(\tau) + \tilde{g}(x(t_0), w(t_0)), 0, u_f(\tau), w(t_0)) \tag{6}$$

where x and w can be considered as frozen to their initial values in the fast time-scale since their change in this time-scale is of order ϵ. Note that $\bar{y}(\tau)$ and $u_f(\tau)$ change in the fast time-scale τ and this is why on the right-hand side of Eq. (6) we allow them to vary with τ.

Remark 1. The difference between $y(t)$ and $\bar{y}(\tau)$ is: $y(t)$ is the deviation between the singularly perturbed system state $z(t)$ (Eq. (1) with $\epsilon > 0$) and the solution to the algebraic equation $g(x, \bar{z}, 0, 0, 0) = 0$ denoted as $\bar{z}(t) = \tilde{g}(x(t), w(t))$. The variable $\bar{y}(\tau)$ is used to denote the solution to the fast subsystem obtained from Eq. (6) where x and w are frozen to their initial values and $\epsilon = 0$; the initial condition of the ODE of Eq. (6) is $\bar{y}(0) = y(t_0) = z(t_0) - \tilde{g}(x(t_0), w(t_0))$.

2.4. Stabilizability assumption

We assume that there exists a Lyapunov-based locally Lipschitz feedback controller $u_s = h_s(\bar{x})$ which, under continuous implementation, renders the origin of the nominal closed-loop slow subsystem of Eq. (4) asymptotically stable while satisfying the input constraints for all the states $\bar{x}$ inside a given stability region. Such an explicit controller can be designed using Lyapunov-based control techniques [17,6]. Using converse Lyapunov theorems [22,17,6], this assumption implies that there exist functions $\alpha_{s_i}(\cdot)$, $i = 1, 2, 3, 4$ of class $\mathcal{K}$ and a continuously differentiable Lyapunov function $V_s(\bar{x})$ for the nominal closed-loop slow subsystem that satisfy the following inequalities:

$$\begin{aligned}
&\alpha_{s_1}(|\bar{x}|) \le V_s(\bar{x}) \le \alpha_{s_2}(|\bar{x}|) \\
&\frac{\partial V_s(\bar{x})}{\partial \bar{x}} f_s(\bar{x}, h_s(\bar{x}), 0) \le -\alpha_{s_3}(|\bar{x}|) \\
&\left|\frac{\partial V_s(\bar{x})}{\partial \bar{x}}\right| \le \alpha_{s_4}(|\bar{x}|) \\
&h_s(\bar{x}) \in U_s
\end{aligned} \tag{7}$$

for all $\bar{x} \in D_s \subseteq R^n$ where D_s is an open neighborhood of the origin. We denote the region $\Omega_{\rho^s} \subseteq D_s$ as the stability region of the closed-loop slow subsystem under the Lyapunov-based controller $h_s(\bar{x})$. By continuity, the smoothness property assumed for the vector fields $f_s(\bar{x}, u_s, w)$ and taking into account that the manipulated input u_s and the disturbance w are bounded in convex sets, there exists a positive constant M_s such that

$$|f_s(\bar{x}, u_s, w)| \le M_s \tag{8}$$

for all $\bar{x} \in \Omega_{\rho^s}$, $u_s \in U_s$, and $w \in W$. In addition, by the continuous differentiable property of the Lyapunov function $V_s(\bar{x})$ and the smoothness property assumed for the vector field $f_s(\bar{x}, u_s, w)$, there exist positive constants $L_{\bar{x}}$ and L_{w_s} such that

$$\left|\frac{\partial V_s}{\partial \bar{x}} f_s(\bar{x}, u_s, w) - \frac{\partial V_s}{\partial \bar{x}} f_s(\bar{x}', u_s, w)\right| \le L_{\bar{x}}|\bar{x} - \bar{x}'|, \quad \left|\frac{\partial V_s}{\partial \bar{x}} f_s(\bar{x}, u_s, w) - \frac{\partial V_s}{\partial \bar{x}} f_s(\bar{x}, u_s, w')\right| \le L_{w_s}|w - w'| \tag{9}$$

for all $\bar{x}, \bar{x}' \in \Omega_{\rho^s}$, $u_s \in U_s$, and $w, w' \in W$.

Assumption 2. There exists a feedback controller $u_f = p(x)\bar{y} \in U_f$ where $p(x)$ is a sufficiently smooth vector function in x, such that the origin of the closed-loop fast subsystem:

$$\frac{d\bar{y}}{d\tau} = g(x, \bar{y} + \tilde{g}(x, w), 0, p(x)\bar{y}, w) \tag{10}$$

is globally asymptotically stable, uniformly in $x \in R^n$ and $w \in R^l$, in the sense that there exists a class $\mathcal{KL}$ function β_y such that for any $\bar{y}(0) \in R^m$:

$$|\bar{y}(\tau)| \le \beta_y(|\bar{y}(0)|, \tau) \tag{11}$$

for $\tau \ge 0$.

This assumption implies that there exist functions $\alpha_{f_i}(\cdot)$, $i = 1, 2, 3$ of class $\mathcal{K}$ and a continuously differentiable Lyapunov function $V_f(\bar{y})$ for the nominal closed-loop fast subsystem which satisfy the following inequalities:

$$\begin{aligned}
&\alpha_{f_1}(|\bar{y}|) \le V_f(\bar{y}) \le \alpha_{f_2}(|\bar{y}|)\\
&\frac{\partial V_f(\bar{y})}{\partial \bar{y}}(g(x, \bar{y} + \tilde{g}(x, 0), 0, p(x)\bar{y}, 0)) \le -\alpha_{f_3}(|\bar{y}|)\\
&p(x)y \in U_f
\end{aligned} \tag{12}$$

for all $y \in D_f \subseteq R^m$ where D_f is an open neighborhood of the origin. We denote the region $\Omega_{\rho^f} \subseteq D_f$ as the stability region of the closed-loop fast subsystem under the nonlinear controller $p(x)\bar{y}$.

By continuity, the smoothness property assumed for the vector function $g(x, \bar{y}, 0, u_f, w)$ and taking into account that the manipulated input u_f and the disturbance w are bounded in convex sets, there exists a positive constant M_f such that

$$|g(x, \bar{y} + \tilde{g}(x, w), 0, u_f, w)| \le M_f \tag{13}$$

for all $\bar{y} \in \Omega_{\rho^f}$, $u_f \in U_f$, and $w \in W$. Furthermore, by the continuous differentiable property of the Lyapunov function $V_f(\bar{y})$ and the smoothness property assumed for the vector function $g(x, \bar{y}, 0, u_f, w)$, there exist positive constants $L_{\bar{y}}$ and L_{w_f} such that

$$\begin{aligned}
&\left|\frac{\partial V_f}{\partial \bar{y}} g(x, \bar{y}, 0, u_f, w) - \frac{\partial V_f}{\partial \bar{y}} g(x, \bar{y}', 0, u_f, w)\right| \le L_{\bar{y}}|\bar{y} - \bar{y}'|\\
&\left|\frac{\partial V_f}{\partial \bar{y}} g(x, \bar{y}, 0, u_f, w) - \frac{\partial V_f}{\partial \bar{y}} g(x, \bar{y}, 0, u_f, w')\right| \le L_{w_f}|w - w'|
\end{aligned} \tag{14}$$

for all $\bar{y}, \bar{y}' \in \Omega_{\rho^f}$, $u_f \in U_f$, and $w, w' \in W$. Also, by the smoothness property assumed for the vector function $f(x, z, \epsilon, u_s, w)$, there exist positive constants L_z, L_ϵ, L_x, $\bar{L}_z$, $\bar{L}_\epsilon$, and $\bar{L}_w$ such that

$$|f(x, z, \epsilon, u_s, w) - f(x, z', \epsilon', u_s, w)| \le L_z|z - z'| + L_\epsilon|\epsilon - \epsilon'| \tag{15}$$

$$\left|\frac{\partial V_s}{\partial x} f(x, z, \epsilon, u_s, w) - \frac{\partial V_s}{\partial x} f(x', z', \epsilon', u_s, w')\right| \le L_x|x - x'| + \bar{L}_z|z - z'| + \bar{L}_\epsilon|\epsilon - \epsilon'| + \bar{L}_w|w - w'| \tag{16}$$

for all $x, x' \in \Omega_{\rho^s}$, $z - \tilde{g}(x, w), z' - \tilde{g}(x, w') \in \Omega_{\rho^f}$, $u_f \in U_f$, and $w, w' \in W$

Remark 2. Ω_{ρ^s} and Ω_{ρ^f} denote the stability regions for the closed-loop slow and fast subsystems under the controllers $u_s = h_s(\bar{x})$ and $u_f = p(x)\bar{y}$, respectively, in the sense that the closed-loop states of the fast and slow subsystems, starting in Ω_{ρ^s} and Ω_{ρ^f}, remain in these sets thereafter. Regarding the construction of Ω_{ρ^s}, we have estimated it through the following procedure: $\dot{V}_s(\bar{x})$ is evaluated for different values of $\bar{x}$ while $h_s(\bar{x})$ is applied to the nominal system subject to the input constraint $h_s(\bar{x}) \in U_s$. Then, we estimated Ω_{ρ^s} as the largest level set of the Lyapunov function $V_s(\bar{x})$ where $\dot{V}_s(\bar{x}) \le 0$. The region Ω_{ρ^f} can be estimated in a similar fashion using the fast subsystem and the controller $p(x)\bar{y}$. For the type of quadratic Lyapunov functions used in our example calculations in Section 4, the Ω sets are also convex sets.

3. LEMPC of nonlinear singularly perturbed systems

In this section, we consider the design of EMPC for nonlinear singularly perturbed systems. First, a "fast" MPC is used to stabilize the fast dynamics. Then, a possible time-varying operation is dictated by a two-mode LEMPC to address economic considerations as well as closed-loop stability of the slow subsystem. The two-mode LEMPC of [10] is incorporated in the MPC design of the slow subsystem. We assume that over the operation period $[t_0, t')$ we deal with economic considerations in the LEMPC of the slow subsystem while after time t' we deal with enforcing convergence of the closed-loop subsystem state to an economically optimal steady-state. In operation period $[t_0, t')$, the predicted slow subsystem state along the finite prediction horizon is maintained in the corresponding stability region $\Omega_{\rho_e} \subset \Omega_{\rho^s}$, in order to account for the process disturbance as well as the fast dynamics effects, while it allows the slow system state to optimize the economic cost function by dictating a possible time-varying operation and maintaining the closed-loop system slow state in the stability region Ω_{ρ^s}; thus, it can take advantage of the closed-loop stability properties of the Lyapunov-based controller $h_s(\cdot)$. If the current slow subsystem state is in the region $\Omega_{\rho^s} \setminus \Omega_{\rho_e}$, the LEMPC first drives the slow subsystem state to the region Ω_{ρ_e} and then maximizes the cost function within Ω_{ρ_e}. In the second operation mode which corresponds to $t \ge t'$, LEMPC ensures that the slow subsystem state is driven to a neighborhood of the economically optimal steady-state.

3.1. Implementation strategy

The above described implementation strategy corresponding to the fast subsystem can be described as follows:

1. LMPC receives fast subsystem state $\bar{y}(t_{k_f})$ through the sensors.
2. LMPC obtains its manipulated input trajectory which ensures that the fast subsystem state is steered to the neighborhood of the equilibrium slow manifold of the fast state.
3. LMPC sends the first step value of the manipulated input to the corresponding actuators.
4. Go to Step 1 ($k \leftarrow k+1$).

Similarly, the implementation strategy corresponding to the slow subsystem can be described as follows

1. The LEMPC receives $\bar{x}(t_{k_s})$ from the sensors.
2. If $t_{k_s} < t'$, go to Step 3. Else, go to Step 4.
3. If $\bar{x}(t_{k_s}) \in \Omega_{\rho_e}$, go to Step 3.1. Else, go to Step 3.2.
   3.1. The controller maximizes the economic cost function within Ω_{ρ_e}. Go to Step 5.
   3.2. The controller drives the slow subsystem state to the region Ω_{ρ_e} and then maximizes the economic cost function within Ω_{ρ_e}. Go to Step 5.
4. The controller drives the slow subsystem state to a small neighborhood of the origin.
5. Go to Step 1 ($k \leftarrow k+1$).

Remark 3. The time t' used to denote the time the LEMPC switches from first operation mode to the second operation mode is chosen based on practical considerations like preventing excessive wear on the control actuators. For the case when closed-loop economic performance with LMPC and LEMPC operating in the first mode (i.e., potentially time-varying operation) is better than steady-state operation, one would pick this time to manage the trade-off between the practical considerations and better closed-loop economic performance. We note that t' can be made arbitrary large if one always wants the LEMPC to operate in the first operation mode as is the case in the "Application to a chemical process example" section.

3.2. Fast LMPC formulation

Referring to the fast subsystem of Eq. (6), the fast LMPC at sampling time t_{k_f} is formulated as follows

$$\min_{u_f \in S(\Delta_f)} \int_{t_{k_f}}^{t_{k_f}+N_f\Delta_f} [\tilde{y}^T(\hat{\tau})Q_f\tilde{y}(\hat{\tau}) + u_f^T(\hat{\tau})R_f u_f(\hat{\tau})]d\hat{\tau} \tag{17a}$$

$$\text{s.t.}\frac{d\tilde{y}(\hat{\tau})}{d\hat{\tau}} = g(x(t_{k_s}), \tilde{y}(\hat{\tau}) + \tilde{g}(x(t_{k_s}), 0), 0, u_f(\hat{\tau}), 0) \tag{17b}$$

$$u_f(\hat{\tau}) \in U_f \quad \forall \hat{\tau} \in [t_{k_f}, t_{k_f} + N_f\Delta_f) \tag{17c}$$

$$\tilde{y}(t_{k_f}) = \bar{y}(t_{k_f}) \tag{17d}$$

$$\frac{\partial V_f(\bar{y}(t_{k_f}))}{\partial \bar{y}} g(x(t_{k_s}), \bar{y}(t_{k_f}) + \tilde{g}(x(t_{k_s}), 0), 0, u_f(t_{k_f}), 0) \le \frac{\partial V_f(\bar{y}(t_{k_f}))}{\partial \bar{y}} g(x(t_{k_s}), \bar{y}(t_{k_f}) + \tilde{g}(x(t_{k_s}), 0), 0, p(x(t_{k_s}))\bar{y}(t_{k_f}), 0) \tag{17e}$$

where $S(\Delta_f)$ is the family of piece-wise constant functions with sampling period Δ_f, N_f is the prediction horizon of LMPC, Q_f and R_f are positive definite weight matrices that penalize the deviation of the fast subsystem state and manipulated input from their corresponding values at the equilibrium slow manifold, $\bar{y}(t_{k_f})$ is the fast subsystem state measurement obtained at t_{k_f}, $\tilde{y}$ denotes the predicted fast subsystem state trajectory by taking advantage of the nominal fast subsystem model of Eq. (17b) along the finite prediction horizon N_f and subject to the manipulated input constraint of Eq. (17c). The constraint of Eq. (17e) indicates that the amount of reduction in the value of the Lyapunov function $V_f(\cdot)$ when the manipulated input u_f computed by the LMPC of Eq. (17) is applied, is at least at the level when the Lyapunov-based controller $p(x)y$ is applied in a sample-and-hold fashion. Due to the fact that the LMPC of Eq. (17) obtains the manipulated input trajectory u_f every Δ_f, $x(t_{k_s})$ is the last available measurement of the slow process state, i.e., $t_{k_s} \le t_{k_f}$. The optimal solution to this optimization problem is defined by $u_f^*(\hat{\tau}|t_{k_f}) \forall \hat{\tau} \in [t_{k_f}, t_{k_f} + N_f\Delta_f)$ and the manipulated input of the closed-loop fast subsystem under the LMPC of Eq. (17) is defined as follows:

$$u_f(t) = u_f^*(t|t_{k_f}), \quad \forall t \in [t_{k_f}, t_{k_f} + \Delta_f). \tag{18}$$

Proposition 1 characterizes the closed-loop stability properties of the LMPC of Eq. (17).

Proposition 1 (c.f. [23]). *Consider the fast subsystem of Eq. (6) in closed-loop under the LMPC of Eqs. (17) and (18) based on the feedback controller $p(x)y$ that satisfies the conditions of Eq. (12). Let $\epsilon_{w_f} > 0$, $\Delta_f > 0$ and $\rho^f > \rho_s^f > 0$, $\theta > 0$ satisfy the following constraint:*

$$-\alpha_{f_3}(\alpha_{f_2}^{-1}(\rho_s^f)) + L_{\bar{y}}M_f\Delta_f + (L_{\bar{y}}L_{\tilde{g}} + L_{w_f})\theta \le -\epsilon_{w_f}/\Delta_f. \tag{19}$$

Then, there exists a class $\mathcal{KL}$ function β_y and a class $\mathcal{K}$ function γ_y such that if $\bar{y}(t_0) \in \Omega_{\rho^f}$, then $\bar{y}(t) \in \Omega_{\rho^f}$ for all $t \ge t_0$ and

$$|\bar{y}(t)| \le \beta_y\left(|\bar{y}(t_0)|, \frac{t-t_0}{\epsilon}\right) + \gamma_y(\rho_f^*) \tag{20}$$

with $\rho_f^* = \max\{V_f(\bar{y}(t+\Delta_f)) : V_f(\bar{y}(t)) \le \rho_s^f\}$, *uniformly in* $x \in \Omega_{\rho^s}$ *and* $w \in W$.

Proof. We write the time derivative of the Lyapunov function along the state trajectory $\bar{y}(t)$ of system of Eq. (6) in $t \in [t_{k_f}, t_{k_f} + \Delta_f)$ as follows:

$$\dot{V}_f(\bar{y}(t)) = \frac{\partial V_f(\bar{y})}{\partial \bar{y}} g(x(t_{k_s}), \bar{y}(t) + \tilde{g}(x(t_{k_s}), w), 0, u_f^*(t_{k_f}|t_{k_f}), w) \tag{21}$$

Adding and subtracting the term $\frac{\partial V_f(\bar{y}(t_{k_f}))}{\partial \bar{y}} g(x(t_{k_s}), \bar{y}(t_{k_f}) + \tilde{g}(x(t_{k_s}), 0), 0, u_f^*(t_{k_f}|t_{k_f}), 0)$ to the right-hand-side of Eq. (21) and taking Eq. (12) into account, we obtain the following inequality:

$$\dot{V}_f(\bar{y}(t)) \le -\alpha_{f_3}(|\bar{y}(t_{k_f})|) + \frac{\partial V_f(\bar{y})}{\partial \bar{y}} g(x(t_{k_s}), \bar{y}(t) + \tilde{g}(x(t_{k_s}), w), 0, u_f^*(t_{k_f}|t_{k_f}), w) - \frac{\partial V_f(\bar{y}(t_{k_f}))}{\partial \bar{y}} g(x(t_{k_s}), \bar{y}(t_{k_f}) + \tilde{g}(x(t_{k_s}), 0), 0, u_f^*(t_{k_f}|t_{k_f}), 0) \tag{22}$$

From Eq. (14), Assumption 1 and the inequality of Eq. (22), the following inequality is obtained for all $\bar{y}(t_{k_f}) \in \Omega_{\rho^f} \backslash \Omega_{\rho_s^f}$:

$$\dot{V}_f(\bar{y}(t)) \le -\alpha_{f_3}(\alpha_{f_2}^{-1}(\rho_s^f)) + L_{\bar{y}}|\bar{y}(t) - \bar{y}(t_{k_f})| + (L_{\bar{y}}L_{\tilde{g}} + L_{w_f})\theta. \tag{23}$$

Taking into account Eq. (13) and the continuity of $\bar{y}(t)$, the following bound can be written for all $t \in [t_{k_f}, t_{k_f} + \Delta_f)$, $|\bar{y}(t) - \bar{y}(t_{k_f})| \le M_f\Delta_f$. Using this expression, we obtain the following bound on the time derivative of the Lyapunov function for $t \in [t_{k_f}, t_{k_f} + \Delta_f)$, $\forall \bar{y}(t_{k_f}) \in \Omega_{\rho^f} \backslash \Omega_{\rho_s^f}$:

$$\dot{V}_f(\bar{y}(t)) \le -\alpha_{f_3}(\alpha_{f_2}^{-1}(\rho_s^f)) + L_{\bar{y}}M_f\Delta_f + (L_{\bar{y}}L_{\tilde{g}} + L_{w_f})\theta.$$

Since the condition of Eq. (19) is satisfied, then $\forall \bar{y}(t_{k_f}) \in \Omega_{\rho^f} \backslash \Omega_{\rho_s^f}$ we can obtain:

$$\dot{V}_f(\bar{y}(t)) \le \frac{-\epsilon_{w_f}}{\Delta_f}, \quad \forall t \in [t_{k_f}, t_{k_f} + \Delta_f).$$

Integrating this bound on $t \in [t_{k_f}, t_{k_f} + \Delta_f)$, we obtain that:

$$\begin{aligned} &V_f(\bar{y}(t_{k_f} + \Delta_f)) \le V_f(\bar{y}(t_{k_f})) - \epsilon_{w_f} \\ &V_f(\bar{y}(t)) \le V_f(\bar{y}(t_{k_f})), \forall t \in [t_{k_f}, t_{k_f} + \Delta_f) \end{aligned} \tag{24}$$

for all $\bar{y}(t_{k_f}) \in \Omega_{\rho^f} \backslash \Omega_{\rho_s^f}$. Using Eq. (24) recursively, it can be proved that, if $x(t_{k_f}) \in \Omega_{\rho^f} \backslash \Omega_{\rho_s^f}$, the state converges to $\Omega_{\rho_s^f}$ in a finite number of sampling times without leaving the stability region. Once the state converges to $\Omega_{\rho_s^f} \subseteq \Omega_{\rho_f^*}$, it remains inside $\Omega_{\rho_f^*}$ for all times. This statement holds because of the definition of ρ_f^*. This proves that the closed-loop system under the fast LMPC design is ultimately bounded in $\Omega_{\rho_f^*}$. Thus, due to the continuity of the Lyapunov function $V_f(\cdot)$, it can be concluded that there exists a class $\mathcal{KL}$ function β_y and a class $\mathcal{K}$ function γ_y such that if $\bar{y}(t_0) \in \Omega_{\rho^f}$, then $\bar{y}(t) \in \Omega_{\rho^f}$ for all $t \ge t_0$ and

$$|\bar{y}(t)| \le \beta_y\left(|\bar{y}(t_0)|, \frac{t - t_0}{\epsilon}\right) + \gamma_y(\rho_f^*) \tag{25}$$

□

Remark 4. Note that the purpose of the fast LMPC scheme is to stabilize fast subsystem dynamics while economic considerations are addressed through the slow LEMPC. Depending on the application and certain optimality specifications, the fast LMPC is desired in processes where the fast time-scale is large enough to warrant the use of MPC to achieve optimal performance compared to explicit fast feedback controllers that achieve fast dynamics stabilizability without necessary optimal performance. However, when the fast time-scale is short, the explicit feedback control is needed to ensure sufficiently fast computation of the "fast" control action; please see the example section.

3.3. Slow LEMPC formulation

Referring to the slow subsystem of Eq. (4), the slow LEMPC at sampling time t_{k_s} is formulated as follows

$$\max_{u_s \in S(\Delta_s)} \int_{t_{k_s}}^{t_{k_s}+N_s\Delta_s} L(\tilde{x}(\tilde{\tau}), u_s(\tilde{\tau}))d\tilde{\tau} \tag{26a}$$

$$\text{s.t.} \frac{d\tilde{x}(\tilde{\tau})}{d\tilde{\tau}} = f_s(\tilde{x}(\tilde{\tau}), u_s(\tilde{\tau}), 0) \tag{26b}$$

$$u_s(\tilde{\tau}) \in U_s, \quad \tilde{\tau} \in [t_{k_s}, t_{k_s} + N_s\Delta_s) \tag{26c}$$

$$\tilde{x}(t_{k_s}) = \bar{x}(t_{k_s}) \tag{26d}$$

$$V_s(\tilde{x}(\tilde{\tau})) \le \rho_e, \quad \forall \tilde{\tau} \in [t_{k_s}, t_{k_s} + N_s\Delta_s), \text{ if } \ t_{k_s} \le t' \ \text{ and } \ V_s(\bar{x}(t_{k_s})) \le \rho_e \tag{26e}$$

$$\frac{\partial V_s(\bar{x}(t_{k_s}))}{\partial \bar{x}} f_s(\bar{x}(t_{k_s}), u_s(t_{k_s}), 0) \le \frac{\partial V_s(\bar{x}(t_{k_s}))}{\partial \bar{x}} f_s(\bar{x}(t_{k_s}), h_s(\bar{x}(t_{k_s})), 0), \quad \text{if} \quad t_{k_s} > t' \quad \text{or} \quad \rho_e < V_s(x(t_{k_s})) \le \rho^s \tag{26f}$$

where $S(\Delta_s)$ is the family of piece-wise constant functions with sampling period Δ_s, N_s is the prediction horizon of LEMPC and $L(\tilde{x}(\tilde{\tau}), u_s(\tilde{\tau}))$ denotes an economic (typically non-quadratic) cost function. $\tilde{x}$ denotes the predicted slow subsystem state trajectory obtained by utilizing the nominal slow subsystem model of Eq. (26b) initialized by the state feedback of Eq. (26d). For mode one operation which corresponds to sampling times $t_{k_s} \le t'$, the constraint of Eq. (26e) maintains the predicted slow state within Ω_{ρ_e} if $V_s(x(t_{k_s})) \le \rho_e$. If $\rho_e < V_s(x(t_{k_s})) \le \rho^s$ or the LEMPC operates at mode 2, the constraint of Eq. (26f) ensures that the amount of reduction in the value of the Lyapunov function $V_s(\cdot)$ when u_s computed by the LEMPC of Eq. (26) is applied, is at least at the level when the feedback controller $h_s(\cdot)$ is applied in a sample-and-hold fashion. The optimal solution to this optimization problem is defined by $u_s^*(\tilde{\tau}|t_{k_s})\, \forall \tilde{\tau} \in [t_{k_s}, t_{k_s} + N_s\Delta_s)$ and the manipulated input of the closed-loop slow subsystem under the LEMPC of Eq. (26) is defined as follows:

$$u_s(t) = u_s^*(t|t_{k_s}), \quad \forall t \in [t_{k_s}, t_{k_s} + \Delta_s). \tag{27}$$

Proposition 2 characterizes the closed-loop stability properties of the LEMPC of Eq. (3.2).

Proposition 2 (c.f. [10]). *Consider the slow subsystem of Eq. (4) in closed-loop under the LEMPC design of Eqs. (26) and (27) based on a controller $h_s(\cdot)$ that satisfies the conditions of Eq. (7). Let $\epsilon_{w_s} > 0$, $\Delta_s > 0$, $\rho^s > \rho_e > 0$ and $\rho^s > \rho_s^s > 0$ satisfy*

$$\rho_e \le \rho^s - f_V(f_W(\Delta_s)) \tag{28}$$

where

$$f_V(s) = \alpha_{s_4}(\alpha_{s_1}^{-1}(\rho^s))s + M_v s^2 \tag{29}$$

with M_v being a positive constant and

$$f_W(s) = \frac{L_{w_s}\theta}{L_{\bar{x}}}(e^{L_{\bar{x}}s} - 1). \tag{30}$$

and

$$-\alpha_{s_3}(\alpha_{s_2}^{-1}(\rho_s^s)) + L_{\bar{x}} M_s \Delta_s + L_{w_s}\theta \le -\frac{\epsilon_{w_s}}{\Delta_s}. \tag{31}$$

If $\bar{x}(t_0) \in \Omega_{\rho^s}$, $\rho_s^s \le \rho_e$, $\rho_s^* \le \rho^s$ and $N_s \ge 1$ then the state $\bar{x}(t)$ of the closed-loop slow subsystem is always bounded in Ω_{ρ^s}. Furthermore, there exists a class $\mathcal{KL}$ function β_x and a class $\mathcal{K}$ function γ_x such that

$$|\bar{x}(t)| \le \beta_x(|\bar{x}(t^*)|, t - t^*) + \gamma_x(\rho_s^*) \tag{32}$$

with $\rho_s^* = \max\{V_s(\bar{x}(t + \Delta_s)) : V_s(\bar{x}(t)) \le \rho_s^s\}$, $\forall \bar{x}(t^*) \in B_\delta \subset \Omega_{\rho^s}$ and $\forall t \ge t^* > t'$ where t^* is chosen such that $\bar{x}(t^*) \in B_\delta$ and $B_\delta = \{x \in R^n : |x| \le \delta\}$.

The proof includes similar steps as the proof of Proposition 1. For the specific details, please refer to [10].

3.4. Closed-loop stability

The closed-loop stability of the system of Eq. (1) under the LMPC of Eq. (17) and LEMPC of Eq. (26) is established in the following theorem under appropriate conditions.

Theorem 1. *Consider the system of Eq. (1) in closed-loop with u_f and u_s computed by the LMPC of Eq. (17) and LEMPC of Eq. (26) based on the Lyapunov-based controllers $p(x)y$ and $h_s(\cdot)$ that satisfy the conditions of Eqs. (12) and (7), respectively. Let also Assumptions 1 and 2 and the conditions of Propositions 1 and 2 hold and*

$$\rho_e + (M_s + L_z M_y + L_\epsilon \epsilon)\Delta_s \alpha_{s_4}(\alpha_{s_1}^{-1}(\rho^s)) < \rho^s \tag{33}$$

and

$$-\alpha_{s_3}(\alpha_{s_1}^{-1}(\rho_s^s)) + d_1 < 0 \tag{34}$$

where L_z, M_y and d_1 are positive constants to be defined in the proof. Then there exist functions β_x of class $\mathcal{KL}$ and γ_x of class $\mathcal{K}$, a pair of positive real numbers (δ_x, d) and $\epsilon^* > 0$ such that if $\max\{|x(t_0)|, |y(t_0)|, ||w||, ||\dot{w}||\} \le \delta_x$ and $\epsilon \in (0, \epsilon^*]$, then $x(t) \in \Omega_{\rho^s}$ and $y(t) \in \Omega_{\rho^f}\, \forall t \ge t_0$ and

$$|x(t)| \le \beta_x(|x(t^*)|, t - t^*) + \gamma_x(\rho_s^*) + d \tag{35}$$

for all $t \ge t^* > t'$ where t^* has been defined in Proposition 2.

Proof. When $u_f = u_f^*$ and $u_s = u_s^*$ are determined by the LMPC of Eq. (17) and LEMPC of Eq. (26), respectively, the closed-loop system takes the following form:

$$\begin{aligned} \dot{x} &= f(x, z, \epsilon, u_s^*, w) \\ \epsilon\dot{z} &= g(x, z, \epsilon, u_f^*, w). \end{aligned} \tag{36}$$

We will first compute the slow and fast closed-loop subsystems. Setting $\epsilon = 0$ in Eq. (36) and taking advantage of the fact that $u_f^* = 0$ when $\epsilon = 0$, we obtain:

$$\begin{aligned} \frac{d\bar{x}}{dt} &= f(\bar{x}, \bar{z}, 0, u_s^*, w) \\ 0 &= g(\bar{x}, \bar{z}, 0, 0, w). \end{aligned} \tag{37}$$

Using Assumption 1, we can re-write Eq. (37) as follows:

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \tilde{g}(\bar{x}, w), 0, u_s^*, w) = f_s(\bar{x}, u_s^*, w) \tag{38}$$

For fast subsystem using $\tau = (t - t_0)/\epsilon$ and $y = z - \tilde{g}(x, w)$, the closed-loop system of Eq. (36) can be written as:

$$\begin{aligned} \frac{dx}{d\tau} &= \epsilon f(x, y + \tilde{g}(x, w), \epsilon, u_s^*, w) \\ \frac{dy}{d\tau} &= g(x, y + \tilde{g}(x, w), \epsilon, u_f^*, w) - \epsilon \frac{\partial \tilde{g}}{\partial w}\dot{w} - \epsilon \frac{\partial \tilde{g}}{\partial x} f(x, y + \tilde{g}(x, w), u_s^*, w) \end{aligned} \tag{39}$$

Setting $\epsilon = 0$, the following closed-loop fast subsystem is obtained:

$$\frac{d\bar{y}}{d\tau} = g(x, \bar{y} + \tilde{g}(x, w), 0, u_f^*, w) \tag{40}$$

Now we focus on the singularly perturbed system of Eq. (39). Considering the fast subsystem state $y(t)$ of Eq. (39) and assuming that $x(t)$ is bounded in Ω_{ρ^s} (which will be proved later), it can be obtained using a Lyapunov argument that there exist positive constants δ_{x_1} and ϵ_1 such that if $\max\{|x(t_0)|, |y(t_0)|, ||w||, ||\dot{w}||\} \le \delta_{x_1}$ and $\epsilon \in (0, \epsilon_1]$, there exists a positive constant k_1 such that $\forall t \ge t_0$:

$$|z - \tilde{g}(x, w)| = |y(t)| \le \beta_y\left(\delta_{x_1}, \frac{t - t_0}{\epsilon}\right) + \gamma_y(\rho_f^*) + k_1 \tag{41}$$

We consider $t \in (t_0, t_0 + \Delta_s]$ and $t \ge t_0 + \Delta_s$ separately and prove that if the conditions stated in Theorem 1 are satisfied, the boundedness of the states of the system of Eq. (39) is ensured. When $x(t_0) \in B_{\delta_{x_2}} \subset \Omega_{\rho_e} \subset \Omega_{\rho^s}$, where δ_{x_2} is a positive real number, considering the closed-loop x-subsystem of Eq. (39) state trajectory:

$$\dot{x}(t) = f(x, y + \tilde{g}(x, w), \epsilon, u_s^*, w), \quad \forall t \in (t_0, t_0 + \Delta_s]$$

and considering the facts that $|f(x, z, \epsilon, u_s^*, w)| \le |f_s(x, u_s^*, w)| + |f(x, z, \epsilon, u_s^*, w) - f_s(x, u_s^*, w)|$, $|f_s(x, u_s^*, w)| \le M_s$ and $|y(t)| \le \beta_y(\delta_{x_2}, 0) + \gamma_y(\rho_f^*) + k_1 < M_y$ where M_y is a positive constant such that

$$|f(x, z, \epsilon, u_s^*, w) - f_s(x, u_s^*, w)| = |f(x, z, \epsilon, u_s^*, w) - f(x, \tilde{g}(x, w), 0, u_s^*, w)| \le L_z|z - \tilde{g}(x, w)| + L_\epsilon \epsilon \le L_z M_y + L_\epsilon \epsilon \tag{42}$$

and

$$V_s(x(t)) = V_s(x(t_0)) + \int_{t_0}^{t} \dot{V}_s(x(\tau))d\tau = V_s(x(t_0)) + \int_{t_0}^{t} \frac{\partial V_s(x(\tau))}{\partial x}\dot{x}(\tau)d\tau \le \rho_e + (M_s + L_z M_y + L_\epsilon \epsilon)\Delta_s \alpha_{s_4}(\alpha_{s_1}^{-1}(\rho^s)) \tag{43}$$

Thus, there exists Δ_1 and ϵ_2 such that if $\Delta_s \in (0, \Delta_1]$ and $\epsilon \in (0, \epsilon_2]$, Eq. (33) holds and

$$V_s(x(t)) < \rho^s, \quad \forall t \in (t_0, t_0 + \Delta_s] \tag{44}$$

Picking $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ ensures that $\forall t \in [t_0, t_0 + \Delta_s)$, $x(t) \in \Omega_{\rho^s}$ and $y(t) \in \Omega_{\rho^f}$.

For $t \ge t_0 + \Delta_s$, considering Eq. (41), there exists a positive real number $\overline{M}_y$ such that

$$|y(t)| \le \beta_y\left(\delta_{x_2}, \frac{\Delta_s}{\epsilon}\right) + \gamma_y(\rho_f^*) + k_1 \le \overline{M}_y \tag{45}$$

and we can write the time derivative of the Lyapunov function $V_s(\cdot)$ along the closed-loop system state of Eq. (1) under the LEMPC of Eq. (26) $\forall t \in [t_{k_s}, t_{k_s+1})$ (assuming without loss of generality that $t_{k_s} = t_0 + \Delta_s$) as follows

$$\dot{V}_s(x(t)) = \frac{\partial V_s(x(t))}{\partial x} f(x(t), z(t), \epsilon, u_s^*(t_{k_s}), w(t)) \tag{46}$$

Adding and subtracting the terms $\frac{\partial V_s(x(t_{k_s}))}{\partial x} f_s(x(t_{k_s}), u_s(t_{k_s}), 0)$ and $\frac{\partial V_s(x(t))}{\partial x} f_s(x(t), u_s(t_{k_s}), w(t))$ to/from the above inequality and taking advantage of Eqs. (7) and (26f) and the fact that $|f(x(t), z(t), \epsilon, u_s^*(t_{k_s}), w(t))| \le |f_s(x(t), u_s^*(t_{k_s}), 0)| + |f(x(t), z(t), \epsilon, u_s^*(t_{k_s}), w(t)) - f_s(x(t), u_s^*(t_{k_s}), w(t))| + |f_s(x(t), u_s^*(t_{k_s}), w(t)) - f_s(x(t_{k_s}), u_s^*(t_{k_s}), 0)|$ and considering

$$\left|\frac{\partial V_s(x(t))}{\partial x} f_s(x, u_s^*, w)\right| \le \alpha_{s_4}(\alpha_{s_1}^{-1}(\rho^s))M_s \tag{47}$$

$$\left|\frac{\partial V_s(x(t))}{\partial x} f(x(t), z(t), \epsilon, u_s^*(t_{k_s}), w(t)) - \frac{\partial V_s(x(t))}{\partial x} f_s(x(t), u_s^*(t_{k_s}), w(t))\right| \le L_{\overline{z}}|z - \tilde{g}(x, w)| + L_{\overline{\epsilon}}\epsilon \tag{48}$$

$$|z - \tilde{g}(x, w)| \le \overline{M}_y \tag{49}$$

$$\left|\frac{\partial V_s(x(t))}{\partial x} f_s(x(t), u_s^*(t_{k_s}), w(t)) - \frac{\partial V_s(x(t))}{\partial x} f_s(x(t_{k_s}), u_s^*(t_{k_s}), 0)\right| \le L_{\overline{x}}|x(t) - x(t_{k_s})| + L_{w_s}\theta \tag{50}$$

$$|x(t) - x(t_{k_s})| \le M_s \Delta_s \tag{51}$$

we can obtain

$$\dot{V}_s(x(t)) \leq -\alpha_{s_3}(\alpha_{s_1}^{-1}(\rho_s^s)) + d_1 \tag{52}$$

where

$$d_1 = \alpha_{s_4}(\alpha_{s_1}^{-1}(\rho^s))M_s + L_{\bar{z}}\overline{M}_y + L_{\bar{\epsilon}}\epsilon + L_{\bar{x}}M_s\Delta_s + L_{w_s}||w|| \tag{53}$$

where d_1 is a positive constant. Picking δ_{x_2}, ϵ_4 and Δ_2 such that $\forall\epsilon \in (0, \epsilon_4]$, $\max\{|x(t_0)|, |y(t_0)|, ||w||, ||\dot{w}||\} \leq \delta_{x_2}$ and $\forall\Delta_s \in (0, \Delta_2]$, Eq. (34) is satisfied, the closed-loop system state $x(t)$ is bounded in Ω_{ρ^s}, $\forall t \geq t_0 + \Delta_s$. Finally, using similar arguments to the proof of Theorem 1 in [8], we have that there exist a class $\mathcal{KL}$ function β_x and a class $\mathcal{K}$ function γ_x, positive real numbers (δ_x, d) where $\delta_x < \min\{\delta_{x_1}, \delta_{x_2}\}$, and $0 < \epsilon^* < \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ and $0 < \Delta^* < \min\{\Delta_1, \Delta_2\}$ such that if $\max\{|x(t_0)|, |y(t_0)|, ||w||, ||\dot{w}||\} \leq \delta_x$, $\epsilon \in (0, \epsilon^*]$ and $\Delta_s \in (0, \Delta^*]$, then, the bound of Eq. (35) holds for all $t \geq t^* > t'$. □

Remark 5. It should be emphasized that in Theorem 1, it has been indicated that for operation periods corresponding to LEMPC mode 1 operation, both of fast and slow reduced order subsystem states are bounded in invariant sets (i.e., $\Omega_{\tilde{\rho}^s}$ and $\Omega_{\tilde{\rho}^f}$) to ensure that the actual states of the system are bounded in certain stability regions (i.e., Ω_{ρ^s} and Ω_{ρ^f}) through restricting their corresponding initial states. On the other hand, for operation periods corresponding to LEMPC mode 2 operation, both of system states are asymptotically bounded in a small invariant set containing the origin.

Remark 6. While the present work focuses on nonlinear singularly perturbed systems and general (non-convex) economic cost functions, the results of this work are novel and apply to the case of linear singularly perturbed systems; however, in the linear case, the verification of the assumption that there is an isolated equilibrium manifold for the fast dynamics (Assumption 1), the construction of the explicit control laws for the slow and fast subsystems imposed in Section 2.4 and the computation of the associated closed-loop stability regions, and the solution of the LEMPC and LMPC optimization problems when convex economic cost functions are used simplify significantly, given the availability of robust and efficient tools for matrix calculations and convex optimization.

Remark 7. Within the context of the integration of economic process optimization and model predictive control, the separation of time-scales of various phenomena is used throughout the literature to design integrated systems for process control and optimization. To this end, it is important to note that in the present work we take advantage of the co-existence of dynamic physicochemical phenomena evolving on different time-scale in the process model to develop a two-time-scale framework; however, this framework should not be viewed as a two-layer approach like real-time optimization (RTO) where optimal operating set-points are computed in the upper layer and sent down to the lower process control layer to force the system to track these set-points [21]. Below, we compare our approach to various dynamic RTO (D-RTO) frameworks in an attempt to further clarify this point.

In [26], a nested optimization problem was proposed to compute optimal process design parameters and operating steady-states using a steady-state process model. In contrast, in the present work, we do not address the process design problem, and we design the two MPCs using reduced-order dynamic models associated with the fast and slow time-scales. In [12], a D-RTO structure was designed based on the assumption that there exists a significant time-scale separation of the time-varying evolution of the disturbances and of the optimal process trajectory. This is further explored in [14] where the authors assume that there exists time-scale separation between the different types of disturbances. In the present work on the other hand, we do not consider upper-layer data reconciliation and consider the class of process systems that are in standard singularly perturbed form with bounded uncertainties and disturbances. In the single-point technique proposed in [18], the RTO and MPC layers are separate, but executed at the same rate; while, in [25], the authors propose a D-RTO framework that resolves the upper-layer at an intermediate rate between (1) the rate traditional RTO schemes are resolved (i.e., once steady-state is reached) and (2) the rate at which the lower-layer MPC is resolved (i.e., at each sampling times). In this manner, a two-layer approach to process optimization and control is maintained. On the other hand, in our work, the LEMPC and LMPC are solved separately in parallel but at different rates. In [20], RTO and MPC are combined by creating one optimization problem which computes both set-points and manipulated inputs. With our approach, the slow LEMPC operates the process in a time-varying fashion around a steady-state as to optimize the economic cost and the fast LMPC works to drive the fast states to the equilibrium slow manifold. Furthermore, the LEMPC can be constructed with a general cost function accounting directly for the process economics and this cost function does not need to have quadratic terms like the ones employed in conventional MPC.

4. Application to a chemical process example

4.1. Process description

Consider a well-mixed, non-isothermal continuous stirred tank reactor (CSTR) where an irreversible, second-order, endothermic reaction $A \rightarrow B$ takes place, where A is the reactant and B is the desired product. The feed to the reactor consists of the reactant A and an inert gas at flow rate F, temperature T_0 and molar concentration C_{A0}. Due to the non-isothermal nature of the reactor, a jacket is used to provide heat to the reactor. The dynamic equations describing the behavior of the reactor, obtained through material and energy balances under standard modeling assumptions, are given below:

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{A0} - C_A) - k_0 e^{-E/RT} C_A^2 \tag{54a}$$

$$\sigma c_P \frac{dT}{dt} = \frac{F\sigma c_P}{V}(T_0 - T) - \Delta H k_0 e^{-E/RT} C_A^2 + \frac{Q}{V} \tag{54b}$$

where C_A denotes the concentration of the reactant A, T denotes the temperature of the reactor, Q denotes the rate of heat supply to the reactor, V represents the volume of the reactor, ΔH, k_0 and E denote the enthalpy, pre-exponential constant and activation energy of the reaction, respectively, and c_P and σ denote the heat capacity and the density of the fluid in the reactor, respectively. The values of the

Table 1
Parameter values.

$T_0 = 300$	K	$F = 5$	m^3/h
$V = 1.0$	m^3	$E = 5 \times 10^4$	kJ/kmol
$k_0 = 8.46 \times 10^6$	1/h	$\Delta H = -19.91$	kJ/kmol
$c_P = 0.02$	kJ/kg K	$R = 8.314$	kJ/kmol K
$\sigma = 20$	kg/m^3	$C_{As} = 1.95$	$kmol/m^3$
$T_s = 401.87$	K	$C_{A0s} = 4$	$kmol/m^3$
$Q_s = 0$	kJ/h		

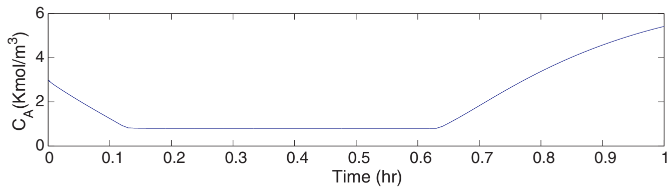

Fig. 1. Closed-loop reactant concentration for initial condition ($C_A(0)$, $T(0)$ = 3 kmol/m^3, 400 K).

process parameters used in the simulations are shown in Table 1. The process model of Eq. (54) is numerically simulated using an explicit Euler integration method with integration step $h_c = 10^{-6}$ h.

The process model has one unstable steady-state in the operating range of interest. The control objective is to optimize the process operation in a region around the unstable steady-state (C_{As}, T_s) to maximize the average production rate of B through manipulation of the concentration of A in the inlet to the reactor, C_{A0}. The steady-state input value associated with the steady-state point is denoted by C_{A0s}. Defining $\epsilon = \sigma c_P$, the following nonlinear state-space model can be obtained

$$\begin{aligned} \dot{x}(t) &= f(x(t), z(t), \epsilon, u_s(t), 0) \\ \epsilon \dot{z}(t) &= g(x(t), z(t), \epsilon, u_f(t), 0) \end{aligned} \tag{55}$$

where $x = C_A - C_{As}$ and $z = T - T_s$ are the states, $u_s = C_{A0} - C_{A0s}$ and $u_f = Q - Q_s$ are the inputs and f and g are scalar functions. The inputs are subject to constraints as follows: $|u_s| \leq 3.5$ kmol/m^3 and $|u_f| \leq 5 \times 10^5$ kJ/h. The economic measure considered in this example is as follows [24]:

$$L_e(x, u) = \frac{1}{t_N} \int_0^{t_N} k_0 e^{-E/RT(\tau)} C_A^2(\tau) d\tau \tag{56}$$

where $t_N = 1$ h is the time duration of a reactor operating period. This economic objective function highlights the maximization of the average production rate over a process operation period for $t_N = 1$ h (of course, different, yet finite, values of t_N can be chosen). We also consider that there is limitation on the amount of reactant material which can be used over the period t_N. Specifically, the control input trajectory of u_s should satisfy the following constraint:

$$\frac{1}{t_N} \int_0^{t_N} u_s(\tau) d\tau = 1 \quad \text{kmol/m}^3. \tag{57}$$

This constraint means that the available amount of reactant material over one period is fixed. For the sake of simplicity and without loss of generality, we will refer to Eq. (57) as the material constraint.

In terms of the Lyapunov-based controllers, feedback linearization techniques are utilized for the design of explicit controllers for the fast and slow reduced-order subsystems subject to input constraints and quadratic Lyapunov functions $V_s(x) = x^2$ and $V_f(y) = y^2$ are used to compute the stability regions. Through feedback linearization and evaluating $\dot{V}_s(\cdot)$ subject to the input constraint, $\dot{V}_s(x) \leq 0$ when $x \in \Omega_{\rho^s}$ and $\rho^s = 4$. Furthermore, to guarantee that $C_A > 0$ and $T \leq 480$ K, the corresponding stability Ω_{ρ^s} is defined as $\Omega_{\rho^s} = \{x | -1.15 \leq x \leq 3.95\}$.

We consider design of a slow LEMPC to regulate the slow subsystem state and maximize the economic objective function and a fast feedback linearizing controller which stabilize the fast dynamics. With respect to the fast feedback linearizing controller, the deviation variable $y(t)$ is defined as $z(t) - \bar{z}(t)$ where $\bar{z}(t)$ is the unique root of the algebraic equation $g(x(t), \bar{z}(t), 0, 0, 0) = 0$ given $x(t)$. For the purpose of simulation, this unique root has been approximated through a 10th order polynomial. Furthermore, we assume that the state measurements are available every $\Delta_f = 10^{-6}$ h and the manipulated input u_f is obtained every Δ_f such that

$$g(x(t), y + \tilde{g}(x, 0), 0, u_f(t), 0) = -\mu y \tag{58}$$

where $\mu = 100$. Regarding the slow dynamics, LEMPC obtains its manipulated input trajectory u_s every $\Delta_s = 10^{-2}$ h by optimizing the economic cost function of Eq. (56) using the one dimensional slow subsystem which is independent of ϵ. As a result, the slow subsystem used in the economic MPC is well conditioned and is integrated with time step 10^{-3} h resulting in a nearly three order of magnitude improvement in the computational time needed to compute the control action for u_s. Specifically, LEMPC operates only at mode 1 to highlight the effect of economic optimization. Considering the material constraint which needs to be satisfied through one period of process operation, a decreasing LEMPC horizon sequence $N_0, \ldots, N_{99}$ where $N_i = 100 - i$ and $i = 0, \ldots, 99$ is utilized at the different slow sampling times.

Figs. 1–5 illustrate closed-loop state and manipulated input trajectories of the chemical process of Eq. (54) under the mode one operation of LEMPC design of Eq. (26) and feedback linearization of Eq. (58). As it can be seen in these figures, by dictating time-varying operation by LEMPC, the economic cost of Eq. (56) is optimized while considering the material constraint of Eq. (57). Furthermore, u_f through feedback

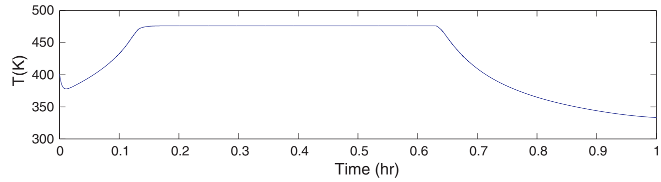

Fig. 2. Closed-loop temperature profile for initial condition ($C_A(0)$, $T(0)$ = 3 kmol/m^3, 400 K).

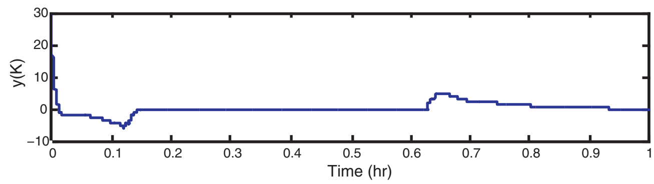

Fig. 3. Closed-loop profile of $y = z - \tilde{g}(x, 0)$ for initial condition ($C_A(0)$, $T(0)$ = 3 kmol/m^3, 400 K).

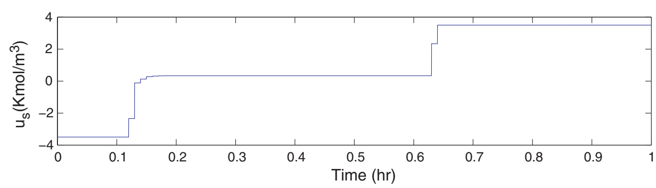

Fig. 4. Profile of the manipulated input u_s under the LEMPC for initial condition ($C_A(0)$, $T(0)$ = 3 kmol/m^3, 400 K).

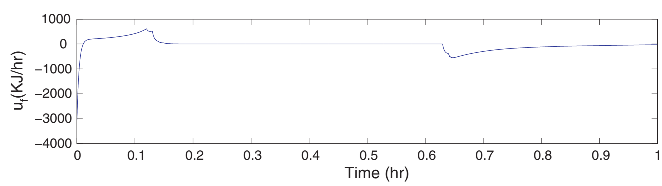

Fig. 5. Profile of the manipulated input u_f under the feedback linearizing controller of Eq. (58) for initial condition ($C_A(0)$, $T(0)$ = 3 kmol/m^3, 400 K).

linearization ensures that the fast subsystem state $y(t)$ goes to zero. We point out that either the open-loop or closed-loop dynamics can evolve on different time-scales. In our example, we use feedback linearization with a gain chosen to drive the deviation variable y to zero fast relative to the slow state C_A as observed in Fig. 3 while satisfying the bound on the available control energy as seen in Fig. 5. Therefore, this illustrative example possesses two-time-scale behavior.

We have also performed a set of simulations to compare the economic closed-loop performance of the proposed method versus the case that the input material is fed at a constant rate (i.e., $u_s(t) = 1\ \forall\, t \in [0, 1\ \text{h}]$). To carry out this comparison, we have computed the total cost of each scenario based on the index of the following form:

$$J = \frac{1}{t_N}\sum_{i=0}^{N}[k_0 e^{-E/RT(t_i)} C_A^2(t_i)]$$

where $t_0 = 0$ h, $t_N = 1$ h and $N = 100$. By comparing the performance index J for these two cases, we find that that the proposed LEMPC through a time-varying operation achieves a higher cost value (13.12 vs. 5.92) compared to the case that the reactant material is uniformly in time fed to the process (i.e., $u_s(t) = 1\ \forall\, t \in [0, 1\ \text{h}]$).

5. Conclusions

This work focused on the design of economic MPC for a class of nonlinear singularly perturbed systems which is capable of optimizing closed-loop performance with respect to a general economic objective function. Under appropriate stabilizability assumptions, fast sampling of the fast states and slow sampling of the slow states, the proposed EMPC system dictates a possible time-varying operation to address economic considerations while guaranteeing closed-loop stability. Closed-loop stability was addressed through singular perturbation arguments while a chemical process example was used to demonstrate the proposed method.

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