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Robust moving horizon estimation based output feedback economic model predictive control

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ABSTRACT

In this work, we develop an economic model predictive control scheme for a class of nonlinear systems with bounded process and measurement noise. In order to achieve fast convergence of the state estimates to the actual system state as well as the robustness of the observer to measurement and process noise, a deterministic (high-gain) observer is first applied for a small time period with continuous output measurements to drive the estimation error to a small value; after this initial small time period, a robust moving horizon estimation scheme is used on-line to provide more accurate and smoother state estimates. In the design of the robust moving horizon estimation scheme, the deterministic observer is used to calculate reference estimates and confidence regions that contain the actual system state. Within the confidence regions, the moving horizon estimation scheme is allowed to optimize its estimates. The output feedback economic model predictive controller is designed via Lyapunov techniques based on state estimates provided by the deterministic observer and the moving horizon estimation scheme. The stability of the closed-loop system is analyzed rigorously and conditions that ensure the closed-loop stability are derived. Extensive simulations based on a chemical process example illustrate the effectiveness of the proposed approach.

1. Introduction

In recent years, significant efforts have been devoted to the development of economic model predictive control (EMPC) designs due to the pursuit of higher process operation efficiency (e.g., [1–5]). EMPC is different from the traditional two-layer real-time optimization structure and addresses economic objectives directly within the framework of model predictive control (MPC) by replacing the conventional MPC quadratic cost function with a general economic cost function (which is not quadratic in general). Therefore, EMPC may, in general, lead to time-varying process operation policies instead of steady-state operation.

Various results of EMPC have been developed. In [6], a design that combines steady-state optimization and a linear MPC was proposed. In [2], an EMPC scheme for nonlinear systems that requires the closed-loop system state settles to a steady-state at the end of the prediction horizon was developed. The application of EMPC to cyclic processes as well as a closed-loop stability analysis was discussed in [3]. In [4], a two-mode Lyapunov-based EMPC (LEMPC) design for nonlinear systems was developed. The LEMPC is capable of handling asynchronous and delayed measurements and can be implemented in a distributed fashion [7]. All of the above mentioned EMPC schemes were developed under the assumption of state feedback. However, this assumption may not hold in many applications. In order to address this issue, in [8], an output feedback EMPC was proposed based on a high-gain observer [9,10]. However, in [8], process disturbances and measurement noise were not taken into account explicitly. When measurement noise is present, the performance of a high-gain observer may decrease significantly due to its sensitivity to measurement noise [11].

In order to improve the robustness of the high-gain observer to model mismatch and uncertainties while reducing its sensitivity to measurement noise significantly, in this work, we propose a robust moving horizon estimation (RMHE) based output feedback EMPC design. The idea of RMHE was initially developed in [12] which integrates deterministic observer techniques and optimization-based estimation techniques in a unified framework. Specifically,

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in the RMHE, an auxiliary deterministic nonlinear observer that is able to asymptotically track the nominal system state is taken advantage of to calculate a confidence region. In the calculation of the confidence region, bounded process and measurement noise are taken into account. The RMHE is only allowed to optimize its state estimates within the confidence region. By this approach, it was proved that the RMHE gives bounded estimation error in the case of bounded process noise. It was also shown to compensate for the error in the arrival cost approximation and could be used together with different arrival cost approximation techniques to further improve the state estimate. The RMHE has been applied to the design of a robust output feedback Lyapunov-based MPC [13] and has also been extended to estimate the state of large-scale systems in a distributed manner [14].

In the present work, we consider EMPC of nonlinear systems with bounded process and measurement noise. In order to achieve fast convergence of the state estimates to the actual system state (thus an effective separation principle between the observer and controller designs) and the robustness of the system to process and measurement noise, a deterministic (high-gain) observer is first applied for a small time period with continuous output measurements to drive the estimation error to a small value; after this initial small time period, a RMHE based on the deterministic observer is used on-line to provide more accurate and smooth state estimates. In the design of the RMHE, the deterministic observer is used to calculate the reference estimate and the confidence region for the state estimate. The output feedback EMPC is designed via the LEMPC technique based on state estimates provided by the deterministic observer and the RMHE. The stability of the closed-loop system is rigorously analyzed, and conditions that ensure the closed-loop stability are derived. Extensive simulations based on a chemical process example illustrate the effectiveness of the proposed approach.

2. Preliminaries

2.1. Notation

The operator $|\cdot|$ denotes the Euclidean norm of a scalar or a vector while $|\cdot|_Q^2$ indicates the square of the weighted Euclidean norm of a vector, defined as $|x|_Q^2 = x^T Q x$ where Q is a positive definite square matrix. A function $f(x)$ is said to be locally Lipschitz with respect to its argument x if there exists a positive constant L_f^x such that $|f(x') - f(x'')| \leq L_f^x |x' - x''|$ for all x' and x'' in a given region of x and L_f^x is the associated Lipschitz constant. A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. A function $\beta(r, s)$ is said to be a class $\mathcal{KL}$ function if for each fixed s, $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed r, it is decreasing with respect to s, and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. The symbol $\mathrm{diag}([v])$ denotes a diagonal matrix whose diagonal elements are the elements of vector v. The symbol '\' denotes set subtraction such that $\mathbb{A} \setminus \mathbb{B} := \{x \in \mathbb{A}, x \notin \mathbb{B}\}$. Finally, x^T denotes the transpose of the vector x.

2.2. System description

We consider nonlinear systems described by the following state-space model:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) + l(x(t))w(t) \\ y(t) &= h(x) + v(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state vector, $u \in \mathbb{R}^p$ denotes the control (manipulated) input vector, $w \in \mathbb{R}^m$ denotes the disturbance vector, $y \in \mathbb{R}^q$ denotes the measured output vector and $v \in \mathbb{R}^q$ is the measurement noise vector. The control input vector is restricted to be in a nonempty convex set $\mathbb{U} \subseteq \mathbb{R}^p$ such that $\mathbb{U} := \{u \in \mathbb{R}^p : |u| \leq u^{\max}\}$ where $u^{\max}$ is the magnitude of the input constraint. It is assumed that the noise vectors are bounded such as $w \in \mathbb{W}$ and $v \in \mathbb{V}$ where

$$\begin{aligned} \mathbb{W} &:= \{w \in \mathbb{R}^m : |w| \leq \theta_w,\ \theta_w > 0\} \\ \mathbb{V} &:= \{v \in \mathbb{R} : |v| \leq \theta_v,\ \theta_v > 0\} \end{aligned}$$

with θ_w and θ_v being known positive real numbers. Moreover, it is assumed that the output measurement vector y of the system is continuously available at all times. It is further assumed that f, g, l and h are sufficiently smooth functions and $f(0) = 0$ and $h(0) = 0$.

Remark 1. The model of Eq. (1) describes a large number of processes arising in the context of the chemical process industry. For example, one may express the model of the benzene alkylation process network considered in [7] in this form.

2.3. Stabilizability and observability assumptions

It is assumed that there exists a state feedback controller $u = k(x)$ that renders the origin of the nominal system of Eq. (1) (i.e., the system of Eq. (1) with $w(t) \equiv 0$) asymptotically stable while satisfying the input constraint for all the states x inside a given compact set containing the origin. This assumption implies that there exist class $\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$ and a continuously differentiable Lyapunov function $V(x)$ for the closed-loop nominal system, that satisfy the following inequalities [15,16] :

$$\begin{aligned} &\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \\ &\frac{\partial V(x)}{\partial x}(f(x) + g(x)k(x)) \leq -\alpha_3(|x|) \\ &\left| \frac{\partial V(x)}{\partial x} \right| \leq \alpha_4(|x|) \end{aligned} \tag{2}$$

and $k(x) \in \mathbb{U}$ for all $x \in \mathbb{D} \subseteq \mathbb{R}^n$ where $\mathbb{D}$ is an open neighborhood of the origin. We denote the level set of $V(x)$, $\Omega_\rho \subseteq \mathbb{D}$, as the stability region of the closed-loop system under the controller $k(x)$.

It is also assumed that there exists a deterministic observer that takes the following general form:

$$\dot{z}(t) = F(\epsilon, z, y) \tag{3}$$

where z is the observer state which is an estimate of the state of system of Eq. (1), y is the output measurement vector and ϵ is a positive parameter. This observer together with the state feedback controller $u = k(x)$ form an output feedback controller: $\dot{z} = F(\epsilon, z, y)$, $u = k(z)$ which satisfies the following assumptions:

(1) there exist positive constants θ_w^*, θ_v^* such that for each pair $\{\theta_w, \theta_v\}$ with $\theta_w \leq \theta_w^*, \theta_v \leq \theta_v^*$, there exist $0 < \rho_1 < \rho$, $e_{m0} > 0$, $\epsilon_L^* > 0$, $\epsilon_U^* > 0$ such that if $x(t_0) \in \Omega_{\rho_1}$, $|z(t_0) - x(t_0)| \leq e_{m0}$ and $\epsilon \in (\epsilon_L^*, \epsilon_U^*)$, the trajectories of the closed-loop system are bounded in Ω_ρ for all $t \geq t_0$;
(2) and there exists $e_m^* > 0$ such that for each $e_m \geq e_m^*$, there exists t_b such that $|z(t) - x(t)| \leq e_m$ for all $t \geq t_b(\epsilon)$.

Note that a type of observer that satisfies the above assumptions is a high-gain observer [11]. From an estimate error convergence speed point of view, it is desirable to pick the observer parameter ϵ as small as possible; however, when the parameter ϵ is too small (i.e., the observer gain is too large), it will make the observer very sensitive to measurement noise. In the observer assumptions, a key idea is to pick the gain ϵ in a way that balances the estimate error convergence speed to zero and the effect of the noise. In the remainder of this work, the estimate given by the observer F will be denoted as z.

Remark 2. It is important to point out the difference between the positive constants θ_w^* and θ_v^* and the bounds θ_w and θ_v. Specifically, the positive constants θ_w^* and θ_v^* correspond to theoretical bounds on the noise such that the closed-loop system under the output feedback controller: $\dot{z} = F(\epsilon, z, y)$, $u = k(z)$ is maintained in Ω_ρ. The constants θ_w^* and θ_v^* depend on the stability properties of a given system under the output feedback controller. On the other hand, the bounds θ_w and θ_v correspond to the actual bound on the process and measurement noise for a given (open-loop) system.

2.4. Robust MHE

In order to take advantage of the tunable convergence speed of the observer presented in the previous subsection while significantly reducing its sensitivity to measurement noise, an RMHE scheme developed in [12] will be adopted in this work.

The RMHE is evaluated at time instants $\{t_{k\geq 0}\}$ with $t_k = t_0 + k\Delta$, $k = 0, 1, \ldots$ where t_0 is the initial time. In the RMHE scheme, the deterministic observer F will be used to calculate a reference state estimate at each sampling time based on continuous output measurements. Based on the reference state estimate, the RMHE determines a confidence region for the actual system state; within the confidence region, the RMHE optimizes the state estimate based on a sequence of previous output measurements, system model and bounds information of the process and measurement noise. Specifically, the robust MHE scheme at time instant t_k is formulated as follows:

$$\min_{\tilde{x}(t_{k-N_e}),\ldots,\tilde{x}(t_k)} \sum_{i=k-N_e}^{k-1} |w(t_i)|^2_{Q_m^{-1}} + \sum_{i=k-N_e}^{k} |v(t_i)|^2_{R_m^{-1}} + \hat{V}_T(t_{k-N_e}) \tag{4a}$$

$$\text{s.t. } \dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))u(t) + l(\tilde{x}(t))w(t_i),$$
$$t \in [t_i, t_{i+1}] \tag{4b}$$

$$v(t_i) = y(t_i) - h(\tilde{x}(t_i)) \tag{4c}$$

$$w(t_i) \in \mathbb{W}, \qquad v(t_i) \in \mathbb{V}, \qquad \tilde{x}(t) \in \Omega_\rho \tag{4d}$$

$$|\tilde{x}(t_k) - z(t_k)| \leq \kappa |y(t_k) - h(z(t_k))| \tag{4e}$$

where N_e is the estimation horizon, Q_m and R_m are the estimated covariance matrices of w and v respectively, $\hat{V}_T(t_{k-N_e})$ denotes the arrival cost which summarizes past information up to t_{k-N_e}, $\tilde{x}$ is the predicted state x in the above optimization problem, $y(t_i)$ is the output measurement at t_i, $z(t_k)$ is an estimate given by the observer F based on continuous measurements of y, and κ is a positive constant which is a design parameter.

Once the optimization problem of Eq. (4) is solved, an optimal trajectory of the system state, $\tilde{x}^*(t_{k-N}), \ldots, \tilde{x}^*(t_k)$, is obtained. The optimal estimate of the current system state is denoted:

$$\hat{x}^*(t_k) = \tilde{x}^*(t_k). \tag{5}$$

Note that in the optimization problem of Eq. (4), w and v are assumed to be piece-wise constant variables with sampling time Δ to ensure that (4) is a finite dimensional optimization problem.

In the optimization problem of Eq. (4), $z(t_k)$ is a reference estimate calculated by the observer F. Based on the reference estimate and the current output measurement (i.e., $y(t_k)$), a confidence region that contains the actual system state is constructed (i.e., $\kappa|y(t_k) - h(z(t_k))|$). The estimate of the current state provided by the RMHE is only allowed to be optimized within the confidence region. This approach ensures that the RMHE inherits the robustness of the observer F and gives estimates with bounded errors.

Remark 3. In order to account for the effect of historical data outside the estimation window, an arrival cost which summarizes the information of those data is included in the cost function of an MHE optimization problem. The arrival cost plays an important role in the performance and stability of an MHE scheme. Different methods have been developed to approximate the arrival cost including Kalman filtering and smoothing techniques for linear systems [17], extended Kalman filtering for nonlinear systems [18], and particle filters for constrained systems [19].

3. Output feedback LEMPC

The proposed design of the output feedback LEMPC for nonlinear systems is presented in this section. Without loss of generality, it is assumed that the LEMPC is evaluated at time instants $\{t_{k\geq 0}\}$ with sampling time Δ as used in the RMHE. In the proposed LEMPC design, we will take advantage of both the fast convergence rate of the observer F and the robustness of the RMHE to measurement noise.

3.1. Implementation strategy

In the proposed approach, the observer F is applied for a short period at the initial time to drive the estimate of the observer to a small neighborhood of the actual system state; once the estimate has converged to a small neighborhood of the actual system state, the RMHE takes over the estimation task and provides smoother and optimal estimate to the LEMPC. Without loss of generality, we assume that t_b is a multiple integer of the sampling time Δ. Specifically, in the first b sampling periods, the observer F is applied with continuously output measurements and provides state estimates to the LEMPC at every sampling time; that is, from t_0 to t_b with $b \geq 1$, the observer F is evaluated continuously and provides state estimates to the LEMPC at time instants t_i with $i = 0, \ldots, b-1$. Starting from t_b, the RMHE is activated. The RMHE is evaluated and provides an optimal estimate of the system state to the LEMPC at every sampling time. The LEMPC evaluates its optimal input trajectory based on either the estimates provided by the observer F or the estimates from the RMHE.

The two-mode operation scheme in [4] is adopted in the design of the proposed LEMPC. Specifically, we assume that from the time t_0 up to a specific time t_s, the LEMPC operates in the first operation mode to maximize the economic cost function while maintaining the closed-loop system state in the stability region Ω_ρ. In this operation mode, in order to account for the uncertainties in state estimates and process noise, a region Ω_{ρ_e} with $\rho_e < \rho$ is used. If the estimated current state is in the region Ω_{ρ_e}, the LEMPC maximizes the cost function within the region Ω_{ρ_e}; if the estimated current state is in the region $\Omega_\rho \setminus \Omega_{\rho_e}$, the LEMPC first drives the system state to the region Ω_{ρ_e} and then maximizes the cost function within Ω_{ρ_e}. After time t_s, the LEMPC operates in the second operation mode and calculates the inputs in a way that the state of the closed-loop system is driven to a neighborhood of the desired steady-state. The above described implementation strategy of the proposed output feedback LEMPC can be summarized as follows:

Algorithm 1. Output feedback LEMPC implementation algorithm

1. Initialize the observer F with $z(t_0)$ and run the observer F continuously based on the output measurements y.
2. At a sampling time t_k, if $t_k < t_b$, go to Step 2.1; otherwise, go to Step 2.2.
   2.1. The LEMPC gets a sample of the estimated system state $z(t_k)$ at t_k from the observer F, and go to Step 3.
   2.2. Based on the estimate $z(t_k)$ provided by the observer F and output measurements at the current and previous N_e sampling instants (i.e., $y(t_i)$ with $i = k - N_e, \ldots, k$), the RMHE calculates the optimal state estimate $\hat{x}^*(t_k)$. The estimate $\hat{x}^*(t_k)$ is sent to the LEMPC.

3. If $t_k < t_s$ and if $z(t_k) \in \Omega_{\rho_e}$ (or if $\hat{x}^*(t_k) \in \Omega_{\rho_e}$), go to Step 3.1. Otherwise, go to Step 3.2.
   3.1. Based on $z(t_k)$ or $\hat{x}^*(t_k)$, the LEMPC calculates its input trajectory to maximize the economic cost function within Ω_{ρ_e}. The first value of the input trajectory is applied to the system. Go to Step 4.
   3.2. Based on $z(t_k)$ or $\hat{x}^*(t_k)$, the LEMPC calculates its input trajectory to drive the system state towards the origin. The first value of the input trajectory is applied to the system.
4. Go to Step 2 ($k \longleftarrow k+1$).

In the remainder, we will use $\hat{x}$ to denote the state estimate used in the LEMPC. Specifically, $\hat{x}$ at time t_k is defined as follows:

$$\hat{x}(t_k) = \begin{cases} z(t_k), & \text{if } t_k < t_b \\ \hat{x}^*(t_k), & \text{if } t_k \geq t_b. \end{cases} \tag{6}$$

Remark 4. In the implementation Algorithm 1 as well as in the RMHE design of Eq. (4), the observer F provides state estimate to the RMHE at every sampling time and is evaluated independently from the RMHE. In order to improve the quality of the estimates provided by the observer F, the state of the observer F may be set to the estimate of the RMHE at every sampling time since the estimates obtained from the RMHE are expected to be more accurate. That is, at Step 2.2, the estimate $\hat{x}^*(t_k)$ is also sent to the observer F and the observer F resets its state to $z(t_k) = \hat{x}^*(t_k)$.

3.2. LEMPC design

The LEMPC is evaluated every sampling time to obtain the future input trajectories based on estimated state $\hat{x}(t_k)$ provided by the observer F or the RMHE. Specifically, the optimization problem of the LEMPC is formulated as follows:

$$\max_{u \in S(\Delta)} \int_{t_k}^{t_{k+N}} L(\tilde{x}(\tau), u(\tau))d\tau \tag{7a}$$

$$\text{s.t. } \dot{\tilde{x}}(\tau) = f(\tilde{x}(\tau)) + g(\tilde{x}(\tau))u(\tau) \tag{7b}$$

$$u(\tau) \in \mathbb{U}, \qquad \tau \in [t_k, t_{k+N}) \tag{7c}$$

$$\tilde{x}(t_k) = \hat{x}(t_k) \tag{7d}$$

$$V(\tilde{x}(t)) \leq \rho_e, \quad \forall t \in [t_k, t_{k+N}), \text{ if } t_k \leq t_s \text{ and } V(\hat{x}(t_k)) \leq \rho_e \tag{7e}$$

$$\frac{\partial V(\hat{x}(t_k))}{\partial x} g(\hat{x}(t_k))u(0) \leq \frac{\partial V(\hat{x}(t_k))}{\partial x} g(\hat{x}(t_k))k(\hat{x}(t_k)),$$
$$\text{if } t_k > t_s \text{ or } \rho_e < V(\hat{x}(t_k)) \leq \rho \tag{7f}$$

where N is the control prediction horizon, $L(\cdot,\cdot)$ is the general economic cost function that is maximized, $\tilde{x}$ is the predicted trajectory of the system with control inputs calculated by this LEMPC and $S(\Delta)$ is the family of piecewise continuous functions with period Δ. Constraint (7b) is the nominal system model used to predict the future evolution of the system subject to input constraint (7c). Constraint (7e) is active only for operation mode 1 which requires that the economic cost is maximized within the region defined by Ω_{ρ_e}. Constraint (7f) is active for operation mode 2 as well as operation mode 1 when the estimated system state is out of Ω_{ρ_e}. This constraint forces the LEMPC to generate control actions that drive the closed-loop system state towards the origin. The optimal solution to this optimization problem is denoted by $u^*(t|t_k)$, which is defined for $t \in [t_k, t_{k+N})$. The manipulated input of the LEMPC is defined as follows:

$$u(t) = u^*(t|t_k), \quad \forall t \in [t_k, t_{k+1}). \tag{8}$$

3.3. Stability analysis

The stability of LEMPC of Eq. (7) based on state estimates obtained following Eq. (6) is analyzed in this subsection. A set of sufficient conditions is derived under which the closed-loop system state trajectory is ensured to be maintained in the region Ω_ρ and ultimately bounded in an invariant set.

In the remainder of this subsection, we first present two propositions and then summarize the main results in a theorem. Proposition 1 characterizes the continuity property of the Lyapunov function V. Proposition 2 characterizes the effects of bounded state estimation error and process noise.

Proposition 1 (*C.f.* [20]). *Consider the Lyapunov function $V(\cdot)$ of system of Eq. (1). There exists a quadratic function $f_V(\cdot)$ such that*

$$V(x) \leq V(\hat{x}) + f_V(|x-\hat{x}|) \tag{9}$$

for all $x, \hat{x} \in \Omega_\rho$ with

$$f_V(s) = \alpha_4(\alpha_1^{-1}(\rho))s + M_v s^2 \tag{10}$$

where M_v is a positive constant.

Proposition 2. *Consider the systems*

$$\begin{aligned} \dot{x}_a(t) &= f(x_a) + g(x_a)u(t) + l(x_a)w(t) \\ \dot{x}_b(t) &= f(x_b) + g(x_b)u(t) \end{aligned} \tag{11}$$

with initial states $|x_a(t_0) - x_b(t_0)| \leq \delta_x$. There exists a function $f_W(\cdot,\cdot)$ such that

$$|x_a(t) - x_b(t)| \leq f_W(\delta_x, t - t_0) \tag{12}$$

for all $x_a(t), x_b(t) \in \Omega_\rho$ and $u \in \mathbb{U}$, $w \in \mathbb{W}$ with:

$$f_W(s,\tau) = \left(s + \frac{M_l\theta_w}{L_f + L_g u^{\max}}\right) e^{(L_f + L_g u^{\max})\tau} - \frac{M_l\theta_w}{L_f + L_g u^{\max}} \tag{13}$$

where L_f, L_g, M_l are positive constants associated with functions f, g, l.

Proof. Define $e_x = x_a - x_b$. The time derivative of e_x is given by:

$$\dot{e}_x(t) = f(x_a) + g(x_a)u(t) + l(x_a)w(t) - f(x_b) - g(x_b)u(t). \tag{14}$$

By continuity and the smooth property assumed for f, g, there exist positive constants L_f, L_g such that:

$$|\dot{e}_x(t)| \leq L_f|e_x(t)| + L_g u(t)|e_x(t)| + |l(x_a)w(t)|. \tag{15}$$

By the boundedness of x_a and the smooth property assumed for l as well as the boundedness of u and w, there exist positive constants M_l such that:

$$|\dot{e}_x(t)| \leq (L_f + L_g u^{\max})|e_x(t)| + M_l\theta_w. \tag{16}$$

Integrating the above inequality and taking into account that $|e_x(t_0)| \leq \delta_x$, it is obtained that:

$$|e_x(t)| \leq \left(\delta_x + \frac{M_l\theta_w}{L_f + L_g u^{\max}}\right) e^{(L_f + L_g u^{\max})(t-t_0)} - \frac{M_l\theta_w}{L_f + L_g u^{\max}}. \tag{17}$$

This proves Proposition 2. □

Theorem 1 summarizes the stability properties of the output feedback LEMPC. The stability of the closed-loop system is based on the observer F and controller k pair with F implemented continuously and k implemented in a sample-and-hold fashion.

Theorem 1. *Consider system of Eq. (1) in closed loop under LEMPC of Eq. (7) with state estimates determined following Eq. (6) based on an observer and controller pair satisfying the assumptions in Section 2.3. Let $\theta_w \le \theta_w^*$, $\theta_v \le \theta_v^*$, $\epsilon \in (\epsilon_L^*, \epsilon_U^*)$ and $|z(t_0) - x(t_0)| \le e_{m0}$. Also, let $\epsilon_w > 0$, $\Delta > 0$ and $\rho > \rho_1 > \rho_e > \rho^* > \rho_s > 0$ and $\kappa \ge 0$ satisfy the following conditions:*

$$\rho_e \le \rho - \max\{f_V(f_W(\delta_x, \Delta)) + f_V(\delta_x),\\ M\max\{\Delta, t_b\}\alpha_4(\alpha_1^{-1}(\rho))\}, \tag{18}$$

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + \left(L_V^f + L_V^g u^{\max}\right)\\ \times (M\Delta + \delta_x) + M_V^l\theta_w \le -\epsilon_w/\Delta \tag{19}$$

where $\delta_x = (\kappa L_h + 1)e_m + \kappa\theta_v$, L_V^f, L_V^g are Lipschitz constants associated with $\frac{\partial V}{\partial x}f$ and $\frac{\partial V}{\partial x}g$, respectively, M is a constant that bounds the time derivative of x (i.e., $|\dot{x}| \le M$) and M_V^l is a constant that bounds $\left|\frac{\partial V}{\partial x}l\right|$ for $x \in \Omega_\rho$. If $x(t_0) \in \Omega_{\rho_e}$, then $x(t) \in \Omega_\rho$ for all $t \ge t_0$ and is ultimately bounded in an invariant set.

Proof. In this proof, we consider $t \in [t_0, \max\{\Delta, t_b\})$ and $t \ge \max\{\Delta, t_b\}$ separately and prove that if the conditions stated in Theorem 1 are satisfied, the boundedness of the closed-loop state is ensured. Specifically, the proof consists of three parts. In *Part* I, we prove that the closed-loop state trajectory is contained in Ω_ρ for $t \in [t_0, \max\{\Delta, t_b\})$; in *Part* II, we prove that the boundedness of the closed-loop state trajectory under the first operation mode of the LEMPC for $t \ge \max\{\Delta, t_b\}$ when the initial state is within Ω_{ρ_e}; and in *Part* III, we prove that the closed-loop state trajectory is bounded for the first operation mode when the initial state is within $\Omega_\rho \setminus \Omega_{\rho_e}$ and is ultimately bounded in an invariant set for the second operation mode for $t \ge \max\{\Delta, t_b\}$.

Part I: First, we consider the case that $t \in [t_0, \max\{\Delta, t_b\})$. The closed-loop system state can be described as follows:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + l(x(t))w(t) \tag{20}$$

with $u(t)$ determined by the LEMPC with $\hat{x} = z$. The Lyapunov function of the state trajectory can be evaluated as follows:

$$V(x(t)) = V(x(t_0)) + \int_{t_0}^{t} \dot{V}(x(t))d\tau = V(x(t_0))\\ + \int_{t_0}^{t} \frac{\partial V(x(\tau))}{\partial x}\dot{x}(\tau)d\tau. \tag{21}$$

Using condition of Eq. (2) and the boundedness of $\dot{x}$ in the region of interest, if $x(t_0) \in \Omega_{\rho_e} \subset \Omega_{\rho_1} \subset \Omega_\rho$, it can be written for all $t \in [t_0, \max\{\Delta, t_b\})$ that:

$$V(x(t)) \le \rho_e + M\max\{\Delta, t_b\}\alpha_4(\alpha_1^{-1}(\rho)) \tag{22}$$

with M a positive constant which bounds $\dot{x}$ in Ω_ρ (i.e., $|\dot{x}| \le M$). If ρ_e is defined as in Theorem 1 (Eq. (18)), then

$$V(x(t)) < \rho, \quad \forall t \in [t_0, \max\{\Delta, t_b\}). \tag{23}$$

Part II: In this part, we consider the case that $t \ge \max\{\Delta, t_b\}$. In this case, we have that $|x(t) - z(t)| \le e_m$. We consider that the LEMPC is operated in the first operation mode and focus on the evolution of the state trajectory from t_k to t_{k+1}. Moreover, we consider $\tilde{x}(t_k) = \hat{x}(t_k) \in \Omega_{\rho_e}$. In this case, the LEMPC will optimize the economic cost while keeping $\tilde{x}(t)$ within Ω_{ρ_e}. We prove that if $\tilde{x}(t_k) \in \Omega_{\rho_e}$, then $x(t_{k+1}) \in \Omega_\rho$ and $\hat{x}(t_{k+1}) \in \Omega_\rho$.

From t_k to t_{k+1}, the worst case scenario is as shown in Fig. 1. At time t_k, the estimate of the state $\hat{x}(t_k) = \tilde{x}(t_k)$ is on the boundary of Ω_{ρ_e} while the actual system state is outside of Ω_{ρ_e} and on the boundary of another set Ω_{ρ_2} due to uncertainty in $\hat{x}$. The LEMPC will keep $\tilde{x}(t)$ inside Ω_{ρ_e} from t_k to t_{k+1}. However, due to the initial error in $\tilde{x}(t_k)$ and the presence of process noise, the actual system state $x(t_{k+1})$ may diverge to a point (on the boundary of Ω_{ρ_2} in Fig. 1) that is further away of Ω_{ρ_e}. The distance between $\tilde{x}(t_{k+1})$ and $x(t_{k+1})$, however, is bounded. Specifically, from Proposition 2, it can be obtained that:

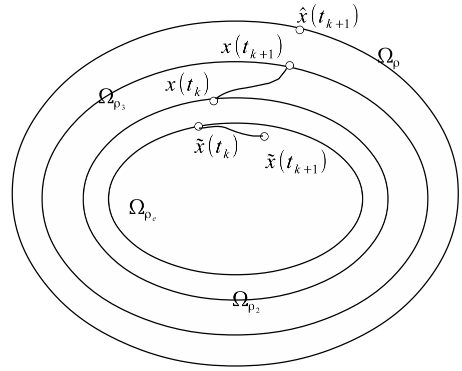

Fig. 1. Worst case scenario of the evolution of $\tilde{x}$ and x from t_k to t_{k+1} in the first operation mode.

$$|\tilde{x}(t_{k+1}) - x(t_{k+1})| \le f_W(|\hat{x}(t_k) - x(t_k)|, \Delta). \tag{24}$$

Recall that when $t \ge t_b$, all the estimates are provided by the RMHE. From the design of the RMHE, it can be written that:

$$|\hat{x}(t_k) - z(t_k)| \le \kappa|y(t_k) - h(z(t_k))|. \tag{25}$$

Using the relation that $|\hat{x} - x| \le |\hat{x} - z| + |z - x|$, it can be obtained that:

$$|\hat{x}(t_k) - x(t_k)| \le \kappa|y(t_k) - h(z(t_k))| + |z(t_k) - x(t_k)|. \tag{26}$$

Noticing that $|z(t_k) - x(t_k)| \le e_m$ and $|y(t_k) - h(z(t_k))| = |h(x(t_k)) + v(t_k) - h(z(t_k))|$, and using the Lipschitz property of h, the boundedness of v, the following inequality can be written:

$$|\hat{x}(t_k) - x(t_k)| \le (\kappa L_h + 1)e_m + \kappa\theta_v. \tag{27}$$

From Eqs. (24) to (27), it can be obtained that:

$$|\tilde{x}(t_{k+1}) - x(t_{k+1})| \le f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta). \tag{28}$$

This implies that if $\tilde{x}$ is maintained in Ω_{ρ_e}, the actual system state x is ensured to be within the set Ω_{ρ_2} with $\rho_2 = \rho_e + f_V(f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta))$ which can be obtained from Proposition 1.

Taking into account Eq. (27) again for $t = t_{k+1}$, the estimate of x obtained at t_{k+1} could be outside the region Ω_{ρ_2} but the distance is bounded as follows:

$$|\hat{x}(t_{k+1}) - x(t_{k+1})| \le (\kappa L_h + 1)e_m + \kappa\theta_v. \tag{29}$$

In order to ensure that $\hat{x}(t_{k+1})$ is within Ω_ρ which is required for the feasibility of the LEMPC of Eq. (7), the following inequality should be satisfied:

$$\rho \ge \rho_e + f_V(f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta))\\ + f_V((\kappa L_h + 1)e_m + \kappa\theta_v) \tag{30}$$

which implies that ρ_e should be picked to satisfy the following condition:

$$\rho_e \le \rho - f_V(f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta))\\ - f_V((\kappa L_h + 1)e_m + \kappa\theta_v). \tag{31}$$

If ρ_e is defined as in Theorem 1 (Eq. (18)), the above condition is satisfied.

Part III: Next, we consider the case that $\hat{x}(t_k) = \tilde{x}(t_k) \in \Omega_\rho \setminus \Omega_{\rho_e}$ in the first operation mode or $t_k \ge t_s$ (i.e., the second operation mode). In this case, constraint (7f) will be active. The time derivative of the Lyapunov function can be evaluated as follows:

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x}(f(x(t)) + g(x(t))u(t_k) + l(x(t))w(t)) \quad (32)$$

for $t \in [t_k, t_{k+1})$. Adding and subtracting the term $\frac{\partial V(\hat{x}(t_k))}{\partial x}(f(\hat{x}(t_k)) + g(\hat{x}(t_k))u(t_k))$ to/from the above equation and considering constraint (7f) as well as condition (2), it is obtained that:

$$\dot{V}(x(t)) \leq -\alpha_3(|\hat{x}(t_k)|) + \frac{\partial V(x(t))}{\partial x}(f(x(t)) + g(x(t))u(t_k) + l(x(t))w(t)) - \frac{\partial V(\hat{x}(t_k))}{\partial x}(f(\hat{x}(t_k)) + g(\hat{x}(t_k))u(t_k)). \quad (33)$$

By the smooth properties of V, f, g and l, the boundedness of x, u and w, there exist positive constants L_V^f, L_V^g, M_V^l such that:

$$\dot{V}(x(t)) \leq -\alpha_3(|x(t_k)|) + \left(L_V^f + L_V^g u^{\max}\right) \times |x(t) - \hat{x}(t_k)| + M_V^l \theta_w \quad (34)$$

for all $x \in \Omega_\rho$. Noticing that $|x(t) - \hat{x}(t_k)| \leq |x(t) - x(t_k)| + |x(t_k) - \hat{x}(t_k)|$, it is obtained that:

$$|x(t) - \hat{x}(t_k)| \leq |x(t) - x(t_k)| + (\kappa L_h + 1)e_m + \kappa\theta_v. \quad (35)$$

By the continuity and smoothness properties of f, g, l and the boundedness of x, u and w, there exists positive constant M such that $|\dot{x}| \leq M$. From the above inequalities, it can be obtained that:

$$\dot{V}(x(t)) \leq -\alpha_3(\alpha_2^{-1}(\rho_s)) + \left(L_V^f + L_V^g u^{\max}\right) \times (M\Delta + (\kappa L_h + 1)e_m + \kappa\theta_v) + M_V^l \theta_w \quad (36)$$

for all $x \in \Omega_\rho \setminus \Omega_{\rho_s}$. If condition (19) is satisfied, it can be obtained from Eq. (36) that:

$$V(x(t_{k+1})) \leq V(x(t_k)) - \epsilon_w. \quad (37)$$

This means that $V(x)$ decreases in the first operation mode if $\tilde{x}(t_k) = \hat{x}(t_k)$ is outside of Ω_{ρ_e}. This implies that $\hat{x}(t_k)$ will eventually enter Ω_{ρ_e}. This also implies that in the second operation mode, $V(x)$ decreases every sampling time and x will eventually enter Ω_{ρ_s}. Once $x \in \Omega_{\rho_s} \subset \Omega_{\rho^*}$, it will remain in Ω_{ρ^*} because of the definition of ρ^*. This proves Theorem 1. □

Remark 5. Part I of Theorem 1 essentially treats the input as a disturbance. Given that the input and the noise are bounded, a bound is derived for how large the Lyapunov function may increase over time t_b (which is small). This follows from the fact that the initial estimation error of the deterministic observer and actual state are both bounded in a region containing the origin.

Remark 6. Parts II and III prove that if the current state $x(t_k) \in \Omega_\rho$ and if the current estimate $\hat{x}(t_k) \in \Omega_\rho$, the actual state and the estimated state at the next sampling period are also within Ω_ρ. Since Part II considers mode 1 operation of the LEMPC (i.e., may dictate time-varying operation), the worst case scenario is considered (Fig. 1). Part III considers mode 2 operation of the LEMPC (i.e., convergence of the state to a small neighborhood of the origin). While the theoretical developments and corresponding bounding inequalities contained in this section are conservative, they do provide valuable insight and guidelines for selecting the parameters of the state feedback controller $k(x)$, the deterministic observer, the RMHE, and the output feedback LEMPC such that the closed-loop system of Eq. (1) under the output feedback LEMPC of Eq. (7) is stable with bounded process and measurement noise.

Remark 7. One could potentially apply the RMHE for t_0 to t_b instead of using the deterministic observer. However, it is difficult to prove closed-loop stability for this case owing to the fact that the estimation error may not have decayed to a small value over this time period with the RMHE (i.e., it is difficult to show that the RMHE satisfies the observability assumptions of Section 2.3).

Table 1
Parameter values.

Symbol	Description	Value
F	Inlet flow rate	$5.0\ \mathrm{m^3\ h^{-1}}$
T_0	Inlet temperature	300 K
V	Reactor volume	$1.0\ \mathrm{m^3}$
Q_s	Heat rate supplied to the reactor	$1.73 \times 10^5\ \mathrm{kJ\ h^{-1}}$
ΔH	Heat of reaction	$1.16 \times 10^4\ \mathrm{kJ\ kmol^{-1}}$
k_0	Pre-exponential factor	$13.93\ \mathrm{h^{-1}}$
E	Activation energy	$5.0 \times 10^3\ \mathrm{kJ\ kmol^{-1}}$
C_p	Heat capacity	$0.231\ \mathrm{kJ\ kg^{-1}\ K^{-1}}$
R	Gas constant	$8.314\ \mathrm{kJ\ kmol^{-1}\ K^{-1}}$
ρ_L	Liquid density	$1000\ \mathrm{kg\ m^3}$
C_{As}	Steady-state reactant concentration	$2.44\ \mathrm{kmol\ m^{-3}}$
T_s	Steady-state temperature	321.95 K
C_{A0s}	Steady-state inlet reactant concentration	$5.0\ \mathrm{kmol\ m^{-3}}$

4. Application to a chemical process example

Consider a well-mixed, non-isothermal continuous stirred tank reactor (CSTR) where a second-order reaction of the form $A \to B$ takes place. The species B is the desired product. Since the reaction is endothermic and irreversible, thermal energy is supplied to the reactor through a heating jacket at a constant heat rate Q_s. The feedstock consists of the reactant A in an inert solvent and does not contain any of the product B. The feedstock volumetric flow rate F and temperature T_0 are constant; while, the inlet concentration C_{A0} can be manipulated. Due to the constant volumetric flow rate and the assumption that the liquid in the reactor has a constant density ρ_L, the liquid volume V in the CSTR is constant. Applying first principles and other standard modeling assumptions, a dynamic model of the CSTR can be derived and is given by the following ODEs:

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{A0} - C_A) - k_0 e^{-E/RT} C_A^2 \quad (38a)$$

$$\frac{dT}{dt} = \frac{F}{V}(T_0 - T) - \frac{\Delta H k_0}{\rho_L C_p} e^{-E/RT} C_A^2 + \frac{Q_s}{\rho_L C_p V} \quad (38b)$$

where C_A and T denote the reactant concentration and temperature in the reactor, respectively, C_p denotes the heat capacity of the liquid in the CSTR, and ΔH, k_0, and E denote the enthalpy, pre-exponential factor and activation energy of the reaction, respectively. The process parameters are given in Table 1. The two states are C_A and T which are denoted as x_1 and x_2, respectively, the input u is the inlet reactant concentration (i.e., $u = C_{A0}$) and the output measurement y is the reactor temperature which is measured continuously. For the input u, the available actuation is given by the following convex set: $\mathbb{U} = \{u(t) \in \mathbb{R} \,|\, 0.5\ \mathrm{kmol\ m^{-3}} \leq u(t) \leq 7.5\ \mathrm{kmol\ m^{-3}}\}$.

The control objective is to manipulate the inlet reactant concentration C_{A0} in an economically optimal manner to maximize the reaction rate. To accomplish this objective, an LEMPC scheme (i.e., mode 1 of the scheme detailed in [4,8]) is formulated with the following economic cost function:

$$L(x, u) = k_0 e^{-E/Rx_2} x_1^2. \quad (39)$$

The input trajectory that maximizes this cost dictates feeding the maximum allowable material to the reactor for all times. However, feeding the maximum amount of material to the reactor for all times may not be practical from an economic perspective. Instead, we consider the economically optimal time-varying distribution of a fixed amount of reactant material to the reactor that maximizes the reaction rate. Given that the inlet flow rate is constant, the constraint added to the formulation of the LEMPC scheme that

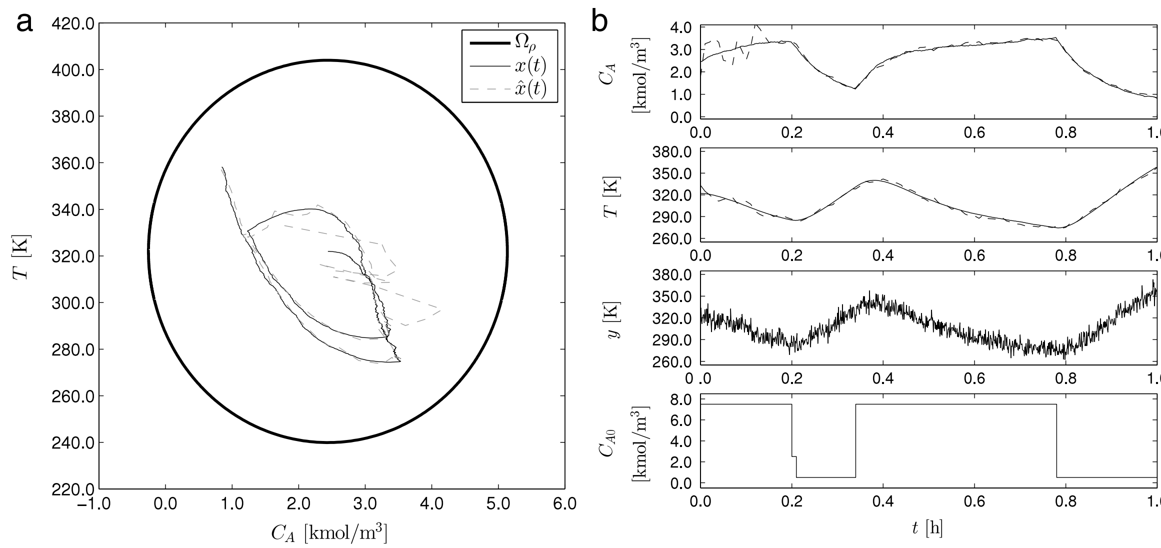

Fig. 2. The evolution of the closed-loop CSTR with the RMHE scheme used to compute an estimate of the state from the noisy output measurement y and with the LEMPC scheme used to compute the control action from the state estimate provided by the RMHE shown in state-space (left plot) and as a function of time (right plots). The solid line is the actual state trajectory $x(t)$; while, the dashed line is the estimated state $\hat{x}(t)$.

limits the available amount of reactant material has the following integral form:

$$\frac{1}{t_f}\int_{t_0}^{t_f} C_{A0}(\tau)\,d\tau = C_{A0s} \tag{40}$$

which imposes the time-averaged reactant material A be equal to C_{A0s} over a finite operating window t_f and thus, the time-averaged reactant material usage is C_{A0s} over the entire length of operation which could be a large multiple of t_f. There is only one feasible steady-state in the operating range of interest that satisfies the reactant material constraint of Eq. (40). Therefore, the open-loop asymptotically stable steady-state denoted as x_s is (trivially) the economically optimal steady-state and is given in Table 1. Using the economic cost function of Eq. (39), the dynamic model of Eq. (38), the constraints on control action, and the reactant material constraint of Eq. (40), we formulate an LEMPC scheme which is given by the following optimization problem:

$$\begin{aligned}
&\underset{u\in S(\Delta)}{\text{maximize}} \int_{t_k}^{t_f} \left[k_0 e^{-E/R\tilde{x}_2(\tau)}\tilde{x}_1^2(\tau)\right] d\tau\\
&\text{subject to } \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0)\\
&\tilde{x}(t_k) = \hat{x}(t_k)\\
&0.5 \le u(t) \le 7.5, \quad \forall\, t \in [t_k,\ t_f)\\
&\frac{1}{t_f}\left(\int_{t_0}^{t_k} u^*(\tau)\,d\tau + \int_{t_k}^{t_f} u(\tau)\,d\tau\right) = 5.0\\
&\tilde{x}(t)^T P\tilde{x}(t) \le \rho
\end{aligned} \tag{41}$$

where the sampling period $\Delta = 0.01$ h, the positive definite matrix in the quadratic Lyapunov function is $P = \text{diag}([110.11,\ 0.12])$ and the bound on the Lyapunov function $\rho = 800$ (i.e., value of level set in which process operation is constrained by the EMPC). Since the integral material constraint is enforced of over a finite operating window $t_f = 1$ h, the LEMPC of Eq. (41) is formulated with a shrinking horizon: $N_k = 100 - k$ (k is reset to zero at the beginning of each operating window). The optimization problem of Eq. (41) is initialized through a state estimate obtained at sampling time t_k.

To estimate the process state from the noisy temperature measurements, the RMHE scheme is used. The weighting matrices of the RMHE are given by $Q_e = \text{diag}([\sigma_{w_1}^2\ \sigma_{w_2}^2])$ and $R_e = \sigma_v^2$ where σ denotes the standard deviation of the process or measurement noise. The design parameter of the RMHE is $\kappa = 0.4$, the sampling period is the same as the LEMPC (i.e., $\Delta_e = 0.01$ h), and the estimation horizon of the RMHE is $N_e = 15$. The robust constraint of the RMHE is based on a high-gain observer as in [8]. For the first 15 sampling periods, the high-gain observer is used to provide the LEMPC with a state estimate. At each subsequent sampling periods, the LEMPC is initialized using the state estimate from the RMHE. To solve the optimization problems of the LEMPC and the RMHE at each sampling period, the open-source software Ipopt [21] was used. The process model of Eq. (38) is numerically simulated using an explicit Euler integration method with integration step $h_c = 10^{-3}$ h. To simulate the process and measurement noise, new random numbers are generated and applied over each integration step. The process noise is assumed to enter the system additively to the right-hand side of the ODEs of Eq. (38). The random numbers are generated from a zero-mean, bounded Gaussian distribution.

Square bounds of $w_{\max} = [20.0\ 50.0]$ and $v_{\max} = 20.0$ (i.e., $w_1 \in [-20.0,\ 20.0]$) are used to bound the process and measurement noise, respectively, and the standard deviation of the noise terms are $\sigma_w = [7.0\ 20.0]$ and $\sigma_v = 7.0$, respectively. The CSTR is initialized at $x_0^T = [2.44\,\text{kmol m}^{-3}\ \ 320.0\,\text{K}]$ (i.e., the economically optimal steady-state). The evolution of the closed-loop CSTR under the RMHE and LEMPC is shown in Fig. 2. Initially, the estimated concentration is significantly affected by the measurement noise which is expected since the estimate comes from the high-gain observer. After the RMHE is activated, the estimated state trajectories are nearly overlapping with the actual state trajectories. Furthermore, the LEMPC computes a periodic-like input profile to optimize the process economics over the 1 h period of operation.

The average reaction rate over this 1 h period of operation is 13.59 kmol m^{-3}. If, instead, the CSTR was maintained at the economically optimal steady-state (x_0) without process and measurement noise (nominal operation), the average reaction rate over this 1 h operation would be 12.80 kmol m^{-3}. This is a 6.2% improvement in the economic cost of the closed-loop system under RMHE/LEMPC with process and measurement noise over nominal steady-state operation. We note that the economic performance of the closed-loop system under LEMPC with full state feedback and nominal operation over 1 h operation is 13.60 kmol m^{-3} which is a 6.3% economic performance improvement. To assess the estimation performance of the RMHE, another simulation is performed

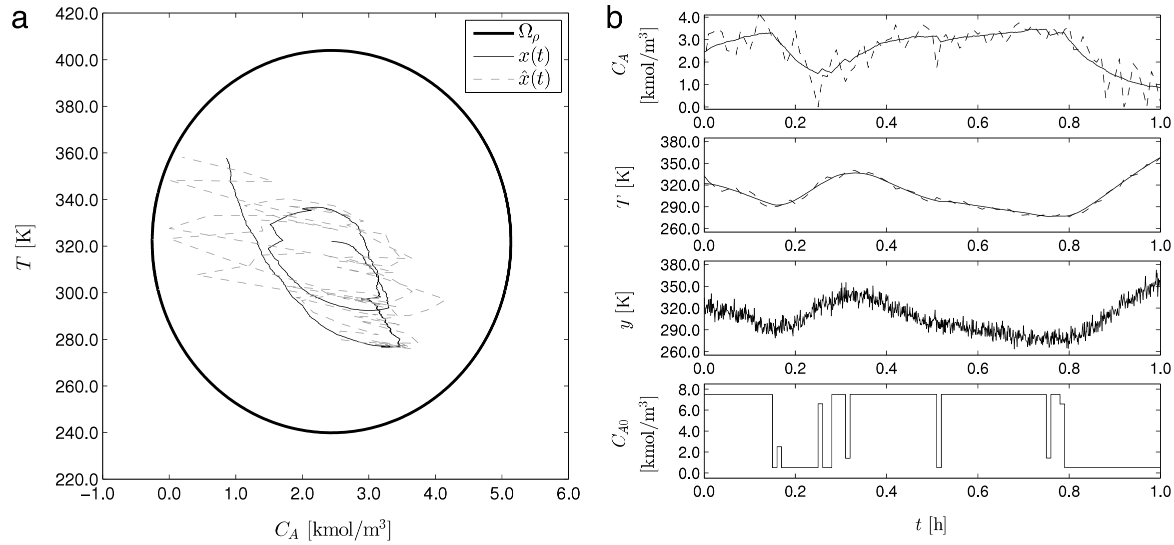

Fig. 3. The evolution of the closed-loop CSTR with the high-gain observer of [8] used to compute an estimate of the state from the noisy output measurement y and with the LEMPC scheme used to compute the control input from the state estimate provided by the high-gain observer shown in state-space (left plot) and as a function of time (right plots). The solid line is the actual state trajectory $x(t)$; while, the dashed line is the estimated state $\hat{x}(t)$.

with the same realization of the process and measurement noise and with the high-gain observer presented in [8]. The evolution of the closed-loop CSTR under the high-gain observer and LEMPC is shown in Fig. 3. Not only does the noise impact the estimates provided by the high-gain observer in this case, but also, it impacts the computed input profile (Fig. 3(b)). Comparing Figs. 2 and 3, the RMHE is able to provide estimates of the state within a small neighborhood of the actual process states; while, the high-gain observer is not able to estimate the concentration as well as the RMHE. Furthermore, since the RMHE provides better (smoother) estimates of the states, the operation of the closed-loop system under RMHE/LEMPC is smoother which can be observed in the input trajectories.

Several additional closed-loop simulations with various bounds and standard deviations on the process and measurement noise and initial conditions are performed to further assess the estimation performance of RMHE compared to the one of the high-gain observer of [8]. An estimation performance index which is defined as

$$J = \sum_{k=0}^{99} \left| \hat{x}(t_k) - x(t_k) \right|_S^2 \tag{42}$$

is used to assess the estimation performance where the matrix S is a positive definite weighting matrix given by $S = \mathrm{diag}([50\ 1])$. The matrix S has been chosen to account for the different numerical ranges of the concentration and temperature. In addition to the assessment on the estimation performance, the total economic performance index over the length of the simulation is defined as

$$J_e = \frac{1}{99} \sum_{k=0}^{100} k_0 e^{-E/RT(t_k)} C_A^2(t_k) \tag{43}$$

which is the time-averaged reaction rate over the simulation. From the results displayed in Table 2, the RMHE consistently provides significantly better estimates of the state than the high-gain observer which demonstrates the robustness of the RMHE to process and measurement noise. However, the estimation performance does not translate into a significant closed-loop average economic performance improvement of the closed-loop system with RMHE/LEMPC over the closed-loop system with the high-gain observer and LEMPC. This relationship is due to the fact that the closed-loop average economic performance over one operation period is not strongly dependent on the initial condition of the LEMPC optimization problem (i.e., $\hat{x}(t_k)$) for this particular example. In other words, providing the LEMPC with an estimate of the actual state anywhere in a neighborhood around the actual state will return the optimal input trajectory that leads to nearly the same economic cost for the closed-loop systems. For systems that are more sensitive to the estimate of the current state, it is expected that there would also be improved closed-loop economic performance with RMHE/LEMPC in addition to improved estimation performance.

Table 2
Estimation performance comparison of the closed-loop CSTR with various bounds and standard deviation of the disturbances and noise and initial conditions under the high-gain observer and LEMPC and under the RMHE and LEMPC (ordered below by increasing bounds and standard deviation). The J column refers to the performance index of Eq. (42), the "SSE of C_A" column denotes the sum of squared errors of the concentration C_A estimation, and the J_e column refers to the economic performance index of Eq. (43).

	High-gain observer			RMHE		
	J	SSE of C_A	J_e	J	SSE of C_A	J_e
1	310.5	4.450	13.04	104.0	1.277	13.04
2	528.5	7.781	14.19	310.1	4.169	14.19
3	271.6	3.669	13.47	88.1	0.440	13.47
4	506.4	7.066	13.06	181.9	1.476	13.07
5	583.2	8.097	14.20	354.4	3.888	14.20
6	482.1	6.397	13.48	137.7	0.372	13.48
7	592.4	7.821	13.09	257.1	1.734	13.09
8	572.8	8.519	14.23	252.6	3.425	14.23
9	616.4	8.579	13.51	168.6	1.126	13.52
10	992.0	13.700	13.00	429.9	4.355	13.08
11	1079.8	14.871	14.14	888.7	12.076	14.21
12	1012.5	14.304	13.42	552.0	5.817	13.43
13	1643.6	22.606	13.02	665.3	3.523	12.99
14	1758.5	23.396	14.24	771.2	5.492	14.27
15	1591.0	21.740	13.51	561.5	1.717	13.55

5. Conclusions

In this work, we considered the design of an output feedback EMPC for a class of nonlinear systems with bounded process and measurement noise. In order to achieve fast convergence of the state estimate to the actual system state as well as the robustness of the estimator to measurement and process noise, a high-gain observer and a RMHE scheme were used to estimate the system states. In particular, the high-gain observer was first applied for a small time period with continuous output measurements to drive the estimation error to a small value. Once the estimation error had converged to a small neighborhood of the origin, the RMHE was activated to provide more accurate and smoother state estimates. In the design of the RMHE, the high-gain observer was used to provide reference estimates based on which confidence regions were calculated. The RMHE was only allowed to optimize the estimates within the confidence regions. The output feedback EMPC was designed via Lyapunov techniques based on state estimates provided by the high-gain observer and the RMHE. The application of the proposed design to a chemical reactor demonstrated the applicability and effectiveness of the proposed approach and its ability to deal with measurement noise.

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