Economic Model Predictive Control of Nonlinear Time-Delay Systems: Closed-Loop Stability and Delay Compensation

Matthew Ellis
Dept. of Chemical and Biomolecular Engineering, University of California, Los Angeles, CA 90095-1592

Panagiotis D. Christofides
Dept. of Chemical and Biomolecular Engineering, University of California, Los Angeles, CA 90095-1592

Dept. of Electrical Engineering, University of California, Los Angeles, CA 90095-1592

DOI 10.1002/aic.14964
Published online August 3, 2015 in Wiley Online Library (wileyonlinelibrary.com)

Closed-loop stability of nonlinear time-delay systems under Lyapunov-based economic model predictive control (LEMPC) is considered. LEMPC is initially formulated with an ordinary differential equation model and is designed on the basis of an explicit stabilizing control law. To address closed-loop stability under LEMPC, first, we consider the stability properties of the sampled-data system resulting from the nonlinear continuous-time delay system with state and input delay under a sample-and-hold implementation of the explicit controller. The steady-state of this sampled-data closed-loop system is shown to be practically stable. Second, conditions such that closed-loop stability, in the sense of boundedness of the closed-loop state, under LEMPC are derived. A chemical process example is used to demonstrate that indeed closed-loop stability is maintained under LEMPC for sufficiently small time-delays. To cope with performance degradation owing to the effect of input delay, a predictor feedback LEMPC methodology is also proposed. The predictor feedback LEMPC design employs a predictor to compute a prediction of the state after the input delay period and an LEMPC scheme that is formulated with a differential difference equation (DDE) model, which describes the time-delay system, initialized with the predicted state. The predictor feedback LEMPC is also applied to the chemical process example and yields improved closed-loop stability and economic performance properties. © 2015 American Institute of Chemical Engineers *AIChE J*, 61: 4152–4165, 2015
Keywords: economic model predictive control, nonlinear time-delay systems, process control, process economics, process optimization

Introduction

Within the context of control of chemical process systems, time-delays resulting from computation and communication delays are enviable albeit these time-delays may be small and insignificant depending on the magnitude of the time-delay relative to the time-constants of the process dynamics. From a modeling perspective, time-delays are often employed to describe and/or to approximate the dynamics of transportation of material through the process system (e.g., flow through pipes), control actuator dynamics, measurement sensor dynamics, and high-order dynamic behavior. Thus, robustness with respect to closed-loop stability and performance of control systems to time-delays is an important consideration. Moreover, many chemical process systems have significant nonlinear behavior owing to complex reaction mechanisms, Arrhenius reaction-rate dependence on temperature, and thermodynamic relationships which adds additional complexity in considering the closed-loop behavior of the system resulting from a nonlinear time-delay system under a control law.

Correspondence concerning this article should be addressed to P. D. Christofides at pdc@seas.ucla.edu.

© 2015 American Institute of Chemical Engineers

Dynamic models of systems that involve nonlinearities and time-delays are systems of nonlinear differential difference equations (DDEs). Systems described by DDEs are fundamentally different than systems described by ordinary differential equations (ODEs) (see e.g., Ref. 1 for more details on this point). One important difference is that a dynamic system with an arbitrarily small delay is an infinite-dimensional system even though the dimension of the state vector may be finite. For nonlinear DDEs, there are typically two approaches employed to analyze stability that are analogous to Lyapunov stability theory employed to assess stability of equilibria and solutions of ODEs. The first method uses Lyapunov–Krasovski functionals, which is the direct analog to Lyapunov functions for ODEs, and the second method uses Lyapunov–Razumikhin functions.[1] While Razumikhin theorems are typically more conservative than Krasovski theorems, Razumikhin theorems require the construction of a function which is typically less challenging than the construction of a functional. Various extensions of stability theory for DDE systems exist including input-to-state stability using Razumikhin-type arguments and extending the notion of control Lyapunov functions to control Lyapunov–Krasovski functionals and control Lyapunov–Razumikhin functions.[2–6]

One approach to design a controller for a time-delay system is to neglect the delays in the model used for controller design and employ controller design methods used for systems described by ODEs. This may lead to acceptable closed-loop properties especially when the time-delays are small. However, employing these methods, in general, may pose unacceptable limitations on the achievable control quality and performance (e.g., sluggish response, oscillations) and even instability.[7] Perhaps, one of the most well-known results to cope with time-delay in the control input or sensor measurement is the classical Smith predictor structure designed for linear time-delay systems which eliminates the delay from the characteristic polynomial of the closed-loop system and allows for larger controller gains to be used.[8] Many extensions of the Smith predictor exist in the literature including extensions to nonlinear systems.[9,10] With respect to control design and/or input delay compensation for nonlinear systems, many results exist in the literature including state and output feedback control designs for nonlinear DDE systems,[11–14] input delay compensation for nonlinear systems,[15,16] and nonlinear sampled-data systems with time-delays[17–20] (see, also, the reviews[6,21] for more references on control of time-delay systems).

In addition to nonlinearities and time-delays, another important consideration within the context of chemical process control is process constraints and performance considerations. To deal with constraints and performance considerations, model predictive control (MPC) is often used because it can control multiple-input multiple-output nonlinear systems in an optimal manner while accounting for state and control constraints.[22] The traditional MPC methodology (i.e., tracking MPC) is formulated with a quadratic stage cost that attains a global minimum at a desired set point. Recently, economic MPC (EMPC), which is formulated with a stage cost representing the process economics, has been proposed which merges economic considerations with feedback control[23–26] (see, also, the survey papers[27,28] and the references therein). The use of a general stage cost introduces additional complexity in assessing closed-loop stability. Nonetheless, key economic performance benefits have been demonstrated.[28] While there has been some work completed on tracking MPC of nonlinear time-delay systems,[29–33] to the best of our knowledge, no work on EMPC for time-delay systems has been completed.

In this work, we first consider the robustness of Lyapunov-based EMPC (LEMPC),[25] formulated with an ODE model, for nonlinear systems with state and input delays in the sense that closed-loop stability (to be made precise below) will be maintained when the state and input delays are sufficiently small. LEMPC is an EMPC methodology with stability constraints designed from an explicit stabilizing nonlinear control law. The nonlinear control law is designed for the continuous-time ODE system using standard techniques.[13,34,35] However, EMPC is a control methodology that typically applies control actions in sample-and-hold fashion. Thus, to address closed-loop stability under LEMPC for the time-delay system, we first consider the closed-loop stability of the nonlinear sampled-data time-delay system resulting from the continuous-time delay system under the nonlinear control law applied in a sample-and-hold fashion. Leveraging the aforementioned results, closed-loop stability is shown for the closed-loop time-delay system under LEMPC when the time-delays are sufficiently small. Using a chemical process example, we demonstrate that the LEMPC maintains closed-loop stability when the time-delays in both inputs and states are sufficiently small. To address economic performance deterioration due to time-delays, in the second part, we develop a predictor feedback-based LEMPC scheme, formulated with a DDE model, that compensates for the effect of the input delay.

Preliminaries

Notation

The notation $x(t) \in \mathbb{R}^n$ is a time dependent vector and $x_d(t) : [-t_d, 0] \to \mathbb{R}^n$ is a time-dependent function where $x_d(t)(\tau) := x(t+\tau)$ for $\tau \in [-t_d, 0]$. The explicit dependence of t on $x(t)$ and $x_d(t)$ may be omitted and we may simply write x and x_d when convenient. The symbol $|\cdot|$ denotes the Euclidean norm of a real vector and the symbol $||\cdot||$ denotes the norm of a function ϕ

$$||\phi|| := \max_{a \le s \le b} |\phi(s)|$$

where $\phi \in C([a,b], \mathbb{R}^n)$ and $C([a,b], \mathbb{R}^n)$ is a space of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$ (see Appendix for more technical details regarding this space). The floor and ceiling functions, denoted as $\lfloor a \rfloor$ and $\lceil a \rceil$ ($a \in \mathbb{R}$), respectively, are the largest integer not greater than a and the smallest integer not less than a, respectively. The symbol B_δ denotes a norm-ball of radius δ, and the symbol Ω_ρ is a level set or level surface of a scalar positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ (i.e., $\Omega_\rho := \{x \in \mathbb{R}^n : V(x) \le \rho\}$). The family of piecewise constant, right-continuous functions with period Δ (defined over the appropriate interval) is denoted as $S(\Delta)$. For a detailed summary of the other notation and mathematical concepts used in this article, the interested reader is referred to the Appendix.

Class of nonlinear time-delay systems

The class of nonlinear time-delay systems considered in the present work are described by a system of differential difference equations (DDEs), which are also commonly referred to as delay differential equations, and have the following form

$$\dot{x}(t) = f(x(t), x(t-d_1), u(t-d_2)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{U} \subset \mathbb{R}^m$ is the bounded control input, $d_1 \ge 0$ and $d_2 \ge 0$ are the state and input delays, respectively, and f is a locally Lipschitz vector function of its arguments with $f(0,0,0)=0$. The set that bounds the available control action is given by $\mathbb{U} := \{u \in \mathbb{R}^m : |u_i| \le u_{\max,i},\ i = 1, \ldots, m\}$. The initial time is taken to be zero ($t_0=0$), the initial data is given by the function ϕ_x where $\phi_x \in C([-d_1, 0], \mathbb{R}^n)$, and the initial input function, which is denoted as ϕ_u, is defined over the interval $[-d_2, 0]$, takes values in $\mathbb{U}$ and is therefore, bounded. Moreover, ϕ_u is assumed to be piecewise continuous over its domain. Full state feedback of the system of Eq. 1 is assumed.

We will design a controller for the system of Eq. 1 on the basis of the ODE model (i.e., Eq. 1 with $d_1=d_2=0$). We impose the following stabilizability assumption on the system of Eq. 1. For the system of Eq. 1 with $d_1=d_2=0$, we assume that there exists a locally Lipschitz control law $h_c : \mathbb{R}^n \to \mathbb{U}$ where $h_c(0)=0$ such that the origin of the system $\dot{x}=f(x, x, h_c(x))$ is asymptotically stable. This assumption implies that there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that the following inequalities hold[35]

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{2a}$$

$$\frac{\partial V}{\partial x}(x) f(x, x, h_c(x)) \leq -\alpha_3(|x|) \tag{2b}$$

$$\left|\frac{\partial V}{\partial x}(x)\right| \leq \alpha_4(|x|) \tag{2c}$$

for all $x \in D \subset \mathbb{R}^n$ where D is an open neighborhood of the origin and $\alpha_i \in \mathcal{K}$, $i = 1, 2, 3, 4$. Standard techniques exist for designing a stabilizing control law for various classes of continuous-time nonlinear systems (see, for instance, Refs. 13, 34, and 35 as well as the references contained therein). In the remainder, we will use the notation $h_c(x)$ when referring to the stabilizing control law.

Remark 1. *While we restrict our focus to nonlinear systems described by differential difference equations with constant delay, the extension of the results to systems described by more general functional differential equations, bounded time-varying delays, and multiple state and input delays is conceptually possible.*

Controller emulation design

Given the stability properties of the control law $h_c(x)$, one may consider designing stability constraints based on the stabilizing control law $h_c(x)$ to be imposed within a MPC framework. However, MPC is typically implemented in a sample-and-hold fashion; that is, after receiving a state feedback measurement, a control action is computed and applied over a finite-time interval, called the sampling period. The control action for the next sampling period is computed from a new state measurement at the next sampling time.

To design stabilizing constraints based on the control law $h_c(x)$ for MPC, one must first consider the stabilizing properties of the control law $h_c(x)$ when applied to the system of Eq. 1 (with time-delays) in a sample-and-hold fashion. When a controller is applied to the continuous-time system of Eq. 1 in a sample-and-hold fashion, the resulting closed-loop system is a sampled-data time-delay system and is given by

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-d_1), h_\Delta(x, t-d_2)) \\ h_\Delta(x, t) &= h_c(x(k\Delta)), \ \ t \in [k\Delta, (k+1)\Delta) \\ k &= 0, \ 1, \ 2, \ldots \end{aligned} \tag{3}$$

where the $\Delta > 0$ is the sampling period and ϕ_x is the initial data. The initial data is assumed to be continuous and defined over the appropriate interval. We assume that the initial input function is determined by $h_\Delta(x, s-d_2)$ for $s \in [0, d_2]$ which implies that the initial data must be defined over a prolonged interval when $\Delta + d_2 > d_1$. Provided the time-delays and sampling rate are sufficiently small and the initial data are restricted to a ball of radius δ_2 (i.e., $\phi_x(s) \in B_{\delta_2}$ for all $s \in [-\max\{d_1, \Delta + d_2\}, 0]$), it can be shown that the origin of the closed-loop system is uniformly input-to-state stable (ISS) which is stated in the proposition below. The result follows the ideas of Ref. 17. Specifically, the system of Eq. 3 is treated as a perturbed form of the closed-loop system $\dot{x} = f(x, x, h_c(x))$. By employing Razumikhin-type arguments, we consider solutions to the sampled-data system where the solution at time t may be bounded by the state at time t to show that the origin of the closed-loop system is locally uniformly ISS.

Proposition 1. *Consider the sampled-data system of Eq.* 3. *Let* $\tilde{t}_d = \max\{d_1, \Delta + d_2\}$, *then there exists a* t_d^* *such that for all* $\tilde{t}_d \in (0, t_d^*)$, *the origin is uniformly ISS with offset* δ_1 *and restriction* δ_2 *where* $\delta_1 < \delta_2$ *and* $\{x \in \mathbb{R}^n : |x| \leq \delta_2\} \subset D$.

For appropriately chosen offset δ_1 and restriction δ_2, one can find a level set of the Lyapunov (or Lyapunov–Razumikhin) function, Ω_ρ, such that $B_{\delta_1} \subset \Omega_\rho \subseteq B_{\delta_2}$ and for all initial data satisfying $\phi_x(s) \in \Omega_\rho$ for all $s \in [-\tilde{t}_d, 0]$ the closed-loop state trajectory of the system of Eq. 3 under the control law $h_c(x)$ applied in a sample-and-hold fashion will be (uniformly) ultimately bounded in a ball of radius δ_1. Moreover, the closed-loop state is always bounded in Ω_ρ as a consequence of the construction of Ω_ρ and Proposition 1. This gives practical stability of the origin of the system of Eq. 3. The set Ω_ρ is an estimate of the region of attraction, and is used to design a region constraint in LEMPC. Thus, Ω_ρ is referred to as the stability region for the remainder.

Remark 2. *While the system of Eq.* 1 *is autonomous and thus, explicitly referring to the stability properties as uniform may not appear to be needed, the sampled-data system of Eq.* 3 *is a periodically time-varying system and hence, the fact that stability holds uniformly is needed.*

Robustness of LEMPC to Small Time-Delays

In this section, the closed-loop stability of the time-delay system of Eq. 1 under LEMPC, formulated with an ODE model of the system of Eq. 1, is analyzed. The sequence $\{t_k\}_{k \geq 0}$ denotes the sequence of sampling times where $t_k := k\Delta$ and $k \in \mathbb{N}$.

Formulation and implementation

We assume that the system of Eq. 1 is equipped with a stage cost, $l_e : \Omega_\rho \times \mathbb{U} \to \mathbb{R}$, that is a measure of the instantaneous economic cost of Eq. 1. The economic stage cost is assumed to be continuous over its domain of definition and to depend on the current state and input (i.e., at time t, the economic stage cost is $l_e(x(t), u(t))$). For systems described by nonlinear ODEs, a two-mode LEMPC scheme has been designed.[25] We will show that applying the LEMPC scheme, formulated with an ODE model of the system of Eq. 1, to the system of Eq. 1 will guarantee closed-loop stability (in a sense to be made precise below) for sufficiently small time-delays. Instead of applying the two-mode LEMPC scheme proposed in Ref. 25 directly to the time-delay system of Eq. 1, we consider only mode one operation of the LEMPC. As we only use mode one operation of LEMPC, we drop this distinction and simply refer to this as LEMPC for the remainder.

LEMPC may dictate a time-varying operation strategy that optimizes the economics while maintaining the closed-loop state trajectory in Ω_ρ. The motivation of this type of control strategy is that forcing the state to converge to a small neighborhood of the steady state and maintaining the state within this neighborhood thereafter (i.e., steady state operation) may not necessarily be the best operation strategy from an economical perspective. For example, the best operating strategy may be some periodic operating policy or time-varying operating policy.[25,28] Thus, LEMPC may dictate some bounded time-varying operating policy that optimizes the economics.

Given that plant-model mismatch affects the accuracy of the prediction of any MPC scheme, a subset of Ω_ρ is used whereby the LEMPC may dictate a time-varying operating policy. The subset of Ω_ρ, denoted as $\Omega_{\hat{\rho}}$ ($B_{\delta_1} \subset \Omega_{\hat{\rho}} \subset \Omega_\rho$), is designed such that if the current state $x(t_k) \in \Omega_{\hat{\rho}}$ and the predicted state at the next sampling time $z(t_{k+1}) \in \Omega_{\hat{\rho}}$ ($z(t_{k+1})$ denotes the predicted state at the next sampling time), then the actual state at the next sampling time plus the input delay time will be in Ω_ρ. In this case, plant-model mismatch arises from the use of an ODE model to predict the behavior of the time-delay system of Eq. 1. For any state in $\Omega_\rho \setminus \Omega_{\hat{\rho}}$, we use the stabilizing control law $h_c(x)$ to force the state back to $\Omega_{\hat{\rho}}$. However, in most practical cases, the amount of time that the stabilizing control law is applied to the system is insignificant relative to the amount of time that the LEMPC is applied for good closed-loop performance. It is important to note that the LEMPC proposed in Ref. 25 uses a second mode of operation to force the state to $\Omega_{\hat{\rho}}$, utilizing a contractive Lyapunov-based constraint. Potentially, one could consider using such a contractive constraint within the context of LEMPC for time-delay systems. Nonetheless, the closed-loop stability properties of LEMPC with a contractive constraint for time-delay systems remain an open problem from a theoretical standpoint.

The optimal control problem (OCP) that defines LEMPC is given by the following optimization problem

$$\min_{v \in S(\Delta)} \int_{t_k}^{t_k+T} l_e(z(\tau), v(\tau))\ d\tau + V_f(z(t_k+T)) \tag{4a}$$

$$\text{s.t.} \quad \dot{z}(t) = \bar{f}(z(t), v(t)) \tag{4b}$$

$$z(t_k) = x(t_k) \tag{4c}$$

$$v(t) \in \mathbb{U}\ \forall\ t \in [t_k, t_k+T) \tag{4d}$$

$$z(t) \in \Omega_{\hat{\rho}}\ \forall\ t \in [t_k, t_k+T] \tag{4e}$$

where $\bar{f}(z, v) := f(z, z, v)$, $T = N\Delta$ ($N \in \mathbb{N}$) is the prediction horizon, z denotes the predicted state trajectory over the prediction horizon, and v is the piecewise constant input trajectory over the prediction horizon which is the decision variable of the OCP.

The objective function of Eq. 4a consists of the economic cost functional with a terminal cost. The design of the terminal cost is beyond the scope of the present work because it is not needed to prove closed-loop stability (for terminal cost design techniques for EMPC, see, for example, Refs. 23 and 26). A model (Eq. 4b) of the system of Eq. 1 with $d_1 = d_2 = 0$ is used to predict the future behavior of the system under the input trajectory computed by the LEMPC over the prediction horizon. The model is initialized with a state measurement at the current sampling time (Eq. 4c). Equation 4d constrains the computed input trajectory to take values within the set of admissible control action values. The constraint of Eq. 4e bounds the predicted state trajectory be in $\Omega_{\hat{\rho}}$.

The optimal input trajectory of Eq. 4 at sampling time t_k is denoted as $v^*(t|t_k)$ and defined over $t \in [t_k, t_k+T)$. The control action computed at the kth sampling period is $u(t) = v^*(t_k|t_k)$ for $t \in [t_k, t_{k+1})$. At the next sampling time, t_{k+1}, the LEMPC (assuming $x(t_{k+1}) \in \Omega_{\hat{\rho}}$) is reinitialized with an updated state measurement and it computes an optimal input trajectory over a shifted horizon (i.e., LEMPC is applied in a standard receding horizon fashion). The algorithm below summarizes the implementation of LEMPC.

Algorithm 1. LEMPC implementation.

1. At sampling time t_k, the LEMPC receives a state measurement $x(t_k)$.
2. If $x(t_k) \in \Omega_{\hat{\rho}}$, go to Step 2.1. Else, go to Step 2.2.
   - 2.1. Solve the OCP of Eq. 4 to compute the optimal input trajectory $v^*(t|t_k)$ defined for $t \in [t_k, t_k+T)$. Go to Step 3.
   - 2.2. Compute the control action from the stabilizing control law $v^*(t_k|t_k) = h_c(x(t_k))$ Go to Step 3.
3. Send the control action $v^*(t_k|t_k)$ to the control actuators which will be applied to the system from $t_k + d_2$ to $t_{k+1} + d_2$ (i.e., $u(t) = v^*(t_k|t_k)$ for $t \in [t_k, t_{k+1})$). Go to Step 4.
4. Set $k \leftarrow k+1$ and go to Step 1.

The main tuning parameter of LEMPC is $\hat{\rho}$ and does not need to be chosen so that $\Omega_{\hat{\rho}}$ is the largest subset of Ω_ρ such that the state is guaranteed to be in Ω_ρ under LEMPC. The parameter $\hat{\rho}$ governs the set of points (in state-space) where the LEMPC can operate the system in a time-varying fashion to optimize the process economics. It is chosen to manage a potential trade-off between robustness and performance of the closed-loop system. In this case, closed-loop robustness is defined as the ability to maintain the closed-loop state inside an invariant state-space set in the presence of uncertainty. The uncertainty considered here is time-delay that is not included in the process model of the LEMPC. Conversely, operating over a larger region in state-space may improve closed-loop performance. If there is little plant-model mismatch (i.e., the delay is small), one can take it to be almost the size of ρ because the LEMPC can very accurately predicted the behavior of the process. If there is significant plant-model mismatch, $\hat{\rho}$ will need to be much smaller than ρ. This will be extensively investigated in the "Application to a chemical process example" subsection of this section below.

Closed-loop stability analysis

In this subsection, sufficient conditions such that closed-loop stability of the time-delay system of Eq. 1 under the LEMPC of Eq. 4 in the sense that the state trajectory is bounded in Ω_ρ are derived. We will show that the state will be maintained in Ω_ρ when the control action is computed by LEMPC and if the state is in $\Omega_\rho \setminus \Omega_{\hat{\rho}}$, the state will be converge to $\Omega_{\hat{\rho}}$ in finite-time when the control action is computed by the explicit stabilizing control law.

The following proposition bounds the difference between the state trajectory of the DDE system of Eq. 1 and of the ODE system of Eq. 1 with $d_1 = d_2 = 0$ over a finite-time interval.

Proposition 2. *Consider the following two systems*

$$\dot{x}(t) = f(x(t), x(t-d_1), u(t-d_2)) \tag{5}$$

$$\dot{z}(t) = f(z(t), z(t), v(t)) \tag{6}$$

where the initial time is t_0, the initial data of Eq. 5 is given by $\eta_x \in C([t_0-d_1, t_0], \mathbb{R}^n)$ where $\eta_x(\theta) \in \Omega_\rho$ for $\theta \in [t_0-d_1, t_0]$, $u(t) \in \mathbb{U}$ for all $t \geq t_0-d_2$, $v(t) \in \mathbb{U}$ for all $t \geq t_0$, and $x(t_0) = z(t_0) \in S$ where S is a compact subset of Ω_ρ. Let $t_1 > 0$ be such that the state trajectories of the systems of Eqs. 5 and 6 are bounded in Ω_ρ for all $t \in [t_0, t_0+t_1]$. There exists $\bar{\alpha} \in \mathcal{K}$ such that

$$|x(t) - z(t)| \leq \bar{\alpha}(t-t_0) \tag{7}$$

for all $t \in [t_0, t_0+t_1]$.

The difference of the Lyapunov function values evaluated at any two points in Ω_ρ can be bounded by a quadratic function because V is continuously differentiable and Ω_ρ is a compact set. This is stated in the following proposition.

Proposition 3 (c.f. Ref. 36). *For all x_1, $x_2 \in \Omega_\rho$, there exists $\beta > 0$ such that*

$$V(x_1) - V(x_2) \leq f_V(|x_1 - x_2|) \tag{8}$$

where the quadratic function $f_V(\cdot)$ is given by

$$f_V(s) := \alpha_4(\alpha_1^{-1}(\rho))s + \beta s^2 \tag{9}$$

The following result gives sufficient conditions such that the closed-loop state trajectory of the system Eq. 1 under the LEMPC implemented according to Algorithm 1 is always bounded in Ω_ρ.

Theorem 1. *Consider the system of Eq. 1 under the LEMPC of Eq. 4 designed via a control law that satisfies Eq. 2 which is implemented according to Algorithm 1. Let the initial data $\phi_x \in C([-d_1, 0], \mathbb{R}^n)$ be such that $\phi_x(s) \in \Omega_\rho$ for all $s \in [-d_1, 0]$ and the initial input ϕ_u be such that the state over the interval $[0, d_2]$ is bounded in Ω_ρ for $t \in [0, d_2]$ in the sense that $x(t) \in \Omega_\rho$ for all $t \in [0, d_2]$. Let $\tilde{t}_d \in (0, t_d^*)$ and $\hat{\rho} > 0$ be such that*

$$\hat{\rho} < \rho - f_V(\bar{\alpha}(\Delta + d_2)) \tag{10}$$

and $B_{\delta_1} \subseteq \Omega_{\hat{\rho}} \subset \Omega_\rho \subseteq B_{\delta_2}$. If $T \geq \Delta$, then the closed-loop state trajectory of Eq. 1 under the LEMPC is bounded in Ω_ρ for all $t \geq 0$.

Remark 3. *Given the generality of the class of nonlinear time-delay systems considered, it is difficult to explicitly characterize the functions used in the analysis. The results and conditions of this section are constructive in the sense that they provide insight into the relationship of system parameters (e.g., nonlinearities and delay size, etc.) and design parameters of the control system (e.g., sampling period, choice of stabilizing control law $h_c(x)$, and region constraint size ($\Omega_{\hat{\rho}}$)).*

Remark 4. *In the proof of Proposition 2, we bound $|u(t-d_2)-v(t)|$ by $2|u_{\max}|$ (Eq. A14) which may appear overly conservative. Even if $v(t)=u(t)$ for all t, the input trajectories are piecewise continuous. For a given t, the control actions $u(t-d_2)$ (control action implemented at time t) and u(t) (control action computed at time t) could be computed at different sampling times and $u(t) \neq u(t-d_2)$. Within the context of EMPC, it is conceivable that $|u(t-d_2)-u(t)| = 2|u_{\max}|$ as is the case when EMPC computes a bang-bang input trajectory. One could consider imposing rate of change constraints within the context of EMPC that limits how much the computed control action may change at each sampling time (i.e., $|u(t_{k+1})-u(t_k)| \leq \delta u$). This may allow one to upper bound $|u(t-d_2)-u(t)|$ by δu. Nonetheless, this may make the EMPC problem infeasible and/or may limit the achievable closed-loop performance when rate of change constraints are imposed in the EMPC problem.*

Remark 5. *Stabilization at a steady state typically will provide a degree of robustness to plant-model mismatch. In contrast, LEMPC may dictate a time-varying operating strategy. While time-varying operation may lead to better closed-loop economic performance, it may lead to a decrease in the robustness to uncertainty (see the "Application to a chemical process example" subsection below for a demonstration of this point).*

Remark 6. *The sampling period Δ plays a role in stability and performance. Sufficiently fast sampling is required to maintain closed-loop stability (Proposition 1 and Theorem 1). Potentially, faster sampling may also improve closed-loop performance because there are more degrees of freedom to optimize over. However, in the example considered below, little performance benefit is achieved through faster sampling. From a practical standpoint, there may also be limitations on the hardware that may limit how fast one can sample.*

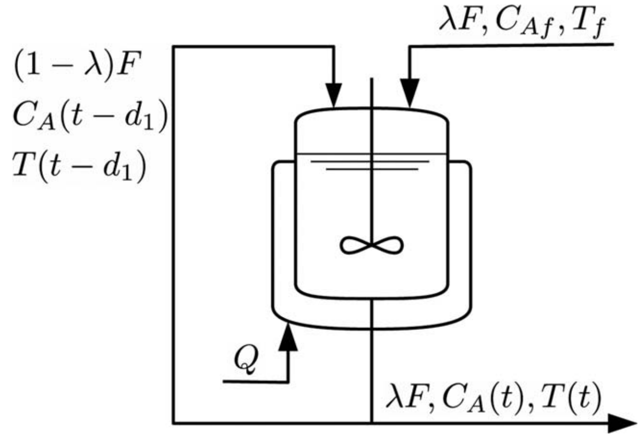

Figure 1. Process flow diagram of the CSTR with recycle.

Application to a chemical process example

Consider a well-mixed, nonisothermal continuous stirred-tank reactor (CSTR) where an irreversible, elementary second order, and exothermic reaction takes place that converts a reactant A to a desired product B ($A \rightarrow B$). A process flow diagram of the CSTR is given in Figure 1. The inlet stream of the reactor with flow rate λF (constant density of the inlet flow stream is assumed), temperature T_f, and reactant concentration C_{Af} feeds the reactor with the reactant A in an inert solvent D. The outlet stream of the reactor is split with a constant splitting fraction λ ($\lambda \in [0, 1]$), and the fraction λ of the outlet stream is the product stream. A recycle stream, which contains a fraction of $1-\lambda$ of the reactor outlet, is used to recover unreacted A. A transportation lag, which is modeled as a time-delay with magnitude d_1, is considered in the recycle stream leading to a state delay in the process model. To supply or remove thermal energy to/from the reactor, the CSTR is outfitted with a jacket. The manipulated inputs to the CSTR are the feed concentration of the reactant C_{Af} and the heat rate Q supplied to/removed from the reactor.

To construct a first-principles model of the reactor, constant liquid hold-up, density, and heat capacity are assumed. In addition to the state delay, the control actuators are assumed to operate with dead-time which gives rise to input delay. The input delay is assumed to model, for instance, the actuator dynamics and/or to model the possible computation delay resulting from solving the EMPC optimization problem online. The dead-times of the two manipulated inputs are assumed to be the same for simplicity of the presentation and are denoted as d_2. Using a mass balance of the reactant A and energy balance along with standard modeling assumptions, a dynamic model can be constructed of the CSTR with recycle and is given by the following system of DDEs

$$\dot{C}_A(t) = \frac{(1-\lambda)F}{V_R} C_A(t-d_1) + \frac{\lambda F}{V_R} C_{Af}(t-d_2) - \frac{F}{V_R} C_A(t) - k_0 e^{-E/RT(t)} C_A^2(t) \tag{11a}$$

$$\dot{T}(t) = \frac{(1-\lambda)F}{V_R} T(t-d_1) + \frac{\lambda F}{V_R} T_f - \frac{F}{V_R} T(t) - \frac{\Delta H k_0}{\rho_L C_p} e^{-E/RT(t)} C_A^2(t) + \frac{Q(t-d_2)}{V_R \rho_L C_p} \tag{11b}$$

where the state variables are the concentration of the reactant C_A in the reactor and the reactor temperature T (i.e., $x^T = [C_A \ T]$).

The remaining notation and process parameter values are defined and summarized in Table 1. The admissible input values for each input are bounded in the following sets: $C_{A0} \in [0.5, 7.5]$ kmol m^{-3} and $Q \in [-80.0, 80.0]$ MJ h^{-1}. The CSTR model of Eq. 11 has a steady state in the operating range of interest: $x_s^T := [C_{As} \quad T_s] = [2.96 \text{ kmol m}^{-3} \quad 320.0 \text{ K}]$ corresponding to the inputs: $C_{A0s} = 4.0$ kmol m^{-3} and $Q_s = 11.5$ MJ h^{-1}. The steady state x_s is open-loop asymptotically stable.

The control objective is to operate the CSTR around the steady state to maximize the average production rate of the product B while satisfying two process constraints. The process constraints are: (1) to maintain the closed-loop state within a bounded and well-defined state-space set and (2) the time-averaged amount of reactant A that may be fed to the CSTR must be equal to $\lambda F C_{A0s}$ or mathematically

$$\frac{1}{t_f}\int_0^{t_f} C_{A0}(t)\; dt = C_{A0s} \tag{12}$$

where t_f is the length of operation. To accomplish the control objective while satisfying the process constraints, LEMPC is applied to the CSTR. In the closed-loop simulations of this section, robustness of LEMPC to time-delay is demonstrated. More precisely, we illustrate robustness in the sense that the closed-loop state trajectory of Eq. 11 under an LEMPC system formulated with an ODE model of the CSTR will be maintained in a compact state-space set (in $\mathbb{R}^2$) for sufficiently small time-delays.

To design an LEMPC system with the ODE model of the CSTR (Eq. 11 with $d_1 = d_2 = 0$), an explicit stabilizing control law is designed and a Lyapunov function for the closed-loop system under the control law is constructed. The explicit stabilizing control law $h_c : \mathbb{R}^2 \to U$ is given component-wise with $h_c^T(x) = [h_{c,1}(x) \;\; h_{c,2}(x)]$ where $h_{c,1}(x) = C_{A0s}$ to satisfy the input average constraint of Eq. 12 and feedback linearization for $h_2(x)$. Specifically, within the range where the input bounds are satisfied, the following nonlinear control law makes the evolution of T to be described by a linear model

$$\bar{h}_{c,2}(x) = -V_R \rho_L C_p \left(-\frac{\lambda F}{V_R} T_f + \frac{\lambda F}{V_R} T + \frac{\Delta H k_0}{\rho_L C_p} e^{-E/RT} C_A^2 + K(T - T_s) \right) \tag{13}$$

where K is the gain of the control law. To account for the bounds on the admissible input, saturation is accounted for in $h_{c,2}$ which is given by

$$h_2(x) = \begin{cases} Q_{\min} & \text{if } \bar{h}_2(x) < Q_{\min} \\ \bar{h}_2(x) & \text{if } Q_{\min} \le \bar{h}_2(x) \le Q_{\max} \\ Q_{\max} & \text{if } \bar{h}_2(x) > Q_{\max} \end{cases} \tag{14}$$

where $Q_{\min}$ and $Q_{\max}$ is the minimum and maximum admissible heat rate value. In this example, the gain of the controller is tuned to $K = 2.0$. A quadratic Lyapunov function for the closed-loop system under the stabilizing control law is constructed, which has the form

$$V(x) = (x - x_s)^T P (x - x_s) \tag{15}$$

where

$$P = \begin{bmatrix} 500 & 20 \\ 20 & 1 \end{bmatrix}$$

With the control law and Lyapunov function, the stability region is estimated to be a level set of the Lyapunov function Ω_ρ with $\rho = 1200$ by taking it to be points in state-space where $\dot{V} < 0$.

Table 1. Notation and Parameter Values of the CSTR with Recycle

Density	$\rho_L = 1.0 \times 10^3$ kg m^{-3}
Heat capacity	$C_p = 4.18$ kJ kg^{-1} K^{-1}
Flow rate	$F = 6.0$ m^3 h^{-1}
Reactor volume	$V_R = 1.0$ m^3
Heat of reaction	$\Delta H = -7.8 \times 10^4$ kJ kmol^{-1}
Activation energy	$E/R = -5.7 \times 10^4 / 8.314$ K
Feed temperature	$T_f = 300.0$ K
Reaction rate constant	$k_0 = 1.0 \times 10^9$ m^3 kmol^{-1} h^{-1}
Splitting fraction	$\lambda = 0.70$
Concentration of chemical A	C_A
Reactor temperature	T
Heat removal rate from reactor	Q
Feed concentration	C_{Af}

The following economic stage cost is used in the LEMPC

$$l_e(x, u) = -k_0 e^{-E/RT} C_A^2 + q_T (T - T_s)^2 \tag{16}$$

where the first term credits the production rate of B and the second term penalizes temperature deviations from the steady-state temperature T_s. The justification for the second term is the stability region of this system is large and operating over a large temperature range may be impractical. Thus, the second term penalizes large deviations of the temperature from the steady state temperature. In this case, the stage cost parameter is $q_T = 0.05$ which has been tuned on the basis of the delay-free system such that the closed-loop temperature is maintained near the steady state temperature.

To integrate forward in time the DDEs of Eq. 11, the standard Runge–Kutta (4,5) method was used with an absolute tolerance set at 10^{-7}. The remaining parameters of the LEMPC and implementation details of the LEMPC are as followed: the sampling period is $\Delta = 0.01$ h, the number of sampling periods in the prediction horizon is $N = 70$ (i.e., $T = 0.70$ h), and the average constraint must be satisfied over each operating interval of 0.65 h (i.e., the average constraint of Eq. 12 must be satisfied every 0.65 h which guarantees that the average constraint is satisfied over the entire length of operation). For the remainder, the operating period will refer to an interval of length 0.65 h that the average constraint is imposed. The prediction horizon and operating period length have been chosen so that the asymptotic average closed-loop performance of the CSTR without time-delays leads to better closed-loop performance compared with that of the operation at the steady state. Orthogonal collocation with three Radau collocation points per sampling period is used to integrate the ODE model within the LEMPC OCP. Ipopt[37] was employed to solve the OCP of the LEMPC at each sampling time. Analytical first-order and second-order derivative information was supplied to the solver. The simulations were completed on a desktop computer with an Intel® Core™ 2 Quad 2.66 GHz processor running an Ubuntu Linux operating system. In the simulations below, the computation time to solve the OCP at each sampling time was less than 1% of the sampling period ($\Delta = 0.01$ h $= 36$ s) on average.

In the first set of simulations, the closed-loop stability and performance properties of the CSTR without delays under LEMPC are considered. The level set $\Omega_{\hat{\rho}}$ is such that $\hat{\rho} = 1000$. The closed-loop CSTR under LEMPC without delay is simulated over six operating periods, and the closed-loop trajectories are given in Figure 2. From Figure 2, the LEMPC dictates

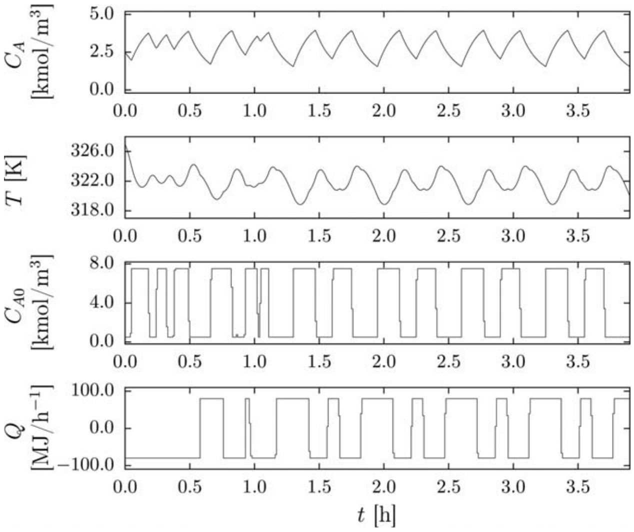

Figure 2. The closed-loop trajectories of the CSTR under the LEMPC without time-delays ($d_1=d_2=0$).

a cyclical operating policy. Throughout the length of operation, the state trajectory is maintained in Ω_ρ. The asymptotic average production rate of B (i.e., average production rate after the effect of the initial condition becomes insignificant) was 4.820 kmol m^{-3}. As a comparison, the average production rate of B at the steady state is 4.354 kmol m^{-3} and the closed-loop (asymptotic) production rate under LEMPC is 10.70% better than the steady state production rate. Moreover, we will use the average closed-loop economic stage cost index (given and explained below) to assess the closed-loop performance of the CSTR with time-delay. The asymptotic average economic stage cost index for this case study is 4.572, while it is 4.354 for operation at the steady state (performance under LEMPC as assessed through this metric is 5.01% better than that at the steady state).

In the second set of simulations, we consider the effect of the time-delay on performance and stability. In all cases, the state and input delays are taken to be equal in magnitude and the time-delay is denoted as d ($d=d_1=d_2$). Figure 3 shows the closed-loop state evolution of the CSTR under the LEMPC for $d=0.05$ h and $d=0.10$ h, respectively. With the time-delay, the CSTR is still operated in a cyclical fashion, but as the time-delay increases, the CSTR operates over a larger region of state-space. The closed-loop state is maintained in Ω_ρ for $d=0.05$ h, but for $d=0.10$ h, the state is not bounded in Ω_ρ over the length of operation. To assess the closed-loop performance, the average economic stage cost performance index is used which is given by

$$\bar{L}_e=\frac{1}{t_f}\int_0^{t_f} l_e(x(t),u(t-d_2))\ dt \tag{17}$$

The average economic stage cost is computed with the closed-loop state and input. The reason for using the metric of Eq. 17 as opposed to the average production rate of B to assess the closed-loop performance is as the magnitude of the time-delay increases, the CSTR operates over a larger temperature range (Figure 3). The temperature is greater than the steady-state temperature on average for the cases with time-delay. As the production rate scales with temperature, the production rate of B increases with the size of the time-delay. Moreover, the LEMPC does not directly optimize the production rate of B, but rather the stage cost of Eq. 16. Thus, we use the metric of Eq. 17 to assess the performance because it also accounts for operation over a larger temperature range.

Table 2 summarizes the closed-loop performance and closed-loop stability properties of the CSTR under LEMPC for several closed-loop simulations each over six operating periods with varying time-delays. Closed-loop stability is defined as the closed-loop state remaining bounded in Ω_ρ over the length of the simulated operation. From Table 2, it follows that the closed-loop performance deteriorates as the time-delay increases. Moreover, for time-delays greater than 0.06 h, the closed-loop stability of the CSTR is not maintained. It is important to note that the state trajectory of the closed-loop system under the stabilizing control law remains bounded in Ω_ρ for all the magnitudes of the time-delay used in Table 2 which suggests that steady state type operation (i.e., stabilization at a steady state) is more robust to time-delays than time-varying-type operation.

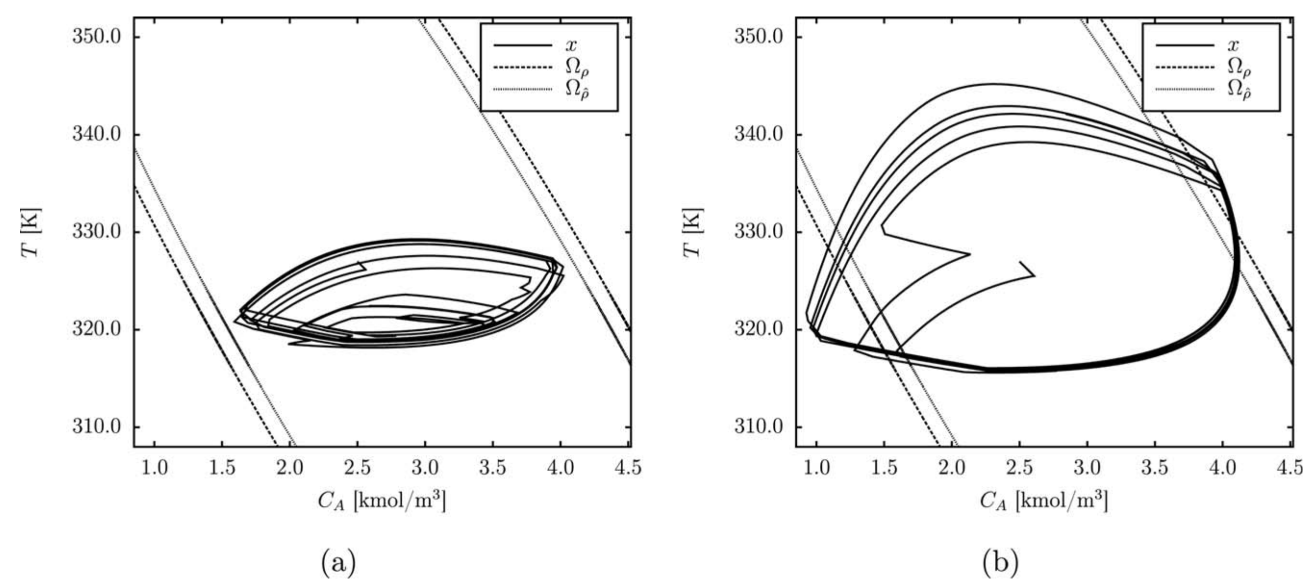

Figure 3. The state-space evolution of the closed-loop CSTR states under LEMPC with (a) $d=0.05$ h and (b) $d=0.10$ h.

Table 2. Closed-Loop Performance Relative to the Performance at the Steady State and Closed-Loop Stability Properties of the CSTR under LEMPC

d	$\bar{L}_e$	Diff. (%)	Stability
0	4.668	7.21	Yes
0.01	4.654	6.88	Yes
0.02	4.618	6.07	Yes
0.03	4.518	3.77	Yes
0.04	4.441	1.99	Yes
0.05	4.220	−3.09	Yes
0.06	3.995	−8.26	Yes
0.07	2.763	−36.55	No
0.08	3.002	−31.06	No
0.09	1.801	−58.64	No
0.10	0.565	−87.03	No

The average stage cost index for operation at the steady state is 4.354. The column "Diff." is the percent difference of the average stage cost index relative to the steady state stage cost index.

The available tuning parameter of LEMPC that can be manipulated to make the closed-loop system more robust to uncertainty is $\hat{\rho}$ (i.e., the level set where LEMPC operates). In the last set of simulations, the effect of the size of $\hat{\rho}$ on closed-loop stability and performance is evaluated. Several closed-loop simulations were performed each over six operating periods with varying $\hat{\rho}$ and d. The average economic stage cost indices for these simulations are given in Figure 4. Only the closed-loop simulations that led to a stable operation were included in Figure 4. Figure 4 shows that the closed-loop performance degrades with larger time-delays. For small time-delays, the closed-loop performance is better as the size of $\Omega_{\hat{\rho}}$ increases because the LEMPC may operate the system over a larger state-space set with greater $\hat{\rho}$. For small time-delays, the ODE model can capture enough of the behavior of the system to improve the performance. For larger time-delays, the closed-loop performance is worse than that achieved at the steady state.

Time-Delay Compensation for Improved Closed-Loop Performance

Larger time-delay may lead to significant performance degradation when the control system does not explicitly account for the time-delays as demonstrated in the example of the previous section. In this section, a methodological framework for compensating state and input delay within the context of EMPC is presented. A closed-loop predictor is used to compensate the adverse effect of the input delay, and a DDE model is used within the EMPC to predict the behavior of the system of Eq. 1. We restrict our attention to nominally operated systems of the form of Eq. 1.

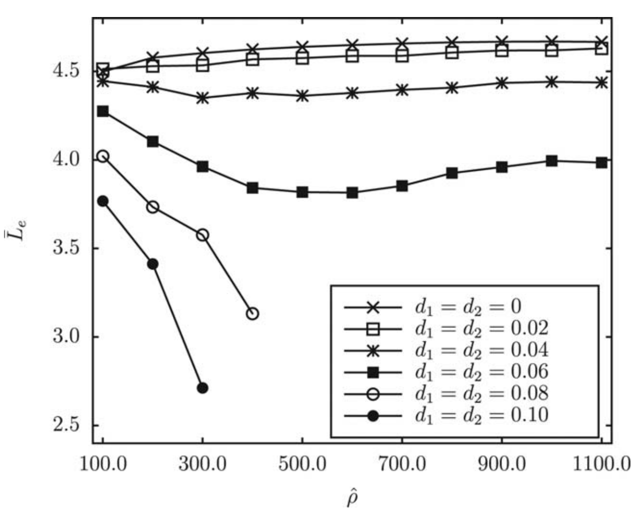

Figure 4. A comparison of the closed-loop performance with the tuning parameter $\hat{\rho}$ and magnitude of the time-delay.

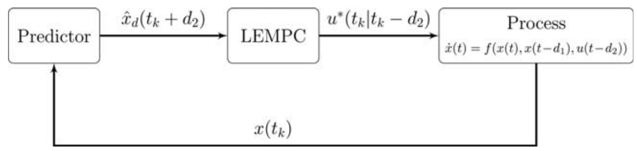

Figure 5. Flow diagram of the predictor feedback LEMPC scheme.

Predictor feedback LEMPC methodology and implementation

A block diagram of the predictor feedback LEMPC methodology is shown in Figure 5. At a sampling time t_k, a predictor is used to predict the state at t_k+d_2 utilizing the past measurements of the state and the previously computed input trajectory over t_k to t_k+d_2 to compensate the effect of the input delay. Then, the LEMPC system is initialized with the predicted state and solves for the optimal control action that will be implemented on the system from t_k+d_2 to $t_{k+1}+d_2$. Moreover, instead of using an ODE model within the LEMPC, the DDE model is used to account for the state delay. Thus, the predictor must also generate the initial data used to initialize the DDE model in the LEMPC from $t_k+d_2-d_1$ to t_k+d_2. The predicted state (initial data) for a given sampling time is denoted as $\hat{x}_d(t_k + d_2) \in C([-d_1, 0], \mathbb{R}^n)$.

Given that we consider nominal operation, the predictor may simply consist of solving the DDEs forward in time which is what we employ in the example below. The predictor is a closed-loop predictor in the sense that the predictor is reinitialized with a new state measurement at each sampling time. The closed-loop nature of the predictor allows for the potential use of the predictor feedback LEMPC on open-loop unstable processes as opposed to open-loop predictors (e.g., the classical Smith predictor) which require the steady state solution be open-loop asymptotically stable. Other types of time-delay compensators or predictors may potentially be used to possibly increase the robustness of the closed-loop system to plant-model mismatch (e.g., the predictor proposed in Ref. 10), but we note that most, if not all, of time-delay compensators have been designed for systems with input-delay only. Thus, appropriate modifications may need to be made to these other types of time-delay compensators to account for state delay.

A shifted sampling time sequence is defined as $\{\bar{t}_k\}_{k\geq 0}$ where $\bar{t}_k=k\Delta+d_2$, $k=0, 1, \ldots$ which is a time sequence corresponding to when control actions are applied to the system (i.e., the control action computed at t_k is applied to the system from $\bar{t}_k$ to $\bar{t}_{k+1}$). The OCP that defines the predictor feedback LEMPC is

$$\min_{v\in S(\Delta)} \int_{\bar{t}_k}^{\bar{t}_k+T} l_e(z(\tau), v(\tau))\ d\tau + V_f(z(\bar{t}_k+T)) \tag{18a}$$

$$\text{s.t.}\quad \dot{z}(t)=f(z(t), z(t-d_1), v(t)) \tag{18b}$$

$$z_d(\bar{t}_k)=\hat{x}_d(\bar{t}_k) \tag{18c}$$

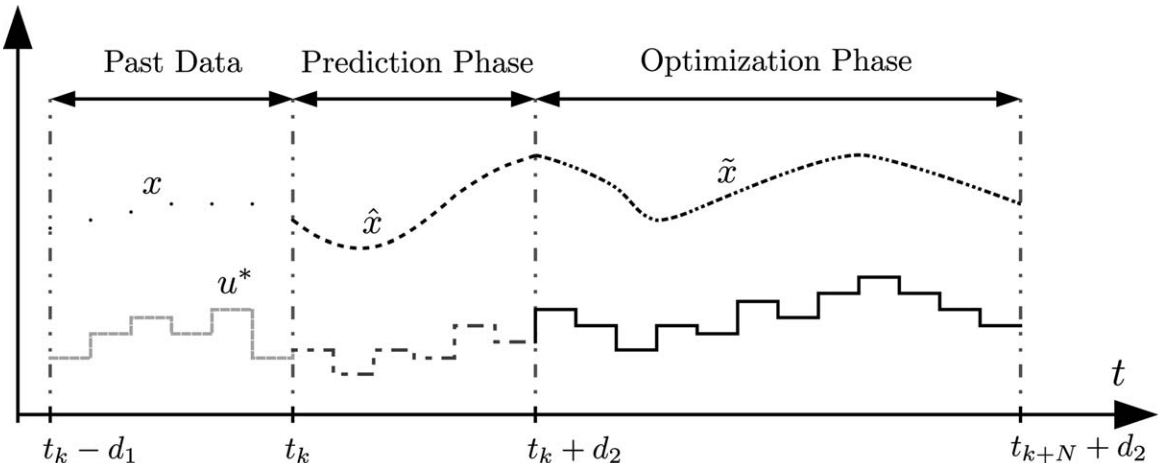

Figure 6. An illustration of the phases of the predictor feedback LEMPC scheme.

$$v(t) \in \mathbb{U} \;\; \forall \;\; t \in [\bar{t}_k, \bar{t}_k + T) \tag{18d}$$

$$z(t) \in \Omega_{\hat{\rho}} \;\; \forall \;\; t \in [\bar{t}_k, \bar{t}_k + T] \tag{18e}$$

where $\hat{x}(t_k) = \hat{x}_d(\bar{t}_k)(0)$. The main difference between the predictor feedback LEMPC and the LEMPC of Eq. 4 is the DDE model, which is initialized with the initial data computed by the predictor and provides a prediction of the future evolution of the system in the predictor feedback LEMPC. The LEMPC of Eq. 4 uses an ODE model and is initialized with a current state measurement.

An illustration of the implementation of the predictor feedback LEMPC methodology is given in Figure 6. At a sampling time t_k, the predictor (i.e., prediction phase) uses past state data and the control actions that will be applied from t_k to $t_k + d_2$ (these control actions are computed at previous sampling times) to predict the state trajectory over t_k to $t_k + d_2$. In the optimization phase, the predictor feedback LEMPC solves for the optimal input trajectory over $t_k + d_2$ to $t_k + d_2 + T$. This implementation is summarized in the following algorithm:

Algorithm 2. Predictor feedback LEMPC implementation.

1. At sampling time t_k, the predictor receives a state measurement $x(t_k)$. Go to Step 2.
2. The predictor computes the predicted data $\hat{x}_d(\bar{t}_k)$. Go to Step 3.
3. If $\hat{x}(\bar{t}_k) \in \Omega_{\hat{\rho}}$, go to Step 3.1. Else, go to Step 3.2.
   3.1. Solve the OCP of Eq. 18 to compute the optimal input trajectory $v^*(t|\bar{t}_k - d_2)$ defined for $t \in [\bar{t}_k, \bar{t}_{k+N})$. Go to Step 4.
   3.2. Compute the control action from the stabilizing control law $v^*(\bar{t}_k|\bar{t}_k - d_2) = h_c(\hat{x}(\bar{t}_k))$ Go to Step 4.
4. Send the computed control action $v^*(\bar{t}_k|\bar{t}_k - d_2)$, to the control actuators to be applied from $\bar{t}_k$ to $\bar{t}_{k+1}$ (i.e., $u(t + d_2) = v^*(t_k + d_2|t_k)$ for $t \in [t_k, t_{k+1})$). Go to Step 5.
5. Set $k \leftarrow k+1$ and go to Step 1.

Remark 7. *In the design of the LEMPC of Eq. 18, we leverage the results of Proposition 1 to again design an explicit stabilizing control law and utilize it to characterize a region constraint that is imposed in the LEMPC problem. This design methodology allows for standard control techniques developed for systems described by nonlinear ODEs be applied to design stabilizing control laws for nonlinear time-delay systems. Conversely, the results and size of delays that may be handled in the closed-loop systems may be limited owing to the fact that the delays are neglected in the design of the stabilizing control law. However, design of stabilizing control laws for nonlinear time-delay systems is by no means a trivial task. Moreover, a complete and rigorous stability analysis of such a closed-loop system is a challenging and potentially intractable task given the degree of complexity (e.g., state and input delay, sampling, and nonlinearities).*

Remark 8. *It is important to point out that the extension of numerical methods used to obtain solutions of ODEs to obtaining solutions of differential difference equations (DDEs) is not straightforward.*[38] *Thus, when selecting the numerical method used to solve the predictor, the practitioner must be aware of potential numerical issues (e.g., "ghost solutions," loss of injectivity, and nonuniqueness of solution).*

Application to a chemical process example

The predictor feedback LEMPC methodology is applied to the CSTR example of Eq. 11. The LEMPC design and parameters are the same as that used in the previous section (in all simulations below $\hat{\rho} = 1000$). To solve the DDEs of Eq. 11 embedded in the LEMPC optimization problem (i.e., the constraint of Eq. 18b), orthogonal collocation with three Radau collocation points per sampling period was employed (see, for example, Refs. 38 and 39 for stability and convergence analysis of collocation methods applied to DDEs). As a qualitative comparison, a single-shooting implementation of the OCP of Eq. 18a using the explicit Euler method with first-order derivatives approximated through finite-difference and quasi-Newton with the Broyden-Fletcher-Goldforb-Shanno (BFGS) method for the second-order derivatives required greater than 50 times more time to solve the optimization problem at each sampling time compared with the implementation with orthogonal collocation. Moreover, the average computation time required to solve the OCP with the collocation implementation was approximately 2% of the sampling period. It is important to point out that while the success of orthogonal collocation used to solve OCPs with ODEs has been well documented,[40] fewer cases of employing orthogonal collocation within the context of OCPs formulated with DDE models have reported in the literature especially in the EMPC literature.

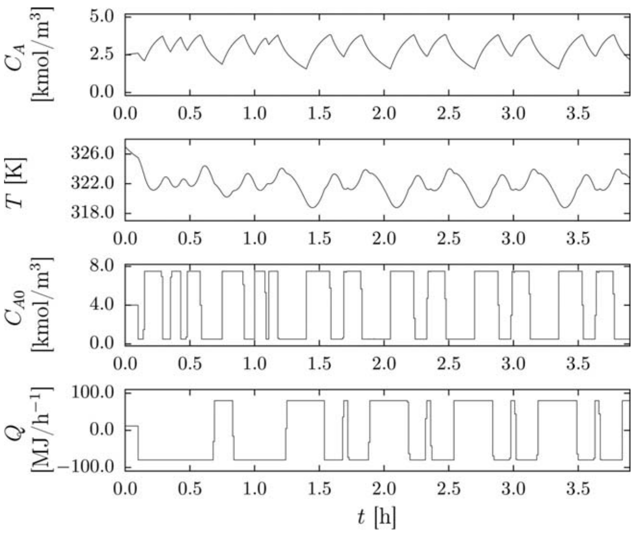

Figure 7. The closed-loop trajectories of the CSTR under the LEMPC with time-delay of $d=0.10$ h.

The input trajectories shown in the plots correspond to the input values applied to the system at each time.

Several closed-loop simulations were completed of the CSTR under the predictor feedback LEMPC with varying magnitudes of the time-delays. Figure 7 gives the closed-loop trajectories of the CSTR with $d=0.10$ h. In comparison with the CSTR under LEMPC formulated with an ODE model (Figure 3b), the predictor feedback LEMPC operates the CSTR over a smaller temperature range (Figure 7). To further emphasize the differences between the evolution of the CSTR under the predictor feedback LEMPC and the LEMPC of the previous section, Figure 8 gives the state space evolution of the CSTR with $d=0.05$ h and $d=0.10$ h, respectively. Comparing the evolution of the two cases shown in Figure 7, less differences in the evolution between the two cases are observed compared with the two cases of Figure 3.

The closed-loop performance under the predictor feedback LEMPC is considered with respect to the magnitude of the time-delay. Table 3 summarizes the average economic stage cost of Eq. 17 of six operating period simulations. Interestingly, the closed-loop performance improves with larger time-delay. The performance improvement is associated with the state delay in the stream recycle (given that the predictor effectively deals with the effect of the input delay on the closed-loop system). In all cases, the closed-loop performance under the predictor feedback LEMPC was at least 7% better than that achieved at the steady state.

Table 3. Closed-Loop Performance of the CSTR under the Predictor Feedback LEMPC Relative to the Performance at the Steady State

d	$\bar{L}_e$	Diff. (%)
0	4.668	7.21
0.01	4.675	7.36
0.02	4.682	7.54
0.03	4.690	7.71
0.04	4.697	7.87
0.05	4.703	8.01
0.06	4.710	8.17
0.07	4.705	8.07
0.08	4.722	8.45
0.09	4.728	8.58
0.10	4.733	8.70

The average stage cost index for operation at the steady state is 4.354. The column "Diff." is the percent difference of the average stage cost index relative to the steady state stage cost index.

Conclusion

In this work, closed-loop stability and performance of systems described by nonlinear DDEs under LEMPC was considered. First, conditions such that closed-loop stability for systems with sufficiently small state and input delays under LEMPC, formulated with an ODE model of the system, were derived. A chemical process example demonstrated that indeed closed-loop stability is maintained under LEMPC for sufficiently small time-delays in both the states and the inputs. However, closed-loop performance significantly degraded for larger input delays. This motivated designing a predictor feedback LEMPC methodology. The predictor feedback LEMPC

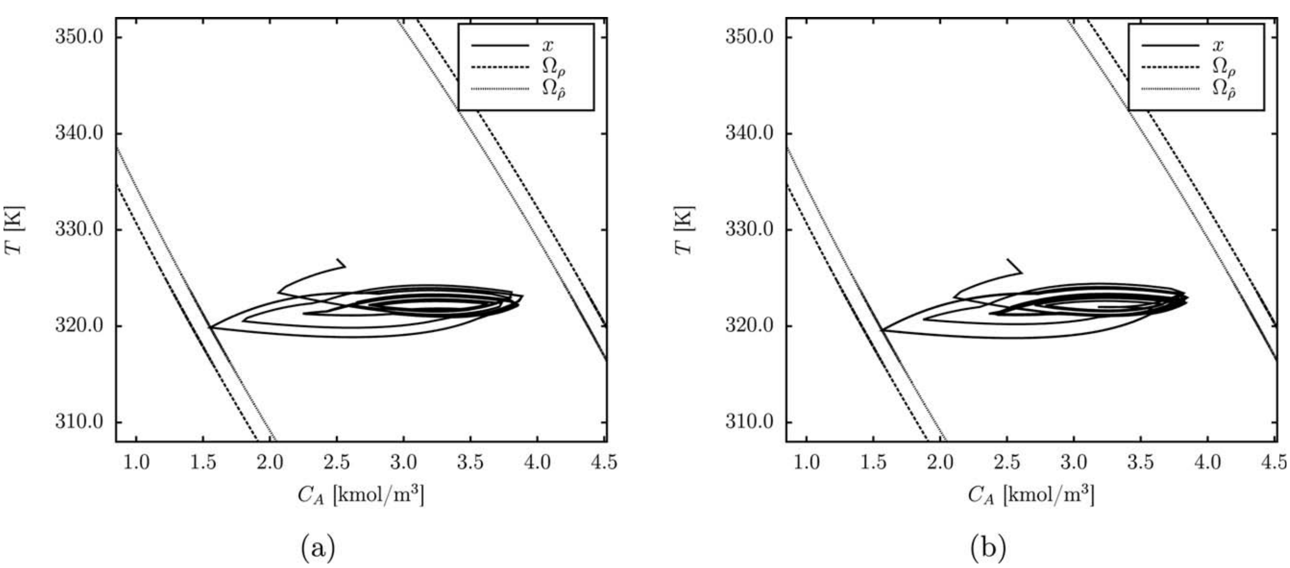

Figure 8. The state-space evolution of the closed-loop CSTR states under LEMPC with (a) $d=0.05$ h and (b) $d=0.10$ h.

design employs a predictor to compute a prediction of the state after the input delay and an LEMPC scheme, formulated with a DDE model. The predicted state from the predictor is used to initialize the DDE model. The predictor feedback LEMPC was applied to the chemical process example and resulted in better closed-loop stability and performance properties compared with the LEMPC, formulated with an ODE approximation of the nonlinear time-delay system.

Acknowledgment

Financial support from the National Science Foundation and the Department of Energy is gratefully acknowledged.

Literature Cited

1. Hale JK, Lunel SMV. *Introduction to Functional Differential Equations*. New York: Springer-Verlag, 1993.
2. Xu B, Liu Y. An improved Razumikhin-type theorem and its applications. *IEEE Trans Autom Control*. 1994;39:839–841.
3. Teel AR. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Trans Autom Control*. 1998; 43:960–964.
4. Jankovic M. Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems. *IEEE Trans Autom Control*. 2001; 46:1048–1060.
5. Jankovic M. Control of nonlinear systems with time delay. In *Proceedings of the 42nd IEEE Conference on Decision and Control, Vol. 5*. Maui, HI: IEEE, 2003:4545–4550.
6. Richard JP. Time-delay systems: an overview of some recent advances and open problems. *Automatica*. 2003;39:1667–1694.
7. Stephanopoulos G. *Chemical Process Control: An Introduction to Theory and Practice*. Englewood Cliffs, NJ: Prentice Hall, 1984.
8. Smith OJM. Closer control of loops with dead time. *Chem Eng Prog*. 1957;53:217–219.
9. Kravaris C, Wright RA. Deadtime compensation for nonlinear processes. *AIChE J*. 1989;35:1535–1542.
10. Henson MA, Seborg DE. Time delay compensation for nonlinear processes. *Ind Eng Chem Res*. 1994;33:1493–1500.
11. Antoniades C, Christofides PD. Feedback control of nonlinear differential difference equation systems. *Chem Eng Sci*. 1999;54:5677–5709.
12. Antoniades C, Christofides PD. Robust control of nonlinear time-delay systems. *Int J Appl Math Comput Sci*. 1999;9:811–837.
13. Christofides PD, El-Farra NH. *Control of Nonlinear and Hybrid Process Systems: Designs for Uncertainty, Constraints and Time-Delays*. Berlin, Germany: Springer-Verlag, 2005.
14. Pepe P. Input-to-State stabilization of stabilizable, time-delay, control-affine, nonlinear systems. *IEEE Trans Autom Control*. 2009;54: 1688–1693.
15. Krstic M. Input delay compensation for forward complete and strict-feedforward nonlinear systems. *IEEE Trans Autom Control*. 2010;55: 287–303.
16. Karafyllis I. Stabilization by means of approximate predictors for systems with delayed input. *SIAM J Control Optim*. 2011;49:1100–1123.
17. Teel AR, Nešić D, Kokotovic PV. A note on input-to-state stability of sampled-data nonlinear systems. In *Proceedings of the 37th IEEE Conference on Decision and Control, Vol. 3*. Tampa, FL: IEEE, 1998:2473–2478.
18. Hu L, Cao Y, Cheng C, Shao H. Sampled-data control for time-delay systems. *J Franklin Inst*. 2002;339:231–238.
19. Castillo-Toledo B, Di Gennaro S, Castro GS. Stability analysis for a class of sampled nonlinear systems with time-delay. In *Proceedings of the 49th IEEE Conference on Decision and Control*. Altanta, GA: IEEE, 2010:1575–1580.
20. Karafyllis I, Krstic M. Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold. *IEEE Trans Autom Control*. 2012;57:1141–1154.
21. Gu K, Niculescu SI. Survey on recent results in the stability and control of time-delay systems. *J Dyn Syst Meas Control*. 2003;125: 158–165.
22. Mayne DQ, Rawlings JB, Rao CV, Scokaert POM. Constrained model predictive control: stability and optimality. *Automatica*. 2000; 36:789–814.
23. Amrit R, Rawlings JB, Angeli D. Economic optimization using model predictive control with a terminal cost. *Annu Rev Control*. 2011;35:178–186.
24. Huang R, Harinath E, Biegler LT. Lyapunov stability of economically oriented NMPC for cyclic processes. *J Process Control*. 2011; 21:501–509.
25. Heidarinejad M, Liu J, Christofides PD. Economic model predictive control of nonlinear process systems using Lyapunov techniques. *AIChE J*. 2012;58:855–870.
26. Liu S, Zhang J, Liu J. Economic MPC with terminal cost and application to an oilsand primary separation vessel. *Chem Eng Sci*. In press. DOI: 10.1016/j.ces.2015.01.041.
27. Rawlings JB, Angeli D, Bates CN. Fundamentals of economic model predictive control. In *Proceedings of the 51st IEEE Conference on Decision and Control*. Maui, Hawaii: IEEE, 2012:3851–3861.
28. Ellis M, Durand H, Christofides PD. A tutorial review of economic model predictive control methods. *J Process Control*. 2014;24:1156–1178.
29. Normey-Rico JE, Camacho EF. *Control of Dead-time Processes. Advanced Textbooks in Control and Signal Processing*. London: Springer-Verlag London, 2007.
30. Raff T, Angrick C, Findeisen R, Kim JS, Allgöwer F. Model predictive control for nonlinear time-delay systems. In *Proceedings of the 7th IFAC Symposium on Nonlinear Systems*. Pretoria, South Africa: IFAC, 2007:60–65.
31. Mahboobi Esfanjani R, Nikravesh SKY. Stabilising predictive control of non-linear time-delay systems using control Lyapunov-Krasovskii functionals. *IET Control Theory Appl*. 2009;3:1395–1400.
32. Reble M, Mahboobi Esfanjani R, Nikravesh SKY, Allgöwer F. Model predictive control of constrained non-linear time-delay systems. *IMA J Math Control Inf*. 2011;28:183–201.
33. Dashkovskiy S, Naujok L. Input-to-state stability for model predictive control of single systems and networks with time-delays. In *Proceedings of the 51st IEEE Conference on Decision and Control*. Maui, Hawaii: IEEE, 2012:2599–2604.
34. Kokotović P, Arcak M. Constructive nonlinear control: a historical perspective. *Automatica*. 2001;37:637–662.
35. Khalil HK. *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.
36. Muñoz de la Peña D, Christofides PD. Lyapunov-based model predictive control of nonlinear systems subject to data losses. *IEEE Trans Autom Control*. 2008;53:2076–2089.
37. Wächter A, Biegler LT. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math Program*. 2006;106:25–57.
38. Bellen A, Zennaro M. *Numerical Methods for Delay Differential Equations*. New York: Oxford University Press, 2003.
39. Bellen A. One-step collocation for delay differential equations. *J Comput Appl Math*. 1984;10:275–283.
40. Biegler LT. *Nonlinear Programming: Concepts, Algorithms, and Applications to Chemical Processes*. Philadelphia, PA: SIAM, 2010.

Appendix

Notation and preliminary results

A function $\gamma : [0, a) \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{G}$ if it is a nondecreasing continuous function and $\gamma(0)=0$. A function $\alpha : [0, a) \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing, and it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$, $a=\infty$, and $\alpha(r)$ as $r \rightarrow \infty$. A function $\beta : [0, a)\times\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is monotonically decreasing to 0 for each $s \geq 0$. The symbol $C([a, b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with topology of uniform convergence and equipped with the norm

$$||\phi|| := \max_{a \leq s \leq b} |\phi(s)|$$

where $\phi \in C([a, b], \mathbb{R}^n)$.

To present some preliminary definitions and results, consider the class of systems described by functional differential equations

$$\dot{x}(t)=f(x_d(t),w_d(t)),\quad x_d(t_0)=\xi \tag{A1}$$

where $f:C([-t_d,0],\mathbb{R}^n)\times C([-t_d,0],\mathbb{R}^m)\rightarrow\mathbb{R}^n$ with $f(0,0)=0$ and the initial data is $\xi\in C([-t_d,0],\mathbb{R}^n)$, and w is a bounded, piecewise continuous input. For each initial data and initial time $t_0\geq 0$, we suppose there exists $T_f\geq 0$ and a unique maximal solution $x(\cdot)$ defined on $[t_0-t_d,t_0+T_f]$. We define the following norm

$$||x_d||_{t_0}:=\sup_{s\geq t_0}||x_d(s)||$$

We will make use of the following definition and results on input-to-state stability of the zero solution of the system of Eq. A1 from Ref. 3. The definition and results are stated here for the reader's convenience.

Definition (Ref. 3). *Let $\gamma\in\mathcal{G}$, $v\in\mathbb{R}_{\geq 0}$ and Δ_x, $\Delta_w\in\mathbb{R}_{\geq 0}$. The zero solution of Eq.* A1 *is said to be uniform ISS with gain γ, offset v, and restriction (Δ_x, Δ_w) if $||x_d(t_0)||<\delta_x$ imply $T_f=\infty$ and that the following properties hold uniformly in t_0: (1) for each $\epsilon>0$ there exists $\delta>0$ such that $||x_d(t_0)||\leq\delta$ implies $||x_d||_{t_0}\leq\max\{\epsilon,\gamma(||w_d||_{t_0}),v\}$ and (2) for each $\epsilon>0$ and $\eta_x\in(0,\Delta_x)$ and $\eta_w\in(0,\Delta_w)$ there exists $T>0$ such that $||x_d(t_0)||\leq\eta_x$ and $||w_d||_{t_0}\leq\eta_w$ imply $||x_d||_{t_0+T}\leq\max\{\epsilon,\gamma(||w_d||_{t_0}),v\}$.*

If $V:[-t_d,\infty)\times\mathbb{R}^n\rightarrow\mathbb{R}_{\geq 0}$ is a continuous function, then we use $\dot{V}(t,x(t))$ to denote the upper right-hand derivative of V along the solution of Eq. A1 and is defined as

$$\dot{V}(t,x(t)):=\limsup_{h\rightarrow 0^+}\frac{V(t+h,x(t+h)-V(t,x(t))}{h}\ . \tag{A2}$$

The following two results are needed

Lemma 1 (Refs. 3 and 35). *Let $\mu\geq 0$ and $\alpha\in\mathcal{K}$. If $V(t)\geq\mu$ implies $\dot{V}(t)\leq-\alpha(V(t))$, then there exists $\beta\in\mathcal{KL}$ (independent of μ) with $\beta(s,0)\geq s$ such that $V(t)\leq\max\{\beta(V(t_0),t-t_0),\mu\}$.*

The following stability result for the system of Eq. A1 is from Ref. 3 which is a Razumikhin-type theorem for input-to-state stability of functional differential equations.

Theorem 2 (Ref. 3, Theorem 2). *Suppose there exist a continuous function $V:[-t_d,\infty)\times\mathbb{R}^n\rightarrow\mathbb{R}_{\geq 0}$, α_1, $\alpha_2\in\mathcal{K}_\infty$, $\alpha_3\in\mathcal{K}$, γ_x, $\gamma_w\in\mathcal{G}$, and nonnegative real numbers $\delta<\Delta$ such that*

1. $\alpha_1(|x|)\leq V(t,x)\leq\alpha_2(|x|)$;
2. $|x(t)|\geq\max\{\gamma_x(||x_d(t)||),\ \gamma_w(||w_d(t)||)\}\Rightarrow\dot{V}(t,x(t))\leq-\alpha_3(|x(t)|)$;
3. $\alpha_1^{-1}\circ\alpha_2\circ\gamma_x(s)<s$ *for* $\delta<s<\Delta$.

Let $\beta\in\mathcal{KL}$ be as in the conclusion of Lemma 1 when $\alpha=\alpha_3\circ\alpha_2^{-1}$. Then, the origin of the system of Eq. A1 *is uniformly ISS with gain $\tilde{\gamma}_w:=\alpha_1^{-1}\circ\alpha_2\circ\gamma_w$, offset δ, and restriction (Δ_x, Δ_w) such that $\max\{\alpha_1^{-1}(\beta(\alpha_2(s_1),0)),\tilde{\gamma}_w(s_2)\}<\Delta$ when $s_1<\Delta_x$, $s_2<\Delta_w$.*

Proof of Proposition 1

The sampled-data system of Eq. 3 can be written as a time-delay system with time-varying delay to account for sampling

$$\dot{x}(t)=f(x(t),x(t-d_1),h_c(x(t-\tau(t)))) \tag{A3}$$

where $\tau(t)=t-\lfloor(t-d_2)/\Delta\rfloor\Delta$ and $\tau(t)\leq(d_2/\Delta+1)\Delta=:N_d\Delta$ ($N_d:=(d_2/\Delta+1)$). In the sampled-data system with input delay setting that we consider, the quantity $t-\tau(t)$ represents the sampling time instance that feedback is received to compute the control action applied to the system at time t. The maximum amount of time in the past that the feedback measurement used to compute the control action applied to the system at the current time (the maximum of $\tau(t)$) is the sampling time (Δ) plus the input delay (d_2).

The system of Eq. A3 may be analyzed as a perturbed form of the system without delays and sampling (i.e., the system $\dot{x}=f(x,x,h_c(x))$). In other words, consider the following system

$$\begin{aligned}\dot{x}(t)&=f(x(t),x(t)+\xi_1(t),h_c(x(t))+\xi_2(t))\\ \xi_1(t)&=x(t-d_1)-x(t)\\ \xi_2(t)&=h_c(x(t-\tau(t)))-h_c(x(t))\end{aligned} \tag{A4}$$

We define the following: $\xi^T:=[\xi_1^T\ \xi_2^T]$ with $\xi(t)\in D\times\mathbb{U}\subset\mathbb{R}^{n+m}$, $g(x,\xi):=f(x,x+\xi_1,h_c(x)+\xi_2)$, and $t_d:=\max\{2d_1,2N_d\Delta\}$. The perturbation term $\xi(t)$ is bounded. From the triangle inequality, we have

$$|\xi(t)|=|[\xi_1^T(t)\ 0^T]^T+[0\ \xi_2^T(t)]^T|\leq|\xi_1(t)|+|\xi_2(t)| \tag{A5}$$

Owing to the fact that f and h_c are locally Lipschitz vector functions, there exists a $\gamma_1^*\in\mathcal{K}$ such that

$$\begin{aligned}|\xi_1(t)|&=\left|\int_{t-d_1}^{t}f(x(s),x(s-d_1),h_c(x(s-\tau(s))))\ ds\right|\\ &\leq d_1\gamma_1^*(||x_d(t)||)\end{aligned} \tag{A6}$$

where $||x_d(t)||=\max_{s\in[-t_d,0]}|x(t+s)|$. Again, by the locally Lipschitz properties assumed for f and h_c, there exist functions L_h, $\gamma_2^*\in\mathcal{K}$ such that

$$\begin{aligned}|\xi_2(t)|&\leq L_h(||x_d(t)||)|x(t-\tau(t))-x(t)|\\ &\leq N_d\Delta L_h(||x_d(t)||)\gamma_2^*(||x_d(t)||)\end{aligned} \tag{A7}$$

From the inequalities of Eqs. A5–A7, there exists a $\gamma^*\in\mathcal{K}$ such that

$$|\xi(t)|\leq\tilde{t}_d\gamma^*(||x_d(t)||) \tag{A8}$$

where $\tilde{t}_d=\max\{d_1,N_d\Delta\}$.

The time-derivative of V along the state trajectory of Eq. 3 is

$$\begin{aligned}\dot{V}&=\frac{\partial V}{\partial x}(x(t))g(x(t),0)+\frac{\partial V}{\partial x}(x(t))[g(x(t),\xi(t))-g(x(t),0)]\\ &\leq-\alpha_3(|x(t)|)+\alpha_4(|x(t)|)|g(x(t),\xi(t))-g(x(t),0)|\end{aligned} \tag{A9}$$

for all $x(t)\in D$ where the inequality follows from Eqs. 2b and 2c. As g is a locally Lipschitz vector function (this follows from the fact that f and h_c are locally Lipschitz), there exists a function $L_\xi\in\mathcal{K}$ such that

$$\begin{aligned}\dot{V}&\leq-\alpha_3(|x(t)|)+\alpha_4(|x(t)|)L_\xi(|\xi(t)|)\\ &\leq-\alpha_3(|x(t)|)+\alpha_4(|x(t)|)L_\xi(\tilde{t}_d\gamma^*(||x_d(t)||))\end{aligned} \tag{A10}$$

where the second inequality follows from Eq. A8. For some pair of strictly positive real numbers δ_1 and δ_2 such that $\delta_1<\delta_2$ and $\{x\in\mathbb{R}^n\ :\ |x|\leq\delta_2\}\subset D$, there exists $t_d^*>0$ such that $\tilde{t}_d\in(0,t_d^*)$ implies

$$\begin{aligned}&|x(t)|\geq\alpha_3^{-1}(\alpha_4(\delta_2)L_\xi(\tilde{t}_d\gamma^*(||x_d(t)||))/q)\\ &\Rightarrow\dot{V}\leq-(1-q)\alpha_3(|x(t)|)\end{aligned} \tag{A11}$$

for some $q\in(0,1)$ and

$$\alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}\left(\frac{\alpha_4(\delta_2)L_\xi(\tilde{t}_d\gamma^*(s))}{q}\right) < s \tag{A12}$$

for all $s \in (\delta_1, \delta_2)$. We note that the size of δ_2, which governs the restriction on the initial condition, and $\tilde{t}_d$, which accounts for the magnitude of delays and the sampling period size, are restricted to be sufficiently small so that the arguments of the inverted class $\mathcal{K}$ functions in Eqs. A11 and A12 are within the domain of the functions. From Theorem 2 (taking $\gamma_x(\cdot)=\alpha_3^{-1}(\alpha_4(\delta_2)L_\xi(\tilde{t}_d\tilde{\gamma}^*(\cdot)/q))$, the origin of the sampled-data time-delay system of Eq. 3 is uniformly ISS with offset δ_1 and restriction δ_2 for all $\tilde{t}_d \in (0, t_d^*)$.

Proof of Proposition 2

Owing to the fact that f is locally Lipschitz, there exist positive constants L_{x_1}, L_{x_2}, and L_u such that

$$\begin{aligned} &|f(x,y,u)-f(z,z,v)| \\ &\leq L_{x_1}|x-z|+L_{x_2}|y-z|+L_u|u-v| \end{aligned} \tag{A13}$$

for all $x,\ y,\ z \in \Omega_\rho$ and $u,\ v \in \mathbb{U}$. Consider the difference between the state, $x(t)$, of Eq. 5 and the state, $z(t)$ of Eq. 6, for $t \in [t_0, t_0+t_1]$ where $t_1 > 0$ is such that $x(t)$ and $z(t)$ are bounded in Ω_ρ for all $t \in [t_0, t_0+t_1]$ (the existence of $t_1 > 0$ follows from continuity arguments and the fact that $S \subset \Omega_\rho$ is compact). Let $e(t) := x(t)-z(t)$ which has dynamics $\dot{e}(t)=\dot{x}(t)-\dot{z}(t)$. The error dynamics can be bounded using Eq. A13

$$\begin{aligned} |\dot{e}(t)| &= |f(x(t),x(t-d_1),u(t-d_2))-f(z(t),z(t),v(t))| \\ &\leq L_{x_1}|x(t)-z(t)|+L_{x_2}|x(t-d_1)-z(t)|+L_u|u(t-d_2)-v(t)| \\ &\leq L_{x_1}|e(t)|+L_{x_2}|x(t-d_1)-x(t)+x(t)-z(t)|+L_u|u(t-d_2)-v(t)| \\ &\leq (L_{x_1}+L_{x_2})|e(t)|+L_{x_2}|x(t-d_1)-x(t)|+2L_u|u_{\max}| \end{aligned} \tag{A14}$$

for all $t \in [t_0, t_0+t_1]$ where $x(t),\ x(t-d_1),\ z(t) \in \Omega_\rho$ and $u(t-d_2),\ v(t) \in \mathbb{U}$. The last inequality of Eq. A14 follows from the triangle inequality and the fact that $u(t-d_2)$ and $v(t)$ are bounded in $\mathbb{U}$. For all $x(s) \in \Omega_\rho$ for $s \in [t-d_1, t]$, we can bound the difference between $x(t-d_1)$ and $x(t)$

$$\begin{aligned} |x(t-d_1)-x(t)| &\leq d_1||x_d(t)|| \\ &\leq d_1\alpha_1^{-1}(\rho) \end{aligned} \tag{A15}$$

where $||x_d(t)||$ is a slight abuse of notation and it denotes the max-norm of $x_d(t) \in C([-d_1, 0], \mathbb{R}^n)$ (i.e., $||x_d(t)|| = \max_{\theta\in[-d_1,0]}|x(t-\theta)|$) and the last inequality follows from Eq. 2a. From Eqs. A14 and A15, the error dynamics can be bounded by

$$|\dot{e}(t)| \leq (L_{x_1}+L_{x_2})|e(t)|+L_{x_2}d_1\alpha_1^{-1}(\rho)+2L_u|u_{\max}| \tag{A16}$$

The error is bounded for all $t \in [t_0, t_0+t_1]$ which can be shown by integrating the bound of Eq. A16 and noting $e(t_0)=0$ to derive the following bound

$$|e(t)| \leq \frac{L_{x_2}d_1\alpha_1^{-1}(\rho)+2L_u|u_{\max}|}{(L_{x_1}+L_{x_2})}[\exp((L_{x_1}+L_{x_2})(t-t_0))-1] \tag{A17}$$

for all $t \in [t_0, t_0+t_1]$. Taking

$$\bar{a}(s) := \frac{L_{x_2}d_1\alpha_1^{-1}(\rho)+2L_u|u_{\max}|}{(L_{x_1}+L_{x_2})}[\exp((L_{x_1}+L_{x_2})s)-1] \tag{A18}$$

completes the proof.

Proof of Theorem 1

To prove the theorem, we need to show the following: (1) the LEMPC is feasible for all $x(t_k) \in \Omega_{\hat{\rho}}$, under the LEMPC, (2) the state is bounded in Ω_ρ under the LEMPC, and (3) the closed-loop state starting from $\Omega_\rho \setminus \Omega_{\hat{\rho}}$ will converge to $\Omega_{\hat{\rho}}$ in a finite number of sampling times without coming out of Ω_ρ. To help the readability of the proof, we divide the proof into three parts corresponding to the three results that we need to show, respectively.

Part 1: Feasibility of the LEMPC is proved provided that the state at the current sampling time is in $\Omega_{\hat{\rho}}$. If $\tilde{t}_d \in (0, t_d^*)$ (where $\tilde{t}_d$ and t_d^* are defined according to Proposition 1), the input trajectory computed by the stabilizing control law $h_c(x)$ applied in a sample-and-hold fashion is a feasible solution to the OCP of Eq. 4. Specifically, let $\hat{z}(t)$ and $\hat{v}(t)$ denote the solution and input trajectory, respectively, to the system

$$\dot{\hat{z}}(t)=\bar{f}(\hat{z}(t),\hat{v}(t))\hat{v}(t)=h_c(\hat{z}(j\Delta)),\ \ t \in [t_j, t_{j+1}) \tag{A19}$$

for $j=k,\ k+1,\ \ldots,\ k+N-1$ with the initial condition $\hat{z}(t_k)=x(t_k) \in \Omega_{\hat{\rho}}$. Owing to the properties of the stabilizing control law $h_c(x)$, $\hat{v}(t) \in \mathbb{U}$ for all $t \in [t_k, t_k+T)$. Moreover, from Proposition 1, the input trajectory $\hat{v}$ is a feasible input trajectory to the OCP of Eq. 4. This statement holds as $\Omega_{\hat{\rho}}$ is forward invariant for the sampled-data system of Eq. A19 if $B_{\delta_1} \subseteq \Omega_{\hat{\rho}}$ and if the sampling period is sufficiently small such that Proposition 1 holds (i.e., the proof of forward invariance of the set $\Omega_{\hat{\rho}}$ for the system of Eq. A19 follows that of Proposition 1 with $d_1=d_2=0$).

Part 2: We consider boundedness of the state in Ω_ρ under LEMPC. The LEMPC computes an input trajectory such that the predicted state will be maintained in $\Omega_{\hat{\rho}}$ over the prediction horizon. However, the system of Eq. 1 does not evolve according to the ODE model of Eq. 4b and the control actions applied to the system over the time t_k to t_k+d_1 may be computed by either the LEMPC or the stabilizing control law at previous sampling times. Specifically, the predicted state, $z(t)$, over the prediction horizon will be bounded in $\Omega_{\hat{\rho}}$ for all $t \in [t_k, t_k+T)$ under the computed input trajectory which is guaranteed by the constraint of Eq. 4e. Owing to the input delay the control action computed at t_k is applied to the system from t_k+d_2 to $t_{k+1}+d_2$. We need to show that $x(t) \in \Omega_\rho$ for all $t \in [t_k, t_{k+1}+d_2]$ when $x(t_k) \in \Omega_{\hat{\rho}}$.

Let $\hat{\rho}$ satisfy Eq. 10. We proceed by contradiction and assume there exists a time $\tau^* \in [t_k, t_{k+1}+d_2]$ such that $V(x(\tau^*)) > \rho$ (the case that $x(t)$ is not defined for some $\tau \in [t_k, t_{k+1}+d_2]$ is also covered by this assumption). We define $\tau_1 := \inf\{\tau \in [t_k, t_{k+1}+d_2] : V(x(\tau)) > \rho\}$. A standard continuity argument in conjunction with the fact that $V(x(t_k)) \leq \hat{\rho} < \rho$ shows that $\tau_1 \in (t_k, t_{k+1}+d_2]$, $V(x(t)) \leq \rho$ for all $t \in [t_k, \tau_1]$ with $V(x(\tau_1))=\rho$, and $V(x(t)) > \rho$ for some $t \in [\tau_1, t_{k+1}+d_2]$. If $\hat{\rho}$ satisfies Eq. 10, we have

$$\begin{aligned} \rho=V(x(\tau_1)) &\leq V(z(\tau_1))+f_V(\bar{\alpha}(\tau_1)) \\ &\leq \hat{\rho}+f_V(\bar{\alpha}(\Delta+d_2)) < \rho \end{aligned} \tag{A20}$$

where the first inequality follows from Propositions 2 and 3 and the second inequality follows from the fact that $f_V \circ \bar{\alpha} \in \mathcal{K}$ and $\tau_1 \leq \Delta+d_2$. Equation A20 leads to a contradiction. Thus, $x(t_{k+1}+d_1) \in \Omega_\rho$ if Eq. 10 is satisfied (regardless of whether the LEMPC or the stabilizing control $h_c(x)$ is used to compute the input trajectory over t_k to t_k+d_1).

Part 3: We prove that the state will converge to $\Omega_{\hat{\rho}}$ for any state starting in $\Omega_\rho \setminus \Omega_{\hat{\rho}}$. When the current state $x(t_k) \in \Omega_\rho \setminus \Omega_{\hat{\rho}}$, the control action is computed from the stabilizing control law $h_c(x)$. First, consider $t_k \in [0, d_2]$, by assumption of the initial data and initial input value, the past data from $t_k - d_1$ to t_k is in Ω_ρ. The stabilizing control law will force the state to converge to $\Omega_{\hat{\rho}}$ without coming out of Ω_ρ in finite-time if $\tilde{t}_d \in (0, t_d^*)$. This follows from Proposition 1.

Now, let us consider the case that the LEMPC previously computed the control actions for the system. Owing to the plant-model mismatch, the state may come outside of $\Omega_{\hat{\rho}}$. From Part 2, the state will still remain bounded in Ω_ρ if indeed it comes outside of $\Omega_{\hat{\rho}}$. Again, once the stabilizing control law $h_c(x)$ starts to compute control actions for the system to force the state to converge to $\Omega_{\hat{\rho}}$ all of the assumptions of Proposition 1 are satisfied (assuming $\tilde{t}_d \in (0, t_d^*)$) and thus, the state will be forced back to $\Omega_{\hat{\rho}}$ in finite-time without coming out of Ω_ρ when $x(t_k) \in \Omega_\rho \setminus \Omega_{\hat{\rho}}$ which completes the proof.

Manuscript received Mar. 13, 2015, and revision received July 16, 2015.