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State-estimation-based economic model predictive control of nonlinear systems

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ABSTRACT

In this work, we focus on a class of nonlinear systems and design an estimator-based economic model predictive control (MPC) system which is capable of optimizing closed-loop performance with respect to general economic considerations taken into account in the construction of the cost function. Working with the class of full-state feedback linearizable nonlinear systems, we use a high-gain observer to estimate the nonlinear system state using output measurements and a Lyapunov-based approach to design an economic MPC system that uses the observer state estimates. We prove, using singular perturbation arguments, that the closed-loop system is practically stable provided the observer gain is sufficiently large. We use a chemical process example to demonstrate the ability of the state-estimation-based economic MPC to achieve process time-varying operation that leads to a superior cost performance metric compared to steady-state operation using the same amount of reactant material.

1. Introduction

The development of optimal operation and control policies for chemical process systems aiming at optimizing process economics has always been an important research subject with major practical implications. Traditionally, economic considerations are addressed via a two-layer approach in which the upper layer carries steady-state process optimization to obtain economically optimal process operating points (steady states) while the lower layer utilizes appropriate feedback control systems to steer the process state to an economically optimal steady state. Model predictive control (MPC) is widely used in the lower layer to obtain optimal manipulated input values by minimizing a (typically) quadratic cost function which usually penalizes the deviation of the system state and manipulated inputs from their economically optimal steady-state values subject to input and state constraints [1,2]; however, this two-layer approach usually limits process operation around a steady state. The economic model predictive control (EMPC) framework deals with a reformulation of the conventional MPC quadratic cost function in which an economic (not necessarily quadratic) cost function is used directly as the cost in MPC, and thus, it may, in general, lead to time-varying process operation policies (instead of steady-state operation), which directly optimize process economics.

With respect to recent results on economic MPC, efforts have focused on combination of steady-state optimization and linear MPC [3], stability of economic MPC of nonlinear systems through employing a terminal constraint which requires that the closed-loop system state settles to a steady state at the end of the prediction horizon [4] and economic MPC of cyclic processes (including closed-loop stability analysis using a suitable terminal constraint) [5]. In a previous work [6], we presented a two-mode Lyapunov-based economic MPC (LEMPC) design for nonlinear systems which is also capable of handling asynchronous and delayed measurements and extended it in the context of distributed MPC [7]. Currently, all economic MPC schemes including the ones above have been developed under the assumption of state feedback. State estimation in certain classes of nonlinear systems can be carried out within the framework of high-gain observers (e.g., [8,9]), however, at this stage these estimation techniques have not been used in conjunction with economic MPC schemes.

Motivated by this, in this work, we focus on a class of nonlinear systems and design an estimator-based EMPC system. Working with the class of full-state feedback linearizable nonlinear systems, we use a high-gain observer to estimate the nonlinear system state using output measurements and a Lyapunov-based approach to design an EMPC system that uses the observer state estimates. We prove, using singular perturbation arguments, that the closed-loop system is practically stable provided the observer gain is sufficiently large. We use a chemical process example to demonstrate the ability of the state-estimation based EMPC to achieve process time-varying operation that leads to a superior cost performance metric compared to steady-state operation. In the example, the high-gain observer is used to obtain estimates

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of the reactant concentration from temperature measurements; a meaningful case in process control practice.

2. Preliminaries

2.1. Notation

The notation $|\cdot|$ is used to denote the Euclidean norm of a vector. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$, and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. The symbol Ω_r is used to denote the set $\Omega_r := \{x \in R^{n_x} : V(x) \le r\}$ where V is a sufficiently smooth, positive definite scalar function and $r > 0$, and the operator '/' denotes set subtraction, that is, $A/B := \{x \in R^{n_x} : x \in A, x \notin B\}$. The notation $L_f^k h(\cdot)$ denotes the standard kth order Lie derivative of a scalar function $h(\cdot)$ with respect to the vector function $f(\cdot)$. The notation $L_g L_f h(\cdot)$ denotes the mixed Lie derivative of a scalar function $h(\cdot)$, with respect to vector functions $f(\cdot)$ and $g(\cdot)$. The symbol $\mathrm{diag}(v)$ denotes a matrix whose diagonal elements are the elements of vector v and all the other elements are zeros. $sat(\cdot)$ denotes the standard saturation function. I_n and 0_n are the identity matrix and a vector of zeros of dimension n, respectively. Also, the ball B_δ with radius $\delta > 0$ is defined as $B_\delta = \{x \in R^{n_x} : |x| \le \delta\}$.

2.2. Class of nonlinear systems

We consider single-input single-output nonlinear systems described by the following state-space model:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in R^{n_x}$ denotes the vector of state variables of the system, $x(t_0) = x(0) = x_0$, $u \in R$ is the manipulated input and $y \in R$ is the measured output. The manipulated input is restricted to be in a nonempty convex set $U \subseteq R$, which is defined as $U := \{u \in R : |u| \le u^{\max}\}$ where $u^{\max}$ is the magnitude of the input constraint. We assume that f, g and h are sufficiently smooth functions and that the origin is an equilibrium point of the unforced nominal system (i.e., system of Eq. (1) with $u(t) \equiv 0$) which implies that $f(0) = 0$. Without loss of generality, in this work we focus on single input, single output systems; however, the proposed approach can be extended to multi-input multi-output systems in a conceptually straightforward manner. We assume that the output measurement y of the system is continuously available at all times. We also assume that the system in Eq. (1) is full-state feedback linearizable. Thus, the relative degree of the output with respect to the input is n. Assumption 1 states this requirement.

Assumption 1. There exists a set of coordinates

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \tag{2}$$

such that the system of Eq. (1) can be written as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u \\ y &= z_1 \end{aligned}$$

where $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in R^{n_x}$.

Using Assumption 1, the system of Eq. (1) can be rewritten in the following compact form:

$$\begin{aligned} \dot{z} &= Az + B[L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u] \\ y &= Cz \end{aligned}$$

where

$$A = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}^T \end{bmatrix}, \qquad B = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ 0_{n-1} \end{bmatrix}^T.$$

Remark 1. We note that Assumption 1 imposes certain practical restrictions on the applicability of the method, however, this should be balanced with the nature of the results achieved by the output feedback controller (please see Theorem 1) in the sense that for a sufficiently large observer gain, the closed-loop system under the output feedback controller approaches the closed-loop stability region and performance of the state feedback controller (essentially a nonlinear separation-principle that is achieved because of Assumption 1 and the use of a high-gain observer). This is an assumption imposed in all previous works that use high-gain observers for state estimation, starting from the early work of Khalil and co-workers [10]. With respect to practical restrictions, our example demonstrates that the method is applicable to a class of chemical reactor models. We note that the requirement of full state linearizability can be relaxed by allowing for inverse dynamics (the case where the relative degree, r, is smaller than the system dimension n; i.e., input/output linearizable systems) at the expense of having an additional observer to estimate the state of the inverse dynamics; please see [11] for a detailed development of this case.

2.3. Stabilizability assumption

We assume that there exists a state feedback controller $u = k(x)$, which renders the origin of the closed-loop system asymptotically stable while satisfying the input constraints for all the states x inside a given stability region. Using converse Lyapunov theorems [12,13], this assumption implies that there exist class $\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$ and a continuously differentiable Lyapunov function $V(x)$ for the closed-loop system, that satisfy the following inequalities:

$$\begin{aligned} &\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \\ &\frac{\partial V(x)}{\partial x}(f(x) + g(x)k(x)) \le -\alpha_3(|x|) \\ &\left|\frac{\partial V(x)}{\partial x}\right| \le \alpha_4(|x|) \\ &k(x) \in U \end{aligned} \tag{3}$$

for all $x \in D \subseteq R^{n_x}$ where D is an open neighborhood of the origin. We denote the region $\Omega_\rho \subseteq D$ as the stability region of the closed-loop system under the controller $k(x)$. Using the smoothness assumed for the f and g, and taking into account that the manipulated input u is bounded, there exists a positive constant M such that

$$|f(x) + g(x)u| \le M \tag{4}$$

for all $x \in \Omega_\rho$ and $u \in U$. In addition, by the continuous differentiable property of the Lyapunov function $V(x)$ and the smoothness of f and g, there exist positive constants L_x, L_u, C_x, $C_{g'}$ and C_g

such that

$$\begin{aligned}
&\left|\frac{\partial V}{\partial x}f(x)-\frac{\partial V}{\partial x}f(x')\right|\le L_x|x-x'|\\
&\left|\frac{\partial V}{\partial x}g(x)-\frac{\partial V}{\partial x}g(x')\right|\le L_u|x-x'|\\
&|f(x)-f(x')|\le C_x|x-x'|\\
&|g(x)-g(x')|\le C_{g'}|x-x'|\\
&\left|\frac{\partial V}{\partial x}g(x)\right|\le C_g
\end{aligned} \tag{5}$$

for all $x, x' \in \Omega_\rho$ and $u \in U$.

Remark 2. Note that in the present work, we use the level set Ω_ρ of the Lyapunov function $V(x)$ to estimate the stability region (i.e., domain of attraction) of the closed-loop system under the controller $k(x)$. Specifically, an estimate of the domain of attraction of the closed-loop system is computed as follows: first, a controller (e.g., $k(x)$) is designed that makes the time-derivative of a Lyapunov function, $V(x)$, along the closed-loop system trajectory negative definite in an open neighborhood around the equilibrium point; then, an estimate of the set where $\dot{V}$ is negative is computed, and finally, a level set (ideally the largest) of V (denoted by Ω_ρ in the present work) embedded in the set where $\dot{V}$ is negative, is computed; see Section 5 for an application of this approach.

2.4. State estimation via high gain observer

The state-estimation-based EMPC takes advantage of a high-gain observer [10,14], which obtains estimates of the output derivatives up to order $n-1$ and consequently, provides estimates of the transformed system state z, to obtain the estimated state of the system $\hat{x}$ through the inverse transformation $T^{-1}(\cdot)$. Proposition 1 defines the high-gain observer equations and establishes precise conditions under which the combination of the high-gain observer and of the controller $k(x)$ together with appropriate saturation functions to eliminate wrong estimates enforce asymptotic stability of the origin in the closed-loop system for sufficiently large observer gain. The proof of the proposition follows from the results in [11,9].

Proposition 1. *Consider the nonlinear system of Eq. (1) for which Assumption 1 holds. Also, assume that there exists a $k(x)$ for which Eq. (3) holds and it enforces local exponential stability of the origin in the closed-loop system. Consider the nonlinear system of Eq. (1) under the output feedback controller*

$$u = k(\hat{x}) \tag{6}$$

where

$$\hat{x} = T^{-1}(sat(\hat{z})) \tag{7}$$

and

$$\dot{\hat{z}} = A\hat{z} + L(y - C\hat{z}) \tag{8}$$

with

$$L = \left[\frac{a_1}{\epsilon} \quad \frac{a_2}{\epsilon^2} \quad \cdots \quad \frac{a_n}{\epsilon^n}\right]^T,$$

and the parameters a_i are chosen such that the roots of

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \tag{9}$$

are in the open left-half of the complex plane. Then given δ, there exists ϵ^* such that if $\epsilon \in (0, \epsilon^*]$, $|\hat{z}(0)| \le z_m$, $x(0) \in \Omega_\delta$ with z_m being the maximum of the vector $\hat{z}$ for $|\hat{z}| \le \beta_z(\delta_z, 0)$ where β_z is a class $\mathcal{KL}$ function and $\delta_z = \max\{|T(x)|, x \in \Omega_\delta\}$; the origin of the closed-loop system is asymptotically stable. This stability property implies that given $\epsilon \in (0, \epsilon^*]$ and some positive constant $e_m > 0$ there exists a positive real constant $t_b > 0$ such that if $x(0) \in \Omega_\delta$ and $|\hat{z}(0)| \le z_m$, then $|x(t) - \hat{x}(t)| \le e_m$ for all $t \ge t_b$.

Remark 3. Note that in Proposition 1, the saturation function, $sat(\cdot)$, is used to eliminate the peaking phenomenon associated with the high-gain observer, see for example [10]. Note also that it is considered that the estimated state $\hat{x}$ has converged to the actual state x, when the estimation error $|x - \hat{x}|$ is less than or equal to a given bound e_m. The time needed to converge, is given by t_b which is proportional to ϵ. During this transient, the value of the Lyapunov function $V(x)$ may increase. Finally, we note that for nonlinear MPC designs, computational complexity of the state estimation scheme is very critical. A high-gain observer, as adopted in this work, can be solved very fast and it could be more suitable in the context of output feedback control. Other state observers (e.g., moving horizon estimation) may also be used to estimate the system state but the closed-loop stability for this case needs to be studied carefully.

3. State-estimation-based economic MPC

In this section, we consider the design of an estimation-based Lyapunov-based EMPC (LEMPC) for nonlinear systems. We assume that the output measurements are continuously available. Also, LEMPC is evaluated at synchronous time instants $\{t_{k\ge 0}\}$ with $t_k = t_0 + k\Delta,\ k = 0, 1, \ldots$ where $t_0 = 0$ is the first time that LEMPC is evaluated while the high gain observer has converged and Δ is the LEMPC sampling time.

3.1. Implementation strategy

The high gain observer of Eq. (8) receives output measurements (i.e., y) and provides estimated system states (i.e., $\hat{x}$) continuously. At each sampling time t_k, the LEMPC obtains the estimated system state $\hat{x}(t_k)$ from the observer. Based on $\hat{x}(t_k)$, the LEMPC takes advantage of the nominal system model to predict the future evolution of the system over a finite prediction horizon while maximizing a cost function that accounts for specific economic considerations.

The two-mode operation architecture in [6] is adopted in the design of the LEMPC. Specifically, we assume that from time t_0 up to a specific time t' where without loss of generality t' is assumed to be a multiple of LEMPC sampling time, the LEMPC operates in the first operation mode to maximize the economic cost function while maintaining the closed-loop system state in the stability region Ω_ρ. In this operation mode, in order to account for the high gain observer effect, we consider another region Ω_{ρ_e} with $\rho_e < \rho$. If the estimated current state is in the region Ω_{ρ_e}, the LEMPC maximizes the cost function within the region Ω_{ρ_e}; if the estimated current state is in the region $\Omega_\rho/\Omega_{\rho_e}$, the LEMPC first drives the system state to the region Ω_{ρ_e} and then maximizes the cost function within Ω_{ρ_e}.

After time t', the LEMPC operates in the second operation mode and calculates the inputs in a way that the state of the closed-loop system is driven to a neighborhood of the desired steady state through the knowledge of the Lyapunov-based controller $k(x)$.

The above described implementation strategy of the proposed LEMPC can be summarized as follows:

1. Based on the output measurements $y(t)$, the high gain observer estimates continuously the system state $\hat{x}(t)$. The LEMPC gets a sample of the estimated system state at t_k from the observer.

2. If $t_k < t'$, go to Step 3. Else, go to Step 4.
3. If $\hat{x}(t_k) \in \Omega_{\rho_e}$, go to Step 3.1. Else, go to Step 3.2.
   3.1. The controller maximizes the economic cost function within Ω_{ρ_e}. Go to Step 5.
   3.2. The controller drives the system state to the region Ω_{ρ_e} and then maximizes the economic cost function within Ω_{ρ_e}. Go to Step 5.
4. The controller drives the system state to a small neighborhood of the origin.
5. Go to Step 1 ($k \longleftarrow k+1$).

Remark 4. The two-mode operation in the design and implementation of the proposed output feedback LEMPC is adopted in order to reconcile two objectives: (1) time-varying operation (off steady-state operation) of the process that optimizes a given economic cost function, ensuring boundedness for the closed-loop system state within a well-defined stability region (mode 1), and (2) eventual convergence of the closed-loop system state to an economically optimal steady-state (mode 2). We note that it is not necessary to adopt a two-mode operation strategy and it is possible to operate the process under mode 1 for arbitrarily large period of time (i.e., t' can be made arbitrarily large). The operation in mode 2, where the state of the closed-loop system state converges eventually to a steady state (potentially economically optimal) is very often dictated by practical considerations which require time-invariant operation at steady state to minimize wear and tear on the control actuators. Possible reasons for picking t' (i.e., duration of operation in mode 1) in practice may include acceptable time to operate the process in time-varying fashion given actuator specifications and economic considerations.

3.2. LEMPC formulation

The LEMPC is evaluated to obtain the future input trajectories based on estimated state $\hat{x}(t_k)$ provided by the high gain observer. Specifically, the optimization problem of the proposed LEMPC is as follows:

$$\max_{u\in S(\Delta)} \int_{t_k}^{t_{k+N}} L(\tilde{x}(\tau), u(\tau))d\tau \tag{10a}$$

$$\text{s.t. } \dot{\tilde{x}}(\tau) = f(\tilde{x}(\tau)) + g(\tilde{x}(\tau))u(\tau) \tag{10b}$$

$$u(\tau) \in U, \quad \tau \in [t_k, t_{k+N}) \tag{10c}$$

$$\tilde{x}(t_k) = \hat{x}(t_k) \tag{10d}$$

$$V(\tilde{x}(t)) \le \rho_e, \quad \forall t \in [t_k, t_{k+N}), \text{ if } t_k \le t' \text{ and } V(\hat{x}(t_k)) \le \rho_e \tag{10e}$$

$$L_g V(\hat{x}(t_k))u(0) \le L_g V(\hat{x}(t_k))k(\hat{x}(t_k)), \quad \text{if } t_k > t' \text{ or } \rho_e < V(\hat{x}(t_k)) \le \rho \tag{10f}$$

where $\tilde{x}$ is the predicted trajectory of the system with control input calculated by this LEMPC and $S(\Delta)$ is the family of piecewise continuous functions with period Δ which allows us to obtain an optimization problem to be solved at each sampling time with a finite number of decision variables. The constraint of Eq. (10b) is the system model used to predict the future evolution of the system subject to the input constraint of Eq. (10c). The constraint of Eq. (10e) is associated with the operation mode 1 which restricts the predicted system state to be in the set Ω_{ρ_e} while the constraint of Eq. (10f) is associated with the operation mode 2 and the operation mode 1 when the estimated system state is out of the predefined set Ω_{ρ_e}. This constraint makes sure that the amount of reduction of the Lyapunov function value when the first step of LEMPC input is applied is at least at the level achieved by applying $k(x)$. The optimal solution to this optimization problem is denoted by $u^*(t|t_k)$, which is defined for $t \in [t_k, t_{k+N})$. The manipulated input of the LEMPC of Eq. (10) is defined as follows:

$$u(t) = u^*(t|t_k), \quad \forall t \in [t_k, t_{k+1}). \tag{11}$$

4. Closed-loop stability analysis

To state our main closed-loop stability result, we need the following proposition.

Proposition 2 (*Cf.* [6]). *Consider the system of Eq. (1) in closed loop under the LEMPC of Eq. (10) with state feedback (i.e., $\tilde{x}(t_k) = x(t_k)$) based on a controller $k(\cdot)$ that satisfies the conditions of Eq. (3). Let $\epsilon_w > 0$, $\Delta > 0$ and $\rho > \rho_s > 0$ satisfy the following constraint:*

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + L_x M\Delta \le -\epsilon_w/\Delta. \tag{12}$$

If $x(0) \in \Omega_\rho$, then $x(t) \in \Omega_\rho$, $\forall t \ge 0$. Furthermore, there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that

$$|x(t)| \le \beta(|x(t^*)|, t-t^*) + \gamma(\rho^*) \tag{13}$$

with $\rho^* = \max\{V(x(t+\Delta)) : V(x(t)) \le \rho_s\}$, $\forall x(t^*) \in B_\delta \subset \Omega_\rho$ and $\forall t \ge t^* > t'$ where t^* is chosen such that $x(t^*) \in B_\delta$.

Theorem 1 provides sufficient conditions under which the state-estimation-based LEMPC of Eq. (10) with the high-gain observer of Eq. (8) guarantees that the state of the closed-loop system of Eq. (1) is always bounded and is ultimately bounded in a small region containing the origin. To state Theorem 1, we need the following definitions:

$$e_i = \frac{1}{\epsilon^{n-i}}(y^{(i-1)} - \hat{z}_i), \quad i = 1, \dots, n, \tag{14}$$

$$e = [e_1\, e_2\, \dots\, e_n]^T \tag{15}$$

and

$$A^* = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 0 & 1 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

where $y^{(i-1)}$ is the $(i-1)$th derivative of the output measurement y and $\hat{z}_i$ is the ith component of $\hat{z}$.

Theorem 1. *Consider the system of Eq. (1) in closed-loop with u computed by the state-estimation-based LEMPC of Eqs. (7), (8) and (10) based on a feedback controller $k(\cdot)$ that satisfies the conditions of Eq. (3). Let Assumption 1, Eqs. (12) and (14)–(16) hold and choose the parameters a_i ($i = 1, \dots, n$) such that the roots of Eq. (9) are in the open left-half of the complex plane. Then there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ, a pair of positive real numbers (δ_x, d_x), $0 < \rho_e < \rho$, $\epsilon^* > 0$ and $\Delta^* > 0$ such that if $\max\{|x(0)|, |e(0)|\} \le \delta_x$, $\epsilon \in (0, \epsilon^*]$, $\Delta \in (0, \Delta^*]$,*

$$-\alpha_3(\alpha_1^{-1}(\rho_s)) + (M\Delta + e_m)(L_x + L_u u^{\max}) < 0 \tag{17}$$

and

$$\rho_e \le \rho - \alpha_4(\alpha_1^{-1}(\rho))M\max\{t_b(\epsilon), \Delta\} \tag{18}$$

with t_b defined in Proposition 1, then $x(t) \in \Omega_\rho \forall t \ge 0$. Furthermore, $\forall t \ge t^* > t'$, the following bound holds:

$$|x(t)| \le \beta(|x(t^*)|, t-t^*) + \gamma(\rho^*) + d_x. \tag{19}$$

Proof. When $u = u^*$ is obtained from the state-estimation-based LEMPC of Eqs. (7), (8) and (10), the closed-loop system takes the following singularly perturbed form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u^*(\hat{x}) \\ \epsilon\dot{e} &= A^*e + \epsilon b L_f^n h(T^{-1}(z)) + \epsilon b L_g L_f^{n-1} h(T^{-1}(z))u^*(\hat{x}). \end{aligned} \tag{20}$$

First, we compute the reduced-order slow and fast closed-loop subsystems related to Eq. (20) and prove the closed-loop stability of the slow and fast subsystems.

Setting $\epsilon = 0$ in Eq. (20), we obtain the corresponding slow subsystem as follows:

$$\dot{x} = f(x) + g(x)u^*(\hat{x}) \tag{21a}$$

$$A^* e = 0. \tag{21b}$$

Taking into account the fact that A^* is non-singular and $e = [0\,0\,\ldots\,0]^T$ is the unique solution of Eq. (21b), we can obtain $\hat{z}_i = y^{(i-1)}, i = 1, \ldots, n$ and $x(t) = \hat{x}(t)$. This means that the closed-loop slow subsystem is reduced to the one studied in Proposition 2 under state feedback. According to Proposition 2, if $x(0) \in B_\delta \subset \Omega_\rho$, then $x(t) \in \Omega_\rho, \forall t \geq 0$ and $\forall t \geq t^* > t'$, the following bound holds:

$$|x(t)| \leq \beta(|x(t^*)|, t - t^*) + \gamma(\rho^*) \tag{22}$$

where ρ^* and t^* have been defined in Proposition 2.

Introducing the fast time scale $\bar{\tau} = \frac{t}{\epsilon}$ and setting $\epsilon = 0$, the closed-loop fast subsystem can be represented as follows:

$$\frac{de}{d\bar{\tau}} = A^* e. \tag{23}$$

Since A^* is Hurwitz, the closed-loop fast subsystem is also stable. Moreover, there exist $k_e \geq 1$ and $a_e > 0$ such that:

$$|e(\bar{\tau})| \leq k_e |e(0)| e^{-a_e \bar{\tau}}, \quad \forall \bar{\tau} \geq 0. \tag{24}$$

Next, we consider $t \in (0, \max\{\Delta, t_b\}]$ and $t \geq \max\{\Delta, t_b\}$ separately and prove that if the conditions stated in Theorem 1 are satisfied, the boundedness of the state is ensured. Note that t_b decreases as ϵ decreases.

When $x(0) \in B_{\delta_x} \subset \Omega_{\rho_e} \subset \Omega_\rho$, and $\delta_x < \delta$, considering the closed-loop system state trajectory:

$$\dot{x}(t) = f(x(t)) + g(x(t))u^*(\hat{x}(0)), \quad \forall t \in (0, \max\{\Delta, t_b\}]$$

and using Eqs. (4) and (3), we can obtain that for all $t \in (0, \max\{\Delta, t_b\}]$:

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \\ &= V(x(0)) + \int_0^t \frac{\partial V(x(\tau))}{\partial x}\dot{x}(\tau)d\tau \\ &\leq \rho_e + M \max\{\Delta, t_b(\epsilon)\}\alpha_4(\alpha_1^{-1}(\rho)). \end{aligned} \tag{25}$$

Since t_b decreases as ϵ decreases, there exist Δ_1 and ϵ_1 such that if $\Delta \in (0, \Delta_1]$ and $\epsilon \in (0, \epsilon_1]$, Eq. (18) holds and thus,

$$V(x(t)) < \rho, \quad \forall t \in (0, \max\{\Delta, t_b\}]. \tag{26}$$

For $t \geq \max\{\Delta, t_b\}$, we have that $|x(t) - \hat{x}(t)| \leq e_m$ (this follows from Proposition 1 and e_m decreases as ϵ decreases), and we can write the time derivative of the Lyapunov function along the closed-loop system state of Eq. (1) under the state-estimation-based LEMPC of Eqs. (7), (8) and (10) for all $t \in [t_k, t_{k+1})$ (assuming without loss of generality that $t_k = \max\{\Delta, t_b\}$) as follows

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x}(f(x(t)) + g(x(t))u^*(\hat{x}(t_k))). \tag{27}$$

Adding and subtracting the term $\frac{\partial V(\hat{x}(t_k))}{\partial x}(f(\hat{x}(t_k)) + g(x(t_k))u^*(\hat{x}(t_k)))$ to/from the above inequality and taking advantage of Eqs. (3) and (10f), we can obtain

$$\begin{aligned} \dot{V}(x(t)) \leq & -\alpha_3(\alpha_1^{-1}(\rho_s)) + \frac{\partial V(x)}{\partial x}(f(x(t)) - f(\hat{x}(t_k)) \\ & + u^*(\hat{x}(t_k))(g(x(t)) - g(\hat{x}(t_k)))). \end{aligned} \tag{28}$$

Using the smoothness properties of V, f, g and Eq. (5), we can obtain

$$\dot{V}(x(t)) \leq -\alpha_3(\alpha_1^{-1}(\rho_s)) + (L_x + L_u u^{\max})|x(t) - \hat{x}(t_k)|. \tag{29}$$

By taking advantage of $|x(t) - \hat{x}(t_k)| \leq |x(t) - x(t_k)| + |x(t_k) - \hat{x}(t_k)| \leq M\Delta + e_m$ (using Eq. (4)) and the fact that the estimation error is bounded by e_m for $t \geq \max\{\Delta, t_b\}$, we have

$$\dot{V}(x(t)) \leq -\alpha_3(\alpha_1^{-1}(\rho_s)) + (L_x + L_u u^{\max})(M\Delta + e_m). \tag{30}$$

Picking ϵ_2 and Δ_2 such that $\forall \epsilon \in (0, \epsilon_2]$ and $\forall \Delta \in (0, \Delta_2]$, Eq. (17) is satisfied, the closed-loop system state $x(t)$ is bounded in $\Omega_\rho, \forall t \geq \max\{\Delta, t_b\}$. Finally, using similar arguments to the proof of Theorem 1 in [8], we have that there exist class $\mathcal{KL}$ function β, positive real numbers (δ_x, d_x) (note that the existence of $\delta_x < \delta$ such that $|x(0)| \leq \delta_x$ follows from the smoothness of $V(x)$), and $0 < \epsilon^* < \min\{\epsilon_1, \epsilon_2\}$ and $0 < \Delta^* < \min\{\Delta_1, \Delta_2\}$ such that if $\max\{|x(0)|, |e(0)|\} \leq \delta_x, \epsilon \in (0, \epsilon^*]$ and $\Delta \in (0, \Delta^*]$, then, the bound of Eq. (19) holds for all $t \geq t^*$. □

Remark 5. It needs to be clarified that under state feedback LEMPC, the closed-loop system state is always bounded in Ω_ρ for both mode 1 and mode 2 operation; however, for mode 2 operation, after time t^* the closed-loop system state enters the ball B_δ, and the closed-loop system state can be bounded by Eq. (22). On the other hand, in state-estimation-based LEMPC, the closed-loop system state is always bounded in Ω_ρ, if the initial system state belongs in $B_{\delta_x} \subset \Omega_{\rho_e} \subset \Omega_\rho$.

Remark 6. In the present work, we consider that there is no measurement noise in the process output and assume that the full system model is available. We can consider a smaller stability region, say $\Omega_{\tilde{\rho}}$, which takes into account the effect of measurement noise as well as a lower observer gain to deal better with measurement noise. Please refer to Chapter 6 of [2] (see also [7]) for a detailed discussion on how to determine the stability region Ω_ρ in the presence of measurement noise and to the example section for an evaluation of the closed-loop performance of the proposed output feedback controller under measurement noise.

Remark 7. If the initial condition $x(t_0)$ (and the following $\hat{x}(t_k)$ estimate) is outside of the stability region Ω_ρ, we cannot take advantage of the stability properties of the nonlinear controller $k(x)$. However, since Ω_ρ is an estimate of the stability region, it is possible to achieve closed-loop stability under the proposed LEMPC design for states outside of Ω_ρ. In the case where $x(t_k)$ is outside of Ω_ρ, the proposed LEMPC mode 1 can be made feasible by removing the constraint of Eq. (10f) at the expense of losing closed-loop stability guarantees.

Remark 8. The major motivation for taking advantage of the nonlinear controller $k(x)$ arises from the need for formulating an a priori feasible economic MPC problem for a well-defined set of initial conditions. The control action of $k(x)$ is always a feasible candidate for the proposed LEMPC design (even though the LEMPC via optimization is free to choose a different control action) and the LEMPC can take advantage of $k(x)$ to characterize its own corresponding stability region. In addition, the closed-loop system state is always bounded in the invariant stability region of $k(x)$.

5. Application to a chemical process example

Consider a well-mixed, non-isothermal continuous stirred tank reactor (CSTR) where an irreversible, second-order, endothermic reaction $A \rightarrow B$ takes place, where A is the reactant and B is the desired product. The feed to the reactor consists of pure A at flow

Table 1
Parameter values.

$T_0 = 300$	K
$V = 1.0$	m^3
$k_0 = 13.93$	$\frac{1}{\mathrm{h}}$
$C_p = 0.231$	$\frac{\mathrm{kJ}}{\mathrm{kgK}}$
$\sigma = 1000$	$\frac{\mathrm{kg}}{\mathrm{m}^3}$
$T_s = 350$	K
$Q_s = 1.73 \times 10^5$	$\frac{\mathrm{kJ}}{\mathrm{h}}$
$F = 5$	$\frac{\mathrm{m}^3}{\mathrm{h}}$
$E = 5 \times 10^3$	$\frac{\mathrm{kJ}}{\mathrm{kmol}}$
$\Delta H = 1.15 \times 10^4$	$\frac{\mathrm{kJ}}{\mathrm{kmol}}$
$R = 8.314$	$\frac{\mathrm{kJ}}{\mathrm{kmolK}}$
$C_{As} = 2$	$\frac{\mathrm{kmol}}{\mathrm{m}^3}$
$C_{A0s} = 4$	$\frac{\mathrm{kmol}}{\mathrm{m}^3}$

rate F, temperature T_0 and molar concentration C_{A0}. Due to the non-isothermal nature of the reactor, a jacket is used to provide heat to the reactor. The dynamic equations describing the behavior of the reactor, obtained through material and energy balances under standard modeling assumptions, are given below:

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT}} C_A^2 \tag{31a}$$

$$\frac{dT}{dt} = \frac{F}{V}(T_0 - T) + \frac{-\Delta H}{\sigma C_p} k_0 e^{\frac{-E}{RT}} C_A^2 + \frac{Q_s}{\sigma C_p V} \tag{31b}$$

where C_A denotes the concentration of the reactant A, T denotes the temperature of the reactor, Q_s denotes the steady-state rate of heat supply to the reactor, V represents the volume of the reactor, ΔH, k_0, and E denote the enthalpy, pre-exponential constant and activation energy of the reaction, respectively, and C_p and σ denote the heat capacity and the density of the fluid in the reactor, respectively. The values of the process parameters used in the simulations are shown in Table 1. The process model of Eq. (31) is numerically simulated using an explicit Euler integration method with integration step $h_c = 10^{-3}$ h.

The process model has one stable steady state in the operating range of interest. The control objective is to economically optimize the process in a region around the stable steady state (C_{As}, T_s) to maximize the average production rate of B through manipulation of the concentration of A in the inlet to the reactor, C_{A0}. The steady-state C_{A0} value associated with the steady-state point is denoted by C_{A0s}. The process model of Eq. (31) belongs to the following class of nonlinear systems:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

where $x^T = [x_1\ x_2] = [C_A - C_{As}\ \ T - T_s]$ is the state, $u = C_{A0} - C_{A0s}$ is the input, $f = [f_1\ f_2]^T$ and $g_i = [g_{i1}\ g_{i2}]^T$ $(i = 1, 2)$ are vector functions. The input is subject to constraint as follows: $|u| \leq 3.5$ kmol/m^3. There is an economic measure considered in this example as follows [15]:

$$L(x, u) = \frac{1}{t_f} \int_0^{t_f} k_0 e^{-\frac{E}{RT(\tau)}} C_A^2(\tau) d\tau \tag{32}$$

where t_f is the time duration of the reactor operation; we note that in an ideal case a CSTR system can operate indefinitely, however, in practice the operation time is finite. In the simulations presented below, we consider different operating time scenarios to demonstrate that the application of the proposed economic MPC method is not dependent on t_f. In fact, the CSTR can operate indefinitely either under mode 1 economic MPC operation (which leads to time-varying operation) or under mode 2 economic MPC operation which leads to steady-state operation. The economic objective function of Eq. (32) describes the average production rate over the entire process operation. We also consider that there is a limitation on the amount of reactant material which can be used over a specific period $t_p = 1$ h; this is a standard constraint in a practical setting where the amount of reactant material is always finite. Specifically, $u = C_{A0} - C_{A0s}$ should satisfy the following constraint:

$$\frac{1}{t_p} \int_0^{t_p} u(\tau) d\tau = 1 \text{ kmol/m}^3. \tag{33}$$

It should be emphasized that due to the second-order dependence of the reaction rate on the reactant concentration, the production rate can be improved through switching between the upper and lower bounds of the manipulated input [15], as opposed to steady-state operation via steady in time distribution of the reactant in the feed. In this section we will design an estimation-based LEMPC to manipulate the C_{A0} subject to the material constraint. In the first set of simulations, we assume that state feedback information is available at synchronous time instants while in the second set of simulations we take advantage of a high-gain observer to estimate the reactant concentration from temperature measurements.

In terms of the Lyapunov-based controller, we use a proportional controller (P controller) in the form $u = -\gamma_1 x_1 - \gamma_2 x_2$ subject to input constraints and the quadratic Lyapunov function $V(x) = x^T P x$ where $\gamma_1 = 1.6$, $\gamma_2 = 0.01$, $P = \mathrm{diag}([110.11, 0.12])$ and $\rho = 430$. It should be emphasized that Ω_ρ has been estimated through evaluation of $\dot{V}$ when we apply the proportional controller. We assume that the full system state $x = [x_1\ x_2]^T$ is measured and sent to the LEMPC at synchronous time instants $t_k = k\Delta$, $k = 0, 1, \ldots,$ with $\Delta = 0.01$ h $= 36$ s in the first set of simulations while for output feedback LEMPC only temperature (x_2) is available to LEMPC and a high-gain observer is utilized to estimate the reactant concentration from temperature measurements. Considering the material constraint which needs to be satisfied through one period of process operation, a decreasing LEMPC horizon sequence $N_0, \ldots, N_{99}$ where $N_i = 100 - i$ and $i = 0, \ldots, 99$ is utilized at the different sampling times. At each sampling time t_k, LEMPC with horizon N_k takes into account the leftover amount of reactant material and adjusts its horizon to predict future system state up to time $t_p = 1$ h to maximize the average production rate. Since the LEMPC is evaluated at discrete-time instants during the closed-loop simulation, the material constraint is enforced as follows:

$$\sum_{i=0}^{M-1} u(t_i) = \frac{t_p}{\Delta} \tag{34}$$

where $M = 100$. As LEMPC proceeds at different sampling times, this constraint is adjusted according to the optimal manipulated input at previous sampling times. Specifically, the state feedback LEMPC formulation for the chemical process example in question has the following form:

$$\max_{u \in S(\Delta)} \frac{1}{N_k \Delta} \int_{t_k}^{t_{k+N_k}} \left[k_0 e^{-\frac{E}{RT(\tau)}} C_A^2(\tau) \right] d\tau \tag{35a}$$

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))u(t) \quad t \in [t_k, t_{k+N_k}] \tag{35b}$$

$$\sum_{i=k}^{k+N_k-1} u(t_i|t_k) = \zeta_k \tag{35c}$$

$$\tilde{x}(t_k) = x(t_k) \tag{35d}$$

$$V(\tilde{x}(t)) \leq \rho \quad t \in [t_k, t_{k+N_k}] \tag{35e}$$

$$u(t) \in U \quad t \in [t_k, t_{k+N_k}] \tag{35f}$$

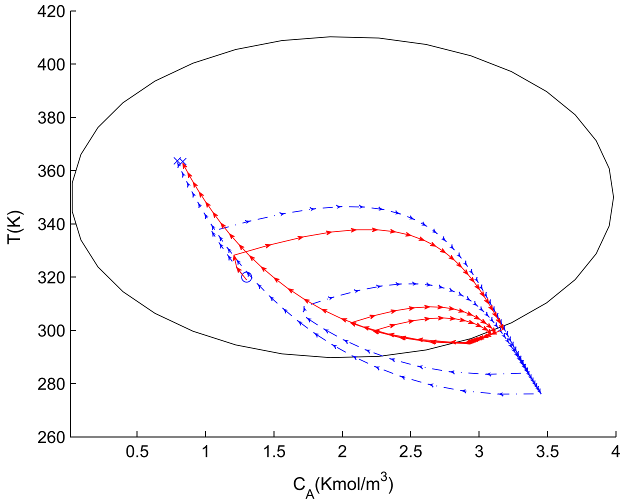

Fig. 1. Ω_ρ and state trajectories of the process under the LEMPC design of Eq. (35) with state feedback and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation with (solid line) and without (dash–dotted line) the constraint of Eq. (35e). The symbols $\circ$ and $\times$ denote the initial ($t = 0$ h) and final ($t = 1$ h) state of these closed-loop system trajectories, respectively.

where $x(t_k)$ is the process state measurement at sampling time t_k and the predicted system state along the LEMPC horizon is restricted in the invariant set Ω_ρ through enforcement of the constraint of Eq. (35e) subject to the manipulated input constraint of Eq. (35f). The constraint of Eq. (35c) implies that the optimal values of u along the prediction horizon should be chosen to satisfy the material constraint where the explicit expression of ζ_k can be computed based on Eq. (34) and the optimal manipulated input values prior to sampling time t_k. In other words, this constraint indicates the amount of the remaining reactant material at each sampling time. Thus, it ensures that the material constraint is enforced through one period of process operation. In terms of the initial guess for solving the optimization problem of Eq. (35), at the first sampling time we take advantage of the Lyapunov-based controller while for the subsequent sampling times, a shifted version of the optimal solution of the previous sampling time is utilized. The simulations were carried out using Java programming language in a Pentium 3.20 GHz computer and the optimization problems were solved using the open source interior point optimizer Ipopt [16]. The purpose of the following set of simulations is to demonstrate that: (I) the proposed LEMPC design subject to state and output feedback restricts the system state in an invariant set; (II) the proposed LEMPC design maximizes the economic measure of Eq. (35a); and (III) the proposed LEMPC design achieves a higher objective function value compared to steady-state operation with equal distribution in time of the reactant material. We have also performed simulations for the case where the constraint of Eq. (35e) is not included in the LEMPC design of Eq. (35). In this case, the process state is not constrained to be in a specific invariant set.

In the first set of simulations, we take the CSTR operation time $t_f = t_p = 1$ h. Figs. 1–3 illustrate the process state profile in state space (temperature T versus concentration C_A) considering the stability region Ω_ρ, the time evolution of the process state and the manipulated input profile for the LEMPC formulation of Eq. (35) with and without the state constraint of Eq. (35e), respectively. In both cases the initial process state is $(1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$. For both cases, the material constraint is satisfied while in the unconstrained state case, there is more freedom to compute the optimal input trajectory to maximize the average production rate. It needs to be emphasized that the process state trajectory under the LEMPC design of Eq. (35) subject to the constraint of Eq. (35e) never leaves the invariant level set Ω_ρ when this constraint is enforced. We have also compared the time-varying operation through LEMPC of Eq. (35) to steady-state operation where the reactant material is uniformly distributed in the feed to the reactor over the process operation time (1 h), from a closed-loop performance point of view. To carry out this comparison, we have computed the total cost of each operating scenario based on an index of the following form:

$$J = \frac{1}{t_M} \sum_{i=0}^{M} \left[k_0 e^{-\frac{E}{RT(t_i)}} C_A^2(t_i) \right]$$

where $t_0 = 0$ h, $t_M = 1$ h and $M = 100$. To be consistent in comparison, both of the simulations have been initialized from the steady-state point $(2\ \frac{\text{kmol}}{\text{m}^3}, 350\ \text{K})$. We find that through time-varying LEMPC operation, there is approximately 7% improvement with respect to steady-state operation. Specifically, in the case of LEMPC operation with $\rho = 430$ the cost is 13.48, in the case of LEMPC operation with $\rho = \infty$ (LEMPC of Eq. (35) without the state constraint of Eq. (35e)) the cost is 13.55 and in the case of steady-state operation the cost is 12.66.

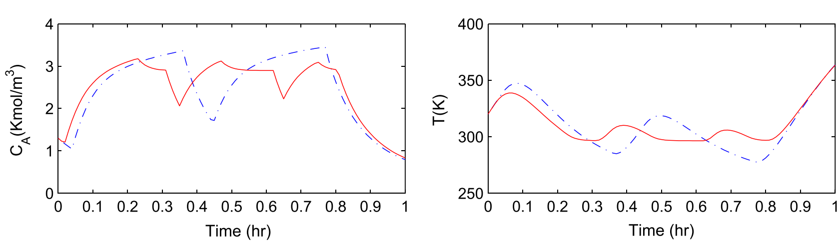

Fig. 2. State trajectories of the process under the LEMPC design of Eq. (35) with state feedback and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation with (solid line) and without (dash–dotted line) the constraint of Eq. (35e).

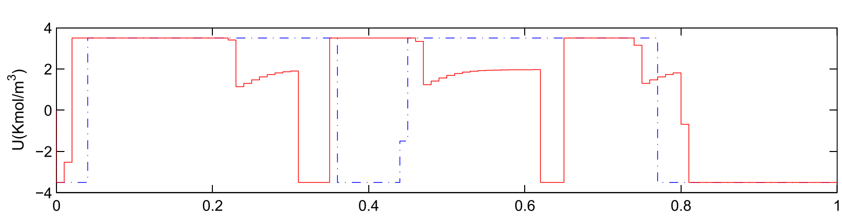

Fig. 3. Manipulated input trajectory under the LEMPC design of Eq. (35) with state feedback and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation with (solid line) and without (dash–dotted line) the constraint of Eq. (35e).

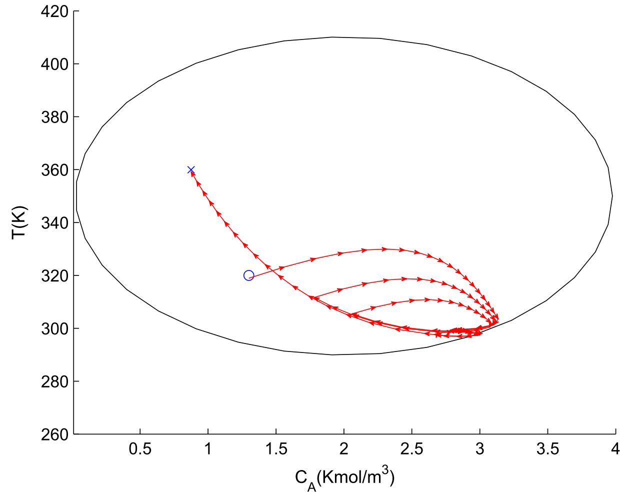

Fig. 4. Ω_ρ and state trajectory of the process under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation subject to the constraint of Eq. (35e). The symbols ∘ and × denote the initial ($t = 0$ h) and final ($t = 1$ h) state of this closed-loop system trajectory, respectively.

We have also performed closed-loop simulation with the state-estimation-based LEMPC (again, $t_f = t_p = 1$ h). For this set of simulation the high-gain observer parameters are $\epsilon = 0.01$, $a_1 = a_2 = 1$, $\rho_e = 400$ and $z_m = 1685$; the high-gain observer is of the form of Eq. (8) with $n = 2$. In this case, the LEMPC formulation at each sampling time is initialized by the estimated system state $\hat{x}(t_k)$ while the output (temperature) measurement is continuously available to the high-gain observer. To ensure that the actual system state is restricted in Ω_ρ, we set $\rho_e = 400$. Figs. 4–6 illustrate the process state profile in state space (temperature T versus concentration C_A) considering the stability region Ω_ρ, the time evolution of process states and the manipulated input profile for the LEMPC formulation of Eq. (35) using high-gain observer and with the state constraint of Eq. (35e), respectively. Similar to the state feedback case, the initial process state is $(1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$. Through LEMPC implementation, the material constraint is satisfied while the closed-loop system state is restricted inside the stability region Ω_ρ. The cost is 12.98 which is higher than the one for steady-state operation (12.66).

Also, we performed a set of simulations to compare LEMPC with the Lyapunov-based controller from an economic closed-loop performance point of view for operation over two consecutive one hour periods (i.e., $t_f = 2$ h and $t_p = 1$ h). To be consistent

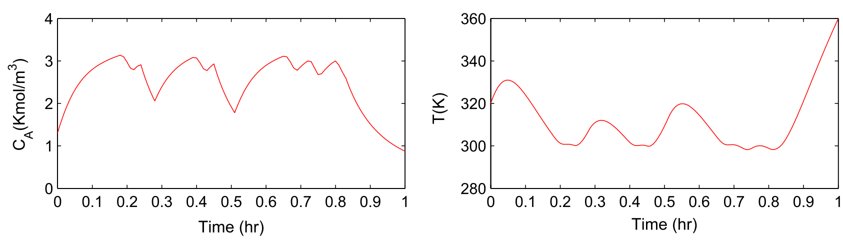

Fig. 5. State trajectories of the process under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation subject to the constraint of Eq. (35e).

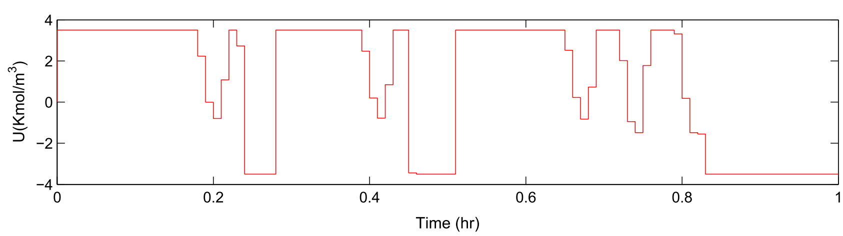

Fig. 6. Manipulated input trajectory under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation subject to the constraint of Eq. (35e).

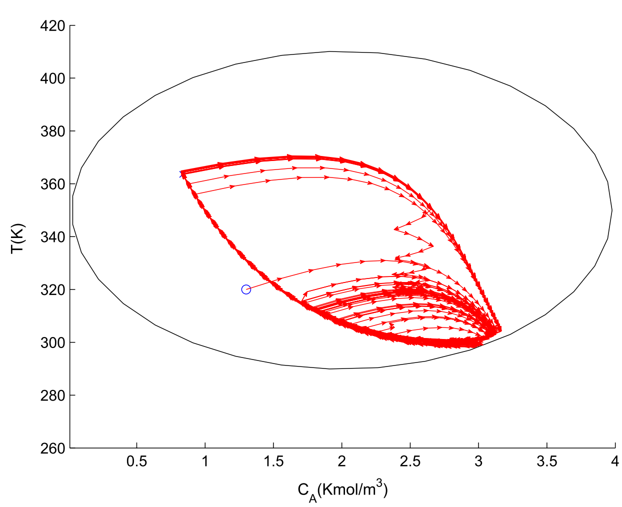

Fig. 7. Ω_ρ and state trajectory of the process under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for 10 h operation in mode 1, then 10 h of operation in mode 2 and finally 10 h of operation in mode 1. The symbols ∘ and × denote the initial ($t = 0$ h) and final ($t = 30$ h) state of this closed-loop system trajectory, respectively.

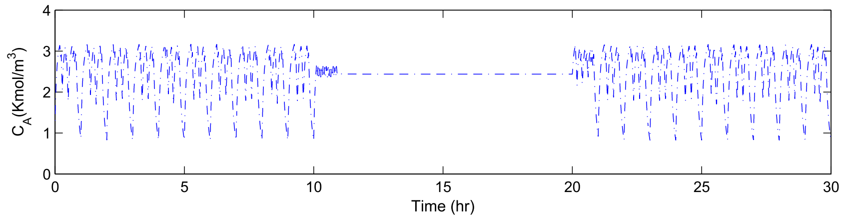

Fig. 8. Reactant concentration trajectory of the process under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for 10 h operation in mode 1, then 10 h of operation in mode 2 and finally 10 h of operation in mode 1.

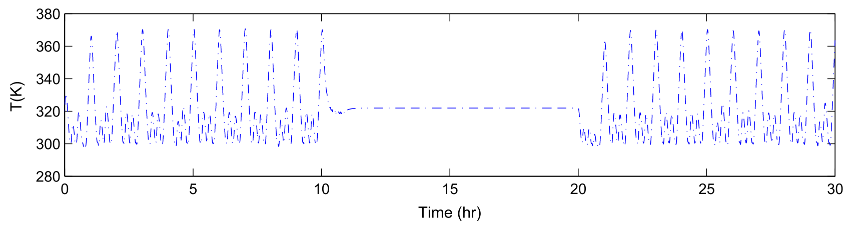

Fig. 9. Temperature trajectory of the process under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for 10 h operation in mode 1, then 10 h of operation in mode 2 and finally 10 h of operation in mode 1.

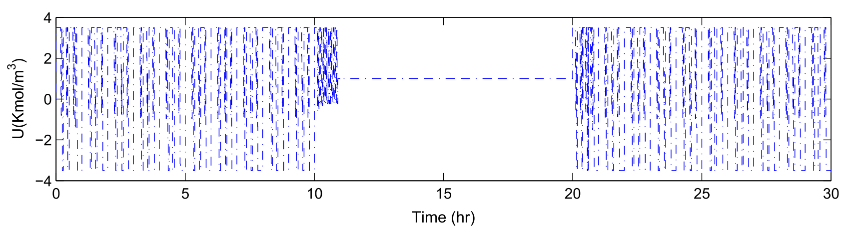

Fig. 10. Manipulated input trajectory under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for 10 h operation in mode 1, then 10 h of operation in mode 2 and finally 10 h of operation in mode 1.

in this comparison in the sense that both the LEMPC and the Lyapunov-based controller use the same, available amount of reactant material, we start the simulation in both cases from the same initial condition $(2.44\ \frac{\text{kmol}}{\text{m}^3}, 321.96\ \text{K})$, which corresponds to the steady state of the process when the available reactant material is uniformly distributed over each period of operation. The objective of the Lyapunov-based controller is to drive the system state at this steady state, while the output feedback LEMPC leads to time-varying operation that optimizes directly the economic cost. The corresponding economic costs for this two-hour operation are 26.50 for the LEMPC and 25.61 for the Lyapunov-based controller.

Furthermore, to demonstrate long-term reactor operation (i.e., $t_f = 30$ h and $t_p = 1$ h), we operate the process in a time-varying fashion to optimize the economic cost in mode 1 for the first 10 h, then switch to mode 2 to drive the closed-loop state to the steady state corresponding to $u = 1$ (i.e., equal distribution with time of the reactant material) for the next 10 h, and finally operate the process in mode 1 for the last 10 h. Figs. 7–10 display the results for this case, where the closed-loop system successfully alternates between the two different types (time-varying versus steady-state) of operation.

Finally, we performed a set of simulations to evaluate the effect of bounded measurement noise. Figs. 11–13 display the closed-loop system state and manipulated input of the state-estimation-based LEMPC subject to bounded output (temperature) measurement noise whose absolute value is bounded by 1 K. As can be seen in Figs. 11–13, the controller can tolerate the effect of measurement noise; in this case, Ω_{ρ_e} was reduced to 370 to improve the robustness margin of the controller to measurement noise. Economic closed-loop performance in this case is 12.95.

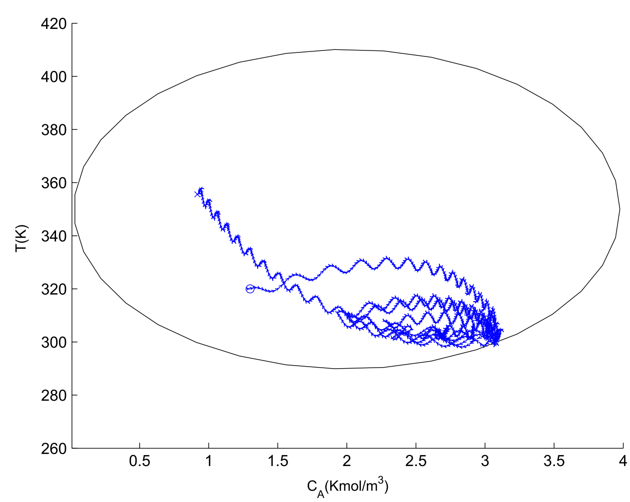

Fig. 11. Ω_ρ and state trajectory of the process under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation subject to the constraint of Eq. (35e) and bounded measurement noise. The symbols $\circ$ and $\times$ denote the initial ($t = 0$ h) and final ($t = 1$ h) state of this closed-loop system trajectory, respectively.

6. Conclusions

In this work, we designed an estimator-based EMPC for the class of full-state feedback linearizable nonlinear systems. A high-gain observer is used to estimate the nonlinear system state using output measurements and a Lyapunov-based approach is

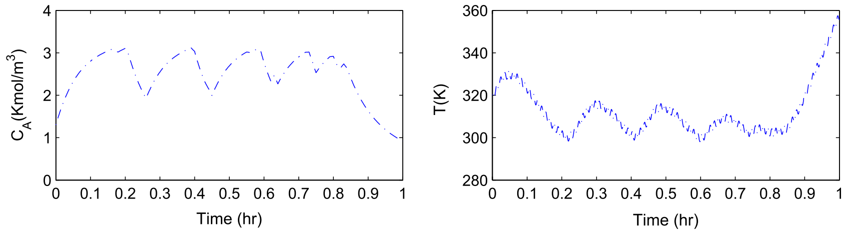

Fig. 12. State trajectories of the process under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation subject to the constraint of Eq. (35e) and bounded measurement noise.

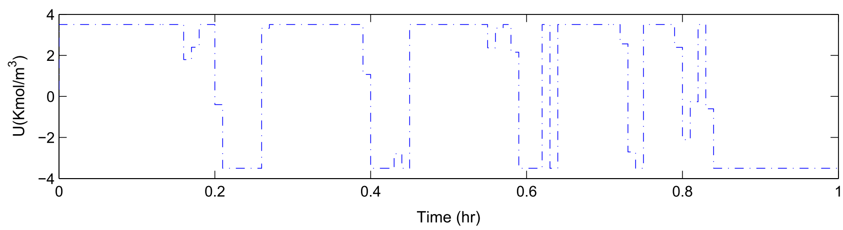

Fig. 13. Manipulated input trajectory under state-estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\ \frac{\text{kmol}}{\text{m}^3}, 320\ \text{K})$ for one period of operation subject to the constraint of Eq. (35e) and bounded measurement noise.

adopted to design the EMPC that uses the observer state estimates. It was proved, using singular perturbation arguments, that the closed-loop system is practically stable provided the observer gain is sufficiently large. A chemical process example was used to demonstrate the ability of the state-estimation-based economic MPC to achieve time-varying process operation that leads to a superior cost performance metric compared to steady-state operation using the same amount of reactant material.

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