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Multirate Lyapunov-based distributed model predictive control of nonlinear uncertain systems

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ABSTRACT

In this work, we consider the design of a network-based distributed model predictive control system using multirate sampling for large-scale nonlinear uncertain systems composed of several coupled subsystems. Specifically, we assume that the states of each local subsystem can be divided into fast sampled states (which are available every sampling time) and slowly sampled states (which are available every several sampling times). The distributed model predictive controllers are connected through a shared communication network and cooperate in an iterative fashion at time instants in which full system state measurements (both fast and slow) are available, to guarantee closed-loop stability. When local subsystem fast sampled state information is only available, the distributed controllers operate in a decentralized fashion to improve closed-loop performance. In the proposed control architecture, the controllers are designed via Lyapunov-based model predictive control techniques taking into account bounded measurement noise, process disturbances and communication noise. Sufficient conditions under which the state of the closed-loop system is ultimately bounded in an invariant region containing the origin are derived. The theoretical results are demonstrated through a nonlinear chemical process example.

1. Introduction

With the rapid growth in the area of network technology, augmentation of local process control systems with additional networked sensors and actuators has become a subject of increasing importance. Such an augmentation can significantly improve the efficiency, flexibility, robustness and fault tolerance of an industrial control system (e.g., Refs. [1–3]) at the cost of coordination and design/redesign of the various control systems employed in the new control architecture. Motivated by this trend towards network-based control systems in a variety of engineering applications, significant efforts over the last ten years have led to results on analysis and design of networked control systems using centralized control architectures (e.g., Refs. [4,5]).

Model predictive control (MPC) is an appropriate framework to deal with the design and coordination of networked control systems because of its ability to account for process/controller interactions in the calculation of the control actions while handling input/state constraints. Typically, MPC is studied within a centralized control architecture in which all the manipulated inputs are calculated in a single MPC [6]. In the evaluation of MPC control actions, the evaluation time of the MPC strongly depends on the number of manipulated inputs because online optimization problems need to be solved. As the number of manipulated inputs increases as it is the case in the context of networked control systems, the evaluation time of a centralized MPC increases significantly. This may impede the ability of networked centralized MPC to carry out real-time calculations within the limits imposed by process dynamics and operating conditions. Moreover, a centralized control system for large-scale systems may be difficult to organize and maintain and is vulnerable to potential process faults. To overcome these issues, decentralized and/or distributed MPC can be utilized. While in a decentralized control architecture [7], individual controllers make their decisions based on local information, in a distributed framework, controllers communicate with each other to coordinate their actions.

Distributed MPC (DMPC) has attracted a lot of attention in the design of cooperative networked control systems. In a DMPC architecture, the manipulated inputs are computed by solving more than one control (optimization) problems in separate processors in a coordinated fashion. In the literature, several DMPC methods have been proposed; please see Refs. [8–14] for results in this

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area. Specifically, in Ref. [10], a DMPC scheme for coupled nonlinear systems subject to decoupled constraints was designed. In Ref. [11], a robust DMPC design was developed for linear systems, and in Ref. [14], a cooperative DMPC scheme was developed for linear systems with guaranteed stability of the closed-loop system and convergence of the cost to its corresponding, centralized optimal value. In our previous work [15,16], two different DMPC architectures, namely, a sequential DMPC architecture and an iterative DMPC architecture, were designed for nonlinear systems via Lyapunov techniques and were extended to account for asynchronous and delayed measurements [17]. However, the results in Refs. [15–17] were obtained under the assumptions of noise-free communication and availability of noise-free measurements of system states to all the distributed controllers at each sampling time.

In the present work, we consider the design of a network-based DMPC system using multirate sampling for large-scale nonlinear uncertain systems composed of several coupled subsystems. This problem formulation is important in the context of large-scale networks of heterogeneous components that involve variables that exhibit dynamics and are sampled in significantly different time-scales, for example, energy/water networks as well as chemical process networks. Specifically, we assume that the states of each local subsystem can be divided into fast sampled states and slowly sampled states. Furthermore, we assume that there is a distributed controller associated with each subsystem and the distributed controllers are connected through a shared communication network. We propose to design the distributed controllers via Lyapunov-based MPC (LMPC) and coordinate their actions in an iterative fashion to guarantee closed-loop stability when full system state measurements (both fast and slow) are available. The transmitted information over the shared communication network is subject to communication channel noise. When only fast sampled states are available, the distributed controllers operate in a decentralized fashion to improve closed-loop performance. Sufficient conditions under which the state of the closed-loop system is ultimately bounded in an invariant region containing the origin are derived. The theoretical results are demonstrated through a nonlinear chemical process example.

2. Preliminaries

2.1. Notation and class of nonlinear systems

The operator $|\cdot|$ is used to denote Euclidean norm of a vector while $|\cdot|_Q$ refers to the square of the weighted Euclidean norm, defined by $|x|_Q = x^TQx$. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. The symbol Ω_r is used to denote the set $\Omega_r := \{x \in R^{n_x} : V(x) \leq r\}$ where V is a scalar positive definite, continuous differentiable function and $V(0) = 0$, and the operator '/' denotes set subtraction, that is, $A/B := \{x \in R^{n_x} : x \in A,\ x \notin B\}$. The symbol diag($v$) denotes a square diagonal matrix whose diagonal elements are the elements of vector v. We consider a class of nonlinear systems composed of m interconnected subsystems where each of the subsystems can be described by the following state-space model:

$$\dot{x}_i(t) = f_i(x) + g_{si}(x)u_i(t) + k_i(x)w_i(t) \quad (1)$$

where $i = 1, \ldots, m$, $x_i(t) \in R^{n_{x_i}}$ denotes the vector of state variables of subsystem i, $u_i(t) \in R^{n_{u_i}}$ and $w_i(t) \in R^{n_w}$ denote the set of control (manipulated) inputs and disturbances associated with subsystem i, respectively. The variable $x \in R^{n_x}$ denotes the state of the entire nonlinear system which is composed of the states of the m subsystems, that is $x = [x_1^T \cdots x_i^T \cdots x_m^T]^T \in R^{n_x}$. The dynamics of x can be described as follows:

$$\dot{x}(t) = f(x) + \sum_{i=1}^{m} g_i(x)u_i(t) + k(x)w(t) \quad (2)$$

where $f = [f_1^T \cdots f_i^T \cdots f_m^T]^T$, $g_i = [\mathbf{0}^T \cdots g_{si}^T \cdots \mathbf{0}^T]^T$ with $\mathbf{0}$ being the zero matrix of appropriate dimensions, k is a matrix composed of k_i ($i = 1, \ldots, m$) and zeros whose explicit expression is omitted for brevity, and $w = [w_1^T \cdots w_i^T \cdots w_m^T]^T$ is assumed to be bounded, that is, $w(t) \in W$ with $W := \{w \in R^{n_w} : |w| \leq \theta, \theta > 0\}$. The m sets of inputs are restricted to be in m nonempty convex sets $U_i \subseteq R^{m_{u_i}}$, $i = 1, \ldots, m$, which are defined as $U_i := \{u_i \in R^{n_{u_i}} : |u_i| \leq u_i^{\max}\}$ where $u_i^{\max}$, $i = 1, \ldots, m$, is the magnitude of the constraint on the inputs of the i-th subsystem. We will design m controllers to compute the m sets of control inputs u_i, $i = 1, \ldots, m$, respectively. We will refer to the controller computing u_i as controller i. We assume that f, g_i, $i = 1, \ldots, m$, and k are locally Lipschitz vector functions and that the origin is an equilibrium point of the unforced nominal system (i.e., system of Eq. (2) with $u_i(t) = 0$, $i = 1, \ldots, m$, $w(t) = 0$ for all t) which implies that $f(0) = 0$.

2.2. Modeling of measurements and communication

We assume that the states of each of the m subsystems, x_i ($i = 1, \ldots, m$), are divided into two parts: $x_{f,i}$, states that can be measured at each sampling time (e.g., temperatures and pressures) and $x_{s,i}$, states which are sampled at a relatively slow rate (e.g., species concentrations). Specifically, we assume that $x_{f,i}$, are available at synchronous time instants $t_p = t_0 + p\Delta$, $p = 0, 1, \ldots$, where t_0 is the initial time and Δ is the sampling time; and assume that $x_{s,i}$, are available every T sampling times (i.e., $x_{s,i}$, are available at t_k with $k = 0, T, 2T, \ldots$). Note that, in order to simplify the development, we assume that the slowly sampled states of different subsystems are all available at the same time instants. This modeling of measurements is relevant to systems involving heterogeneous measurements, which have different sampling rates; please see the example in Section 4. We also assume that for each subsystem its local sensors, actuators and controller are connected using point-to-point links, which implies that $x_{f,i}$ and $x_{s,i}$ are available without delay to controller i once they are measured and that the controllers for different subsystems are connected through a shared communication network and communicate when the full (fast and slow) system state is available. We consider measurement noise and communication network noise. Specifically, we consider measurement noise caused by the lack of complete accuracy of measurement sensors. This type of noise is defined as the difference between the reading value of a state from a sensor and the true value of the state. We assume that the sensor reading values of states $x_{f,i}$ and $x_{s,i}$ are $\breve{x}_{f,i}^s$ and $\breve{x}_{s,i}^s$, respectively; and $\breve{x}_{f,i}^s$ and $\breve{x}_{s,i}^s$ are modeled as follows: $\breve{x}_{f,i}^s = x_{f,i} + n_{x_{f,i}}^s$, $\breve{x}_{s,i}^s = x_{s,i} + n_{x_{s,i}}^s$ where $n_{x_{f,i}}^s$ and $n_{x_{s,i}}^s$ are the measurement noise terms associated with $x_{f,i}$ and $x_{s,i}$, respectively. The measurement noise is assumed to be bounded; that is, $|n_{x_{f,i}}^s| \leq \theta_{x_{f,i}}^s$ and $|n_{x_{s,i}}^s| \leq \theta_{x_{s,i}}^s$ with $\theta_{x_{f,i}}^s$ and $\theta_{x_{s,i}}^s$ being positive real numbers. It should be mentioned that this assumption on the type of measurement noise is meaningful from a practical standpoint due to the limit on the accuracy of the measurement sensors and the fact that measurement noise is usually modeled as a percentage of the actual value.

At t_k with $k = 0, T, 2T, \ldots$, when fast and slowly sampled states are available to each controller, the distributed controllers exchange information, which is subject to communication channel noise. Specifically, we assume that controller i sends $\breve{x}_i^s = [\breve{x}_{f,i}^{s,T}\ \breve{x}_{s,i}^{s,T}]^T$ as well as its control input trajectory u_i to the other controllers; and

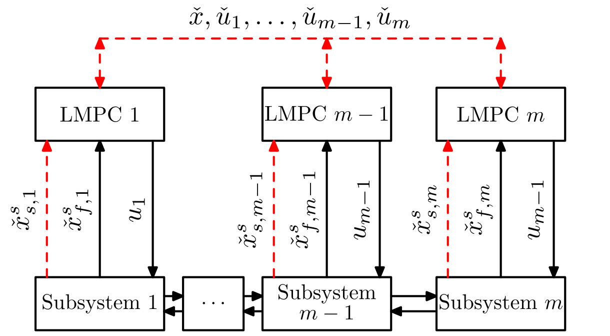

Fig. 1. Distributed LMPC control architecture (solid line denotes fast state sampling and/or point-to-point links; dashed line denotes slow state sampling and/or shared communication networks).

the values received by controller j ($j \neq i$), $\check{x}_i^j$ and $\check{u}_i^j$, are modeled as follows: $\check{x}_i^j = \check{x}_i^s + n_{x_i}^{c,j}$, $\check{u}_i^j = u_i + n_{u_i}^j$ where $n_{x_i}^{c,j}$ and $n_{u_i}^j$ are the communication noise terms. The communication noise terms are also assumed to be bounded; that is, $|n_{x_i}^{c,j}| \leq \theta_{x_i}^{c,j}$ and $|n_{u_i}^j| \leq \theta_{u_i}$ with $\theta_{x_i}^{c,j}$ and θ_{u_i} being positive real numbers. According to the above modeling, at time t_k with $k = 0, T, 2T, \ldots$ when fast and slowly sampled states are available, the state information received by controller i ($i = 1, \ldots, m$) is described as follows:

$$\check{x}^i(t_k) = [\check{x}_1^i, \ldots, \check{x}_{i-1}^i, \check{x}_i^s, \check{x}_{i+1}^i, \ldots, \check{x}_m^i] = x(t_k) + n_x^i \tag{3}$$

where $n_x^i \in R^{n_x}$ denotes combined communication and measurement noise and $|n_x^i| \leq \theta_x^i$ with θ_x^i being a suitable composition of $\theta_{x_{f,i}}^s$, $\theta_{x_{s,i}}^s$ and $\theta_{x_j}^{c,i}$ ($j \neq i$). This class of systems is relevant to the case of large-scale chemical processes that are controlled by distributed control systems that exchange information over a shared communication network through which it is not cost-effective to communicate at every sampling time. Instead, in order to achieve closed-loop stability and good closed-loop performance, the controllers communicate every several sampling times. Please see Fig. 1 for a schematic of such type of DMPC system with the local controllers designed via Lyapunov-based MPC techniques.

2.3. Lyapunov-based controller

We assume that there exists a locally Lipschitz Lyapunov-based controller $h(x) = [h_1^T(x) \cdots h_m^T(x)]^T$ such that when the m control inputs are determined as $u_i = h_i(x)$, $i = 1, \ldots, m$, the origin of the nominal interconnected closed-loop system is asymptotically stable and the input constraints are satisfied for all x inside a compact set. This assumption implies that there exist a continuously differentiable Lyapunov function $V(x)$ for the nominal closed-loop system, a class $\mathcal{K}$ function $\alpha_1(\cdot)$ which bounds the value of the Lyapunov function from above and a class $\mathcal{K}$ function $\alpha_2(\cdot)$ which bounds the time derivative of the Lyapunov function from above [18,19]. Specifically, from this assumption, we have the following inequalities:

$$\begin{aligned} & V(x) \leq \alpha_1(|x|), \; h_i(x) \in U_i, \quad i = 1, \ldots, m \\ & \frac{\partial V(x)}{\partial x}\left(f(x) + \sum_{i=1}^{m} g_i(x) h_i(x)\right) \leq -\alpha_2(|x|) \end{aligned} \tag{4}$$

for all $x \in \Omega_\rho$ where Ω_ρ denotes the stability region of the closed-loop system under $h(x)$. The set Ω_ρ is usually chosen to be a level set of $V(x)$.

Since the manipulated inputs u_i, $i = 1, \ldots, m$, and the disturbance w are bounded in closed sets and the vector fields f, g_i, $i = 1, \ldots, m$, k are locally Lipschitz, we can have the following inequality for all the states within the stability region (i.e., $x \in \Omega_\rho$):

$$\left| f(x) + \sum_{i=1}^{m} g_i(x) u_i + k(x) w \right| \leq M \tag{5}$$

where M is a positive constant. Moreover, if we take into account the continuous differentiable property of the Lyapunov function $V(x)$, we can write the following inequalities:

$$\begin{aligned} & \left| \frac{\partial V(x)}{\partial x} f(x) - \frac{\partial V(x')}{\partial x} f(x') \right| \leq L_x |x - x'| \\ & |f(x) - f(x')| \leq C_x |x - x'| \\ & \left| \frac{\partial V(x)}{\partial x} g_i(x) - \frac{\partial V(x')}{\partial x} g_i(x') \right| \leq L_{u_i} |x - x'| \\ & \left| \frac{\partial V(x)}{\partial x} k(x) \right| \leq L_w, \; \left| \frac{\partial V(x)}{\partial x} g_i(x) \right| \leq C_{g_i} \\ & |h_i(x) - h_i(x')| \leq C_{h_i} |x - x'|, \; |g_i(x)| \leq M_{g_i} \\ & |g_i(x) - g_i(x')| \leq C_{u_i} |x - x'|, \; |k(x)| \leq M_w \end{aligned} \tag{6}$$

with L_x, L_{u_i}, C_x, C_{h_i}, C_{u_i}, C_{g_i}, M_{g_i}, M_W $i = 1, \ldots, m$, and L_w being positive constants for all $x, x' \in \Omega_\rho$, $u_i \in U_i$, $i = 1, \ldots, m$, and $w \in W$. Note that the inequalities of Eqs. (4)–(6) are derived from the basic assumptions (i.e., Lipschitz vector fields and existence of stabilizing Lyapunov-based controller) used in this work. The various constants involved in the upper bounds are not assumed to be arbitrarily small.

Remark 1. The construction of Lyapunov functions can be carried out in a number of ways using techniques like, for example, sum-of-squares methods. For broad classes of nonlinear systems arising in the context of chemical process control applications, quadratic Lyapunov functions are widely used and provide very good estimates of closed-loop stability regions; please see example in Section 4.

3. Multirate DMPC

3.1. Multirate DMPC implementation strategy

In this work, the m controllers manipulating the m sets of inputs will be designed through LMPC techniques. For the LMPC associated with controller i, $i = 1, \ldots, m$, we will refer to it as LMPC i. A schematic of the control system is shown in Fig. 1. At a sampling time in which slowly and fast sampled states are available, the distributed controllers coordinate their actions and predict future input trajectories which, if applied until the next instant that both slowly and fast sampled states are available, guarantee closed-loop stability. At a sampling time in which only fast sampled states are available, each distributed controller tries to further optimize the input trajectories calculated at the last instant in which the controllers communicated, within a constrained set of values to improve the closed-loop performance with the help of the available fast sampled states of its subsystem.

The proposed implementation strategy of the DMPC architecture at time instants in which fast and slowly sampled states are available is as follows:

1. At t_k with $k = 0, T, 2T, \ldots$, all the controllers first broadcast their local subsystem states to the other controllers and then evaluate their future input trajectories in an iterative fashion with initial input guesses generated by $h(\cdot)$.
2. At iteration c ($c \geq 1$)
   2.1. Each controller evaluates its own future input trajectory based on $\check{x}^i(t_k)$ (noisy version of $x(t_k)$) and the last received

control input trajectories (initial input guesses generated by $h(\cdot)$ when $c=1$).

2.2. All the distributed controllers exchange their latest future input trajectories. Based on the input information, each controller calculates and stores the corresponding value of the cost.

2.3. If a termination condition is satisfied, each controller sends its entire future input trajectory corresponding to the smallest value of the cost to its actuators; if the termination condition is not satisfied, go to step 2.1 ($c \leftarrow c+1$).

The proposed implementation strategy of the DMPC architecture at time instants when only local fast sampled states are available is as follows:

1. Controller i, $i=1, \ldots, m$, receives its local fast sampled states, $\breve{x}^s_{f,i}$.
2. Each controller i estimates the current full system state and evaluates its future input trajectory and sends the first step input value to its actuators.

3.2. Multirate DMPC formulation

Before presenting the design of the LMPCs, we define a nominal sampled trajectory for each subsystem $x^i_h(\tau|t_k)$, $k=0, T, 2T, \ldots$, which will be employed in the construction of the stability constraint of LMPC i ($i=1, \ldots, m$). This nominal sampled trajectory is obtained by integrating recursively, for $t\in[t_k, t_{k+T})$ and $k=0, T, 2T, \ldots$, the following equation:

$$\dot{x}^i_h(\tau|t_k) = f(x^i_h(\tau|t_k)) + \sum_{i=1}^{m} g_i(x^i_h(\tau|t_k))h_i(x^i_h(l\Delta|t_k)),\quad \forall\tau\in[l\Delta,(l+1)\Delta) \tag{7}$$

with initial condition $x^i_h(0|t_k)=\breve{x}^i(t_k)$ where $l=0, \ldots, T-1$, $\breve{x}^i(t_k)$ is the system state received by controller i at t_k. Based on this sampled trajectory, we define the following input trajectories:

$$u^i_{h,j}(\tau|t_k) = h_j(x^i_h(l\Delta|t_k)),\ \forall\tau\in[l\Delta,(l+1)\Delta) \tag{8}$$

where $j=1, \ldots, m$. This sampled trajectory, $x^i_h(\tau|t_k)$, will be used in the LMPC i formulation.

At time t_k, $k=0, T, 2T, \ldots$, the LMPCs are evaluated in an iterative fashion to obtain the future input trajectories. Specifically, the optimization problem of LMPC j at iteration c is as follows:

$$\min_{u_j\in S(\Delta)} \int_0^{N\Delta}\left[|\tilde{x}^j(\tau)|_{Q_c} + \sum_{i=1}^{m}|u_i(\tau)|_{R_{ci}}\right]d\tau \tag{9a}$$

$$\text{s.t.}\ \dot{\tilde{x}}^j(\tau) = f(\tilde{x}^j(\tau)) + \sum_{i=1}^{m} g_i(\tilde{x}^j(\tau))u_i \tag{9b}$$

$$u_i(\tau) = \breve{u}^{*,c-1}_i(\tau|t_k),\quad \forall i\neq j \tag{9c}$$

$$\left|u_j(\tau) - u^{*,c-1}_j(\tau|t_k)\right| \le \Delta u_j,\quad \forall\tau\in[0, T\Delta] \tag{9d}$$

$$u_j(\tau)\in U_j \tag{9e}$$

$$\tilde{x}^j(0) = \breve{x}^j(t_k) \tag{9f}$$

$$\frac{\partial V(\tilde{x}^j(\tau))}{\partial x}\left(\frac{1}{m}f(\tilde{x}^j(\tau)) + g_j(\tilde{x}^j(\tau))u_j(\tau)\right) \le \frac{\partial V(x^j_h(\tau|t_k))}{\partial x} \times \left(\frac{1}{m}f(x^j_h(\tau|t_k)) + g_j(x^j_h(\tau|t_k))u^j_{h,j}(\tau|t_k)\right),\ \forall\tau\in[0, T\Delta] \tag{9g}$$

where $S(\Delta)$ is the family of piece-wise constant functions with sampling period Δ, N is the prediction horizon, Q_c and R_{ci}, $i=1, \ldots, m$, are positive definite weight matrices, the state $\tilde{x}^j$ is the predicted trajectory of the nominal system with u_j computed by the LMPC of Eq. (9) and all the other inputs are received from the other controllers (i.e., $\breve{u}^{*,c-1}_i(\tau|t_k)$ which is a noisy version of $u^{*,c-1}_i(\tau|t_k)$). The optimal solution to this optimization problem is denoted by $u^{*,c}_j(\tau|t_k)$ which is defined for $\tau\in[0, N\Delta)$. Accordingly, we define the final optimal input trajectory of LMPC j (that is, the optimal trajectories computed at the last iteration) as $u^{*,f}_j(\tau|t_k)$ which is also defined for $\tau\in[0, N\Delta)$. Note that for the first iteration of each distributed LMPC, the input trajectories defined in Eq. (8) are used as the initial input trajectory guesses; that is, $u^{*,0}_i = u^j_{h,i}$ with $i=1, \ldots, m$.

The constraint of Eq. (9d) imposes a limit on the input change between two consecutive iterations. Note that this constraint does not restrict the input to be in a small region and as the iteration number increases, the final optimal input could be quite different from the initial guess. This constraint is enforced to make sure that the predicted future evolutions of the system state in the distributed controllers are close enough so that their actions are coordinated and they work together to improve the closed-loop performance. For LMPC j (i.e., u_j), the magnitude of input change between two consecutive iterations is restricted to be smaller than a positive constant Δu_j. The constraint of Eq. (9g) is used to guarantee the closed-loop stability. The manipulated inputs of the proposed control design for $t\in[t_k, t_{k+1})$ ($k=0, T, 2T, \ldots$) are defined as follows:

$$u_i(t) = u^{*,f}_i(t-t_k|t_k),\quad i=1,\ldots,m. \tag{10}$$

For the iterations in the design of Eq. (9), the number of iterations c may be restricted to be smaller than a maximum iteration number $c_{\max}$ (i.e., $c\le c_{\max}$) or/and the iterations may be terminated when a maximum computational time is reached. In order to improve the performance, between two slow sampling times, each controller uses the available local fast sampled measurements to adjust its control input based on the calculated optimal input trajectory for the current time obtained at the last time instant in which fast and slowly sampled states were available. In order to guarantee closed-loop stability, the maximum deviation of the adjusted inputs from the optimal input trajectory at each time step is bounded. Between two slow sampling times, each controller estimates the current full system state using an observer based on the system model and the available information. Specifically, the observer for controller i takes the following form for $t\in[t_{l-1}, t_l)$:

$$\dot{\hat{x}}^i(t) = f(\hat{x}^i(t)) + \sum_{j=1}^{m, j\neq i} g_j(\hat{x}^i(t))\breve{u}^{*,i}_j(t-t_k|t_k) + g_i(\hat{x}^i(t))u^*_i(t) \tag{11}$$

with initial condition $\hat{x}^i(t_{l-1}) = x^i_e(t_{l-1})$ where $\hat{x}^i$ is the state of this observer, $\breve{u}^{*,i}_j(\tau|t_k)$ is the optimal input trajectory of LMPC j ($j=1, \ldots, m$, $j\neq i$) received by LMPC i, $u^*_i(t)$ is the actual input that has been applied to subsystem i, and $x^i_e(t_{l-1})$ is the full state estimate obtained at t_{l-1}. The state estimate $x^i_e(t_l)$, $l\neq 0, T, 2T, \ldots$, is a combination of the state of the observer of Eq. (11) and of the available local state information $\breve{x}^s_{f,i}(t_l)$ as follows: $x^i_e(t_l) = [\hat{x}^i_1(t_l)^T\cdots\breve{x}_i(t_l)^T\cdots\hat{x}^i_m(t_l)^T]^T$ where $\breve{x}_i(t_l)^T = [\breve{x}^{s,T}_{f,i}\ \hat{x}^T_{s,i}]$. The optimization problem of LMPC j for a time instant t_l, $l\neq 0, T, 2T, \ldots$ is as follows:

$$\min_{u_j\in S(\Delta)} \int_0^{N\Delta}\left[|\tilde{x}^j(\tau)|_{Q_c} + \sum_{i=1}^{m}|u_i(\tau)|_{R_{ci}}\right]d\tau \tag{12a}$$

$$\text{s.t.}\ \dot{\tilde{x}}^j(\tau) = f(\tilde{x}^j(\tau)) + \sum_{i=1}^{m} g_i(\tilde{x}^j(\tau))u_i \tag{12b}$$

$$u_i(\tau) = \tilde{u}_i^{*,j}(t_l - t_k + \tau|t_k), \quad \forall i \neq j, \quad \tau \in [0, t_k + N\Delta - t_l) \tag{12c}$$

$$u_i(\tau) = h_i(\tilde{x}^j(\tau)), \quad \forall i \neq j, \quad \tau \in [t_k + N\Delta - t_l, N\Delta) \tag{12d}$$

$$\left| u_j(\tau) - u_j^{*,f}(t_l - t_k + \tau|t_k) \right| \leq \Delta u_j, \quad \tau \in [0, t_k + N\Delta - t_l) \tag{12e}$$

$$u_j(\tau) \in U_j \tag{12f}$$

$$\tilde{x}^j(0) = x_e^j(t_l) \tag{12g}$$

where t_k is the last time instant in which both fast and slowly sampled states are available, the state $\tilde{x}^j$ is the predicted trajectory of the nominal system with u_j computed by the LMPC of Eq. (12) and all the other inputs are determined by the constraints of Eqs. (12c) and (12d). In this optimization problem, the input u_j is restricted to be within a bounded region around the reference input trajectories given by $u_i^{*,f}(\tau|t_k)$ and $h(x)$. The optimal solution to this optimization problem is denoted by $u_j^{*,l}(\tau|t_l)$ which is defined for $\tau \in [0, N\Delta)$. The manipulated inputs of the control design of Eq. (12) for $t \in [t_l, t_{l+1})$ $(l \neq 0, T, 2T, \ldots)$ are defined as follows:

$$u_i(t) = u_i^{*,l}(t - t_l|t_l), \quad i = 1, \ldots, m. \tag{13}$$

In the design of Eqs. (9) and (10) and (12) and (13), the closed-loop stability of the system of Eq. (2) is guaranteed by the design of Eqs. (9) and (10) at each sampling time t_k, $k = 0, T, 2T, \ldots$, when the full state measurements are available. The design of Eqs. (12) and (13) takes advantage of the predicted input trajectories $u_i^{*,f}$, $i = 1, \ldots, m$, at sampling times t_k, $k = 0, T, 2T, \ldots$, and the additional available fast-sampling state measurements to adjust the predicted inputs, $u_i^{*,f}$, to improve the closed-loop performance.

Remark 2. Note that in the case of linear systems and flawless communication, at each iteration, the input given by LMPC j of Eq. (9) may be defined as a convex combination of the current optimal input trajectory and the previous one; for example,

$$u_j^c(\tau|t_k) = \sum_{i=1}^{m, i\neq j} w_i u_j^{c-1}(\tau|t_k) + w_j u_j^{*,c}(\tau|t_k) \tag{14}$$

where $\sum_{i=1}^{m} w_i = 1$ with $0 < w_i < 1$, $u_j^{*,c}$ is the current solution given by the optimization problem of Eq. (9) and u_j^{c-1} is the convex combination of the solutions obtained at iteration $c-1$. By doing this, it is possible to prove that the optimal cost of the distributed LMPC of Eq. (9) converges to its optimal value [20,14]. We also note that in this case, the constraint of Eq. (9d) can be removed and the stability of the proposed DMPC architecture is still ensured. Please see Corollary 1 below. We further note that for nonlinear systems it is not possible to prove the convergence of the optimal cost of the distributed optimization problem of Eq. (9) to the cost of the centralized LMPC [21] because the distributed LMPC does not solve the centralized LMPC in a distributed fashion due to the way the Lyapunov-based constraint of the centralized LMPC is broken down into constraints imposed on the individual LMPCs (i.e., Eq. (9g)).

Remark 3. Note that when there is no measurement noise or communication noise, the implementation strategy at t_k $(k = 0, T, 2T, \ldots)$ guarantees that at each sampling time the optimal cost of the distributed optimization of Eq. (9) is upper bounded by the cost of the Lyapunov-based controller $h(\cdot)$.

3.3. Stability analysis

The stability of the closed-loop system is achieved due to the constraints of Eq. (9g) incorporated in each LMPC. The stability property is presented in Theorem 1 below. To prove this theorem, we need the following definitions and propositions. Specifically, we first define the stability region of the closed-loop system under Lyapunov-based control, and certain state trajectories of the closed-loop system accounting for the effect of noise. Subsequently, we state four propositions that bound the discrepancy between various closed-loop system solutions for finite-time under Lyapunov-based control that will be used to state the conditions and prove the closed-loop stability result under multirate DMPC of Theorem 1.

Definition 1. We define Ω_{ρ_n} as follows:

$$\rho_n = \max\{V(x) : (x + n) \in \Omega_\rho, |n| \leq \theta_x\} \tag{15}$$

where $\theta_x = \max_{1\leq i\leq m}\{\theta_x^i\}$ defines the upper bound on the noise n. The region Ω_{ρ_n} will be used as the stability region of the system under the Lyapunov-based controller $h(x)$ in the presence of measurement noise, process disturbances and communication noise.

Definition 2. The closed-loop state trajectory of the nominal system for $t \in [t_k, t_{k+1})$ under $h(x)$ based on actual system state, $x(t_k)$, and applied in sample and hold fashion is denoted by $x_{h,2}(t)$ which is obtained by integrating, for $t \in [t_k, t_{k+1})$, the following equation:

$$\dot{x}_{h,2}(t) = f(x_{h,2}(t)) + \sum_{i=1}^{m} g_i(x_{h,2}(t))h_i(x_{h,2}(t_k)) \tag{16}$$

where $x_{h,2}(t_k) \in \Omega_{\rho_n}$.

Definition 3. The closed-loop state trajectory of the nominal system for $t \in [t_k, t_{k+1})$ under $h(x)$ based on noisy system states and applied in sample and hold fashion is denoted by $x_h(t)$ which is obtained by integrating, for $t \in [t_k, t_{k+1})$, the following equation:

$$\dot{x}_h(t) = f(x_h(t)) + \sum_{i=1}^{m} g_i(x_h(t))h_i(\check{x}_h(t_k)) \tag{17}$$

where $x_h(t_k) \in \Omega_{\rho_n}$, $\check{x}_h(t_k) = x_{h,2}(t_k) + n(t_k)$, $|n| \leq \theta_x$.

Proposition 1 below bounds the difference between the state trajectories starting from two different initial conditions in Ω_{ρ_n} (which is in the stability region Ω_ρ of the control law $h(x)$) under noise with control inputs generated by $h(x)$.

Proposition 1. *Consider the systems:*

$$\dot{x}_a(t) = f(x_a(t)) + \sum_{i=1}^{m} g_i(x_a(t))h_i(\check{x}_a(0))$$

$$\dot{x}_b(t) = f(x_b(t)) + \sum_{i=1}^{m} g_i(x_b(t))h_i(\check{x}_b(0))$$

where the initial states $x_a(0), x_b(0) \in \Omega_{\rho_n}$, $|x_a(0) - x_b(0)| \leq \theta_{ab}$, $|x_a(0) - \check{x}_a(0)| \leq \theta_a$ *and* $|x_b(0) - \check{x}_b(0)| \leq \theta_b$. *If* $0 < \rho_n < \rho$, *then there exists a function* $f_E(\cdot, \cdot, \cdot, \cdot)$ *such that* $|x_a(t) - x_b(t)| \leq f_E(\theta_{ab}, \theta_a, \theta_b, t)$ *for all* $x_a(t), x_b(t) \in \Omega_{\rho_n}$ *with* $f_E(\theta_{ab}, \theta_a, \theta_b, t) = (\theta_{ab} + \frac{L_2}{L_1})e^{L_1 t} - \frac{L_2}{L_1}$ *where* $L_1, L_2, \theta_{ab}, \theta_a$ *and* θ_b *are positive real numbers.*

Proof. If we define $e(t) = x_a(t) - x_b(t)$, then the derivative of $e(t)$ can be calculated as $\dot{e}(t) = \dot{x}_a - \dot{x}_b$. Adding/subtracting $\sum_{i=1}^{m} g_i(x_a(t))h_i(\check{x}_b(0))$ to/from the expression of $\dot{e}(t)$ and using the conditions defined in Eq. (6) obtained by the local Lipschitz properties and the fact that $h_i(\cdot)$ satisfies input constraints, we can obtain the following inequality:

$$|\dot{e}(t)| \le C_x|x_a(t)-x_b(t)| + \sum_{i=1}^{m} M_{g_i}C_{h_i}|\check{x}_a(0)-\check{x}_b(0)| + \sum_{i=1}^{m} u_i^{\max}C_{u_i}|x_a(t)-x_b(t)|. \quad (18)$$

Using that $|\check{x}_a(0)-\check{x}_b(0)| \le \theta_a+\theta_b+\theta_{ab}$ and defining $L_1 = C_x + \sum_{i=1}^{m} u_i^{\max}C_{u_i}$ and $L_2 = (\theta_a+\theta_b+\theta_{ab})\sum_{i=1}^{m} M_{g_i}C_{h_i}$, we obtain $|\dot{e}(t)| \le L_1|e(t)| + L_2$. Integrating $|\dot{e}(t)|$ with initial condition $|e(0)| \le \theta_{ab}$, we can obtain $|e(t)| \le (\theta_{ab} + \frac{L_2}{L_1})e^{L_1 t} - \frac{L_2}{L_1}$ which proves Proposition 1. □

The following proposition provides sufficient conditions that ensure that $h(\cdot)$ can achieve closed-loop stability of the nominal system in the presence of bounded measurement and communication noise.

Proposition 2. *Consider the closed-loop nominal sampled trajectory $x_h(t)$ of the system of Eq. (2) as defined in Definition 3. Let $\Delta, \epsilon_s, \theta_x > 0$ and $0 < \rho_s < \rho_n < \rho$ satisfy:*

$$\left(L_x + \sum_{i=1}^{m} u_i^{\max}L_{u_i}\right)\left(f_E(0,0,\theta_x,\Delta) + M\Delta\right) + \theta_x\sum_{i=1}^{m} C_{g_i}C_{h_i} - \alpha_2\left(\alpha_1^{-1}(\rho_s)\right) \le -\epsilon_s/\Delta. \quad (19)$$

where f_E is defined in Proposition 1. For any k, if $x_h(t_k) \in \Omega_{\rho_n}/\Omega_{\rho_s}$, then $V(x_h(t_{k+1})) \le V(x_h(t_k)) - \epsilon_s$ and $V(x_h(t)) \le V(x_h(t_k))$ for $t \in [t_k, t_{k+1})$. Also, if $\rho_{min} \le \rho_n$ where $\rho_{min} = \max\{V(x_h(t+\Delta)) : V(x_h(t)) \le \rho_s\}$ and $x_h(t_0) \in \Omega_{\rho_n}$, then we also have $V(x_h(t_k)) \le \max\{V(x_h(t_0)) - k\epsilon_s, \rho_{min}\}$ and $V(x_h(t)) \le \max\{V(x_h(t_k)), \rho_{min}\}$ for $t \in [t_k, t_{k+1})$.

Proof. Following Definition 3, the time derivative of the Lyapunov function along the nominal sampled trajectory $x_h(t)$ of the system of Eq. (2) for $t \in [t_k, t_{k+1})$ is given by $\dot{V}(x_h(t)) = \frac{\partial V(x_h(t))}{\partial x}\dot{x}_h(t)$. Adding/subtracting $\frac{\partial V(x_{h,2}(t_k))}{\partial x}\left(f(x_{h,2}(t_k)) + \sum_{i=1}^{m} g_i(x_{h,2}(t_k))h_i(x_{h,2}(t_k))\right)$ to/from the expression describing $\dot{V}(x_h(t))$, and then adding/substracting $\frac{\partial V(x_h(t))}{\partial x}\sum_{i=1}^{m} g_i(x_h(t))h_i(x_{h,2}(t_k))$ to/from the resulting inequality, we can obtain the following inequality by the conditions of Eqs. (4) and (6):

$$\dot{V}(x_h(t)) \le \left(L_x + \sum_{i=1}^{m} u_i^{\max}L_{u_i}\right)|x_h(t) - x_{h,2}(t_k)| + \sum_{i=1}^{m} C_{g_i}C_{h_i}|\check{x}_h(t_k) - x_{h,2}(t_k)| - \alpha_2\left(\alpha_1^{-1}(\rho_s)\right) \quad (20)$$

for all $x_{h,2}(t_k) \in \Omega_{\rho_n}/\Omega_{\rho_s}$. Using the triangular inequality, we obtain $|x_h(t) - x_{h,2}(t_k)| \le |x_h(t) - x_{h,2}(t)| + |x_{h,2}(t) - x_{h,2}(t_k)|$ for $t \in [t_k, t_{k+1})$. Taking into account the condition of Eq. (5), the continuity of $x_{h,2}(t)$, the fact that $|\check{x}_h(t_k) - x_{h,2}(t_k)| \le \theta_x$, and applying Proposition 1, we obtain from Eq. 20 the following bound on the time derivative of the Lyapunov function for $t \in [t_k, t_{k+1})$, for all initial states $x_h(t_k) \in \Omega_{\rho_n}/\Omega_{\rho_s}$:

$$\dot{V}(x_h(t)) \le \theta_x\sum_{i=1}^{m} C_{g_i}C_{h_i} - \alpha_2\left(\alpha_1^{-1}(\rho_s)\right) + \left(L_x + \sum_{i=1}^{m} u_i^{\max}L_{u_i}\right)\left(f_E(0,0,\theta_x,\Delta) + M\Delta\right). \quad (21)$$

If the condition of Eq. (19) is satisfied, then $\dot{V}(x_h(t)) \le -\epsilon_s/\Delta$. Integrating this bound on $t \in [t_k, t_{k+1})$, we obtain that $V(x_h(t_{k+1})) \le V(x_h(t_k)) - \epsilon_s$ and $V(x_h(t)) \le V(x_h(t_k))$. Applying this result recursively, it is easy to verify that $V(x_h(t_k)) \le \max\{V(x_h(t_0)) - k\epsilon_s, \rho_{min}\}$ and $V(x_h(t)) \le \max\{V(x_h(t_k)), \rho_{min}\}$. □

Proposition 2 ensures that if the nominal system under the control $h(x)$ implemented in a sample-and-hold fashion starts in Ω_{ρ_n}, then it is ultimately bounded in $\Omega_{\rho_{\min}}$.

Proposition 3. *Consider the systems*

$$\dot{x}_a(t) = f(x_a(t)) + \sum_{i=1}^{m} g_i(x_a(t))u_i^c(t)$$

$$\dot{x}_b(t) = f(x_b(t)) + \sum_{i=1}^{m,i\neq j} g_i(x_b(t))\check{u}_i^{c-1}(t) + g_j(x_b(t))u_j^c(t)$$

where $\check{u}_i^{c-1}(t) = u_i^{c-1}(t) + n_{u_i}$ with initial states $x_b(t_0) = x_a(t_0) + n_x^j \in \Omega_\rho$, $x_a(t_0) \in \Omega_{\rho_n}$, $|n_x^j| \le \theta_x^j$ and $|n_{u_i}| \le \theta_{u_i}$. There exists a function $f_{X,j}(\cdot,\cdot)$ such that

$$|x_a(t) - x_b(t)| \le f_{X,j}(\theta_x^j, t - t_0) \quad (22)$$

for all $x_a(t), x_b(t) \in \Omega_\rho$, and $u_i^c(t), u_i^{c-1} \in U_i$ and $|u_i^c(t) - \check{u}_i^{c-1}(t)| \le \Delta u_i$, $i = 1, \ldots, m$ and $f_{X,j}(\tau) = \left(\frac{C_{2,j}}{C_{1,j}} + \theta_x^j\right)e^{C_{1,j}(\tau)} - \frac{C_{2,j}}{C_{1,j}}$ with $C_{2,j}$ and $C_{1,j}$ are positive constants.

Proof. Let $e(t) = x_a(t) - x_b(t)$. The derivative of $e(t)$ can be calculated as $\dot{e}(t) = \dot{x}_a(t) - \dot{x}_b(t)$. Adding/subtracting $\sum_{i=1}^{m,i\neq j} g_i(x_a(t))\check{u}_i^{c-1}(t)$ to/from the expression of $e(t)$, and then using the fact $\check{u}_i^{c-1}(t) \le u_i^{\max} + \theta_{u_i}$ and the conditions defined in Eq. (6), we obtain the following inequality:

$$|\dot{e}(t)| \le \left(C_x + u_j^{\max}C_{u_j} + \sum_{i=1}^{m,i\neq j}(u_i^{\max} + \theta_{u_i})C_{u_i}\right)|e(t)| + \sum_{i=1}^{m,i\neq j} M_{g,i}\Delta u_i$$

Defining $C_{1,j} = C_x + u_j^{\max}C_{u_j} + \sum_{i=1}^{m,i\neq j}(u_i^{\max} + \theta_{u_i})C_{u_i}$ and $C_{2,j} = \sum_{i=1}^{m,i\neq j} M_{g_i}\Delta u_i$, from the above inequality, we have $|\dot{e}(t)| \le C_{1,j}|e(t)| + C_{2,j}$. Since the initial condition, $e(t_0)$, satisfies $|e(t_0)| \le \theta_x^j$ (recall $x_b(t_0) = x_a(t_0) + n_x^j$ where $|n_x^j| \le \theta_x^j$), we can obtain $|e(t)| \le \left(\frac{C_{2,j}}{C_{1,j}} + \theta_x^j\right)e^{C_{1,j}(t-t_0)} - \frac{C_{2,j}}{C_{1,j}}$. This proves Proposition 3. □

Proposition 3 bounds the difference between the nominal state trajectory under the optimized control inputs and the predicted nominal state trajectory generated in each LMPC optimization problem.

Proposition 4. *Consider the systems*

$$\dot{x}_a(t) = f(x_a(t)) + \sum_{i=1}^{m} g_i(x_a(t))u_i(t) + k(x_a(t))w(t)$$

$$\dot{x}_b(t) = f(x_b(t)) + \sum_{i=1}^{m} g_i(x_b(t))u_i(t)$$

with initial states $x_b(t_k) = x_a(t_k) + n \in \Omega_\rho$, $x_a(t_k) \in \Omega_\rho$, and $|n| \le \theta_x$. There exists a function $f_W(\cdot,\cdot)$ such that

$$|x_a(t) - x_b(t)| \le f_W(\theta_x, t - t_k), \quad (23)$$

for all $x_a(t)$, $x_b(t) \in \Omega_\rho$ and all $w(t) \in W$ with $f_W(\theta_x, \tau) = \left(\theta_x + \frac{\Gamma_2}{\Gamma_1}\right)e^{\Gamma_1\tau} - \frac{\Gamma_2}{\Gamma_1}$ where Γ_1, Γ_2 are positive real numbers.

Proof. Define $e(t) = x_a(t) - x_b(t)$, then $\dot{e}(t) = \dot{x}_a(t) - \dot{x}_b(t)$. Using the condition of Eq. (6), we obtain the following inequality:

$$|\dot{e}(t)| \leq \left(C_x + \sum_{i=1}^{m} u_i^{\max} C_{u_i} \right) |e(t)| + M_w \theta. \tag{24}$$

Defining $\Gamma_1 = C_x + \sum_{i=1}^{m} u_i^{\max} C_{u_i}$ and $\Gamma_2 = M_w \theta$, and accounting for that $|e(t_k)| \leq \theta_x$, we obtain $|e(t)| \leq \left(\theta_x + \frac{\Gamma_2}{\Gamma_1}\right) e^{\Gamma_1 (t-t_k)} - \frac{\Gamma_2}{\Gamma_1}$. This proves Proposition 4. □

Proposition 4 provides an upper bound on the deviation of the state trajectory obtained using the nominal model, from the actual state trajectory when the same control actions are applied. Proposition 5 bounds the difference between the magnitudes of the Lyapunov function of two states in Ω_ρ.

Proposition 5 (c.f. [16]). *Consider the Lyapunov function $V(\cdot)$ of the system of* Eq. (2). *There exists a quadratic function $f_V(\cdot)$ such that $V(x) \leq V(x') + f_V(|x - x'|)$ for all $x, x' \in \Omega_\rho$.*

In Theorem 1 below, we provide sufficient conditions under which the DMPC of Eqs. (9) and (10) and (12) and (13) guarantees that the state of the closed-loop system is ultimately bounded in a region that contains the origin. To simplify the proof of Theorem 1, we define new functions $f_H(\tau)$ and $f_X(\tau)$ based on f_E and $f_{X,i}$ ($i = 1, \ldots, m$), respectively, as follows:

$$f_H(\tau) = \sum_{i=2}^{m} \left(\frac{1}{m} L_x + M_{g_i} C_{h_i} + u_i^{\max} L_{u_i} \right) \left(\frac{1}{L_1} f_E(\theta_x^i + \theta_x^1, 0, 0, \tau) - \frac{L_2 \tau + \theta_x^i + \theta_x^1}{L_1} \right),$$

$$f_X(\tau) = \left(\frac{1}{m} L_x + L_{u_1} u_1^{\max} \right) \left(\frac{1}{C_{1,1}} f_{X,1}(0, \tau) - \frac{C_{2,1}}{C_{1,1}} \tau \right) + \sum_{i=2}^{m} \left(\frac{1}{m} L_x + L_{u_i} u_i^{\max} \right) \left(\frac{1}{C_{1,i}} f_{X,i}(\theta_x^i + \theta_x^1, \tau) - \frac{C_{2,i}}{C_{1,i}} \tau - \frac{\theta_x^i + \theta_x^1}{C_{1,i}} \right).$$

It is easy to verify that $f_H(\tau)$ and $f_X(\tau)$ are strictly increasing and convex functions of their arguments.

Theorem 1. *Consider the system of* Eq. (2) *in closed-loop with the DMPC design of* Eqs. (9) and (10) and (12) and (13) *based on the controller $h(x)$ that satisfies the conditions of* Eq. (4) *with class $\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, 2$. If there exist $\Delta > 0$, $\epsilon_s > 0$, $\theta_x > 0$, $\rho > \rho_n > \rho_{\min} > 0$, $\rho > \rho_n > \rho_s > 0$ and $N \geq T \geq 1$ that satisfy the conditions of* Eq. (19) *and the following inequality:*

$$f_X(T\Delta) + f_V(f_W(\theta_x, T\Delta)) + f_V(f_W(\theta_x, 0)) + f_H(T\Delta) + \sum_{i=1}^{m} C_{g,i} \Delta u_i (T-1)\Delta - T\epsilon_s < 0, \tag{25}$$

and if the initial state of the closed-loop system $x(t_0) \in \Omega_{\rho_n}$, then $x(t)$ is ultimately bounded in $\Omega_{\rho_b} \subseteq \Omega_{\rho_n}$ where

$$\rho_b = \rho_{\min} + f_V(f_W(\theta_x, 0)) u_i (T-1)\Delta + \sum_{i=1}^{m} C_{g,i} \Delta + f_H(T\Delta) + f_V(f_W(\theta_x, T\Delta)) + f_X(T\Delta).$$

Proof. We first consider two consecutive time instants in which both fast and slowly sampled states are available: t_k and t_{k+T} ($k = 0, T, 2T, \ldots$). We will prove that the Lyapunov function of the system is decreasing from t_k to t_{k+T}. In the following, we denote the trajectory of the nominal system of Eq. (2) under the DMPC of Eqs. (9) and (10) and (12) and (13) starting from $\check{x}^1(t_k)$ (which is the state received by LMPC 1 at t_k) as $\tilde{x}$, and we also denote the predicted nominal system trajectory in the evaluation of the LMPC of Eq. (9) at the final iteration as $\tilde{x}^j$ with $j = 1, \ldots, m$. It should be mentioned that the initial condition for the nominal sampled trajectory $\tilde{x}$ under the implementation of u_i^* can be $\tilde{x}(t_k) = x_h^i(0|t_k)$ for any $i = 1, \ldots, m$. Without loss of generality, we assume that $\tilde{x}(t_k) = \check{x}^1(t_k) = x_h^1(0|t_k)$; use of any $i = 2, \ldots, m$ in $\tilde{x}(t_k) = x_h^i(0|t_k)$ would simply require an appropriate modification in the definitions of $f_X(\cdot)$ and $f_H(\cdot)$.

The derivative of the Lyapunov function of the nominal system of Eq. (2) under the DMPC of Eqs. (9) and (10) and (12) and (13) from t_k to t_{k+T} can be expressed as follows:

$$\dot{V}(\tilde{x}(\tau)) = \frac{\partial V(\tilde{x}(\tau))}{\partial x} \left(f(\tilde{x}(\tau)) + \sum_{i=1}^{m} g_i(\tilde{x}(\tau)) u_i^*(\tau) \right) \tag{26}$$

where $\tilde{x}(t_k) = \check{x}^1(t_k) = x_h^1(0|t_k)$ and $u_i^*(\tau)$ is the actual input applied to the system and defined as follows:

$$u_i^*(\tau) = \begin{cases} u_i^{*,f}(\tau|t_k), & \tau \in [0, \Delta) \\ u_i^{*,l}(\tau|t_l), & \tau \in [0, \Delta), l = k+1, \ldots, k+T-1. \end{cases}$$

Combining Eq. (26) and the inequality constraints of Eq. (9g) ($i = 1, \ldots, m$), and adding/subtracting $\frac{\partial V(x_h^1(\tau|t_k))}{\partial x} \left(f(x_h^1(\tau|t_k)) + \sum_{i=1}^{m} g_i(x_h^1(\tau|t_k)) u_{h,i}^1(\tau|t_k) \right)$ to/from the righthand side of the resulting inequality, we can obtain the following inequality for all $\tau \in [0, T\Delta]$ by taking into account the conditions of Eq. (6):

$$\begin{aligned}
\dot{V}(\tilde{x}(\tau)) &\leq \dot{V}(x_h^1(\tau|t_k)) + \sum_{i=1}^{m} C_{g_i} \left(u_i^*(\tau) - u_i^{*,f}(\tau|t_k) \right) \\
&+ \left(\frac{1}{m} L_x + L_{u_1} u_1^{*,f}(\tau|t_k) \right) \left| \tilde{x}(\tau) - \tilde{x}^1(\tau) \right| + \ldots \\
&+ \left(\frac{1}{m} L_x + L_{u_m} u_m^{*,f}(\tau|t_k) \right) \left| \tilde{x}(\tau) - \tilde{x}^m(\tau) \right| \\
&+ \left(\frac{1}{m} L_x + u_2^{\max} L_{u_2} \right) \left| x_h^2(\tau|t_k) - x_h^1(\tau|t_k) \right| + \ldots \\
&+ \left(\frac{1}{m} L_x + u_m^{\max} L_{u_m} \right) \left| x_h^m(\tau|t_k) - x_h^1(\tau|t_k) \right| \\
&+ M_{g_2} C_{h_2} \left| x_h^2(\tau|t_k) - x_h^1(\tau|t_k) \right| + \ldots \\
&+ M_{g_m} C_{h_m} \left| x_h^m(\tau|t_k) - x_h^1(\tau|t_k) \right|
\end{aligned} \tag{27}$$

Applying Propositions 3 and 1 to the inequality of Eq. (27), and then integrating the resulting inequality from $\tau = 0$ to $\tau = T\Delta$ and taking into account that $\tilde{x}(t_k) = x_h^1(0|t_k)$, the constraints of Eqs. (9d) and (12e) and the definitions of $f_X(\cdot)$, $f_H(\cdot)$ and $u^*(\tau)$, the following inequality can be obtained:

$$V(\tilde{x}(t_{k+T})) \leq V(x_h^1(T\Delta|t_k)) + f_X(T\Delta) + f_H(T\Delta) + \sum_{i=1}^{m} C_{g,i} \Delta u_i (T-1)\Delta. \tag{28}$$

Since $V(x_h^1(T\Delta|t_k)) \leq \max\{V(x_h^1(0|t_k)) - T\epsilon_s, \rho_{min}\}$ from Proposition 2, $\tilde{x}(t_k) = x_h^1(0|t_k)$ and $|V(\tilde{x}(t_k)) - V(x(t_k))| \leq f_V(f_W(\theta_x, 0))$ and $|V(\tilde{x}(t_{k+T})) - V(x(t_{k+T}))| \leq f_V(f_W(\theta_x, T\Delta))$ from

Propositions 4 and 5, we can obtain the following inequality from Eq. (28):

$$V(x(t_{k+T})) \le \max\{V(x(t_k)) - T\epsilon_s, \rho_{\min}\} + f_X(T\Delta) + f_H(T\Delta) + f_V(f_W(\theta_x, T\Delta)) + f_V(f_W(\theta_x, 0)) + \sum_{i=1}^{m} C_{g,i}\Delta u_i(T-1)\Delta. \quad (29)$$

If there exist $\Delta > 0$, $\epsilon_s > 0$, $\theta_x > 0$, $\rho > \rho_n > \rho_{\min} > 0$, $\rho > \rho_n > \rho_s > 0$ and $N \ge T \ge 1$ that satisfy the conditions of Eqs. (19) and 25, then there exists $\epsilon_w > 0$ such that the following inequality holds

$$V(x(t_{k+T})) \le \max\{V(x(t_k)) - \epsilon_w, \rho_b\} \quad (30)$$

which implies that if $x(t_k) \in \Omega_{\rho_n}/\Omega_{\rho_b}$, then $V(x(t_{k+T})) < V(x(t_k))$, and if $x(t_k) \in \Omega_{\rho_b}$, then $V(x(t_{k+T})) \le \rho_b$.

Because the upper bound on the difference between the Lyapunov function of the actual trajectory x and the nominal trajectory $\tilde{x}$ (see Eq. (30)) is a strictly increasing function of T, the inequality of Eq. (30) also implies that:

$$V(x(t)) \le \max\{V(x(t_k)) - \epsilon_w, \rho_b\}, \quad \forall t \in [t_k, t_{k+T}]. \quad (31)$$

Using the inequality of Eq. (31) recursively, it can be proved that if $x(t_0) \in \Omega_{\rho_n}$, then the closed-loop trajectories of the system of Eq. (2) under the proposed DMPC design stay in Ω_{ρ_n} for all times (i.e., $x(t) \in \Omega_{\rho_n}$ for all t). Moreover, if $x(t_0) \in \Omega_{\rho_n}$, the closed-loop trajectories of the system of Eq. (2) under the proposed iterative DMPC design satisfy $\limsup_{t\to\infty} V(x(t)) \le \rho_b$. This proves Theorem 1. □

In addition to the stability result of Theorem 1, we note that because the closed-loop states of the system of Eq. (2) under the proposed DMPC scheme are guaranteed to be bounded in a compact set containing the origin and the manipulated inputs are bounded for all times (this follows from the practical stability of the closed-loop system), the cost along the closed-loop system trajectory over finite time (which only depends on the absolute values of the magnitude of the system states and the manipulated inputs) is also bounded. We also note that in the context of linear systems and noise-free measurements and communication, the distributed optimization problem of Eq. (9) is convex. Furthermore, if the inputs of the distributed controllers are defined as convex combinations of their current and previous solutions as described in Eq. (14), as the iteration number c increases, the optimal cost given by the distributed optimization problem of Eq. (9) converges to its corresponding centralized optimal value. This property is summarized in the following Corollary 1.

Corollary 1. *Consider a class of linear time-invariant systems:*

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (32)$$

with

$$\dot{x}_i = A_{ii}x_i + \sum_{j\ne i} A_{ij}x_j + B_i u_i(t) \quad (33)$$

where A, B, A_{ii}, A_{ij} and B_i are constant matrices with appropriate dimensions. If we define the inputs of the distributed controllers at iteration c as in Eq. (14), *then at a sampling time t_k, as the iteration number $c \to \infty$, the optimal cost of the distributed optimization problem of* Eq. (9) *converges to the optimal cost of the corresponding centralized control system. If $x(0) \in \Omega_\rho$ and the corresponding centralized MPC asymptotically stabilizes the origin of the closed-loop system, the DMPC of* Eq. (9) *also asymptotically stabilizes the origin of the closed-loop system and the closed-loop performance of the DMPC converges to the one given by the centralized control system.*

Proof. In this proof, we focus on a simplified case: (1) a linear system composed of two subsystems, and (2) full state feedback, x, is available every sampling time. We first prove that, at each sampling time, the optimal cost of the distributed optimization problem of Eq. (9) converges to the optimal cost of the corresponding centralized control system as the iteration number increases, and then prove that if the corresponding centralized MPC asymptotically stabilizes the origin of the closed-loop system, then the DMPC of Eq. (9) also asymptotically stabilizes the origin of the closed-loop system. This proof can be extended in a straightforward manner to include general linear systems with measurements available every T ($T \le N$) sampling times.

For a linear system, it is easy to verify that the constraints of Eqs. (9a)–(9f) are convex. We will focus on the proof of the convexity of the constraint of Eq. (9g). Specifically, using a quadratic Lyapunov function $V(x) = x^T P x$ where P is a positive definite symmetric matrix, the constraint of Eq. (9g) takes the following form:

$$\left(\frac{1}{2}\tilde{x}^j(\tau)^T A^T + u_j(\tau)^T B_j^T\right) P\tilde{x}^j(\tau) + \tilde{x}^j(\tau)^T P\left(\frac{1}{2}A\tilde{x}^j(\tau) + B_j u_j(\tau)\right) \le \left(\frac{1}{2}x_h^j(\tau|t_k)^T A^T + u_{h,j}^j(\tau|t_k)^T B_j^T\right) P\tilde{x}_h^j(\tau|t_k) + x_h^j(\tau|t_k)^T P\left(\frac{1}{2}Ax_h^j(\tau|t_k) + B_j u_{h,j}^j(\tau|t_k)\right) \quad (34)$$

where $j = 1, 2$, $\tau \in [0, \Delta]$ and $\tilde{x}^j$ is the predicted trajectory of the nominal system with u_j computed by the LMPC of Eq. (9) and the other input is received from the other controller. The right hand side of Eq. (34) has no dependence on u_j or $\tilde{x}^j$ and can be considered as a constant. If we take into account that the input trajectories are piece-wise constant and that $\tilde{x}^j(\tau) = e^{A\tau}\tilde{x}^j(0) + \int_0^\tau e^{A(\tau-s)}Bu(s)ds$, for $\tau \in [0, \Delta]$, we can obtain that:

$$\tilde{x}^j(\tau) = C^j(t_k, \tau) + D^j(t_k, \tau)u_j \quad (35)$$

where $C^j(t_k, \tau)$ and $D^j(t_k, \tau)$ are matrices that depend only on τ. As it can be seen from Eq. (35), u_j appears linearly. Taking into account Eq. (35) and the fact that the right hand side of Eq. (34) can be considered as a constant, we can re-write Eq. (34) in a quadratic form with respect to u_j as follows:

$$u_j^T E^j(t_k, \tau)u_j + F^j(t_k, \tau)u_j \le G^j(t_k, \tau) \quad (36)$$

where $E^j(t_k, \tau)$, $F^j(t_k, \tau)$ and $G^j(t_k, \tau)$ are matrices that depend only on τ. This proves that the constraint of Eq. (34) is convex. Therefore, the optimization problem of Eq. (9) for the linear system with two subsystems is convex. If the inputs of the distributed controllers at each iteration c are defined as in Eq. (14), then the convergence of the cost given by the distributed optimization problem to the corresponding centralized control system can be proved following similar strategies used in Refs. [20,14] for a specific sampling time t_k. If $x(0) \in \Omega_\rho$ and the centralized MPC can asymptotically stabilize the origin of the closed-loop system, using the above arguments recursively for each sampling time, if $c \to \infty$ for each sampling time, it follows that the DMPC also asymptotically stabilizes the origin of the closed-loop system and the closed-loop cost converges to the one given by the centralized control system. □

Remark 4. Referring to the open-loop nature of the estimator of Eq. (11), it is important to note that it does not pose any restrictions on the open-loop stability of the processes in which the proposed multirate DMPC method can be applied. The reason is that this estimator is used to provide "short-term" (within the slow sampling period upper bound) estimates of plant states which are used in the fast sampling-time DMPCs applied in the various subsystems; therefore, if the upper bound on the slow sampling time is sufficiently small as required by Theorem 1, the stability of the closed-loop system under the proposed multirate DMPC scheme is guaranteed.

Table 1
Disturbance parameters.

	σ_p	ϕ	θ_p		σ_p	ϕ	θ_p
C_{A1}	0.1	0.7	0.09	C_{A2}	0.1	0.7	0.09
C_{B1}	0.02	0.7	0.01	C_{B2}	0.1	0.7	0.03
C_{C1}	0.02	0.7	0.01	C_{C2}	0.1	0.7	0.01
T_1	10	0.7	1.17	T_2	10	0.7	1.35
C_{A3}	0.1	0.7	0.09	C_{B3}	0.1	0.7	0.02
C_{C3}	0.02	0.7	0.01	T_3	10	0.7	1.35

Remark 5. Even though the conditions of Theorem 1 are conservative in nature in order to guarantee closed-loop stability, they do provide insight into the relationship between the various variables characterizing the controller, process and measurement sampling components of the closed-loop system and can be used to properly tune the overall control system. The degree of conservativeness of the conditions of Theorem 1 can be assessed in practice via closed-loop simulations.

Remark 6. Note that if all the distributed controllers have access to the whole system state vector measurements at each slow sampling instant, the controllers do not have to communicate and can make their calculation in a decentralized fashion without loss of the closed-loop stability because of the design of the stability constraints. The communication and iteration of the distributed controllers, however, can improve the overall closed-loop system performance significantly. It is also important to note that the DMPC system operating at the slow sampling time can utilize alternative communication strategies between the distributed controllers like, for example, sequential communication or local (nearest-neighbor) communication, provided appropriate conditions are satisfied that ensure stability of the closed-loop system in each case. Furthermore, ideas from the quasi-decentralized control framework for multi-unit plants developed in Ref. [12] where suitable models are used in each controller to estimate state variables of the other subsystems, can be adopted in the proposed DMPC framework. Finally, we note that if measurements of some of the state variables are not available, networked state estimation schemes [22] may be used within the proposed multirate DMPC framework.

4. Application to a chemical process

The process considered in this study is a three vessel, reactor-separator system consisting of two continuously stirred tank reactors (CSTRs) and a flash tank separator. The reactions $A \to B$ and $A \to C$ (referred to as 1 and 2, respectively) take place in the two CSTRs before the effluent from CSTR 2 is fed to a flash tank. The detailed description and modeling of the process can be found in Ref. [23]. The process is numerically simulated using a standard Euler integration method. Process noise was added to simulate disturbances/model uncertainty and it is generated as autocorrelated noise of the form $w_k = \phi w_{k-1} + \xi_k$ where $k = 0, 1, \ldots$ is the discrete time step of 0.001 h, ξ_k is generated by a normally distributed random variable with standard deviation σ_p, and ϕ is the autocorrelation factor and w_k is bounded by θ_p, that is $|w_k| \le \theta_p$. Table 1 contains the parameters used in generating the process noise. The process is divided into three subsystems corresponding to the first CSTR, the second CSTR and the separator, respectively. For the three subsystems, we will refer to them as subsystem 1, subsystem 2 and subsystem 3, respectively. The state of subsystem 1 is defined as the deviations of the temperature and species concentrations in the first CSTR from their desired steady-state; that is, $x_1^T = [x_{f,1}^T, x_{s,1}^T]$ where $x_{f,1} = T_1 - T_{1s}$ and $x_{s,1}^T = [C_{A1} - C_{A1s}\, C_{B1} - C_{B1s}\, C_{C1} - C_{Cs}]$ denote fast sampled and slowly sampled measurements of subsystem 1, respectively. Due to the simplicity of temperature measurement at each sampling time, we denote the temperature as the fast sampled measurement of each subsystem. The states of subsystems 2 and 3 are defined similarly; they are $x_2^T = [T_2 - T_{2s}\, C_{A2} - C_{A2s}\, C_{B2} - C_{B2s}\, C_{C2} - C_{C2s}]$ and $x_3^T = [T_3 - T_{3s}\, C_{A3} - C_{A3s}\, C_{B3} - C_{B3s}\, C_{C3} - C_{C3s}]$. The values of the desired steady state are shown in Table 2. Accordingly, the state of the whole process is defined as a combination of the states of the three subsystems; that is, $x^T = [x_1^T\, x_2^T\, x_3^T]$.

Table 2
Steady-state values for x_s.

C_{A1s}	3.31 [kmol/m^3]	C_{A2s}	2.75 [kmol/m^3]
C_{B1s}	0.17 [kmol/m^3]	C_{B2s}	0.45 [kmol/m^3]
C_{C1s}	0.04 [kmol/m^3]	C_{C2s}	0.11 [kmol/m^3]
T_{1s}	369.53 [K]	T_{2s}	435.25 [K]
C_{A3s}	2.88 [kmol/m^3]	C_{B3s}	0.50 [kmol/m^3]
C_{C3s}	0.12 [kmol/m^3]	T_{3s}	435.25 [K]

The process has one unstable and two stable steady states. The control objective is to regulate the process at the unstable steady state x_s corresponding to the operating point defined by $Q_{1s} = 0$ kJ/h, $Q_{2s} = 0$ kJ/h and $Q_{3s} = 0$ kJ/h, respectively. Each of the tanks has an external heat input which is the control input associated with each subsystem, that is, $u_1 = Q_1 - Q_{1s}$, $u_2 = Q_2 - Q_{2s}$ and $u_3 = Q_3 - Q_{3s}$. The inputs are subject to constraints as follows: $|u_1| \le 5 \times 10^4$ kJ/h, $|u_2| \le 1.5 \times 10^5$ kJ/h, and $|u_3| \le 2 \times 10^5$ kJ/h. Three distributed MPC controllers (controller 1, controller 2 and controller 3) will be designed to manipulate each one of the three inputs in the three subsystems, respectively. The process model (see Ref. [23]) belongs to the following class of nonlinear systems:

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{3} g_i(x(t))u_i(t) + w(x(t))$$

where the explicit expressions of f, g_i ($i = 1, 2, 3$), are omitted for brevity. We assume that $x_{f,1}$, $x_{f,2}$, $x_{f,3}$ are measured and sent to controller 1, controller 2 and controller 3, respectively, at synchronous time instants $t_l = l\Delta$, $l = 0, 1, \ldots$, with $\Delta = 0.01$ h = 36 s while we assume that each controller receives $x_{s,i}$ every $T = 4$ sampling times. The three subsystems exchange their states at $t_k = kT\Delta$, $k = 0, 1, \ldots$; that is, the full system state x is sent to all the controllers every $T = 4$ sampling times. In the simulations, we consider a quadratic Lyapunov function $V(x) = x^T P x$ with $P = \text{diag}([20\, 10^3\, 10^3\, 10^3\, 20\, 10^3\, 10^3\, 10^3\, 20\, 10^3\, 10^3\, 10^3])$. We design the Lyapunov-based controller $h(x)$ following the continuous bounded control law [24,19] as follows:

$$h(x) = -p(x)(L_G V)^T \tag{37}$$

where

$$p(x) = \begin{cases} \dfrac{L_f V + \sqrt{\left(L_f V\right)^2 + \left(u^{\max}|L_G V^T|\right)^4}}{|L_G V^T|^2 \left[1 + \sqrt{1 + \left(u^{\max}|L_G V^T|\right)^2}\right]}, & L_G V \neq 0 \\ 0, & L_G V = 0 \end{cases}$$

with $L_f V = \frac{\partial V}{\partial x} f(x)$ and $L_G V = \frac{\partial V}{\partial x} G(x)$ where $G = [g_1\, g_2\, g_3]$ being the Lie derivatives of the scalar function V with respect to the vector fields f and G, respectively. To estimate the stability region Ω_ρ, extensive simulations were carried out to get an estimate of the region of the closed-loop system under Lyapunov-based control $h(x)$ where the time-derivative of the Lyapunov function is negative, and then Ω_ρ is defined as a level set of the Lyapunov function $V(x)$ embedded within this region.

Based on the Lyapunov-based controller $h(x)$ and $V(x)$, we design the three LMPCs following Eqs. (9) and (10) and (12) and (13) and refer to them as LMPC 1, LMPC 2 and LMPC 3. For each LMPC, we also design a state observer following Eq. (11). In the design of the LMPC controllers, the weighting matrices are chosen to be $Q_c = \text{diag}([20\,10^3\,10^3\,10^3\,20\,10^3\,10^3\,10^3\,20\,10^3\,10^3\,10^3])$, $R_1 = R_2 = R_3 = 10^{-6}$. The prediction horizon for the optimization problem is $N = 5$ with a sampling time of $\Delta = 0.01$ h. In the simulations, we put a maximum iteration number $c_{\max}$ on the DMPC evaluation and the maximum iteration number is chosen to be $c_{\max} = 2$. Also, we set Δu_i as 10% percent of $u_i^{\max}$ ($i = 1, 2, 3$). The optimization problems are solved by the open source interior point optimizer Ipopt [25]. The initial condition which is utilized to carry out simulations is $x(0)^T = [360.69\,3.19\,0.15\,0.03\,430.91\,2.76\,0.34\,0.08\,430.42\,2.79\,0.38\,0.08]$. We set the bound on the measurement noise to be 1% of the instantaneous value of the signal measured by sensors. The communication channel noise is generated using gaussian random variables with variances σ_n and σ_u bounded by θ_n and θ_u for state values and control inputs, respectively. These values are shown in Table 3.

We first carried out simulations to illustrate that the proposed multirate DMPC achieves practical closed-loop stability. Fig. 2 shows the temperature and concentration trajectories of the process under the DMPC design of Eqs. (9) and (10) and (12) and (13), respectively. As it can be seen from the figure, the proposed DMPC system can steer the system state to a neighborhood of the desired steady state. It should be emphasized that the inequalities of Eqs. (19) and (25) have been confirmed through simulations (Fig. 3).

We also carried out a set of simulations to demonstrate the optimality of the closed-loop performance of the proposed multirate DMPC compared with different control schemes. Specifically, we compared the proposed multirate DMPC with five different control schemes from a performance point of view for the case in which there is no communication and measurement noise. The five control schemes considered are as follows: (1) the proposed DMPC

Table 3
Communication noise parameters.

	σ_n	θ_n		σ_n	θ_n
C_{A1}	1	0.033	C_{A2}	1	0.027
C_{B1}	1	0.001	C_{B2}	1	0.004
C_{C1}	1	0.001	C_{C2}	1	0.001
T_1	10	3.695	T_2	10	4.352
	σ_n	θ_n		σ_u	θ_u
C_{A3}	1	0.028	u_1	10	7.39
C_{B3}	1	0.005	u_2	30	22.17
C_{C3}	1	0.001	u_3	40	29.56
T_3	10	4.352			

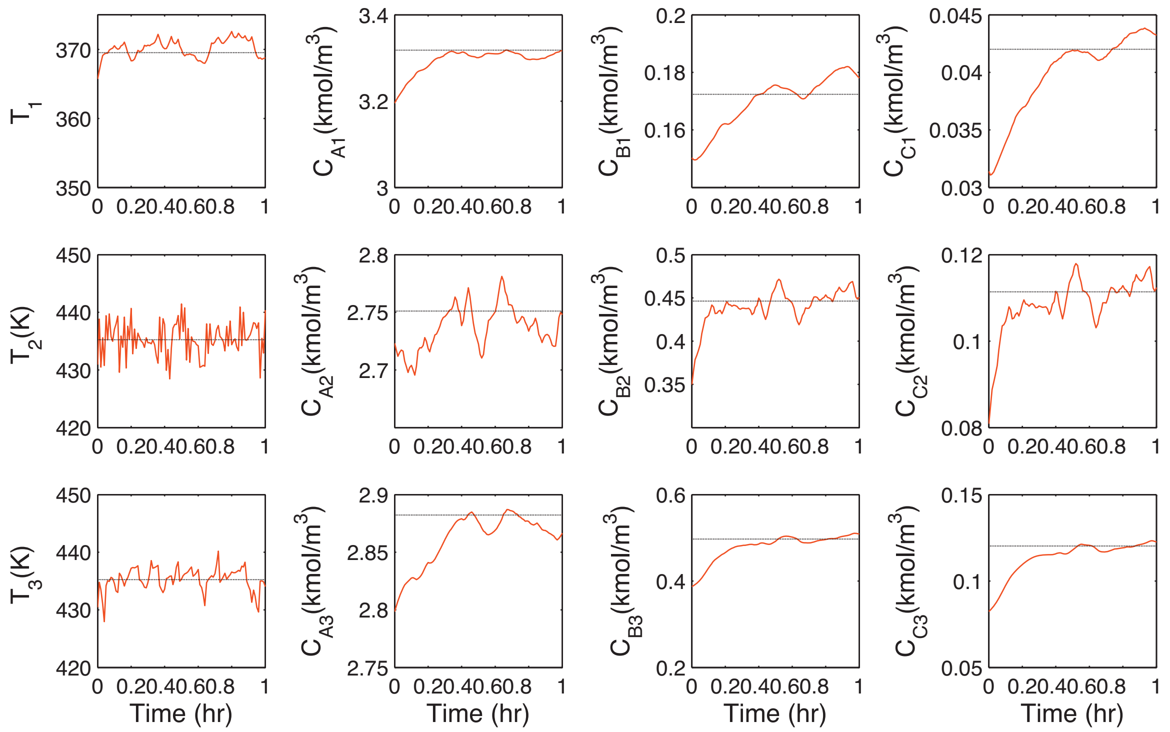

Fig. 2. State trajectories of the process under the DMPC design of Eqs. (9) and (10) and (12) and (13) with noise.

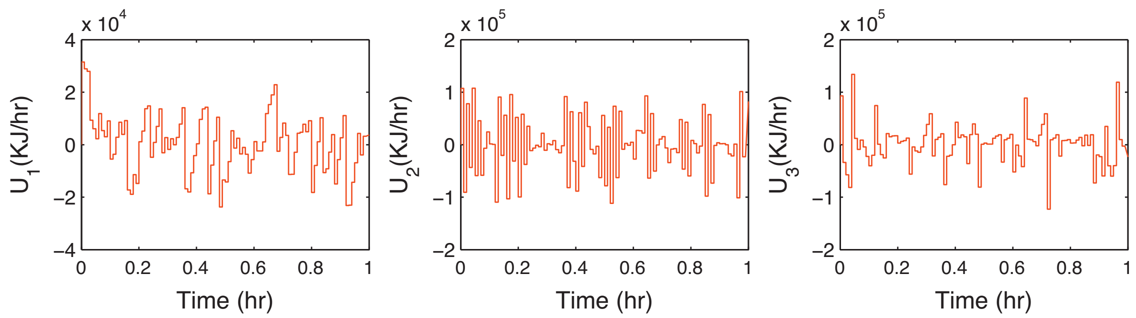

Fig. 3. Manipulated input trajectories under the DMPC design of Eqs. (9) and (10) and (12) and (13) with noise.

Table 4
Total performance cost comparison along the closed-loop system trajectories in 10 different runs under: (1) the proposed multirate DMPC design; (2) a DMPC design with LMPCs formulated as in Eq. (9) evaluated only at time instants in which full system states are available and the inputs are implemented in open-loop fashion between two full system state measurements; (3) the proposed DMPC design but without communication between the distributed controllers and each controller estimating the full system states and the actions of the other controllers based on the process model and $h(x)$; (4) the DMPC design as in (2) but without communication between the distributed controllers and each controller estimating the full system states and actions of the other controllers based on the process model and $h(x)$; (5) $h(x)$ applied in sample-and-hold; (6) the centralized LMPC [26].

(1)	(2)	(3)	(4)	(5)	(6)
43963	633589	72200	812903	1116578	27057
21512	606628	28079	743874	1095819	7370
23041	604148	27407	706319	1084445	15112
24681	613289	30211	720131	1104045	8838
31440	618649	36290	723598	1106508	18654
21775	654268	25950	859380	1079984	15287
28553	667143	34209	879852	1109976	13168
28974	659250	34565	865643	1109363	13424
28228	672756	33949	891549	1110884	12991
23929	668499	29688	887300	1106623	11903

design of Eqs. (9) and (10) and (12) and (13); (2) a DMPC design with LMPCs formulated as in Eq. (9) which are only evaluated at time instants in which full system states are available and the inputs are implemented in open-loop fashion between two full system state measurements (in this case, the additional fast sampled measurements are not used to improve the closed-loop performance); (3) the proposed DMPC design but without communication between the distributed controllers and each controller estimating the full system states and the actions of the other controllers based on the process model and $h(x)$ (in this case, a distributed LMPC in the DMPC design takes advantage of both fast and slowly sampled measurements of its own local subsystem but does not receive any input or state information from the other subsystems); (4) the DMPC design as in (2) but without communication between the distributed controllers and each controller estimating the full system states and actions of the other controllers based on the process model and $h(x)$; (5) $h(x)$ applied in sample-and-hold; (6) the centralized LMPC [26]. We perform these simulations under different initial conditions and different process noise/disturbances. To carry out this comparison, we have computed the total cost of each simulation based on the index of the following form:

$$J=\sum_{i=0}^{M}\left[x(t_i)^T Q_c x(t_i)+\sum_{j=1}^{3}u_j(t_i)^T R_{cj} u_j(t_i)\right]$$

where $t_0 = 0$ is the initial time of the simulations and $t_M = 1$ h is the end of the simulations. Table 4 shows the total cost computed for 10 different closed-loop simulations under the six different control schemes. From Table 4, we see that the centralized LMPC gives the best performance and the proposed DMPC design gives the second best performance in all the simulations. Also, Table 4 demonstrates that when there is communication between controllers or there is MPC implementation when there is only partial state information in each controller (fast sampled state), the closed-loop performance is improved. It should be mentioned that the Lyapunov-based controller is a feasible solution to the DMPC problem; however, the DMPC solution can substantially improve closed-loop performance while it inherits closed-loop stability from the Lyapunov-based controller. All of the DMPC designs yield improvement in performance compared to the Lyapunov-based controller.

In the final set of simulations, we demonstrated that the proposed multirate DMPC has a reduced computational complexity with respect to a corresponding centralized scheme. Specifically, we compared the evaluation time of the centralized LMPC [26] with the one of the proposed DMPC design in the case that there is no noise in communication or measurements. We consider the case where each controller evaluates the input trajectories every $T = 4$ sampling times (both in the centralized and the distributed architectures) and evaluate the computational time of the LMPC optimization problems for 2500 independent closed-loop simulation runs. We consider only the sampling times in which controllers have access to full system states including fast and slowly sampled states. We found that the mean evaluation time of the centralized LMPC is 0.267 s and the mean evaluation time of the DMPC is 0.235 s which is the maximum time among the three distributed controllers (LMPC 1: 0.215 s, LMPC 2: 0.235 s and LMPC 3: 0.206 s). From this set of simulations, we see that the proposed DMPC design leads to about 12% reduction in the controller evaluation time.

5. Conclusions

In this work, we designed a DMPC system using multirate sampling for large-scale nonlinear uncertain systems composed of several coupled subsystems. In the proposed control architecture, the controllers were designed via LMPC techniques taking into account bounded measurement and communication noise and process disturbances. Sufficient conditions under which the state of the closed-loop system is ultimately bounded in an invariant region containing the origin were derived. Finally, the applicability and performance of the proposed DMPC scheme were demonstrated through a nonlinear chemical process example.

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