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Economic model predictive control of switched nonlinear systems

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ABSTRACT

We focus on the development of a Lyapunov-based economic model predictive control (LEMPC) method for a class of switched nonlinear systems for which the mode transitions take place according to a prescribed switching schedule. In contrast to steady-state operation of conventional model predictive control (MPC) methods which use a quadratic objective function in their formulations, LEMPC utilizes a general (non-quadratic) cost function which may directly address economic considerations and may lead to time-varying closed-loop operation. Appropriate stabilizability assumptions for the switched nonlinear system are made and suitable constraints are imposed on the proposed LEMPC formulation to guarantee closed-loop stability of the switched nonlinear system and ensure satisfaction of the prescribed switching schedule policy while dictating time-varying operation that optimizes the economic cost function. The proposed control method is demonstrated through a chemical process example described by a switched nonlinear system.

1. Introduction

The development of optimal operation and control policies for chemical process systems aiming at optimizing process economics has always been an important research subject with major practical implications. Typically, in the context of process control methodologies, the economic optimization considerations of a plant are usually performed through a two-layer architecture [1] which usually limits process operation around a steady-state. The upper layer addresses the calculation of optimal process operation set-points while considering economic constraints and taking advantage of steady-state process models, while the lower layer (i.e., process control layer) utilizes appropriate feedback control laws to steer the process state to track the set-points computed by the upper layer. Model predictive control (MPC) is usually employed in the process control layer because of its ability to satisfy manipulated input and state constraints in the context of computing online an optimization-based solution. Generally, in a conventional MPC setting, a quadratic performance index (cost function), which penalizes the deviation of the predicted state and input along the MPC prediction horizon from the optimal set-points computed by the upper layer, is used which results in steady-state operation [2]. In order to address general economic optimization considerations, the quadratic cost function used in conventional MPC should be replaced by an economics-based cost function which may result in closed-loop system time-varying operation. Consequently, the conventional MPC scheme should be re-formulated in an appropriate way to guarantee closed-loop stability. Thus, optimizing closed-loop performance with respect to general economic considerations for nonlinear systems results in a tighter integration of the process economics and process control layers and gives rise to the subject of economic MPC.

Recently, economic MPC has received considerable attention. In particular, integration of linear MPC and steady-state optimization layers [3], enforcing a terminal constraint in economic MPC formulation to achieve closed-loop stability [4], economic MPC of cyclic processes (including closed-loop stability analysis using a suitable terminal constraint) [5], as well as energy reduction in the context of economic MPC [6], have been studied. In a previous work [7], we presented a two-mode Lyapunov-based economic MPC (LEMPC) design for nonlinear systems which is also capable of handling asynchronous and delayed measurements and extended it in the context of output feedback [3] and distributed MPC [8]. In this two-mode method, the first mode addresses economic considerations by enforcing time-varying economically-optimal operation while maintaining the closed-loop system state in a predefined invariant set, and the second mode deals with convergence to an economically optimal steady state.

On the other hand, in the context of control of switched nonlinear systems with scheduled mode transitions, mode transition

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situations need to be handled explicitly to achieve closed-loop stability. In the context of process control applications, mode transitions may arise due to feedback changes, phase changes and actuator/sensor faults to name a few. In this direction, analysis and control of switched systems have been studied using approaches based on Lyapunov functions (e.g., [9–13]) as well as optimal control theory (e.g., [14,15]). MPC can also be utilized to follow a prescribed switching schedule policy subject to input constraint [16]. However, economic model predictive control of switched nonlinear systems with scheduled mode transitions has not been studied.

Motivated by this, the current work presents an LEMPC method for a broad class of nonlinear switched systems with scheduled mode transitions. Appropriate stabilizability assumptions for the switched nonlinear system are made and suitable constraints are imposed on the proposed LEMPC formulation to guarantee closed-loop stability of the switched nonlinear system and ensure satisfaction of the prescribed switching schedule policy while dictating time-varying operation that optimizes the economic cost function. The proposed control method is demonstrated through a chemical process example described by a switched nonlinear system.

2. Preliminaries

2.1. Notation

The notation $|\cdot|$ is used to denote the Euclidean norm of a vector. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. The symbol Ω_r is used to denote the set $\Omega_r := \{x \in R^{n_x} : V(x) \leq r\}$ where V is a continuously differentiable, positive definite scalar function and $r > 0$, and the operator '/' denotes set subtraction, that is, $A/B := \{x \in R^{n_x} : x \in A, x \notin B\}$. The symbol $\mathrm{diag}(v)$ denotes a matrix whose diagonal elements are the elements of vector v and all the other elements are zeros.

2.2. Class of switched nonlinear systems

We consider switched nonlinear systems which are composed of p modes (i.e., finite-number of switching modes) described by the following state-space model:

$$\dot{x}(t) = f_{\sigma(t)}(x) + g_{\sigma(t)}(x)u_{\sigma(t)}(t) \tag{1}$$

where $x(t) \in R^{n_x}$ denotes the vector of state variables of the system and $u_{\sigma(t)}(t) \in R$ is the control (manipulated) input affecting the σ mode. $\sigma : [0, \infty) \to \mathcal{I}$ denotes the switching signal which is assumed to be a piecewise continuous from the right function of time, i.e., $\sigma(t_k) = \lim_{t\to t_k^+} \sigma(t)$ for all k, implying that only a finite number of switches is allowed over any finite interval of time. The switching signal takes its values in a finite index set $\mathcal{I} = \{1, 2, \ldots, p\}$. The input is restricted to be in nonempty convex sets $U_{\sigma(t)} \subseteq R$, which is defined as $U_{\sigma(t)} := \{u_{\sigma(t)} \in R : |u_{\sigma(t)}| \leq u^{\max}_{\sigma(t)}\}$ where $u^{\max}_{\sigma(t)}$ is the magnitude of the input constraint. We will design a controller to compute the control input $u_{\sigma(t)}$ and will refer to it as controller at mode $\sigma(t)$. Through the rest of this paper, $t_{k_r^{in}}$ and $t_{k_r^{out}}$ denote the time when, for the r^{th} time, the system of Eq. (1) has switched in and out of the k^{th} mode, respectively, i.e., $\sigma(t^+_{k_r^{in}}) = \sigma(t^-_{k_r^{out}}) = k$. So, for $t_{k_r^{in}} \leq t < t_{k_r^{out}}$, the system of Eq. (1) is represented by $\dot{x} = f_k(x) + g_k(x)u_k$.

We assume that the vector functions $f_k(\cdot)$ and $g_k(\cdot) \forall k \in \mathcal{I}$ are locally Lipschitz vector functions and that the origin is an equilibrium point of the unforced system (i.e., system of Eq. (1) with $u_k(t) = 0$, for all $t, k \in \mathcal{I}$) which implies that $f_k(0) = 0, \forall k \in \mathcal{I}$. We further assume that during the system operation at mode k for r^{th} time, i.e., $t_{k_r^{in}} \leq t < t_{k_r^{out}}$, the system state measurements are available and sampled at synchronous time instants $t_q = t_{k_r^{in}} + q\Delta_{k_r}$, $q = 0, 1, 2, \ldots, N_{k_r}$ where Δ_{k_r} is the sampling time. Without loss of generality, we assume that N_{k_r} is a positive integer.

2.3. Stabilizability assumption

Consider the system of Eq. (1), at a fixed switching mode $\sigma(t) = k$ where $k \in \mathcal{I}$. We assume that there exists a feedback controller $u_k = h_k(x)$, which renders the origin of the closed-loop system at mode k asymptotically stable while satisfying the input constraints for all the states x inside a given stability region. Using converse Lyapunov theorems [17,18], this assumption implies that there exist class $\mathcal{K}$ functions $\alpha_{l_k}(\cdot)$, $l = 1, 2, 3, 4$ and a continuously differentiable Lyapunov function $V_k(x)$ for the closed-loop system, that satisfy the following inequalities:

$$\begin{aligned}
&\alpha_{1_k}(|x|) \leq V_k(x) \leq \alpha_{2_k}(|x|)\\
&\frac{\partial V_k(x)}{\partial x}(f_k(x) + g_k(x)h_k(x)) \leq -\alpha_{3_k}(|x|)\\
&\left|\frac{\partial V_k(x)}{\partial x}\right| \leq \alpha_{4_k}(|x|)\\
&h_k(x) \in U_k
\end{aligned} \tag{2}$$

for all $x \in D_k \subseteq R^{n_x}$ where D_k is an open neighborhood of the origin. We denote the region $\Omega_{\rho_k} \subseteq D_k$ as the stability region of the closed-loop system at mode k under the controller $h_k(x)$. Using the smoothness assumed for the $f_k(\cdot)$ and $g_k(\cdot)$, and taking into account that the manipulated inputs u_k is bounded, there exists a positive constant M_k such that

$$|f_k(x) + g_k(x)u_k| \leq M_k \tag{3}$$

for all $x \in \Omega_{\rho_k}$, $u_k \in U_k$ and $k \in \mathcal{I}$. In addition, by the continuous differentiable property of the Lyapunov function $V_k(x)$ and the smoothness of f_k and g_k, there exist positive constants L_{x_k} and L_{u_k} such that

$$\begin{aligned}
&\left|\frac{\partial V_k}{\partial x}f_k(x) - \frac{\partial V_k}{\partial x}f_k(x')\right| \leq L_{x_k}|x - x'|\\
&\left|\frac{\partial V_k}{\partial x}g_k(x) - \frac{\partial V_k}{\partial x}g_k(x')\right| \leq L_{u_k}|x - x'|
\end{aligned} \tag{4}$$

for all $x, x' \in \Omega_{\rho_k}$, $u_k \in U_k$ and $k \in \mathcal{I}$.

Proposition 1 characterizes the closed-loop stability properties of the feedback controller $h_k(x)$ while Proposition 2 provides sufficient conditions that ensure that the closed-loop system state under implementation of the controller $h_k(\cdot)$ in a sample-and-hold fashion enters the corresponding stability region of the subsequent mode once the system switches to that mode according to the prescribed switching schedule policy. For the sake of simplicity and without loss of generality, we define the following sampled state trajectory when the controller $h_k(x)$ is applied in a sample-and-hold fashion at mode k for $t_0 = t_{k_r^{in}} \leq \tau < t_{k_r^{out}}$ as follows

$$\begin{aligned}
&\dot{\hat{x}}(\tau) = f_k(\hat{x}(\tau)) + g_k(\hat{x}(\tau))h_k(\hat{x}(t_l)),\\
&\quad l = 0, 1, \ldots, N_{k_r} - 1,\ \hat{x}(t_0) = x(t_{k_r^{in}}).
\end{aligned} \tag{5}$$

Proposition 1 below ensures that if the closed-loop system at mode k controlled by $h_k(x)$, implemented in a sample-and-hold fashion, starts in Ω_{ρ_k} and stays in mode k for all times, then it is ultimately bounded in $\Omega_{\rho_{\min_k}}$. It characterizes the closed-loop stability region corresponding to each mode.

Proposition 1 (*C.f.* [2]). *Consider the closed-loop system of Eq.* (5) *and assume it operates at mode k for all times. Let* $\Delta_k, \epsilon_{s_k} > 0$ *and* $\tilde{\rho}_k > \rho_{s_k} > 0$ *satisfy:*

$$-\alpha_{3_k}\left(\alpha_{2_k}^{-1}(\rho_{s_k})\right) + (L_{x_k} + L_{uk}u_k^{\max})M_k\Delta_k \leq -\epsilon_{s_k}/\Delta_k. \tag{6}$$

Then, if $\hat{x}(t_0) \in \Omega_{\rho_k}$ *and* $\rho_{\min_k} < \rho_k$ *where* $\rho_{\min_k} = \max\{V_k(\hat{x}(t+\Delta_{k_r})) : V_k(\hat{x}(t)) \leq \rho_{s_k}\}, \forall \Delta_{k_r} \in (0, \Delta_k]$ *the following inequality holds:* $V_k(\hat{x}(t)) \leq V_k(\hat{x}(t_q)), \ \forall t \in [t_q, t_{q+1})\ (q = 0, 1, \ldots)$ *and* $V_k(\hat{x}(t_q)) \leq \max\{V_k(\hat{x}(t_0)) - q\epsilon_{s_k}, \rho_{\min_k}\}$. *Since* $V_k(\cdot)$ *is a continuous function,* $V_k(\hat{x}) \leq \rho_{\min_k}$ *implies* $|\hat{x}| \leq d_k$ *where* d_k *is a positive constant and therefore,* $\limsup_{t\to\infty} |\hat{x}(t)| \leq d_k$.

For each mode $k \in \mathcal{I}$, we assume there exist a set of initial conditions Ω_{ρ_k}, which is estimated as the level set of the Lyapunov function at mode k ($V_k(\cdot)$) and a positive real number ρ_k^* such that under implementation of the Lyapunov-based controller $h_k(\cdot)$ in a sample-and-hold fashion, the state of Eq. (5) satisfies

$$\dot{V}_k(\hat{x}(\tau)) \leq -\rho_k^* V_k(\hat{x}(\tau)), \quad \hat{x}(\tau) \in \Omega_{\rho_k/\rho_{s_k}}, \ t_{k_r^{in}} \leq \tau < t_{k_r^{out}}. \tag{7}$$

Proposition 2. *Consider the closed-loop sampled trajectory* $\hat{x}(t)$ *defined in Eq.* (5). *Given that* $t_{k_r^{in}} \leq t < t_{k_r^{out}} = t_{f_w^{in}}$, *and* $\hat{x}(t_{k_r^{in}}) \in \Omega_{\rho_k}$, *if there exist* $\rho_k > 0$, $\rho_k^* > 0$, $N_{k_r} > 0$ *and* $\Delta_{k_r} > 0 \forall k \in \mathcal{I}$ *such that*

$$\alpha_{2_f}(\alpha_{1_k}^{-1}(\rho_k\, \mathrm{e}^{-\rho_k^* N_{k_r}\Delta_{k_r}})) \leq \rho_f, \tag{8}$$

then $\hat{x}(t_{f_w^{in}}) \in \Omega_{\rho_f}$.

Proof. It can be obtained from Eq. (7) that

$$V_k(\hat{x}(t_{k_r^{out}})) \leq V_k(\hat{x}(t_{k_r^{in}}))\, \mathrm{e}^{-\rho_k^* N_{k_r}\Delta_{k_r}}. \tag{9}$$

Since $\hat{x}(t_{k_r^{in}}) \in \Omega_{\rho_k}$, we have

$$V_k(\hat{x}(t_{k_r^{out}})) \leq \rho_k\, \mathrm{e}^{-\rho_k^* N_{k_r}\Delta_{k_r}}. \tag{10}$$

From Eq. (2) we can obtain $|\hat{x}(t_{k_r^{out}})| \leq \alpha_{1_k}^{-1}(\rho_k\, \mathrm{e}^{-\rho_k^* N_{k_r}\Delta_{k_r}})$. If Eq. (8) is satisfied, using Eq. (2) for the Lyapunov-based controller at mode f, it can be concluded that $V_f(\hat{x}(t_{f_w^{in}})) \leq \rho_f$ which implies that $\hat{x}(t_{f_w^{in}}) \in \Omega_{\rho_f}$. □

Remark 1. Note that the stability region Ω_{ρ_k} characterizes the set of initial conditions for the nonlinear switched system at mode $k \in \mathcal{I}$ starting from where, the closed-loop system state enters the corresponding stability region of the subsequent mode according to the prescribed switching schedule policy at the time of the switch. From a feasibility point of view, the Lyapunov-based controller satisfying Eq. (8) yields a feasible solution to the prescribed switching schedule. It should be emphasized that the purpose of the LEMPC formulation in this paper is to take advantage of this feasible solution to address optimization of an economic (non-quadratic) cost function. Furthermore, a necessary condition for Proposition 2 is that the stability regions of two subsequent switching modes according to the switching schedule policy (i.e., Ω_{ρ_k} and Ω_{ρ_f}) should have a non-empty intersection; otherwise, it is not possible to steer the state of the system to the stability region corresponding to the subsequent switching mode.

3. LEMPC of switched nonlinear systems

In this section, we consider the design of Lyapunov-based economic MPC (LEMPC) for nonlinear switched systems. When there is an economic cost function which directly addresses economic considerations of the system (e.g., $L_e(x(t), u(t))$), steady-state operation may not yield optimal economic closed-loop performance. For the sake of clarity in presentation and without loss of generality, we assume that over the operation period $[t_0, t')$, we deal with economic considerations in LEMPC design through different switching modes while after time t', when the system operates in a specific switching mode $z \in \mathcal{I} \forall t \geq t'$, we deal with enforcing closed-loop stability in terms of convergence to a steady-state point at switching mode z. In operation period $[t_0, t')$ and at each switching mode $k \in \mathcal{I}$, the predicted system state along the finite prediction horizon is maintained in the corresponding stability region Ω_{ρ_k} while it allows the system state to optimize the economic cost function by dictating a possible time-varying operation; however, due to the fact that the system switches through different modes according to the prescribed switching schedule policy, an appropriate constraint needs to be included in the LEMPC formulation to make sure that the closed-loop system state at the end of each switching interval enters the stability region corresponding to the subsequent switching mode. Thus, it can take advantage of the closed-loop stability properties of the Lyapunov-based controller of every specific mode. It should be mentioned that it has been assumed that at the beginning of the prescribed switching schedule that the initial system state is within the stability region of the first switching mode according to the prescribed switching schedule policy.

3.1. Implementation strategy

At sampling time $t_q \in [t_0, t')$ where $t_{k_r^{in}} \leq t_q < t_{k_r^{out}} = t_{f_w^{in}}$, sampled state feedback $x(t_q)$ is received through measurement sensors. The predicted system state is maintained at the stability region Ω_{ρ_k} while the LEMPC formulation enforces appropriate constraint to ensure that at the end of the current switching interval the closed-loop system state enters the stability region of the subsequent mode Ω_{ρ_f}. If $t_q \geq t'$, the LEMPC enforces appropriate constraint to make sure it achieves practical closed-loop stability by steering the closed-loop system state in a small neighborhood of the origin corresponding to the switching mode z (i.e., z is the active mode at $t = t'$). The implementation strategy of the proposed LEMPC can be summarized as follows:

1. At sampling time t_q, the state measurement $x(t_q)$ is received. If $t_q \in [t_0, t')$ go to step 2; otherwise, go to step 3.
2. The LEMPC maintains the predicted system state in the stability region Ω_{ρ_k} while it makes sure that at the end of the current switching mode interval, the system state enters the stability region Ω_{ρ_f} where $t_{k_r^{in}} \leq t_q < t_{k_r^{out}} = t_{f_w^{in}}$.
3. The LEMPC steers the closed-loop system state to a small neighborhood of the origin at switching mode z.

3.2. LEMPC formulation

Depending on sampling time t_q, there are two LEMPC formulations. Specifically, if $t_q \in [t_0, t')$ and $t_{k_r^{in}} \leq t_q < t_{k_r^{out}} = t_{f_w^{in}}$, the LEMPC is formulated as follows:

$$\max_{u_k \in S(\Delta_{k_r})} \int_{t_q}^{t_{k_r^{out}}} L_e(\tilde{x}(\tau), u_k(\tau))\, \mathrm{d}\tau \tag{11a}$$

$$\text{s.t. } \dot{\tilde{x}}(\tau) = f_k(\tilde{x}(\tau)) + g_k(\tilde{x}(\tau))u_k(\tau) \tag{11b}$$

$$u_k(\tau) \in U_k, \quad \tau \in [t_q, t_{k_r^{out}}) \tag{11c}$$

$$\tilde{x}(t_q) = x(t_q) \tag{11d}$$

$$V_k(\tilde{x}(\tau)) \leq \rho_k, \quad \forall \tau \in [t_q, t_{k_r^{out}}) \tag{11e}$$

$$V_f(\tilde{x}(t_{k_r^{out}})) \leq \rho_f \tag{11f}$$

where $\tilde{x}$ is the predicted state trajectory of the system with control input calculated by the LEMPC of Eq. (11) and $S(\Delta_{k_r})$ is the set of piecewise constant functions with period Δ_{k_r}. Eq. (11b) uses a nominal system model at switching mode $k \in \mathcal{I}$ to predict the future evolution of the system state initialized at $x(t_q)$ Eqs. (11c) and (11d) denotes the constraint on the manipulated input at switching mode k. The constraint of Eq. (11e) maintains the predicted system state along the prediction horizon in the invariant set Ω_{ρ_k}, and within this set, the LEMPC addresses economic considerations by optimizing the economic cost function of Eq. (11a). The constraint of Eq. (11f) ensures that the closed-loop system state at the end of the switching interval $[t_{k_r^{in}}, t_{k_r^{out}})$ enters the stability region Ω_{ρ_f} corresponding to switching mode f. It should be emphasized that at every sampling time t_q the finite prediction horizon N_{k_r} is chosen in a way that $N_{k_r} = \frac{t_{k_r^{out}} - t_q}{\Delta_{k_r}}$. The optimal solution to this optimization problem is denoted by $u_k^*(t|t_q)$, which is defined for $t \in [t_q, t_{k_r^{out}})$. The manipulated input of the LEMPC of Eq. (11) is defined as follows:

$$u_k(t) = u_k^*(t|t_q), \quad \forall t \in [t_q, t_{q+1}). \tag{12}$$

If $t_q \geq t'$, the system operates at switching mode $z \in \mathcal{I}$ and the LEMPC is formulated as follows:

$$\max_{u_z \in S(\Delta)} \int_{t_q}^{t_{q+N}} L_e(\tilde{x}(\tau), u_z(\tau))\, d\tau \tag{13a}$$

$$\text{s.t. } \dot{\tilde{x}}(\tau) = f_z(\tilde{x}(\tau)) + g_z(\tilde{x}(\tau))u_z(\tau) \tag{13b}$$

$$u_z(\tau) \in U_z, \quad \tau \in [t_q, t_{q+N}) \tag{13c}$$

$$\tilde{x}(t_q) = x(t_q) \tag{13d}$$

$$\frac{\partial V_z(x(t_q))}{\partial x} g_z(x(t_q))u_z(t_q) \leq \frac{\partial V_z(x(t_q))}{\partial x} g_z(x(t_q))h_z(x(t_q)). \tag{13e}$$

The constraint of Eq. (13e) ensures that the amount of reduction in the value of the Lyapunov function at switching mode z is at least at the level when the Lyapunov-based controller at mode z is applied in a sample-and-hold fashion. It should be emphasized that $\forall t_q \geq t'$, LEMPC is formulated with a fixed horizon N and sampling time Δ at switching mode z. The optimal solution to this optimization problem is denoted by $u_z^*(t|t_q)$, which is defined for $t \in [t_q, t_{q+N})$. The manipulated input of the LEMPC of Eq. (13) is defined as follows:

$$u_z(t) = u_z^*(t|t_q), \quad \forall t \in [t_q, t_{q+1}). \tag{14}$$

Remark 2. Note that there are two types of state constraints in the LEMPC formulation of Eq. (11). The constraint of Eq. (11e) ensures that through the system operation at switching mode $k \in \mathcal{I}$, the closed-loop system state is maintained in the corresponding invariant set Ω_{ρ_k} to address economic optimization by allowing the switched system to operate in a possible time-varying fashion while it inherits the boundedness properties of the sampled state trajectory of the Lyapunov-based controller $h_k(\cdot)$ when it is applied in a sample-and-hold-fashion. On the other hand, to ensure that the subsequent switching mode according to the prescribed switching schedule policy can also take advantage of the Lyapunov-based controller at the subsequent switching mode, the constraint of Eq. (11f) has been included in the LEMPC formulation of Eq. (11). Thus, it ensures that economic considerations for the subsequent switching mode will be addressed. It should be emphasized that these type of constraints are different from multiple Lyapunov function (MLF) constraints which were incorporated to ensure closed-loop stability of the switched systems in an asymptotic sense [14,16].

Remark 3. The constraint of Eq. (13e) in the LEMPC formulation of Eq. (13) ensures certain level of reduction in the value of the Lyapunov function corresponding to the last switching mode according to the prescribed switching schedule policy. It should be mentioned that once the switched system enters the last mode according to the switching schedule, it will operate thereafter with a finite fixed prediction horizon to address practical closed-loop stability through steering the closed-loop system state to an invariant small neighborhood of the origin corresponding to the last switching mode $z \in \mathcal{I}$. Note that depending on the prescribed switching schedule policy, the system may switch in and switch out to mode z before the last time that it enters the mode z to address economic optimization considerations. Please see Section 4 for an example of this scenario.

Remark 4. It should be emphasized that the LEMPC of Eq. (11) is not implemented in the context of conventional receding horizon scheme. Based on the prescribed switching schedule policy, at each time interval that the system is supposed to operate in a specific mode, it uses a prediction horizon from the current time until the time that the system is supposed to switched out from that mode. However, for the LEMPC of Eq. (13), since the system operates in the switching mode $z \in \mathcal{I}$ $\forall t \geq t'$, a fixed prediction horizon is utilized.

3.3. Closed-loop stability

The following theorem characterizes the closed-loop stability properties of the proposed LEMPC of Eqs. (11) and (13).

Theorem 1. *Consider the system of Eq. (1) in closed-loop under the LEMPC of Eqs. (11) and (13) and assume that there exists Lyapunov-based controllers $h_k(\cdot)$, $\forall k \in \mathcal{I}$ satisfying Eqs. (2) and (7). Then, given a positive real number $d^{\max}$, if there exist $\Delta_k, \epsilon_{s_k} > 0$ and $\rho_k > \rho_{s_k} > 0$ $\forall k \in \mathcal{I}$ such that Eqs. (6) and (8) are satisfied and $\Delta_{k_r} \in (0, \Delta^*]$ where $\Delta^* = \min_{k\in\mathcal{I}} \Delta_k$, then $x(t)$ is bounded and* $\limsup_{t\to\infty} |x(t)| \leq d_{\max}$.

Proof. First we prove that the optimization problems of Eqs. (11) and (13) are feasible and then we proceed with the closed-loop stability analysis. Regarding the optimization problem of Eq. (11), the Lyapunov-based controller $h_k(\tilde{x}(t_q + l\Delta_{kr}))$ where $l = 0, 1, \ldots, N_{kr} - 1$ is a feasible solution. It satisfies the control input constraint of Eq. (11c) due to Eq. (2) while the constraints of Eqs. (11e) and (11f) are satisfied through the Eq. (6) in Proposition 1 and Eq. (8) in Proposition 2. On the other hand, the Lyapunov-based controller $h_z(\tilde{x}(t_q + l\Delta_{kr}))$ where $l = 0, 1, \ldots, N - 1$ is feasible solution for the optimization problem of Eq. (13). It satisfies the control input constraint of Eq. (13c) due to Eq. (2) while it satisfies the constraint of Eq. (13e).

Given the radius of the ball around the origin, $d^{\max}$, the values of $\rho_{\min_k}$ and $\Delta_k \forall k \in \mathcal{K}$ are computed based on Propositions 2 and 1. Then, for the purpose of LEMPC implementation, a value of $\Delta_{k_r} \in (0, \Delta^*]$ is chosen where $\Delta^* = \min_{k\in\mathcal{I}} \Delta_k$ and $t_{k_r^{out}} - t_{k_r^{in}} = l_{k_r}\Delta_{k_r}$ for some integer $l_{k_r} > 0$.

Regarding the boundedness of the closed-loop system state, up to time t', at each switching mode $k \in \mathcal{I}$, the closed-loop system state is bounded in the invariant Ω_{ρ_k}. After time t', since the system operates at switching mode $z \in \mathcal{I}$, we can write the time derivative of the Lyapunov function along the state trajectory $x(t)$ of system of Eq. (1) in $t \in [t_q, t_{q+1})$ as follows:

$$\dot{V}_z(x(t)) = \frac{\partial V_z(x)}{\partial x}(f_z(x(t)) + g_z(x(t))u_z^*(t_q|t_q)). \tag{15}$$

Adding and subtracting $\frac{\partial V_z(x(t_q))}{\partial x}(f_z(x(t_q)) + g_z(x(t_q))u_z^*(t_q|t_q))$ to the right-hand-side of Eq. (15) and taking Eq. (2) into account, we obtain the following inequality:

$$\begin{aligned}\dot{V}_z(x(t)) \leq\ & -\alpha_{3_z}(|x(t_q)|) \\ & + \frac{\partial V_z(x)}{\partial x}(f_z(x(t)) + g_z(x(t))u_z^*(t_q|t_q)) \\ & - \frac{\partial V_z(x(t_q))}{\partial x}(f_z(x(t_q)) + g_z(x(t_q))u_z^*(t_q|t_q)).\end{aligned} \quad (16)$$

From Eq. (4) and the inequality of Eq. (16), the following inequality is obtained for all $x(t_q) \in \Omega_{\rho_z} \setminus \Omega_{\rho_{sz}}$:

$$\begin{aligned}\dot{V}_z(x(t)) \leq\ & -\alpha_{3_z}(\alpha_2^{-1}(\rho_{s_z})) \\ & + (L_{x_z} + L_{uz}u_z^*(t_q|t_q))|x(t) - x(t_q)|.\end{aligned} \quad (17)$$

Taking into account Eq. (3) and the continuity of $x(t)$, the following bound can be written for all $t \in [t_q, t_{q+1})$, $|x(t) - x(t_q)| \leq M_z\Delta$ where $\Delta \in (0, \Delta^*]$. Using this expression, we obtain the following bound on the time derivative of the Lyapunov function for $t \in [t_q, t_{q+1})$, for all initial states $x(t_q) \in \Omega_{\rho_z} \setminus \Omega_{\rho_{sz}}$:

$$\dot{V}_z(x(t)) \leq -\alpha_{3_z}(\alpha_{2_z}^{-1}(\rho_{s_z})) + (L_{x_z} + L_{uz}u_z^{\max})M_z\Delta.$$

Since the condition of Eq. (6) is satisfied, then $\forall x(t_q) \in \Omega_{\rho_z} \setminus \Omega_{\rho_{sz}}$ we can obtain:

$$\dot{V}_z(x(t)) \leq -\epsilon_{s_z}/\Delta, \quad \forall t = [t_q, t_{q+1}).$$

Integrating this bound on $t \in [t_q, t_{q+1})$, we obtain that:

$$\begin{aligned}& V_z(x(t_{q+1})) \leq V(x(t_q)) - \epsilon_{s_z} \\ & V_z(x(t)) \leq V_z(x(t_q)), \quad \forall t \in [t_q, t_{q+1})\end{aligned} \quad (18)$$

for all $x(t_q) \in \Omega_{\rho_z} \setminus \Omega_{\rho_{sz}}$. Using Eq. (18) recursively, it can be proved that, if $x(t_q) \in \Omega_{\rho_z} \setminus \Omega_{\rho_{sz}}$, the state converges to $\Omega_{\rho_{sz}}$ in a finite number of sampling times without leaving the stability region. Once the state converges to $\Omega_{\rho_{sz}} \subseteq \Omega_{\rho_{\min_z}}$, it remains inside $\Omega_{\rho_{\min_z}}$ for all times. This statement holds because of the definition of $\rho_{\min_z}$. This proves that the closed-loop system under the LEMPC design is ultimately bounded in $\Omega_{\rho_{\min_z}}$ from which it follows that $\limsup_{t\to\infty} |x(t)| \leq d^{\max}$. □

Remark 5. Note that the Lyapunov-based controllers $h_k(\cdot)\forall k \in \mathcal{I}$ as a feasible solution to the LEMPC of Eqs. (11) and (13) satisfy the prescribed switching schedule policy. The purpose of the LEMPC is to take advantage of these feasible solutions to address economic considerations as well as closed-loop stability. Thus, LEMPC closed-loop performance is lower bounded by the closed-loop performance of the set of Lyapunov-based controllers corresponding to different switching modes.

Remark 6. Referring to the issue of output feedback implementation of the proposed economic MPC scheme, we note that high-gain observers may be used to compute estimates of unmeasured process states from output measurements assuming certain structure on the modes of the switched nonlinear system (e.g., feedback linearizable). In this direction, one possible approach is to separately design high-gain observers for each switching mode; however, detailed development is outside of the scope of this work.

4. Application to a chemical process example

Consider a well-mixed, non-isothermal continuous stirred tank reactor (CSTR) where an irreversible, second-order, endothermic reaction $A \to B$ takes place, where A is the reactant and B is the desired product. The operation schedule requires switching between two available inlet streams consisting of pure reactant at different flow rates, concentrations and temperatures. At mode $\sigma = 1, 2$, the feed to the reactor consists of pure A at flow rate F, temperature T_{0_σ} and molar concentration C_{A0_σ}. Due to the non-isothermal nature of the reactor, a jacket is used to provide heat to the reactor. The dynamic equations describing the behavior of the reactor, obtained through material and energy balances under standard modeling assumptions, are given below:

Table 1
Parameter values.

$T_{0_1} = 300$, $T_{0_2} = 295$	K	$F = 5$	$\frac{\text{m}^3}{\text{h}}$
$V = 1.0$	m^3	$E = 5 \times 10^3$	$\frac{\text{kJ}}{\text{kmol}}$
$k_0 = 13.93$	$\frac{1}{\text{h}}$	$\Delta H = 1.15 \times 10^4$	$\frac{\text{kJ}}{\text{kmol}}$
$C_p = 0.231$	$\frac{\text{kJ}}{\text{kg K}}$	$R = 8.314$	$\frac{\text{kJ}}{\text{kmol K}}$
$\varrho = 1000$	$\frac{\text{kg}}{\text{m}^3}$	$C_{As_1} = 2$, $C_{As_2} = 1.53$	$\frac{\text{kmol}}{\text{m}^3}$
$T_{s_1} = 350$, $T_{s_2} = 378.54$	K	$C_{A0s_1} = 4$, $C_{A0s_2} = 2.86$	$\frac{\text{kmol}}{\text{m}^3}$
$Q = 1.73 \times 10^5$	$\frac{\text{KJ}}{\text{h}}$		

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{A0_\sigma} - C_A) - k_0 e^{\frac{-E}{RT}} C_A^2 \quad (19a)$$

$$\frac{dT}{dt} = \frac{F}{V}(T_{0_\sigma} - T) + \frac{-\Delta H}{\varrho C_p} k_0 e^{\frac{-E}{RT}} C_A^2 + \frac{Q}{\varrho C_p V}. \quad (19b)$$

A detailed description of this chemical process example can be found in [3]. The values of the process parameters used in the simulations are shown in Table 1. The process model of Eq. (19) is numerically simulated using an explicit Euler integration method with integration step $h_c = 10^{-3}$ h.

The process model has one stable steady-state in the operating range of interest at each switching mode. The control objective is to optimize the process operation in a region around the stable steady-state $(C_{As_\sigma}, T_{s_\sigma})$ to maximize the average production rate of B through manipulation of the concentration of A in the inlet to the reactor, C_{A0_σ}. The steady-state input value associated with the steady-state point is denoted by C_{A0s_σ}. The process model of Eq. (19) belongs to the following class of nonlinear systems:

$$\dot{x}(t) = f_\sigma(x(t)) + g_\sigma(x(t))u_\sigma(t)$$

where $x^T = [x_1\ x_2] = [C_A - C_{As_\sigma}\ \ T - T_{s_\sigma}]$ is the state, $u = C_{A0} - C_{A0s_\sigma}$ is the input, and $f_\sigma = [f_{1_\sigma}\ f_{2_\sigma}]^T$ and $g_\sigma = [g_{1_\sigma}\ g_{2_\sigma}]^T$ are vector functions. The inputs at different switching modes are subject to constraints as follows: $|u_1| \leq 3.5$ kmol/m^3 and $|u_2| \leq 4$ kmol/m^3. The economic measure considered in this example is as follows [19]:

$$L_e(x, u_\sigma) = \frac{1}{t_N}\int_0^{t_N} k_0 e^{-\frac{E}{RT(\tau)}} C_A^2(\tau)\, d\tau \quad (20)$$

where $t_N = 1$ h is the time duration of the reactor operation. This economic objective function highlights the maximization of the average production rate over process operation for $t_N = 1$ h (of course, different, yet finite, values of t_N can be chosen). The fact that inlet streams contain pure species A means that they include only species A and not any other species, in particular, species B which is the product of the reaction taking place in the reactor. The fact that the two inlet streams contain only species A does not mean that the concentration of A (i.e., how many moles of A are included in a liter of inlet stream) in these streams cannot be varied with time; this is actually often done in practice and this is why C_{A0_σ} is chosen here as the manipulated input.

In terms of the Lyapunov-based feedback controller, we use a proportional controller in the form of $u_\sigma = -\gamma_{1_\sigma}x_1 - \gamma_{2_\sigma}x_2$ subject to input constraints and a quadratic Lyapunov function $V_\sigma(x) = x^TP_\sigma x$ where $\gamma_{1_1} = 1.6$, $\gamma_{1_2} = 5.9$, $\gamma_{2_1} = 0.01$, $\gamma_{2_2} = 0.05$, $P_1 = \text{diag}([110.11\ 0.12])$, $P_2 = \text{diag}([63.69\ 0.02])$, $\rho_1 = 430$ and $\rho_2 =$

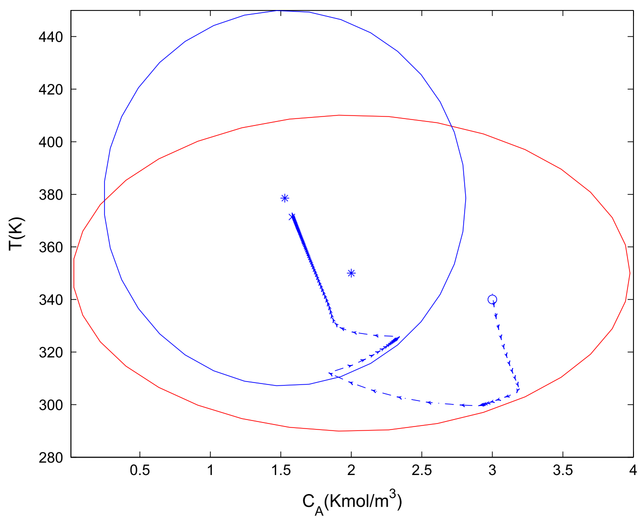

Fig. 1. Ω_{ρ_1}, Ω_{ρ_2} and state trajectories of the process under LEMPC and initial state $(C_A(0), T(0)) = \left(3\,\frac{\text{kmol}}{\text{m}^3}, 340\text{ K}\right)$. The symbols $\circ$ and $\times$ denote the initial ($t = 0$ h) and final ($t = 1$ h) state of this closed-loop system trajectories, respectively. Steady-state points corresponding to both switching modes are denoted by $\star$ symbols.

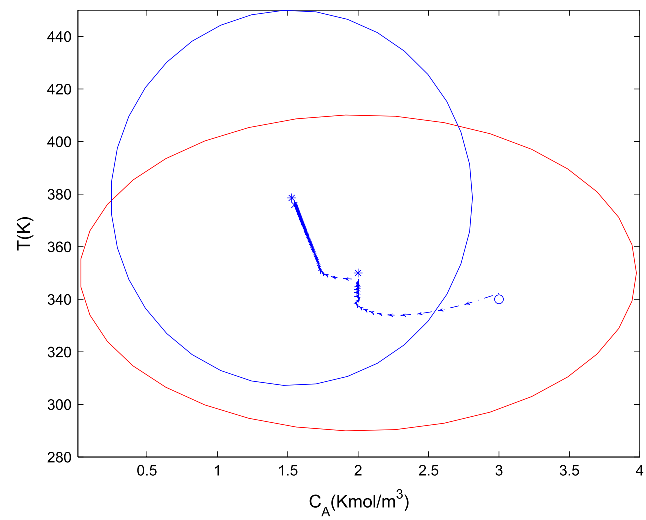

Fig. 4. Ω_{ρ_1}, Ω_{ρ_2} and state trajectories of the process under LMPC and initial state $(C_A(0), T(0)) = \left(3\,\frac{\text{kmol}}{\text{m}^3}, 340\text{ K}\right)$. The symbols $\circ$ and $\times$ denote the initial ($t = 0$ h) and final ($t = 1$ h) state of this closed-loop system trajectories, respectively. Steady-state points corresponding to both switching modes are denoted by $\star$ symbols.

105. It should be emphasized that Ω_{ρ_σ} has been estimated through evaluation of $\dot{V}_\sigma$ when we apply the proportional controller. We assume that the full system state $x = [x_1\ x_2]^T$ at both switching modes is measured and sent to the LEMPC at synchronous time instants $t_k = k\Delta$, $k = 0, 1, \ldots,$ with $\Delta = 0.01$ h $= 36$ s. As a scheduling policy, we assume that from time $t = 0$ to $t = 0.24$ h, the system operates at mode 1 while from time $t = 0.24$ h to $t = 0.39$ h it operates at mode 2 and from time $t = 0.39$ h to $t = 1$ h it stabilizes the system at the steady-state of mode 2. The simulations were carried out using Java programming language in a Pentium 3.20 GHz computer. The optimization problems in MPC were solved using the open-source interior point optimizer Ipopt.

Figs. 1–3 display the closed-loop state and manipulated input profiles for the proposed LEMPC design. From time $t = 0$ to $t = 0.24$ h and from time $t = 0.24$ h to $t = 0.39$ h, the LEMPC obtains its optimal manipulated input trajectory according to the optimization problem of Eq. (11). It dictates a time-varying operation to maximize the economic cost function while it forces the state trajectory to enter the stability region Ω_{ρ_2} at the end of operation in mode 1 ($t = 0.24$ h). From time $t = 0.24$ h to $t = 0.39$ h, the closed-loop system state is maintained at Ω_{ρ_2} to maximize the economic cost function through a time-varying operation enforced by the LEMPC of Eq. (11) while after $t = 0.39$ h, steady-state operation is enforced to steer the closed-loop system state to a small neighborhood of the steady-state point corresponding to mode 2 by the LEMPC of Eq. (13).

Also, we compare the time-varying operation of the proposed LEMPC and the closed-loop operation under the Lyapunov-based MPC (LMPC) method of [16] from an economic cost function point of view. The LMPC scheme at mode $k \in \mathcal{I}$ employs a quadratic cost function $L_s(\tilde{x}(\tau), u_k(\tau))$ (instead of economic cost function $L_e(\tilde{x}(\tau), u(\tau))$) in MPC formulation as follows

$$L_s(\tilde{x}(\tau), u_k(\tau)) = \tilde{x}^T(\tau) Q \tilde{x}(\tau) + u_k(\tau) R_u u_k(\tau) \qquad (21)$$

where Q and R_u denote the weighting matrix and the weighting factor employed to penalize the deviation of the predicted state

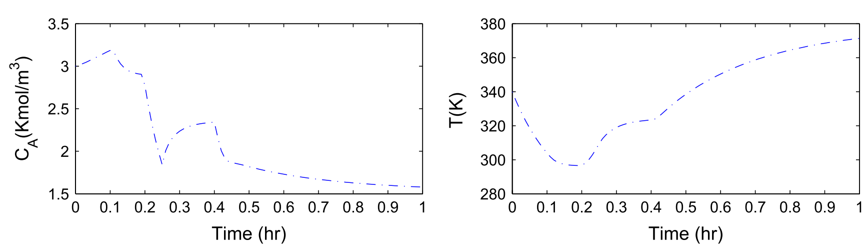

Fig. 2. State trajectories of the process under LEMPC and initial state $(C_A(0), T(0)) = \left(3\,\frac{\text{kmol}}{\text{m}^3}, 340\text{ K}\right)$.

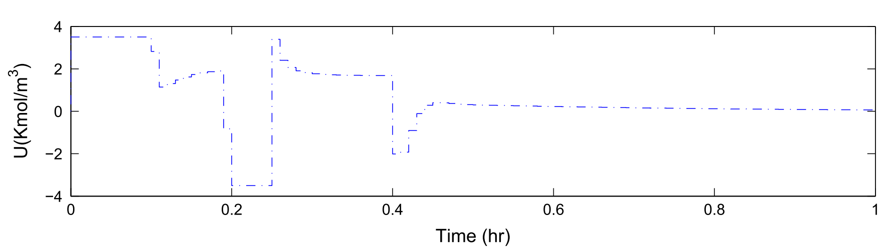

Fig. 3. Manipulated input trajectory under LEMPC and initial state $(C_A(0), T(0)) = \left(3\,\frac{\text{kmol}}{\text{m}^3}, 340\text{ K}\right)$.

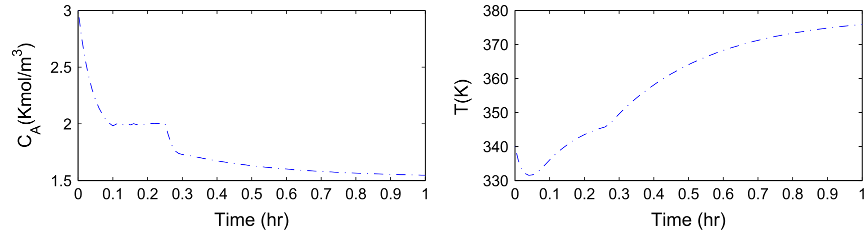

Fig. 5. State trajectories of the process under LMPC and initial state $(C_A(0), T(0)) = \left(3\,\frac{\text{kmol}}{\text{m}^3}, 340\text{ K}\right)$.

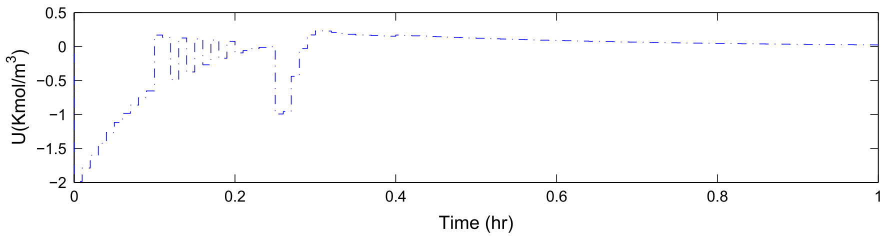

Fig. 6. Manipulated input trajectory under LMPC and initial state $(C_A(0), T(0)) = \left(3\,\frac{\text{kmol}}{\text{m}^3}, 340\text{ K}\right)$.

and input from their corresponding steady-state values, respectively, and T denotes matrix transpose operator. The LMPC at each switching mode enforces appropriate Lyapunov-based constraint in the LMPC formulation to achieve a steady-state operation while according to the prescribed switching schedule policy, it forces the state of the system to enter the stability region of the subsequent mode. For a detailed description, please refer to [16]. To carry out this comparison, we have computed the total cost of each operating scenario based on an index of the following form:

$$J = \frac{1}{t_{100}} \sum_{i=0}^{100} \left[k_0 \, e^{-\frac{E}{RT(t_i)}} C_A^2(t_i) \right] \tag{22}$$

where $t_0 = 0$ h and $t_{100} = 1$ h. Furthermore, we assume $Q = \text{diag}([1\ 0.01])$ and $R_u = 1$. The LMPC method is implemented as follows: From time $t = 0$ to $t = 0.24$ h, it operates at mode 1 in a steady-state manner and it forces the closed-loop system state to enter Ω_{ρ_2} at time $t = 0.24$ h while after $t = 0.24$ h, steady-state operation is enforced by steering the closed-loop system state to a small neighborhood of the steady-state point corresponding to mode 2. Figs. 4–6 display the closed-loop state and manipulated input profiles for the LMPC [16]. The proposed LEMPC achieves an economic cost function value 10 217.930 while the LMPC attains 7966.235, according to the cost defined in Eq. (22). This comparison indicates that through time-varying operation achieved by the proposed LEMPC design, the optimal cost has been approximately 28% improved for this chemical process example compared to the operation under the LMPC scheme. Regarding economic performance improvement with respect to steady-state operation through taking advantage of economic MPC, the proposed design maintains the system state in an invariant set while it does not force the process state to settle to a steady-state and allows for time-varying operation. Thus, it provides more degrees of freedom to the evolution of process state to meet economic optimization criteria.

5. Conclusions

This work focused on the design of an LEMPC scheme for a class of switched nonlinear systems which are capable of optimizing closed-loop performance with respect to a general objective function that may directly address economic considerations subject to a prescribed switching schedule policy. Under appropriate stabilizability assumptions, the proposed LEMPC designs may dictate time-varying operation to optimize an economic (typically non-quadratic) cost function in contrast to conventional LMPC designs which typically include a quadratic objective function and regulate a process at a steady-state. The proposed scheme incorporated appropriate Lyapunov-based constraint in its formulation to meet the switching schedule. Up to a certain amount of time, the LEMPC deals with economic optimization while after that time it enforces convergence to a steady-state. A chemical process example was used to demonstrate the proposed LEMPC scheme.

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