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Brief Paper

Analysis and control of parabolic PDE systems with input constraints☆

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Abstract

This paper develops a general framework for the analysis and control of parabolic partial differential equations (PDE) systems with input constraints. Initially, Galerkin's method is used for the derivation of ordinary differential equation (ODE) system that capture the dominant dynamics of the PDE system. This ODE systems are then used as the basis for the synthesis, via Lyapunov techniques, of stabilizing bounded nonlinear state and output feedback control laws that provide an explicit characterization of the sets of admissible initial conditions and admissible control actuator locations that can be used to guarantee closed-loop stability in the presence of constraints. Precise conditions that guarantee stability of the constrained closed-loop parabolic PDE system are provided in terms of the separation between the fast and slow eigenmodes of the spatial differential operator. The theoretical results are used to stabilize an unstable steady-state of a diffusion-reaction process using constrained control action.

Keywords: Parabolic PDE systems; Input constraints; Nonlinear bounded control; Static output feedback control; Transport-reaction processes

1. Introduction

Transport-reaction processes with significant diffusive and convective phenomena are typically characterized by strong nonlinearities and spatial variations, and are naturally described by quasi-linear parabolic partial differential equations (PDEs). Examples include tubular reactors, packed-bed reactors, and chemical vapor deposition reactors. Parabolic PDE systems typically involve spatial differential operators whose eigenspectrum can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement (Friedman, 1976). This implies that the dynamic behavior of such systems can be approximately described by finite-dimensional systems. Therefore, the standard approach to the control of parabolic PDEs involves the application of Galerkin's method to the PDE system to derive ordinary differential equation (ODE) systems that describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for the synthesis of finite-dimensional controllers (e.g., Balas, 1979; Ray, 1981). More recently, research efforts on control of parabolic PDE systems have focused on the problem of synthesizing low-order controllers on the basis of ODE models obtained through combination of Galerkin's method with approximate inertial manifolds (see the recent book (Christofides, 2001) for details and references). In addition to this work, other important recent contributions in control of PDE systems include controller design based on the infinite-dimensional system and subsequent use of approximation theory to design and compute low-order finite dimensional compensators (Burns & King, 1998), and stabilization of PDE systems using boundary control (Byrnes, Gilliam, & Shubov, 1994; Balogh & Krstic, 2002).

Although the above methods lead to the systematic design of nonlinear controllers for transport-reaction processes, they do not address the practical problem of input constraints. Virtually all physical and chemical control systems are subject to hard constraints typically arising from the finite capacity of control actuators. These constraints limit our ability to modify the process dynamics and, if not appropriately accounted for at the stage of controller design, can lead to significant performance deterioration and even cause closed-loop instability. The ill-effects due to actuator

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constraints have consequently motivated many recent studies on the dynamics and control of systems subject to input constraints. Most of the research in this area, however, has focused on lumped parameter systems modeled by ordinary differential equation systems (see, e.g., Kurtz & Henson, 1997; Alonso & Banga, 1998; Kothare, Campo, Morari, & Nett, 1994; Valluri & Soroush, 1998; Kapoor, Teel & Daoutidis, 1998; El-Farra & Christofides, 2001).

Compared with lumped-parameter systems, the problem of constrained control of distributed parameter systems has received much less attention. The few results available on this problem have focused primarily on linear distributed parameter systems (see, e.g., Chernousko, 1992; Bounit & Hammouri, 1997; Diblasio, 1991), with virtually no results on their nonlinear counterparts. A host of issues therefore remain to be resolved, including the synthesis of nonlinear controllers that account explicitly for input constraints, as well as the explicit characterization of the limitations imposed by input constraints on the set of admissible initial conditions and admissible actuator locations that can be used for stabilization. In contrast to the case of lumped parameters systems, constraints on the manipulated input of distributed parameter systems that exhibit spatial variations impose an additional limitation on *where* the control actuators can be placed to guarantee closed-loop stability.

Motivated by these problems, we develop in this work a general framework for the analysis and control of quasi-linear parabolic PDE systems with input constraints. Initially, finite-dimensional systems, that captures the dominant dynamics of the PDE system, are constructed using Galerkin's method. The ODE systems is then used as the basis for the synthesis, via Lyapunov techniques, of bounded nonlinear feedback controllers that enforce closed-loop stability in the and provide, simultaneously, an explicit characterization of the limitations imposed by input constraints on both the admissible initial states and admissible control actuator locations that can be used to guarantee closed-loop stability. The proposed analysis and controller synthesis results have been used to stabilize an unstable steady-state of a diffusion-reaction process (section 4) and an unstable steady-state of the Kuramsto-Sivashinsky equation (Armaou & Christofides, 2001).

2. Preliminaries

2.1. Class of systems

We consider quasi-linear parabolic PDE systems of the form:

$$\begin{aligned}
\frac{\partial \bar{x}}{\partial t} &= A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2} + w\sum_{i=1}^{m} b_i(z)u_i + f(\bar{x}),\\
y_c^i &= \int_\alpha^\beta c^i(z)k\bar{x}(z,t)\,\mathrm{d}z, \qquad i=1,\ldots,m,\\
y_m^\kappa &= \int_\alpha^\beta s^\kappa(z)\omega\bar{x}(z,t)\,\mathrm{d}z, \qquad \kappa=1,\ldots,p,
\end{aligned} \tag{1}$$

subject to the boundary conditions:

$$\begin{aligned}
C_1\bar{x}(\alpha,t) + D_1\frac{\partial \bar{x}}{\partial z}(\alpha,t) &= R_1,\\
C_2\bar{x}(\beta,t) + D_2\frac{\partial \bar{x}}{\partial z}(\beta,t) &= R_2
\end{aligned} \tag{2}$$

and the initial condition

$$\bar{x}(z,0) = \bar{x}_0(z), \tag{3}$$

where $\bar{x}(z,t) = [\bar{x}_1(z,t)\ \cdots\ \bar{x}_n(z,t)]^{\mathrm{T}} \in \mathbb{R}^n$ denotes the vector of state variables, $z \in [\alpha,\beta] \subset \mathbb{R}$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $u_i \in [u_{i,\min}, u_{i,\max}] \subset \mathbb{R}$ denotes the ith constrained manipulated input, $y_c^i \in \mathbb{R}$ denotes the ith controlled output, and $y_m^\kappa \in \mathbb{R}$ denotes the κth measured output. $\partial\bar{x}/\partial z, \partial^2\bar{x}/\partial z^2$ denote the first- and second-order spatial derivatives of $\bar{x}$, $f(\bar{x})$ is a nonlinear vector function, w, k are constant vectors, A, B, C_1, D_1, C_2, D_2 are constant matrices, R_1, R_2 are column vectors, and $\bar{x}_0(z)$ is the initial condition. $b_i(z)$ is a known smooth function of z which describes how the control action $u_i(t)$ is distributed in the interval $[\alpha,\beta]$, $c^i(z)$ is a known smooth function of z which is determined by the desired performance specifications in the interval $[\alpha,\beta]$, and $s^\kappa(z)$ is a known smooth function of z which depends on the shape (point or distributing sensing) of the measurement sensors in the interval $[\alpha,\beta]$. Whenever the control action enters the system at a single point z_0, with $z_0 \in [\alpha,\beta]$ (i.e. point actuation), the function $b_i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_0-\varepsilon, z_0+\varepsilon]$, where ε is a small positive real number, and zero elsewhere in $[\alpha,\beta]$. Throughout the paper, we will use the order of magnitude notation $O(\varepsilon)$. In particular, $\delta(\varepsilon) = O(\varepsilon)$ if there exist positive real numbers k_1 and k_2 such that: $|\delta(\varepsilon)| \leqslant k_1|\varepsilon|$, $\forall|\varepsilon| < k_2$.

To precisely characterize the class of parabolic PDE systems considered in this work, we formulate the system of Eq. (1) as an infinite dimensional system in the Hilbert space $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$, with $\mathscr{H}$ being the space of n-dimensional vector functions defined on $[\alpha,\beta]$ that satisfy the boundary conditions of Eq. (2), with inner product and norm:

$$(\omega_1,\omega_2) = \int_\alpha^\beta (\omega_1(z),\omega_2(z))_{\mathbb{R}^n}\,\mathrm{d}z, \quad \|\omega_1\|_2 = (\omega_1,\omega_1)^{1/2}, \tag{4}$$

where ω_1, ω_2 are two elements of $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function x on $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$ as

$$x(t) = \bar{x}(z,t), \qquad t > 0,\ z \in [\alpha,\beta], \tag{5}$$

the operator $\mathscr{A}$ in $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$ as

$$\mathscr{A}x = A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2},$$

$$x \in D(\mathscr{A}) = \left\{ x \in \mathscr{H}([\alpha,\beta];\mathbb{R}^n) : C_1\bar{x}(\alpha,t) + D_1 \frac{\partial \bar{x}}{\partial z}(\alpha,t) = R_1,\ C_2\bar{x}(\beta,t) + D_2\frac{\partial \bar{x}}{\partial z}(\beta,t) = R_2 \right\}$$

and the input, controlled output, and measured output operators as

$$\mathscr{B}u = w\sum_{i=1}^{m} b_i u_i, \quad \mathscr{C}x = (c, kx), \quad \mathscr{S}x = (s, \omega x) \tag{6}$$

the system of Eqs. (1)–(3) takes the form

$$\dot{x} = \mathscr{A}x + \mathscr{B}u + f(x), \quad x(0) = x_0, \qquad y_c = \mathscr{C}x, \quad y_m = \mathscr{S}x, \tag{7}$$

where $f(x(t)) = f(\bar{x}(z,t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear terms $f(x)$ are locally Lipschitz with respect to their arguments and satisfy $f(0)=0$. For $\mathscr{A}$, the eigenvalue problem is defined as

$$\mathscr{A}\phi_j = \lambda_j \phi_j, \quad j = 1,\ldots,\infty, \tag{8}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction; the eigenspectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, is defined as the set of all eigenvalues of $\mathscr{A}$, i.e. $\sigma(\mathscr{A}) = \{\lambda_1, \lambda_2, \ldots,\}$. Assumption 1 that follows states that the eigenspectrum of $\mathscr{A}$ can be partitioned into a finite part consisting of m slow eigenvalues and a stable infinite complement containing the remaining fast eigenvalues, and that the separation between the slow and fast eigenvalues of $\mathscr{A}$ is large. This assumption is satisfied by the majority of diffusion-convection-reaction processes (Christofides, 2001).

Assumption 1.

(1) $Re\{\lambda_1\} \geqslant Re\{\lambda_2\} \geqslant \cdots \geqslant Re\{\lambda_j\} \geqslant \cdots$, where $Re\{\lambda_j\}$ denotes the real part of λ_j.
(2) $\sigma(\mathscr{A})$ can be partitioned as $\sigma(\mathscr{A}) = \sigma_1(\mathscr{A}) + \sigma_2(\mathscr{A})$, where $\sigma_1(\mathscr{A})$ consists of the first m (with m finite) eigenvalues, i.e. $\sigma_1(\mathscr{A}) = \{\lambda_1, \ldots, \lambda_m\}$, and $|Re\{\lambda_1\}|/|Re\{\lambda_m\}| = O(1)$.
(3) $Re\,\lambda_{m+1} < 0$ and $|Re\{\lambda_m\}|/|Re\{\lambda_{m+1}\}| = O(\varepsilon)$ where $\varepsilon < 1$ is a small positive number.

2.2. Galerkin's method

In this section, we apply standard Galerkin's method to the system of Eq. (1) to derive an approximate finite-dimensional system. Let $\mathscr{H}_s$, $\mathscr{H}_f$ be modal subspaces of $\mathscr{A}$, defined as $\mathscr{H}_s = span\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathscr{H}_f = span\{\phi_{m+1}, \phi_{m+2}, \ldots,\}$ (the existence of $\mathscr{H}_s$, $\mathscr{H}_f$ follows from Assumption 1). Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq. (7) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x. \tag{9}$$

Applying P_s and P_f to the system of Eq. (7) and using the above decomposition for x, the system of Eq. (7) can be equivalently written in the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathscr{A}_s x_s + \mathscr{B}_s u + f_s(x_s, x_f), \\ \frac{\partial x_f}{\partial t} &= \mathscr{A}_f x_f + \mathscr{B}_f u + f_f(x_s, x_f), \\ y_c &= \mathscr{C}x_s + \mathscr{C}x_f, \quad y_m = \mathscr{S}x_s + \mathscr{S}x_f, \\ x_s(0) &= P_s x(0) = P_s x_0, \quad x_f(0) = P_f x(0) = P_f x_0, \end{aligned} \tag{10}$$

where $\mathscr{A}_s = P_s\mathscr{A}$, $\mathscr{B}_s = P_s\mathscr{B}$, $f_s = P_s f$, $\mathscr{A}_f = P_f\mathscr{A}$, $\mathscr{B}_f = P_f\mathscr{B}$ and $f_f = P_f f$ and the partial derivative notation in $\partial x_f/\partial t$ is used to denote that the state x_f belongs to an infinite-dimensional space. In the above system, $\mathscr{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathscr{A}_s = diag\{\lambda_j\}$, $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are Lipschitz vector functions, and $\mathscr{A}_f$ is an unbounded differential operator which is exponentially stable (following from part 3 of Assumption 1 and the selection of $\mathscr{H}_s, \mathscr{H}_f$). Neglecting the fast and stable infinite-dimensional x_f-subsystem in the system of Eq. (10), the following m-dimensional slow system is obtained:

$$\frac{d\bar{x}_s}{dt} = \mathscr{A}_s\bar{x}_s + \mathscr{B}_s u + f_s(\bar{x}_s, 0), \qquad \bar{y}_c = \mathscr{C}\bar{x}_s, \quad \bar{y}_m = \mathscr{S}\bar{x}_s, \tag{11}$$

where the bar symbol in $\bar{x}_s$, $\bar{y}_c$ and $\bar{y}_m$ denotes that these variables are associated with a finite-dimensional system.

Remark 1. We note that the above model reduction procedure which led to the approximate ODE system of Eq. (11) can also be used, when empirical eigenfunctions of the system of Eq. (1) computed through Karhunen-Loéve expansion (see Atwell & King, 2001 & Christofides, 2001 for details) are used as basis functions in $\mathscr{H}_s$ and $\mathscr{H}_f$ instead of the eigenfunctions of $\mathscr{A}$.

Remark 2. Although the finite-dimensional system of Eq. (11) was obtained through standard Galerkin's method, the results of this paper can be generalized to the case where the finite-dimensional approximation of the system of Eq. (10) is obtained through combination of Galerkin's method with approximate inertial manifolds (Christofides, 2001). This approach can be used to further reduce the dimension of the system of Eq. (11) and ensure that it is of an appropriately low-order suitable for controller design and analysis.

3. Bounded nonlinear control of constrained parabolic PDE systems

3.1. State feedback control

In this section, we use the constrained m-dimensional ODE system of Eq. (11) as the basis for the synthesis of bounded nonlinear state feedback control laws of the general form:

$$u = p(\bar{x}_s, \bar{v}, u_{\max}), \tag{12}$$

where $p(\cdot)$ is a bounded nonlinear vector function (i.e. $|u| \leqslant u_{\max}$, where $|\cdot|$ is the standard Euclidean norm and $u_{\max}$ is the maximum norm of the vector of manipulated inputs allowed by the constraints), and $\bar{v}$ is a vector of the form $\bar{v} = [\bar{v}_1^{\mathrm{T}} \cdots \bar{v}_m^{\mathrm{T}}]^{\mathrm{T}}$ where $\bar{v}_i = [v_i \; v_i^{(1)}, \ldots, v_i^{(r_i)}]$ is a smooth vector function, $v_i^{(k)}$ is the kth time derivative of the external reference input v_i (which is assumed to be a smooth function of time) and r_i is the relative order of the controlled output y_c^i with respect to the vector of manipulated inputs (which is assumed to be well-defined). The requested controller design specifications include: (a) enforcing asymptotic stability (and local exponential stability) as well as reference input tracking in the constrained infinite-dimensional closed-loop system, (b) characterizing explicitly the admissible initial states that can be used, for a given actuator location, to guarantee closed-loop stability, and (c) identifying explicitly the set of admissible control actuator locations that can be used, for a given initial condition, to stabilize the constrained closed-loop system.

To proceed with the controller synthesis, we need to introduce some notation and definitions that will be used in stating our results. We initially transform the system of Eq. (11), by means of an invertible coordinate change of the form $\bar{x}_s = T^{-1}(\zeta, \eta)$, into the following partially linear form:

$$\begin{aligned}
\dot{\zeta}_1^{(i)} &= \zeta_2^{(i)} \\
&\;\vdots \\
\dot{\zeta}_{r_i-1}^{(i)} &= \zeta_{r_i}^{(i)}, \\
\dot{\zeta}_{r_i}^{(i)} &= L_{\tilde{f}}^{r_i}\tilde{h}_i(\bar{x}_s) + \sum_{k=1}^{m} L_{\tilde{g}_k} L_{\tilde{f}}^{r_i-1}\tilde{h}_i(\bar{x}_s) u_k, \\
\dot{\eta} &= \Psi(\zeta, \eta), \\
\bar{y}_c^i &= \zeta_1^{(i)}, \quad i = 1, \ldots, m,
\end{aligned} \tag{13}$$

where $\bar{y}_c^i = \tilde{h}_i(\bar{x}_s)$, $\tilde{f}(\bar{x}_s) = \mathscr{A}_s\bar{x}_s + f_s(\bar{x}_s, 0)$, $\tilde{g}_k$ is the ith column of the matrix $\mathscr{B}_s$, $\zeta_k^{(i)} = L_{\tilde{f}}^{k-1}\tilde{h}_i(\bar{x}_s)$, $\zeta = [\zeta^{(1)^{\mathrm{T}}} \cdots \zeta^{(m)^{\mathrm{T}}}]^{\mathrm{T}}$, $\eta = [\eta_1 \cdots \eta_{m-\sum_i r_i}]^{\mathrm{T}}$. Defining the tracking error $e_k^{(i)} = \zeta_k^{(i)} - v_i^{(k-1)}$ and introducing the vector notation $e^{(i)} = [e_1^{(i)} \; e_2^{(i)} \cdots e_{r_i}^{(i)}]^{\mathrm{T}}$, $e = [e^{(1)^{\mathrm{T}}} \; e^{(2)^{\mathrm{T}}} \cdots e^{(m)^{\mathrm{T}}}]^{\mathrm{T}}$, where $i = 1, \ldots, m$, $k = 1, \ldots, r_i$, the ζ-subsystem of Eq. (13) can be further transformed into the following more compact form:

$$\dot{e} = \bar{f}(e, \eta) + \sum_{k=1}^{m} \bar{g}_k(e, \eta) u_k, \tag{14}$$

where $\bar{f}(e, \eta) = Fe + Kl(T^{-1}(e, \eta))$, F and K are constant matrices, $l(\cdot)$ is a smooth vector function, $\bar{g}_k$ the ith column of the matrix $\bar{G}(e, \eta) = KC(T^{-1}(e, \eta))$, and $C(\bar{x}_s)$ is the characteristic matrix of the system of Eq. (11) which, for simplicity of presentation, is assumed to be nonsingular, uniformly in $\bar{x}_s$. Motivated by the requirement of output tracking and local exponential stability, we impose the following assumption which allows us then to carry out the controller synthesis task on the basis of the system of Eq. (14) which describes the input/output dynamics of the system of Eq. (11).

Assumption 2. The η-subsystem of Eq. (13) is input-to-state stable with respect to e and locally exponentially stable when $e = 0$.

Theorem 1 below provides an explicit synthesis formula of the desired state feedback control law and states precise conditions that guarantee closed-loop stability and asymptotic reference-input tracking in the presence of input constraints. The proof is given in the appendix.

Theorem 1. (1) *Consider the system of Eq.* (11), *for which Assumption* 2 *holds, under the control law*

$$u = -\frac{1}{2} R^{-1}(\bar{x}_s)(L_{\bar{g}}V)^{\mathrm{T}}, \tag{15}$$

where

$$\frac{1}{2} R^{-1}(\bar{x}_s) = \frac{L_{\bar{f}}^{*}V + \sqrt{(L_{\bar{f}}^{*}V)^2 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^4}}{|(L_{\bar{g}}V)^{\mathrm{T}}|^2[1 + \sqrt{1 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]}, \tag{16}$$

$L_{\bar{f}}^{*}V = L_{\bar{f}}V + \rho|e|^2$, $\rho > 0$, $L_{\bar{g}}V = [L_{\bar{g}_1}V \cdots L_{\bar{g}_m}V]$ *is a row vector*, $V = e^{\mathrm{T}}\Phi e$, Φ *is a positive definite matrix that satisfies the Riccati inequality* $F^{\mathrm{T}}\Phi + \Phi F - \Phi KK^{\mathrm{T}}\Phi < 0$. *Let* δ_s *be a positive real number such that the compact set* $\Omega = \{\bar{x}_s \in \mathscr{H}_s : |\bar{x}_s| \leqslant \delta_s\}$ *is the largest invariant set embedded within the region described by the following inequality*:

$$L_{\bar{f}}^{*}V \leqslant u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|. \tag{17}$$

Then, for any $\bar{x}_s(0)$ *with* $|\bar{x}_s(0)| \leqslant \delta_s$, *the closed-loop system is asymptotically stable in the sense that there exists a function* β *of class KL such that* $|\bar{x}_s(t)| \leqslant \beta(|\bar{x}_s(0)|, t)$, $\forall t \geqslant 0$.

(2) *Consider the parabolic PDE system of Eq.* (1), *for which Assumption* 1 *holds, under the state feedback controller of Eqs.* (15) and (16). *Then given any pair of positive real numbers* (d, δ_b) *such that* $\beta(\delta_b, 0) + d \leqslant \delta_s$, *and*

given any positive real number δ_f, *there exists* $\varepsilon^* > 0$ *such that if* $\varepsilon \in (0, \varepsilon^*]$, $|x_s(0)| \leqslant \delta_b$, $\|x_f(0)\|_2 \leqslant \delta_f$, *the infinite-dimensional closed-loop system is asymptotically (and locally exponentially) stable, and the outputs of the closed-loop system satisfy a relation of the form*:

$$\limsup_{t\to\infty} |y_c^i(t) - v_i(t)| = O(\varepsilon). \tag{18}$$

Remark 3. The inequality of Eq. (17) characterizes explicitly the region in state space where the control action satisfies the constraints and the time-derivative of the Lyapunov function is negative-definite along the trajectories of the finite-dimensional closed-loop "slow" system of Eqs. (11), (15) and (16). To guarantee that trajectories starting within this region do not leave, the initial conditions are confined within the largest invariant set, Ω, embedded within the region, in order to guarantee closed-loop stability. The size of Ω is fixed by δ_s and represents the state feedback estimate of the set of admissible initial "slow" states starting from where stability of the constrained finite dimensional closed-loop "slow" system is guaranteed (region of guaranteed stability) (see El-Farra & Christofides (2001) and Khalil (1996, Chap. 4) for how to compute this estimate). Owing to the large separation between the slow and fast eigenmodes of the spatial differential operator of the PDE system of Eq. (1), this estimate of the closed-loop stability region remains preserved for the constrained closed-loop PDE system, in the sense that given any initial "slow" state that belongs to any compact subset (size of this subset is fixed by δ_b), and given any initial "fast state", there always exists ε sufficiently small such that the controller of Eqs. (15) and (16) continues to enforce stability in the constrained infinite-dimensional closed-loop system. According to Theorem 1, selection of the initial slow state within a compact subset of Ω with radius δ_b, ensures that, during the evolution of x, the slow states remain within Ω. An estimate of ε^* can be extracted from the stability proof of theorem 1. However, this estimate is typically conservative and thus it is useful to check its appropriateness through computer simulations.

Remark 4. Owing to the dependence of the operator $\mathscr{B}_s$ in Eq. (11) on the actuator distribution function, $b_i(z)$, the inequality of Eq. (17) (and hence the stability region Ω) is parameterized by the actuator locations, and can therefore be used to explicitly identify the admissible locations where the control actuators can be placed to guarantee stability of the constrained closed-loop system. The inequality of Eq. (17) captures, quantitatively, the interplay between the limitations imposed by input constraints on the admissible initial states and on the admissible control actuator locations. To see this, note that this inequality can be used in two ways. For a given actuator location (i.e., fixed $\mathscr{B}_s$), the inequality provides the set of admissible initial states that guarantee closed-loop stability. Alternatively, for a given initial condition (i.e., fixed δ_s), the inequality provides the admissible actuator locations.

Remark 5. The control law in Eqs. (15) and (16) involves a modification of the controller design proposed in Lin and Sontag (1991) by including the term $-\rho|e|^2$, where ρ is a strictly positive tuning parameter, which ensures that the control law enforces exponential stability in the closed-loop finite-dimensional system. With $\rho = 0$, it can be shown that the control law of Eqs. (15) and (16) enforces only asymptotic stability. We therefore prefer to set $\rho > 0$ and have exponential stability because of its robustness to bounded perturbations, which are always present in most practical applications. This also ensures that the closed-loop PDE system is locally exponentially stable. Note that in order to use the inequality of Eq. (17) to compute an estimate of the stability region, as described in Remark 3, a value of ρ must be specified.

3.2. Output feedback control

The nonlinear controller of Eqs. (15) and (16) was derived under the assumption that measurements of the state variables, $\bar{x}(z,t)$, are available at all positions and times. In this section, we address the problem of synthesizing bounded nonlinear output feedback controllers of the general form

$$u(t) = \mathscr{F}(y_m, \bar{v}, u_{\max}), \tag{19}$$

where $\mathscr{F}(y_m, \bar{v}, u_{\max})$ is a bounded nonlinear vector function and y_m is the vector of measured outputs, that: (a) enforce stability and output tracking in the constrained closed-loop PDE system, and (b) recover the state feedback region of guaranteed closed-loop stability. The synthesis of the controller of Eq. (19) will be achieved by combining the state feedback controller of Eq. (15) with a procedure proposed in Christofides (2001) for obtaining estimates for the states of the approximate ODE model of Eq. (11) from the measurements. To this end, we need to impose the following requirement on the number of measured outputs in order to obtain estimates of the states x_s of the finite-dimensional system of Eq. (11), from the measurements y_m^κ, $\kappa = 1, \ldots, p$.

Assumption 3. $p = m$ (i.e., the number of measurements is equal to the number of slow modes), and the inverse of the operator $\mathscr{S}$ exist, so that $\hat{x}_s = \mathscr{S}^{-1} y_m$.

We note that the requirement that the inverse of the operator $\mathscr{S}$ exists can be achieved by appropriate choice of the location of the measurement sensors (i.e., the functions $s^\kappa(z)$).

Theorem 2 that follows establishes that the proposed output feedback controller enforces stability and reference-input tracking in the constrained infinite-dimensional closed-loop system and practically preserves the region of guaranteed closed-loop stability obtained under state feedback.

Theorem 2. *Consider the system of Eq.* (11) *for which Assumption* 3 *holds. Consider also the parabolic PDE system of Eq.* (1), *for which Assumptions* 1 *and* 2 *hold, under the nonlinear output feedback controller*

$$u = -\frac{1}{2} R^{-1}(\hat{x}_s)(L_{\bar{g}} V)^T, \tag{20}$$

where $\hat{x}_s = \mathscr{S}^{-1} y_m$, $R^{-1}(\cdot)$ *and* V *as defined in Theorem* 1. *Then given any pair of positive real numbers* (d, δ_b), *defined in Theorem* 1, *and given any positive real number* δ_f, *there exists* $\varepsilon^{**} > 0$ *such that if* $\varepsilon \in (0, \varepsilon^{**}]$, $|x_s(0)| \leqslant \delta_b$, $\|x_f(0)\|_2 \leqslant \delta_f$, *the infinite-dimensional closed-loop system is asymptotically stable* (*and locally exponentially stable*) *and its outputs satisfy Eq.* (18).

Remark 6. We note that the controller of Eq. (20) uses static feedback of the measured outputs y_m^κ, $\kappa = 1, \ldots, p$, and thus, it feeds back both x_s and x_f (this is in contrast to the state feedback controller of Eq. (15) and (16) which only uses feedback of the slow state x_s). However, even though the use of x_f feedback could lead to destabilization of the stable fast subsystem, the large separation of the slow and fast modes of the spatial differential operator (i.e., the assumption that ε is sufficiently small) and the fact that the controller does not include terms of the form $O(1/\varepsilon)$ do not allow such a destabilization to occur.

Remark 7. Owing to the estimation error, a discrepancy exists between the state and output feedback stability regions. This discrepancy, however, can be made small by increasing the order of the ODE approximation and including more measurements. In the asymptotic limit (as $\varepsilon \to 0$), the stability region for both the state and output feedback problems (set of admissible initial "slow" states) approach that for the constrained finite-dimensional system of Eq. (11). This result is important because it is only the finite-dimensional system that is used to design the controller and carry out any practical computations for the region of closed-loop stability.

4. Application to a diffusion–reaction process

Consider a long, thin rod in a reactor. The reactor is fed with pure species A and a zeroth order exothermic catalytic reaction of the form $A \to B$ takes place on the rod. Since the reaction is exothermic, a cooling medium in contact with the rod is used for cooling. Under standard assumptions, the spatiotemporal evolution of the dimensionless rod temperature is described by the following parabolic PDE:

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T e^{-\gamma/(1+\bar{x})} + \beta_U (b(z)u(t) - \bar{x}) - \beta_T e^{-\gamma}$$

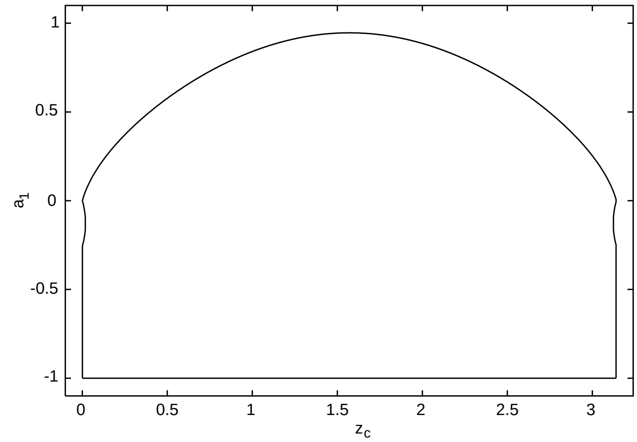

Fig. 1. Dependence of the set of feasible initial conditions, for a_1, on actuator location for $u_{max} = 1.4$.

subject to the boundary and initial conditions:

$$\bar{x}(0,t) = 0, \quad \bar{x}(\pi,t) = 0, \quad \bar{x}(z,0) = \bar{x}_0(z), \tag{21}$$

where $\bar{x}$ denotes a dimensionless rod temperature, β_T denotes a dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, $u(t)$ denotes the vector of manipulated inputs and $b(z)$ the vector of the corresponding actuator distribution functions. The following typical values are given to the process parameters: $\beta_T = 50.0, \beta_U = 2.0, \gamma = 4.0$. For these values, it was verified that the operating steady state $\bar{x}(z,t) = 0$ is an unstable one. The control objective therefore is to stabilize the rod temperature profile at the unstable steady state $\bar{x}(z,t) = 0$ by manipulating the temperature of the cooling medium, which is subject to hard constraints. To achieve this objective, the controlled output is defined as

$$y_c(t) = \int_0^\pi \sqrt{\frac{2}{\pi}} \sin(z) \bar{x}(z,t)\, dz \tag{22}$$

and one point control actuator is assumed to be available for stabilization. The eigenvalue problem for the spatial differential operator of the process can be solved analytically and its solution yields

$$\lambda_j = -j^2, \quad \phi_j(z) = \sqrt{\frac{2}{\pi}} \sin(j\,z), \; j = 1, \ldots, \infty. \tag{23}$$

For this system, we consider the first eigenvalue as the dominant one and use standard Galerkin's method to derive an ODE which is used for controller synthesis. The controllers are then implemented on a 30th order Galerkin discretization of the parabolic PDE system (higher order discretizations led to identical results).

Before we proceed with the design and implementation of the controllers, we use the inequality of Eq. (17) first to explicitly characterize the region of guaranteed closed-loop stability. Fig. 1 depicts the region of guaranteed closed-loop

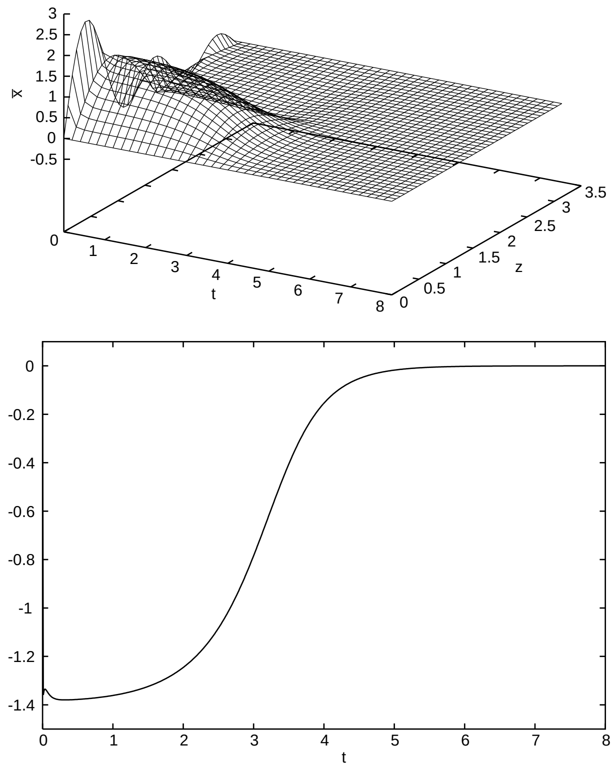

Fig. 2. Closed-loop temperature (top) and manipulated input (bottom) profiles for $u_{\max} = 1.4$, $z_c = 0.5\pi$, $a_1(0) = 0.7$, under state feedback.

stability for a value of $u_{\max} = 1.4$, $\rho = 0.0001$. This region, which has a dome-like shape, describes how the set of admissible initial conditions for the amplitude of the first eigenmode $a_1(t)$ varies as a function of control actuator location, z_c, along the rod length. It is clear from the figure that the size of this set peaks at the middle of the rod and diminishes, as expected, as we get closer to the edges of the rod, owing to the zeros of the first eigenfunction at the boundaries. Guided by the information contained in Fig. 1, we now proceed with the implementation of the controllers. The state feedback results are presented first. Fig. 2 shows the evolution of the closed-loop rod temperature and manipulated input profiles under the controller of Eqs. (15) and (16) when the actuator is placed in the middle of the rod and $a_1(0) = 0.7$. For this initial condition, it can be easily verified from Fig. 1 that the location $z_c = 0.5\pi$ is an admissible one when $u_{\max} = 1.4$. Clearly, the controller successfully stabilizes the temperature profile at the desired steady-state. In the next simulation run, the actuator location is moved to $z_c = 0.25\pi$ but the initial condition is kept at $a_1(0) = 0.7$. In this case, it is clear from Fig. 1 that the $(a_1(0), z_c)$ pair falls outside the region of guaranteed closed-loop stability. The simulation results for this case are depicted in Fig. 3 where we observe that the controller is unable to stabilize the spatially uniform steady-state and the system moves to the spatially non-uniform steady state. For the case of output feedback control, we used a single point sensor located at $z = 0.33\pi$ to obtain estimates of the first eigenmode and then used the synthesis formula of Eq. (20) to design the output feedback controller. The simulation results for this case are shown in Figs. 4 and 5, for the same combinations of initial conditions and actuator locations considered in the state feedback case. We observe that the output feedback controller successfully stabilizes the constrained closed-loop system when the (initial condition, actuator location) pair lies inside the stability region (Fig. 4) and fails to do so otherwise (Fig. 5).

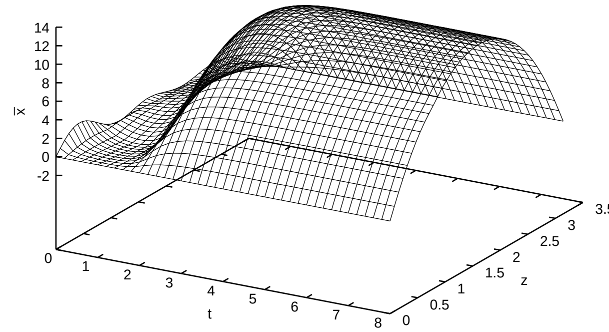

Fig. 3. Closed-loop temperature profile for $u_{\max} = 1.4$, $z_c = 0.25\pi$, $a_1(0) = 0.7$ under state feedback.

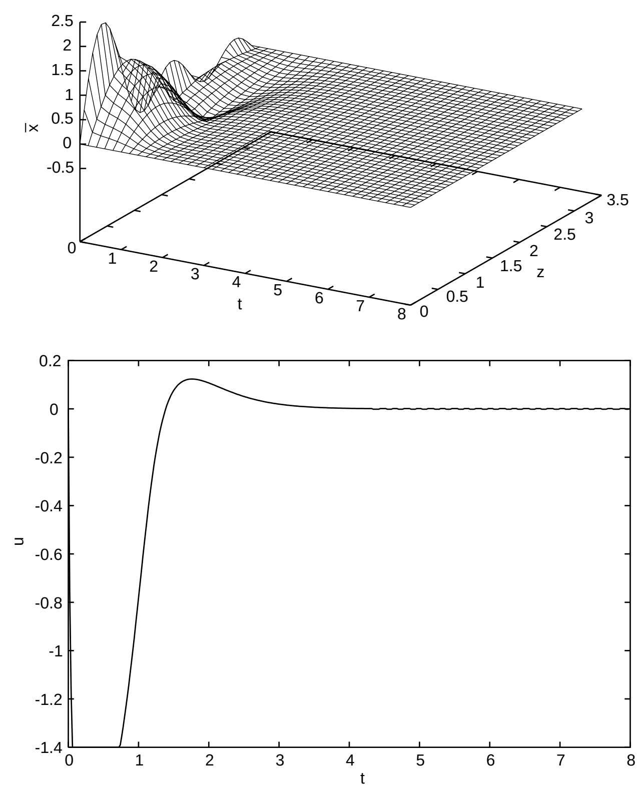

Fig. 4. Closed-loop temperature (top) and manipulated input (bottom) profiles for $u_{\max} = 1.4$, $z_c = 0.5\pi$, $a_1(0) = 0.7$, under output feedback.

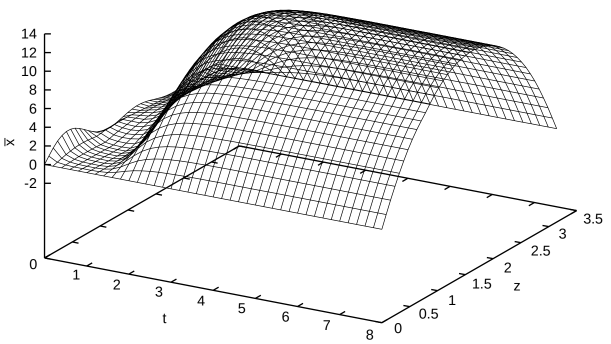

Fig. 5. Closed-loop temperature profile for $u_{\max} = 1.4$, $z_c = 0.25\pi$, $a_1(0) = 0.7$, under output feedback.

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Appendix A.

Proof of Theorem 1. *Part* 1: Consider the representation of the system of Eq. (11), for which Assumption 2 holds, in the e, η coordinates introduced in Eqs. (13) and (14). Evaluating the time-derivative of the Lyapunov function candidate $V = e^{\mathrm{T}}\Phi e$, defined in Theorem 1, along the trajectories of the e-subsystem in Eq. (14), we obtain

$$\dot{V} = L_{\bar{f}}V + L_{\bar{g}}Vu. \tag{A.1}$$

Substituting the controller of Eqs. (15) and (16) into the above equation yields, after some algebraic manipulations:

$$\dot{V} = \frac{-\rho|e|^2 + L_{\bar{f}}V\sqrt{1+(u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}}{[1+\sqrt{1+(u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]} - \frac{\sqrt{(L_{\bar{f}}^*V)^2 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^4}}{[1+\sqrt{1+(u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]}. \tag{A.2}$$

Using the fact that $L_{\bar{f}}^*V = L_{\bar{f}}V + \rho|e|^2$, where $\rho > 0$, it is clear that whenever $L_{\bar{f}}^*V \leqslant 0$, we have $L_{\bar{f}}V \leqslant 0$ and therefore

$$\dot{V} \leqslant \frac{-\rho|e|^2}{[1+\sqrt{1+(u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]} < 0 \quad \forall e \neq 0. \tag{A.3}$$

For the case when $0 < L_{\bar{f}}^*V \leqslant u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|$, we have $(L_{\bar{f}}^*V)^2 \leqslant u_{\max}^2|(L_{\bar{g}}V)^{\mathrm{T}}|^2$ and consequently

$$-\sqrt{(L_{\bar{f}}^*V)^2 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^4} \leqslant -L_{\bar{f}}^*V\sqrt{1+(u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2} \leqslant -L_{\bar{f}}V\sqrt{1+(u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2} \tag{A.4}$$

Substituting the last estimate, obtained above, into Eq. (A.2), we conclude that $\dot{V}$ satisfies Eq. (A.3). Summarizing, if the inequality of Eq. (17) holds, we have that $\dot{V}$ satisfies Eq. (A.3). Since Ω is the largest invariant set embedded within the region described by the inequality of Eq. (17), it is clear then that starting from any $\bar{x}_s(0) \in \Omega$, the evolution of the closed-loop trajectory obeys Eq. (17) and $\dot{V}$ satisfies Eq. (A.3), which implies that the closed-loop e-subsystem is asymptotically stable, i.e. there exists a function β_e of class KL such that the e states of the closed-loop system satisfy

$$|e(t)| \leqslant \beta_e(|e(0)|, t) \quad \forall t \geqslant 0. \tag{A.5}$$

Since the η subsystem, with e as input, is ISS (from Assumption 2), the η states of the closed-loop system satisfy the following inequality

$$|\eta(t)| \leqslant K_\eta|\eta(0)|\mathrm{e}^{-at} + \gamma_\eta(\|e\|) \quad \forall t \geqslant 0 \tag{A.6}$$

for some $K_\eta \geqslant 1$, $a > 0$, where γ_η is a class K function. Using the inequalities of Eqs. (A.5) and (A.6), it can be shown by means of a small gain argument that, given any initial condition such that $|\bar{x}_s(0)| \leqslant \delta_s$, the closed-loop e-η interconnection, and hence the closed-loop system of Eqs. (11), (15) and (16), is asymptotically stable. Therefore, there exists a function β of class KL such that $|\bar{x}_s(t)| \leqslant \beta(|\bar{x}_s(0)|, t)\ \forall\, t \geqslant 0$.

Part 2: From the stability result obtained in part 1, it is clear that the state $\bar{x}_s$ is bounded and therefore the denominator term (henceforth denoted by $D(\bar{x}_s)$) in Eq. (A.3), which is a continuous function of $\bar{x}_s$, is also bounded by some positive real number k_s, where $k_s = \max_{|\bar{x}_s| \leqslant \delta_s} D(\bar{x}_s)$, and there exist real numbers $k_e = \rho/k_s > 0$, $k_1 \geqslant 1$, $a_1 = k_e/2\lambda_{\min}(\Phi) > 0$, where $\lambda_{\min}(\Phi)$ is the minimum eigenvalue of the matrix Φ, such that if $|\bar{x}_s(0)| \leqslant \delta_s$, $\dot{V}$ satisfies $\dot{V} \leqslant -k_e|e|^2$ and the e states of the closed-loop system satisfy

$$|e(t)| \leqslant k_1|e(0)|\mathrm{e}^{-a_1 t} \quad \forall t \geqslant 0 \tag{A.7}$$

which shows that the origin of the e-subsystem is exponentially stable. From this result and the fact that the η-subsystem (with $e = 0$) is locally exponentially stable, we have that the closed-loop e-η interconnection, and hence the closed-loop system of Eqs. (11), (15) and (16), is locally exponentially stable (see Khalil, 1996 for details). Therefore, given the positive real number δ_s, there exists $b > 0$ such that if $|\bar{x}_s(0)| \leqslant \delta_s$, the following bound holds

$$|\bar{x}_s(t)| \leqslant k_2|\bar{x}_s(0)|\mathrm{e}^{-a_2 t} \quad \forall t \geqslant 0 \tag{A.8}$$

for all $|\bar{x}_s(t)| \leqslant b$, for some $k_2 \geqslant 1$, $a_2 > 0$. Substituting the controller of Eqs. (15) and (16) into Eq. (10) and using the fact that $\varepsilon = |Re\,\lambda_1|/|Re\,\lambda_{m+1}|$, the closed-loop PDE system can be written in the following form (Christofides and Daoutidis, 1997):

$$\frac{dx_s}{dt} = \mathscr{A}_s x_s - \tfrac{1}{2}\mathscr{B}_s R^{-1}(x_s)(L_{\bar{g}}V)^T + f_s(x_s, 0) + [f_s(x_s, x_f) - f_s(x_s, 0)],$$

$$\varepsilon\frac{\partial x_f}{\partial t} = \mathscr{A}_{f\varepsilon} x_f + \varepsilon \bar{f}_f(x_s, x_f), \tag{A.9}$$

where $\mathscr{A}_{f\varepsilon}$ is an unbounded differential operator defined as $\mathscr{A}_{f\varepsilon} = \varepsilon\mathscr{A}_f$, and $\bar{f}_f(x_s, x_f) = -\frac{1}{2}\mathscr{B}_s R^{-1}(x_s)(L_{\bar{g}}V)^T + f_f(x_s, x_f)$. Note that $f_f(x_s, x_f)$ is independent of ε. Since ε is a small positive number less than unity (Assumption 1, part 3), the system of Eq. (A.9) is in the standard singularly perturbed form, with x_s being the slow states and x_f being the fast states. Introducing the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq. (A.9):

$$\frac{\partial \bar{x}_f}{\partial \tau} = \mathscr{A}_{f\varepsilon}\bar{x}_f, \tag{A.10}$$

where the bar symbol in $\bar{x}_f$, denotes that the state $\bar{x}_f$ is associated with the approximation of the fast x_f-subsystem. From the fact that $Re\,\lambda_{m+1} < 0$ and the definition of ε, we have that the above system is globally exponentially stable. Therefore there exists real numbers $k_3 \geqslant 1$, $a_3 > 0$ such that

$$\|\bar{x}_f(\tau)\|_2 \leqslant k_3 \|\bar{x}_f(0)\|_2 e^{-a_3\tau} \quad \forall\, t \geqslant 0. \tag{A.11}$$

Setting $\varepsilon = 0$ in the system of Eq. (A.9) and using the fact that the operator $\mathscr{A}_{f\varepsilon}$ is invertible, we have that $\bar{x}_f = 0$. The finite-dimensional closed-loop slow system therefore reduces to the one analyzed in part 1 of the proof where we have already shown that it is asymptotically (and locally exponentially) stable for all initial conditions within Ω and that its states are bounded. By exploiting the stability properties of the fast and slow systems, it can be shown, with the aid of calculations similar to those performed in Christofides and Teel, 1996, that the inequalities of Eqs. (A.11), (A.6), and (A.7) continue to hold, for the states of the infinite-dimensional closed-loop system, up to an arbitrarily small offset d for initial conditions in large compact subsets ($\Omega_b \subset \Omega$) where $\Omega_b = \{x_s \in \mathscr{H}_s : |x_s| \leqslant \delta_b\}$ and $\beta(\delta_b, 0) + d \leqslant \delta_s$, provided that the singular perturbation parameter ε is sufficiently small. The requirement that $\beta(\delta_b, 0) + d \leqslant \delta_s$ guarantees that the "slow" states of the closed-loop system remain within the invariant region Ω. Therefore, given the pair (δ_b, d), and given any δ_f, there exists $\varepsilon^{(1)} > 0$ such that if $\varepsilon \in (0, \varepsilon^{(1)}]$, $|x_s(0)| \leqslant \delta_b$, $\|x_f(0)\|_2 \leqslant \delta_f$, then, for all $t \geqslant 0$, the states of the closed-loop singularly perturbed system satisfy

$$|x_s(t)| \leqslant k_2|x_s(0)|e^{-a_2 t} + d,$$
$$\|x_f(t)\|_2 \leqslant k_3\|x_f(0)\|_2 e^{-a_3(t/\varepsilon)} + d. \tag{A.12}$$

The above inequalities imply that the trajectories of the closed-loop singularly perturbed system will be bounded. Furthermore, as t increases, they will be ultimately bounded with an ultimate bound that depends on d. Since d is arbitrary, we can choose it small enough such that after a sufficiently large time, say $\tilde{t}$, the trajectories of the closed-loop system are confined within a small compact neighborhood of the origin of the closed-loop system. Obviously, $\tilde{t}$ depends on both the initial condition and the desired size of the neighborhood, but is independent of ε. Choose $d = b/2$ and let $\tilde{t}$ be the smallest time such that $\max\{k_2|x_s(0)|e^{-a_2\tilde{t}}, k_3\|x_f(0)\|_2 e^{-a_3(\tilde{t}/\varepsilon)}\} \leqslant d$. Then it can be easily verified that

$$|x_s(t)| \leqslant b, \ \|x_f(t)\|_2 \leqslant b \ \forall t \geqslant \tilde{t}. \tag{A.13}$$

Recall from Eqs. (A.11) and (A.8) that both the fast and slow subsystems are exponentially stable within the ball of Eq. (A.13). It follows then from Proposition 4.1 in Christofides (2001) that there exists $\varepsilon^{(2)}$ such that if $\varepsilon \leqslant \varepsilon^{(2)}$, the singularly perturbed closed-loop system of Eq. (A.9) is locally exponentially stable and, therefore, once inside the ball of Eq. (A.13), the closed-loop trajectories converge to the origin as $t \to \infty$. To summarize, we have that given the pair of positive real numbers (δ_b, d) such that $\beta(\delta_b, 0) + d \leqslant \delta_s$, and given any positive real number δ_f, there exists $\varepsilon^* \equiv \min\{\varepsilon^{(1)}, \varepsilon^{(2)}\}$ such that if $|x_s(0)| \leqslant \delta_b$, $\|x_f(0)\|_2 \leqslant \delta_f$, and $\varepsilon \in (0, \varepsilon^*]$, the closed-loop trajectories are bounded and converge to the origin as time tends to infinity, i.e. the closed-loop system is asymptotically stable. Taking the limsup of both sides of Eq. (A.7) as $t \to \infty$, we have that

$$\limsup_{t\to\infty} |e_1^i(t)| = \limsup_{t\to\infty} |\bar{y}_c^i(t) - v_i(t)| = 0 \tag{A.14}$$

for $i = 1, \ldots, m$, where $\bar{y}_c^i(t)$ is the ith output of the closed-loop slow system (with $\varepsilon = 0$). From the exponential stability of the closed-loop system, it follows that there exists $t_b > 0$ such that, for all $t \geqslant t_b$, the following estimates hold:

$$x_s(t) = \bar{x}_s(t) + O(\varepsilon),$$
$$x_f(t) = O(\varepsilon). \tag{A.15}$$

Using the above estimates together with the notation $y_c = y_s + y_f$, $y_s = \mathscr{C}x_s$, $y_f = \mathscr{C}x_f$, it is clear that

$$\begin{aligned}\limsup_{t\to\infty} &|\bar{y}_c^i(t) - v_i(t)| \\ &= \limsup_{t\to\infty} |y_s^i(t) + y_f^i(t) - v_i(t)| \\ &= \limsup_{t\to\infty} |\bar{y}_c^i(t) - v_i(t)| + O(\varepsilon) \\ &= O(\varepsilon), \quad i = 1, \ldots, m. \end{aligned} \tag{A.16}$$

This completes the proof of the theorem. □

Proof of Theorem 2. The proof of this theorem is similar to the proof of Theorem 1. We will only highlight the differences. Under the output feedback controller of Eq. (20), the closed-loop system takes the form

$$\frac{\mathrm{d}x_{\mathrm{s}}}{\mathrm{d}t} = \mathscr{A}_{\mathrm{s}}x_{\mathrm{s}} - \tfrac{1}{2}\mathscr{B}_{\mathrm{s}}R^{-1}(\hat{x}_{\mathrm{s}})(L_{\bar{g}}V)^{\mathrm{T}} + f_{\mathrm{s}}(x_{\mathrm{s}},0) + [f_{\mathrm{s}}(x_{\mathrm{s}},x_{\mathrm{f}}) - f_{\mathrm{s}}(x_{\mathrm{s}},0)],$$

$$\varepsilon\frac{\partial x_{\mathrm{f}}}{\partial t} = \mathscr{A}_{f\varepsilon}x_{\mathrm{f}} + \varepsilon\bar{f}^{*}_{\mathrm{f}}(x_{\mathrm{s}},x_{\mathrm{f}}), \tag{A.17}$$

where $\hat{x}_{\mathrm{s}} = \mathscr{S}^{-1}y_m = x_{\mathrm{s}} + x_{\mathrm{f}}$, $\mathscr{A}_{f\varepsilon}$ is an unbounded differential operator defined as $\mathscr{A}_{f\varepsilon} = \varepsilon\mathscr{A}_{\mathrm{f}}$, and $\bar{f}^{*}_{\mathrm{f}}(x_{\mathrm{s}},x_{\mathrm{f}}) = -\frac{1}{2}\mathscr{B}_{\mathrm{s}}R(\hat{x}_{\mathrm{s}})(L_{\bar{g}}V)^{\mathrm{T}} + f_{\mathrm{f}}(x_{\mathrm{s}},x_{\mathrm{f}})$. Since ε is a small positive number less than unity (Assumption 1, part 3), the system of Eq. (A.17) is in the standard singularly perturbed form, with x_{s} being the slow states and x_{f} being the fast states. Note that the term $\bar{f}^{*}_{\mathrm{f}}(x_{\mathrm{s}},x_{\mathrm{f}})$ does not contain any terms of the form $O(1/\varepsilon)$. Therefore, by performing two-time scale decomposition, we obtain the same fast and slow systems obtained under state feedback (analyzed in the proof of Theorem 1). It follows then from the result of this theorem that given the positive real numbers (δ_b, d) as defined in Theorem 1, and given any positive real number δ_{f}, there exists $\varepsilon^{**} > 0$ such that if $\varepsilon \in (0,\varepsilon^{**}]$, $|x_{\mathrm{s}}(0)| \leqslant \delta_b$, $\|x_{\mathrm{f}}(0)\|_2 \leqslant \delta_{\mathrm{f}}$, then the infinite-dimensional closed-loop system is asymptotically stable (and locally exponentially stable) and its output satisfies Eq. (18). This completes the proof of the theorem.

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