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Computers and Chemical Engineering 28 (2004) 111–128

Computers
& Chemical
Engineering

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Coordinating feedback and switching for control of spatially distributed processes

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Abstract

This work proposes a methodology for coordinating feedback controller synthesis and actuator configuration switching in control of spatially-distributed processes, described by highly dissipative partial differential equations (PDEs) with actuator constraints. Under the assumption that the eigenspectrum of the spatial differential operator can be partitioned into a finite slow set and an infinite stable fast complement, Galerkin's method is initially used to derive a finite-dimensional system (set of ordinary differential equations (ODEs) in time) that captures the dominant dynamics of the PDE system. Using this ODE system, a stabilizing nonlinear feedback controller is designed, for a given actuator configuration, and an explicit characterization of the corresponding stability region is obtained in terms of the size of actuator constraints and the spatial locations of the actuators. Switching laws are then derived, on the basis of the stability regions, to orchestrate the transition between multiple, spatially-distributed control actuator configurations, in a way that respects actuator constraints, accommodates multiple (possibly conflicting) control objectives and guarantees closed-loop stability. Precise conditions that guarantee stability of the constrained closed-loop PDE system under switching are provided, and the proposed approach is successfully applied to the problem of constrained, fault-tolerant stabilization of unstable steady-states of a representative diffusion-reaction process and a non-isothermal tubular reactor with recycle.

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Keywords: Partial differential equations; Highly dissipative systems; Galerkin's method; Bounded nonlinear control; Actuator configuration switching; Actuator constraints; Transport-reaction processes

1. Introduction

Processes that exhibit significant spatial variations owing to the underlying physical phenomena, such as diffusion, convection, and phase-dispersion, are ubiquitous among modern-day chemical processes essential in making important industrial products. Examples include the catalytic packed-bed reactors used to convert methanol to formaldehyde, the Czochralski crystallization of high-purity crystals and the chemical vapor deposition of thin films for microelectronics manufacturing, as well as the aerosol-based production of nanoparticles used in medical applications. For these processes, the distinguishing attribute of the control problem is that it involves the regulation of distributed variables using spatially-distributed control actuators

and measurement sensors. Many of these systems are also naturally modeled by highly dissipative partial differential equation (PDE) systems, such as parabolic PDE systems (transport-reaction processes) whose spatial differential operators are characterized by an eigenspectrum that can be partitioned into a finite slow part and an infinite stable fast complement (Friedman, 1976), which implies that the dominant dynamics of these systems can be captured by finite-dimensional systems. The traditional approach to the control of parabolic PDEs involves the application of Galerkin's method to the PDE system to derive ordinary differential equation (ODE) systems that describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for the synthesis of finite-dimensional controllers (e.g. see Balas, 1979; Ray, 1981). A potential drawback of this approach is that the number of modes that should be retained to derive an ODE system that yields the desired degree of accuracy may be very large, leading to

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complex controller design and high dimensionality of the resulting controllers.

Motivated by these considerations, significant recent work on the analysis and control of distributed process systems has brought together concepts from the dynamics of infinite dimensional systems, model reduction, and nonlinear control theory, leading to the development of a general framework for the synthesis of low-order nonlinear controllers for broad classes of nonlinear distributed parameter systems that arise in the modeling of spatially distributed processes, on the basis of ODE models obtained through combination of Galerkin's method with approximate inertial manifolds (see the book Christofides, 2001 for details and references). In addition to this work, other advances on control of distributed systems have been made, including, for example, distributed controller design using generalized invariants (Palazoglu & Karakas, 2000) and concepts from passivity and thermodynamics (Alonso & Ydstie, 2001), the development of low-order control-relevant models for distributed systems (Hoo & Zheng, 2001), analysis and control of parabolic PDE systems with actuator saturation (El-Farra, Armaou & Christofides, 2003), reduced spatial order model reference adaptive control of distributed systems (Bentsman & Orlov, 2001), and state observation and adaptive control of distributed chemical reactors (Dochain, 2001).

While the above research efforts have led to the development of a number of systematic approaches for distributed controller design, an underlying theme of the available control approaches is the use of a fixed (with respect to spatial location) control actuator/measurement sensor configuration to accomplish the desired control objectives. There are many practical situations, however, where it may be desirable, and sometimes even necessary, to consider multiple, spatially-distributed actuator/sensor configurations and switch between them in a specific manner, in order to achieve the desired control objectives. An important problem where this approach is necessary is that of coping with control actuator failure. In this case, upon the detection of faults in the operating control actuator configuration, it is often necessary to switch to an alternative, well-functioning actuator configuration, with a different spatial arrangement of the control actuators, in order to preserve closed-loop stability. Switching between spatially-distributed actuator configurations in this case provides the means for fault-tolerant control. In other instances, switching between actuator configurations can be motivated by some additional performance objectives, such as the desire to optimize a given performance criterion for the control system, or accommodate inherently competing control objectives that cannot be reconciled using a single control actuator configuration. For example, when the performance index associated with the control system penalizes the

placement of control actuators at different spatial locations differently, the actuator configuration that minimizes this cost will provide the best performance. However, owing to the presence of actuator constraints, it may not be possible to use this actuator configuration (at least initially) especially if the desired initial condition is infeasible. This conflict can be resolved by considering a number of pre-determined, spatially-distributed actuator configurations and switching between them.

Motivated by the above considerations, we focus in this work on the problem of coupling feedback and switching in the control of spatially-distributed processes described by highly dissipative PDE systems with actuator constraints. The rest of the paper is organized as follows. In Section 2, we present some mathematical preliminaries to characterize the class of spatially-distributed processes considered and formulate precisely the switching problem of interest. Then in Section 3, we initially use Galerkin's method to derive an ODE system that captures the dominant dynamics of the PDE system. This ODE system is then used as the basis for the integrated synthesis, via Lyapunov techniques, of stabilizing nonlinear feedback controllers together with stabilizing switching laws that orchestrate the switching between the admissible control actuator configurations, in a way that respects the actuator constraints, accommodates inherently conflicting control objectives, and guarantees closed-loop stability at the same time. Precise conditions that guarantee stability of the constrained closed-loop PDE system under switching are provided. Finally, in Sections 4 and 5, we demonstrate, through numerical simulations, applications of the proposed methodology of coordinating feedback and switching to the problem of fault-tolerant, constrained stabilization of unstable steady-states of a typical diffusion-reaction process and a non-isothermal tubular reactor with recycle.

2. Preliminaries

2.1. Class of spatially-distributed processes

In this work, we focus on spatially-distributed processes, described by highly dissipative (in a sense made precise in Assumption 1 below) infinite-dimensional systems. This class of systems arises often in the modeling of transport-reaction processes and various classes of fluid dynamic systems. To provide a precise specification of the control problem, we consider an important class of processes that fit this category, namely quasi-linear parabolic PDE systems of the form:

$$\frac{\partial \bar{x}}{\partial t} = A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2} + w \sum_{i=1}^m b_i(z) u_i + f(\bar{x})$$

$$y_c^i = \int_{\alpha}^{\beta} c^i(z) k \bar{x}(z, t) dz \quad i = 1, \dots, m \quad (1)$$

subject to the boundary conditions:

$$C_1 \bar{x}(\alpha, t) + D_1 \frac{\partial \bar{x}}{\partial z}(\alpha, t) = R_1$$

$$C_2 \bar{x}(\beta, t) + D_2 \frac{\partial \bar{x}}{\partial z}(\beta, t) = R_2 \quad (2)$$

and the initial condition:

$$\bar{x}(z, 0) = \bar{x}_0(z) \quad (3)$$

where $\bar{x}(z, t) = [\bar{x}_1(z, t) \dots \bar{x}_n(z, t)]^T \in \mathbb{R}^n$ denotes the vector of state variables, $z \in [\alpha, \beta] \subset \mathbb{R}$ is the spatial coordinate, $t \in [0, \infty)$ is the time, $u_i \in [-u_{i,\max}, u_{i,\max}] \subset \mathbb{R}$ denotes the i th constrained manipulated input, $y_{ci} \in \mathbb{R}$ denotes the i th controlled output, $\partial \bar{x} / \partial z$, $\partial^2 \bar{x} / \partial z^2$ denote the first and second-order spatial derivatives of \bar{x} , respectively, $f(\bar{x})$ is a nonlinear vector function, w is a constant column vector, k is a constant row vector, A , B , C_1 , D_1 , C_2 , D_2 are constant matrices, R_1 , R_2 are column vectors, and $\bar{x}_0(z)$ is the initial condition. The function $b_i(z)$ is a known smooth function of z that describes how the control action, $u_i(t)$, is distributed in the finite interval $[\alpha, \beta]$ (actuator distribution function), and $c^i(z)$ is a known smooth function of z that is determined by the desired performance specifications in the interval $[\alpha, \beta]$. Whenever the control action enters the system at a single point, z_0 , with $z_0 \in [\alpha, \beta]$ (i.e. point actuation), the function $b_i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_0 - \mu, z_0 + \mu]$, where μ is a small positive real number, and zero elsewhere in $[\alpha, \beta]$.

Throughout the paper, the order of magnitude $O(\epsilon)$ and Lie derivative notations will be used. In particular, $\delta(\epsilon) = O(\epsilon)$ if there exist positive real numbers k_1 and k_2 such that: $|\delta(\epsilon)| \leq k_1 |\epsilon|$, $\forall |\epsilon| \leq k_2$. $L_f \bar{h}$ denotes the standard Lie derivative of a scalar function, $\bar{h}(\cdot)$, with respect to the vector function $f(\cdot)$, $L_f^k \bar{h}$ denotes the k th order Lie derivative and $L_g L_f^{k-1} \bar{h}$ denotes the mixed Lie derivative where $g(\cdot)$ is a vector function. Furthermore, the notation, $|\cdot|$, will be used to denote the standard Euclidean norm, while the notation $|\cdot|_2$ and $\|\cdot\|_2$ will be used to denote the L_2 -norms of vectors belonging in a finite-dimensional and an infinite-dimensional Hilbert spaces, respectively.

For a precise characterization of the class of PDE systems considered in this work, we formulate the system of Eq. (1) as an infinite-dimensional system in the Hilbert space $\mathcal{H}([\alpha, \beta]; \mathbb{R}^n)$, with \mathcal{H} being the space of sufficiently smooth n -dimensional vector functions

defined on $[\alpha, \beta]$ that satisfy the boundary conditions of Eq. (2), with inner product and norm:

$$(\omega_1, \omega_2) = \int_{\beta}^{\alpha} (\omega_1(z), \omega_2(z))_{\mathbb{R}^n} dz, \quad (4)$$

$$\|\omega_1\|_2 = (\omega_1, \omega_1)^{\frac{1}{2}}$$

where ω_1, ω_2 are two elements of $\mathcal{H}([\alpha, \beta]; \mathbb{R}^n)$ and the notation $(\cdot, \cdot)_{\mathbb{R}^n}$ denotes the standard inner product in \mathbb{R}^n . Defining the state function x on $\mathcal{H}([\alpha, \beta]; \mathbb{R}^n)$ as:

$$x(t) = \bar{x}(z, t), \quad t > 0, \quad z \in [\alpha, \beta], \quad (5)$$

the operator \mathcal{A} in $\mathcal{H}([\alpha, \beta]; \mathbb{R}^n)$ as:

$$\mathcal{A}x = A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2},$$

$$x \in \mathcal{D}(\mathcal{A})$$

$$= \left\{ x \in \mathcal{H}([\alpha, \beta]; \mathbb{R}^n) : C_1 \bar{x}(\alpha, t) + D_1 \frac{\partial \bar{x}}{\partial z}(\alpha, t) = R_1, \right. \\ \left. C_2 \bar{x}(\beta, t) + D_2 \frac{\partial \bar{x}}{\partial z}(\beta, t) = R_2 \right\}$$

and the input and controlled output operators as:

$$\mathcal{B}u = w \sum_{i=1}^m b_i u_i, \quad \mathcal{C}x = (c, kx) \quad (6)$$

the system of Eqs. (1)–(3) takes the form:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + f(x), \quad x(0) = x_0.$$

$$y_c = \mathcal{C}x \quad (7)$$

where $f(x(t)) = f(\bar{x}(z, t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear terms $f(x)$ are locally Lipschitz with respect to their arguments and satisfy $f(0) = 0$. For \mathcal{A} , the eigenvalue problem is defined as:

$$\mathcal{A}\phi_j = \lambda_j \phi_j, \quad j = 1, \dots, \infty \quad (8)$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction. The eigenspectrum of \mathcal{A} , $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of \mathcal{A} , i.e. $\sigma(\mathcal{A}) = \{\lambda_1, \lambda_2, \dots\}$. **Assumption 1** that follows states that the eigenspectrum of \mathcal{A} can be partitioned into a finite part consisting of m slow eigenvalues and a stable infinite complement containing the remaining fast eigenvalues, and that the separation between the slow and fast eigenvalues of \mathcal{A} is large.

Assumption 1.

- 1) $Re\{\lambda_1\} \geq Re\{\lambda_2\} \geq \dots \geq Re\{\lambda_j\} \geq \dots$, where $Re\{\lambda_j\}$ denotes the real part of λ_j .
- 2) $\sigma(\mathcal{A})$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) + \sigma_2(\mathcal{A})$, where $\sigma_1(\mathcal{A})$ consists of the first m (with m finite) eigenvalues, i.e. $\sigma_1(\mathcal{A}) = \{\lambda_1, \dots, \lambda_m\}$, and $|Re\{\lambda_m\}| / |Re\{\lambda_m\}| = O(1)$.

- 3) $Re\{\lambda_{m+1}\} < 0$ and $|Re\{\lambda_m\}|/|Re\{\lambda_{m+1}\}| = O(\epsilon)$ where $\epsilon < 1$ is a small positive number.

The assumption of finite number of unstable eigenvalues is always satisfied for parabolic PDE systems (Friedman, 1976), while the assumption of discrete eigenspectrum and the assumption of existence of only a few dominant modes that describe the dynamics of the parabolic PDE system are usually satisfied by the majority of diffusion–convection–reaction processes (see the examples in Sections 4 and 5).

2.2. Problem formulation and solution overview

Consider the spatially-distributed process of Eq. (1) and assume that N (where N is finite) spatially-distinct control actuator configurations are available (installed) for feedback control purposes. Each configuration is characterized by: (1) a unique spatial arrangement of the actuators, \bar{z}^k , $k = 1, \dots, N$, where \bar{z}^k is an m -dimensional vector whose components represent the corresponding spatial locations of the actuators associated with the k th control actuator configuration, and (2) hard actuator constraints, $-u_{i,\max}^k \leq u_i^k \leq u_{i,\max}^k$. The superscript k in \bar{z}^k is a discrete index that denotes which of the N actuator configurations is active at any given time. Only one configuration can be engaged for control at any time instance and, to ensure controllability of the system, we allow only a finite number of configuration switchings over any finite time-interval. The problem under consideration is how to coordinate switching between the N different control actuator configurations in a way that respects actuator constraints and guarantees closed-loop stability. To address this problem, we formulate the following three objectives:

- 1) Initially, a model reduction scheme based on Galerkin's method is used to derive a nonlinear finite-dimensional system that accurately reproduces the solutions and dynamics of the parabolic PDE system of Eq. (1).
- 2) Next, the finite-dimensional approximation of the system of Eq. (1) is used as the basis for the synthesis, via Lyapunov-based control techniques, of bounded nonlinear feedback controllers of the general form:

$$u^k = p(x_s, \bar{v}, u_{\max}^k, \bar{z}^k) \quad (9)$$

that enforce asymptotic stability and reference-input tracking in the constrained closed-loop system and provide an explicit characterization of the constrained stability region, associated with each control actuator configuration. Referring to Eq. (9), $p(\cdot)$ is a bounded nonlinear vector function (i.e. $|u^k| \leq u_{\max}^k$, where $|\cdot|$ is the standard Euclidean

norm and u_{\max}^k is the maximum norm of the vector of manipulated inputs allowed by the constraints associated with the k th actuator configuration), x_s is the vector of slow states, and \bar{v} is a generalized reference input (which is assumed to be a smooth function of time).

- 3) Finally, a set of switching rules is derived to orchestrate the transition between the available control actuator configurations, and an upper bound on the separation between the slow and fast eigenvalues, which guarantees stability of the switched closed-loop infinite-dimensional system, is computed. The switching laws determine which actuator configuration can be activated at a given moment, i.e. the value of the index k :

$$k(t) = \phi(x_s(t), t) \quad (10)$$

which can be viewed as a piecewise-constant function of time that depends on the slow states.

3. Hybrid control of spatially-distributed processes

3.1. Galerkin's method

In this section, we apply standard Galerkin's method to the system of Eq. (7) to derive an approximate finite-dimensional system. Let \mathcal{H}_s , \mathcal{H}_f be modal subspaces of \mathcal{A} , defined as $\mathcal{H}_s = \text{span}\{\phi_1, \phi_2, \dots, \phi_m\}$ and $\mathcal{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \dots\}$ (the existence of \mathcal{H}_s , \mathcal{H}_f follows from Assumption 1). Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq. (7) can be decomposed as:

$$x = x_s + x_f = P_s x + P_f x \quad (11)$$

Applying P_s and P_f to the system of Eq. (7) and using the above decomposition for x , the system of Eq. (7) can be equivalently written in the following form:

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f)$$

$$\frac{\partial x_f}{\partial t} = \mathcal{A}_f x_f + \mathcal{B}_f u + f_f(x_s, x_f)$$

$$y_c = \mathcal{C}_s x_s + \mathcal{C}_f x_f$$

$$x_s(0) = P_s x(0) = P_s x_0, \quad x_f(0) = P_f x(0) = P_f x_0 \quad (12)$$

where $\mathcal{A}_s = P_s \mathcal{A}$, $\mathcal{B}_s = P_s \mathcal{B}$, $f_s = P_s f$, $\mathcal{A}_f = P_f \mathcal{A}$, $\mathcal{B}_f = P_f \mathcal{B}$ and $f_f = P_f f$ and the partial derivative notation in $\partial x_f / \partial t$ is used to denote that the state x_f belongs in an infinite-dimensional space. In the above system, \mathcal{A}_s is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = \text{diag}\{\lambda_j\}$, $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are Lipschitz vector

functions, and \mathcal{A}_f is an unbounded differential operator which is exponentially stable (following from part 3 of Assumption 1 and the selection of $\mathcal{H}_s, \mathcal{H}_f$). Neglecting the fast and stable infinite-dimensional x_f -subsystem in the system of Eq. (12), the following m -dimensional slow system is obtained:

$$\frac{d\bar{x}_s}{dt} = \mathcal{A}_s \bar{x}_s + \mathcal{B}_s u + f_s(\bar{x}_s, 0)$$

$$\bar{y}_c = \mathcal{C} \bar{x}_s \tag{13}$$

where the bar symbol in \bar{x}_s, \bar{y}_c denotes that these variables are associated with a finite-dimensional system.

Remark 1. We note that the above model reduction procedure which led to the approximate ODE system of Eq. (13) can also be used when empirical eigenfunctions of the system of Eq. (1), computed through Karhunen–Loeve (KL) expansion, are used as basis functions in \mathcal{H}_s and \mathcal{H}_f instead of the eigenfunctions of \mathcal{A} . For approaches on how to compute empirical eigenfunctions through KL expansion, the reader is referred to Atwell and King (2001).

Remark 2. Although the finite-dimensional system of Eq. (13) was obtained through standard Galerkin’s method, the results of this paper can be generalized to the case where the finite-dimensional approximation of the system of Eq. (12) is obtained through combination of Galerkin’s method with approximate inertial manifolds (see Christofides, 2001). This approach can be used to further reduce the dimension of the system of Eq. (13) and ensure that it is of an appropriately low-order suitable for controller design and analysis.

3.2. Coordinating feedback and switching

Having obtained a finite-dimensional model that describes the dominant dynamics of the infinite-dimensional system, we proceed in this section to describe the proposed procedure for designing the hybrid control system. To this end, we consider the equivalent representation of the slow system of Eq. (13) in terms of the evolution of the amplitudes of the eigenmodes. This ODE system is given by:

$$\dot{a}_s(t) = Fa_s(t) + G(z^k)u^k(t) + d(a_s(t))$$

$$\tilde{y}_c(t) = Ca_s(t) \tag{14}$$

where $a_s(t) = [a_1(t) \dots a_m(t)]^T \in \mathbb{R}^m$, $a_i(t)$ is the amplitude of the i th eigenmode, $\bar{x}_s(t) = \sum_{j=1}^m a_j(t)\phi_j$, $(\bar{x}_s(t), \phi_j) = a_j(t)(\phi_j, \phi_j)$, F is an $m \times m$ diagonal matrix of the form $F = \text{diag}\{\lambda_j\}$, G is an $m \times m$ matrix (when point actuation is used, the (i, j) element of G is given by

$\phi_i(z_j^k)$), $d(\cdot) = [d_1(\cdot) \dots d_m(\cdot)]^T$ is a vector function and $(f_s(\cdot), \phi_j) = d_j(\cdot)(\phi_j, \phi_j)$, $\tilde{y}_c(t) = [\tilde{y}_c^1(t) \dots \tilde{y}_c^m(t)]^T \in \mathbb{R}^m$, \tilde{y}_c^i is the i th controlled output of the finite-dimensional slow system, C is a constant $m \times m$ matrix. Referring to the system of Eq. (14), we define the relative order of the output \tilde{y}_c^i with respect to the vector of manipulated inputs u as the smallest integer r_i for which:

$$[L_{\tilde{g}_1} L_{\tilde{f}}^{r_i-1} \tilde{h}_i(a_s) \dots L_{\tilde{g}_m} L_{\tilde{f}}^{r_i-1} \tilde{h}_i(a_s)] \neq [0 \dots 0] \tag{15}$$

where \tilde{g}_i is the i th column of the matrix G , $\tilde{f}(a_s) = Fa_s + d(a_s)$, and $\tilde{y}_c^i = \tilde{h}_i(a_s)$ is the i th component of the vector Ca_s , or $r_i = \infty$ if such an integer does not exist. We also define the characteristic matrix:

$$\tilde{C}(a_s) = \begin{bmatrix} L_{\tilde{g}_1} L_{\tilde{f}}^{r_1-1} \tilde{h}_1(a_s) & \dots & L_{\tilde{g}_m} L_{\tilde{f}}^{r_1-1} \tilde{h}_1(a_s) \\ L_{\tilde{g}_1} L_{\tilde{f}}^{r_2-1} \tilde{h}_2(a_s) & \dots & L_{\tilde{g}_m} L_{\tilde{f}}^{r_2-1} \tilde{h}_2(a_s) \\ \vdots & \dots & \vdots \\ L_{\tilde{g}_1} L_{\tilde{f}}^{r_m-1} \tilde{h}_m(a_s) & \dots & L_{\tilde{g}_m} L_{\tilde{f}}^{r_m-1} \tilde{h}_m(a_s) \end{bmatrix} \tag{16}$$

which, to simplify our development, is assumed to be nonsingular uniformly in a_s . This assumption can be relaxed if dynamic state feedback is used instead of static state feedback (see Isidori, 1989 for details). To proceed with controller synthesis, and under the assumption that the relative degree is well-defined, we initially transform the system of Eq. (14), by means of an invertible coordinate change of the form $a_s = T^{-1}(\zeta, \eta)$, into the following partially linear form:

$$\begin{aligned} \dot{\zeta}_1^{(i)} &= \zeta_2^{(i)} \\ &\vdots \\ \dot{\zeta}_{r_i-1}^{(i)} &= \zeta_{r_i}^{(i)} \\ \dot{\zeta}_{r_i}^{(i)} &= L_{\tilde{f}}^{r_i} \tilde{h}_i(a_s) + \sum_{j=1}^m L_{\tilde{g}_j} L_{\tilde{f}}^{r_i-1} \tilde{h}_i(a_s) u_j^k \\ \dot{\eta}_1 &= \Psi_1(\zeta, \eta) \\ &\vdots \\ \dot{\eta}_{m-\sum_i r_i} &= \Psi_{m-\sum_i r_i}(\zeta, \eta) \\ \tilde{y}_{ci} &= \zeta_1^{(i)}, \quad i = 1, \dots, m, \quad k = 1, \dots, N \end{aligned} \tag{17}$$

where $\zeta_j^{(i)} = L_{\tilde{f}}^{j-1} \tilde{h}_i(a_s)$, $\zeta = [\zeta^{(1)T} \dots \zeta^{(m)T}]^T$, $\eta = [\eta_1 \dots \eta_{m-\sum_i r_i}]^T$. Defining the tracking error variables, $e_j^{(i)} = \zeta_j^{(i)} - v_j^{(i)}$, and introducing the vector notation $e^{(i)} = [e_1^{(i)} e_2^{(i)} \dots e_{r_i}^{(i)}]^T$, $e = [e^{(1)T} e^{(2)T} \dots e^{(m)T}]^T$, where $i = 1, \dots, m, j = 1, \dots, r_i$, the system of Eq. (17) can be cast in the following more compact form:

$$\begin{aligned} \dot{e} &= \Lambda e + \Gamma[l(e, \eta, \bar{v}) + \tilde{C}(e, \eta, \bar{v})u^k] \\ \dot{\eta} &= \Psi(e, \eta, \bar{v}) \end{aligned} \tag{18}$$

where $\Lambda = \text{diag}\{\Lambda_i\}$ is an $(\sum_i r_i \times \sum_i r_i)$ block diagonal constant matrix whose constituent blocks are $r_i \times r_i$ matrices of the form:

$$\Lambda_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (19)$$

Γ is an $(\Sigma_{r_i} \times m)$ constant matrix, $l(\cdot)$ is an $(m \times 1)$ smooth vector function. The specific forms of Γ and $l(\cdot)$ are omitted for brevity. Finally, we define $\tilde{f}(e, \eta, \bar{v}) = \Lambda e + \Gamma l(e, \eta, \bar{v})$ and denote by \bar{g}_i the i th column of the matrix $\bar{G} = \Gamma \bar{C}$.

Assumption 2. The η -subsystem of Eq. (18) is input-to-state stable with respect to e , uniformly in \bar{v} , and locally exponentially stable when $e = 0$.

We are now ready to state the main result of this work. Theorem 1 below provides both the state feedback control law (see the discussion in Remark 11 and the example in Section 4 for output feedback controller design and implementation) as well as the necessary switching laws, and states precise conditions that guarantee closed-loop stability and asymptotic reference-input tracking in the switched closed-loop system. The proof is given in Appendix A.

Theorem 1.

1) Consider the system of Eq. (14) and its transformation of Eq. (18), for which Assumption 2 holds, under the feedback control law:

$$u^k = -r(a_s, u_{\max}^k, \bar{z}^k)(L_{\bar{g}}^k V)^T(\bar{z}^k) \quad (20)$$

where

$$r(a_s, u_{\max}^k, \bar{z}^k) = \frac{L_f^* V + \sqrt{(L_f^* V)^2 + (u_{\max}^k |(L_{\bar{g}}^k V)^T(\bar{z}^k)|)^4}}{|(L_{\bar{g}}^k V)^T(\bar{z}^k)|^2 [1 + \sqrt{1 + (u_{\max}^k |(L_{\bar{g}}^k V)^T(\bar{z}^k)|)^2}]} \quad (21)$$

$\bar{z}^k = [z_1^k z_2^k \dots z_m^k]^T$, $k = 1, \dots, N$, $L_f^* V = L_f V + \rho |e|^2$, $\rho > 0$, $L_{\bar{g}}^k V$ is a row vector of the form $[L_{\bar{g}_1}^k V \dots L_{\bar{g}_m}^k V]$, $V = e^T \Phi e$, Φ , is a positive definite matrix that satisfies the Riccati inequality $\Lambda^T \Phi + \Phi \Lambda - \Phi \Gamma \Gamma^T \Phi < 0$. Let δ_s^k be a positive real number such that the compact set $\Omega(u_{\max}^k, \bar{z}^k) = \{a_s \in \mathbb{R}^m : a_s^T \Phi a_s \leq \delta_s^k\}$ is the largest invariant set embedded within the region described by the following inequality:

$$L_f^* V \leq u_{\max}^k |(L_{\bar{g}}^k V)^T(\bar{z}^k)| \quad (22)$$

Without loss of generality, assume that $\bar{z}^{k(0)} = \bar{z}^1$ and $a_s(0) \in \Omega(u_{\max}^1, \bar{z}^1)$. If, at any given time T , the condition:

$$a_s(T) \in \Omega(u_{\max}^j, \bar{z}^j) \quad (23)$$

holds, for some $j \in \{1, \dots, N\}$, then setting $\bar{z}^{k(t)} = \bar{z}^j$ for all $t > T$ guarantees that the switched closed-loop system is asymptotically stable.

2) Consider the parabolic PDE system of Eq. (1) under the control law of Eqs. (20) and (21) and the switching law of Eq. (23). Let $\Theta_s^1 := \{x_s(0) \in \mathcal{H}_s : |x_s(0)|_2 \leq \bar{\delta}_s^1\}$ be the set of all $x_s(0)$ for which $a_s^T(0) \Phi a_s^T(0) \leq \delta_s^1$. Then given any pair of positive real numbers $(d, \bar{\delta}_b^1)$ such that $\bar{\delta}_b^1 + d \leq \bar{\delta}_s^1$, and given any positive real number δ_f , there exists $\epsilon^* > 0$ such that if $\epsilon \in (0, \epsilon^*]$, $|x_s(0)|_2 \leq \bar{\delta}_b^1$, $\|x_f(0)\|_2 \leq \delta_f$:

2.1 The infinite-dimensional closed-loop system is asymptotically (and locally exponentially) stable.

2.2 The outputs of the closed-loop system satisfy a relation of the form:

$$\limsup_{t \rightarrow \infty} |y_c^j(t) - v_i(t)| = O(\epsilon) \quad (24)$$

Remark 3. Owing to the dependence of the input operator, \mathcal{B}_s , in Eq. (13) on the spatial locations of the control actuators (through the actuator distribution functions $b_i(z)$), the inequality of Eq. (22) is parameterized by the actuator locations and can, therefore, be used to explicitly identify the admissible control actuator configurations (\bar{z}^k) for which stability of the constrained closed-loop system is guaranteed under constraints. For a given actuator configuration (fixed u_{\max}^k and \bar{z}^k), the inequality of Eq. (22) describes a region of the state-space where the control action satisfies the constraints and the time-derivative of the Lyapunov function is negative-definite, along the trajectories of the finite-dimensional closed-loop slow system. Therefore, by computing an invariant set, $\Omega(u_{\max}^k, \bar{z}^k)$, (preferably the largest) within this region, we obtain an estimate of the stability region associated with each control actuator configuration (see chapter 4 in Khalil, 1996; El-Farra & Christofides, 2001 for how to compute this estimate). The requirement that the set $\Omega(u_{\max}^k, \bar{z}^k)$ be invariant is needed to ensure that the closed-loop trajectories do not leave the region described by Eq. (22), under a given actuator configuration.

Remark 4. Due to the large separation between the slow and fast eigenvalues of the spatial differential operator of the PDE system of Eq. (1), the characterization (or estimate) of the constrained stability region obtained for the approximate, finite-dimensional closed-loop slow subsystem remains practically preserved for the slow subsystem in the infinite-dimensional closed-loop system, in the sense that given any initial slow state that belongs to any compact subset of $\Omega(u_{\max}^k, \bar{z}^k)$, and

given any initial fast state, there always exists ϵ sufficiently small such that the controller of Eqs. (20) and (21) continues to enforce stability and reference-input tracking in the constrained infinite-dimensional closed-loop system. In the asymptotic limit (as $\epsilon \rightarrow 0$), the stability region of the slow subsystem is fully recovered. It should be noted here that this result is fundamentally different from its counter part in the linear case (El-Farra & Christofides, 2003) where the stability region of the closed-loop slow subsystem is exactly preserved (i.e. the slow stability region is the same for the finite- and infinite-dimensional closed-loop systems) due to the fact that the evolution of the slow subsystem in the linear case is completely independent of the fast subsystem. As discussed in El-Farra and Christofides (2003), this decoupling property is a consequence of: (1) the absence of the nonlinear terms, $f_s(x_s, x_f)$, $f_f(x_s, x_f)$ (see Eq. (12)) which introduce an interconnection between the two subsystem, and (2) the fact that the control input is only a function of the slow states.

Remark 5. The switching law of Eq. (23) orchestrates the transition between the N actuator configurations in a way that respects their constraints and guards against any potential instability that may arise due to switching. The basic problem here owes to the limitations imposed by constraints on the set of feasible initial conditions that can be used, for a given actuator configuration, to stabilize the closed-loop system. Different actuator configurations possess different stability regions and, depending on where the state is at a given moment in time, a switch from one configuration to another may land the state outside the stability region of the target configuration, thus leading to instability. To guard against this possibility, the switching law of Eq. (23) monitors the evolution of the slow state in time (fast states can be neglected since ϵ is sufficiently small) and allows switching to take place only when the state is within the stability region of the desired actuator configuration. This condition determines, implicitly, the earliest time for which switching is safe.

Remark 6. The results of Theorem 1 can be useful for dealing with a variety of distributed control problems. One example is the problem of actuator failure. In this case, it is often necessary to switch from the failed actuator configuration to an alternative, well-functioning, configuration that may already be installed on the process, in order to preserve closed-loop stability. The switching scheme of Theorem 1 can be used in this case to determine when the switching can take place and which of the alternative actuator configurations can be used (see Sections 4 and 5 for simulation examples). Note, however, that in other control problems, switching may not be needed to maintain closed-loop stability,

but is motivated, instead, by the desire to achieve some higher objective, such as the desire to optimize a given performance criterion for the control system or to accommodate conflicting control objectives that cannot be reconciled using a single control actuator configuration. For example, when the performance index associated with the control system includes penalty on the actuator location, the actuator configuration (or locations) that minimizes this cost provides the best performance. However, owing to the presence of different constraints for actuators placed at different locations, it may not be possible to use this actuator configuration if the desired initial condition lies outside its stability region. To resolve this conflict, one may initially use another actuator configuration, for which the initial condition is admissible, and later switch to the desired configuration, that provides better performance, when the state of the system enters its stability region.

Remark 7. Note that it is possible for more than one control actuator configuration (z^k) to satisfy the switching rule given in Eq. (23) at a given time. This can happen when the slow state lies in the intersection of several stability regions. In this case, Theorem 1 only guarantees that a switch from the current configuration to any of these configurations is safe, as far as closed-loop stability is concerned. The decision to select a particular configuration should then be made on the basis of some other objective as discussed in Remark 6. For example, in the event that the operating actuator fails and it becomes necessary to switch to some backup configuration, then if more than one backup configuration is feasible, one might choose the one that utilizes the smallest control effort or minimizes some other performance criterion (see Section 5 for an example) in order to minimize any performance losses resulting from the failure of the primary actuator configuration. In Theorem 1, however, the selection of the N actuator configurations is assumed to be pre-determined and not necessarily based on any optimality criteria. For some results on the problem of optimal placement of control actuators and measurement sensors for distributed systems, the reader is referred to Demetriou (1999), Antoniadis and Christofides (2001a, 2002).

Remark 8. Owing to the spatially distributed nature of the control problem, the switching scheme proposed in this paper is conceptually different from the classical switching schemes studied in the context of hybrid controller design for lumped-parameter systems. For lumped systems, the idea is to achieve the specified control objectives by switching between members of an a priori specified family of feedback functions, whereas the switching scheme presented here seeks to achieve the desired control objectives by switching between members of an a priori specified family of control actuator

configurations (or locations), using the same feedback function (parameterized by \bar{z}^k). For each actuator configuration, the temporal evolution of the state is governed by a different set of differential equations which define a particular mode. The closed-loop system can, therefore, be viewed as multi-modal even though the unforced dynamics themselves are not hybrid in nature. The area of switched and hybrid controller design for lumped-parameter systems is currently an active area of research (e.g. see El-Farra & Christofides, 2002 and the references therein for results in this area).

Remark 9. The control law of Eqs. (20) and (21) involves a modification of the controller design originally proposed in Lin and Sontag (1991), by including the term $-\rho|e|^2$ which is made to ensure that the control law enforces exponential stability in the closed-loop PDE system (see the proof of Theorem 1 in Appendix A). With $\rho = 0$, it can be shown that the control law of Eqs. (20) and (21) enforces only asymptotic closed-loop stability. It is, therefore, preferable to set $\rho > 0$ and achieve exponential stability because of its robustness to bounded perturbations, which are always present in most practical applications (Balas, 1991). We also note that even though a single (common) Lyapunov function is used to design the controller and characterize the stability region for all actuator configurations, it is possible to employ multiple Lyapunov functions (one for each configuration) instead, to achieve greater flexibility in switching (e.g. larger stability regions). In this case, however, additional switching rules, that impose restrictions on the growth of each Lyapunov function at transition times, are needed to ensure closed-loop stability (see El-Farra & Christofides, 2002 for further details on this issue).

Remark 10. Referring to the practical applications of the result of Theorem 1, one must initially identify the candidate control actuator configurations to be used. A single quadratic Lyapunov function is then used to: (1) synthesize, via Eqs. (20) and (21), the necessary bounded nonlinear controller, and (2) construct, with the aid of Eq. (22), the region of guaranteed closed-loop stability $\Omega(\nu_{\max}^k, \bar{z}^k)$ associated with each actuator configuration, on the basis of the finite-dimensional model that captures the dominant dynamics of the distributed system. Once this is done, one can proceed with the implementation of the control strategy by initializing the closed-loop system within the stability region of the desired initial actuator configuration and implementing the feedback controller using this configuration. Then, the switching law of Eq. (23) is checked on-line to determine if it is possible to switch to another actuator configuration at any given time. If the condition is satisfied for some configuration, then this configuration can be safely activated. Otherwise, the current actuator

configuration remains active. Note that when switching is necessitated by the failure of the operating actuator, then if none of the backup configurations is feasible at failure time (due to tight constraints), then a process shutdown may be unavoidable. This possibility can be minimized by increasing the number of backup actuators and their capacities.

Remark 11. The nonlinear controller of Eqs. (20) and (21) was derived under the assumption that measurements of the state variables $\bar{x}(z, t)$ are available at all positions and times. From a practical point of view, measurements of the state variables are available only at a finite number of spatial positions, while in addition, there are many applications where measurements of state variables cannot be obtained on-line. To deal with this problem, an output feedback controller that uses only measurements of the outputs, y_m , can be constructed under the assumption that the number of measurements is equal to the number of slow modes and that the inverse of the measurement operator \mathcal{S} exists, so that $\hat{x}_s = \mathcal{S}^{-1}y_m$ where \hat{x}_s is the estimate of the slow state and y_m is the measured output. The invertibility of the measurement operator can be ensured by appropriate choice of the location of the measurement sensors. The synthesis of the output feedback controller can be carried out by combining the state feedback controller of Eqs. (20) and (21) with a procedure proposed in Christofides (2001) for obtaining estimates for the states of the approximate ODE model of Eq. (13) from the measurements. While the estimation error leads to some loss in the size of the stability region obtained under state feedback, this loss can be made small by increasing the order of the ODE approximation and including more measurements. This approach allows us to asymptotically (as $\epsilon \rightarrow 0$) recover the stability region associated with each control actuator configuration.

Remark 12. An a priori estimate of the minimum necessary separation between the slow and fast eigenvalues (i.e. a value for ϵ^*) can, in principle, be extracted from the stability proof via singular perturbation techniques. However, as is the case with most singular perturbation techniques, this estimate is typically conservative and, therefore, it is useful to check its appropriateness through computer simulations. To this end, an initial value for m (the number of slow eigenmodes) is chosen by the user (thus fixing the value of ϵ) and Galerkin's method is applied to derive a finite-dimensional ODE system that describes the dynamics of these m slow modes, which is then used to design the feedback and switching laws of Theorem 1 which, in turn, are applied to the parabolic PDE system. Closed-loop stability is then checked through computer simulations and, if not achieved, the initial choice for m is

revised, by including more slow eigenvalues and increasing the order of the ODE approximation.

4. Application to a diffusion-reaction process

In this section, we illustrate through computer simulations how the concept of coupling feedback and switching, proposed in Theorem 1, can be used to deal with the problem of control actuator failure in control of a diffusion-reaction process. To this end, consider a long, thin rod in a reactor. The reactor is fed with pure species A and a zeroth order exothermic catalytic reaction of the form $A \rightarrow B$ takes place on the rod. Since the reaction is exothermic, a cooling medium in contact with the rod is used for cooling. Under standard assumptions, the spatiotemporal evolution of the dimensionless rod temperature is described by the following parabolic PDE:

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T e^{-\frac{\gamma}{1+\bar{x}}} + \beta_U (b(z)u(t) - \bar{x}) - \beta_T e^{-\gamma}$$

subject to the boundary and initial conditions:

$$\bar{x}(0, t) = 0, \quad \bar{x}(\pi, t) = 0, \quad \bar{x}(z, 0) = \bar{x}_0(z) \quad (25)$$

where \bar{x} denotes the dimensionless temperature in the reactor, β_T denotes a dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, $u(t)$ denotes the vector of manipulated inputs and $b(z)$ the vector of the corresponding actuator locations. The following typical values are given to the process parameters: $\beta_T = 50.0$, $\beta_U = 2.0$, $\gamma = 4.0$.

For the above values, it was verified that the operating steady-state $\bar{x}(z, t) = 0$ is an unstable one. The control objective, therefore, is to stabilize the rod temperature profile at this unstable steady-state by manipulating the temperature of the cooling medium, subject to hard constraints on the manipulated input. To achieve this objective, the controlled output is defined as:

$$y_c(t) = \int_0^\pi \sqrt{\frac{2}{\pi}} \sin(z)x(z, t) dz \quad (26)$$

The eigenvalue problem for the spatial differential operator of the process can be solved analytically and its solution is:

$$\lambda_j = -j^2, \quad \phi_j(z) = \sqrt{\frac{2}{\pi}} \sin(jz), \quad j = 1, \dots, \infty \quad (27)$$

For this system, we consider the first eigenvalue as the dominant one and use standard Galerkin's method to derive an ODE that describes the temporal evolution of the amplitude, $a_1(t)$, of the first eigenmode, where

$x_s(t) = a_1(t)\phi_1(z)$. This ODE is used for the synthesis of the controller, using Eqs. (20) and (21), which is then implemented on a 30th order Galerkin discretization of the parabolic PDE system (higher order discretizations led to identical results).

In order to demonstrate the utility of the switching scheme, we consider the following two problems where switching is needed to preserve closed-loop stability. In the first problem, two point control actuators A and B, placed at two different locations $z_A = 0.5\pi$ and $z_B = 0.12\pi$, respectively, are assumed to be available for stabilization. Both actuators have the same magnitude constraints of $u_{\max} = 1.4$ but only one actuator is to be used for control at any given moment. The question to be addressed in this problem is: when is it possible to switch between the two actuators, to preserve closed-loop stability, in case the operating actuator fails. In the second problem, we assume that three point control actuators, A, B, and C, located at $z_A = 0.5\pi$, $z_B = 0.23\pi$, and $z_C = 0.36\pi$, respectively, are available for stabilization. The three actuators have different constraints of $u_{\max}^A = u_{\max}^B = 2.5$ and $u_{\max}^C = 0.5$. Again, only one actuator is to be active at any given moment. The question in this problem is how to decide which of the two backup actuators can be used to maintain stability once the third actuator fails. The first problem basically deals with the issue of identifying the appropriate switching times that ensure fault-tolerance for a given actuator re-configuration policy, while the second problem addresses the issue of how to select the appropriate actuator re-configuration policy among a family of possible choices.

We proceed with the state feedback results first. For the first problem, we initially use Eq. (22) with $\rho = 0.2$ to compute the stability region as a function of control actuator location. To simplify the presentation of our results, we plot in Fig. 1(a) the variation of the set of admissible initial conditions for the amplitude, $a_1(0)$, of the first eigenmode, with actuator location (note that $x_s(0) = a_1(0)\phi_1(z)$). From this figure, it is clear that for the initial condition $a_1(0) = 0.78$, only actuator A can be used initially since the initial condition is outside the stability region of actuator B. Fig. 1(b–c) depict, respectively, the closed-loop temperature and manipulated input profiles corresponding to this initial condition, using actuator A. Clearly, the controller successfully stabilizes the temperature profile at the desired steady-state. Now, suppose that sometime after process startup, a fault is detected in actuator A. From Fig. 1(a), we conclude that switching to actuator B may or may not solve the problem depending on when the failure of A actually occurs. For example, consider the case when actuator A fails at $t = 1.25$. By tracking the slow state a_1 in time, we find that $a_1(1.25) = 0.5$, which is outside the stability region of actuator B, and therefore, a switch to actuator B is not expected to preserve closed-

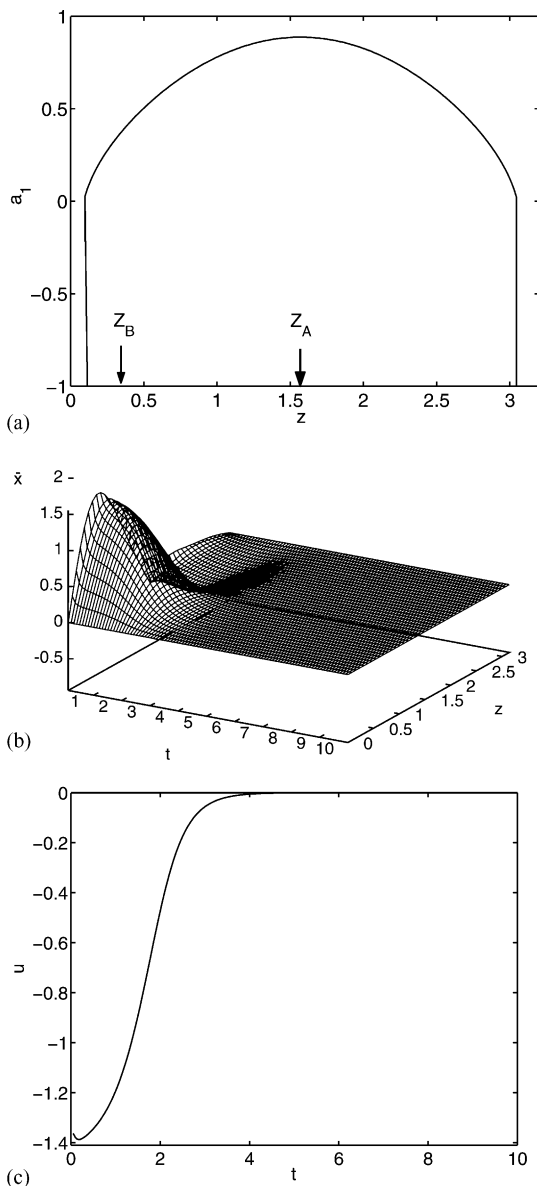


Fig. 1. (a) Stability region as a function of actuator location for $u_{\max} = 1.4$, (b) closed-loop dimensionless temperature profile (state feedback) for $a_1(0) = 0.78$ when actuator A ($u_{\max} = 1.4$, $z_A = 0.5\pi$) is used without switching, (c) corresponding manipulated input profile.

loop stability. This is confirmed by the closed-loop temperature profile in Fig. 2(a) and the corresponding manipulated input profile in Fig. 2(c) (solid profile). Note that the input stays saturated for all times after switching. Suppose now that actuator A fails at $t = 1.75$ instead. By tracking the slow state, we find that $a_1(1.75) = 0.29$, which is well within the stability region of actuator B, and, therefore, according to the switching scheme of Theorem 1, closed-loop stability can be preserved by activating actuator B. This is illustrated in Figs. 2b and 2c (dashed profile) which show that the controller successfully stabilizes the closed-loop temperature when actuator B is activated at the time that

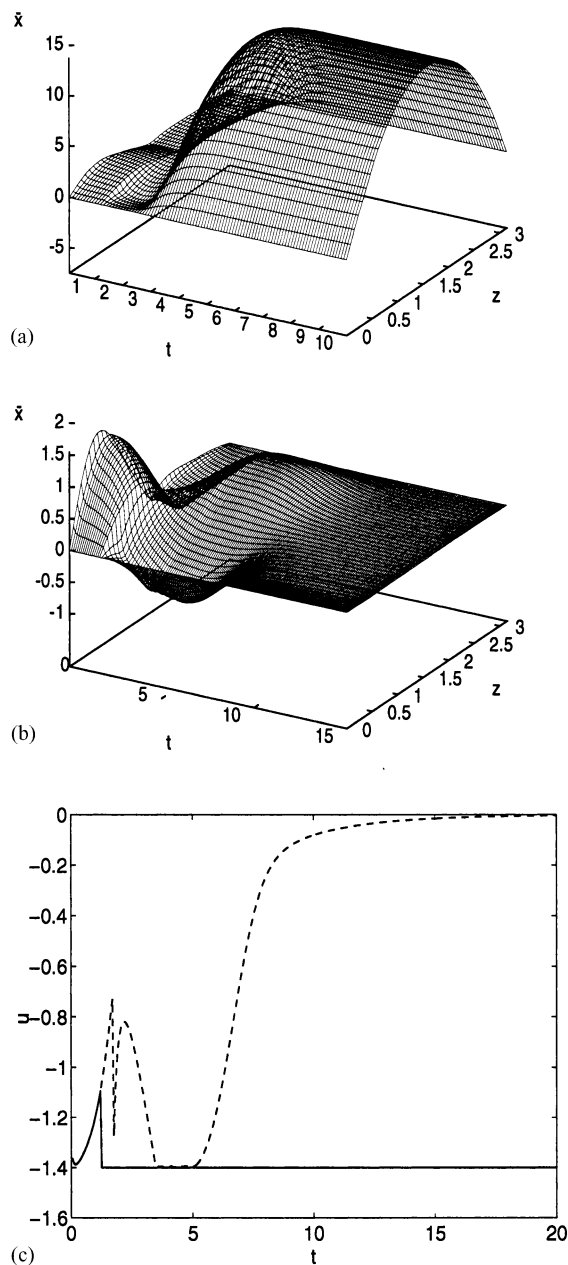


Fig. 2. (a) Closed-loop dimensionless temperature profile (state feedback) for $a_1(0) = 0.78$ when actuator A fails at $t = 1.25$ and actuator B ($u_{\max} = 1.4$, $z_B = 0.12\pi$) is activated, (b) closed-loop dimensionless temperature profile (state feedback) for $a_1(0) = 0.78$ when actuator A fails at $t = 1.75$ and actuator B is activated, (c) corresponding manipulated input profiles for case a (solid) and case b (dashed).

actuator A fails. From these results we conclude that a fault in actuator A can be tolerated after $t = 1.75$ by using the proposed switching scheme.

We now turn our attention to the second problem. Using Eq. (22) with $\rho = 0.02$, the stability regions for $u_{\max} = 2.5$ (solid line) and for $u_{\max} = 0.5$ (dashed line) are computed and shown in Fig. 3(a). From this figure, it is easy to see that for the initial condition $a_1(0) = 1.3$, only actuator A ($z_A = 0.5\pi$, $u_{\max}^A = 2.5$) can be used

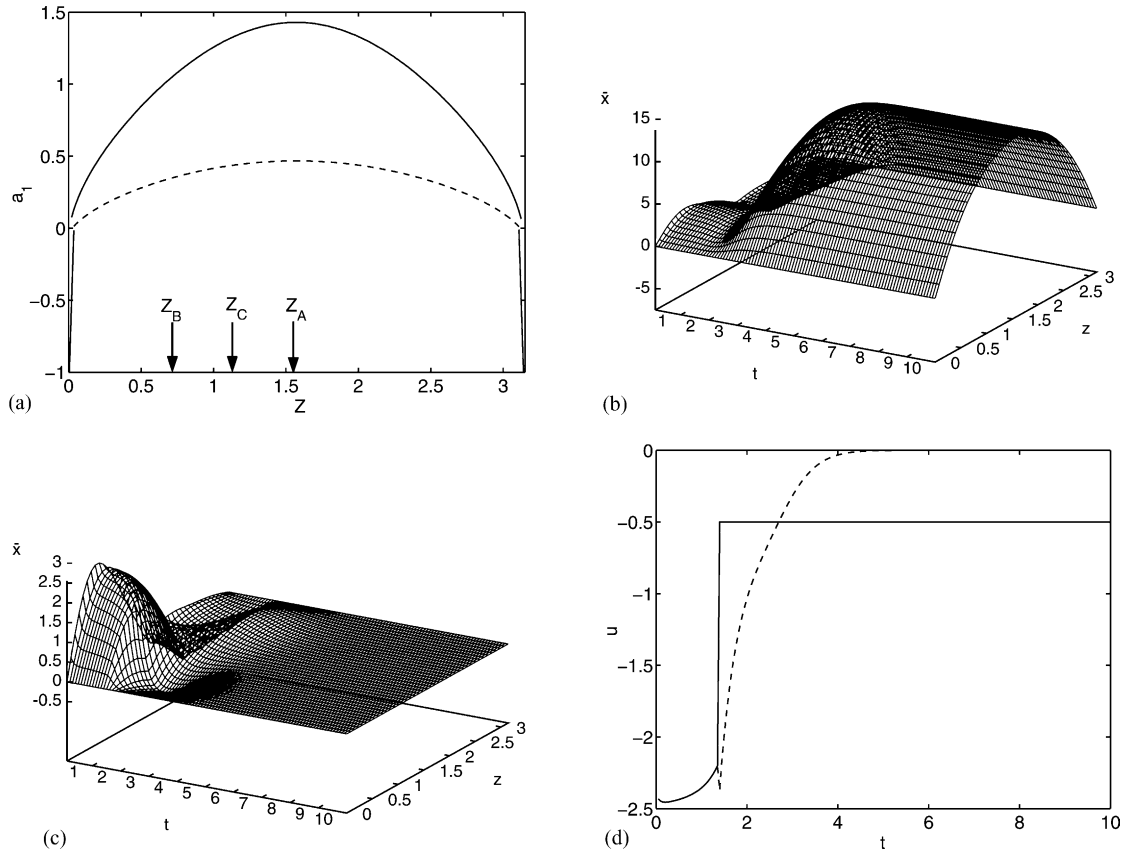


Fig. 3. (a) Stability region as a function of actuator location for $u_{\max} = 2.5$ (solid) and $u_{\max} = 0.5$ (dashed), (b) closed-loop dimensionless temperature profile (state feedback) for $a_1(0) = 1.3$ when actuator A ($u_{\max} = 2.5$, $z_A = 0.5\pi$) fails at $t = 1.4$ and actuator C ($u_{\max} = 0.5$, $z_C = 0.36\pi$) is activated, (c) closed-loop dimensionless temperature profile (state feedback) for $a_1(0) = 1.3$ when actuator A fails at $t = 1.4$ and actuator B ($u_{\max} = 2.5$, $z_B = 0.23\pi$) is activated, (d) corresponding manipulated input profiles for case b (solid) and case c (dashed).

initially since the initial condition is outside the stability regions of both actuator B ($z_B = 0.23\pi$, $u_{\max}^B = 2.5$) and actuator C ($z_C = 0.36\pi$, $u_{\max}^C = 0.5$). Now, suppose that sometime after process startup, say $t = 1.4$, a fault is detected in actuator A and it becomes necessary to switch to another actuator. Without using the switching law of Theorem 1, it is not clear whether actuator B or C should be activated at this time. Figs. 3b and 3d (solid profile) depict, respectively, the resulting closed-loop temperature and manipulated input profiles when actuator C is chosen. We see in this case that the controller is unable to stabilize the system at the desired steady-state. In contrast, using the proposed switching scheme, we find that the slow state at the time that actuator A fails is $a_1(1.4) = 0.62$, which is inside the stability region of actuator B and outside the stability region of actuator C. So, we choose to activate actuator B. Figs. 3c and 3d (dashed line) depict the results for this case which show that the controller successfully stabilizes the closed-loop system.

For the case of output feedback, we use a single point sensor located at $z = 0.33\pi$, to obtain estimates of the first eigenmode, which are then used for implementing the output feedback controller. Owing to the small,

$O(\epsilon)$, discrepancy between the stability regions obtained under state and output feedback (due to estimation errors), the switching law of Theorem 1 should be used only as an approximate law of switching between the different actuators in the case of output feedback (see Remark 11). The simulation results for this case are consistent with the state feedback results for both problems. For brevity, we present only the results for the second problem. These results are given in Fig. 4 which shows that the closed loop system becomes unstable when configuration C is activated at the time that actuator A fails at $t = 1.4$ (Figs. 4a and 4c (solid line)), whereas closed-loop stability is maintained by activating configuration B instead (Figs. 4b and 4d (dashed line)).

5. Application to a non-isothermal tubular reactor with recycle

We consider the non-isothermal tubular reactor studied in Antoniadis and Christofides (2001b), where an irreversible first-order reaction of the form $A \rightarrow B$ takes place. The reaction is exothermic and a cooling

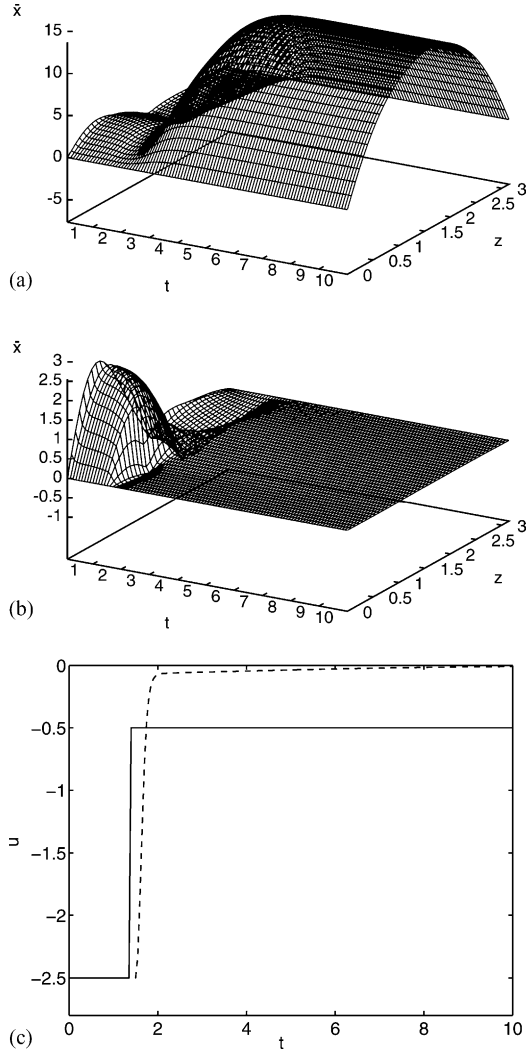


Fig. 4. (a) Closed-loop dimensionless temperature profile (output feedback) for $a_1(0) = 1.3$ when actuator A ($u_{\max} = 2.5$, $z_A = 0.5\pi$) fails at $t = 1.4$ and actuator C ($u_{\max} = 0.5$, $z_C = 0.36\pi$) is activated, (b) closed-loop dimensionless temperature profile (output feedback) for $a_1(0) = 1.3$ when actuator A fails at $t = 1.4$ and actuator B ($u_{\max} = 2.5$, $z_B = 0.23\pi$) is activated, (c) corresponding manipulated input profiles for case a (solid) and case b (dashed).

jacket is used to remove heat from the reactor. The outlet of the reactor is fed to a separator where the unreacted species A is separated from the product B and then fed back to the reactor through a recycle loop. Under standard modeling assumptions, the dynamic model of the process can be derived from mass and energy balances and takes the following dimensionless form:

$$\frac{\partial \bar{x}_1}{\partial t} = -\frac{\partial \bar{x}_1}{\partial z} + \frac{1}{Pe_T} \frac{\partial^2 \bar{x}_1}{\partial z^2} + B_T B_C \exp^{\frac{\gamma \bar{x}_1}{1 + \bar{x}_1}} (1 + \bar{x}_2) + \beta_T (b(z)u(t) - \bar{x}_1)$$

$$\frac{\partial \bar{x}_2}{\partial t} = -\frac{\partial \bar{x}_2}{\partial z} + \frac{1}{Pe_C} \frac{\partial^2 \bar{x}_2}{\partial z^2} - B_C \exp^{\frac{\gamma \bar{x}_1}{1 + \bar{x}_1}} (1 + \bar{x}_2) \quad (28)$$

subject to the boundary conditions:

$$\frac{\partial \bar{x}_1(0, t)}{\partial z} = Pe_T (\bar{x}_1(0, t) - (1-r)\bar{x}_{1f}(t) - r\bar{x}_1(1, t))$$

$$\frac{\partial \bar{x}_2(0, t)}{\partial z} = Pe_C (\bar{x}_2(0, t) - (1-r)\bar{x}_{2f}(t) - r\bar{x}_2(1, t))$$

$$\frac{\partial \bar{x}_1(1, t)}{\partial z} = 0, \quad \frac{\partial \bar{x}_2(1, t)}{\partial z} = 0 \quad (29)$$

where \bar{x}_1 and \bar{x}_2 denote dimensionless temperature and concentration of species A in the reactor, respectively, \bar{x}_{1f} and \bar{x}_{2f} denote dimensionless inlet temperature and inlet concentration of species A in the reactor, respectively, Pe_T and Pe_C are the heat and thermal Peclet numbers, respectively, B_T and B_C denote a dimensionless heat of reaction and a dimensionless pre-exponential factor, respectively, r is the recirculation coefficient (it varies from zero to one, with one corresponding to total recycle with zero fresh feed, and zero corresponding to no recycle), γ is a dimensionless activation energy, β_T is a dimensionless heat transfer coefficient, u is a dimensionless jacket temperature (chosen to be the manipulated input), and $b(z)$ is the actuator distribution function. For the purposes of our analysis here, we will assume that there is no recycle loop dead-time.

In order to transform the boundary condition of Eq. (29) to a homogeneous one, we insert the non-homogeneous part of the boundary condition into the differential equation and obtain the following PDE representation of the process:

$$\frac{\partial \bar{x}_1}{\partial t} = -\frac{\partial \bar{x}_1}{\partial z} + \frac{1}{Pe_T} \frac{\partial^2 \bar{x}_1}{\partial z^2} + B_T B_C \exp^{\frac{\gamma \bar{x}_1}{1 + \bar{x}_1}} (1 + \bar{x}_2) + \beta_T (b(z)u(t) - \bar{x}_1) + \delta(z-0) \times ((1-r)\bar{x}_{1f} + r\bar{x}_1(1, t))$$

$$\frac{\partial \bar{x}_2}{\partial t} = -\frac{\partial \bar{x}_2}{\partial z} + \frac{1}{Pe_C} \frac{\partial^2 \bar{x}_2}{\partial z^2} - B_C \exp^{\frac{\gamma \bar{x}_1}{1 + \bar{x}_1}} (1 + \bar{x}_2) + \delta(z-0) \times ((1-r)\bar{x}_{2f} + r\bar{x}_2(1, t)) \quad (30)$$

where $\delta(\cdot)$ is the standard Dirac function, subject to the homogeneous boundary conditions:

$$\frac{\partial \bar{x}_1(0, t)}{\partial z} = Pe_T \bar{x}_1(0, t), \quad \frac{\partial \bar{x}_2(0, t)}{\partial z} = Pe_C \bar{x}_2(0, t);$$

$$\frac{\partial \bar{x}_1(1, t)}{\partial z} = 0, \quad \frac{\partial \bar{x}_2(1, t)}{\partial z} = 0 \quad (31)$$

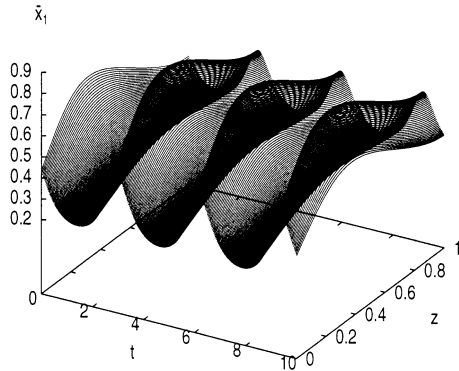


Fig. 5. Spatiotemporal evolution of the dimensionless reactor temperature profile in the open-loop system.

The following values for the process parameters were used in our calculations: $Pe_T = 7.0$, $Pe_C = 7.0$, $B_C = 0.1$, $B_T = 2.5$, $\beta_T = 2.0$, $\gamma = 10.0$, $r = 0.5$, $\bar{x}_{1f} = \bar{x}_{2f} = 0$. For these values, the operating steady-state of the open-loop system is unstable (the linearization around the steady-state possesses one real unstable eigenvalue and infinitely many stable eigenvalues). This instability is shown by the open-loop temperature profile in Fig. 5 where it is seen that starting close to the unstable steady-state, the temperature profiles moves to an asymptotically stable periodic (limit cycle) spatially non-uniform steady-state. We note that in the absence of recycle-loop (i.e. $r = 0$), the above process parameters correspond to a unique, globally asymptotically stable spatially non-uniform steady-state for the open-loop system (see Antoniadis & Christofides, 2001b for a detailed study of the effect of the recycle ratio on the dynamics of the tubular reactor).

The control problem is to stabilize the reactor at a spatially-nonuniform steady-state where the production of species B is desirable and the ‘hot-spot’ temperature is acceptable, by manipulating the jacket temperature, $u(t)$, which is subject to hard constraints and possible actuator failures. To achieve this control objective, the controlled output is defined as $y_c(t) = \int_0^1 e^{-Pe z} \phi_{11}(z) \bar{x}_1 dz$, and the actuator distribution function is taken to be $b(z) = \delta(z - z_{act})$ (point control actuator placed at $z = z_{act}$). The desired reference input value was set at $v = 0.10$ (this value of $y(t)$ corresponds to a steady-state with the desired characteristics of a high production rate with relatively low ‘hot-spot’ temperature). Furthermore, we assume that there is available a large number of point measurements of the temperature throughout the reactor so that $y_c(t)$ is known with sufficient accuracy.

The spatial differential operator of the system of Eq. (30) is of the form:

$$\begin{aligned} \mathcal{A} \bar{x} &= \begin{bmatrix} \mathcal{A}_1 \bar{x}_1 & 0 \\ 0 & \mathcal{A}_2 \bar{x}_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{Pe_T} \frac{\partial^2 \bar{x}_1}{\partial z^2} - \frac{\partial \bar{x}_1}{\partial z} & 0 \\ 0 & \frac{1}{Pe_C} \frac{\partial^2 \bar{x}_2}{\partial z^2} - \frac{\partial \bar{x}_2}{\partial z} \end{bmatrix} \end{aligned} \quad (32)$$

The solution of the eigenvalue problem for \mathcal{A}_i can be obtained by utilizing standard techniques from linear operator theory (see, for example, Ray, 1981) and is of the form:

$$\lambda_{ij} = \frac{\bar{a}_{ij}^2}{Pe} + \frac{Pe}{4}, \quad i = 1, 2, j = 1, \dots, \infty$$

$$\phi_{ij}(z) = B_{ij} e^{\frac{Pe z}{2}} \left(\cos(\bar{a}_{ij} z) + \frac{Pe}{2\bar{a}_{ij}} \sin(\bar{a}_{ij} z) \right),$$

$$i = 1, 2, j = 1, \dots, \infty$$

$$\bar{\phi}_{ij}(z) = e^{-Pe z} \phi_{ij}(z), \quad i = 1, 2, j = 1, \dots, \infty \quad (33)$$

where $Pe = Pe_T = Pe_C$, and λ_{ij} , ϕ_{ij} , $\bar{\phi}_{ij}$, denote the eigenvalues, eigenfunctions and adjoint eigenfunctions of \mathcal{A}_i , respectively. \bar{a}_{ij} , \mathcal{B}_{ij} can be calculated from the following formulas:

$$\tan(\bar{a}_{ij}) = \frac{Pe \bar{a}_{ij}}{\bar{a}_{ij}^2 - \left(\frac{Pe}{2}\right)}, \quad i = 1, 2, j = 1, \dots, \infty$$

$$\mathcal{B}_{ij} = \left\{ \int_0^1 \left(\cos(\bar{a}_{ij} z) + \frac{Pe}{2\bar{a}_{ij}} \sin(\bar{a}_{ij} z) \right)^2 dz \right\}^{-\frac{1}{2}}, \quad (34)$$

$$i = 1, 2, j = 1, \dots, \infty$$

For this system, we take as the slow modes of the process the first temperature mode and use Galerkin’s method to derive an ODE that describes the temporal evolution of the amplitude of the first eigenmode. This ODE is employed for the synthesis of the controller, using Eqs. (20) and (21), which is then implemented on a 400th order Galerkin discretization of the system of Eqs. (30) and (31) (further increases in the order of the Galerkin truncation were found to give negligible improvement on the accuracy of the results). To deal with possible actuator faults, we consider switching between three point control actuators placed at $z_A = 0$ (constraints magnitude: $u_{\max}^A = 0.06$), $z_B = 0.2$ (constraints magnitude: $u_{\max}^B = 0.06$), and $z_C = 0.8$ (constraints magnitude: $u_{\max}^C = 0.02$).

In the first set of simulation runs, we evaluate the ability of the constrained controller to stabilize the reactor at the desired steady-state in the absence of actuator faults. To this end, we design the bounded nonlinear controller of Eqs. (20) and (21) with $\rho = 0.001$ and implement it using actuator A placed at the

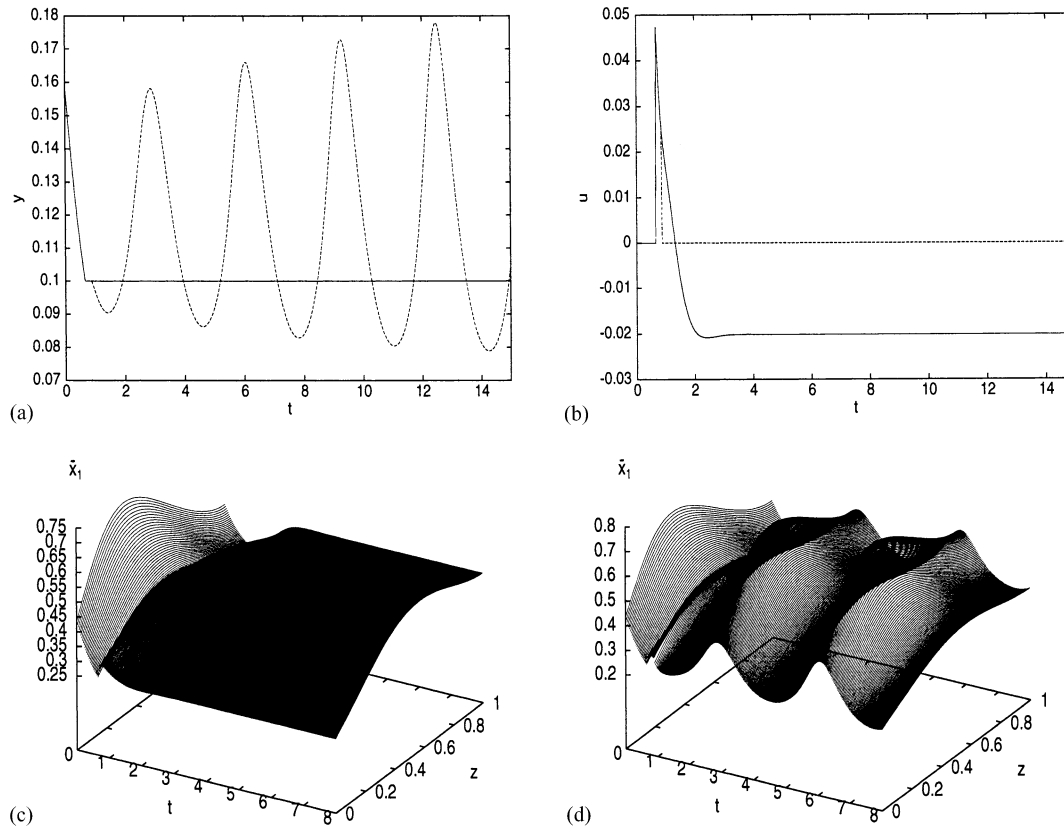


Fig. 6. (a) Controlled output profiles when actuator A ($u_{\max} = 0.06$, $z_A = 0$) operates without failure (solid) and when it fails at $t = 0.875$ without activating any of the backup actuators (dashed), (b) corresponding manipulated input profiles, (c) closed-loop dimensionless temperature profile when actuator A operates without failure, (d) closed-loop dimensionless temperature profile when actuator A fails at $t = 0.875$ without activating any of the backup actuators.

entrance of the reactor. The result is shown by the solid profiles in Figs. 6a and 6b which depict the evolution of the controlled output and manipulated input, respectively. It is seen that the controller successfully stabilizes the output at the desired reference input while satisfying the constraints. The corresponding temperature profile is shown in Fig. 6(c).

In the second set of simulation runs, we demonstrate how actuator switching can be used to deal with the problem of actuator failure. To this end, the closed-loop system is initialized at the same initial condition, considered earlier, under control actuator A which is assumed to fail at $t = 0.875$. If neither of the two backup actuators, B or C, is activated at this time, we see from the dashed profiles in Fig. 6(a–b) that the system reverts to its open-loop mode where the output moves away from the desired steady-state and exhibits a limit cycle behavior. This instability is also reflected in the temperature profile shown in Fig. 6(d). When actuator B is activated, however, it is clear from the resulting controlled output (Fig. 7(a) solid line) and temperature (Fig. 7(c)) profiles that the controller successfully preserves closed-loop stability and achieves the desired steady-state despite the failure of actuator A. In contrast, when actuator C is activated instead of B, we see

from the dashed profile in Fig. 7(a) that the controller is unable to stabilize the output at the desired steady-state, leading to transient oscillatory behavior and offset. The resulting poor performance is also evident in the closed-loop temperature profile depicted in Fig. 7(d). The difference in outcome between the two switching sequences (activation of B following failure of A vs. activation of C following failure of A) can be explained from the corresponding input profiles shown in Fig. 7(b). In particular, when actuator C (closer to the reactor exit) is activated, the controller requests large control action to drive the system to the desired steady-state, which is not available due to the tight constraints of this actuator, leading to prolonged periods of actuator saturation that result in poor performance and offset. In the case of actuator B, however, the control action requested is within the allowable range for this actuator, and therefore, the desired stability and performance properties are preserved successfully upon switching. These results are consistent with the solution of the optimal actuator placement problem for this tubular reactor, obtained in Antoniadis and Christofides (2001a), which show that actuators placed closer to the inlet of the reactor utilize smaller control effort than those placed closer to the exit.

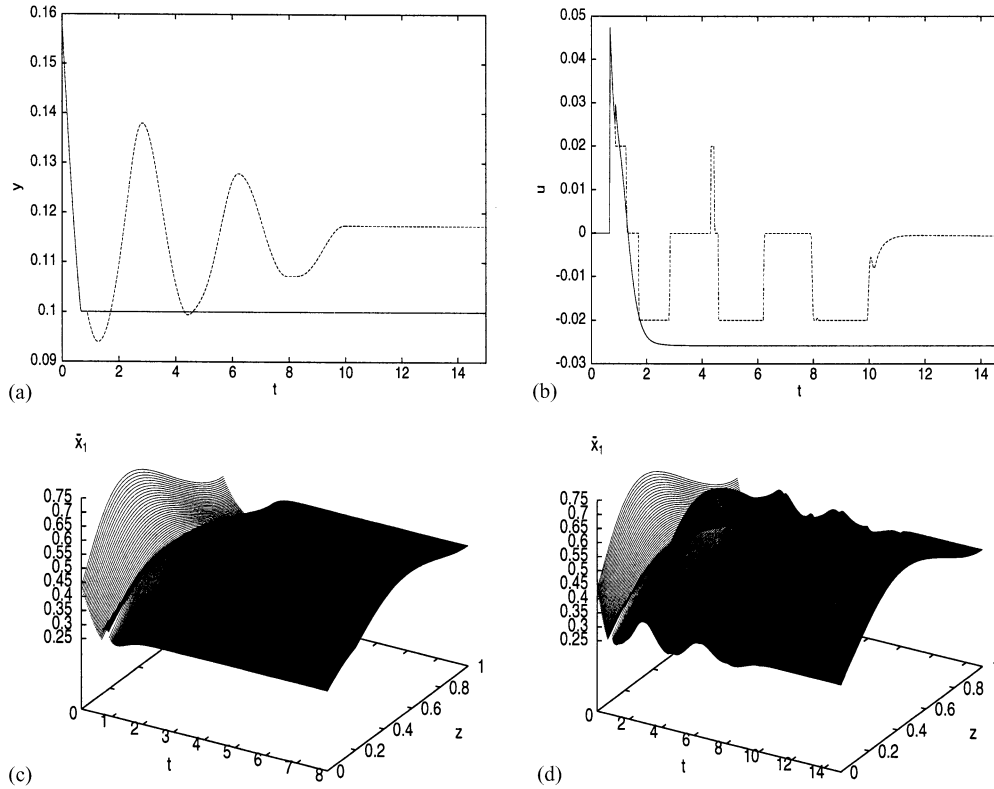


Fig. 7. (a) Controlled output profiles for the case when actuator A ($u_{\max} = 0.06, z_A = 0$) fails at $t = 0.875$ and actuator B ($u_{\max} = 0.06, z_B = 0.2$) is activated (solid) and for the case when actuator C ($u_{\max} = 0.02, z_C = 0.8$) is activated (dashed) instead of B, (b) corresponding manipulated input profiles, (c) closed-loop dimensionless temperature profile when actuator A fails at $t = 0.875$ and actuator B is activated, (d) closed-loop dimensionless temperature profile when actuator A fails at $t = 0.875$ and actuator C is activated.

6. Conclusions

A methodology for coordinating feedback and switching in control of spatially-distributed processes described by highly dissipative, constrained PDE systems was proposed. Under the assumption that the eigenspectrum of the spatial differential operator can be partitioned into a finite slow set and an infinite stable fast complement, Galerkin’s method was initially used to derive a finite-dimensional ODE system that captures the dominant dynamics of the PDE system. The ODE system was then used as the basis for the integrated synthesis, via Lyapunov techniques, of a stabilizing nonlinear feedback controller, together with a switching law that orchestrates the switching between the admissible control actuator configurations, in a way that respects actuator constraints, accommodates conflicting control objectives, and guarantees closed-loop stability at the same time. Precise conditions that guarantee stability of the constrained closed-loop PDE system under switching were provided. Finally, the proposed methodology was successfully applied to stabilize an unstable steady-state of a diffusion-reaction process and a non-isothermal tubular reactor with recycle, under constraints and control actuator failure.

Acknowledgements

Financial support from NSF, CTS-0129571, is gratefully acknowledged.

Appendix A

Proof of Theorem 1. The proof of this theorem is divided into three parts. In the first part, we show that, for each fixed actuator configuration, \bar{z}^k , (without switching), the finite-dimensional closed-loop system is asymptotically (and locally exponentially) stable for any initial condition $a_s(0) \in \Omega(u_{\max}^k, \bar{z}^k)$. In the second part, we show that switching among the N actuator configurations, using the switching rule of Eq. (23), guarantees that the switched finite-dimensional closed-loop system remains also asymptotically (and locally exponentially) stable. Finally, in the third part, we use the stability results obtained for the finite-dimensional closed-loop system to prove closed-loop stability and reference-input tracking for the infinite-dimensional closed-loop system.

Part 1: Consider first the finite-dimensional system of

Eq. (14) in the transformed (e, η) coordinates introduced in Eqs. (17) and (18), subject to the control law of Eqs. (20) and (21). Using a standard Lyapunov argument, it can be shown that whenever Eq. (22) is satisfied, for a given actuator configuration \bar{z}^k , the time-derivative of the Lyapunov function, along the trajectories of the closed-loop e -subsystem, satisfies:

$$\dot{V} \leq \frac{-\rho|e|^2}{[1 + \sqrt{1 + (u_{\max}^k |(L_{\bar{g}} V(\bar{z}^k))^T)|^2}]} < 0 \quad (35)$$

$\forall e \neq 0, k = 1, \dots, N$

Since the function $D(u_{\max}^k, \bar{z}^k) = [1 + \sqrt{1 + (u_{\max}^k |(L_{\bar{g}} V(\bar{z}^k))^T)|^2}]$ is continuous everywhere, then given any compact set $\Omega(u_{\max}^k, \bar{z}^k)$, there exists a positive real number $v(u_{\max}^k, \bar{z}^k) > 0$ such that $D(u_{\max}^k, \bar{z}^k) \leq v(u_{\max}^k, \bar{z}^k)$ for all $a_s \in \Omega(u_{\max}^k, \bar{z}^k)$, which implies that $\dot{V} \leq -\alpha(u_{\max}^k, \bar{z}^k)|e|^2 < 0$ for all $e \neq 0, k = 1, \dots, N$, where $\alpha(u_{\max}^k, \bar{z}^k) = \rho/v(u_{\max}^k, \bar{z}^k) > 0$. Consider now any initial condition such that $a_s(0) \in \Omega(u_{\max}^k, \bar{z}^k)$. Since the set $\Omega(u_{\max}^k, \bar{z}^k)$ is an invariant set within the region described by Eq. (22), then starting from $a_s(0)$, the closed-loop state satisfies Eq. (22) for all time and, consequently from Eq. (35) above, we have that V decreases monotonically according to the dissipation inequality $\dot{V} \leq -\alpha(u_{\max}^k, \bar{z}^k)|e|^2 < 0$, which implies that the closed-loop e -subsystem is exponentially stable under the k th actuator configuration. Therefore, there exists positive real numbers $\sigma_1^k \geq 1, \phi_1^k > 0$ such that the e states of the finite-dimensional closed-loop system satisfy:

$$|e(t)| \leq \sigma_1^k |e(0)| e^{-\phi_1^k t} \quad \forall t \geq 0 \quad (36)$$

Since the η subsystem, with e as input, is ISS (from Assumption 2), uniformly in \bar{v} , the η states of the closed-loop system satisfy the following inequality:

$$|\eta(t)| \leq \sigma_2 |\eta(0)| e^{-\phi_2 t} + \gamma_\eta (\|e\|) \quad \forall t \geq 0 \quad (37)$$

for some $\sigma_2 \geq 1, \phi_2 > 0$, where γ_η is a class \mathcal{K} function of its argument (i.e. continuous, increasing and zero at zero) and $\|e\| = \text{ess. sup} |e(t)|, \forall t \geq 0$. Using the inequalities of Eqs. (36) and (37), it can be shown by means of a small gain argument that the closed-loop (e, η) interconnected system, and hence that of Eqs. (14), (20) and (21), is asymptotically stable. Furthermore, from this result and the fact that the η -subsystem (with $e = 0$) is locally exponentially stable, we have that the closed-loop system of Eqs. (14), (20) and (21) is locally exponentially stable (see Khalil, 1996 for details).

Part 2: Without loss of generality, assume that $a_s(0) \in \Omega(u_{\max}^1, \bar{z}^1)$. Since the set $\Omega(u_{\max}^1, \bar{z}^1)$ is invariant, then for

all times that the configuration $\bar{z}^k = \bar{z}^1$ is active, the closed-loop state satisfies Eq. (22) with $\bar{z}^k = \bar{z}^1$ and V decreases monotonically according to the dissipation inequality $\dot{V} \leq -\alpha(u_{\max}^1, \bar{z}^1)|e|^2 < 0$. Now, if at any time T , such that $a_s(T) \in \Omega(u_{\max}^j, \bar{z}^j)$ (Eq. (23)) for some $j \in \{2, \dots, N\}$, configuration \bar{z}^j is activated (and \bar{z}^1 switched out), then it is clear from the invariance of $\Omega(u_{\max}^j, \bar{z}^j)$ that, for all times that the configuration $\bar{z}^k = \bar{z}^j$ remains active, the closed-loop state will satisfy Eq. (22) with $\bar{z}^k = \bar{z}^j$ and, consequently from Eq. (35) above, we have that V continues to decrease monotonically, but with a different dissipation rate $\dot{V} \leq -\alpha(u_{\max}^j, \bar{z}^j)|e|^2 < 0$. Therefore, we conclude that, as long as the k th configuration is switched in at time T when $a_s(T) \in \Omega(u_{\max}^k, \bar{z}^k)$, the time-derivative of the Lyapunov function (for the switched e -subsystem) will always satisfy the following worst-case dissipation inequality for all $t \geq 0$:

$$\dot{V} \leq \max_{k=1, \dots, N} \{-\alpha(u_{\max}^k, \bar{z}^k)\} |e|^2 < 0 \quad (38)$$

which implies that the switched closed-loop e -subsystem is exponentially stable and that the switched closed-loop system of Eqs. (14), (20)–(23) is asymptotically stable (by invoking the ISS property of the η -subsystem similar to part 1 of the proof). Therefore, there exists a function $\bar{\beta}$ of class \mathcal{KL} such that $|a_s(t)| \leq \bar{\beta}(|a_s(0)|, t) \forall t \geq 0$ (a function $\bar{\beta}(s, t)$ is said to be of class \mathcal{KL} if, for each fixed t , the function $\bar{\beta}(s, \cdot)$ is continuous, increasing, and zero at zero and, for each fixed s , the function $\bar{\beta}(\cdot, t)$ is nonincreasing and tends to zero at infinity). Using the fact that $\bar{x}_s = \sum_{j=1}^m a_j(t) \phi_j(z)$, and the definition of the L_2 -norm associated with the finite-dimensional Hilbert space, \mathcal{H}_s , we have:

$$\|\bar{x}_s\|_2 = \int_\alpha^\beta \sqrt{a_1^2(t) \phi_1^2(z) + \dots + a_m^2(t) \phi_m^2(z)} dz \quad (39)$$

Since the functions $\phi_j(\cdot)$ are continuous on the closed interval $[\alpha, \beta]$, then there exist real numbers $\varphi_j \geq 0, j = 1, \dots, m$ such that $\phi_j^2 \leq \varphi_j$, and consequently:

$$\begin{aligned} \|\bar{x}_s\|_2 &\leq \int_\alpha^\beta (\varphi^* \sqrt{a_1^2(t) + \dots + a_m^2(t)}) dz \\ &= \int_\alpha^\beta (\varphi^* |a_s(t)|) dz \leq \varphi^* (\beta - \alpha) \bar{\beta}(|a_s(0)|, t) \end{aligned} \quad (40)$$

where $\varphi^* = \sqrt{\varphi}$, $\varphi = \max_{j=1, \dots, m} \{\varphi_j\}$, which implies that the state \bar{x}_s is bounded and $\|\bar{x}_s\|_2 \rightarrow 0$ as $t \rightarrow \infty$. Therefore, the origin of the switched closed-loop slow subsystem of Eqs. (13), (20)–(23) is asymptotically stable, i.e. there exists a class \mathcal{KL} function, β_s , such that:

$$\|\bar{x}_s\|_2 \leq \beta_s(\|\bar{x}_s(0)\|_2, t) \quad \forall t \geq 0 \quad (41)$$

Furthermore, from the exponential stability of the switched closed-loop e -subsystem (see Eq. (38)) and the local exponential stability of the η -subsystem (Assumption 2), we have that the switched closed-loop system of Eqs. (14), (20)–(23) (hence that of Eqs. (13), (20)–(23)) is locally exponentially stable. Therefore, given the positive real number δ_s^1 (or $\bar{\delta}_s^1$), there exists $b > 0$ such that if $|a_s(0)| \leq \delta_s^1$ (or $|\bar{x}_s(0)| \leq \bar{\delta}_s^1$), the following bound holds:

$$|\bar{x}_s(t)|_2 \leq \sigma_3 |\bar{x}_s(0)|_2 e^{-\phi_3 t} \quad \forall t \geq 0 \quad (42)$$

for all $|\bar{x}_s(t)|_2 \leq b$, for some $\sigma_3 \geq 1$, $\phi_3 > 0$.

Part 3: Substituting the controller of Eqs. (20) and (21) into Eq. (12) and using the fact that $\epsilon = |Re\{\lambda_1\}| / |Re\{\lambda_{m+1}\}| < 1$, the closed-loop system can be written in the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s - \mathcal{B}_s r(x_s, u_{\max}^k, \bar{z}^k) (L_g V)^T + f_s(x_s, 0) \\ &\quad + [f_s(x_s, x_f) - f_s(x_s, 0)] \\ \epsilon \frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon} x_f + \epsilon \bar{f}_f(x_s, x_f) \end{aligned} \quad (43)$$

where $\mathcal{A}_{f\epsilon}$ is an unbounded differential operator defined as $\mathcal{A}_{f\epsilon} = \epsilon \mathcal{A}_f$, and the function \bar{f}_f defined as $\bar{f}_f(x_s, x_f) = -\mathcal{B}_s r(x_s, u_{\max}^k, \bar{z}^k) (L_g V)^T + f_f(x_s, x_f)$. The system of Eq. (43) is in the standard singularly perturbed form, with x_s being the slow states and x_f being the fast states. Introducing the fast time-scale $\tau = t/\epsilon$ and setting $\epsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq. (43):

$$\frac{\partial \bar{x}_f}{\partial \tau} = \mathcal{A}_{f\epsilon} \bar{x}_f \quad (44)$$

where the bar symbol in \bar{x}_f , denotes that the state \bar{x}_f is associated with the approximation of the fast x_f -subsystem. From the fact that $Re\{\lambda_{m+1}\} < 0$ and the definition of ϵ , we have that the above system is globally exponentially stable. Therefore, there exists real numbers $\sigma_4 \geq 1$, $\phi_4 > 0$ such that:

$$\|\bar{x}_f(\tau)\|_2 \leq \sigma_4 \|\bar{x}_f(0)\|_2 e^{-\phi_4 \tau} \quad \forall \tau \geq 0 \quad (45)$$

Setting $\epsilon = 0$ in the system of Eq. (43) and using that the operator $\mathcal{A}_{f\epsilon}$ is invertible, we have that $\bar{x}_f = 0$ and thus the finite-dimensional closed-loop slow system takes the form of Eqs. (13), (20)–(23). We have already shown in part 2 of the proof that this slow system is asymptotically (and locally exponentially) stable starting from any initial condition $\bar{x}_s(0) \in \Theta_s(u_{\max}^1, \bar{z}^1)$ (equivalently, from any initial condition $a_s(0) \in \Omega(u_{\max}^1, \bar{z}^1)$) and under the switching rule of Eq. (23) and that its states

are bounded. Therefore, exploiting the stability properties of the fast and slow systems, it can be shown, using techniques similar to those performed in Christofides and Teel (1996) for the analysis of singularly perturbed systems, that the inequalities of Eqs. (41) and (45) continue to hold, for the states of the infinite-dimensional singularly perturbed closed-loop system, up to an arbitrarily small offset, d , for initial conditions in large compact subsets, $\Theta_b(u_{\max}^1, \bar{z}^1) \subset \Theta_s(u_{\max}^1, \bar{z}^1)$, (equivalently, $\Omega_b(u_{\max}^1, \bar{z}^1) \subset \Omega(u_{\max}^1, \bar{z}^1)$) where $\Theta_b(u_{\max}^1, \bar{z}^1) = \{x_s(0) \in \mathcal{H}_s : |x_s(0)|_2 \leq \delta_b^{-1}\}$. Therefore, given any pair of positive real numbers $(\bar{\delta}_b^1, d)$, such that $\bar{\delta}_b^1 + d \leq \bar{\delta}_s^1$, and given any δ_f , there exists $\epsilon^{(1)} > 0$ such that if $\epsilon \in (0, \epsilon^{(1)})$, $|x_s(0)|_2 \leq \bar{\delta}_b^1$, $\|x_f(0)\|_2 \leq \delta_f$, then, for all $t \geq 0$, the states of the closed-loop singularly perturbed system satisfy:

$$\begin{aligned} |x_s(t)|_2 &\leq \beta_s(|x_s(0)|_2, t) + d \\ \|x_f(t)\|_2 &\leq \sigma_4 \|x_f(0)\|_2 e^{-\phi_4 \frac{t}{\epsilon}} + d \end{aligned} \quad (46)$$

The above inequalities imply that the trajectories of the switched closed-loop singularly perturbed system will be bounded. Furthermore, as t increases, they will be ultimately bounded with an ultimate bound that depends on d . Since d is arbitrary, we can choose it small enough such that after a sufficiently large time, say \tilde{t} , the trajectories of the closed-loop system are confined within a small compact neighborhood of the origin of the closed-loop system. Obviously, \tilde{t} depends on both the initial condition and the desired size of the neighborhood, but is independent of ϵ . Let $d = b/2$ and \tilde{t} be the smallest time such that $\max\{\beta_s(|x_s(0)|_2, \tilde{t}), \sigma_4 \|x_f(0)\|_2 e^{-\phi_4 \tilde{t}/\epsilon}\} \leq d$. Then it can be easily verified that:

$$|x_s(t)|_2 \leq b, \quad \|x_f(t)\|_2 \leq b \quad \forall t \geq \tilde{t} \quad (47)$$

Recall from Eqs. (42) and (45) that both the fast and slow subsystems are exponentially stable within the ball of Eq. (47). Therefore, there exists $\epsilon^{(2)} > 0$ such that if $\epsilon \leq \epsilon^{(2)}$, the singularly perturbed closed-loop system of Eq. (43) is locally exponentially stable and, therefore, once inside the ball of Eq. (47), the closed-loop trajectories converge to the origin as $t \rightarrow \infty$. The details of this argument can be found in the proof of proposition 4.1 in Christofides (2001) and will not be repeated here. To summarize, we have that given any pair of positive real numbers $(\bar{\delta}_b^1, d)$ such that $\bar{\delta}_b^1 + d \leq \bar{\delta}_s^1$, and given any positive real number δ_f , there exists $\epsilon^* \equiv \min\{\epsilon^{(1)}, \epsilon^{(2)}\}$ such that if $|x_s(0)|_2 \leq \bar{\delta}_b^1$, $\|x_f(0)\|_2 \leq \delta_f$, and $\epsilon \in (0, \epsilon^*)$, the closed-loop trajectories are bounded and converge to the origin as time tends to infinity, i.e. the closed-loop system is asymptotically stable. By

taking the lim sup of both sides of Eq. (36) as $t \rightarrow \infty$, we have that $\limsup_{t \rightarrow \infty} |e_1^i(t)| = 0$ and, therefore, $\limsup_{t \rightarrow \infty} |\bar{y}_c^i(t) - v_i(t)| = 0$, $i = 1, \dots, m$. Owing to the exponential stability of the infinite-dimensional closed-loop system, standard results from singular perturbation theory on closeness of solutions can be used (see Christofides, 2001) to show that there exists $t_b > 0$ such that, for all $t \geq t_b$, the following estimates hold:

$$\begin{aligned} x_s(t) &= \bar{x}_s(t) + O(\epsilon) \\ x_f(t) &= O(\epsilon) \end{aligned} \quad (48)$$

Using the above estimates together with the definition of y_c , given in Eq. (12), we get:

$$y_c = \mathcal{C}x_s + \mathcal{C}x_f = \mathcal{C}\bar{x}_s + O(\epsilon) = \bar{y}_c + O(\epsilon) \quad (49)$$

where we have used the fact that $\bar{y}_c = \mathcal{C}\bar{x}_s$ to derive the last equality. From Eq. (49), we can write:

$$y_c^i = \bar{y}_c^i + O(\epsilon), \quad i = 1, \dots, m \quad (50)$$

and, therefore:

$$\begin{aligned} \limsup_{t \rightarrow \infty} |y_c^i(t) - v_i(t)| &= \limsup_{t \rightarrow \infty} |\bar{y}_c^i(t) - v_i(t)| + O(\epsilon), \\ i &= 1, \dots, m \end{aligned} \quad (51)$$

Since the limit on the right hand side of Eq. (51) is zero, we finally have:

$$\limsup_{t \rightarrow \infty} |y_c^i(t) - v_i(t)| = O(\epsilon), \quad i = 1, \dots, m \quad (52)$$

This completes the proof of the theorem.

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