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Integrating robustness, optimality and constraints in control of nonlinear processes☆

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Abstract

This work focuses on the development of a unified practical framework for control of single-input–single-output nonlinear processes with uncertainty and actuator constraints. Using a general state-space Lyapunov-based approach, the developed framework yields a direct nonlinear controller design method that integrates robustness, optimality, and explicit constraint-handling capabilities, and provides, at the same time, an explicit and intuitive characterization of the state-space regions of guaranteed closed-loop stability. This characterization captures, quantitatively, the limitations imposed by uncertainty and input constraints on our ability to steer the process dynamics in a desired direction. The proposed control method leads to the derivation of explicit analytical formulas for bounded robust optimal state feedback control laws that enforce stability and robust asymptotic reference-input tracking in the presence of active input constraints. The performance of the control laws is illustrated through the use of a chemical reactor example and compared with existing process control strategies. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Nonlinearity; Model uncertainty; Input constraints; Lyapunov-based control; Inverse optimality; Bounded control; Chemical processes

1. Introduction

Most chemical processes are inherently nonlinear and cannot be effectively controlled using controllers designed on the basis of approximate linear or linearized process models. The limitations of traditional linear control methods in dealing with nonlinear chemical processes have become increasingly apparent as chemical processes may be required to operate over a wide range of conditions due to large process upsets or set-point changes. Motivated by this, the area of nonlinear process control has been one of the most active research areas within the chemical engineering community over the last 15 years. In this area, important contributions have been made including the synthesis of state feedback controllers (Hoo & Kantor, 1985; Kravaris & Chung, 1987; Kravaris & Kantor, 1990a; Kazantzis & Kravaris, 1999a), the design of state estimators (Soroush, 1997; Kazantzis & Kravaris, 1999b) and output feedback controllers (Daoutidis & Kravaris, 1994; Soroush, 1995; Kurtz & Henson, 1997a; Christofides, 2000), and the analysis and control of nonlinear systems using functional expansions (Batigun, Harris, & Palazoglu, 1997; Harris & Palazoglu, 1997, 1998). Reviews of results in the area of nonlinear process control can be found in Kravaris and Kantor (1990a, b), Kravaris and Arkun (1991), Bequette (1991), Allgöwer and Doyle (1997), Rawlings, Meadows, and Muske (1994), Lee (1997), Henson and Seborg (1997).

In addition to nonlinear behavior, many industrial process models are characterized by the presence of time-varying uncertainty such as unknown process parameters and external disturbances which, if not accounted for in the controller design, may cause performance deterioration and even closed-loop instability. Motivated by the problems caused by model uncertainty on the closed-loop behavior, the problem of designing controllers for nonlinear systems with uncertain variables, that enforce stability and output tracking in the closed-loop system, has received significant attention in the past. For feedback linearizable nonlinear processes with constant uncertain variables, a linear controller with "integral"

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action is often employed around the static feedback linearizing controller for rejection of unmeasured disturbances (Daoutidis & Kravaris, 1994). On the other hand, for feedback linearizable nonlinear processes with time-varying uncertain variables, Lyapunov's direct method has been used to design robust state feedback controllers that enforce boundedness of the states and an arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output (e.g., Kravaris & Palanki, 1988; Arkun & Calvet, 1992; Qu, 1993; Allgöwer, Rehm, & Gilles, 1994; Christofides, Teel, & Daoutidis, 1996). More recently, robust output feedback controllers have been also designed through combination of robust state feedback controllers with high-gain observers (Khalil, 1994; Mahmoud & Khalil, 1996; Christofides, 2000).

While the above works provide systematic methods for nonlinear and robust controller design, they do not lead, in general, to controllers that are optimal with respect to a meaningful cost and therefore do not guarantee achievement of the control objectives with the smallest possible control action. This is an important limitation of these methods, especially in view of the fact that the capacity of control actuators used to regulate chemical processes is almost always limited. Such limitations may arise either due to the finite capacity of control actuators (e.g., bounds on the magnitude of the opening of valves) or may be imposed on the process to ensure safe operation, meet environmental regulations, or maintain desired product quality specifications. The presence of input constraints restricts our ability to freely modify the dynamic behavior of a chemical process and compensate for the effect of model uncertainty through high-gain feedback. The ill-effects due to actuator constraints manifest themselves, for example, in the form of sluggishness of response and loss of stability. Additional problems that arise in the case of dynamic controllers include undesired oscillations and overshoots, a phenomenon usually referred to as "windup". The problems caused by input constraints have consequently motivated many recent studies on the dynamics and control of chemical processes subject to input constraints. Notable contributions in this regard include controller design and stability analysis within the model predictive control framework (Kurtz & Henson, 1997b; Coulibaly, Maiti, & Brosilow, 1995; Valluri, Soroush, & Nikravesh, 1998; Rao & Rawlings, 1999; Scokaert & Rawlings, 1999 Schwarm & Nikolaou 1999), constrained linear (Chmielewski & Manousiouthakis, 1996) and nonlinear (Chmielewski & Manousiouthakis, 1998) quadratic-optimal control, the design of "anti-windup" schemes in order to prevent excessive performance deterioration of an already designed controller when the input saturates (Kothare, Campo, Morari, & Nett, 1994; Kapoor, Teel, & Daoutidis, 1998; Kendi & Doyle, 1995; Oliveira, Nevistic, & Morari, 1995; Calvet & Arkun, 1991; Valluri & Soroush, 1998; Nikolaou & Cherukuri, submitted; Kapoor & Daoutidis, 1999a), the study of the nonlinear bounded control problem for a class of two- and three-state chemical reactors (Alvarez, Alvarez, & Suarez, 1991; Alvarez, Alvarez, Barron, & Suarez, 1993), the characterization of regions of closed-loop stability under static state feedback linearizing controllers (Kapoor & Daoutidis, 1998), and some general results on the dynamics of constrained nonlinear systems (Kapoor & Daoutidis, 1999b). However, these control methods do not explicitly account for robust uncertainty attenuation.

At this stage, existing process control methods lead to the synthesis of controllers that can deal with either model uncertainty or input constraints, but not simultaneously or effectively with both. This clearly limits the achievable control quality and closed-loop performance, especially in view of the commonly encountered co-presence of uncertainty and constraints in chemical processes. Therefore, the development of a unified framework for control of nonlinear systems that explicitly accounts for the presence of model uncertainty and input constraints is expected to have a significant impact on process control.

A natural approach to resolve the apparent conflict between the need to compensate for model uncertainty through high-gain control action and the presence of input constraints that limit the availability of such action is the design of robust optimal controllers which expend minimal control effort to achieve stabilization and uncertainty attenuation. Within an analytical setting, one approach to design robust optimal controllers is within the nonlinear H_∞ control framework (e.g., van der Schaft, 1992; Pan & Basar, 1993). However, the practical applicability of this approach is still questionable because the explicit construction of the controllers requires the analytic solution of the steady-state Hamilton–Jacobi–Isaacs (HJI) equation which is not a feasible task except for simple problems. An alternative approach to robust optimal controller design which avoids the unwieldy task of solving the HJI equation is the inverse optimal approach proposed by Kalman (1964) and introduced recently in the context of robust stabilization in Freeman and Kokotovic (1996). The central idea of the inverse optimal approach is to compute a robust stabilizing control law together with the appropriate penalties that render the cost functional well defined and meaningful in some sense. This approach is well motivated by the fact that the closed-loop robustness achieved as a result of controller optimality is largely independent of the specific choice of the cost functional (Sepulchre, Jankovic, & Kokotovic, 1997) as long as it is a meaningful one.

In a previous work (El-Farra & Christofides, 1999), we addressed the problem of robust optimal controller design for a broad class of nonlinear systems with time-varying uncertain variables and synthesized, through Lyapunov's direct method, robust optimal nonlinear controllers that enforce stability and asymptotic output tracking with attenuation of the effect of the uncertain variables on the output of the closed-loop system. Utilizing the inverse optimal approach, the controllers were shown to be optimal with respect to physically meaningful costs that include penalty on the control effort. These controllers, although better equipped to handle input constraints than feedback linearizing controllers, do not explicitly account for input constraints, and therefore, offer no a priori guarantees regarding the desired closed-loop stability and performance in the presence of arbitrary input constraints.

In this paper, we extend our previous work and develop a unified framework for control of constrained uncertain nonlinear processes that integrates robustness, optimality, and explicit constraint-handling capabilities in the controller synthesis and provides, simultaneously, an explicit and intuitive characterization of the regions of guaranteed closed-loop stability. Using a general state-space Lyapunov approach, the developed framework yields a direct nonlinear controller design method that accounts explicitly and simultaneously for closed-loop stability and performance in the presence of model uncertainty and active input constraints. The basic idea is the development of a scaling procedure, inspired by the results in Lin and Sontag (1991), that bounds the robust optimal controllers synthesized in El-Farra and Christofides (1999). This leads to the derivation of explicit analytical formulas for continuous state feedback bounded robust optimal controllers with well-characterized stability and performance properties. For processes with vanishing uncertainty, the developed controllers are shown to guarantee asymptotic stability and robust asymptotic set-point tracking with an arbitrary degree of attenuation of the effect of uncertainty on the output of the closed-loop system in the presence of active input constraints. For processes with nonvanishing uncertainty, the same controllers are shown to ensure boundedness of the states and robust asymptotic output tracking in the presence of active input constraints. The proposed control method is illustrated through the use of a chemical reactor example and compared with existing process control strategies.

2. Preliminaries

2.1. System description

We consider the class of continuous-time single-input–single-output nonlinear processes with uncertain variables with the following state-space description:

$$\dot{x} = f(x) + g(x)\mathrm{sat}(u) + \sum_{k=1}^{q} w_k(x)\theta_k(t), \qquad y = h(x), \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}$ denotes the manipulated input, $\theta_k(t) \in \mathcal{W} \subset \mathbb{R}$ denotes the kth uncertain (possibly time-varying) but bounded variable taking values in a nonempty compact convex subset $\mathcal{W}$ of $\mathbb{R}$, $y \in \mathbb{R}$ denotes the output to be controlled, and sat refers to the standard saturation nonlinearity. The uncertain variable $\theta_k(t)$ may describe time-varying parametric uncertainty and/or exogenous disturbances. To simplify the presentation of our results, we assume, without loss of generality, that the origin is the only equilibrium point of the nominal (i.e., $u(t) = \theta_k(t) \equiv 0$) system of Eq. (1). The vector functions $f(x)$, $w_k(x)$ and $g(x)$, and the scalar function $h(x)$ are assumed to be sufficiently smooth. In the remainder of this paper, for simplicity, we will suppress the time dependence in the notation of the uncertain variable $\theta_k(t)$. We remark here that the class of systems described by Eq. (1) is general enough to be of practical interest (see, for instance, the chemical reactor example in Section 2.3), yet specific enough to allow the meaningful synthesis of controllers.

2.2. Motivation

Most of nonlinear analytical controllers emanating from the area of geometric process control are input–output linearizing and induce a linear input–output response in the absence of constraints (Kravaris & Kantor, 1990a; Isidori, 1989). For the class of processes modeled by equations of the form of Eq. (1) with $\theta_k \equiv 0$ and relative order r and under the minimum phase assumption, the appropriate linearizing state feedback controller is given by

$$u = \frac{1}{L_g L_f^{r-1} h(x)}(v - L_f^r h(x) - \beta_1 L_f^{r-1} h(x) - \cdots - \beta_{r-1} L_f h(x) - \beta_r h(x)) \tag{2}$$

and induces the linear input–output dynamics given by

$$\frac{\mathrm{d}^r y}{\mathrm{d}t} + \beta_1 \frac{\mathrm{d}^{r-1} y}{\mathrm{d}t} + \cdots + \beta_{r-1}\frac{\mathrm{d}y}{\mathrm{d}t} + \beta_r y = v, \tag{3}$$

where the tunable parameters $\beta_1 \cdots \beta_r$ are essentially closed-loop time constants that influence and shape the output response. The nominal stability of the process is guaranteed by placing the roots of the polynomial $s^r + \beta_1 s^{r-1} + \cdots + \beta_{r-1}s + \beta_r$ in the open left-half of the complex plane. Since the control law of Eq. (2) does not possess the ability to reject unmeasured disturbances or track exogenous signals, an external linear controller with "integral" action is typically employed around the static state feedback linearizing controller to ensure an offsetless response in the presence of constant disturbances and model errors. In the event of input saturation, however, the controller dynamics will result in the problem of windup, causing deterioration in the closed-loop performance and stability properties of the controller due to the wrong update of the states of the linear controller around the linearizing controller. Motivated by these problems, recent research efforts on the analytical control of constrained nonlinear processes have focused on devising various compensatory schemes and modifications of the above approach to offer greater flexibility in obtaining a desirable closed-loop output response in the presence of constraints and compensate for windup (Valluri & Soroush, 1998; Kapoor & Daoutidis, 1999a,b).

Despite the improvement that may be achieved in the closed-loop performance using the above approaches, these schemes continue to face two fundamental limitations in their ability to effectively handle the co-presence of model uncertainty and input constraints. The first such limitation is the use of input–output linearizing control techniques to synthesize the necessary control laws. Despite being a powerful control strategy in the absence of constraints, a major drawback of input–output linearizing control techniques, in general, is their inherent lack of direct constraint handling capabilities. One fundamental reason for this is the fact that these techniques rely on cancelling out process nonlinearities using feedback and may therefore generate unnecessarily large control effort to cancel beneficial process nonlinearities. The cost of cancelling the nonlinearities can be large enough to unnecessarily saturate the control input, potentially leading to closed-loop instability or performance deterioration. Modifying such controllers a posteriori to account for the presence of active input constraints may result in an unnecessary restriction of operation in a relatively small region around the desired set-point and makes the stability analysis of the closed-loop system quite a challenging task. The second limitation of existing control techniques relates to the continued use of integral or integral-like action to compensate for disturbances. While maybe suitable for constant disturbances, these techniques may yield unsatisfactory robustness in the presence of time-varying disturbances and prompt the controller to generate large control action in its attempt to compensate for the disturbances, further confounding the problems of input saturation.

Realization of the above limitations clearly motivates the need for an alternative and direct controller design strategy for nonlinear processes with time-varying uncertainty and input constraints, that avoids the above-mentioned shortcomings altogether and guarantees, a priori, the desirable performance and stability properties of the closed-loop system in the presence of arbitrary disturbances and input constraints. Before presenting the details of this strategy, let us first highlight, by means of a benchmark chemical process example, some of the limitations of existing process control strategies.

2.3. Illustrative example

To illustrate some of the detrimental effects of time-varying uncertainty and input constraints on the closed-loop stability and performance under input–output linearizing controllers, we simulate in this subsection the startup performance of a nonisothermal chemical reactor under the generalized mixed error- and state-feedback controller recently proposed in theorem 1 in Valluri and Soroush (1998), which employs input–output linearizing control techniques and inherently compensates for wind-up. The same example is used later to show the superior capabilities of the controllers designed in this article and how they avoid the pitfalls of existing wind-up compensation schemes.

Consider a continuous stirred tank reactor where an irreversible, exothermic first-order reaction of the form $A \xrightarrow{k} B$ takes place. The inlet stream consists of pure A at flow rate F, concentration C_{A0} and temperature T_{A0}. Under standard modeling assumptions, the mathematical model for the process takes the form

$$\begin{aligned} V\frac{\mathrm{d}C_A}{\mathrm{d}t} &= F(C_{A0}-C_A) - k_0 \mathrm{e}^{-E/RT} C_A V, \\ V\frac{\mathrm{d}T}{\mathrm{d}t} &= F(T_{A0}-T) + \frac{(-\Delta H)}{\rho c_p} k_0 \mathrm{e}^{-E/RT} C_A V + \frac{Q}{\rho c_p}, \end{aligned} \tag{4}$$

where C_A denotes the concentration of species A, T denotes the temperature of the reactor, V denotes the volume of the reactor, k_0, E, ΔH denote the pre-exponential constant, the activation energy, and the enthalpy of the reaction, c_p and ρ, denote the heat capacity and density of the fluid in the reactor, and Q denotes the rate of heat input to the reactor. The steady-state values and process parameters are given in Table 1.

The control objective is to maintain the temperature of the reactor at the (unstable) steady-state value given in Table 1, in the presence of time-varying disturbances in the feed temperature and uncertainty in the enthalpy of the reaction, by manipulating the rate of heat input.

Table 1
Process parameters and steady-state values

$V = 100.0$ l
$R = 8.314$ J/mol K
$C_{A0} = 1.0$ mol/l
$T_{A0s} = 310.0$ K
$\Delta H = -4.78 \times 10^4$ J/mol
$k_0 = 7.20 \times 10^{10}$ min^{-1}
$E = 8.31 \times 10^4$ J/mol
$c_p = 0.239$ J/g K
$\rho = 1000.0$ g/l
$F = 100.0$ l/min
$C_{As} = 0.577$ mol/l
$T_s = 395.3$ K

Defining $x_1 = C_A$, $x_2 = T$, $u = Q$, $\theta_1 = T_{A0} - T_{A0s}$, $\theta_2 = \Delta H - \Delta H_{\text{nom}}$, $y = x_2$, where the subscript s denotes the steady-state values, the process model of Eq. (4) can be recast in the form of Eq. (1) with

$$f(x) = \begin{bmatrix} \dfrac{F}{V}(C_{A0} - C_A) - k_0 \mathrm{e}^{-E/RT} C_A \\ \dfrac{F}{V}(T_{A0s} - T) + \dfrac{-\Delta H_{\text{nom}}}{\rho c_p} k_0 \mathrm{e}^{-E/RT} C_A \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ \dfrac{1}{\rho c_p V} \end{bmatrix},$$

$$w_1(x) = \begin{bmatrix} 0 \\ \dfrac{F}{V} \end{bmatrix}, \qquad w_2(x) = \begin{bmatrix} 0 \\ k_0 \mathrm{e}^{-E/RT} C_A \end{bmatrix} \tag{5}$$

and $h(x) = x_2$. The following time-varying uncertain variables are considered:

$$\theta_1(t) = 0.1 T_{A0s} \sin(3t), \qquad \theta_2(t) = 0.5(-\Delta H_{\text{nom}}) \sin(3t). \tag{6}$$

The physical motivation for considering sinusoidal variations in the heat of reaction is the fact that heat of reaction is a function of temperature. Therefore, in the presence of sinusoidal variations (disturbances) in the feed temperature, the reactor temperature and heat of reaction will vary sinusoidally in time.

Fig. 1 depicts the reactor temperature and heat input profiles under the generalized mixed error- and state-feedback controller of Theorem 1 in Valluri and Soroush (1998) with $\beta_1 = 0.5$ min, $\gamma_1 = 0.35$ min. The solid line represents the response when there is no limit on the heat input to the reactor (i.e., $|Q| < \infty$) and no disturbances or model errors are present. In this case, the controller successfully stabilizes the process at the desired steady state. The dashed line in Fig. 1 corresponds to the startup performance of the reactor in the presence of the time-varying uncertain variables considered in Eq. (6) but no limits on the rate of heat input are imposed. It is clear from the oscillatory behavior of the response that the controller fails to stabilize the process at the desired steady state or attenuate the effect of uncertainty on the closed-loop output. Note that the controller generates a huge control effort to try to compensate for the effect of uncertainty on the output of the closed-loop system. The controller starts by requesting excessive heat which together with the higher rate of heat production by the reaction at the higher temperatures later prompts the controller to demand excessive cooling. The closed-loop instability is even more profound when limits are imposed on the rate of heat input to the reactor rendering the excessive heating or cooling requirements infeasible. This is indicated by the dotted line in Fig. 1 which

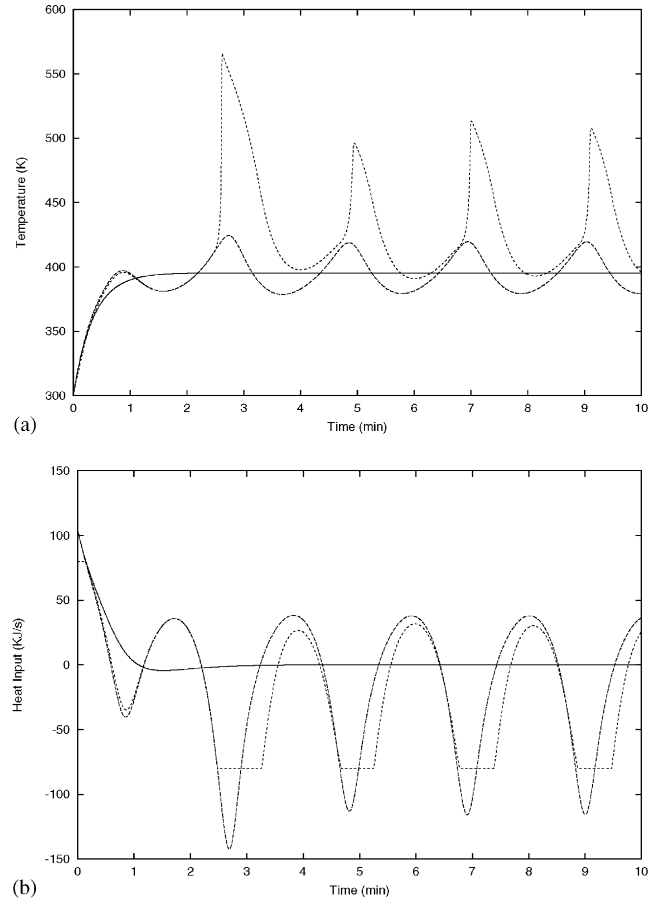

Fig. 1. Startup profiles of the controlled output and manipulated input for the chemical reactor example under the dynamic mixed error-and state-feedback controller with windup compensation proposed in Theorem 1 in (Valluri & Soroush, 1998) in the absence of uncertainty or constraints (solid line), in the presence of uncertainty only (dashed line), and in the presence of both uncertainty and constraints $|Q| \leqslant 80$ kJ/s (dotted line).

corresponds to the response of the reactor in the presence of uncertainty and constraints on the rate of heat input to the reactor ($|Q| \leqslant 80$ kJ/s).

Having demonstrated some of the problems associated with existing control strategies, we are now motivated to develop an alternative systematic controller design strategy that handles directly and effectively the challenges imposed by the co-presence of model uncertainty and input constraints. To this end, we develop, in the next two sections, a Lyapunov-based framework that incorporates robustness, optimality, and explicit constraint-handling capabilities in the controller design; and provides, simultaneously, an explicit characterization of the regions, in state space, starting from where the desired closed-loop stability and performance properties are guaranteed. We begin our development in the next section by presenting first the necessary conceptual tools to address the problem within the Lyapunov framework. Using these tools, we construct robust optimal nonlinear control laws that are better equipped, due to optimality, to handle the co-presence of uncertainty and constraints than feedback linearizing controllers, but do not account explicitly for the presence of arbitrary input constraints. These controllers are presented first to highlight their advantages over existing techniques as well as their inherent limitations in fully addressing the problem, thus motivating the new controller designs in the following section where we show by means of a scaling procedure how the robust optimal design capabilities can be extended quite naturally to encompass all the desired properties of robustness, optimality, and explicit constraint-handling.

3. Robust optimal control of nonlinear processes

It is well known from optimal control theory that optimal feedback systems enjoy several desirable stability and robustness properties as long as the optimality is meaningful (Sepulchre et al., 1997). An additional consequence of optimality is the fact that robust optimal controllers do not waste unnecessary control effort to achieve robust stabilization and are therefore better equipped to cope with the limitations imposed by input constraints on the control action, needed for uncertainty attenuation, than controllers requesting unreasonably large control effort to accomplish this goal. Optimality, therefore, is a key prerequisite for any controller design strategy that tries to effectively address the problem. Furthermore, the benefits of optimality listed above do not depend on the specific choice of the performance index as long as it is a meaningful one, i.e. places sensible penalties on the process states and control input. Such reasoning has motivated the formulation and solution of many inverse optimal control problems (Freeman & Kokotovic, 1996; Sepulchre et al., 1997). This approach has the advantage that it yields optimal control laws without the need to solve the HJI equation which is not a feasible task for real problems.

3.1. Inverse optimality

Preparatory for its use as a tool for robust optimal controller design, we begin this section by reviewing the concept of inverse optimality introduced in the context of robust stabilization in Freeman and Kokotovic (1996). To this end, consider the system of Eq. (1) with $q = 1$ and $w(0)\theta = 0$. Also, let $l(x)$ and $R(x)$ be two continuous scalar functions such that $l(x) \geqslant 0$ and $R(x) > 0$ $\forall x \in \mathbb{R}^n$ and consider the problem of designing a feedback law $u(x)$ that achieves asymptotic stability of the origin and minimizes the cost functional

$$J = \int_0^\infty (l(x) + uR(x)u)\,\mathrm{d}t. \tag{7}$$

The steady-state Hamilton–Jacobi–Isaacs equation associated with the system of Eq. (1) and the cost of Eq. (7) is

$$0 \equiv \inf_{u \in \mathbb{R}} \sup_{\theta \in \mathcal{W}} (l(x) + uR(x)u + L_f V(x) + L_g V(x)u + L_w V(x)\theta). \tag{8}$$

A smooth positive-definite solution $V(x)$ to this equation leads to a continuous optimal state feedback of the form

$$u = -p(x) = -\tfrac{1}{2}R^{-1}(x)L_g V(x). \tag{9}$$

However, such a solution may not exist or may be extremely difficult to compute. Suppose, instead, that we were able to find a smooth positive-definite radially unbounded scalar function V such that

$$\inf_{u \in \mathbb{R}} \sup_{\theta \in \mathcal{W}} (L_f V(x) + L_g V(x)u + L_w V(x)\theta) < 0 \quad \forall x \neq 0. \tag{10}$$

A function V that satisfies the above inequality is known as a robust control Lyapunov function (rclf). It is simply a candidate Lyapunov function whose time derivative, under the worst-case disturbances, can be made negative pointwise by the choice of control values. While the classical Lyapunov function only allows us to conclude the stability of the closed-loop system generated by a predetermined feedback control, the control Lyapunov function allows us to conclude the stabilizability of a system for which a feedback control has not yet been synthesized and can therefore serve as a useful design tool to ensure stability and meet other design specifications. Now, if we can find a meaningful cost (i.e., $l(x) \geqslant 0, R(x) > 0$) such that the given V is the corresponding optimal value function, then we will have indirectly obtained a solution to the HJI equation and can compute the control law of Eq. (9). This motivates following the inverse path where a stabilizing feedback control is designed first and then shown to be optimal with respect to a well-defined cost functional of the form of Eq. (7). The problem is inverse because the weights $l(x)$ and $R(x)$ in the cost functional are a posteriori computed from the chosen stabilizing feedback control law, rather than a priori specified by the designer.

A stabilizing control law $u(x)$ is said to be inverse optimal for the system of Eq. (1) if it can be expressed in the form of Eq. (9) where the negative definiteness of $\dot{V}$ is achieved with the control $u^* = -\frac{1}{2}p(x)$, that is

$$\sup_{\theta \in \mathcal{W}} \dot{V}|_{u^*} = L_f V(x) - \tfrac{1}{2}L_g V(x)p(x) + |L_w V(x)|\theta_b < 0 \quad \forall x \neq 0, \tag{11}$$

where the worst-case uncertainty (i.e., one that maximizes $\dot{V}$) is given by $\theta = \mathrm{sgn}[L_w V(x)]\theta_b$ where $\theta_b = \|\theta\|$ (which is clearly an admissible uncertainty since it is both measurable and bounded). When the function $-l(x)$ is set equal to the right-hand side of Eq. (11), then $V(x)$ is a solution to the following steady-state HJI equation:

$$0 \equiv l(x) + L_f V(x) - \tfrac{1}{4} L_g V(x) R^{-1}(x) L_g V(x) + |L_w V(x)| \theta_b \tag{12}$$

and the optimal (minimal) value of the cost J is $V(x(0))$. The main task in this Lyapunov-based design therefore is the construction of the appropriate Lyapunov functions whose time derivatives in the presence of the worst-case uncertainty can be rendered negative definite by feedback. For the class of processes considered in this article, not only is the existence of such functions guaranteed but their explicit construction, as we will see shortly, is conceptually straightforward.

Finally, we recall the definition of input-to-state stability (ISS) for a system of the form of Eq. (1).

Definition 1 (Sontag, 1989). The system in Eq. (1) (with $u \equiv 0$) is said to be ISS with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_0 \in \mathbb{R}^n$ and for each measurable, essentially bounded input $\theta(\cdot)$ on $[0, \infty)$ the solution of Eq. (1) with $x(0) = x_0$ exists for each $t \geqslant 0$ and satisfies

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|\theta\|) \quad \forall t \geqslant 0. \tag{13}$$

3.2. Control problem formulation

Referring to the nonlinear process of Eq. (1), we consider two separate control problems. In the first problem, we assume that the uncertain variable terms $w_k(x)\theta_k$ in Eq. (1) are vanishing (i.e., $w_k(0)\ \theta_k = 0$) which means that the origin is an equilibrium point of the uncertain system of Eq. (1). We focus on the synthesis of robust optimal nonlinear state feedback controllers of the general form

$$u = \mathscr{P}(x, \bar{v}, t), \tag{14}$$

where $\mathscr{P}(x, \bar{v}, t)$ is a scalar function and $\bar{v} = [v \;\; v^{(1)} \;\; \cdots \;\; v^{(r)}]^{\mathrm{T}}$ is a generalized reference input ($v^{(k)}$ denotes the kth time derivative of the reference input v which is assumed to be a sufficiently smooth function of time), that enforce, in the absence of constraints, global asymptotic stability and asymptotic output tracking with attenuation of the effect of the uncertainty on the output of the closed-loop system, and are optimal with respect to a meaningful infinite time cost functional that places suitable penalty on the control action. In the second control problem, we assume that the uncertain variable terms $w_k(x)\theta_k$ are persistent or nonvanishing (i.e., $w_k(0)\ \theta_k \neq 0$). Under this assumption, the origin is no longer an equilibrium point of the uncertain system of Eq. (1). The design objective is to synthesize robust optimal nonlinear state feedback controllers of the form of Eq. (14) that guarantee, in the absence of constraints, global boundedness of the trajectories and enforce the discrepancy between the output and the reference input to be asymptotically arbitrarily small, and are optimal with respect to a meaningful cost functional defined over a finite time interval. In both control problems, the controller is designed using geometric and Lyapunov techniques. The analysis of the closed-loop system is performed by utilizing the concept of input-to-state stability and nonlinear small gain theorem-type arguments, and optimality is established using the inverse optimal control approach.

3.3. Controller synthesis

Having formulated our control problem in the previous subsection, we now proceed to present its solution. The solution leads to the derivation of explicit analytical formulas for robust optimal state feedback control laws that achieve the desired objectives outlined in the previous subsection. In the remainder of this section, nonlinear processes with vanishing uncertainty are considered first, followed by nonlinear processes with nonvanishing uncertainty.

3.3.1. Case I: vanishing process uncertainty

In order to proceed with the synthesis of the controllers for this case, we will impose the following assumptions on the process of Eq. (1). The first assumption allows transforming the process of Eq. (1) into a partially linear form and is motivated by the requirement of output tracking.

Assumption 1. There exists an integer r and a set of coordinates

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ T_1(x) \\ \vdots \\ T_{n-r}(x) \end{bmatrix}, \tag{15}$$

where $T_1(x), \ldots, T_{n-r}(x)$ are scalar functions such that the system of Eq. (1) takes the form

$$\dot{\zeta}_1 = \zeta_2,$$

$$\vdots$$

$$\dot{\zeta}_{r-1} = \zeta_r,$$

$$\dot{\zeta}_r = L_f^r h(T^{-1}(\zeta,\eta)) + L_g L_f^{r-1} h(T^{-1}(\zeta,\eta))u + \sum_{k=1}^{q} L_{wk} L_f^{r-1}(T^{-1}(\zeta,\eta))\theta_k,$$

$$\dot{\eta}_1 = \Psi_1(\zeta,\eta,\theta),$$

$$\vdots$$

$$\dot{\eta}_{n-r} = \Psi_{n-r}(\zeta,\eta,\theta),$$

$$y = \zeta_1, \tag{16}$$

where $L_g L_f^{r-1} h(x) \neq 0$ for all $x \in \mathbb{R}^n$, $\theta \in \mathcal{W}^q$. Moreover, for each $\theta \in \mathcal{W}^q$, the states ζ, η are bounded if and only if the state x is bounded; and $(\zeta,\eta) \to (0,0)$ if and only if $x \to 0$.

Assumption 1 includes the matching condition of the proposed robust control method; loosely speaking, it states that the uncertain variables cannot have a stronger effect on the controlled output than the manipulated input. This assumption is standard in robust output tracking control methods and is satisfied by many chemical and physical processes (Christofides et al., 1996). We note that the change of variables of Eq. (15) is independent of θ (this is because the vector field, $g(x)$, that multiplies the input u is independent of θ), and is invertible, since, for every x, the variables ζ,η are uniquely determined by Eq. (15). Introducing the notation $e_i = \zeta_i - v^{(i-1)}$, $e = [e_1 \; e_2 \; \cdots \; e_r]^{\mathrm{T}}$, the ζ subsystem of Eq. (16) can be further transformed into the following form:

$$\dot{e} = \bar{f}(e,\eta,\bar{v}) + \bar{g}(e,\eta,\bar{v})u + \sum_{k=1}^{q} \bar{w}_k(e,\eta,\bar{v})\theta_k, \tag{17}$$

where $\bar{f}(e,\eta)$, $\bar{g}(e,\eta)$, and $\bar{w}_k(e,\eta)$ are $r \times 1$ vector fields whose specific form is omitted for brevity. We use the above normal form now to construct the Lyapunov function of Eq. (10). For processes of the form of Eq. (17), this can be done in many different ways. One way, for instance, is to use a quadratic function $V = e^{\mathrm{T}}Pe$ where the positive-definite matrix P is chosen to satisfy the following Ricatti inequality:

$$A^{\mathrm{T}}P + PA - Pbb^{\mathrm{T}}P < 0, \tag{18}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 1 \end{bmatrix} \tag{19}$$

are an $r \times r$ matrix and $r \times 1$ vector, respectively.

Following (Christofides et al., 1996), the requirement of input-to-state stability of the η subsystem of Eq. (16) is imposed to allow the synthesis of a robust state feedback controller that enforces the requested properties in the closed-loop system for any initial condition. Assumption 2 that follows states this requirement.

Assumption 2. The dynamical system

$$\dot{\eta}_1 = \Psi_1(\zeta,\eta,\theta),$$

$$\vdots \tag{20}$$

$$\dot{\eta}_{n-r} = \Psi_{n-r}(\zeta,\eta,\theta)$$

is ISS with respect to ζ uniformly in θ.

In most practical applications, there exists partial information about the uncertain terms of the process model. Information of this kind may result from physical considerations, preliminary simulations, experimental data, etc. In order to achieve attenuation of the effect of the uncertain variables on the output, we will quantify the possible knowledge about the uncertain variables by assuming the existence of known bounds that capture the size of the uncertain variables for all times.

Assumption 3. There exist known positive constants θ_{bk} such that $\|\theta_k(t)\| = \theta_{bk}$.

Theorem 1 below provides a formula of the robust optimal state feedback controller and states precise conditions under which the proposed controller enforces the desired properties in the closed-loop system. The proof of this theorem as well as further details on the controller synthesis can be found in (El-Farra & Christofides, 1999).

Theorem 1. *Consider the uncertain nonlinear process of Eq.* (1), *for which Assumptions* 1–3 *hold, under the static state feedback law*:

$$u = -\tfrac{1}{2}R^{-1}(x)L_{\bar{g}}V = -\left(c_0 + \frac{L_{\bar{f}}V + \alpha\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}}{(L_{\bar{g}}V)^2} + \frac{(\rho + \chi\sum_{k=1}^{q}\theta_{bk}|L_{wk}L_f^{r-1}h(x)|)}{(L_gL_f^{r-1}h(x))^2(|L_{\bar{g}}V|/|L_gL_f^{r-1}h(x)| + \phi)}\right)L_{\bar{g}}V, \tag{21}$$

where $V = e^{\mathrm{T}}Pe$, P is a positive-definite matrix that satisfies Eq. (18), *and c_0, α, ρ, χ, and ϕ are adjustable parameters that satisfy $c_0 > 0$, $\alpha \geqslant 0$, $\rho > 0$, $\chi > 2$, and $\phi > 0$. Further, assume that the uncertain variables in Eq.* (1) *are vanishing in the sense that there exists positive real numbers δ_k such that $|\bar{w}_k(e,\eta)| \leqslant \delta_k|2b^{\mathrm{T}}Pe|$ $\forall e \in D$, where $D = \{e \in \mathbb{R}^r: |2b^{\mathrm{T}}Pe| \leqslant \phi/(\frac{1}{2}\chi - 1)\}$. Then for any initial condition, there exists $\phi^* > 0$ such that if $\phi \leqslant \phi^*$, the following holds in the absence of constraints*:

(1) *The closed-loop system is globally asymptotically stable.*

(2) *The output of the closed-loop system satisfies a relation of the form*

$$\limsup_{t\to\infty} |y(t) - v(t)| = 0. \tag{22}$$

(3) *The static state feedback control law of Eq.* (21) *minimizes the cost functional*

$$J = \int_0^{\infty} (l(e) + uR(x)u)\,\mathrm{d}t, \tag{23}$$

where $R(x) > 0$ *and* $l(e) = -L_{\bar{f}}V + \frac{1}{4}L_{\bar{g}}VR^{-1}(x)L_{\bar{g}}V - \sum_{k=1}^{q}|L_{wk}V|\theta_{bk} \geqslant k|2b^{\mathrm{T}}Pe|^2 \geqslant 0$ *for all* $x \in \mathbb{R}^n$ *where* $k > 0$.

Remark 1. The control law of Eq. (21) possesses four adjustable parameters that directly influence the performance of the output of the closed-loop system and the achievable level of uncertainty attenuation. The non-negative parameter α allows greater flexibility in shaping the dynamic behavior of the closed-loop output response. By computing the time derivative of V along the trajectories of the closed-loop system of Eq. (1), one can easily obtain the following estimate:

$$\dot{V} \leqslant -c_0(L_{\bar{g}}V)^2 - \alpha\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}. \tag{24}$$

From the above equation, it is transparent that the choice of α essentially determines how negative the time derivative of V is along the closed-loop trajectories and can therefore be made so as to make this derivative more or less negative as desired. For instance, a large value for α will make $\dot{V}$ more negative and therefore generate a faster transient response. It is also clear from Eq. (24) that a positive value for c_0 is not necessary for stabilization. However, a positive value for c_0 may be needed to ensure the strict positivity of $R(x)$. Finally, the two parameters χ and ϕ are primarily responsible for achieving the desired degree of attenuation of the effect of uncertainty on the closed-loop output. A significant degree of attenuation can be achieved by selecting ϕ to be sufficiently small and/or χ to be sufficiently large.

Remark 2. In contrast to feedback linearizing controllers, the optimal control law of Eq. (21) has two desirable properties not present in the feedback linearizing design. The first property is the fact that the controller of Eq. (21) recognizes the beneficial effect of the term $L_{\bar{f}}V$ when $L_{\bar{f}}V < 0$ and prevents its unnecessary cancellation. In this case, the term $L_{\bar{f}}V$ is a stabilizing term whose cancellation may generate positive feedback and destabilize the process. This situation can arise whenever the process under consideration is open-loop stable or contains beneficial nonlinearities that help drive the process towards the desired steady state. Under such circumstances, the term $(L_{\bar{f}}V)^2$ in Eq. (21) guards against the wasteful cancellation of such nonlinearities (by essentially deactivating the cancelling term $L_{\bar{f}}V$) and helps the controller avoid the expenditure of large control effort. This is particularly important when the process operating conditions start relatively far from the equilibrium point, in which case the feedback linearizing designs may generate a huge control effort to cancel process nonlinearities. The second property that the controller of Eq. (21) possesses is the fact that it dominates the term $L_{\bar{f}}V$, instead of cancelling it, when $L_{\bar{f}}V > 0$. This scenario arises, for example, when the process is open-loop unstable. In this case, the term $L_{\bar{f}}V$ is a destabilizing one that must be eliminated. The controller of Eq. (21), however, eliminates the term by domination rather than by cancellation. This property guards against the nonrobustness of cancellation designs which increases the risk of instability due to the presence of other uncertainty not taken into account in the controller design. This means that input perturbations (or equivalently, an error in implementing the control law) will be tolerated in the sense that the trajectories will remain bounded.

Remark 3. Note that since $l(e) = -\sup_{\theta\in\mathbb{R}^q} \dot{V}|_{1/2u} \geqslant k|2b^{\mathrm{T}}Pe|^2 \geqslant 0$ where k is a positive constant, the cost functional of Eq. (23) includes penalty only on the tracking error $e = v - y$ and its time derivatives up to order r and not on the full state of the closed-loop system. This is consistent with the requirement of output tracking. In addition, J imposes penalty on the control action. Finally, the term $-\sum_{k=1}^{q}|L_{wk}V|\theta_{bk}$ in Eq. (23) is a direct consequence of the worst-case uncertainty formulation of the HJI equation (see Eq. (8)) which guarantees optimality of the control law of Eq. (21) under the worst disturbances.

Remark 4. Regarding the properties of the cost functional of Eq. (23), we observe that J is well defined according to the formulation of the optimal control problem for nonlinear systems (Sepulchre et al., 1997) because of the following properties of $l(e)$ and $R(x)$. The function $l(e)$ is continuous, positive definite, and bounded from below by a class K function of the norm of e. The function $R(x)$ is continuous, strictly positive, and the term $uR(x)u$ has a unique global minimum at $u = 0$ implying a larger control penalty for u farther away from zero. This guarantees that J imposes sensible penalties and that the controller inherits all the necessary robustness and optimality benefits regardless of the actual interpretations of $l(e)$ and $R(x)$. Moreover, owing to the vanishing nature of the uncertain variables and the global asymptotic stability of the closed-loop system, J is defined on the infinite time interval and is finite.

Remark 5. The linear growth bound on the functions $\bar{w}_k(e,\eta)$ is imposed to ensure that the results of Theorem 1 hold globally and is consistent, at the same time, with the vanishing nature of the uncertain variables, since it implies that the functions $\bar{w}_k(e,\eta)$ vanish when $e=0$. Note, however, that this condition is required to hold only over the set D which contains the origin, and not everywhere. Since the size of this set can be made arbitrarily small by selecting ϕ to be sufficiently small, or equivalently χ to be sufficiently large, the desired bound can be guaranteed to hold by appropriate selection of the controller tuning parameters. Observe that, locally (i.e. in a sufficiently small compact neighborhood of the origin), this growth condition is automatically satisfied due to the local Lipschitz properties of the functions $\bar{w}_k$. Finally, we note that the imposed growth assumption can be readily relaxed by a slight modification of the controller synthesis formula. One such modification, for example, is to replace the term $|L_w L_f^{r-1} h(x)|$ in Eq. (21) by the term $|L_w L_f^{r-1} h(x)|^2$. Using a standard Lyapunov argument, one can show that the resulting controller globally asymptotically stabilizes the system without imposing the linear growth condition on the functions $\bar{w}_k(e,\eta)$ over D. We choose, however, not to modify the original controller design because such modification will increase, unnecessarily, the gain of the controller, prompting the expenditure of unnecessarily large control action.

3.3.2. Case II: nonvanishing process uncertainty

We now turn to address the second control problem where the process uncertain variable terms in Eq. (1) are nonvanishing. In this case, the origin is no longer an equilibrium point of the closed-loop system. To proceed with the design of the controller, we let Assumptions 1 and 3 hold and modify Assumption 2 to the following one.

Assumption 4. The dynamical system of Eq. (20) is ISS with respect to ζ, θ.

Theorem 2 below provides an explicit formula for the construction of the necessary robust optimal feedback controller and states precise conditions under which the proposed controller enforces the desired properties in the closed-loop system. The proof of this theorem can be found in El-Farra and Christofides (1999).

Theorem 2. *Let assumptions* 1, 3, *and* 4 *hold and assume that the uncertain variables in Eq.* (1) *are nonvanishing in the sense that there exist positive real numbers* $\bar{\delta}_k$, μ_k *such that* $|\bar{w}_k(e,\eta)| \leqslant \bar{\delta}_k |2b^{\mathrm{T}} Pe| + \mu_k \ \forall e \in D$. *Consider the uncertain nonlinear system of Eq.* (1) *under the static state feedback law of Eq.* (21) *with* $\rho = 0$. *Then for any initial condition and for every positive real number d, there exists* $\phi^*(d) > 0$ *such that if* $\phi \in (0, \phi^*(d)]$, *the following holds in the absence of constraints*:

(1) *The trajectories of the closed-loop system are bounded.*

(2) *The output of the closed-loop system satisfies a relation of the form*

$$\limsup_{t\to\infty} |y(t) - v(t)| \leqslant d. \tag{25}$$

(3) *The static state feedback control law of Eq.* (21) *minimizes the cost functional*

$$J = \lim_{t\to T_f} V(e(t)) + \int_0^{T_f} (l(e) + uR(x)u)\,\mathrm{d}t, \tag{26}$$

where $R(x) > 0$, $l(e) \geqslant \bar{k}|2b^{\mathrm{T}} Pe|^2 \geqslant 0 \ \forall t \in [0, T_f]$ *where* $\bar{k} > 0$ *and* $T_f = \inf\{T \geqslant 0\text{: } e(t) \in \Gamma \subseteq D \ \forall t \geqslant T\}$.

Remark 6. Note that the cost functional of Eq. (26) is different from that of Eq. (23) in two important ways. First, owing to the persistent nature of the uncertainty in this case and the fact that asymptotic convergence to the equilibrium point of the nominal system (i.e. $\theta(t) \equiv 0$) is no longer possible, J cannot achieve a finite value over the infinite time interval. Therefore, we modify the cost functional of Eq. (23) to be defined only over a finite time interval $[0, T_f]$. The size of this interval is dictated by the time required for the tracking error trajectory to reach and enter a compact ball Γ centered around the origin without ever leaving again. The size of this ball scales with ϕ and, depending on the desired degree of attenuation of the effect of the uncertainty on the output, can be made arbitrarily small by adjusting the controller tuning parameters ϕ and χ. In addition to the introduction of a terminal time T_f, the cost functional of Eq. (26) imposes a terminal penalty given by the term $\lim_{t\to T_f} V(e)$ to penalize the tracking error trajectory for its inability to converge to the origin for $t > T_f$ due to the presence of persistent process disturbances.

Remark 7. Since the error trajectory of the closed-loop system converges to the ball Γ as time tends to T_f, we have that the cost functional of Eq. (26) is meaningful and well defined in the sense that $l(e) \geqslant 0$ and $R(x) > 0 \ \forall t \in [0, T_f]$. The optimality of the control law of Eq. (21) is therefore meaningful in the presence of persistent process disturbances and the controller inherits all the benefits of optimality stated in Remark 2 above.

Remark 8. Referring to the practical applications of the result of Theorem 2, one has to initially verify whether Assumptions 1 and 4 hold for the process under consideration. Then, given the bound θ_{bk}, Eq. (21) can be used to compute the explicit formula for the controller. Finally, given the asymptotic tracking error desired d, which can be chosen arbitrarily close to zero, the value of ϕ^* should be computed (usually through simulations) to achieve $\limsup_{t\to\infty} |y(t) - v(t)| \leqslant d$.

3.4. Illustrative example revisited

Let us now apply the robust optimal controller design proposed in this section to the nonisothermal chemical reactor considered in the preliminaries section earlier. The aim of this application is to evaluate the capabilities of the robust optimal controller in handling the co-presence of model uncertainty and manipulated input constraints and to compare these capabilities with those of existing feedback linearizing designs. To this end, consider the same control objective and uncertain variables described in Section 2.3. Note that the uncertain variables considered are nonvanishing. Therefore, we use the result of Theorem 2 to construct the necessary robust optimal state feedback controller. Using the control Lyapunov function $V = \frac{1}{2}c(T - T_s)^2$ where $c > 0$, the necessary controller takes the form of Eq. (21) where

$$L_{\bar{f}} V = c(T - T_s) \times \left[\frac{F}{V}(T_{A0} - T) + \frac{-\Delta H_{\mathrm{nom}}}{\rho c_p} k_0 \mathrm{e}^{-E/RT} C_A \right],$$

$$L_{\bar{g}} V = c(T - T_s) \frac{1}{\rho c_p V}, \tag{27}$$

$$\mathrm{L}_{w1} h(x) = \frac{F}{V}, \quad L_{w2} h(x) = k_0 \mathrm{e}^{-E/RT} C_A.$$

The following values were used for the controller tuning parameters: $c_0 = 0.1$, $c = 1.0$, $\phi = 0.01$, $\chi = 2.1$, and $\alpha = 1.0$, to guarantee that the output of the closed-loop system satisfies a relation of the form

$$\limsup_{t \to \infty} |T(t) - T_s| \leqslant 0.01. \tag{28}$$

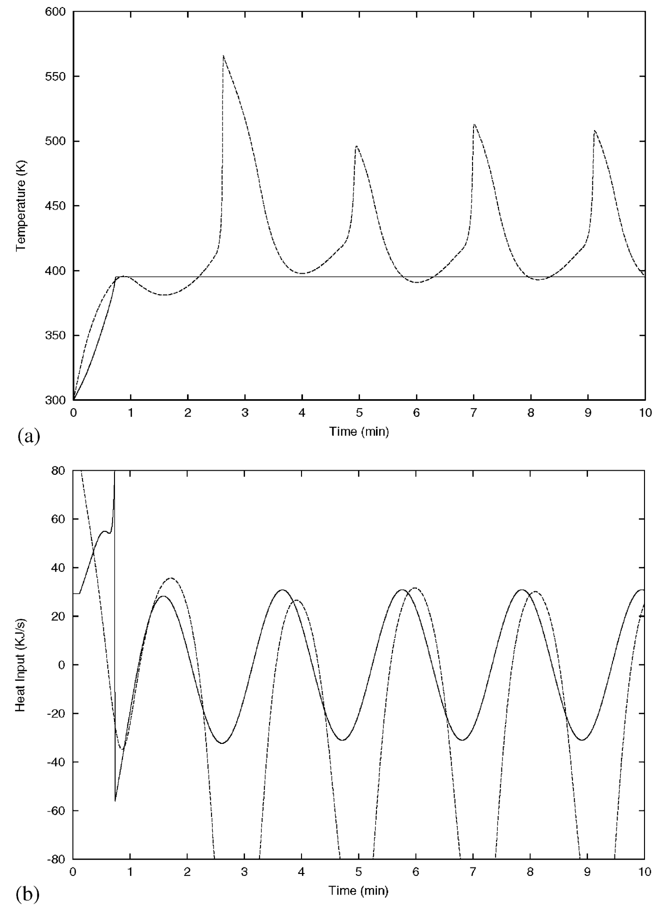

Fig. 2. Controlled output and manipulated input profiles in the presence of both process uncertainty and input constraints $|Q| \leqslant 80$ kJ/s under the controller of Eqs. (21)–(27) (solid line) and the dynamic mixed error-and state feedback controller with windup compensation proposed in Theorem 1 in (Valluri & Soroush, 1998) (dashed line).

Fig. 2 depicts the temperature and rate of heat input profiles for the reactor under the robust optimal controller of Eq. (21) (solid-line) in the presence of both the time-varying uncertainty of Eq. (6) and constraints on the rate of heat input to the reactor ($|Q| \leqslant 80$ kJ/s). For the sake of comparison, the corresponding plots under the input–output linearizing-based controller designed in Valluri and Soroush (1998) (dashed-line) are included. It is clear that while the controller proposed in Valluri and Soroush (1998) fails to stabilize the process at the desired steady state or compensate for the disturbances, the robust optimal controller of Eq. (21) successfully drives the process to the desired steady state and at the same time attenuates the effect of uncertainty on the reactor temperature. The reason for these different outcomes can be understood by referring to the rate of heat input profiles in Fig. 2. Due to its optimality and explicit handling of uncertainty, the robust optimal controller of Eq. (21) does not expend unnecessary control effort or demand infeasible excessive heating or cooling of the reactor to robustly stabilize the reactor temperature. By contrast, the controller proposed in Valluri and Soroush (1998) employs integral action and generates unnecessarily large control action causing the control actuators to saturate. The combination of integral action dynamics and input saturation is responsible for the unstable behavior exhibited under this controller in Fig. 2. Note that the controller of Eq. (21) avoids the problems caused by integral action by adopting a more effective robust controller design through Lyapunov's direct method.

To investigate the significance of the term responsible for uncertainty attenuation in the controller of Eq. (21), we implemented the controller of Eq. (21) on the process without the uncertainty compensation component (i.e., $\chi = 0$). The closed-loop output and manipulated input profiles for this case are given in Fig. 3. It is clear that the effect of process uncertainty is significant, leading to poor transient performance and offset and that uncertainty compensation is required to achieve the control objective.

Note that the robust optimal controller of Eq. (21) successfully achieved the control objective in the presence

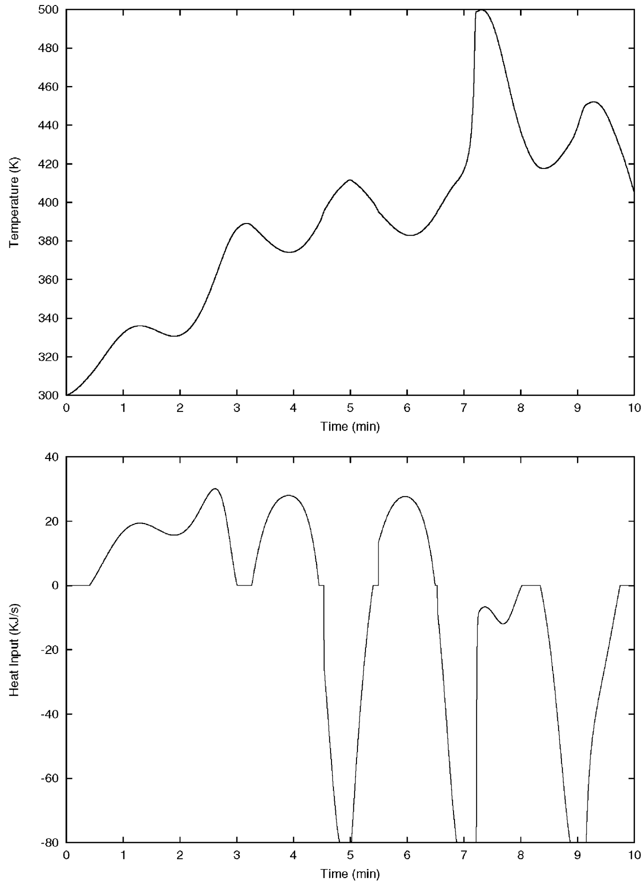

Fig. 3. Controlled output and manipulated input profiles under the controller of Eqs. (21)–(27) without the robust compensation component ($\chi = 0$).

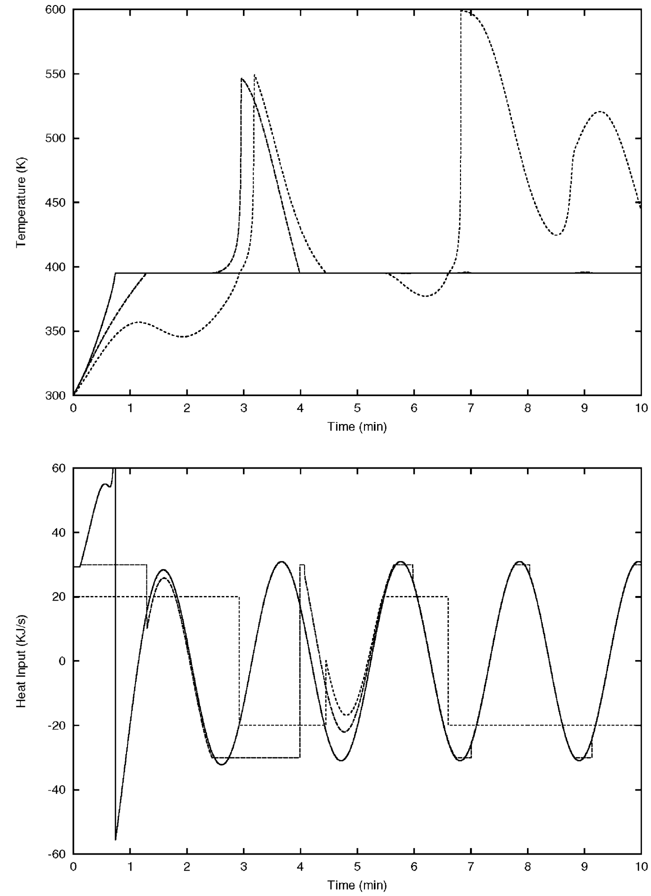

Fig. 4. Closed-loop output and manipulated input profiles under the controller of Eqs. (21)–(27) in the presence of process uncertainty and input constraints of $|Q| \leqslant 60$ kJ/s (solid line), $|Q| \leqslant 30$ kJ/s (dashed line), $|Q| \leqslant 20$ kJ/s (dashed line).

of input constraints despite the fact that the controller was not designed to explicitly deal with the problem of constraints. The inherent capability of the controller to use smaller control effort to robustly stabilize the process assisted the controller in naturally avoiding the imposed limitations on the rate of heat input to the reactor. While such optimal expenditure of control action is clearly a necessary tool that the controller needs to handle input constraints, it might not be sufficient. For example, one can envision situations where the input constraints are tight enough such that the control objectives cannot be met, irrespective of the particular choice of control law, owing to the fundamental limitations imposed by the constraints on the process dynamics. Under such circumstances, it is natural to ask how and whether the robust optimal controller of Eq. (21) will continue to enforce the requested stability and performance specifications in the presence of increasingly tight input constraints.

Fig. 4 provides an answer to this question and shows the temperature and rate of heat input profiles for the cases when $|Q| \leqslant 60$ kJ/s (solid line), $|Q| \leqslant 30$ kJ/s (dashed line) and $|Q| \leqslant 20$ kJ/s (dotted line). It is clear from the figure that although the controller continues to successfully stabilize the process and achieve asymptotic uncertainty attenuation when $|Q| \leqslant 60$ kJ/s, the transient performance of the process begins to deteriorate when $|Q| \leqslant 30$ kJ/s (note the overshoot) and, eventually, the controller is unable to stabilize the process when $|Q| \leqslant 20$ kJ/s. In this case, the available control energy is apparently insufficient to robustly stabilize the reactor temperature at the desired set-point starting from the given initial condition. It is important to note here that this conclusion could not be reached before implementing the controller. The simulation results of Fig. 4 therefore point out the fact that while the robust optimal controller may be better equipped to handle the copresence of uncertainty and constraints in some cases, it provides no explicit or a priori guarantees regarding stability in the presence of arbitrary input constraints. More specifically, the controller design does not provide the necessary knowledge of the set of admissible initial states that guarantee achievement of the desired control objectives in the presence of input constraints. To address this issue, we propose in the next section a direct robust optimal controller design that accounts simultaneously for closed-loop performance and stability in the presence of active input constraints. The controllers designed in the next section are conceptually aligned with those given in this section but have the additional benefits of: (a) integrating robustness, optimality, and explicit constraint handling in the controller design and (b) providing an explicit and a priori characterization of the regions of guaranteed closed-loop stability.

4. Integrating robustness, optimality, and constraints

Having illustrated some the advantages of the robust optimal controller design framework for control of constrained uncertain nonlinear processes, we proceed in this section to extend the capabilities of our robust optimal controllers to incorporate the additional capability of explicitly handling constraints. The key idea behind the new design, to be presented shortly, is that of bounding the robust optimal controllers and is inspired by the results on bounded control presented in Lin and Sontag (1991). In mathematical terms, we seek robust optimal controllers of the form of Eq. (14) that, in addition, satisfy

$$u_{\min} \leqslant u \leqslant u_{\max} \tag{29}$$

and enforce the same desirable closed-loop properties outlined in Section 3.2, including stability, robust asymptotic output tracking, and optimality, in the presence of active input constraints. To simplify our development, it is assumed in the above inequality that $|u_{\min}| = |u_{\max}| = u_{\max}$. Using the same state-space Lyapunov framework developed in the previous section, we derive bounded robust optimal nonlinear control laws that explicitly depend on the magnitude of manipulated input constraints, and generate the necessary control action accordingly. In addition to the explicit incorporation of input constraints, the bounding procedure of the robust optimal controllers yields an explicit characterization of the set of admissible initial conditions, starting from where, the desired control objectives are guaranteed in the presence of uncertainty and constraints (region of closed-loop stability). Therefore, given the inherent constraints on the manipulated input, one can directly ascertain how large the resulting region of closed-loop stability is and check, a priori, whether a certain initial condition belongs to this region. Alternatively, given a desired size of the region of closed-loop stability for a process, one can determine the necessary bounds on the manipulated input.

Similar to the development presented in the previous section, nonlinear processes with vanishing uncertainty are considered first, followed by those with non-vanishing uncertainty.

4.1. Bounded robust optimal controller synthesis under vanishing uncertainty

In this subsection, we consider nonlinear processes of the form of Eq. (1) with vanishing uncertainty and focus on the design of continuous bounded robust optimal nonlinear state feedback control laws that, in the presence of uncertainty and active input constraints: (a) enforce asymptotic stability and asymptotic robust output tracking in the closed-loop system, (b) minimize a meaningful performance index defined on the infinite time interval, and (c) possess a well defined and explicit characterization of their regions of closed-loop stability. Theorem 3 below provides the desired control laws and states precise conditions under which the desired properties hold. The proof of this theorem is given in the appendix.

Theorem 3. *Consider the constrained uncertain nonlinear process of Eq.* (1), *for which Assumptions* 1–3 *hold*, *under the static state feedback law*:

$$u = -\tfrac{1}{2}R^{-1}(x)L_{\bar g}V = -\left(\frac{L_{\bar f}V + (\rho + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar w k}L_f h(x)|)\left(\dfrac{|2b^{\mathrm T}Pe|^2}{|2b^{\mathrm T}Pe| + \phi}\right)}{(L_{\bar g}V)^2[1+\sqrt{1+(u_{\max}L_{\bar g}V)^2}]} + \frac{\sqrt{(L_{\bar f}^{*}V + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar w k}V|)^2 + (u_{\max}L_{\bar g}V)^4}}{(L_{\bar g}V)^2[1+\sqrt{1+(u_{\max}L_{\bar g}V)^2}]}\right)(L_{\bar g}V), \tag{30}$$

where $L_{\bar f}^{*}V = L_{\bar f}V + \rho|2b^{\mathrm T}Pe|$, $V = e^{\mathrm T}Pe$, P *is a positive-definite matrix that satisfies Eq.* (18), *and* ρ, χ *and* ϕ *are adjustable parameters that satisfy* $\rho > 0$, $\chi > 1$ *and* $\phi > 0$. *Assume further that the uncertain variables in Eq.* (1) *are vanishing in the sense defined in Theorem* 1. *Then if the evolution of the closed-loop trajectory satisfies the following inequality*

$$L_{\bar f}^{*}V + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar w k}V| \leqslant u_{\max}|L_{\bar g}V|, \tag{31}$$

there exists $\phi^* > 0$ *such that if* $\phi \in (0,\phi^*]$, *the following holds*:

(1) *The closed-loop system is asymptotically stable.*

(2) *The output of the closed-loop system satisfies a relation of the form*

$$\limsup_{t\to\infty} |y(t) - v(t)| = 0. \tag{32}$$

Furthermore, if the trajectories of the closed-loop system evolve such that $\forall t \geqslant 0$:

$$L_{\bar{f}}^{**}V[2 + 4\sqrt{1 + (u_{\max} L_{\bar{g}} V)^2}] \leqslant \sqrt{(L_{\bar{f}}^{**}V)^2 + (u_{\max} L_{\bar{g}} V)^4}, \tag{33}$$

where $L_{\bar{f}}^{**}V = L_{\bar{f}}^{*}V + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar{w}k}V|$, *and* χ *is chosen to satisfy* $\chi > 2$, *then*:

(3) *The static state feedback control law of Eq.* (30) *is optimal with respect to the cost functional*

$$J = \int_0^\infty (l(e) + uR(x)u)\,\mathrm{d}t, \tag{34}$$

where $R(x) > 0$ *and* $l(e) = -L_{\bar{f}}V + \frac{1}{4}L_{\bar{g}}VR^{-1}(x) L_{\bar{g}}V - \sum_{k=1}^{q}|L_{wk}V|\theta_{bk} \geqslant \sigma|2b^{\mathrm{T}}Pe|^2 \geqslant 0, \sigma > 0$.

Remark 9. Theorem 3 proposes a *direct* robust optimal nonlinear controller design method that accounts explicitly and simultaneously for closed-loop performance and stability in the presence of model uncertainty and active input constraints. Note that the bounded robust optimal control law of Eq. (30) uses explicitly the available knowledge of both the bounds on the uncertain variables (i.e., θ_{bk}) and the manipulated input constraints (i.e., $u_{\max}$) to generate the necessary control action. This is in contrast to the two-step approach typically employed in process control strategies which first involves the design of a controller for the unconstrained process and then accounts for the input constraints through a suitable anti-windup modification to attenuate the adverse effects of improperly handled input constraints.

Remark 10. In addition to proposing a direct controller design strategy, Theorem 3 provides an explicit characterization of the region in state-space where the desired closed-loop stability and set-point tracking properties out-lined in the theorem are guaranteed. This characterization is obtained from the inequality of Eq. (31), which describes explicitly the largest region in state space where the time-derivative of the Lyapunov function is guaranteed to be negative-definite along the trajectories of the closed-loop system under uncertainty and constraints. Any closed-loop trajectory that evolves (i.e. starts and remains) within this region is guaranteed to converge to the desired equilibrium. As will be detailed in the next remark, the inequality of Eq. (31) provides a useful guide for identifying apriori (before implementing the controller) the set of admissible initial conditions, starting from where closed-loop stability is guaranteed. This aspect of the proposed design has important practical implications for efficient process operation since it provides plant operators with a systematic and easy-to-implement guide to identify feasible initial conditions for process operation. Considering the fact that the presence of disturbances and constraints limits the conditions under which the process can be operated safely and reliably, the task of identifying these conditions becomes a central one. This is particularly significant in the case of unstable plants (e.g., exothermic chemical reactor) where lack of such a priori knowledge can lead to undesirable consequences.

Remark 11. Referring to the region described by the inequality of Eq. (31) (which we shall denote by S for notational convenience), it is important to note that even though a trajectory starting in S will move from one Lyapunov surface to an inner Lyapunov surface with lower energy (because $\dot{V} < 0$), there is no guarantee that the trajectory will remain forever in S, since it is not necessarily a region of invariance. Once the trajectory leaves S, however, there is no guarantee that $\dot{V} < 0$. Therefore, in order to use the inequality of Eq. (31) to identify the admissible initial conditions starting from where closed-loop stability is guaranteed, we need to find (or estimate) the largest invariant set within S to guarantee that a trajectory starting in S remains in the region for all future times. A reasonable estimate is provided by the set

$$\Omega_c = \{x \in \mathbb{R}^n : \bar{V}(x) \leqslant c\}$$

when Ω_c is bounded and contained in S and $\dot{\bar{V}}(x) < 0$ over S. This estimate can be made less conservative by selecting the value of c sufficiently large. Referring to the above definition of the invariant set Ω_c, it is important to note that the Lyapunov function $\bar{V}$ is, in general, not the same as the Lyapunov function V given in the statement of Theorem 3. The function $\bar{V}$ is based on the full system (i.e. the ζ, η interconnected system of Eq. (16)) while V is based only on the ζ subsystem. The reason for the difference is the fact that owing to the ISS property of the η subsystem (see assumption 2), only a Lyapunov function based on the ζ subsystem, namely $V = e^TPe$, is needed and used to design a control law that stabilizes the full closed-loop system. However, in constructing Ω_c, we need to guarantee that the evolution of x (and, hence, the evolution of both ζ and η) is confined within S. Therefore, the Lyapunov function in this case must account for the evolution of the η states as well. One possible choice for $\bar{V}$ is a composite Lyapunov function $\bar{V} = V_e + V_\eta$ whose time-derivative is negative-definite at all points satisfying Eq. (31), where V_e is taken to be the same as the Lyapunov function V given in Theorem 3 and V_η is another Lyapunov function for the η subsystem. The latter function is guaranteed to exist, owing to the ISS and asymptotic stability properties of the η subsystem, and can be computed given the particular structure for this system. For nonlinear systems with relative degree equal to n, the choice $\bar{V} = V = e^TPe$ is sufficient.

Remark 12. The inequality of Eq. (31) captures, in an intuitive way, the dependence of the size of region of closed-loop stability on both uncertainty and input constraints. The tighter the input constraints are (i.e., smaller $u_{\max}$), for example, the smaller the resulting closed-loop stability region. Similarly, the larger the plant-model mismatch (i.e., larger θ_{bk}), the smaller the closed-loop stability region. This is consistent with one's intuition, since under such conditions (tighter constraints and larger uncertainty), fewer and fewer initial conditions will satisfy the inequality of Eq. (31) resulting in a smaller closed-loop stability region. Finally, note that, according to the inequality of Eq. (31), the largest region of closed-loop stability under the control law of Eq. (30) is obtained, as expected, in the absence of constraints (i.e., as $u_{\max} \to \infty$).

Remark 13. The inequality of Eq. (31) reveals an interesting interplay between controller design parameters and model uncertainty in influencing the size of the resulting region of guaranteed closed-loop stability. To this end, note the multiplicative appearance of the parameter χ and the bound on the uncertainty θ_{bk} in Eq. (31). In the presence of significant process disturbances, one typically selects a large value for χ to achieve an acceptable level of robust performance of the controller. According to the inequality of Eq. (31), this comes at the expense of obtaining a smaller region of closed-loop stability. Alternatively, if one desires to expand the region of closed-loop stability by selecting a small value for χ, this may be achieved at the expense of obtaining an unsatisfactory degree of uncertainty attenuation or will be limited to cases where the process uncertainty present is not too large (i.e., small θ_{bk}) where a large value for χ is not needed. Therefore, while the presence of the design parameter χ in Eq. (31) offers the possibility of enlarging the region of guaranteed closed-loop stability, a balance must always be maintained between the desired degree of uncertainty attenuation and the desired size of the region of closed-loop stability.

Remark 14. Theorem 3 explains some of the main advantages of using a Lyapunov framework for our development and why it is a natural framework to address the problem within an analytical setting. Owing to the versatility of the Lyapunov approach in serving both as a useful robust optimal design tool (see section 3) and an efficient analysis tool, the proposed controller design method of Theorem 3 integrates explicitly the two seemingly separate tasks of controller design and closed-loop stability analysis into one task. This is in contrast to other controller design approaches where a suitable control law is designed first to meet certain design specifications and then a Lyapunov function is found to analyze the closed-loop stability characteristics. Using a Lyapunov function to examine the closed-loop stability under a predetermined non-Lyapunov-based control law (e.g., a feedback linearizing controller) usually results in a rather conservative stability analysis and yields conservative estimates of the regions of closed-loop stability. In the approach proposed by Theorem 3, however, the Lyapunov function used to characterize the region of guaranteed closed-loop stability under the control law of Eq. (30) is itself the same Lyapunov function used to design the controller and is therefore the only natural Lyapunov function to analyze the closed-loop system.

Remark 15. In the special case when the term $L_{\bar{f}}V$ is negative and no process disturbances are present (i.e., $\theta_{bk} = 0$), the closed-loop stability and performance properties outlined in Theorem 3 will hold globally in the presence of active input constraints. The reason for this is the fact that when $L_{\bar{f}}V < 0$ and $\theta_{bk} = 0$, the inequality of Eq. (31) is automatically satisfied for any initial condition. Consequently, the closed-loop stability and set-point tracking properties are satisfied for any initial condition in the state space; hence the global nature of the result. One implication of this result is that for open-loop stable nonlinear systems of the form of Eq. (1) (where $L_{\bar{f}}V < 0$, e.g., a nonisothermal CSTR with an irreversible endothermic reaction), asymptotic stability and output tracking can always be achieved globally in the presence of arbitrary input constraints using the class of bounded control laws of Eq. (30), either in the presence of no uncertainty or mild enough uncertainty such that Eq. (31) is satisfied.

Remark 16. Despite the conceptual similarities between the robust optimal controller of Eq. (30) and that of Eq. (21), the two control laws differ in several fundamental respects. First, the control law of Eq. (30) is inherently bounded by $u_{\max}$ and generates control action that satisfies the input constraints within the desired region of guaranteed closed-loop stability described by the inequality of Eq. (31). By contrast, the control law of Eq. (21) is unbounded and may compute larger control action that violates the constraints within the same region. Second, the control law of Eq. (30) possesses a well-defined region of guaranteed stability and performance properties in the state space. One can therefore determine, a priori, whether a particular initial condition is feasible. No explicit or analytical characterization of this region is available for the control law of Eq. (21) and therefore no a priori guarantees are available of achieving the desired control objectives. Finally, while the tasks of nominal stabilization and uncertainty attenuation could be conceptually distinguished and assigned separately to the two components of the control law of Eq. (21), this is no longer possible for the controller of Eq. (30) due to the presence of the term $\chi\theta_{bk}|L_{\bar{w}k}V|$ in the radicand, which is a direct consequence of the bounding requirement of the robust optimal controller.

Remark 17. As part of its optimal character and in a somewhat similar fashion to the control law of Eq. (21), the control law of Eq. (30) recognizes the beneficial effects of process nonlinearities and does not expend unnecessary control effort to cancel them. However, unlike the controller of Eq. (21), the control law of Eq. (30) has the additional ability to recognize the extent of the effect of uncertainty on the process and prevent its cancellation if this effect is not significant. To understand this point, recall that the controller of Eq. (21) prevents the unnecessary cancellation of the term $L_{\bar{f}}V$, which does not include the uncertainty, when it is negative. This controller therefore compensates for the presence of any model uncertainty it detects and does not discriminate between small or large uncertainty. In contrast, the controller of Eq. (30) now recognizes the beneficial effect of the entire term $L_{\bar{f}}V + \chi\theta_{bk}|L_{\bar{w}k}V|$ (rather than just the term $L_{\bar{f}}V$) when it is negative and prevents its unnecessary and wasteful cancellation. Note that this term now includes the uncertainty. Therefore, if the plant-model mismatch is not too significant, such that the term $L_{\bar{f}}V + \chi\theta_{bk}|L_{\bar{w}k}V|$ remains negative in the presence of uncertainty, the controller will prevent the expenditure of unnecessary control effort to cancel such uncertainty. Essentially the controller realizes, through the negative sign of this term, that the uncertainty present in the process does not have a strong enough adverse effect to warrant its cancellation. Therefore, the control law of Eq. (30) has the ability to discriminate between small and large model uncertainty and assess the need for optimal control effort accordingly.

Remark 18. Although the controller of Eq. (30) uses small control action to achieve robust stabilization in the presence of constraints, the rigorous proof of optimality of this controller with respect to a cost functional of the form of Eq. (34) requires careful consideration. In contrast to the stability and set-point tracking properties (properties (1) and (2) in Theorem 3), which are guaranteed everywhere within the region of closed-loop stability described by Eq. (31), optimality is guaranteed within a smaller region, contained within the stability region and characterized by Eq. (33). In the proof of Theorem 3, we establish through the inverse optimal approach that the evolution of the trajectories of the closed-loop system within this sub-region renders the cost functional of Eq. (34) meaningful and guarantees that the controller of Eq. (30) is optimal with respect to this cost. The fact that the optimality region is, in general, a subset of the stability region can be understood in light of the bounded nature of the controller of Eq. (30). For example, it was pointed out in Lin and Sontag (1991) that, in contrast to their unbounded counterparts, bounded controls tend to experience a reduction in their stability margins near the boundary of their stability regions. These margins, however, are needed in the inverse optimal approach to establish meaningful optimality. Therefore, to guarantee optimality, one must step back from the boundary of the region of Eq. (31) and restrict the evolution of the trajectories in a smaller region where stability margins are sufficient to render the cost functional meaningful.

Remark 19. Similar to the cost functional of Eq. (23), the cost functional of Eq. (34) places meaningful penalties on the tracking error $e = v - y$ and its time derivatives up to order r, not on the full state of the closed-loop system, and on the control action. Furthermore, J is well defined according to the formulation of the optimal control problem for nonlinear systems because of the following properties of $l(e)$ and $R(x)$. The function $l(e)$ is continuous, positive definite, and bounded from below by a class K function of the norm of e. The function $R(x)$ is continuous, strictly positive, and the term $uR(x)u$ has a unique global minimum at $u = 0$ implying a larger control penalty for u farther away from zero. These properties are the only requirements needed to guarantee that the optimality of the controller of Eq. (30) is meaningful and ensure that the controller enjoys the desired robustness and optimality benefits, regardless of the specific interpretation of what the weights $l(e)$ and $R(x)$ represent. This is one of the basic tenets of the inverse optimal approach. Finally, owing to the vanishing nature of the uncertain variables and the asymptotic stability of the closed-loop system, J is defined on the infinite time-interval and is finite.

4.2. Bounded robust optimal controller synthesis under nonvanishing uncertainty

In this subsection, we consider nonlinear processes of the form of Eq. (1) with nonvanishing uncertainty and focus on the design of continuous bounded robust optimal nonlinear state feedback control laws that, in the presence of active input constraints: (a) enforce boundedness of the closed-loop trajectories and robust output tracking, (b) minimize a meaningful performance index defined on a finite time interval, and (c) possess a well-defined and explicit characterization of their regions of closed-loop stability. Theorem 4 below provides the desired control laws and states precise conditions under which the desired properties hold.

Theorem 4. *Let Assumptions* 1, 3, *and* 4 *hold and assume that the uncertain variables in Eq.* (1) *are non-vanishing in the sense defined in Theorem* 2. *Consider the uncertain nonlinear system of Eq.* (1) *under the static state feedback law of Eq.* (30) *with* $\rho = 0$. *Then if the evolution of the closed-loop trajectory satisfies the following inequality*

$$L_{\bar{f}}V + \chi \sum_{k=1}^{q} \theta_{bk}|L_{\bar{w}k}V| \leqslant u_{\max}|L_{\bar{g}}V| \tag{35}$$

and for every positive real number d, there exists $\phi^*(d) > 0$ *such that if* $\phi \in (0, \phi^*(d)]$, *the following holds*:

(1) *The trajectories of the closed-loop system are bounded.*
(2) *The output of the closed-loop system satisfies a relation of the form*

$$\limsup_{t\to\infty} |y(t) - v(t)| \leqslant d. \tag{36}$$

Furthermore, if the trajectories of the closed-loop system evolve such that $\forall 0 \leqslant t \leqslant T_f$:

$$L'_{\bar{f}}V[2 + 4\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}] \leqslant \sqrt{(L'_{\bar{f}}V)^2 + (u_{\max}L_{\bar{g}}V)^4}, \tag{37}$$

where $L'_{\bar{f}}V = L_{\bar{f}}V + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar{w}k}V|$, *and* χ *is chosen to satisfy* $\chi > 2$, *then*:

(3) *The static state feedback control law of Eq.* (30) *minimizes the cost functional*

$$J = \lim_{t\to T_f} V(e(t)) + \int_0^{T_f} (l(e) + uR(x)u)\,\mathrm{d}t, \tag{38}$$

where $R(x) > 0$, $l(e) \geqslant \bar{\sigma}|2b^{\mathrm{T}}Pe|^2 \geqslant 0\ \forall t \in [0, T_f]$ *where* $T_f = \inf\{T \geqslant 0\colon e(t) \in \Gamma \subseteq D\ \forall t \geqslant T\}$, $\bar{\sigma} > 0$.

Remark 20. Note that the cost functional of Eq. (38) resembles that of Eq. (26) presented in Theorem 2. It is defined only on a finite time interval due to the presence of persistent uncertainty that makes asymptotic convergence to the nominal equilibrium point impossible. In addition, it contains a terminal penalty term to penalize the inability of the tracking error trajectories to converge to the origin. The only difference between the two cost functionals lies in the specific forms of the weights $R(x)$ and $l(e)$. Nonetheless, these weights are well defined and meaningful in both cases rendering the respective control laws optimal with respect to meaningful performance indices.

Remark 21. Referring to the practical applications of the result of Theorem 4, one has to initially verify whether Assumptions 1 and 4 hold for the process under consideration and determine the available bounds on the uncertain variables θ_{bk}. Next, given the available constraints on the manipulated input, the inequality of Eq. (35) should be used, together with the procedure described in Remark 11, to check whether the particular initial condition to start the process is admissible and whether the desirable closed-loop properties in Theorem 4 are guaranteed to hold. Then, Eq. (30) can be used to compute the explicit formula for the controller. Finally, given the asymptotic tracking error desired d (which can be chosen arbitrarily close to zero), the value of ϕ^* should be computed (usually through simulations) to achieve $\limsup_{t\to\infty}|y(t) - v(t)| \leqslant d$. The basic guideline for tuning the parameter ϕ is to select a sufficiently small value (relative to d) for ϕ in order to achieve a small d. Therefore, one initially selects a small value for ϕ and carries out simulations to check if the desired d is achieved. If not, then ϕ must be reduced further until the desired asymptotic error is achieved.

Remark 22. It is important to note that the results of Theorems 3 and 4 can be extended to nonlinear systems with multiple equilibria. In this case, however, caution must be exercised when using the inequalities of Eqs. (31) and (35) to compute the stability region of a certain equilibrium point in the presence of input constraints. Recall that, in the case of a single equilibrium point, these inequalities provide an estimate of the domain of attraction starting from where the trajectories are *guaranteed* to converge to the equilibrium point. Owing to the presence of multiple equilibrium points, however, it is possible for this region to enclose other equilibria and, therefore, portions of the region may be shared by the domains of attraction of these other equilibria. In this case, the inequalities of Eqs. (31)–(35) describe regions in state space starting from where it is just *possible* to steer the process to the corresponding equilibrium point with the available control energy. To guarantee, a priori, convergence to the desired equilibrium point, one can identify, using the techniques reviewed in Khalil (1996), regions of invariance (within the regions described by Eqs. (31)–(35)) around the desired equilibrium point, to exclude the possibility of the system stabilizing at other undesired equilibrium points (see also Kapoor & Daoutidis, 1999a,b for a discussion on this issue).

5. Application to a nonisothermal continuous stirred tank reactor

In this section, we revisit the exothermic continuous stirred tank chemical reactor example considered earlier to illustrate the ability of the bounded robust optimal controllers proposed in the previous section to effectively handle the co-presence of model uncertainty and manipulated input constraints. To this end, consider the same control problem, posed earlier, of regulating the reactor temperature by manipulating the rate of heat input to the reactor. Also consider the same time-varying disturbances and uncertainty in the feed temperature and heat of reaction, respectively, given in Eq. (6). To proceed with the design of the bounded robust optimal controller, we note that the uncertain variables under consideration are non-vanishing and therefore we use the result of Theorem 4 to construct the controller. Using the quadratic Lyapunov function $V = \frac{1}{2}c(T - T_s)^2$, the desired control law takes the form of Eq. (30) where

$$L_{\bar{f}}V + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar{w}k}V| = c(T - T_s)\left[\frac{F}{V}(T_{A0} - T) + \frac{-\Delta H_{\mathrm{nom}}}{\rho c_p}k_0\mathrm{e}^{-E/RT}C_A\right]$$

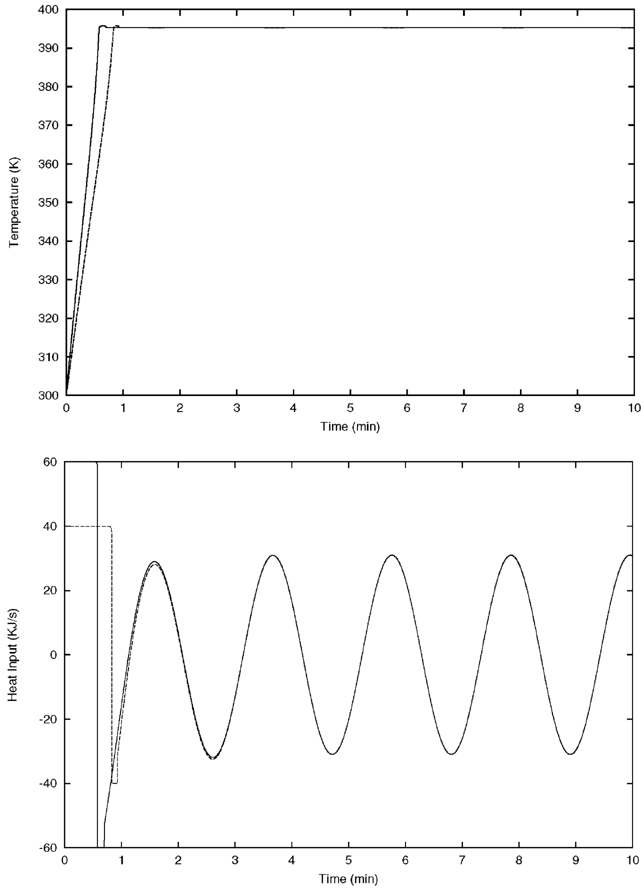

Fig. 5. Closed-loop output and manipulated input profiles under the controller of Eqs. (30)–(39) in the presence of process uncertainty and input constraints of $|Q| \leqslant 60$ kJ/s (solid line) and $|Q| \leqslant 40$ kJ/s (dashed line).

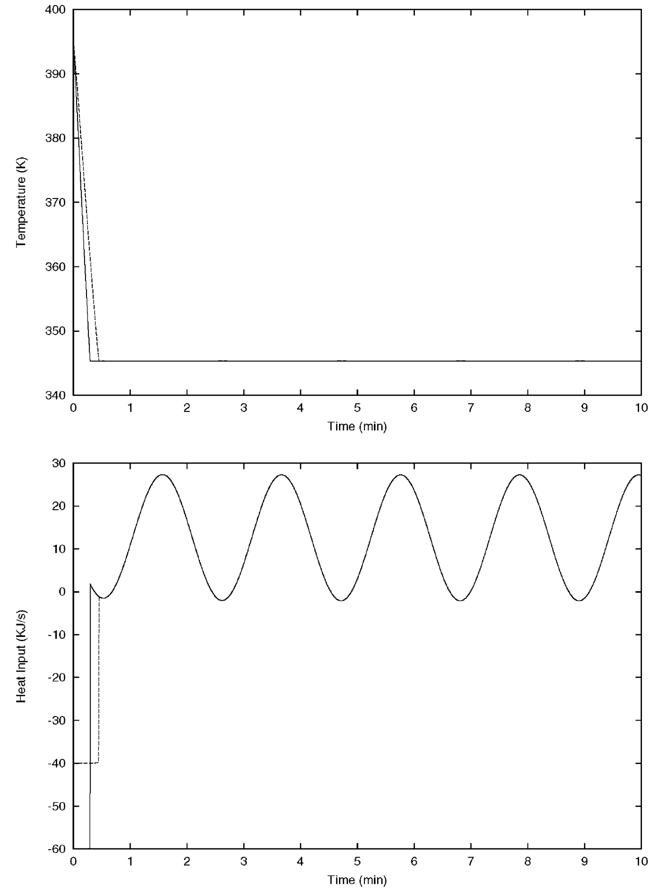

Fig. 6. Closed-loop output and manipulated input profiles for reference input tracking under the controller of Eqs. (30)–(39) in the presence of process uncertainty and input constraints of $|Q| \leqslant 60$ kJ/s (solid line) and $|Q| \leqslant 40$ kJ/s (dashed line).

$$+ \chi\left(\theta_{b1}\frac{F}{V} + \theta_{b2}k_0 e^{-E/RT}C_A\right)c|T - T_s|,$$

$$L_{\bar{g}}V = c(T - T_s)\frac{1}{\rho c_p V}. \tag{39}$$

The following values were used for the controller tuning parameters: $c = 1.0$, $\phi = 0.01$ and $\chi = 2.1$, to guarantee that the output satisfies a relation of the form of Eq. (28). Several sets of simulations were performed to evaluate the performance, robustness, and constraint-handling abilities of the controller of Eq. (30). In the first set of simulations, we examined the startup performance of the reactor and tested the ability of the bounded robust optimal controller to drive and maintain the reactor temperature at the desired (unstable) steady state in the presence of uncertainty and manipulated input constraints. Fig. 5 shows the temperature and rate of heat input profiles for the cases when $|Q| \leqslant 60$ kJ/s (solid line) and $|Q| \leqslant 40$ kJ/s (dashed line). In both cases, one can immediately see that the controller achieves the desired stability and performance properties successfully despite the presence of uncertainty and hard limitations on the rate of heat input. The bounded robust optimal controller uses the information available on the maximum or minimum rate of heat input allowable ($u_{\max}$) to compute the necessary rate of heating or cooling to be provided to the reactor resulting in no discrepancy between the controller output and the actual input to the reactor. Therefore, not only is the control effort generated by the controller of Eq. (30) minimal (due to optimality), but in addition, the controller confines its action within the limits imposed by the input constraints. This feature allows the controller to satisfy the imposed constraints without sacrifice in performance, in contrast to other controllers designed in the absence of constraints and then detuned to satisfy the constraints at the expense of yielding poor performance. Note also that robustness against the presence of time-varying uncertainty is achieved by explicitly incorporating information about the uncertainty in the controller design. This robustness could not be achieved using integral action which confounded the closed-loop instability and performance

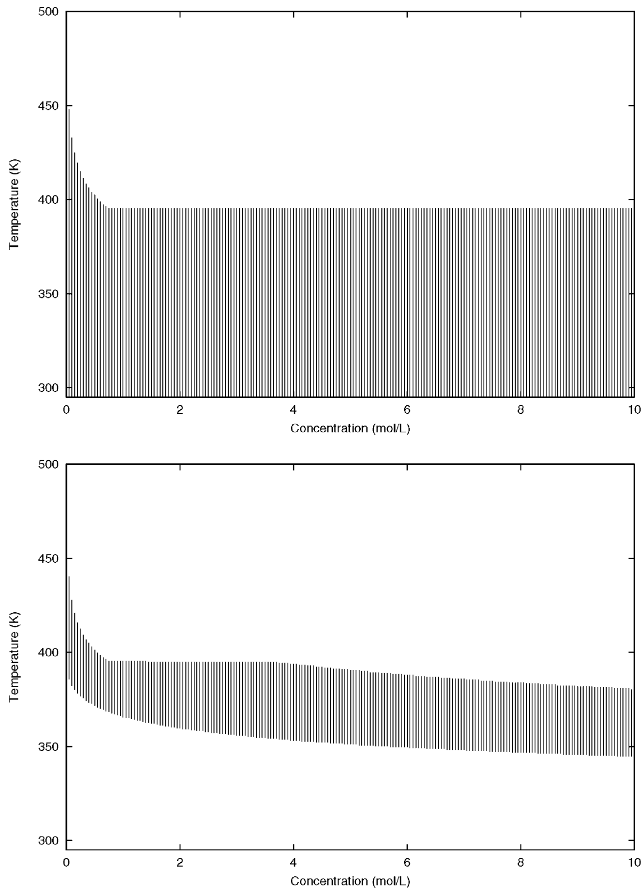

Fig. 7. Regions of guaranteed closed-loop stability under the bounded controller of Eqs. (30)–(39) (top plot) and a robust feedback linearizing controller (bottom plot), $u_{\max} = 50$ kJ/s.

problems in the presence of relatively less tight constraints (see Fig. 1).

In the second set of simulation runs, we tested the robust output tracking capabilities of the robust optimal bounded controller of Eq. (30) in the presence of uncertainty and limitations on the manipulated input. Starting from the steady state given in Table 1, we considered a 50 K decrease in the value of the temperature set point. The resulting temperature and rate of heat input profiles are given in Fig. 6 for the case when $|Q| \leqslant 60$ kJ/s (solid line) and $|Q| \leqslant 40$ kJ/s (dashed line). The figure clearly establishes the ability of the controller to achieve robust output tracking in the presence of input constraints.

Finally, we computed the region of guaranteed closed-loop stability associated with the bounded controller of Eq. (30) using the inequality of Eq. (35) for the case when $u_{\max} = 50$ kJ/s and $\chi = 1.1$. The resulting region is depicted Fig. 7 (top plot). For the sake of comparison, we included in the same figure the estimate associated with a robust input/output linearizing controller designed using the results in Christofides et al. (1996) (bottom plot). The latter estimate was obtained using the procedure proposed in Kapoor and Daoutidis (1998). Both regions basically depict the points in the concentration-temperature space where the input constraints are satisfied and the closed-loop trajectories are guaranteed to stabilize (provided they remain within the region). From this comparison, it is clear that the region of guaranteed closed-loop stability associated with the bounded controller of Eq. (30) is larger. This means that one can safely operate the process and guarantee stability starting from a wider range of initial conditions than that where the feedback linearizing controller is guaranteed to work. The larger region of guaranteed closed-loop stability for the controller of Eq. (30) is a consequence of the fact that the bounded controller is designed to account explicitly for input constraints while the input/output linearizing controller is not. Furthermore, the fact that the input/output linearizing controller may generate unnecessarily large control action to cancel beneficial process nonlinearities renders many of the initial conditions that lie far from the equilibrium point inadmissible, since the control energy required to stabilize the process starting from these conditions is often larger than the control action available from the input constraints. In the case of the bounded controller, however, many of these initial conditions are admissible since the controller avoids the use of unnecessarily large control action to stabilize the process and, consequently, the required control action is more likely to satisfy the imposed constraints. The end result then is a larger region of guaranteed closed-loop stability.

6. Conclusions

A unified framework for control of single-input–single-output constrained uncertain nonlinear processes that integrates robustness, optimality, and explicit constraint-handling capabilities in the controller synthesis was presented. The developed Lyapunov-based framework led to the synthesis of bounded robust optimal state feedback control laws with well-characterized stability and performance properties and provided, at the same time, an explicit and intuitive characterization of the state-space regions of guaranteed closed-loop stability in terms of the input constraints and model uncertainty. For processes with vanishing uncertainty, the developed controllers were shown to guarantee asymptotic stability and asymptotic robust output tracking with attenuation of the effect of uncertainty on the output of the closed-loop system in the presence of active input constraints. For processes with nonvanishing uncertainty, the same controllers were shown to ensure boundedness of the states and robust asymptotic output tracking in the presence of active input constraints. The performance of the control laws was illustrated through the use of a chemical reactor example and compared with existing process control strategies.

Appendix

Notation

- $|\cdot|$ Denotes the standard Euclidean norm, $\text{sgn}(\cdot)$ denotes the sign function, and $\text{sat}(\cdot)$ denotes the saturation function, defined as

$$\text{sat}(u) = \begin{cases} u_{\max} & \text{if } u > u_{\max}, \\ u & \text{if } u_{\min} \leqslant u \leqslant u_{\max}, \\ u_{\min} & \text{if } u < u_{\min}, \end{cases} \tag{A.1}$$

  where $u_{\min}$ and $u_{\max}$ are real numbers.
- $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f. $L_f^k h$ denotes the kth-order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.
- For any measurable (with respect to the Lebesgue measure) function $\theta : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^q$, $\|\theta\|$ denotes ess sup $|\theta(t)|$, $t \geqslant 0$.
- A function $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is said to be positive definite if $W(x) > 0 \ \forall x \neq 0$ and $W(0) = 0$.
- A function $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is said to be proper if $W(x)$ tends to $+\infty$ as $|x|$ tends to $+\infty$.
- A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class K if it is continuous, increasing and is zero at zero. It is of class K_∞, if in addition, it is proper.
- A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.
- For any function γ of class K_∞, its inverse function is well defined and is again of class K_∞.

Proof of Theorem 3. The proof of this theorem is divided into two parts. In the first part, we establish that, if the inequality of Eq. (31) is satisfied, the state feedback controller of Eq. (30) asymptotically stabilizes the closed-loop system and that the closed-loop output satisfies Eq. (32). In the second part, we prove that the controller of Eq. (30) is optimal with respect to a meaningful cost functional of the form defined in Eq. (34). To simplify the development of the proof, we consider only the case of a single uncertain variable ($q = 1$). Extension to the case when $q > 1$ is conceptually straightforward.

Part 1: We initially consider the representation of the closed-loop system in terms of the ζ, η coordinates and introduce the variables $e_i = \zeta_i - v^{(i-1)}$, $i = 1, \ldots, r$ and the notation $\bar{v} = [v \ \ v^{(1)} \ \ \cdots \ \ v^{(r)}]^{\mathrm{T}}$, to write the closed-loop system in the following form:

$$\begin{aligned}
\dot{e}_1 &= e_2, \\
&\vdots \\
\dot{e}_{r-1} &= e_r, \\
\dot{e}_r &= L_f^r h(T^{-1}(e, \bar{v}, \eta)) - v^{(r)} + L_g L_f^{r-1} h(T^{-1}(e, \bar{v}, \eta))u \\
&\quad + L_w L_f^{r-1} h(T^{-1}(e, \bar{v}, \eta))\theta, \\
\dot{\eta}_1 &= \Psi_1(e, \bar{v}, \eta, \theta), \\
&\vdots \\
\dot{\eta}_{n-r} &= \Psi_{n-r}(e, \bar{v}, \eta, \theta), \\
y &= e_1 + v.
\end{aligned} \tag{A.2}$$

We now follow a three-step procedure to establish the asymptotic stability of the closed-loop system of Eq. (A.2). Initially, we use a Lyapunov argument to show that, if the inequality of Eq. (31) is satisfied, the states of the closed-loop e-subsystem converge asymptotically to the origin, and derive bounds that capture the evolution of the states of the e and η subsystems. We then invoke a small gain argument to show that the trajectories of the $e - \eta$ interconnected system remain bounded for all time starting from any initial state satisfying Eq. (31). Finally, we show that the states of the full closed-loop system of Eq. (A.2) converge to the origin and that the output satisfies the relation of Eq. (32).

Step 1: Consider the smooth positive-definite radially unbounded function, $V : \mathbb{R}^r \to \mathbb{R}_{\geqslant 0}$, $V = e^{\mathrm{T}} P e$ as a Lyapunov function candidate for the e-subsystem of Eq. (A.2). Computing the time-derivative of V along the trajectories of the closed-loop e-subsystem, we get

$$\begin{aligned}
\dot{V} = L_{\bar{f}} V + L_{\bar{g}} V u + L_{\bar{w}} V \theta = L_{\bar{f}} V - \Bigg(& \frac{L_{\bar{f}} V + (\rho |2b^{\mathrm{T}} P e| + \chi \theta_b |L_{\bar{w}} V|)(|2b^{\mathrm{T}} P e| / (|2b^{\mathrm{T}} P e| + \phi)}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}} V)^2}]} \\
& + \frac{\sqrt{(L_{\bar{f}}^* V + \chi \theta_b |L_{\bar{w}} V|)^2 + (u_{\max} L_{\bar{g}} V)^4}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}} V)^2}]} \Bigg) + L_{\bar{w}} V \theta.
\end{aligned} \tag{A.3}$$

Using the fact that $L_{\bar{w}} V \theta \leqslant |L_{\bar{w}} V| \theta_b$, we have, after some algebraic manipulations, that

$$\begin{aligned}
\dot{V} \leqslant \Bigg(& \frac{(L_{\bar{f}} V + \chi \theta_b |L_{\bar{w}} V|) \sqrt{1 + (u_{\max} L_{\bar{g}} V)^2}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}} V)^2}]} - \frac{\sqrt{(L_{\bar{f}}^* V + \chi \theta_b |L_{\bar{w}} V|)^2 + (u_{\max} L_{\bar{g}} V)^4}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}} V)^2}]} \\
& + \frac{\theta_b |L_{\bar{w}} V| ((\phi - (\chi - 1)|2b^{\mathrm{T}} P e|) / |2b^{\mathrm{T}} P e| + \phi) - \rho (|2b^{\mathrm{T}} P e|^2 / (|2b^{\mathrm{T}} P e| + \phi))}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}} V)^2}]} \Bigg).
\end{aligned} \tag{A.4}$$

Since $\chi > 1$ and $\rho > 0$, it is clear from the above inequality that, whenever $|2b^{\mathrm{T}}Pe| > \phi/(\chi - 1)$, the last two terms on the right-hand side are strictly negative, and therefore $\dot{V}$ satisfies

$$\dot{V} \leqslant \left(\frac{(L_{\bar{f}}V + \chi\theta_b|L_{\bar{w}}V|)\sqrt{1 + (u_{\max}\bar{g}V)^2}}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} - \frac{\sqrt{(L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V|)^2 + (u_{\max}L_{\bar{g}}V)^4}}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} \right) \equiv K(e,\eta,\bar{v}) \tag{A.5}$$

$\forall |2b^{\mathrm{T}}Pe| > \phi/(\chi - 1)$. To study the behavior of $\dot{V}$ when $|2b^{\mathrm{T}}Pe| \leqslant \phi/(\chi - 1)$, we exploit the fact that the uncertain variables considered in Theorem 3 are vanishing in the sense that the function $\bar{w}(e,\eta,\bar{v}) = L_w L_f^{r-1}h(x)$ satisfies a linear growth bound of the form $|L_w L_f^{r-1}h(x)| \leqslant \delta|2b^{\mathrm{T}}Pe|$, where δ is a positive constant, whenever $|2b^{\mathrm{T}}Pe| \leqslant \phi/(\frac{1}{2}\chi - 1)$ (and hence whenever $|2b^{\mathrm{T}}Pe| \leqslant \phi/(\chi - 1)$). Using this bound together with the explicit expression for the function $L_{\bar{w}}V$ given by

$$L_{\bar{w}}V = \frac{\partial V}{\partial e}\bar{w} = (2b^{\mathrm{T}}Pe)L_w L_f^{r-1}h(x), \tag{A.6}$$

we obtain the following estimates:

$$\begin{aligned} &\theta_b|L_{\bar{w}}V|(\phi - (\chi - 1)|2b^{\mathrm{T}}Pe|) \\ &\leqslant \theta_b|L_{\bar{w}}V|\phi \\ &= \theta_b|L_w L_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|\phi \\ &\leqslant \theta_b\delta|2b^{\mathrm{T}}Pe|^2\phi \quad \forall |2b^{\mathrm{T}}Pe| \leqslant \frac{\phi}{\chi - 1}. \end{aligned} \tag{A.7}$$

Substituting the estimate of Eq. (A.7) directly into Eq. (A.4), we get

$$\dot{V} \leqslant K(e,\eta,\bar{v}) + \left(\frac{(\theta_b\delta\phi - \rho)|2b^{\mathrm{T}}Pe|^2}{(|2b^{\mathrm{T}}Pe| + \phi)[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} \right) \tag{A.8}$$

$\forall |2b^{\mathrm{T}}Pe| \leqslant \phi/(\chi - 1)$. If ϕ is sufficiently small to satisfy the bound $\phi \leqslant \rho/\delta\theta_b \equiv 2\phi^{*}$, then it is clear from Eqs. (A.5)–(A.8) that the inequality of Eq. (A.5) is satisfied, irrespective of the value of $|2b^{\mathrm{T}}Pe|$. From this equation, it is evident that analyzing the sign of $\dot{V}$ further depends on the sign of the term $L_{\bar{f}}V + \chi\theta_b|L_{\bar{w}}V|$. To this end, we have the following two cases:

Case 1: $L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V| \leqslant 0$. Since $L_{\bar{f}}^{*}V = L_{\bar{f}}V + \rho|2b^{\mathrm{T}}Pe|$ and ρ is a positive real number, the fact that $L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V| \leqslant 0$ then implies that $L_{\bar{f}}V + \chi\theta_b|L_{\bar{w}}V| \leqslant 0$. As a result, the first term on the right-hand side of the inequality of Eq. (A.5) is either zero or negative. Therefore, the time derivative of V in this case satisfies the following bounds:

$$\dot{V} \leqslant \frac{-\sqrt{(L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V|)^2 + (u_{\max}L_{\bar{g}}V)^4}}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} \leqslant \frac{-(u_{\max}L_{\bar{g}}V)^2}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}. \tag{A.9}$$

Substituting the expression for the function $L_{\bar{g}}V$ given by

$$L_{\bar{g}}V = \frac{\partial V}{\partial e}\bar{g} = (2b^{\mathrm{T}}Pe)L_g L_f^{r-1}h(x) \tag{A.10}$$

into the last inequality of Eq. (A.9), we get

$$\dot{V} \leqslant \frac{-4(u_{\max}L_g L_f^{r-1}h(x))^2|b^{\mathrm{T}}Pe|^2}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}. \tag{A.11}$$

Since $L_g L_f^{r-1}h(x) \neq 0 \ \forall x \in \mathbb{R}^n$ from Assumption 1, we have that there exists a positive constant k_1 such that $4(u_{\max}L_g L_f^{r-1}h(x))^2 = L(x) \geqslant k_1$ (e.g., set $k_1 = \inf_{x\in\mathbb{R}^n} L(x)$). The above inequality therefore reduces to

$$\dot{V} \leqslant -\frac{k_1|b^{\mathrm{T}}Pe|^2}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}. \tag{A.12}$$

Case 2: $0 < L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V| \leqslant u_{\max}|L_{\bar{g}}V|$. In this case, we have

$$(L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V|)^2 \leqslant (u_{\max}L_{\bar{g}}V)^2 \tag{A.13}$$

and therefore

$$\begin{aligned} &-\sqrt{(L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V|)^2 + (u_{\max}L_{\bar{g}}V)^4} \\ &= -\sqrt{(L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V|)^2 + (u_{\max}L_{\bar{g}}V)^2(u_{\max}L_{\bar{g}}V)^2} \\ &\leqslant -(L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V|)\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}. \end{aligned} \tag{A.14}$$

Substituting the estimate of Eq. (A.14) in the expression for $\dot{V}$ in Eq. (A.5) yields

$$\begin{aligned} \dot{V} &\leqslant \frac{(L_{\bar{f}}V - L_{\bar{f}}^{*}V)\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} \\ &= \frac{-\rho|2b^{\mathrm{T}}Pe|\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} \\ &\leqslant \frac{-\rho u_{\max}|2b^{\mathrm{T}}Pe||L_{\bar{g}}V|}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}. \end{aligned} \tag{A.15}$$

Using the expression for $L_{\bar{g}}V$ in Eq. (A.10), the last inequality can be written as

$$\dot{V} \leqslant \frac{-\rho u_{\max}|L_g L_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|^2}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} \leqslant \frac{-k_2|b^{\mathrm{T}}Pe|^2}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}, \tag{A.16}$$

where $k_2 = 2\rho\sqrt{k_1} > 0$. Summarizing, we have that if $\phi \leqslant \phi^*$ and $L_f^* V + \chi\theta_b|L_{\bar{w}}V| \leqslant u_{\max}|L_{\bar{g}}V|$, $\dot{V}$ satisfies

$$\dot{V} \leqslant \frac{-k_3|b^{\mathrm{T}}Pe|^2}{[1+\sqrt{1+(u_{\max}L_{\bar{g}}V)^2}]}, \tag{A.17}$$

where $k_3 = \max\{k_1, k_2\} > 0$. Note that since the right-hand side of Eq. (A.17) vanishes when $b^{\mathrm{T}}Pe = 0$ (and not just when $e = 0$), Eq. (A.17) by itself allows us to conclude only that $\dot{V}$ is negative-semidefinite. In what follows, however, we exploit the specific properties of the function V to show that $\dot{V}$ cannot vanish except when $e = 0$ and is, therefore, negative-definite. To this end, consider the representation of the e-subsystem of Eq. (A.12) in the following compact form:

$$\dot{e} = Ae + b[m_0(e,\eta,\bar{v}) + m_1(e,\eta,\bar{v})u + m_2(e,\eta,\bar{v})\theta], \tag{A.18}$$

where A and b are the $r\times r$ matrix and $r\times 1$ vector, respectively, defined in Eq. (19), and the functions $m_0(e,\eta,\bar{v}) = L_f^r h(T^{-1}(e,\eta,\bar{v})) - v^r$, $m_1(e,\eta,\bar{v}) = L_g L_f^{r-1}h(T^{-1}(e,\eta,\bar{v}))$, and $m_2(e,\eta,\bar{v}) = L_w L_f^{r-1}h(T^{-1}(e,\eta,\bar{v}))$ are continuous with $m_1(e,\eta,\bar{v})$ nonsingular for all e and η. Direct computation of the time-derivative of the function $V = e^{\mathrm{T}}Pe$ along the trajectories of the system of Eq. (A.18) yields

$$\begin{aligned}\dot{V} &= e^{\mathrm{T}}[A^{\mathrm{T}}P + PA]e + 2b^{\mathrm{T}}Pe[m_0(e,\eta,\bar{v}) \\ &\quad + m_1(e,\eta,\bar{v})u + m_2(e,\eta,\bar{v})\theta].\end{aligned} \tag{A.19}$$

Now we use the fact that the matrix P satisfies the Riccati inequality $A^{\mathrm{T}}P + PA - Pbb^{\mathrm{T}}P < 0$ and choose P, without loss of generality, to be the symmetric positive-definite solution to the Riccati equation

$$A^{\mathrm{T}}P + PA - Pbb^{\mathrm{T}}P + I = 0. \tag{A.20}$$

The time derivative of V then satisfies

$$\begin{aligned}\dot{V} &= -e^{\mathrm{T}}e + b^{\mathrm{T}}Pe[b^{\mathrm{T}}Pe + 2m_0(e,\eta,\bar{v}) \\ &\quad + 2m_1(e,\eta,\bar{v})u + 2m_2(e,\eta,\bar{v})\theta] \\ &\leqslant -e^{\mathrm{T}}e + b^{\mathrm{T}}Pe[b^{\mathrm{T}}Pe + 2m_0(e,\eta,\bar{v}) \\ &\quad + 2m_1(e,\eta,\bar{v})u] + 2|b^{\mathrm{T}}Pe||m_2(e,\eta,\bar{v})|\theta_b.\end{aligned} \tag{A.21}$$

It is clear from the above equation then that whenever $b^{\mathrm{T}}Pe = 0$ and $e \neq 0$, we have $\dot{V} \leqslant -|e|^2 < 0$; and since $\dot{V} < 0$ whenever $b^{\mathrm{T}}Pe \neq 0$ (from Eq. (A.17)), we deduce that $\dot{V} < 0$ $\forall e \neq 0$. Consequently, there exists a function β_e of class KL (see Khalil, 1996 for details) such that the following ISS inequality holds for the e states of the system of Eq. (A.2)

$$|e(t)| \leqslant \beta_e(|e(0)|, t) \quad \forall t \geqslant 0 \tag{A.22}$$

and the origin of the e-subsystem is asymptotically stable whenever Eq. (31) holds. From Assumption 2, we have that the η subsystem of Eq. (A.2) possesses an ISS property with respect to e which implies that there exists a function β_η of class KL and a function γ_η of class K such that the following ISS inequality holds

$$|\eta(t)| \leqslant \beta_\eta(|\eta(0)|, t) + \gamma_\eta(\|e\|) \quad \forall t \geqslant 0 \tag{A.23}$$

uniformly in θ.

Step 2: We now analyze the behavior of the interconnected dynamical system comprised of the e,η states of the system of Eq. (A.2) for which the inequalities of Eq. (A.22) and Eq. (A.23) hold. In order to proceed with our analysis, we need to define the following positive real numbers: $\delta_e = D^e + \phi^*, \delta_\eta = D^\eta + \gamma_\eta(\delta_e) + \phi^*, D^e = \beta_e(\bar{\delta}_e, 0), D^\eta = \beta_\eta(\bar{\delta}_\eta, 0)$, $\phi^* = \rho/2\delta\theta_b$ and $\bar{\delta}_e$, $\bar{\delta}_\eta$ are any positive real numbers that satisfy Eq. (31). Using a contradiction argument similar to that used in Christofides and Teel (1996) and Christofides et al. (1996), we show that if $\phi \in (0, \phi^*)$, then, starting from any initial state that satisfies $|e(0)| \leqslant \bar{\delta}_e$, $|\eta(0)| \leqslant \bar{\delta}_\eta$, the evolution of the states e, η satisfies the following inequalities:

$$|e(t)| \leqslant \delta_e, \quad |\eta(t)| \leqslant \delta_\eta \tag{A.24}$$

for all times and that the states are therefore bounded. Let T be the smallest time such that there is a δ^* so that $t \in (T, T+\delta^*)$ implies either $|e(t)| > \delta_e$ or $|\eta(t)| > \delta_\eta$. Then, for each $t \in [0, T]$ the conditions of Eq. (A.24) hold. Consider the functions $e^{\mathrm{T}}(t), \eta^{\mathrm{T}}(t)$ defined as follows:

$$e^{\mathrm{T}}(t) = \left\{\begin{matrix} e(t) & t \in [0, T] \\ 0 & t \in (T, \infty) \end{matrix}\right\}, \quad \eta^{\mathrm{T}}(t) = \left\{\begin{matrix} \eta(t) & t \in [0, T] \\ 0 & t \in (T, \infty) \end{matrix}\right\}. \tag{A.25}$$

From the fact that $|e(0)| \leqslant \bar{\delta}_e, |\eta(0)| \leqslant \bar{\delta}_\eta$, we have that

$$\begin{aligned}&\sup_{0 \leqslant t \leqslant T} (\beta_\eta(|\eta(0)|, t)) \leqslant \beta_\eta(\bar{\delta}_\eta, 0) =: D^\eta, \\ &\sup_{0 \leqslant t \leqslant T} (\beta_e(|e(0)|, t)) \leqslant \beta_e(\bar{\delta}_e, 0) =: D^e.\end{aligned} \tag{A.26}$$

Combining the above inequalities with Eqs. (A.22) and (A.23), we have that for each $\phi \in (0, \phi^*]$ and for all $t \geqslant 0$,

$$\begin{aligned}&\|e^{\mathrm{T}}\| \leqslant D^e < \delta_e, \\ &\|\eta^{\mathrm{T}}\| \leqslant D^\eta + \gamma_\eta(\|e\|) \leqslant D^\eta + \gamma_\eta(\delta_e) < \delta_\eta.\end{aligned} \tag{A.27}$$

By continuity, we have that there exists some positive real number $\bar{k}$ such that $\|e^{\mathrm{T}+\bar{k}}(t)\| \leqslant \delta_e$ and $\|\eta^{\mathrm{T}+\bar{k}}(t)\| \leqslant \delta_\eta$, $\forall t \in [0, T+\bar{k}]$. This contradicts the definition of T. Hence, Eq. (A.24) holds $\forall t \geqslant 0$ and the states are bounded.

Step 3: In this step, we show that if the η subsystem of Eq. (A.2), with e as input, is ISS and the origin of the e subsystem of Eq. (A.2) is asymptotically stable whenever Eq. (31) holds, then the origin of the interconnected system of Eq. (A.2) is asymptotically stable whenever Eq. (31) holds. We proceed by performing calculations similar to those done in Khalil (1996). From Eqs. (A.22) and (A.23), we have that the solutions of the system of Eq. (A.2) satisfy

$$|\eta(t)| \leqslant \beta_\eta(|\eta(s)|, t-s) + \gamma_\eta\left(\sup_{s \leqslant \tau \leqslant t} |e(\tau)|\right), \tag{A.28}$$

$$|e(t)| \leqslant \beta_e(|e(s)|, t-s) \tag{A.29}$$

whenever Eq. (31) holds, where $t \geqslant s \geqslant 0$. Applying Eq. (A.28) with $s = t/2$, we obtain

$$|\eta(t)| \leqslant \beta_\eta\left(\left|\eta\left(\frac{t}{2}\right)\right|, \frac{t}{2}\right) + \gamma_\eta\left(\sup_{t/2 \leqslant \tau \leqslant t} |e(\tau)|\right). \tag{A.30}$$

To estimate $\eta(t/2)$, we apply Eq. (A.28) with $s = 0$ and t replaced by $t/2$ to obtain

$$\left|\eta\left(\frac{t}{2}\right)\right| \leqslant \beta_\eta\left(|\eta(0)|, \frac{t}{2}\right) + \gamma_\eta\left(\sup_{0 \leqslant \tau \leqslant t/2} |e(\tau)|\right). \tag{A.31}$$

From Eq. (A.29), we have

$$\sup_{0 \leqslant \tau \leqslant t/2} |e(\tau)| \leqslant \beta_e(|e(0)|, 0), \tag{A.32}$$

$$\sup_{t/2 \leqslant \tau \leqslant 0} |e(\tau)| \leqslant \beta_e\left(|e(0)|, \frac{t}{2}\right). \tag{A.33}$$

Substituting Eqs. (A.31)–(A.32) into Eq. (A.30) and using the inequalities $|\eta(0)| \leqslant |[e^{\mathrm{T}}(0)\eta^{\mathrm{T}}(0)]^{\mathrm{T}}|$, $|e(0)| \leqslant |[e^{\mathrm{T}}(0)\eta^{\mathrm{T}}(0)]^{\mathrm{T}}|$ and $|[e^{\mathrm{T}}(t)\eta^{\mathrm{T}}(t)]^{\mathrm{T}}| \leqslant |\eta(0)| + |e(0)|$, we obtain

$$|[e^{\mathrm{T}}(t)\eta^{\mathrm{T}}(t)]^{\mathrm{T}}| \leqslant \beta(|[e^{\mathrm{T}}(0)\eta^{\mathrm{T}}(0)]^{\mathrm{T}}|, t), \tag{A.34}$$

where

$$\beta(r,s) = \beta_\eta\left(\beta_\eta\left(r, \frac{s}{2}\right) + \gamma_\eta(\beta_e(r,0)), \frac{s}{2}\right) + \gamma_\eta\left(\beta_e\left(r, \frac{s}{2}\right)\right) + \beta_e(r,s). \tag{A.35}$$

It can be easily verified that β is a class KL function for all $r \geqslant 0$. Hence the origin of Eq. (A.2) is asymptotically stable. It follows then from Assumption 1 that the closed-loop system is asymptotically stable. The asymptotic output tracking result can be obtained by simply taking the limsup of both sides of Eq. (A.22) which yields

$$\limsup_{t \to \infty} |e(t)| = 0. \tag{A.36}$$

It follows then that

$$\limsup_{t \to \infty} |e_1(t)| = \limsup_{t \to \infty} |y(t) - v(t)| = 0. \tag{A.37}$$

Part 2: In this part, we prove that the control law of Eq. (30) is optimal with respect to a meaningful cost functional defined in Eq. (34). We proceed in two steps. In the first step, we show that the cost functional of Eq. (34) is a meaningful one according to the formulation of the optimal control problem for nonlinear systems. In the second step, we show that the stabilizing control law of Eq. (30) minimizes this cost functional.

Step 1: In this step, we show that the cost functional defined in Eq. (34) is meaningful by proving directly that $l(e)$ defined in Theorem 3 is a positive-definite function, independently of the designed control. From its definition in Theorem 3, $l(e)$ is given by

$$l(e) = -L_{\bar{f}}V + \tfrac{1}{4}L_{\bar{g}}VR^{-1}(x)L_{\bar{g}}V - |L_{\bar{w}}V|\theta_b. \tag{A.38}$$

Direct substitution of the expression for $R^{-1}(x)$ given in Theorem 3 in the above equation yields

$$l(e) = \left(\frac{-\frac{1}{2}L_{\bar{f}}V - (L_{\bar{f}}V + \theta_b|L_{\bar{w}}V|)\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} + \frac{\frac{1}{2}\sqrt{(L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V|)^2 + (u_{\max}L_{\bar{g}}V)^4}}{[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} + \frac{\frac{1}{2}\rho|2b^{\mathrm{T}}Pe|^2 + \theta_b|L_{\bar{w}}V|((\frac{1}{2}\chi - 1)|2b^{\mathrm{T}}Pe| - \phi)}{(|2b^{\mathrm{T}}Pe| + \phi)[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}\right). \tag{A.39}$$

Using the fact that $L_{\bar{f}}V \leqslant L_{\bar{f}}^{*}V + \chi\theta_b|L_{\bar{w}}V| = L_{\bar{f}}^{**}V$, we have the estimate

$$l(e) \geqslant \left(\frac{-L_{\bar{f}}^{**}V(\frac{1}{2} + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}) + \frac{1}{2}\sqrt{(L_{\bar{f}}^{**}V)^2 + (u_{\max}L_{\bar{g}}V)^4}}{1 + [\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} + \frac{\frac{1}{2}\rho|2b^{\mathrm{T}}Pe|^2 + \theta_b|L_{\bar{w}}V|((\frac{1}{2}\chi - 1)|2b^{\mathrm{T}}Pe| - \phi)}{(|2b^{\mathrm{T}}Pe| + \phi)[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}\right) \tag{A.40}$$

which, with the aid of Eq. (33), reduces to

$$l(e) \geqslant \left(\frac{\frac{1}{4}\sqrt{(L_{\bar{f}}^{**}V)^2 + (u_{\max}L_{\bar{g}}V)^4}}{1 + [\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]} + \frac{\frac{1}{2}\rho|2b^{\mathrm{T}}Pe|^2 + \theta_b|L_{\bar{w}}V|((\frac{1}{2}\chi - 1)|2b^{\mathrm{T}}Pe| - \phi)}{(|2b^{\mathrm{T}}Pe| + \phi)[1 + \sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}\right). \tag{A.41}$$

Note that Eq. (33) implies that $L_{\bar{f}}^{**}V \leqslant u_{\max}|L_{\bar{g}}V|$ and therefore the trajectories of the closed-loop system evolve within the stability region described by Eq. (31). From the last inequality and the fact that $\chi > 2$, it is clear that whenever $|2b^{\mathrm{T}}Pe| > \phi/(\frac{1}{2}\chi - 1)$, $l(e)$ satisfies

$$l(e) \geqslant \left(\frac{\frac{1}{4}\sqrt{(L_{\bar{f}}^{**}V)^2 + (u_{\max}L_{\bar{g}}V)^4}}{1 + [\sqrt{1 + (u_{\max}L_{\bar{g}}V)^2}]}\right) \geqslant \alpha_1(|e|), \tag{A.42}$$

where $\alpha_1(\cdot)$ is a class K function of its argument. The second inequality in Eq. (A.42) follows directly from the fact that the left-hand side of this inequality is positive-definite and vanishes only when $e = 0$ (this is because the terms $L_{\bar{g}}V$ and $L_{\bar{f}}^{*}V$ cannot vanish together away from $e=0$ due to Eq. (10)). For the case when $|2b^{\mathrm{T}}Pe| \leqslant \phi/(\frac{1}{2}\chi - 1)$, we substitute the bounds $|L_w L_f^{r-1}h(x)| \leqslant \delta|2b^{\mathrm{T}}Pe|$ and $(\frac{1}{2}\chi-1)|2b^{\mathrm{T}}Pe| - \phi \geqslant -\phi$ together with the definition of $L_{\bar{w}}V$ into Eq. (A.41) to obtain

$$l(e) \geqslant \alpha_1(|e|) + \left(\frac{(\frac{1}{2}\rho - \theta_b\delta\phi)|2b^{\mathrm{T}}Pe|^2}{(|2b^{\mathrm{T}}Pe| + \phi)[1+\sqrt{1+(u_{\max}L_{\bar{g}}V)^2}]}\right) \tag{A.43}$$

$\forall |2b^{\mathrm{T}}Pe| \leqslant \phi/(\frac{1}{2}\chi - 1)$. If ϕ is small enough to satisfy $\phi \leqslant \rho/2\delta\theta_b \equiv \phi^*$, then the inequality of Eq. (A.42) is satisfied irrespective of the value of $|2b^{\mathrm{T}}Pe|$ and we have $l(e) \geqslant \alpha_1(|e|) > 0\ \forall e \neq 0$. Hence, $l(e)$ is positive-definite. In addition, $R(x) > 0$ for all x satisfying Eq. (31). The cost functional defined in Eq. (34) is therefore a meaningful one.

Step 2: In this step, we prove that the control law of Eq. (30) minimizes the cost functional of Eq. (34). Substituting

$$k = u + \tfrac{1}{2}R^{-1}(x)L_{\bar{g}}V \tag{A.44}$$

into Eq. (34) and using the expression for $l(e)$ given in Theorem 3, we get the following chain of equalities

$$\begin{aligned} J &= \int_0^\infty (l(e) + uR(x)u)\,dt \\ &= \int_0^\infty (-L_{\bar{f}}V + \tfrac{1}{2}L_{\bar{g}}VR^{-1}(x)L_{\bar{g}}V - |L_{\bar{w}}V|\theta_b \\ &\qquad - L_{\bar{g}}Vk)\,dt + \int_0^\infty kR(x)k\,dt \\ &= \int_0^\infty (-L_{\bar{f}}V - L_{\bar{g}}Vu - |L_{\bar{w}}V|\theta_b)\,dt + \int_0^\infty kR(x)k\,dt \\ &= -\int_0^\infty \sup_{\theta\in\mathscr{W}\subset\mathbb{R}} \left(\frac{\partial V}{\partial e}(\bar{f} + \bar{g}u + \bar{w}\theta)\right)dt + \int_0^\infty kR(x)k\,dt \\ &= -\int_0^\infty \sup_{\theta\in\mathscr{W}\subset\mathbb{R}} (\dot{V})\,dt + \int_0^\infty kR(x)k\,dt. \end{aligned}$$

Note that the first term on the right-hand side of the last equality is bounded from above since

$$-\int_0^\infty \sup_{\theta\in\mathscr{W}\subset\mathbb{R}} (\dot{V})\,dt \leqslant -\int_0^\infty \dot{V}\,dt = V(e(0)) - \lim_{t\to\infty} V(e(t)). \tag{A.45}$$

It was shown in part 1 of the proof that the control law of Eq. (30) achieves asymptotic stability of the origin of the closed-loop system, for initial conditions satisfying Eq. (31). Hence, $\lim_{t\to\infty} V(e(t)) = 0$ and

$$J^* = -\int_0^\infty \sup_{\theta\in\mathscr{W}\subset\mathbb{R}} (\dot{V})\,dt \leqslant V(e(0)). \tag{A.46}$$

Since $R(x) > 0$, it is clear that the minimum of J is J^*. This minimum is achieved when $k(x) \equiv 0$ which proves that the controller of Eq. (30) minimizes the cost defined in Eq. (34). To complete the proof of optimality, we need to show that $J^* = V(e(0))$. To this end, consider the uncertain variable $\theta_\Delta \in \mathscr{W} \subset \mathbb{R}$ where for every $e_\Delta(0) \in \mathbb{R}^r$, every $u \in \mathbb{R}$, and every $\Delta > 0$, we have

$$\int_0^\infty \dot{V}(e_\Delta, u, \theta_\Delta)\,dt \geqslant \int_0^\infty \sup_{\theta\in\mathscr{W}\subset\mathbb{R}} \dot{V}(e_\Delta, u, \theta)\,dt - \Delta. \tag{A.47}$$

(The existence of θ_Δ follows from the properties of $\theta(t)$.) From the inequality of Eq. (A.47), we obtain

$$\begin{aligned} V(e(0)) &= -\int_0^\infty \dot{V}\,dt \\ &\leqslant -\int_0^\infty \sup_{\theta\in\mathscr{W}\subset\mathbb{R}} \dot{V}(e_\Delta, u, \theta)\,dt + \Delta. \end{aligned} \tag{A.48}$$

Combining the inequalities of Eq. (A.48) and Eq. (A.46), we get

$$V(e(0)) - \Delta \leqslant J^* \leqslant V(e(0)) \tag{A.49}$$

for arbitrarily small Δ. Clearly, if $J^* = V(e(0))$, the proof of optimality is complete. Otherwise, there exists $\mu > 0$ such that $J^* + \mu = V(e(0))$. Since Δ is arbitrary, we can choose it to be sufficiently small such that $\Delta < \mu$ and the inequality of Eq. (A.49) is violated. Thus, the only way to satisfy the inequality of Eq. (A.49) for arbitrary Δ is to set $J^* = V(e(0))$. This completes the proof of the theorem. □

Proof of Theorem 4. Similar to the proof of Theorem 3, we initially show that, if Eq. (35) holds, the trajectories of the closed-loop system remain bounded for all times and that the closed-loop output satisfies the inequality of Eq. (36). Finally, we show that the control law of Eq. (30) with $\rho = 0$ is optimal with respect to a meaningful cost functional of the form of Eq. (38) defined over a finite time interval.

Part 1: We first consider the representation of the closed-loop system in terms of the e, η coordinates given in Eq. (A.2). We show boundedness of the trajectories by first deriving ISS inequalities that capture the evolution of the states e and η, and then, by using a small gain argument.

Step 1: Consider the Lyapunov function candidate $V = e^{\mathrm{T}}Pe$. Setting $\rho = 0$ in the controller of Eq. (30) and evaluating the time derivative of V along the trajectories of the closed-loop e subsystem of Eq. (A.2), we perform similar calculations to those in step 1 of part 1 of the proof of Theorem 3, to conclude that $\dot{V}$ satisfies

$$\dot{V} \leqslant \left(\frac{(L_{\bar{f}}V + \theta_b |L_{\bar{w}}V|)\sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} - \frac{-\sqrt{(L_{\bar{f}}V + \chi\theta_b |L_{\bar{w}}V|)^2 + (u_{\max} L_{\bar{g}}V)^4}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} + \frac{\theta_b |L_w L_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|(\phi - (\chi - 1)|2b^{\mathrm{T}}Pe|)}{(|2b^{\mathrm{T}}Pe| + \phi)[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} \right). \tag{A.50}$$

From the above inequality and the fact that $\chi > 1$, we have that whenever $|2b^{\mathrm{T}}Pe| > \phi/(\chi - 1)$, the last term on the right-hand side of the above equation is negative and, therefore, $\dot{V}$ is bounded from above by the first two terms. For the case when $|2b^{\mathrm{T}}Pe| \leqslant \phi/(\chi - 1)$, we use the fact that the uncertain variables considered in Theorem 4 are nonvanishing in the sense that the function $L_w L_f^{r-1}h(x)$ satisfies a growth bound of the form $|L_w L_f^{r-1}h(x)| \leqslant \bar{\delta}|2b^{\mathrm{T}}Pe| + \mu$, where $\bar{\delta} > 0$ and $\mu > 0$, whenever $|2b^{\mathrm{T}}Pe| \leqslant \phi/(\frac{1}{2}\chi - 1)$, to obtain the following estimates:

$$\begin{aligned} &\frac{\theta_b |L_w L_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|(\phi - (\chi - 1)|2b^{\mathrm{T}}Pe|)}{(|2b^{\mathrm{T}}Pe| + \phi)[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} \\ &\leqslant \theta_b |L_w L_f^{r-1}h(x)||2b^{\mathrm{T}}Pe| \\ &\leqslant \theta_b \bar{\delta}|2b^{\mathrm{T}}Pe|^2 + \theta_b \mu |2b^{\mathrm{T}}Pe| \\ &\leqslant \frac{\theta_b \bar{\delta}\phi^2 + \theta_b \mu\phi(\chi - 1)}{(\chi - 1)^2} \\ &\equiv \varepsilon(\phi) \quad \forall |2b^{\mathrm{T}}Pe| \leqslant \frac{\phi}{\chi - 1}, \end{aligned} \tag{A.51}$$

where we have used the fact that $(|2b^{\mathrm{T}}Pe| + \phi)[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}] \geqslant \phi$ to derive the first inequality. Substituting the estimates of Eq. (A.51) directly into Eq. (A.50) yields that $\dot{V}$ satisfies

$$\dot{V} \leqslant \left(\frac{(L_{\bar{f}}V + \theta_b |L_{\bar{w}}V|)\sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} - \frac{\sqrt{(L_{\bar{f}}V + \chi\theta_b |L_{\bar{w}}V|)^2 + (u_{\max} L_{\bar{g}}V)^4}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} \right) + \varepsilon(\phi) \tag{A.52}$$

irrespective of the value of $|2b^{\mathrm{T}}Pe|$, where $\varepsilon(\phi) > 0$. To analyze the sign of $\dot{V}$ further, we consider the following two cases:

Case 1: $L_{\bar{f}}V + \chi\theta_b |L_{\bar{w}}V| \leqslant 0$. Since $\chi > 0$, then $L_{\bar{f}}V + \chi\theta_b |L_{\bar{w}}V| \leqslant 0$ implies $L_{\bar{f}}V + \theta_b |L_{\bar{w}}V| \leqslant 0$. Analyzing the right-hand side of Eq. (A.52) in this case yields

$$\dot{V} \leqslant -\left(\frac{\sqrt{(L_{\bar{f}}V + \chi\theta_b |L_{\bar{w}}V|)^2 + (u_{\max} L_{\bar{g}}V)^4}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} \right) + \varepsilon(\phi). \tag{A.53}$$

Note that the first term on the right-hand side of the above equation is negative-definite and vanishes only when $e = 0$.

Case 2: $0 < L_{\bar{f}}V + \chi\theta_b |L_{\bar{w}}V| \leqslant u_{\max}|L_{\bar{g}}V|$. In this case, we have that the estimate of Eq. (A.14) holds with $\rho = 0$. Substituting this estimate into Eq. (A.52) yields

$$\dot{V} \leqslant \left(\frac{(1 - \chi)\theta_b |L_{\bar{w}}V|\sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}}{[1 + \sqrt{1 + (u_{\max} L_{\bar{g}}V)^2}]} \right) + \varepsilon(\phi). \tag{A.54}$$

Note that since $\chi > 1$, the first term on the right-hand side of the above equation is strictly negative when $b^{\mathrm{T}}Pe \neq 0$. For the case when $b^{\mathrm{T}}Pe = 0$ and $e \neq 0$, it can also be shown from Eq. (A.21) that $\dot{V}$ is bounded by a negative-definite function. To summarize then, we deduce from Eqs. (A.53) and (A.54) that whenever $L_{\bar{f}}V + \chi\theta_b |L_{\bar{w}}V| \leqslant u_{\max}|L_{\bar{g}}V|$, there exists a function $\alpha_2(\cdot)$ of class K such that

$$\dot{V} \leqslant -\alpha_2(|e|) + \varepsilon(\phi). \tag{A.55}$$

Choosing ε (and hence ϕ) small enough such that $\varepsilon(\phi) \leqslant \frac{1}{2}\alpha_2(|e|)$, we have

$$\dot{V} \leqslant -\tfrac{1}{2}\alpha_2(|e|) < 0 \quad \forall |e| \geqslant \alpha_2^{-1}(2\varepsilon) = \gamma(\varepsilon(\phi)). \tag{A.56}$$

This inequality shows that $\dot{V}$ is negative outside a ball of radius $\gamma(\varepsilon(\phi))$. A direct application then of the result of Theorem 5.1 and its corollaries in Khalil (1996) allows us to conclude that whenever Eq. (35) holds, the following ISS inequality holds for the e states of the system of Eq. (A.2)

$$|e(t)| \leqslant \bar{\beta}_e(|e(0)|, t) + \bar{\gamma}_e(\phi) \quad \forall t \geqslant 0, \tag{A.57}$$

where $\bar{\beta}_e$ is a class KL function and $\bar{\gamma}_e$ is a class K_∞ function. From Assumption 4, we have that the η states of Eq. (A.2) possess an ISS property with respect to e and θ

$$\begin{aligned} |\eta(t)| &\leqslant \bar{\beta}_\eta(|\eta(0)|, t) + \bar{\gamma}_\eta(\|[e^{\mathrm{T}}\,\theta]^{\mathrm{T}}\|) \\ &\leqslant \bar{\beta}_\eta(|\eta(0)|, t) + \bar{\gamma}_{\eta_1}(\|e\|) + \bar{\gamma}_{\eta_2}(\|\theta\|), \end{aligned} \tag{A.58}$$

where $\bar{\gamma}_{\eta_1}$, $\bar{\gamma}_{\eta_2}$ are class K functions defined as $\bar{\gamma}_{\eta_1}(s) = \bar{\gamma}_{\eta_2}(s) = \bar{\gamma}_\eta(2s)$.

Step 2: We now analyze the behavior of the interconnected dynamical system comprised of the states e, η of the system of Eq. (A.2) for which the inequalities of Eq. (A.57) and Eq. (A.58) hold. We first define the following positive real numbers: $\delta_e = D^{\bar{e}} + \phi^*$, $\delta_\eta = D^{\bar{\eta}} + \bar{\gamma}_{\eta^1}(\delta_e) + \phi^*$, $D^{\bar{e}} = \bar{\beta}_e(\bar{\delta}_e, 0) + \bar{\gamma}_e(\phi^*)$, $D^{\bar{\eta}} = \bar{\beta}_\eta(\bar{\delta}_\eta, 0) + \bar{\gamma}_{\eta^2}(\theta_b)$, and $\bar{\delta}_e, \bar{\delta}_\eta$ are any positive real numbers that satisfy Eq. (35). We proceed by contradiction to show that if $\phi \in (0, \phi^*]$ where $\phi^* = \bar{\gamma}_e^{-1}(d)$, the evolution of the states e, η, starting from any initial states that satisfy $|e(0)| \leqslant \bar{\delta}_e$, $|\eta(0)| \leqslant \bar{\delta}_\eta$, where $\bar{\delta}_e$ and $\bar{\delta}_\eta$ are any non-negative real numbers that satisfy Eq. (35), satisfies the following inequalities:

$$|e(t)| \leqslant \delta_e, \quad |\eta(t)| \leqslant \delta_\eta \tag{A.59}$$

for all times. Let T be the smallest time such that there is a δ^* so that $t \in (T, T+\delta^*)$ implies either $|e(t)| > \delta_e$ or $|\bar{\eta}(t)| > \delta_\eta$. Then, for each $t \in [0, T]$ the conditions of Eq. (A.59) hold. Consider the functions $e^{\mathrm{T}}(t)$, $\eta^{\mathrm{T}}(t)$ defined in Eq. (A.25). From the fact that $|e(0)| \leqslant \bar{\delta}_e$, $|\eta(0)| \leqslant \bar{\delta}_\eta$, $\|\theta(t)\| \leqslant \theta_b$, we have that

$$\sup_{0 \leqslant t \leqslant T} (\bar{\beta}_\eta(|\eta(0)|, t) + \bar{\gamma}_{\eta_2}(\|\theta\|)) \leqslant \bar{\beta}_\eta(\bar{\delta}_\eta, 0) + \bar{\gamma}_{\eta_2}(\theta_b) =: D^{\bar{\eta}}, \tag{A.60}$$

$$\sup_{0 \leqslant t \leqslant T} (\bar{\beta}_e(|e(0)|, t) + \bar{\gamma}_e(\phi)) \leqslant \bar{\beta}_e(\bar{\delta}_e, 0) + \bar{\gamma}_e(\phi^*) =: D^{\bar{e}}.$$

Combining the above inequalities with Eq. (A.58), we have that for each $\phi \in (0, \phi^*]$ and for all $t \geqslant 0$

$$\begin{aligned} &\|e^{\mathrm{T}}\| \leqslant D^{\bar{e}} < \delta_e, \\ &\|\eta^{\mathrm{T}}\| \leqslant D^{\bar{\eta}} + \bar{\gamma}_{\eta_1}(\|e\|) \leqslant D^{\bar{\eta}} + \bar{\gamma}_{\eta_1}(\delta_e) < \delta_\eta. \end{aligned} \tag{A.61}$$

By continuity, we have that there exist some positive real number $\bar{k}$ such that $\|e^{T+\bar{k}}(t)\| \leqslant \delta_e$ and $\|\eta^{T+\bar{k}}(t)\| \leqslant \delta_\eta$, $\forall t \in [0, T+\bar{k}]$. This contradicts the definition of T. Hence, Eq. (A.59) holds $\forall t \geqslant 0$. Finally, for $\phi \in (0, \phi^*]$ and for any initial state satisfying Eq. (35), taking the limsup of both sides of Eq. (A.57) as $t \to \infty$, we have

$$\limsup_{t \to \infty} |e(t)| \leqslant \limsup_{t \to \infty} (\bar{\beta}_e(|e(0)|, t) + \bar{\gamma}_e(\phi)) = \bar{\gamma}_e(\phi) \leqslant d. \tag{A.62}$$

Part 2: In this part, we prove that the controller of Eq. (30) is optimal with respect to a meaningful finite-time cost functional defined in Eq. (38). We first prove that this cost functional is meaningful by showing that $l(e)$ is positive definite for all times in the interval $[0, T_f]$. Repeating the same calculations performed in step 1 of part 2 of the proof of Theorem 3 with $\rho = 0$, it can be easily verified that if the trajectories of the closed-loop system evolve such that, $\forall t \in [0, T_f]$, Eq. (37) holds, then there exists a function $\alpha_3(\cdot)$ of class K such that

$$\begin{aligned} &\mathrm{l}(e) \geqslant \alpha_3(|e|) \quad \forall |2b^{\mathrm{T}} Pe| > \frac{\phi}{\frac{1}{2}\chi - 1}, \\ &l(e) \geqslant \alpha_3(|e|) - \theta_b |L_w L_f^{r-1} h(x)| |2b^{\mathrm{T}} Pe| \phi \quad \forall |2b^{\mathrm{T}} Pe| \leqslant \frac{\phi}{\frac{1}{2}\chi - 1}. \end{aligned} \tag{A.63}$$

The first inequality in Eq. (A.63) implies that $l(e) > 0$ outside the set D. Using the fact that $|L_w L_f^{r-1} h(x)| \leqslant \bar{\delta}|2b^{\mathrm{T}} Pe| + \mu$ and $|2b^{\mathrm{T}} Pe| \leqslant \phi/(\frac{1}{2}\chi - 1)$ inside the set D, the second inequality in Eq. (A.63) can be re-written as

$$\begin{aligned} l(e) &\geqslant \alpha_3(|e|) - \frac{\theta_b \bar{\delta} \phi^2 + \theta_b \mu \phi (\frac{1}{2}\chi - 1)}{(\frac{1}{2}\chi - 1)^2} \\ &= \alpha_3(|e|) - \varepsilon'(\phi) \\ &\geqslant \tfrac{1}{2}\alpha_3(|e|) > 0 \quad \forall |e| \geqslant \alpha_3^{-1}(2\varepsilon'(\phi)) \equiv \gamma'(\varepsilon'(\phi)), \end{aligned} \tag{A.64}$$

where $\varepsilon'(\phi) > 0$. The last inequality implies that $l(e)$ is positive outside a compact ball Γ of radius $\gamma'(\varepsilon'(\phi))$. From its definition in Theorem 4, T_f is the minimum time for the trajectories of the closed-loop system to reach and enter this ball without ever leaving again. Therefore, we have that $|e| \geqslant \gamma'(\varepsilon'(\phi))$ $\forall t \in [0, T_f]$ and, hence, $l(e) > 0$ $\forall t \in [0, T_f]$. Note also that $R(x) > 0$. Therefore, the cost functional of Eq. (38) is a meaningful one. To prove that the control law of Eq. (30) with $\rho = 0$ minimizes the cost functional of Eq. (38), we substitute

$$k = u + \tfrac{1}{2}R^{-1}(x) L_{\bar{g}} V \tag{A.65}$$

into Eq. (38) and use the expression given in Theorem 3 for $l(e)$, to get the following chain of equalities

$$\begin{aligned} J &= \lim_{t \to T_f} V(e(t)) + \int_0^{T_f} (l(e) + uR(x)u)\, \mathrm{d}t \\ &= \lim_{t \to T_f} V(e(t)) + \int_0^{T_f} (-L_{\bar{f}} V + \tfrac{1}{2} L_{\bar{g}} V R^{-1}(x) L_{\bar{g}} V \\ &\quad - |L_{\bar{w}} V| \theta_b - L_{\bar{g}} V k)\, \mathrm{d}t + \int_0^{T_f} kR(x)k \\ &= \lim_{t \to T_f} V(e(t)) - \int_0^{T_f} \sup_{\theta \in \mathscr{W} \subset \mathbb{R}} \left(\frac{\partial V}{\partial e} (\bar{f} + \bar{g}u + \bar{w}\theta) \right) \mathrm{d}t \\ &\quad + \int_0^{T_f} kR(x)k\, \mathrm{d}t \\ &= \lim_{t \to T_f} V(e(t)) - \int_0^{T_f} \sup_{\theta \in \mathscr{W} \subset \mathbb{R}} (\dot{V})\, \mathrm{d}t + \int_0^{T_f} kR(x)k\, \mathrm{d}t \end{aligned}$$

Note that the term $\bar{J}^* = \lim_{t \to T_f} V(e(t)) - \int_0^{T_f} \sup_{\theta \in \mathscr{W} \subset \mathbb{R}} (\dot{V})\, \mathrm{d}t$ is bounded from above since

$$\lim_{t \to T_f} V(e(t)) - \int_0^{T_f} \sup_{\theta \in \mathscr{W} \subset \mathbb{R}} (\dot{V})\, \mathrm{d}t \leqslant \lim_{t \to T_f} V(e(t)) - \int_0^{T_f} \dot{V}\, \mathrm{d}t = V(e(0)). \tag{A.66}$$

Since $R(x) > 0$, it is clear that the minimum of J is $\bar{J}^*$. This minimum is achieved when $k(x) \equiv 0$ which proves that the controller of Eq. (30) (with $\rho = 0$) minimizes the cost defined in Eq. (38). To complete the proof of optimality, we need to show that $\bar{J}^* = V(e(0))$. To this end, consider the uncertain variable $\theta_\Delta \in \mathscr{W} \subset \mathbb{R}$ where for every $e_\Delta(0) \in \mathbb{R}^r$, every $u \in \mathbb{R}$, and every $\Delta > 0$, we have

$$\int_0^{T_f} \dot{V}(e_\Delta, u, \theta_\Delta)\,\mathrm{d}t \geqslant \int_0^{T_f} \sup_{\theta \in \mathscr{W} \subset \mathbb{R}} \dot{V}(e_\Delta, u, \theta)\,\mathrm{d}t - \Delta. \tag{A.67}$$

(The existence of θ_Δ follows from the properties of $\theta(t)$.) Using a similar argument to that in part 2 of the proof of Theorem 1, we obtain

$$V(e(0)) - \Delta \leqslant \bar{J}^* \leqslant V(e(0)) \tag{A.68}$$

for arbitrarily small Δ, and then we have $\bar{J}^* = V(e(0))$. This completes the proof of the theorem. □

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