INT. J. CONTROL, 2001, VOL. 74, NO. 2, 133–157

Robust near-optimal output feedback control of non-linear systems

NAEL H. EL-FARRA† and PANAGIOTIS D. CHRISTOFIDES†*

This work proposes a robust near-optimal non-linear output feedback controller design for a broad class of non-linear systems with time-varying bounded uncertain variables. Both vanishing and non-vanishing uncertainties are considered. Under the assumptions of input-to-state stable (ISS) inverse dynamics and vanishing uncertainty, a robust dynamic output feedback controller is constructed through combination of a high-gain observer with a robust optimal state feedback controller synthesized via Lyapunov's direct method and the inverse optimal approach. The controller enforces exponential stability and robust asymptotic output tracking with arbitrary degree of attenuation of the effect of the uncertain variables on the output of the closed-loop system, for initial conditions and uncertainty in arbitrarily large compact sets, provided that the observer gain is sufficiently large. Utilizing the inverse optimal control approach and singular perturbation techniques, the controller is shown to be near-optimal in the sense that its performance can be made arbitrarily close to the optimal performance of the robust optimal state feedback controller on the infinite time-interval by selecting the observer gain to be sufficiently large. For systems with non-vanishing uncertainties, the same controller is shown to ensure boundedness of the states, uncertainty attenuation and near-optimality on a finite time-interval. The developed controller is successfully applied to a chemical reactor example.

1. Introduction

Many industrial processes exhibit highly non-linear behaviour and involve time-varying uncertain variables such as unknown process parameters and external disturbances which, if not accounted for in the controller design, may cause significant deterioration in the closed-loop performance and lead to instability. Recognition of the detrimental effects of uncertainty on the closed-loop behaviour has led to significant research on the problem of designing controllers for non-linear systems with uncertain variables, that enforce output tracking with attenuation of the effect of the uncertain variables on the output. Under the assumption that all process states are available for measurement, a variety of techniques have been developed to address this problem including non-linear adaptive control (Sastry and Isidori 1989, Kanellakopoulos *et al.* 1991, Teel *et al.* 1991, Krstic *et al.* 1995), and non-linear robust control using Lyapunov's direct method (see, for example, Marino and Tomei 1993 c, Qu 1993, Lin *et al.* 1995, Christofides *et al.* 1996, Jiang and Mareels 1997); the reader may also refer to the papers (Corless 1993, Leitman 1993) for a review of results on controller design via Lyapunov's direct method). The controllers designed using these methods, however, are not in general optimal with respect to a meaningful cost and may expend unnecessarily large control effort to achieve their objective. Therefore, their implementation may result in poor closed-loop behaviour owing to the limits frequently imposed on the capacity of control actuators. An approach to address the design of robust optimal controllers is within the nonlinear H_∞ control framework (e.g. Basar and Bernhard 1990, van der Schaft 1992, Pan and Basar 1993). However, the practical applicability of this approach is still questionable because the explicit construction of the controllers requires the analytic solution of the steady-state Hamilton–Jacobi–Isaacs (HJI) equation which is not a feasible task except for simple problems. An appealing approach to robust optimal controller design, which does not require solving the HJI equation, is the inverse optimal approach proposed by Kalman (Kalman 1964) (see also Thau 1967, Moylan and Anderson 1973), for additional results on inverse optimal stabilization) and introduced recently in the context of robust stabilization in Freeman and Kokotovic (1996). The central idea of the inverse optimal approach is to compute a robust stabilizing control law together with the appropriate penalties that render the cost functional well-defined and meaningful in some sense. This approach provides a convenient route for robust optimal controller design and is well-motivated by the fact that the closed-loop robustness achieved as a result of controller optimality is largely independent of the specific choice of the cost functional (Sepulchre *et al.* 1997). This approach has been employed for the design of robust optimal controllers in Freeman and Kokotovic (1996), Krstic and Li (1997), and El-Farra and Christofides (1999).

Unfortunately, even though the above works provide systematic methods for adaptive and robust control design, their practical applicability is limited by the assumption of accessibility of all process states. For

Received 23 November 1999. Accepted 31 May 2000.
* Author for correspondence.
† Department of Chemical Engineering, University of California, Los Angeles, CA 90095-1592, USA.

International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online
http://www.tandf.co.uk/journals

instance, in chemical process control—more so than in most other control areas—the complete state cannot be measured in general (e.g. concentrations of certain species are not accessible on-line) and therefore the above state feedback controllers are not directly suited for practical applications. In the past few years, significant advances have been made in the direction of output feedback controller design for the purpose of robustly stabilizing non-linear systems. Important contributions in this area include general results on robust output feedback stabilization using the controller–observer combination approach (Teel and Praly 1994, 1995), adaptive output feedback control (Marino and Tomei 1993 a, b), and robust output feedback controller design (Khalil 1994, Mahmoud and Khalil 1996, Christofides 2000) for various classes of non-linear systems. In addition to these approaches, other methods for constructing robust stabilizers via output feedback, which do not use combination of the controller with an observer, have been explored including output feedback control within the H_∞ control framework (Isidori and Astolfi 1992, Isidori 1994, Isidori and Kang 1995) and the recursive stabilization scheme recently presented in Isidori (1999).

In this work, we address the problem of synthesizing robust near-optimal *output* feedback controllers for a broad class of non-linear systems with time-varying bounded uncertain variables. Under the assumptions of input-to-state stable (ISS) inverse dynamics and vanishing uncertainty, a dynamic controller is synthesized through combination of a high-gain observer with a robust optimal state feedback controller designed via Lyapunov's direct method and shown through the inverse optimal approach to be optimal with respect to a meaningful cost defined on the infinite time-interval. The dynamic output feedback controller enforces exponential stability and asymptotic output tracking with attenuation of the effect of the uncertain variables on the output of the closed-loop system for initial conditions and uncertainty in arbitrarily large compact sets, as long as the observer gain is sufficiently large. Utilizing the inverse optimal control approach and standard singular perturbation techniques, this approach is shown to yield a near-optimal output feedback design in the sense that the performance of the resulting output feedback controller can be made arbitrarily close to that of the robust optimal state feedback controller on the infinite time-interval, when the observer gain is sufficiently large. For systems with non-vanishing uncertainty, the same controller is shown to ensure boundedness of the states, robust asymptotic output tracking, and near-optimality over a finite time-interval. The developed controller is successfully applied to a chemical reactor example.

2. Notation

- $|\cdot|$ denotes the standard Euclidean norm, $\mathrm{sgn}(\cdot)$ denotes the sign function, and $\mathrm{sat}(\cdot)$ denotes the saturation function, defined as

$$\mathrm{sat}\,(s) = \left\{ \begin{array}{ll} a_m, & \text{if } s \geq a_m \\ s, & \text{if } -a_m < s < a_m \\ -a_m, & \text{if } s \leq -a_m \end{array} \right\} \tag{1}$$

where $s \in \mathbb{R}$ and a_m is a positive real number. For a vector $x \in \mathbb{R}^n$, $\mathrm{sat}\,(x) = [\mathrm{sat}\,(x_1)\ \mathrm{sat}\,(x_2) \cdots \mathrm{sat}\,(x_n)]^{\mathrm{T}}$. $O(\epsilon)$ denotes the standard order of magnitude notation, i.e. $\delta(\epsilon) = O(\epsilon)$ if there exist positive constants k and c such that $|\delta(\epsilon)| \leq k|\epsilon|$, $\forall |\epsilon| < c$. $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f. $L_f^k h$ denotes the kth order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.
- For any measurable (with respect to the Lebesgue measure) function $\theta\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^q$, $\|\theta\|$ denotes ess.sup.$|\theta(t)|$, $t \geq 0$.
- A function $W\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be positive definite if $W(x) > 0\ \forall x \neq 0$ and $W(0) = 0$.
- A function $W\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be proper if $W(x)$ tends to $+\infty$ as $|x|$ tends to $+\infty$.
- A function $\gamma\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class K if it is continuous, increasing and is zero at zero. It is of class K_∞, if in additon, it is proper. A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.
- A matrix A of dimension $n \times n$ is said to be Hurwitz if all of its eigenvalues satisfy $\mathrm{Re}\,[\lambda_i(A)] < 0$, $i = 1, \ldots, n$ where λ_i denotes the ith eigenvalue of the matrix.

3. Preliminaries

We consider single-input single-output non-linear systems with uncertain variables with the following state-space description

$$\left. \begin{array}{l} \dot{x} = f(x) + g(x)u + \displaystyle\sum_{k=1}^{q} w_k(x)\theta_k(t) \\ y = h(x) \end{array} \right\} \tag{2}$$

where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}$ denotes the manipulated input, $\theta_k(t) \in \mathcal{W} \subset \mathbb{R}$ denotes the kth uncertain (possibly time-varying) but bounded variable taking values in a nonempty compact convex subset $\mathcal{W}$ of $\mathbb{R}$, and $y \in \mathbb{R}$ denotes the output to be controlled. The uncertain variable $\theta_k(t)$ may describe time-varying parametric uncertainties and/or exogenous disturbances. The vector functions $f(x)$, $w_k(x)$ and $g(x)$, and the scalar function $h(x)$ are assumed to be sufficiently smooth. In the remainder of this paper, for simplicity, we will suppress the time-dependence in the notation of the uncertain variable $\theta_k(t)$.

We begin by reviewing the concept of inverse optimality introduced in Freeman and Kokotovic (1996) in the context of robust stabilization and used as a tool for robust optimal controller design. To this end, consider the system of equation (2) with $q = 1$ and $w(x_{eq})\theta = 0 \ \forall \theta \in \mathcal{W} \subset \mathbb{R}$ where x_{eq} is the equilibrium point of the system $\dot{x} = f(x)$. Without loss of generality, we assume that $x_{eq} = 0$. Suppose there exists a positive definite radially unbounded C^1 scalar function V such that

$$\inf_{u \in \mathbb{R}} \sup_{\theta \in \mathcal{W}} (L_f V(x) + L_g V(x)u + L_w V(x)\theta) < 0 \qquad \forall x \neq 0 \tag{3}$$

Also, let $l(x)$ and $R(x)$ be two continuous scalar functions such that $l(x) \geq 0$ and $R(x) > 0 \ \forall x \in \mathbb{R}^n$ and consider the cost functional

$$J = \int_0^\infty (l(x) + uR(x)u)\,\mathrm{d}t \tag{4}$$

The steady state Hamilton–Jacobi–Isaacs (HJI) equation associated with the system of equation (2) and the cost of equation (4) is

$$\begin{aligned} 0 \equiv \inf_{u \in \mathbb{R}} \sup_{\theta \in \mathcal{W}} &(l(x) + uR(x)u + L_f V(x) \\ &+ L_g V(x)u + L_w V(x)\theta) \end{aligned} \tag{5}$$

In a direct approach, one would have to solve the above HJI equation to synthesize the desired robust optimal control law. Since the solution of the HJI equation is not a feasible task in general, we take the inverse path instead. In the inverse approach, a stabilizing feedback control law is designed first and then shown to be optimal with respect to a well-defined cost functional of the form of equation (4). The problem is inverse because the functions $l(x)$ and $R(x)$ are *a posteriori* determined by the chosen stabilizing feedback control law, rather than *a priori* specified by the designer.

A stabilizing control law $u(x)$ solves an inverse optimal problem for the system of equation (2) if it can be expressed in the following form

$$u = -p(x) = -\tfrac{1}{2} R^{-1}(x) L_g V(x) \tag{6}$$

where the negative definiteness of $\dot{V}$ is achieved with the control $u^* = -\frac{1}{2}p(x)$, that is

$$\left.\begin{aligned} \sup_{\theta \in \mathcal{W}} \dot{V} &= \sup_{\theta \in \mathcal{W}} (L_f V(x) - \tfrac{1}{2} L_g V(x) p(x) + L_w V(x)\theta) \\ &= L_f V(x) - \tfrac{1}{2} L_g V(x) p(x) + |L_w V(x)|\theta_b \\ &< 0 \qquad \forall x \neq 0 \end{aligned}\right\} \tag{7}$$

where the worst-case uncertainty (i.e. one that maximizes $\dot{V}$) is given by $\theta = \mathrm{sgn}\,[L_w V(x)]\theta_b$ where $\theta_b = \|\theta\|$ (which is clearly an admissible uncertainty since it is both measurable and bounded). When the function $l(x)$ is set equal to

$$l(x) = -L_f V(x) + \tfrac{1}{2} L_g V(x) p(x) - |L_w V(x)|\theta_b \tag{8}$$

then $V(x)$ is a solution to the following steady state HJI equation

$$\begin{aligned} 0 \equiv l(x) + L_f V(x) - \tfrac{1}{4} L_g V(x) R^{-1}(x) L_g V(x) \\ + |L_w V(x)|\theta_b \end{aligned} \tag{9}$$

and the optimal (minimal) value of J is $V(x(0))$. Several investigators solved in a systematic way inverse optimal stabilization problems for various classes of non-linear systems (see, for example, Sepulchre *et al.* 1997). The main task of these design methods is the construction of positive definite functions whose time-derivatives along the trajectories of the closed-loop system can be rendered negative definite by feedback control. In the inverse approach, these functions become optimal value functions.

Finally, we recall the definition of input-to-state stability (ISS) for a system of the form of equation (2).

Definition 1 (Sontag, 1989b)**:** The system in equation (2) (with $u \equiv 0$) is said to be ISS with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_0 \in \mathbb{R}^n$ and for each measurable, essentially bounded input $\theta(\cdot)$ on $[0, \infty)$ the solution of equation (2) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|\theta\|), \quad \forall t \geq 0 \tag{10}$$

4. Robust near-optimal output feedback controller synthesis

4.1. *Control problem formulation*

Referring to the system of equation (2), we consider two control problems with different control objectives. In the first problem, we consider the class of systems described by equation (2) where the uncertain variable terms are assumed to be vanishing (i.e. $w_k(x_{eq})\theta_k = 0$ for any $\theta_k \in \mathcal{W}$ where x_{eq} is the equilibrium point of the system $\dot{x} = f(x)$). In this case, x_{eq} is also an equilibrium point for the uncertain system of equation (2). The objective in this problem is to synthesize robust non-linear dynamic output feedback controllers of the form

$$\left.\begin{aligned} \dot{\omega} &= \mathcal{F}(\omega, y, \bar{v}) \\ u &= \mathcal{P}(\omega, y, \bar{v}, t) \end{aligned}\right\} \tag{11}$$

where $\omega \in \mathbb{R}^5$ is a state, $\mathcal{F}(\omega, y, \bar{v})$ is a vector function, $\mathcal{P}(\omega, y, \bar{v}, t)$ is a scalar function, $\bar{v} = [v \ v^{(1)} \ \cdots \ v^{(r)}]^{\mathrm{T}}$ is a generalized reference input ($v^{(k)}$ denotes the kth time derivative of the reference input v, which is assumed to be a sufficiently smooth function of time), that enforce exponential stability, asymptotic robust output tracking with arbitrary degree of attenuation of the effect of the uncertainty on the output, and are near-optimal with respect to a meaningful cost functional, defined over the infinite time-interval, that imposes penalty on the control action and the worst case uncertainty. The near-optimality property, to be made more precise mathematically in Theorem 1 below, is sought in the sense that the performance of the dynamic output feedback controller can be made to approach the optimal performance of the corresponding robust optimal state feedback controller (i.e. $u = \mathcal{P}(x, \bar{v}, t)$) over the infinite time-interval.

In the second control problem, we again consider the system of equation (2). In this case, however, we assume that the uncertain variable terms are non-vanishing (i.e. $w_k(x)\theta_k \neq 0$). Under this assumption, x_{eq} is no longer an equilibrium point of the system of equation (2). The objective in this problem is to synthesize robust non-linear dynamic output feedback controllers of the form of equation (11) that guarantee boundedness of the closed-loop trajectories, enforce the discrepancy between the output and reference input to be asymptotically arbitrarily small, and are near-optimal with respect to a meaningful cost defined over a finite time-interval, in the sense that their performance can be made arbitrarily close to the optimal performance of the corresponding robust optimal state feedback controllers over the same time interval.

In both control problems, the design of the dynamic controllers is carried out using combination of a high-gain observer and a robust optimal state feedback control design proposed in El-Farra and Christofides (1999). In particular, referring to equation (11), the system $\dot{\omega} = \mathcal{F}(\omega, y, \bar{v})$ is synthesized to provide estimates of the system state variables, while the static component $\mathcal{P}(\omega, y, \bar{v}, t)$ is synthesized to enforce the requested properties in the closed-loop system. The analysis of the closed-loop system employs standard singular perturbation techniques (due to the high-gain nature of the observer) and utilizes the concept of input-to-state stability and non-linear small gain theorem-type arguments. Near-optimality is established through the inverse optimal approach and using standard singular perturbation results. The requested closed-loop properties in both control problems are enforced for arbitrarily large initial conditions and uncertainty provided that the gain of the observer is sufficiently large (semi-global type result).

4.2. *Assumptions*

In order to proceed with the design of the controllers, we need to impose the following three assumptions on the system of equation (2). The first assumption is motivated by the requirement of output tracking and allows transforming the system of equation (2) into a partially linear form.

Assumption 1: *There exists an integer r and a set of coordinates*

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = \mathcal{X}(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ \chi_1(x) \\ \vdots \\ \chi_{n-r}(x) \end{bmatrix} \tag{12}$$

where $\chi_1(x), \ldots, \chi_{n-r}(x)$ are non-linear scalar functions of x, such that the system of equation (2) takes the form

$$\left.\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r \\ \dot{\zeta}_r &= L_f^r h(\mathcal{X}^{-1}(\zeta, \eta)) + L_g L_f^{r-1} h(\mathcal{X}^{-1}(\zeta, \eta)) u \\ &\quad + \sum_{k=1}^{q} L_{wk} L_f^{r-1} h(\mathcal{X}^{-1}(\zeta, \eta)) \theta_k \\ \dot{\eta}_1 &= \Psi_1(\zeta, \eta) \\ &\vdots \\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta, \eta) \\ y &= \zeta_1 \end{aligned}\right\} \tag{13}$$

where $L_g L_f^{r-1} h(x) \neq 0$ for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Moreover, for each $\theta \in \mathbb{R}^q$, the states ζ, η are bounded if and only if the state x is bounded; and $(\zeta, \eta) \to (\zeta_{\mathrm{eq}}, \eta_{\mathrm{eq}})$ if and only if $x \to x_{eq}$ where $(\zeta_{eq}, \eta_{\mathrm{eq}})$ is the equilibrium point of the (ζ, η) system.

We note that the change of variables of equation (12) is independent of θ and invertible, since, for every x, the variables ζ, η are uniquely determined by equation (12). This implies that if we can estimate the values of ζ, η for all times, using appropriate state observers, then we automatically obtain estimates of x for all times. This property will be exploited later to synthesize a state estimator for the system of equation (2) on the basis of the system of equation (13). We also note that Assumption 1 includes the matching condition of our robust control method. In particular, we consider systems of the form equation (2) for which the uncertain variables enter the system in the same equation with the manipulated input. This assumption is more restrictive than the one used in El-Farra and Christofides (1999) to solve the same robust control problem via state feedback and is motivated by our requirement to eliminate the presence of θ in the η subsystem of the system of equation (13). This requirement and the stability requirement of Assumption 2 below will allow including in the controller a replica of the η subsystem of equation (13) which provides estimates of the η states (see Theorem 1).

Introducing the notation

$$\left.\begin{aligned} e &= [e_1 \ \ e_2 \ \ \cdots \ \ e_r]^{\mathrm{T}} \\ \bar{v} &= [v \ \ v^{(1)} \ \ \cdots \ \ v^{(r)}]^{\mathrm{T}} \\ e_i &= \zeta_i - v^{(i-1)} \end{aligned}\right\} \tag{14}$$

where $i = 1, \ldots, r$, $v^{(i)}$ is the ith time dervative of the reference input v, the ζ subsystem of equation (13) can be further transformed into the form

$$\dot{e} = \bar{f}(e, \eta, \bar{v}) + \bar{g}(e, \eta, \bar{v})u + \sum_{k=1}^{q} w_k(e, \eta, \bar{v})\theta_k \tag{15}$$

where

$$\left.\begin{aligned} \bar{f}(e,\eta) &= \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ L_f^r h(\mathcal{X}^{-1}(e,\eta)) - v^{(r)} \end{bmatrix} \\ \bar{g}(e,\eta) &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ L_g L_f^{r-1} h(\mathcal{X}^{-1}(e,\eta)) \end{bmatrix} \\ \bar{w}_k(e,\eta) &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ L_{wk} L_f^{r-1} h(\mathcal{X}^{-1}(e,\eta)) \end{bmatrix} \end{aligned}\right\} \tag{16}$$

are $r \times 1$ vector fields. As stated in the preliminaries, the main task in the design of the controller through the inverse optimal approach is the construction of a positive definite function whose time-derivative can be rendered negative definite via feedback. For systems of the form of equation (15), this can be done in many different ways. One way, for instance, is to use a quadratic function $V = e^{\mathrm{T}}Pe$ where the positive definite matrix P is chosen to satisfy the Ricatti inquality

$$A^{\mathrm{T}}P + PA - Pbb^{\mathrm{T}}P < 0 \tag{17}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{18}$$

are an $r \times r$ matrix and $r \times 1$ vector, respectively.

Assumption 2: *The system*

$$\left.\begin{aligned} \dot{\eta}_1 &= \Psi_1(\zeta, \eta) \\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta, \eta) \end{aligned}\right\} \tag{19}$$

is ISS with respect to ζ with $\beta_\eta(|\eta(0)|, t) = K_\eta|\eta(0)|\,\mathrm{e}^{-at}$ where K_η, a are positive real numbers and $K_\eta \geq 1$.

Following Christofides *et al.* (1996), the requirement of input-to-state stability of the system of equation (19) with respect to ζ is imposed to allow the synthesis of a robust state feedback controller that enforces the requested properties in the closed-loop system for arbitrarily large initial conditions and uncertain variables. On the other hand, the requirement that $\beta_\eta(|\eta(0)|, t) = K_\eta|\eta(0)|\,\mathrm{e}^{-at}$ allows incorporating in the robust output feedback controller a dynamical system identical to the one of equation (19) that provides estimates of the variables η. Assumption 2 is satisfied by many chemical processes (see, for example, Christofides and Daoutidis (1996) and the chemical reactor of § 5).

Finally, in order to attenuate the effect of the uncertain variables on the output, we need to assume the existence of known bounds that capture the size of the uncertain variables for all times.

Assumption 3: *These exist know positive constants θ_{bk} such that $\|\theta_k(t)\| = \theta_{bk}$.*

Now we are in a position to state the main results of this paper. In what follows, we present the results for the case of vanishing uncertainties first and then give the parallel treatment for the non-vanishing uncertainties case.

4.3. *Near-optimality over the infinite horizon*

Theorem 1 below provides a formula for the robust near-optimal output feedback controller and states precise conditions under which the proposed controller enforces the desired properties in the closed-loop system.

Theorem 1: *Consider the uncertain non-linear system of equation (2), for which Assumptions 1, 2 and 3 hold, under the robust output feedback controller*

$$\left.\begin{aligned}
\dot{\tilde{y}} &= \begin{bmatrix} -La_1 & 1 & 0 & \cdots & 0 \\ -L^2a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -L^{r-1}a_{r-1} & 0 & 0 & \cdots & 1 \\ -L^r a_r & 0 & 0 & \cdots & 0 \end{bmatrix}\tilde{y} + \begin{bmatrix} La_1 \\ L^2a_2 \\ \vdots \\ L^{r-1}a_{r-1} \\ L^r a_r \end{bmatrix} y \\
\dot{\omega}_1 &= \Psi_1(\mathrm{sat}(\tilde{y}), \omega) \\
&\vdots \\
\dot{\omega}_{n-r} &= \Psi_{n-r}(\mathrm{sat}(\tilde{y}), \omega) \\
u &= \mathcal{A}_r(\hat{x}, \bar{v}, \phi) \\
&:= -\left(c_0 + \frac{\widehat{L_{\bar{f}}V} + \alpha\sqrt{(\widehat{L_{\bar{f}}V})^2 + (\widehat{L_{\bar{g}}V})^4}}{(\widehat{L_{\bar{g}}V})^2} \right. \\
&\quad \left. + \frac{\rho + \chi\sum_{k=1}^{q}\theta_{bk}|L_{wk}L_f^{r-1}h(\hat{x})|}{(L_gL_f^{r-1}h(\hat{x}))^2\left(\dfrac{|\widehat{L_{\bar{g}}V}|}{|L_gL_f^{r-1}h(\hat{x})|} + \phi\right)} \right)\widehat{L_{\bar{g}}V}
\end{aligned}\right\} \qquad (20)$$

where $\hat{x} = \mathcal{X}^{-1}(\mathrm{sat}(\tilde{y}), \omega))$, $V = e^{\mathrm{T}}Pe$, *P is a positive definite matrix that satisfies the Riccati inequality of equation* (17) *and* c_0, α, χ, ρ, *and* ϕ *are adjustable parameters that satisfy* $c_0 > 0$, $\alpha \geq 0$, $\chi > 2, \rho > 0$ *and* $\phi > 0$. *Further, assume that the functions* $\bar{w}_k(\zeta, \eta)$ *satisfy* $\bar{w}_k(\zeta_{\mathrm{eq}}, \eta) = 0$ *and let* $\epsilon = 1/L$. *Then, for each set of positive real numbers* $\delta_x, \delta_\theta, \delta_{\bar{v}}$, *there exists* $\phi^* > 0$ *and for each* $\phi \in (0, \phi^*)$ *there exists an* $\epsilon^*(\phi) > 0$, *such that if* $\phi \in (0, \phi^*]$, $\epsilon \in (0, \epsilon^*(\phi)]$, $\mathrm{sat}(\cdot) = \min\{1, \zeta_{\max}/|\cdot|\}(\cdot)$ *with* $\zeta_{\max}$ *being the maximum value of the vector* $[\zeta_1\ \zeta_2\ \cdots\ \zeta_r]$ *for* $|\zeta| \leq \beta_\zeta(\delta_\zeta, 0)$ *where* β_ζ *is a class KL function and* δ_ζ *is the maximum value of the vector* $[h(x)\ L_fh(x)\ \cdots\ L_f^{r-1}h(x)]$ *for* $|x| \leq \delta_x$, $|x(0)| \leq \delta_x$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, $|\tilde{y}(0)| \leq \delta_\zeta$, $\omega(0) = \eta(0) + O(\epsilon)$, *the following holds:*

1. *The closed-loop system is exponentially stable.*
2. *The output of the closed-loop system satisfies a relation of the form*

$$\limsup_{t\to\infty} |y(t) - v(t)| = 0 \qquad (21)$$

3. *The output feedback controller of equation* (20) *is near-optimal in the sense that the cost functional associated with the controller of equation* (20) *and the system of equation* (2) *satisfies*

$$J = \int_0^\infty (l(e) + u(\hat{x})R(x)u(\hat{x})\,\mathrm{d}t \to V(e(0)) \quad \text{as } \epsilon \to 0 \qquad (22)$$

where

$$R(x) = \frac{1}{2}\left(c_0 + \frac{L_{\bar{f}}V + \alpha\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}}{(L_{\bar{g}}V)^2} + \frac{\rho + \chi\sum_{k=1}^{q}\theta_{bk}|L_{wk}L_f^{r-1}h(x)|}{(L_gL_f^{r-1}h(x))^2\left(\dfrac{|L_{\bar{g}}V|}{|L_gL_f^{r-1}h(x)|} + \phi\right)} \right)^{-1} \qquad (23)$$

$$l(e) = -L_{\bar{f}}V + \tfrac{1}{4}L_{\bar{g}}VR^{-1}(x)L_{\bar{g}}V - \sum_{k=1}^{q}|L_{\bar{w}k}V|\theta_{bk} \geq 0 \qquad (24)$$

Remark 1: The robust output feedback controller of equation (20) consists of a high gain observer which provides estimates of the derivatives of the output y up to order $r-1$, and thus estimates of the variables $\zeta_1, \ldots, \zeta_r$, an observer that simulates the inverse dynamics of the system of equation (13), and a static state feedback controller (see discussion in Remark 2 below) that attenuates the effect of the uncertain variables on the output and enforces reference input tracking. To eliminate the peaking phenomenon associated with the high-gain observer, we use a standard saturation function, *sat*, to eliminate wrong estimates of the output derivatives for short times. We choose to saturate $\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_{r-1}$ instead of the control action as was proposed in Khalil and Esfandiari (1993), because in most practical applications, it is possible to use knowledge of process operating conditions to derive non-conservative bounds on the actual values of the output derivatives.

Remark 2: Regarding the static component of the controller of equation (20), we note that it was synthesized via Lyapunov's direct method and consists of two components. The first component

$$u_s = -\left(c_0 + \frac{\widehat{L_{\bar{f}}V} + \alpha\sqrt{(\widehat{L_{\bar{f}}V})^2 + (\widehat{L_{\bar{g}}V})^4}}{(\widehat{L_{\bar{g}}V})^2} \right)\widehat{L_{\bar{g}}V} \qquad (25)$$

is responsible for achieving stabilization and output tracking in the nominal (i.e. $\theta(t) \equiv 0$) closed-loop system, while the second component

$$u_r = -\left(\frac{\rho + \chi \sum_{k=1}^{q} \theta_{bk} |L_{wk} L_f^{r-1} h(\hat{x})|}{(L_g L_f^{r-1} h(\hat{x}))^2 \left(\frac{|\widehat{L_{\bar{g}} V}|}{|L_g L_f^{r-1} h(\hat{x})|} + \phi\right)}\right) \widehat{L_{\bar{g}} V} \quad (26)$$

enforces output tracking with arbitrary degree of asymptotic attenuation of the effect of θ on y. We note that the first component is a generalization of Sontag's formula proposed in Sontag (1989 a) where the parameters c_0 and α (not present in the original Sontag's formula) were introduced. While a positive value for the parameter c_0 is not required to achieve stabilization, a choice $c_0 > 0$ may be needed to ensure the strict positivity of $R(x)$ defined in equation (23). The non-negative parameter α, on the other hand, is a tuning parameter that allows greater flexibility in shaping the dynamic behaviour of the closed-loop system as desired. It was shown in El-Farra and Christofides (1999) that the static state feedback controller of equation (20) is optimal with respect to a meaningful cost of the form of equation (22) and that the minimum cost achieved in the state feedback problem is $V(e(0))$.

Remark 3: Note that since $l(e) = \sup_{\theta \in \mathcal{W}^q} \dot{V}|_{u=\frac{1}{2}u^*} \geq ke^2 \geq 0$ where k is a positive constant (see the proof of Theorem 1 in the appendix), the cost functional of equation (22) includes penalty only on the tracking error $e = v - y$ and its time-derivatives up to order r and not on the full state of the closed-loop system. This is consistent with the requirement of output tracking. In addition, J imposes penalty on the control action. Finally, the term $-\sum_{k=1}^{q} |L_{\bar{w}k} V| \theta_{bk}$ in equation (22) is a direct consequence of the worst-case uncertainty formulation of the HJI equation (see equation (5)).

Remark 4: Regarding the properties of the cost functional of equation (22) we observe that J is well-defined according to the formulation of the optimal control problem for non-linear systems (Sepulchre *et al.* 1997) because of the following properties of $l(e)$ and $R(x)$. The function $l(e)$ is continuous, positive definite, and bounded from below by a class K function of the norm of e. The function $R(x)$ is continuous, strictly positive, and the term $uR(x)u$ has a unique global minimum at $u = 0$ implying larger control penalty for u farther away from zero. Moreover, owing to the vanishing nature of the uncertain variables and the exponential stability of the closed-loop system, J is defined on the infinite time-interval and is finite.

Remark 5: The near-optimality property established in Theorem 1 for the output feedback controller of equation (20) is defined in the sense that the cost incurred by implementing this controller to the system of equation (2) tends to the optimal (minimal) cost achieved by implementing the robust optimal state feedback controller (i.e. u of equation (20) with $\hat{x} = x$), for all times, when the gain of the observer is sufficiently large. It is important therefore to realize that near-optimality of the controller of equation (20) is a consequence of two integral and equally-important factors. The first factor is the optimality of the static component of the controller (state feedback problem), and the second factor is the high-gain nature of the observer used which can be exploited to make the performance of the output feedback controller arbitrarily close to that of the robust optimal state feedback controller. Instrumental in this regard is the use of the saturation function (see Remark 1) which allows the use of arbitrarily large values of the observer gain to achieve the desired degree of near-optimality without the detrimental effects of observer peaking. Combining other types of observers which do not possess these properties with the static component of equation (20), therefore, will *not* lead to a near-optimal feedback design in the sense of Theorem 1.

Remark 6: A stronger near-optimality result than the one established in Theorem 1 can be obtained in a sufficiently small neighbourhood of the origin of the system of equation (2), where the functions $l(e)$ and $R(x)$ satisfy the Lipschitz property. Over such a neighbourhood, it can be shown, using the local Lipschitz properties, that the cost computed using the output feedback controller of equation (20) is, in fact, $O(\epsilon)$ close to the optimal cost attained under the optimal state feedback controller (i.e. $J = V(e(0)) + O(\epsilon)$).

Remark 7: Owing to the presence of the fast (high-gain) observer in the dynamical system of equation (20), the closed-loop system can be conveniently regarded as a two-time scale system and, therefore, represented in the following singularly perturbed form, where $\epsilon = 1/L$ is the singular perturbation parameter

$$\left.\begin{aligned} \epsilon \dot{e}_0 &= Ae_0 + \epsilon b \Omega(x, \hat{x}, \theta, \phi, \bar{v}) \\ \dot{\omega}_1 &= \Psi_1(\mathrm{sat}\,(\tilde{y}), \omega) \\ &\vdots \\ \dot{\omega}_{n-r} &= \Psi_{n-r}(\mathrm{sat}\,(\tilde{y}), \omega) \\ \dot{x} &= f(x) + g(x)\mathcal{A}(\hat{x}, \bar{v}, \phi) + \sum_{k=1}^{q} w_k(x)\theta_k(t) \end{aligned}\right\} \quad (27)$$

where e_0 is the vector of the auxiliary error variables $\hat{e}_i = L^{r-i}(y^{(i-1)} - \tilde{y}_i)$ (see part 1 of the proof of Theorem 1 for details of the notation used). It is clear from the above representation that, within the singular perturbation formulation, the observer states e_0, which are directly related to the estimates of the output and its derivatives up to order $r - 1$, constitute the fast states of the singularly perturbed system of equation (27), while the ω states of the observer and the states of the original system of equation (2) represent the slow states. Owing to the dependence of the controller of equation (20) on both the slow and fast states, the control action computed by the static component in equation (20) is not $O(\epsilon)$ close to that computed by the state feedback controller for all times. After the decay of the boundary layer term e_0, however, the static component in equation (20) approximates the state feedback controller to within $O(\epsilon)$.

Remark 8: Although the results of Theorem 1 were developed for the class of systems described by equation (2), these results can be easily extended to the more general two time-scale systems with the following state-space description

$$\left.\begin{aligned} \dot{x} &= f_1(x,\theta(t)) + Q_1(x,\theta(t))z + g_1(x)u \\ \bar{\epsilon}\dot{z} &= f_2(x,\theta(t)) + Q_2(x,\theta(t))z + g_2(x)u \\ y &= h(x) \end{aligned}\right\} \tag{28}$$

where the parameter ϵ introduced in Theorem 1 can now be re-defined as $\epsilon = \max\{1/L, \bar{\epsilon}\}$. In fact, the fast nature of the observer used in equation (20), which allows treatment of the closed-loop system within the singular perturbations framework, makes such an extension quite natural without the need to impose any additional restrictive assumptions. For some related results on near-optimal control of singularly perturbed systems, the reader is referred to Chow and Kokotovic (1978) and Fridman (1996, 1998).

Remark 9: The affine appearance of the control input u in the class of systems considered in equation (2) is satisfied by many practical physical and chemical systems (see, for instance, the chemical reactor example in § 5). Note, however, that the affine appearance of the uncertain variables in equation (2) is used only to simplify our development and is not needed for the results of Theorem 1 to hold. The affine appearance of the variables θ_k can be readily relaxed as shown in equation (28) so long as these variables satisfy the matching condition and an upper bound is available that captures their size.

Remark 10: Although feedback linearization could be used for stabilization of the system of equation (13) with $\theta(t) \equiv 0$, a feedback linearizing controller may unnecessarily cancel beneficial non-linearities and expend unreasonably large control effort (Krstic *et al.* 1995). Consequently, we use the normal form of equation (13) not for the design of the controller but only for the construction of V. The choice of $V = e^{\mathrm{T}}Pe$ is made to simplify the explicit formula of the controller in the context of a specific application. Of course, this choice for V is not unique and many other positive definite functions whose time-derivative along the trajectories of the closed-loop system can be rendered negative definite via feedback could be used.

It is important to point out that the result of Theorem 1 is novel even in the case where $\theta(t) \equiv 0$. The following corollary states the result for this case.

Corollary 1: *Suppose the assumptions of Theorem 1 are satisfied for the non-linear system of equation* (2) *with* $\theta(t) \equiv 0$, *then the closed-loop system properties (1), (2), and (3) given in Theorem 1 hold under the output feedback controller of equation* (20) *with* $\chi = 0$ *and* $\rho = 0$.

4.4. *Near-optimality over the finite horizon*

We now turn to address the second control problem where the uncertain variable terms in equation (2) are non-vanishing. Theorem 2 below provides an explicit formula for the construction of the necessary robust output feedback controller and states precise conditions under which the proposed controller enforces the desired properties in the closed-loop system.

Theorem 2: *Consider the uncertain non-linear system of equation* (2) *for which Assumptions 1, 2, and 3 hold, under the robust output feedback controller*

$$\left.\begin{aligned} \dot{\tilde{y}} &= \begin{bmatrix} -La_1 & 1 & 0 & \cdots & 0 \\ -L^2a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -L^{r-1}a_{r-1} & 0 & 0 & \cdots & 1 \\ -L^r a_r & 0 & 0 & \cdots & 0 \end{bmatrix}\tilde{y} + \begin{bmatrix} La_1 \\ L^2a_2 \\ \vdots \\ L^{r-1}a_{r-1} \\ L^r a_r \end{bmatrix} y \\ \dot{\omega}_1 &= \Psi_1(\mathrm{sat}(\tilde{y}),\omega) \\ &\;\;\vdots \\ \dot{\omega}_{n-r} &= \Psi_{n-r}(\mathrm{sat}(\tilde{y}),\omega) \end{aligned}\right\} \tag{29}$$

$$u = \mathcal{A}_r(\hat{x}, \bar{v}, \phi) := -\left(c_0 + \frac{\widehat{L_{\bar{f}}V} + \alpha\sqrt{(\widehat{L_{\bar{f}}V})^2 + (\widehat{L_{\bar{g}}V})^4}}{(\widehat{L_{\bar{g}}V})^2} + \chi \frac{\sum_{k=1}^{q} \theta_{bk} |L_{wk} L_f^{r-1} h(\hat{x})|}{(L_g L_f^{r-1} h(\hat{x}))^2 \left(\frac{|\widehat{L_{\bar{g}}V}|}{|L_g L_f^{r-1} h(\hat{x})|} + \phi \right)} \right) \widehat{L_{\bar{g}}V} \quad (29)$$

where $\hat{x} = \mathcal{X}^{-1}(\text{sat}(\tilde{y}), \omega))$, $V = e^{\mathrm{T}}Pe$, P *is a positive definite matrix that satisfies the Riccati inequality of equation* (17) *and* c_0, α, χ, *and* ϕ *are adjustable parameters that satisfy* $c_0 > 0$, $\alpha \geq 0$, $\chi > 2$ *and* $\phi > 0$. *Further, assume that the functions* $\bar{w}_k(\zeta, \eta)$ *satisfying* $\bar{w}_k(\zeta_{eq}, \eta_{eq}) \neq 0$ *and let* $\epsilon = 1/L$. *Then, for each set of positive real numbers* $\delta_x, \delta_\theta, \delta_{\bar{v}}, d$, *there exists* $\phi^* > 0$ *and for each* $\phi \in (0, \phi^*]$, *there exists an* $\epsilon^*(\phi) > 0$, *such that if* $\phi \in (0, \phi^*]$, $\epsilon \in (0, \epsilon^*(\phi)]$, $\text{sat}(\cdot) = \min\{1, \zeta_{\max}/|\cdot|\}(\cdot)$ *with* $\zeta_{\max}$ *being the maximum value of the vector* $[\zeta_1\ \zeta_2\ \cdots \zeta_r]$ *for* $|\zeta| \leq \beta_\zeta(\delta_\zeta, 0) + d$ *where* β_ζ *is a class KL function and* δ_ζ *is the maximum value of the vector* $[h(x)\ L_f h(x)\ \cdots\ L_f^{r-1}h(x)]$ *for* $|x| \leq \delta_x$, $|x(0)| \leq \delta_x$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, $|\tilde{y}(0)| \leq \delta_\zeta$, $\omega(0) = \eta(0) + O(\epsilon)$, *the following holds:*

1. *The trajectories of the closed-loop system are bounded.*
2. *The output of the closed-loop system satisfies a relation of the form*

$$\limsup_{t \to \infty} |y(t) - v(t)| \leq d \quad (30)$$

3. *The output feedback controller of equation* (29) *is near optimal in the sense that*

$$J = \lim_{t \to \infty} V(e(t)) + \int_0^{T_f} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t \to V(e(0)) \quad \text{as } \epsilon \to 0 \quad (31)$$

where

$$R(x) = \frac{1}{2}\left(c_0 + \frac{L_{\bar{f}}V + \sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}}{(L_{\bar{g}}V)^2} + \chi \frac{\sum_{k=1}^{q} \theta_{bk} |L_{wk} L_f^{r-1} h(x)|}{(L_g L_f^{r-1} h(x))^2 \left(\frac{|L_{\bar{g}}V|}{|L_g L_f^{r-1} h(x)|} + \phi \right)} \right)^{-1} > 0 \quad (32)$$

$$l(e) = -L_{\bar{f}}V + \tfrac{1}{4} L_{\bar{g}}VR^{-1}(x)L_{\bar{g}}V - \sum_{k=1}^{q} |L_{\bar{w}k}V|\theta_{bk} \geq 0 \quad \forall t \in [0, T_f] \quad (33)$$

$$T_f = \inf\{T \geq 0: e(t) \in \Gamma\ \forall t \geq T\} \quad (34)$$

$$\Gamma = \left\{ e \in \mathbb{R}^r : |2b^{\mathrm{T}}Pe| < \frac{\phi}{\frac{1}{2}\chi - 1} \right\} \quad (35)$$

Remark 11: Note that owing to the persistent nature of the uncertainties in this case, asymptotic convergence of the closed-loop system to the equilibrium point of the nominal system (i.e. $\theta(t) \equiv 0$) is no longer possible. Therefore, the cost functional J cannot achieve a finite value over the infinite time-interval. Instead, J is defined over a finite time-interval $[0, T_f]$ whose size is determined by the time required for the tracking error trajectory to reach and enter the ball Γ of equation (35) without ever leaving again. Depending on the desired degree of uncertainty attenuation, the size of this ball can be tuned by adjusting the parameters ϕ and χ. An additional consequence of the non-vanishing uncertainties is the appearance of the terminal penalty term $\lim_{t \to T_f} V(e(t))$ in the cost functional of equation (31) which accounts for the fact that the closed-loop trajectories do not converge to the origin for $t > T_f$.

Remark 12: In contrast to the output feedback controller of equation (20) proposed in Theorem 1, the controller given in equation (29) in Theorem 2 is near-optimal in the sense that the cost incurred by implementing this controller to the system of equation (2) can be made arbitrarily close to the optimal cost achieved by the corresponding robust optimal state feedback controller (i.e. u of equation (29) with $\hat{x} = x$), for times on the finite interval $[0, T_f]$, by choosing the gain of the observer to be sufficiently large. As a consequence of the non-vanishing uncertainties, near-optimality can be established on the finite time-interval only and not for all times.

Remark 13: It is important to compare the controller of equation (29) with the robust nonlinear output feedback controller proposed in Christofides (2000) with the static component

$$u = \frac{1}{L_g L_f^{r-1} h(\hat{x})} \left\{ \sum_{i=0}^{r} \frac{\beta_i}{\beta_r}(v^{(i)} - L_f^i h(\hat{x})) - \chi \frac{\sum_{k=1}^{q} \theta_{bk} |L_{wk} L_f^{r-1} h(\hat{x})|}{(L_g L_f^{r-1} h(\hat{x})) \left(\frac{|\widehat{L_{\bar{g}}V}|}{|L_g L_f^{r-1} h(\hat{x})|} + \phi \right)} \right\} \quad (36)$$

which guarantees boundedness of the trajectories of the closed-loop system and output tracking with arbitrary degree of asymptotic attenuation of the effect of θ on y. The controller of equation (36) consists of the term

$$u_l = \frac{1}{L_g L_f^{r-1} h(\hat{x})} \left\{ \sum_{i=0}^{r} \frac{\beta_i}{\beta_r} (v^{(i)} - L_f^i h(\hat{x})) \right\} \tag{37}$$

which is used to cancel the non-linearities of the system of equation (13) with $\theta(t) \equiv 0$ that can be cancelled by feedback, and the term

$$u_r = -\left\{ \chi \frac{\sum_{k=1}^{q} \theta_{bk} |L_{wk} L_f^{r-1} h(\hat{x})|}{(L_g L_f^{r-1} h(\hat{x})) \left(\dfrac{|\widehat{L_{\bar{g}} V}|}{|L_g L_f^{r-1} h(\hat{x})|} + \phi \right)} \widehat{L_{\bar{g}} V} \right\} \tag{38}$$

which compensates for the effect of θ on y. We note that the static controller of equation (36) may generate unnecessarily large control action to cancel beneficial non-linearities and, therefore, cannot be expected to be optimal with respect to any reasonable criteria. Consequently, combining the static state feedback of equation (36) with the fast observer used in equation (29) will not lead to an optimal output feedback design.

Remark 14: Referring to the practical applications of the result of Theorem 2, one has to initially verify whether Assumptions 1 and 2 hold for the process under consideration and compute the bounds θ_{bk}. Then, equation (29) can be used to compute the explicit formula for the controller. Finally, given the ultimate uncertainty attenuation level desired d, the values of ϕ^* and the observer gain L should be computed (usually through simulations) to achieve $\lim\sup_{t\to\infty} |y(t) - v(t)| \le d$ and guarantee the near-optimal performance of the controller of equation (29).

5. Application to a chemical reactor

To illustrate the theoretical results presented in this paper, we consider a well-mixed continuous stirred tank reactor where three parallel irreversible elementary endothermic reactions of the form $A \xrightarrow{k_D} D$, $A \xrightarrow{k_U} U$ and $A \xrightarrow{k_R} R$ take place, where A is the reactant species, D is the desired product and U, R are undesired byproducts. The feed to the reactor consists of pure A at flow rate F, molar concentration C_{A0} and temperature T_{A0}. Due to the endothermic nature of the reactions, a heating jacket is used to provide heat to the reactor. Under standard modelling assumptions, a mathematical model of the process can be derived from material and energy balances and takes the following form

$$\left.\begin{aligned}
V\frac{dC_A}{dt} &= F(C_{A0} - C_A) - k_{D0}\,e^{-E_D/RT} C_A V \\
&\quad - k_{U0}\,e^{-E_U/RT} C_A V - k_{R0}\,e^{-E_R/RT} C_A V \\
V\frac{dC_D}{dt} &= -FC_D + K_{D0}\,e^{-E_D/RT} C_A V \\
V\frac{dT}{dt} &= F(T_{A0} - T) + \frac{(-\Delta H_D)}{\rho c_p} k_{D0}\,e^{-E_D/RT} C_A V \\
&\quad + \frac{(-\Delta H_U)}{\rho c_p} k_{U0}\,e^{-E_U/RT} C_A V \\
&\quad + \frac{(-\Delta H_R)}{\rho c_p} k_{R0}\,e^{-E_R/RT} C_A V \\
&\quad + \frac{UA}{\rho c_p}(T_c - T)
\end{aligned}\right\} \tag{39}$$

where C_A and C_D denote the concentrations of the species A and D, T denotes the temperature of the reactor, T_c denotes the temperature of the heating jacket, V denotes the volume of the reactor, ΔH_D, ΔH_U, ΔH_R, k_D, k_U, k_R, E_D, E_U, E_R, denote the enthalpies, pre-exponential constants and activation energies of the three reactions, respectively, c_p and ρ denote the heat capacity and density of the reactor, and U denotes the heat transfer coefficient between the reactor and the heating jacket. The values of the process parameters and the corresponding steady state values are given in Table 1. It was verified that these conditions correspond to a stable equilibrium point of the system of equation (39).

The control problem is formulated as the one of regulating the concentration of the desired product C_D by manipulating the temperature of the fluid in the heat-

V	$= 1.00\ \mathrm{m^3}$
A	$= 6.0\ \mathrm{m^2}$
U	$= 1000.0\ \mathrm{kcal\,hr^{-1}\,m^{-2}\,K^{-1}}$
R	$= 1.987\ \mathrm{kcal\,kmol^{-1}\,K^{-1}}$
C_{A0s}	$= 3.75\ \mathrm{kmol\,m^{-3}}$
T_{A0s}	$= 310.0\ \mathrm{K}$
T_{cs}	$= 320.0\ \mathrm{K}$
ΔH_{D0}	$= 5.4 \times 10^3\ \mathrm{kcal\,kmol^{-1}}$
ΔH_{U0}	$= 5.1 \times 10^3\ \mathrm{kcal\,kmol^{-1}}$
ΔH_{R0}	$= 5.0 \times 10^4\ \mathrm{kcal\,kmol^{-1}}$
k_{D0}	$= 3.36 \times 10^6\ \mathrm{hr^{-1}}$
k_{U0}	$= 7.21 \times 10^6\ \mathrm{hr^{-1}}$
k_{R0}	$= 1.25 \times 10^7\ \mathrm{hr^{-1}}$
E_D	$= 8.0 \times 10^3\ \mathrm{kcal\,kmol^{-1}}$
E_U	$= 9.0 \times 10^3\ \mathrm{kcal\,kmol^{-1}}$
E_R	$= 9.5 \times 10^3\ \mathrm{kcal\,kmol^{-1}}$
c_p	$= 0.231\ \mathrm{kcal\,kg^{-1}\,K^{-1}}$
ρ	$= 900.0\ \mathrm{kg\,m^{-1}}$
F	$= 3.0\ \mathrm{m^3\,hr^{-1}}$
C_{AS}	$= 0.913\ \mathrm{kmol\,m^{-3}}$
C_{DS}	$= 1.66\ \mathrm{kmol\,m^{-3}}$
T_s	$= 302.0\ \mathrm{K}$

Table 1. Process parameters and steady-state values.

ing jacket T_c. Within the posed control problem we distinguish and address the following two scenarios. In the first scenario, the control objective is achieved in the absence of any model uncertainty or exogenous disturbances in the process. By contrast, in the second scenario the control objective is accomplished in the presence of uncertainties. In the latter scenario, the enthalpies of the three reactions ΔH_D, ΔH_U, ΔH_R and the feed temperature T_{A0} are assumed to be the main uncertain variables present. Defining $x_1 = C_A - C_{As}$, $x_2 = C_D - C_{Ds}$, $x_3 = T - T_s$, $u = T_c - T_{cs}$, $\theta_1 = \Delta H_D - \Delta H_{D0}$, $\theta_2 = \Delta H_U - \Delta H_{U0}$, $\theta_3 = \Delta H_R - \Delta H_{R0}$, $\theta_4 = T_{A0} - T_{A0s}$, $y = C_D - C_{D_s}$, where the subscript s denotes the steady state values and $\Delta H_{D_0}, \Delta H_{U0}, \Delta H_{R0}$ are the nominal values for the enthalpies, the process model of equation (39) can be written in the form of equation (2) with (see bottom of page)

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ UA/\rho c_p V \end{bmatrix},$$

$$w_1(x) = \begin{bmatrix} 0 \\ 0 \\ k_{D0}\,\mathrm{e}^{-E_D/R(x_3+T_s)}(x_1 + C_{As}) \end{bmatrix},$$

$$w_2(x) = \begin{bmatrix} 0 \\ 0 \\ k_{U0}\,\mathrm{e}^{-E_U/R(x_3+T_s)}(x_1 + C_{As}) \end{bmatrix}$$

$$w_3(x) = \begin{bmatrix} 0 \\ 0 \\ k_{R0}\,\mathrm{e}^{-E_R/R(x_3+T_s)}(x_1 + C_{As}) \end{bmatrix},$$

$$w_4(x) = \begin{bmatrix} 0 \\ 0 \\ F/V \end{bmatrix}, \qquad h(x) = [x_2]$$

Using the following coordinate transformation

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \eta_1 \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ t(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\dfrac{F}{V}(C_{Ds} + x_2) + k_{D0}\,\mathrm{e}^{-E_D/R(x_3+T_s)}(x_1 + C_{As}) \\ x_1 \end{bmatrix} \quad (40)$$

and the notation $\bar{f}(e,\eta,\bar{v}) = [e_2 \;\; L_f^2 h(x) - v^{(2)}]^{\mathrm{T}}$, $\bar{g}(e,\eta,\bar{v}) = [0 \;\; L_g L_f h(x)]^{\mathrm{T}}$, $\bar{w}_k(e,\eta,\bar{v}) = [0 \;\; L_{wk} L_f h(x)]^{\mathrm{T}}$, $k = 1,2,3,4$, the process model can be cast in the form

$$\left.\begin{aligned} \dot{e} &= \bar{f}(e,\eta,\bar{v}) + g(e,\eta,\bar{v})u + \sum_{k=1}^{4} \bar{w}_k(e,\eta,\bar{v})\theta_k \\ \dot{\eta} &= \Psi(e,\eta,\bar{v}) \\ y &= e_1 + v \end{aligned}\right\} \quad (41)$$

We now proceed with the design of the controller and begin with the first scenario where no uncertainties are present in the process model (i.e. $\theta_k \equiv 0$). Owing to the absence of uncertainty in this case, we use the result of Corollary 1 to design the controller. It can be easily verified that the system of equation (41) satisfies the assumptions of Corollary 1 and therefore the necessary output feedback controller (whose practical implementation requires measurements of C_D only) takes the form

$$\left.\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{y}_2 + La_1(y - \tilde{y}_1) \\ \dot{\tilde{y}}_2 &= L^2 a_2(y - \tilde{y}_1) \\ \dot{\omega}_1 &= \frac{F}{V}(C_{A0} - C_{As} - \hat{x}_1) \\ &\quad - k_{D0}\,\mathrm{e}^{-E_D/R(\hat{x}_3+T_s)}(\hat{x}_1 + C_{As}) \\ &\quad - k_{U0}\,\mathrm{e}^{-E_U/R(\hat{x}_3+T_s)}(\hat{x}_1 + C_{As}) \\ &\quad - k_{R0}\,\mathrm{e}^{-E_R/R(\hat{x}_3+T_s)}(\hat{x}_1 + C_{As}) \\ u &= -\left(c_0 + \frac{\widehat{L_{\bar{f}}V} + \alpha\sqrt{(\widehat{L_{\bar{f}}V})^2 + (\widehat{L_{\bar{g}}V})^4}}{(\widehat{L_{\bar{g}}V})^2}\right)\widehat{L_{\bar{g}}V} \end{aligned}\right\} \quad (42)$$

where

$$f(x) = \begin{bmatrix} \dfrac{F}{V}(C_{A0} - C_{As} - x_1) - k_{D0}\,\mathrm{e}^{-E_D/R(x_3+T_s)}(x_1 + C_{As}) - k_{U0}\,\mathrm{e}^{-E_U/R(x_3+T_s)}(x_1 + C_{As}) - k_{R0}\,\mathrm{e}^{-E_R/R(x_3+T_s)}(x_1 + C_{As}) \\ -\dfrac{F}{V}(C_{Ds} + x_2) - k_{D0}\,\mathrm{e}^{-E_D/R(x_3+T_s)}(x_1 + C_{As}) \\ \dfrac{F}{V}(T_{A0s} - T_s - x_3) + \dfrac{(-\Delta H_{D0})}{\rho c_p} k_{D0}\,\mathrm{e}^{-E_D/R(x_3+T_s)}(x_1 + C_{As}) + \dfrac{(-\Delta H_{U0})}{\rho c_p} + k_{U0}\,\mathrm{e}^{-E_D/R(x_3+T_s)}(x_1 + C_{As}) \\ + \dfrac{(-\Delta H_{R0})}{\rho c_p} + k_{R0}\,\mathrm{e}^{-E_D/R(x_3+T_s)}(x_1 + C_{As}) + \dfrac{UA}{\rho c_p V}(T_{cs} - T_s - x_3) \end{bmatrix}$$

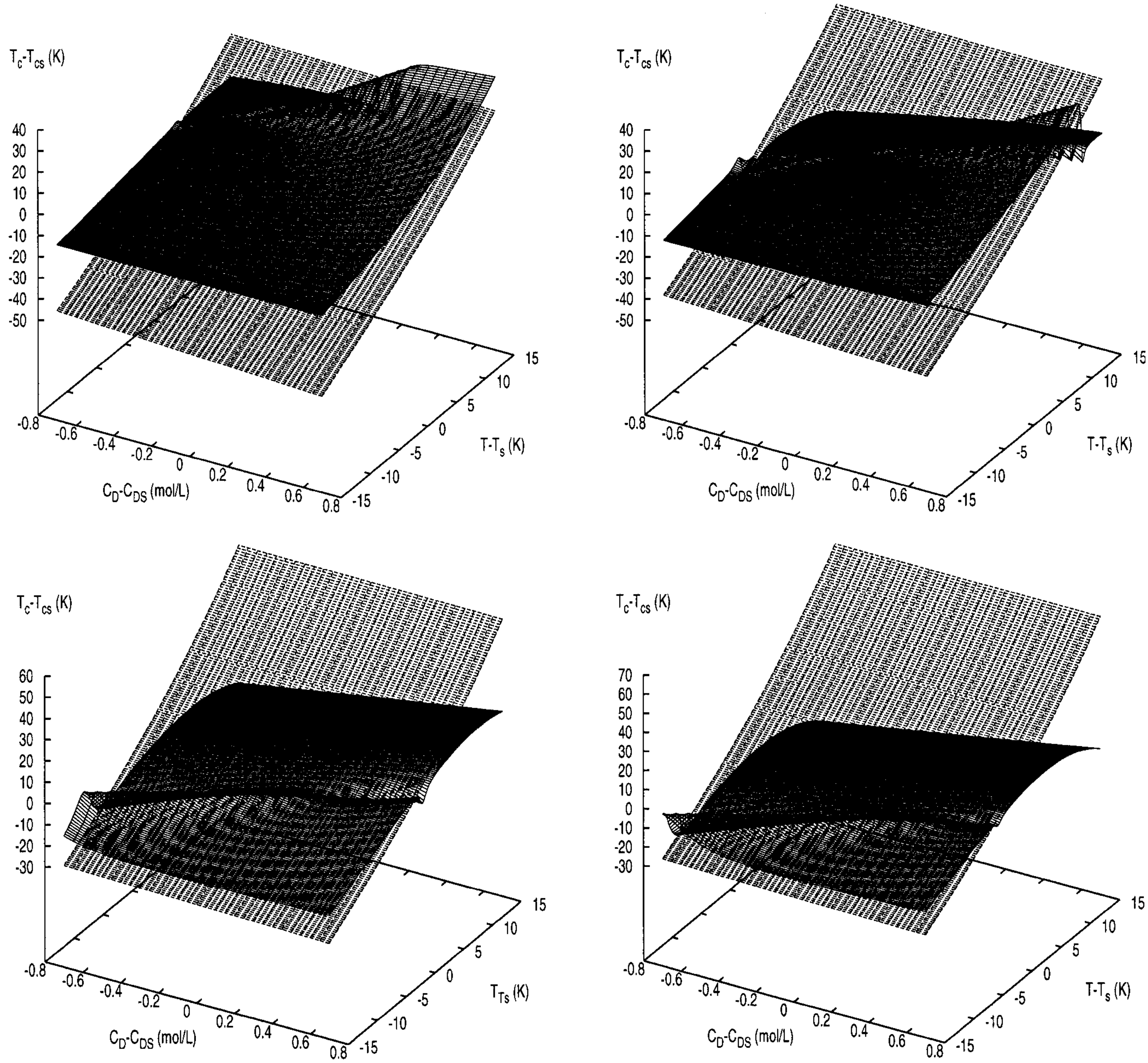

Figure 1. Control actions computed by the static component of the controller of equation (42) (solid lines) and the controller of equation (44) (dashed lines) for $C_A - C_{AS} = -0.4$ (top left), -0.2 (top right), 0.2 (bottom left), 0.4 (bottom right).

$$\left.\begin{aligned} V &= e^{\mathrm{T}}Pe, \qquad P = \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix}, \qquad c \in (0,1) \\ \widehat{L_{\bar{f}}V} &= 2((\hat{x}_2 - v) + c(L_f h(\hat{x}) - v^{(1)}))(L_f h(\hat{x}) - v^{(1)}) \\ &\quad + 2(c(\hat{x}_2 - v) + (L_f h(\hat{x}) - v^{(1)}))(L_f^2 h(\hat{x}) - v^{(2)}) \\ \widehat{L_{\bar{g}}V} &= 2(c(\hat{x}_2 - v) + (L_f h(\hat{x}) - v^{(1)}))L_g L_f h(\hat{x}) \end{aligned}\right\} \tag{43}$$

For the sake of comparison, we consider also the nonlinear output feedback controller synthesized based on the concepts of feedback linearization and proposed in Christofides (2000) with the static component

$$u = -\frac{1}{L_g L_f h(\hat{x})}\left\{\frac{\beta_0}{\beta_2}(\hat{x}_2 - v) + \frac{\beta_1}{\beta_2}(L_f h(\hat{x}) - v^{(1)}) + (L_f^2 h(\hat{x}) - v^{(2)})\right\} \tag{44}$$

Before we present the simulation results for the first scenario, we first establish the fact that the static *state* feedback component of the controller of equation (42) does not expend unnecessary control effort in compar-

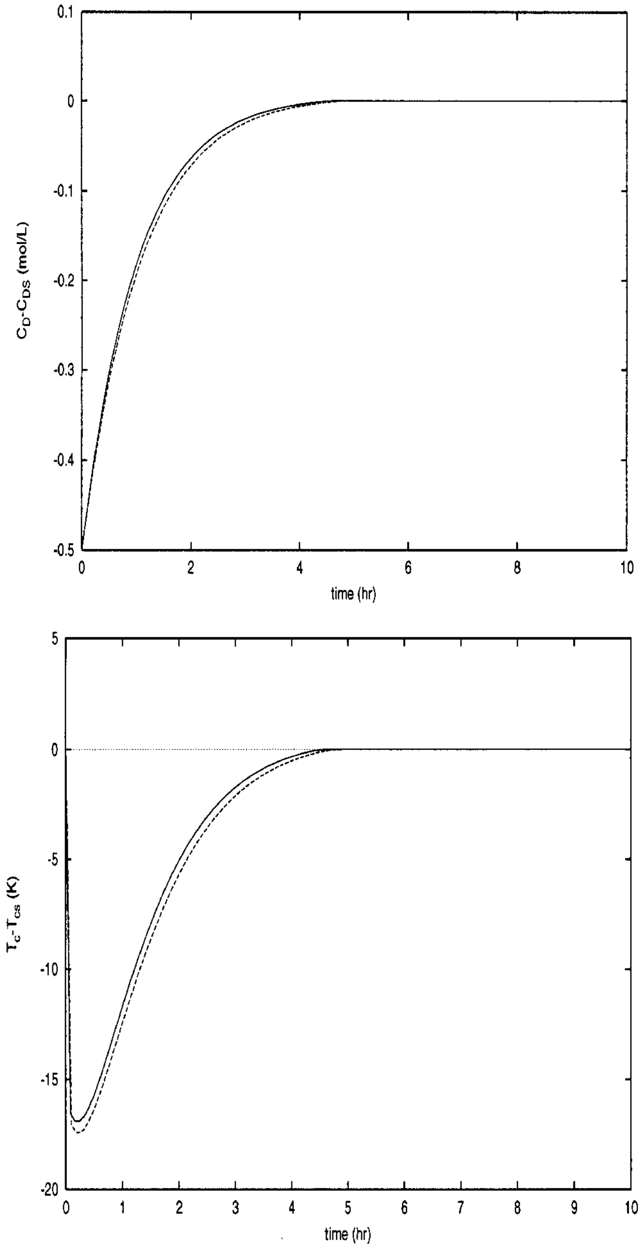

Figure 2. Closed-loop output and manipulated input profiles for stabilization under the output feedback controller of equation (42) (dashed lines) and the optimal state feedback controller (solid lines)—no uncertainty present.

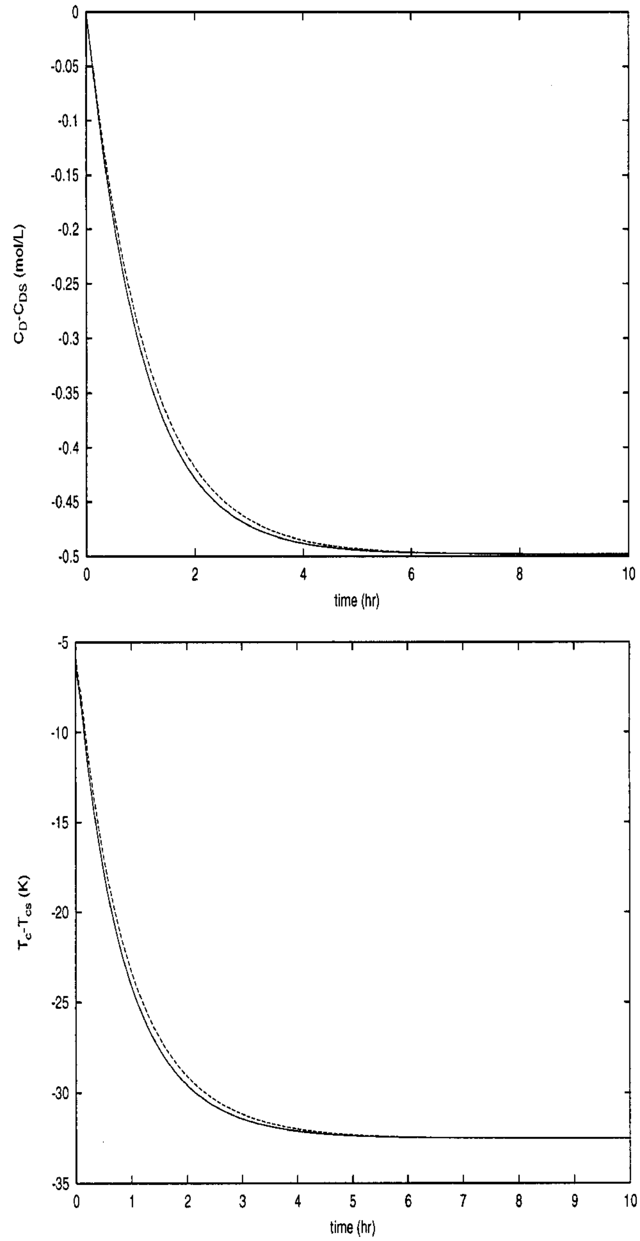

Figure 3. Closed-loop output and manipulated input profiles for reference input tracking under the output feedback controller of equation (42) (dashed lines) and the optimal state feedback controller (solid lines)—no uncertainty present.

ison with the static controller of equation (44). To this end, we compared the control actions generated by the two controllers for a broad range of T and C_D. The results are shown in figure 1 for several values of C_A. Clearly, the controller of equation (42) utilizes smaller control action than that of equation (44).

Two sets of simulation runs were performed to evaluate the performance and near-optimality properties of the dynamic output feedback controller of equation (42). The following values were used for the controller and observer parameters: $c_0 = 0.1$, $\alpha = 1.0$, $c = 0.99$, $L = 3000$, $a_1 = 10$, $a_2 = 20$, $a_m = 1.0$.

In the first set of simulation runs, we evaluated the capability of the controller to steer the process to the steady state given in Table 1, starting from arbitrary initial conditions. Figure 2 shows the controlled output profile and the corresponding manipulated input profile. Clearly, the controller successfully steers the process to the desired steady state. Also shown in the figure is a comparison between the output feedback controller of equation (42) and the corresponding state feedback controller (i.e. the static component of equation (42) with $\hat{x} = x$) synthesized under the assumption that all process states are available for measurement. It was shown in El-Farra and Christofides (1999) that this particular state feedback controller is optimal with respect to a cost of the form of equation (22) with $\hat{x} = x$. From figure 2, one can immediately observe that the controlled output and manipulated input profiles obtained under the output feedback controller are very close to the profiles obtained under the state feedback controller. This comparison then makes clear the fact that the controller of equation (42) is near-optimal in the sense that its performance approaches the optimal performance of the state feedback controller when the observer gain is sufficiently high. To further illustrate the near-optimality result, we compared the costs associated with both the state feedback and output feedback controllers. The costs were computed and found to be 0.250 and 0.257, respectively, further confirming the near-optimality of the controller of equation (42) in the sense that the cost incurred by using the output feedback controller is close to the optimal (minimal) cost produced by the optimal state feedback controller.

In the second set of simulation runs, we tested the output tracking capabilities of the output feedback controller of equation (42). The process was initially assumed to be at the steady state given in table 1 and then a 0.5 mol/L decrease in the value of the reference input was imposed ($v = -0.5$). Figure 3 shows the profiles of the controlled output of the process and of the corresponding manipulated input. One can immediately observe that the controller successfully drives the output to the desired new reference input value. The figure also shows the closeness of the controlled output and manipulated input profiles obtained under the output feedback and state feedback controllers. It is clear that the tracking capabilities of the output feedback controller approach those of the optimal state feedback controller and therefore the controller of equation (42) is near-optimal.

We now consider the second scenario where we account for the presence of uncertainties in the process. The following time-varying uncertainties were considered in all of the simulation runs:

$$\left.\begin{aligned} \theta_1(t) &= 0.4(-\Delta H_{D0})[1 + \sin(2t)] \\ \theta_2(t) &= 0.4(-\Delta H_{U0})[1 + \sin(2t)] \\ \theta_3(t) &= 0.4(-\Delta H_{R0})[1 + \sin(2t)] \\ \theta_4(t) &= 0.07 T_{A0s}[1 + \sin(2t)] \end{aligned}\right\} \tag{45}$$

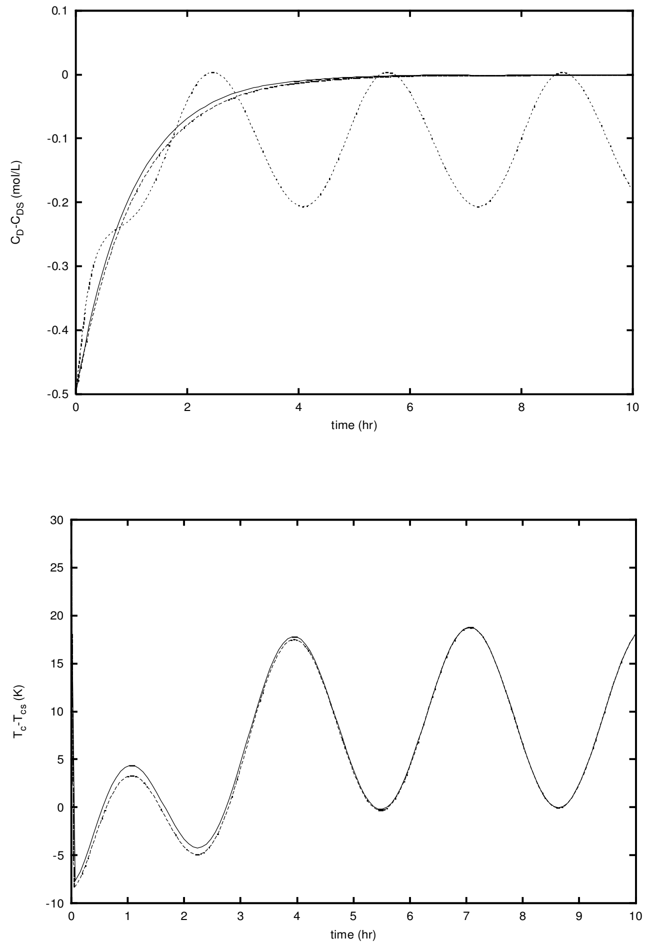

Figure 4. Closed-loop output and manipulated input profiles for stabilization under the robust output feedback controller of equation (47) (dashed lines), the robust optimal state feedback controller (solid lines) and open-loop profile (dotted line).

The upper bounds on the uncertain variables were taken to be $\theta_{b1} = 0.8|(-\Delta H_{D0})|$, $\theta_{b2} = 0.8|(-\Delta H_{U0})|$, $\theta_{b3} = 0.8|(-\Delta H_{R0})|$, $\theta_{b4} = 0.14 T_{A0s}$. Moreover, the following values were used for the tuning parameters of the controller and observer: $c_0 = 0.1$, $\alpha = 1.0$, $c = 0.99$, $\phi = 0.005$, $\chi = 2.1$, $L = 30\,000$, $a_1 = 10$, $a_2 = 20$, $a_m = 1.0$ to guarantee that the output of the closed-loop system satisfies a relation of the form

$$\limsup_{t \to \infty} |y - v| \le 0.005 \tag{46}$$

To design the necessary controller in this case, we note that since the $\bar{w}_k$ functions defined in equation (41) do not vanish at the origin, the uncertainties considered here are non-vanishing. Therefore, we use the result of Theorem 2 to construct the necessary robust output feedback controller which takes the form

$$
\left.
\begin{aligned}
\dot{\tilde{y}}_1 &= \tilde{y}_2 + La_1(y - \tilde{y}_1)\\
\dot{\tilde{y}}_2 &= L^2 a_2(y - \tilde{y}_1)\\
\dot{\omega}_1 &= \frac{F}{V}(C_{A0} - C_{As} - \hat{x}_1)\\
&\quad - k_{D0}\,\mathrm{e}^{-E_D/R(\hat{x}_3+T_s)}(\hat{x}_1 + C_{As})\\
&\quad - k_{U0}\,\mathrm{e}^{-E_U/R(\hat{x}_3+T_s)}(\hat{x}_1 + C_{As})\\
&\quad - k_{R0}\,\mathrm{e}^{-E_R/R(\hat{x}_3+T_s)}(\hat{x}_1 + C_{As})\\
u &= -\left(c_0 + \frac{\widehat{L_{\bar{f}}V} + \alpha\sqrt{(\widehat{L_{\bar{f}}V})^2 + (\widehat{L_{\bar{g}}V})^4}}{(\widehat{L_{\bar{g}}V})^2} \right.\\
&\quad \left. + \chi \frac{\sum_{k=1}^{q} \theta_{bk}|L_{wk}L_f h(\hat{x})|}{(L_g L_f h(\hat{x}))^2 \left(\frac{|\widehat{L_{\bar{g}}V}|}{|L_g L_f h(x)|} + \phi \right)} \right) \widehat{L_{\bar{g}}V}
\end{aligned}
\right\} \quad (47)
$$

where $V, \widehat{L_{\bar{f}}V}, \widehat{L_{\bar{g}}V}$ were defined in equation (43).

Several sets of simulation studies were performed to assess the performance, robustness, and near-optimality properties of the dynamic robust output feedback controller of equation (47). In the first set of simulations, we tested the ability of the controller to drive the output of the process close to the desired steady state starting from arbitrary initial conditions despite the presence of uncertainties. Figure 4 shows the controlled output profile and the manipulated input profile. One can immediately see that the effect of the uncertainty has been significantly reduced (compare with the output of the open-loop system) and the output of the process remains very close to the desired steady state satisfying the requirement of equation (46). Included in the figure also are the controlled output and manipulated input profiles for the process under the optimal state feedback controller (i.e. the static component of equation (47) with $\hat{x} = x$) which was shown in Theorem 2 in El-Farra and Christofides (1999) to be optimal with respect to the cost functional of equation (31) with $\hat{x} = x$. It is clear from the figure that the profiles obtained under the output feedback controller follow closely those obtained under the optimal state feedback controller and, therefore, the robust output feedback controller of equation (47) is near-optimal.

In the second set of simulations, we investigated the significance of the term responsible for uncertainty compensation in the controller of equation (47). To this end, we implemented the controller of equation (47) on the process without the uncertainty compensation component. The closed-loop output and manipulated input profiles for this simulation run are given in Figure 5. It is clear that the effect of the uncertain variables is appreciable, leading to poor transient performance and offset and that uncertainty compensation is required to achieve the control objective.

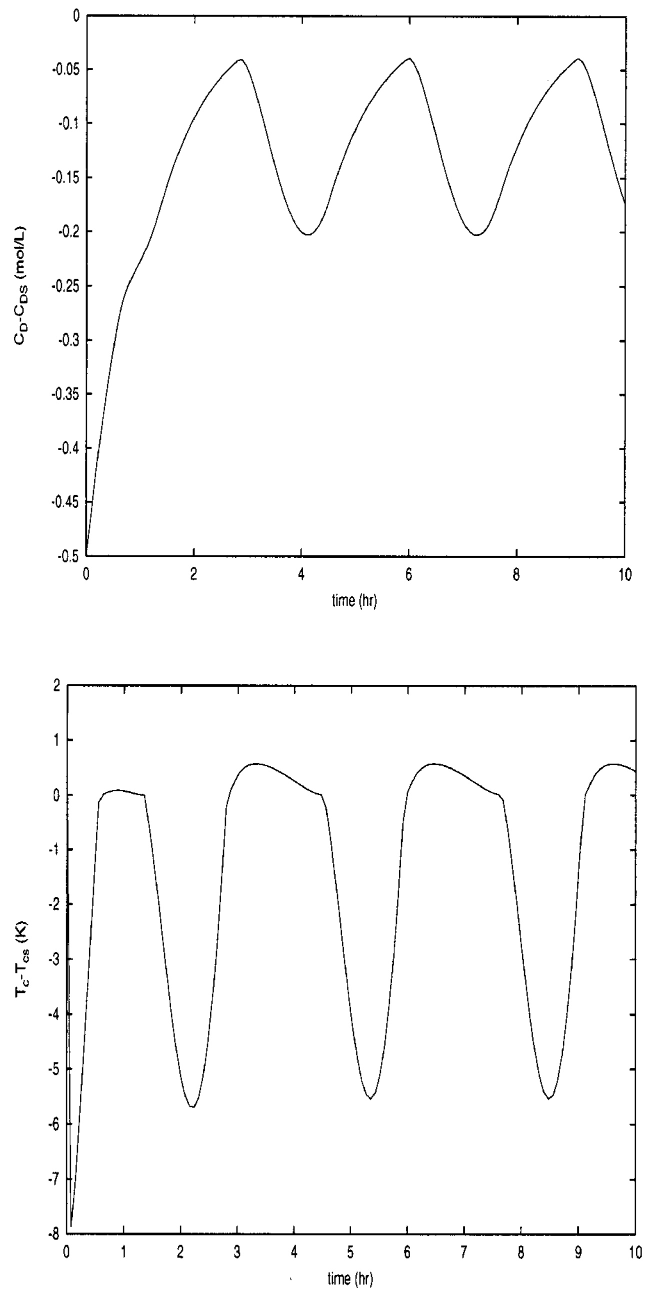

Figure 5. Closed-loop output and manipulated input profiles for stabilization under the output feedback controller of equation (47) with no uncertainty compensation.

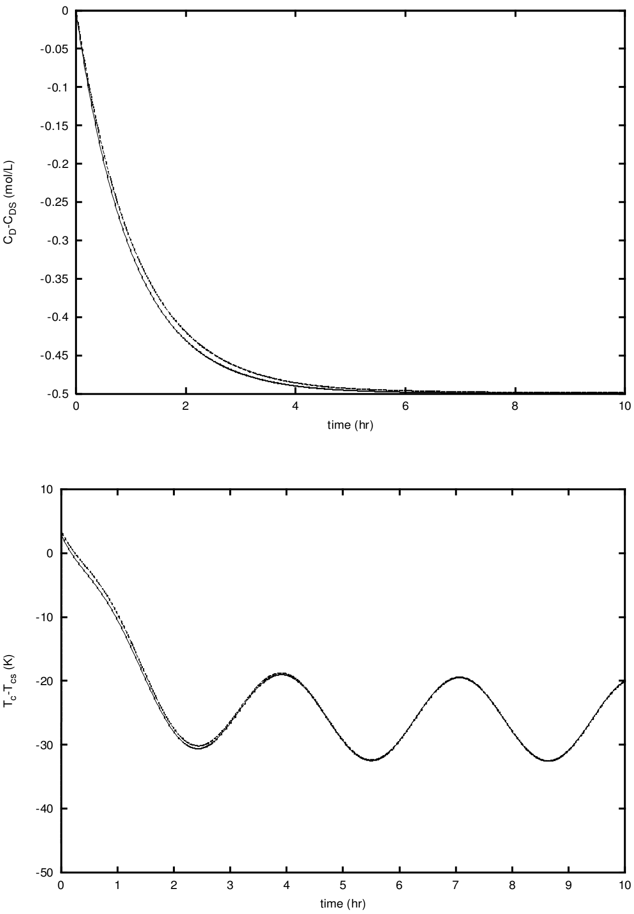

Figure 6. Closed-loop output and manipulated input profiles for reference input tracking under the robust output feedback controller of equation (47) (dashed lines) and the robust optimal state feedback controller (solid lines).

In the last set of simulations performed, we tested the output tracking capabilities of the controller of equation (47) for a 0.5 mol/L decrease in the value of the reference input. The resulting closed-loop output and manipulated input profiles, shown in figure 6, clearly establish the capability of the output feedback controller of equation (47) to enforce the requirement of equation (46) despite the presence of uncertainties. Furthermore, Figure 6 establishes the near-optimality of the output feedback controller within the context of robust output tracking. This is evident from the closeness of the performance of the robust output feedback controller of equation (47) to that of the corresponding robust optimal state feedback controller.

6. Conclusions

This article focused on robust near-optimal output feedback control of input/output linearizable non-linear systems with time-varying bounded uncertainties which possess ISS inverse dynamics. In the presence of vanishing uncertainties and using combination of a high gain observer and a robust optimal state feedback controller, a dynamic robust output feedback controller was synthesized that ensures exponential stability, robust asymptotic output tracking with uncertainty attenuation, and near-optimal performance over the infinite time-interval for arbitrarily large initial conditions and uncertainty, provided that the observer gain is sufficiently large. For systems with persistent uncertainties, the same controllers were shown to guarantee boundedness of the states, robust output tracking with uncertainty attenuation, and near-optimality over a finite-time interval. The developed controllers were successfully tested on a non-isothermal stirred tank reactor with uncertainty.

Acknowledgements

Finncial support in part by UCLA through a Chancellor's Fellowship for N. H. El-Farra and the Petroleum Research Fund, administered by the ACS, is gratefully acknowledged.

Appendix

Proof of Theorem 1: The proof of the theorem consists of three parts. In the first, part, we use a singular perturbation formulation to represent the closed-loop system of equation (2) and show that the resulting fast subsystem is globally exponentially stable. In the second part, we focus on the closed-loop reduced system and derive ISS bounds for its states. Then, using a technical lemma proved in Christofides (2000), we establish that these ISS bounds continue to hold up to an arbitrarily small offset, for arbitrarily large initial conditions and uncertainty. The resulting ISS inequalities are then analysed to establish semi-global boundedness and local exponential stability of the full closed-loop system, which is then used to establish equation (21), provided that ϕ and ϵ are sufficiently small. Finally, in the third part, we establish the near-optimality of the controller of equation (20) in these sense of equation (22).

Part 1: Defining the auxiliary error variables $\hat{e}_i = L^{r-i}(y^{(i-1)} - \tilde{y}_i)$, $i = 1, \ldots, r$, the vector $e_o = [\hat{e}_1\ \hat{e}_2\ \ \hat{e}_r]^{\mathrm{T}}$, the parameter $\epsilon = 1/L$, the matrix $\tilde{A}$ and the vector $\tilde{b}$

$$\tilde{A} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{r-1} & 0 & 0 & \cdots & 1 \\ -a_r & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{48}$$

the system of equation (2) under the controller of equation (20) takes the form

$$\left.\begin{aligned} \epsilon\dot{e}_o &= \tilde{A}e_o + \epsilon\tilde{b}\Omega(x, \hat{x}, \theta, \phi, \bar{v}) \\ \dot{w}_1 &= \Psi_1(\text{sat}\,(\tilde{y}), \omega) \\ &\vdots \\ \dot{\omega}_{n-r} &= \Psi_{n-r}(\text{sat}\,(\tilde{y}), \omega) \\ \dot{x} &= f(x) + g(x)\mathcal{A}(\hat{x}, \bar{v}, \phi) + \sum_{k=1}^{q} w_k(x)\theta_k(t) \end{aligned}\right\} \tag{49}$$

where $\hat{x} = \mathcal{X}^{-1}(\text{sat}\,(y_d - \Delta(\epsilon)e_o), \omega)$ and $y_d = [y^{(0)}\ y^{(1)}\ \cdots\ y^{(r-1)}]^{\mathrm{T}}$, $\Delta(\epsilon)$ is a diagonal matrix whose ith diagonal element is ϵ^{r-i}, and $\Omega(x, \hat{x}, \theta, \phi, \bar{v})$ is a Lipschitz function of its argument. Owing to the presence of the small parameter ϵ that multiplies the time-derivative $\dot{e}_o$, the system of equation (49) is a two-time-scale one. Defining the fast time-scale $\tau = t/\epsilon$ and setting $\epsilon = 0$, the closed-loop fast subsystem takes the form

$$\frac{\mathrm{d}e_o}{\mathrm{d}\tau} = \tilde{A}e_o \tag{50}$$

Since the constant matrix A is Hurwitz, the origin of the system of equation (50) is globally exponentially stable.

Part 2: In this part of the proof, we initially derive ISS bounds, when $\epsilon = 0$, for the states of the system of equation (49), in appropriately transformed coordinates, and then use the result of the Lemma given in Christofides (2000) to show that these bounds hold up to an arbitrarily small offset, for intial conditions and uncertainty in an arbitrarily large compact set, provided that ϵ is sufficiently small. Defining the variables $x = \mathcal{X}^{-1}(\zeta, \eta)$ and using the notation defined in equation (14), the closed-loop system can be written as

$$\left.\begin{aligned} \epsilon\dot{e}_o &= \tilde{A}e_o + \epsilon\tilde{b}\Omega(e, \eta, \hat{x}, \theta, \phi, \bar{v}) \\ \dot{\omega}_1 &= \Psi_1(\text{sat}\,(\tilde{y}), \omega) \\ &\vdots \\ \dot{\omega}_{n-r} &= \Psi_{n-r}(\text{sat}\,(\tilde{y}), \omega) \\ \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_{r-1} &= e_r \\ \dot{e}_r &= L_f^r h(\mathcal{X}^{-1}(e, \bar{v}, \eta)) - v^{(r)} \\ &\quad + L_g L_f^{r-1} h(\mathcal{X}^{-1}(e, \bar{v}, \eta))\mathcal{A}(\hat{x}, \bar{v}, \phi) \\ &\quad + \sum_{k=1}^{q} L_{wk} L_f^{r-1}(\mathcal{X}^{-1}(e, \bar{v}, \eta))\theta_k \\ \dot{\eta}_1 &= \Psi_1(e, \bar{v}, \eta) \\ &\vdots \\ \dot{\eta}_{n-r} &= \Psi_{n-r}(e, \bar{v}, \eta) \\ y &= e_1 + v \end{aligned}\right\} \tag{51}$$

where $\bar{\Psi}_i$, $i = 1, \ldots, n$ are Lipschitz functions of their arguments.

Step 1: Consider the system of equation (51) with $\epsilon = 0$. In order to analyse the dynamic behaviour of the resulting closed-loop slow (reduced) system, we initially need to show that for the system of equation (51), $\eta(0) = \omega(0) + O(\epsilon)$ implies $\eta(t) = \omega(t) + O(\epsilon)$, $\forall t \geq 0$. To this end, consider the following singularly perturbed system

$$\left.\begin{aligned} \epsilon\dot{e}_o &= \tilde{A}e_o + \epsilon\tilde{b}\Omega(e, \eta, \hat{x}, \theta, \phi, \bar{v}) \\ \dot{\omega}_1 &= \Psi_1(\text{sat}\,(\tilde{y}), \omega) \\ &\vdots \\ \dot{\omega}_{n-r} &= \Psi_{n-r}(\text{sat}\,(\tilde{y}), \omega) \\ \dot{\eta}_1 &= \Psi_1(e, \bar{v}, \eta) \\ &\vdots \\ \dot{\eta}_{n-r} &= \Psi_{n-r}(e, \bar{v}, \eta) \end{aligned}\right\} \tag{52}$$

It is straightforward to verify that the above system satisfies the assumptions of Theorem 1 reported in Khalil and Esfandiari (1993). Applying this theorem, we have that there exists a positive real number ϵ_0, such that for any positive real number δ_ω satisfying

$$\delta_\omega \geq \max_{|x| \leq \delta} \left\{ \sum_{\nu=1}^{n-r} |\chi_\nu(x)| \right\} \tag{53}$$

where $\chi_\nu(x)$, $\nu = 1, \ldots, n-r$ are the functions defined in Assumption 1, the states (η, ω) of this system, starting from any initial conditions that satisfy $\eta(0) = \omega(0) + O(\epsilon)$ (with $\max\{\eta(0), \omega(0)\} \leq \delta_\omega$), if $\epsilon \in (0, \epsilon_0]$, satisfy $\eta(t) = \omega(t) + O(\epsilon)$, $\forall t \geq 0$.

Since $\eta(t) = \omega(t)\ \forall t \geq 0$, when $\epsilon = 0$, the closed-loop slow system of equation (51) reduces to the one studied in El-Farra and Christofides (1999). Using $V = e^{\mathrm{T}} P e$ as a Lyapunov function candidate, where P is a positive definite symmetric matrix defined in Theorem 1, it is straightforward to show that there exist positive real numbers ϕ^* and k_e such that if $\phi \in (0, \phi^*]$, $\dot{V}$ satisfies the following properties

$$\dot{V} \leq -k_e |e|^2\ \forall t \geq 0, \quad \dot{V} < 0\ \forall e \neq 0 \tag{54}$$

and, therefore, there exist positive real numbers k_1 and a_1 such that the following inequality holds for the reduced (slow) e subsystem

$$|e(t)| \leq k_1 |e(0)|\, \mathrm{e}^{-a_1 t} \quad \forall t \geq 0 \tag{55}$$

where $a_1 > 0$ and $k_1 \geq 1$. From Assumption 2, we have that the η states of the closed-loop slow system satisfy the following ISS inequality

$$|\eta(t)| \leq K_\eta |\eta(0)|\, \mathrm{e}_\eta^{-at} + \gamma(\|e\|) \quad \forall t \geq 0 \tag{56}$$

where $a > 0$, $K_\eta \geq 1$ and γ_η is a class K function. From the above two inequalities, we have that the origin of the closed-loop reduced e subsystem is globally exponentially stable and that the closed-loop reduced η subsystem, with e as input, is ISS. Realizing that the closed-loop slow system is an interconnection of the e and η subsystems, we show now that the origin of the closed-loop slow system is globally asymptotically stable and locally exponentially stable. From equations (55) and (56), we have that the solutions of the closed-loop slow system satisfy

$$|\eta(t)| \leq K_\eta |\eta(s)|\, \mathrm{e}^{-a(t-s)} + \gamma_\eta\Big(\sup_{s \leq \tau \leq t} |e(\tau)|\Big) \tag{57}$$

$$|e(t)| \leq k_1 |e(s)|\, \mathrm{e}^{-a_1(t-s)} \tag{58}$$

globally, where $t \geq s \geq 0$. Applying equation (57) with $s = t/2$, we obtain

$$|\eta(t)| \leq K_\eta \left|\eta\Big(\frac{t}{2}\Big)\right| \mathrm{e}^{-a(t/2)} + \gamma_\eta\Big(\sup_{t/2 \leq \tau \leq t} |e(\tau)|\Big) \tag{59}$$

To estimate $\eta(t/2)$, we apply equation (57) with $s = 0$ and t replaced by $t/2$ to obtain

$$\left|\eta\Big(\frac{t}{2}\Big)\right| \leq K_\eta |\eta(0)| \mathrm{e}^{-a(t/2)} + \gamma_\eta\Big(\sup_{0 \leq \tau \leq t/2} |e(\tau)|\Big) \tag{60}$$

From equation (58), we have

$$\sup_{0 \leq \tau \leq t/2} |e(\tau)| \leq k_1 |e(0)| \tag{61}$$

$$\sup_{t/2 \leq \tau \leq t} |e(\tau)| \leq k_1 |e(0)| \mathrm{e}^{-a_1(t/2)} \tag{62}$$

Substituting equations (60–62) into equation (59) and using the inequalities $|\eta(0)| \leq |\pi(0)|$, $|e(0)| \leq |\pi(0)|$ and $|\pi(t)| \leq |\eta(t)| + |e(t)|$, where $\pi(0) = [e(0)\ \eta(0)]^{\mathrm{T}}$, we obtain

$$|\pi(t)| \leq \beta(|\pi(0)|, t) \tag{63}$$

where

$$\beta(|\pi(0)|, t) = [K_\eta |\pi(0)| \mathrm{e}^{-a(t/2)} + \gamma_\eta(k_1 |\pi(0)|)]\, \mathrm{e}^{-a(t/2)} + \gamma_\eta(k_1 |\pi(0)|\, \mathrm{e}^{-a_1(t/2)}) + k_1 |\pi(0)|\, \mathrm{e}^{-a_1 t} \tag{64}$$

It can be easily verified that β is a class KL function. Hence the origin of the closed-loop slow (reduced) system is globally asymptotically stable (i.e. $\pi(t) \to 0$ as $t \to \infty$). To prove local exponential stability, we proceed as follows. From equation (64) and the properties of the function γ_η, there exist a finite time $t_1 > 0$ and positive real numbers k_2 and r such that

$$\left.\begin{aligned} &|\pi(t)| \leq r \\ &\gamma_\eta(k_1 |\pi(0)| \mathrm{e}^{-a_1(t/2)}) \leq k_1 |\pi(0)|\, \mathrm{e}^{-a_1(t/2)} \end{aligned}\right\} \tag{65}$$

$\forall t \geq t_1$. The time t_1 represents the smallest time required for the trajectory of the closed-loop slow system $\pi(t)$ to enter a small ball of radius r centred around the origin where the function γ_η satisfies the linear growth bound given in equation (65). Substituting equation (65) directly into equation (64), we conclude that there exist positive real numbers a_2 and $k_3 \geq 1$ such that

$$|\pi(t)| \leq k_3 |\pi(0)|\, \mathrm{e}^{-a_2 t} \quad \forall\, |\pi(t)| \leq r \tag{66}$$

or, equivalently, $\forall t \geq t_1$. Hence, the origin of the closed-loop slow system is locally exponentially stable (note that the same result can be obained from the linearization of the closed-loop slow system around the origin). Finally, we note that since the static component of the controller of equation (20) with $\hat{x} = x$, i.e. $\mathcal{A}(x, v, \phi)$, enforces global asymptotic stability in the closed-loop slow system, the state ζ of the closed-loop slow system satisfies a bound of the following form $\forall t \geq 0$

$$|\zeta(t)| \leq \beta_\zeta(\delta_\zeta, t) \tag{67}$$

where β_ζ is a class KL function and δ_ζ is the maximum value of the vector $[h(x)\ L_f h(x)\ \cdots\ L_f^{r-1} h(x)]$ for $|x| \leq \delta_x$.

Based on the above bound and following the results of Khalil and Esfandiari (1993) and Teel and Praly (1995), we disregard estimates of $\tilde{y}$, obtained from the high-gain observer, with norm $|\tilde{y}| > \beta_\zeta(\delta_\zeta, 0)$. Hence, we set $\mathrm{sat}(\cdot) = \min\{1, \zeta_{\max}/|\cdot|\}(\cdot)$ where $\zeta_{\max}$ is the maximum value of the vector $[\zeta_1\ \zeta_2\ \cdots\ \zeta_r]$ for $|\zeta| \leq \beta_\zeta(\delta_\zeta, 0)$.

Step 2: In order to apply the result of Lemma 1 in Christofides (2000), we need to define a set of positive real numbers $\{\bar{\delta}_e, \bar{\delta}_\eta, \delta_\theta, \delta_{\bar{v}}, \delta_\eta, \delta_e, d, d_e, d_\eta\}$ where δ_θ and $\delta_{\bar{v}}$ were specified in the statement of the theorem, d, d_e and d_η are arbitrary positive real numbers

$$\left.\begin{aligned} &\bar{\delta}_e \geq \max_{|x|\leq\delta_{\bar{x}}, |\bar{v}|\leq\delta_{\bar{v}}} \left\{ \left| \sum_{k=1}^{r} (v^{(k-1)} - L_{\tilde{f}}^{k-1} h(x)) \right| \right\} \\ &\bar{\delta}_\eta \geq \max_{|x|\leq\delta_{\bar{x}}, |\bar{v}|\leq\delta_{\bar{v}}} \left\{ \sum_{\nu=1}^{n-r} |\chi_\nu(x)| \right\} \\ &\delta_e > \bar{\delta}_e, \ \ \delta_\eta > D^\eta + \delta_\eta(\delta_e), \ \ D^\eta := \beta_\eta(\bar{\delta}_\eta, 0) = K_\eta \bar{\delta}_\eta \end{aligned}\right\} \tag{68}$$

First, consider the singularly perturbed system comprised of the states (ζ, e_o) of the closed-loop system of equation (51). This system is in standard form, possesses a globally exponentially stable fast subsystem, and its corresponding reduced system is globally asymptotically stable. These properties allow a direct application of the result of Lemma 1 given in Christofides (2000) with $\delta = \max\{\delta_\zeta, \delta_{e_{\bar{o}}}, \delta_\theta, \delta_{\bar{v}}, \delta_\eta\}$ to obtain the existence of a positive real number ϵ^ζ such that if $\epsilon \in (0, \epsilon^\zeta]$, and $|\zeta(0)| \leq \delta_\zeta$, $|e_o(0)| \leq \delta_{e_o}$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, $\|\eta\| \leq \delta_\eta$, then

$$\left.\begin{aligned} |\zeta(t)| &\leq \beta_\zeta(\delta_\zeta, t) + d \\ |e_o(t)| &\leq k_4 |e_o(0)|\, \mathrm{e}^{-a_3(t/\epsilon)} + d \end{aligned}\right\} \tag{69}$$

where $k_4 \geq 1$ and $a_3 > 0$.

Consider now the singularly perturbed system comprised of the states (e, e_o) of the system of equation (51). This system is in standard form, possesses a globally exponentially stable fast subsystem, and its corresponding reduced system is globally asymptotically stable. These properties allow a direct application of the results of Lemma 1 in Christofides (2000) with $\delta = \max\{\bar{\delta}_e, \delta_{e_o}, \delta_\theta, \delta_{\bar{v}}, \delta_\eta\}$ to obtain the existence of a positive real number ϵ^e such that if $\epsilon \in (0, \epsilon^e]$ and $|e(0)| \leq \bar{\delta}_e$, $|e_o(0)| \leq \delta_{e_o}$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, $\|\eta\| \leq \delta_\eta$, then

$$|e(t)| \leq k_1 |e(0)|\, \mathrm{e}^{-a_1 t} + d_e \tag{70}$$

Similarly, it can be shown that Lemma 1, with $\delta = \max\{\bar{\delta}_\eta, \delta_{e_{\bar{o}}}, \delta_\theta, \delta_{\bar{v}}, \delta_e\}$, can be applied to the system comprised of the states (η, e_o) of the system of equation (51). Thus, we have that there exist positive real numbers ϵ^η such that if $\epsilon \in (0, \epsilon^\eta]$ and $|\eta(0)| \leq \bar{\delta}_\eta$, $|e_o(0)| \leq \delta_{e_o}$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, $\|e\| \leq \delta_e$, then

$$|\eta(t)| \leq K_\eta |\eta(0)|\, \mathrm{e}_\eta^{-at} + \gamma_\eta(\|e\|) + d_\eta \tag{71}$$

Finally, the inequalities of equations (55), (56), (70) and (71) can be manipulated using a small-gain theorem type argument similar to the one used in the proof of Theorem 1 in Christofides *et al.* (1996) to show that given the set of positive real numbers $\{\bar{\delta}_e, \bar{\delta}_\eta, \delta_{e_o}, \delta_\theta, \delta_{\bar{v}}, d, d_e, d_\eta\}$ and with $\phi \in (0, \phi^*]$, there exists $\epsilon^* \in (0, \min\{\epsilon^\zeta, \epsilon^e, \epsilon^\eta\}]$ such that if $\epsilon < (0, \epsilon^*]$, and $|e(0)| \leq \bar{\delta}_e$, $|\eta(0)| \leq \bar{\delta}_\eta$, $|e_o(0)| \leq \delta_{e_o}$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, then the states of the closed-loop system are bounded. For brevity, this argument will not be repeated here.

Step 3: So far in the proof, we have established that the states of the closed-loop system of equation (51) are semi-globally bounded. In this section we show that the origin of the closed-loop system is, in fact, exponentially stable. Referring to the inequalities of equations (69)–(71), we note that the arbitrary positive real numbers d, d_e and d_η can be chosen small enough such that, after a sufficiently large time, say $\tilde{t}$, the trajectories of the closed-loop system lie within a small neighbourhood of the origin. The time $\tilde{t}$ represents the smallest time required for the closed-loop trajectories to enter this neighbourhood without ever leaving again. Obviously, $\tilde{t}$ depends on both δ and the size of the neighbourhood around the origin, but is independent of ϵ. Therefore, given the set of positive real numbers $\{\delta, r_1, r_2, \rho_0\}$, one can pick d, d_e and d_η such that the following inequalities hold

$$\left.\begin{aligned} &|e(t)| \leq k_1 |e(0)|\, \mathrm{e}^{-a_1 \tilde{t}} + d_e \leq r_1 \\ &|\eta(t)| \leq K_\eta |\eta(0)|\, \mathrm{e}^{-a\tilde{t}} + \gamma_\eta(\|e\|) + d_\eta \leq r_2 \\ &|e_o(t)| \leq k_4 |e_o|\, \mathrm{e}^{-a_3(\tilde{t}/\epsilon)} + d \leq \rho_0 \end{aligned}\right\} \tag{72}$$

and we finally have

$$\left.\begin{aligned} |[e(t)\ \ \eta(t)\ \ \omega(t)]^{\mathrm{T}}| &\leq r \\ |e_o(t)| &\leq \rho_0 \end{aligned}\right\} \tag{73}$$

$\forall t \geq \tilde{t}$, where r and ρ_0 are the radii of two small balls centred around the origins of the slow and fast subsystems of the closed-loop system of equation (51), respectively. We require that the size of the neighbourhood around the origin of the closed-loop system (i.e. r and ρ_0) be sufficiently small such that, within this neighbourhood, the right-hand side of equation (51) satisfies the local Lipschitz properties stated in the proof of Theorem 9.3 reported in Khalil (1996). Then, a direct application of the results of this theorem yields that there exists $\bar{\epsilon}$ such that if $\epsilon \in (0, \bar{\epsilon}]$, then:

$$|z(t)| \leq \alpha_1 |z(0)|\, \mathrm{e}^{-\beta_1 t} \tag{74}$$

for some $\alpha_1 \geq 1$ and $\beta_1 > 0$ where $z(t) = [e(t)\ \ \eta(t)\ \ \omega(t)\ \ e_o(t)]^{\mathrm{T}}$ and $\epsilon \in (0, \min\{\bar{\epsilon}, \epsilon^\zeta, \epsilon^e, \epsilon^\eta\}]$. Consequently, the origin of the closed-loop system of equation (51) is locally exponentially stable. The asymptotic output tracking result can now be obtained by simply noting that from equation (74) we have

$$\limsup_{t\to\infty} |e(t)| = 0 \quad (75)$$

and therefore

$$\limsup_{t\to\infty} |e_1(t)| = \limsup_{t\to\infty} |y(t) - v(t)| = 0 \quad (76)$$

Part 3: In this part of the proof, we show that the controller of equation (20) yields near-optimal output feedback design for the class of systems considered in equation (2) and produces a finite cost which tends to the optimal cost for the state feedback problem as $\epsilon \to 0$. To these end, we adopt the following three-step procedure. In the first step, we establish the closeness of solutions of the reduced and full closed-loop systems on the infinite time-interval. In step 2, we show that, for the reduced (slow) system, the controller of equation (20) with $\epsilon = 0$ is optimal with respect to a cost functional J_r of the form of equation (22) with $\hat{x} = x$. Finally, in Step 3, we exploit the closeness of solutions result to show that the cost associated with the full closed-loop system approaches the optimal cost for the reduced system under state feedback when the observer gain is sufficiently large, and, hence, the controller of equation (20) is near-optimal.

Step 1: In this step, we show that starting from an arbitrarily large compact set of initial conditions and for arbitrarily large uncertainties, the following estimates hold

$$\left.\begin{aligned} z(t,\epsilon) - \bar{z}(t) &= O(\epsilon) \qquad \forall t \geq 0 \\ e_o(t,\epsilon) &= O(\epsilon) \qquad \forall t \geq t_b \end{aligned}\right\} \quad (77)$$

for some $t_b > 0$ where $\bar{z}(t) = [\bar{e}(t)\ \bar{\eta}(t)\ \omega(t)]^{\mathrm{T}}$ is the solution of the reduced (slow) problem obtained by setting $\epsilon = 0$ in equation (51). The strategy we shall adopt to prove the above estimates involves, first, the application of Tikhonov's theorem to establish the closeness of solutions on a finite time-interval whose size can be made arbitrarily large by selecting ϵ sufficiently small. Then, once the trajectories of the closed-loop system become confined within an appropriately small ball around the origin, we apply the result of Theorem 9.4 reported in Khalil (1996) to obtain the estimates of equation (77).

Referring to the system of equation (51), it is straightforward to show that this system satisfies the conditions of Theorem 9.1 given in Khalil (1996). Applying the result of this theorem to the system of equation (51), we conclude that there exists a finite time $t_1 > 0$ and positive constants K, L and $\bar{\bar{\epsilon}}$ such that if $\epsilon \in (0, \bar{\bar{\epsilon}}]$ and $|e(0)| \leq \bar{\delta}_e$, $|\eta(0)| \leq \bar{\delta}_\eta$, $|e_o(0)| \leq \delta_{e_o}$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, then

$$|z(t,\epsilon) - \bar{z}(t)| \leq \epsilon K[1 + t_1] \exp(Lt_1) \quad (78)$$

and the estimate

$$z(t,\epsilon) - \bar{z}(t) = O(\epsilon) \quad (79)$$

holds uniformly for $t \in [0, t_1]$. Furthermore, given any $t_b > 0$, there is $\epsilon_b \leq \bar{\bar{\epsilon}}$ such that the estimate

$$e_o(t,\epsilon) = O(\epsilon) \quad (80)$$

holds uniformly for $t \in [t_b, t_1]$ whenever $\epsilon < \bar{\bar{\epsilon}}$. In order for the above estimates to hold on the infinite time-interval, we require that $t_1 \geq \tilde{t}$ where $\tilde{t}$ was defined in Step 2 of Part 2 of the proof. From equation (78), it is clear that t_1 can be made arbitrarily large while maintaining the estimate of equation (79) by selecting ϵ sufficiently small. Therefore there exists a positive constant ϵ^{**} such that if $\epsilon \in (0, \epsilon^{**}]$, we have $t_1 \geq \tilde{t}$ and the trajectories of the closed-loop system lie within the neighbourhood of the origin defined in equation (73). It can be easily verified that, within this neighbourhood, the closed-loop system of equation (51) satisfies the assumptions of Theorem 9.4 reported in Khalil (1996). Applying the results of this theorem, we obtain the existence of a positive constant ϵ' such that if $\epsilon \in (0, \epsilon']$, the estimates of equation (77) hold $\forall |z(t)| \leq r$ and $|e_0(t)| \leq \rho_0$. Finally, we now have that if $\epsilon \in (0, \epsilon^*]$ where $\epsilon^* = \min\{\epsilon', \epsilon^{**}, \bar{\bar{\epsilon}}, \bar{\epsilon}, \epsilon^\zeta, \epsilon^e, \epsilon^\eta, \epsilon_0\}$, then equation (77) is satisfied for $|e(0)| \leq \bar{\delta}_e$, $|\eta(0)| \leq \bar{\delta}_\eta$, $|e_o(0)| \leq \delta_{e_o}$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$.

Step 2: In this step we show that the static component of the controller of equation (20) with $\epsilon = 0$ and $\hat{x} = x$ minimizes the cost functional associated with the slow (reduced) system as defined in equation (22). To this end, consider the closed-loop system of equation (51) with $\epsilon = 0$ where u of equation (20) is applied with $\hat{x} = x$. We note that the resulting slow system is identical to the one studied in El-Farra and Christofides (1999) (state feedback problem) and satisfies the assumptions of Theorem 1 stated therein. Applying the result of this theorem, we obtain that the static component of the controller of equation (20) minimizes the cost functional

$$J_r = \int_0^\infty (l(\bar{e}) + u(\bar{x})R(\bar{x})u(\bar{x}))\,\mathrm{d}t \quad (81)$$

where $\bar{x} = \mathcal{X}^{-1}(\bar{e}, \bar{\eta}, \bar{v})$ refers to the solution of the reduced closed-loop system under state feedback. Following the same treatment presented in Part 3 of the proof of Theorem 1 in El-Farra and Christofides (1999), it can be readily shown that upon substituton of the static component u of equation (20) (which we shall denote from now on by u^* to emphasize that it is optimal) in the above expression, the minimum cost obtained is

$$J_r^* = \int_0^\infty (l(\bar{e}) + u^*(\bar{x})R(\bar{x})u^*(\bar{x}))\,\mathrm{d}t = V(\bar{e}(0)) \quad (82)$$

Step 3: In the last step of this part of the proof, we exploit the closeness of solutions result obtained in Step 1 and combine it with the optimality result established in Step 2 to prove that the output feedback controller of equation (20) is near-optimal in the sense that it produces a cost functional for the full closed-loop sysem that approaches the optimal cost for the reduced (state feedback) problem provided that the gain of the observer is sufficiently large. Towards this end, consider rewriting the cost functional of equation (22) as

$$J = \int_0^{t_b} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t + \int_{t_b}^{\infty} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t \tag{83}$$

From Step 2, we have that the following estimates hold for $t \geq t_b$

$$\left.\begin{aligned} e(t) &= \bar{e}(t) + O(\epsilon) \\ x(t) &= \bar{x}(t) + O(\epsilon) \\ \hat{x}(t) &= \bar{x}(t) + O(\epsilon) \end{aligned}\right\} \tag{84}$$

It follows then from the continuity properties of the functions $l(\cdot)$, $u(\cdot)$, $R(\cdot)$, that for $t \geq t_b$

$$l(e) \to l(\bar{e}), \quad u(\hat{x}) \to u(\bar{x}), \quad R(x) \to R(\bar{x}) \quad \text{as } \epsilon \to 0 \tag{85}$$

and hence

$$\int_{t_b}^{\infty} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t \to \int_{t_b}^{\infty} (l(\bar{e}) + u(\bar{x})R(\bar{x})u(\bar{x})_{\bar{b}})\,\mathrm{d}t \quad \text{as } \epsilon \to 0 \tag{86}$$

From the stability of the closed-loop system established in Part 2 of the proof, there exists a positive real number M that bounds the integrand of equation (22). Using the fact that $t_b = O(\epsilon)$, we have that

$$\begin{aligned} \int_0^{t_b} (l(e) + u(\hat{x})R(x)u(\hat{x})\,\mathrm{d}t) &\leq \int_0^{t_b} M\,\mathrm{d}t \\ &\leq M\epsilon \\ &= O(\epsilon) \end{aligned} \tag{87}$$

Similarly, from the stability of the reduced closed-loop system (state feedback problem) established in Part 2 and the fact that $t_b = O(\epsilon)$, there exists a positive real number M' such that

$$\begin{aligned} \int_0^{t_b} (l(\bar{e}) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t &\leq \int_0^{t_b} M'\,\mathrm{d}t \\ &\leq M'\epsilon \\ &= O(\epsilon) \end{aligned} \tag{88}$$

Combining equations (86–88), we obtain

$$\int_0^{\infty} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t \to \int_0^{\infty} (l(\bar{e}) + u(\bar{x})R(\bar{x})u(\bar{x}))\,\mathrm{d}t \quad \text{as } \epsilon \to 0 \tag{89}$$

and from equation (82), we finally have

$$J = \int_0^{\infty} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t \to V(\bar{e}(0)) \quad \text{as } \epsilon \to 0 \tag{90}$$

This completes the proof of the theorem. □

Proof of Theorem 2: The proof of this theorem follows closely the proof of Theorem 1. We shall point out the differences. In Part 1 we establish the global exponential stability of the closed-loop fast subsystem and derive ISS bounds for the states of the closed-loop reduced system. Then using the technical lemma employed in the proof of Theorem 1, we show that the derived ISS bounds continue to hold up to an arbitrarily small offset, for arbitrarily large initial conditions and uncertainty. In Part 2, the resulting ISS inequalities are studied, using techniques similar to those used in Christofides *et al.* (1996) (see also Jiang *et al.* 1995) to show boundedness of the trajectories and establish the inequality of equation (30). In Part 3, we apply Tikhonov's theorem to obtain the closeness of solutions on the finite time-interval and then use this result to prove near-optimality of the output feedback controller of equation (29).

Part 1: The global exponential stablity of the closed-loop fast subsystem was established in the first step of Part 1 of the proof of Theorem 1 and will not be repeated here. Instead, we focus on the reduced system and derive ISS bounds that capture the evolution of its states. To this end, we consider again the system of equaton (51) with $\epsilon = 0$. Using the same argument presented in Step 2 of Part 1 in the previous proof, one can again show that for the system of equation (51), $\omega(0) = \eta(0) + O(\epsilon)$ implies $\eta(t) = \omega(t) + O(\epsilon)\ \forall t \geq 0$, which yields that $\omega(t) = \eta(t)$ when $\epsilon = 0$. Now consider the Lyapunov function candidate $V = e^{\mathrm{T}}Pe$ where P is the positive definite matrix defined in Theorem 2. To simplify the development of the proof, we consider the case when $q = 1$ in equation (2). Generalization to the case when $q > 1$ is conceptually straightforward. Evaluating the time derivative of V along the trajectories of the closed-loop reduced system and noting that the term $L_{wk}L_f^{r-1}h(x)$ does not vanish at the origin, one can show that

$$\left.\begin{aligned}\dot{V} &\le -c_0(L_{\bar{g}}V)^2 - \alpha\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} \quad \forall |2b^{\mathrm{T}}Pe| \ge \frac{\phi}{\chi - 1}\\ \dot{V} &\le -c_0(L_{\bar{g}}V)^2 - \alpha\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}\\ &\quad + \theta_b|L_w L_f^{r-1}h(x)|\phi \quad \forall |2b^{\mathrm{T}}Pe| < \frac{\phi}{(\chi - 1)}\end{aligned}\right\} \tag{91}$$

A direct application then of the result of Corollary 5.2 in Khalil (1996) can be performed to conclude that the following ISS inequality holds for the e states of the reduced system

$$|e(t)| \le \bar{\beta}_e(|e(0)|, t) + \bar{\gamma}_e(\phi) \qquad \forall t \ge 0 \tag{92}$$

where $\bar{\beta}_e$ is the class KL function and $\bar{\gamma}_e$ is a class K_∞ function. From Assumption 2, we have that the η states of the reduced system possess an ISS property with respect to e

$$|\eta(t)| \le \bar{\beta}_\eta(|\eta(0)|, t) + \bar{\gamma}_\eta(\|e\|) \tag{93}$$

where $\bar{\gamma}_\eta$ is a class K function. Realizing that the reduced system analysed above is identical to the system studied in Theorem 2 in El-Farra and Christofides (1999), we note that that the static component of the controller of equation (29) with $\hat{x} = x$, i.e. $\mathcal{A}(x, \bar{v}, \phi)$ enforces global ultimate boundedness in the closed-loop slow system, and thus, the state ζ of the closed-loop slow system satisfies the bound

$$|\zeta(t)| \le \beta_\zeta(\delta_\zeta, t) + \tilde{\gamma}(\phi) \quad \forall t \ge 0 \tag{94}$$

where β_ζ is a class KL function, $\tilde{\gamma}$ is a class K_∞ function and δ_ζ is the maximum value of the vector $[h(x)\ \ L_f h(x)\ \ \cdots\ \ L_f^{r-1}h(x)]$ for $|x| \le \delta_x$.

Based on the above bound and following the results of Khalil and Esfandiari (1993), and Teel and Praly (1995), we disregard estimates of $\tilde{y}$, obtained from the high-gain observer, with norm $|\tilde{y}| > \beta_\zeta(\delta_\zeta, 0) + d$ where $d > \tilde{\gamma}(\phi)$. Hence, we set $\mathrm{sat}(\cdot) = \{1, \zeta_{\max}/|\cdot|\}(\cdot)$ where $\zeta_{\max}$ is the maximum value of the vector $[\zeta_1\ \ \zeta_2\ \ \cdots \zeta_r]$ for $|\zeta| \le \beta_\zeta(\delta_\zeta, 0) + d$.

Part 2: We are now in a position to apply the result of Lemma 1 in Christofides (2000). To this end, we introduce the same set of positive real numbers defined in Step 2 of Part 2 of the proof of Theorem 1 and note that now we have

$$\left.\begin{aligned}\delta_e > \tilde{\delta}_e + \bar{\gamma}_e(\phi), \quad \delta_\eta > D^\eta + \bar{\gamma}_\eta(\delta_e)\\ D^\eta := \bar{\beta}_\eta(\bar{\delta}_\eta, 0) + d_\eta, \quad d_e = \phi < d\end{aligned}\right\} \tag{95}$$

First, consider the singularly perturbed system comprised of the states (ζ, e_o) of the closed-loop system. This system is in standard form, possesses a globally exponentially stable fast subsystem, and its corresponding reduced system is ISS. These properties allow a direct application of the result of Lemma 1 with $\delta = \max\{\delta_\zeta, \delta_{e_o}, \delta_\theta, \delta_{\bar{v}}, \delta_\eta, \delta_e\}$ and $d = \phi$, to obtain the existence of a positive real number ϵ^ζ such that if $\epsilon \in [0, \epsilon^\zeta]$, and $|\zeta(0)| \le \delta_\zeta$, $|e_o(0)| \le \delta_{e_o}$, $\|\theta\| \le \delta_\theta$, $\|\bar{v}\| \le \delta_{\bar{v}}$, $\|e\| \le \delta_e$, $\|\eta\| \le \delta_\eta$, then

$$|\zeta(t)| \le \beta_\zeta(\delta_\zeta, t) + \tilde{\gamma}(\phi) + \phi \tag{96}$$

Let $\bar{\phi}$ be such that $\tilde{\gamma}(\phi) + \bar{\phi} \le d$. Consider now the singularly perturbed system comprised of the states (e, e_o) of the system of equation (51). This system is in standard form, possesses a globally exponentially stable fast subsystem, and its corresponding reduced system is ISS. These properties allow a direct application of the results of Lemma 1 with $\delta = \max\{\bar{\delta}_e, \delta_{e_o}, \delta_\theta, \delta_{\bar{v}}, \delta_\eta\}$ and $d = d_e = \phi$, to obtain the existence of a positive real number ϵ^e such that if $\epsilon \in (0, \epsilon^e]$, and $|e(0)| \le \bar{\delta}_e$, $|e_o(0)| \le \delta_{e_o}$, $\|\theta\| \le \delta_\theta$, $\|\bar{v}\| \le \delta_{\bar{v}}$, $\|\eta\| \le \delta_\eta$, $\|e\| \le \|\delta_e$, then

$$|e(t)| \le \bar{\beta}_e(|e(0), t) + \bar{\gamma}_e(\phi) + \phi \tag{97}$$

Similarly, it can be shown that Lemma 1, with $\delta = \max\{\bar{\delta}_e, \delta_{e_o}, \delta_\theta, \delta_{\bar{v}}, \delta_e\}$ and $d = d_{\bar{\eta}}$, can be applied to the system comprised of the states (η, e_o) of the system of equation (51).

Thus, we have that there exist positive real numbers ϵ^η such that if $\epsilon \in (0, \epsilon_\eta]$ and $|\eta(0)| \le \bar{\delta}_\eta$, $|e_o(0)| \le \delta_{e_o}$, $\|\theta\| \le \delta_\theta$, $\|\bar{v}\| \le \delta_{\bar{v}}$, $\|e\| \le \delta_e$, then

$$|\eta(t)| \le K_\eta|\eta(0)|\,\mathrm{e}^{-at} + \gamma_\eta(\|e\|) + d_\eta \tag{98}$$

Finally, the inequalities of equations (92), (93), (97) and (98) can be manipulated using a small-gain theorem type argument similar to the one used in the proof of Theorem 1 in El-Farra and Christofides (1999) to show that given the set positive real numbers $\bar{\delta}_e, \bar{\delta}_\eta, \delta_{e_o}, \delta_\theta, \delta_{\bar{v}}, d, d_e, d_\eta$ and with $\phi^* = \min\{\bar{\phi}, d/(1 + \bar{\gamma}_e(\phi))\}$ and with $\phi \in (0, \phi^*]$, there exists $\epsilon^* \in (0, \min\{\epsilon^\zeta, \epsilon^e, \epsilon^\eta\}]$ such that if $\epsilon \in (0, \epsilon^*]$, and $|e(0)| \le \bar{\delta}_e$, $|\eta(0)| \le \bar{\delta}_\eta$, $|e_o(0)| \le \delta_{e_o}$, $|\theta\| \le \delta_\theta$, $\|\bar{v}\| \le \delta_{\bar{v}}$, then $|\zeta(t)| \le \beta_\zeta(\delta_\zeta, 0) + d$ and the remaining states of the closed-loop system are bounded, and its output satisfies the relation of equation (30). For brevity, this argument will not be repeated here.

Part 3: In this part of the proof, we show that the output feedback controller of equation (29) is near-optimal over a finite time-interval for the class of systems considered in equation (2) and produces a finite cost which tends to the optimal cost for the reduced (state feedback) problem as $\epsilon \to 0$. To this end, we adopt a similar three-step procedure to the one employed in Part 3 of the proof of Theorem 1. In the first step, we use Tikhonov's theorem to establish closeness of the solutions of the reduced and full closed-loop systems on the finite time-interval. In Step 2, we show that, for the reduced system, the controller of equation (29) with $\epsilon = 0$ is optimal with respect to a cost functional $\bar{J}_r$ of the form of equation (31). Finally, in Step 3, we combine the results of Steps 1 and 2 to show that the cost associated with the full closed-loop systems tends to the optimal cost for the reduced system as $\epsilon \to 0$.

Step 1: Referring to the system of equation (51), it is straightforward to show that this system satisfies the conditions of Theorem 9.1 given in Khalil (1996). Applying this theorem to the system of equation (51), we conclude that there exists a finite time $t_1 > 0$ and positive constants K, L and $\bar{\bar{\epsilon}}$ such that if $\epsilon \in (0, \bar{\bar{\epsilon}}]$ and $|e(0)| \le \bar{\delta}_e$, $|\eta(0)| \le \bar{\delta}_\eta$, $|e_o(0)| \le \delta_{e_o}$, $\|\theta\| \le \delta_\theta$, $\|\bar{v}\| \le \delta_{\bar{v}}$, then

$$|z(t,\epsilon) - \bar{z}(t)| \le \epsilon K[1 + t_1] \exp(Lt_1) \quad (99)$$

and the estimate

$$z(t,\epsilon) - \bar{z}(t) = O(\epsilon) \quad (100)$$

holds uniformly for $t \in [0, t_1]$ where

$$\bar{z}(t) = [\bar{e}(t)\ \bar{\eta}(t)\ \bar{\omega}(t)]^{\mathrm{T}}$$

is the solution of the reduced (slow) problem. Furthermore, given any $t_b > 0$, there is $\epsilon_b \le \bar{\bar{\epsilon}}$ such that the estimate

$$e_o(t,\epsilon) = O(\epsilon) \quad (101)$$

holds uniformly for $t \in [t_b, t_1]$ whenever $\epsilon < \bar{\bar{\epsilon}}$.

Step 2: In this step we show that the controller of equation (29) with $\epsilon = 0$ and $\hat{x} = x$ minimizes the cost functional associated with the reduced system. To this end, consider the closed-loop system of equation (51) with $\epsilon = 0$ where u of equation (29) is applied with $\hat{x} = x$. We note that the resulting slow system is identical to the one studied in El-Farra and Christofides (1999) under state feedback and satisfies the assumptions of Theorem 2 stated therein. Applying the result of this theorem, we obtain that the static component of the controller of equation (29) minimizes the cost functional

$$\bar{J}_r = \lim_{t\to T_f} V(\bar{e}(t)) + \int_0^{T_f} (l(\bar{e}) + u(\bar{x})R(\bar{x})u(\bar{x}))\,\mathrm{d}t \quad (102)$$

where, as before, $\bar{x} = \mathcal{X}^{-1}(\bar{e}, \bar{\eta}, \bar{v})$ refers to the solution of the reduced closed-loop system under state feedback and T_f is defined in equation (34). Following the same treatment presented in Part 3 of the proof of Theorem 2 in El-Farra and Christofides (1999), it can be easily shown that upon substituton of the static component u of equation (29) (which we shall denote by u^* to emphasize that it is optimal) in the above expression, the minimum cost obtained is

$$\bar{J}_r^* = \lim_{t\to T_f} V(e(t)) + \int_0^{T_f} (l(\bar{e}) + u^*(\bar{x})R(\bar{x})u^*(x))\,\mathrm{d}t = V(\bar{e}(0)) \quad (103)$$

Step 3: In this step, we exploit the closeness of solutions result obtained in Step 1 and combine it with the optimality result established in Step 2 to prove that the output feedback controller of Equation (29) is near-optimal in the sense that the cost functional of equation (31) associated with the full closed-loop system approaches the optimal cost for the reduced (slow) system under state feedback, provided that the gain of the observer is sufficiently large. Note first that in order to guarantee near-optimality, we must require $t_1 \ge T_f$. This requirement guarantees that the solutions of the reduced and full closed-loop systems remain close during the time interval over which the state feedback controller u of equation (29) (with $\epsilon = 0$ and $\hat{x} = x$) is optimal. From equation (99), it is clear that t_1 can be made arbitrarily large while maintaining the estimate of equation (100) by selecting ϵ sufficiently small (or equivalently the gain of the observer sufficiently large). Therefore there exists a positive constant ϵ^{**} such that if $\epsilon \in (0, \epsilon^{**}]$, we have $t_1 \ge T_f$. Therefore, we set ϵ^* in Theorem 2 as

$$\epsilon^* = \min\{\epsilon_0, \epsilon^\zeta, \epsilon^\eta, \epsilon^e, \bar{\bar{\epsilon}}, \epsilon^{**}\}$$

To proceed with the proof of near-optimality, consider rewriting the cost functional of equation (31) as follows

$$J = \lim_{t\to T_f} V(e(t)) + \int_0^{t_b} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t + \int_{t_b}^{T_f} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t \quad (104)$$

From Step 2, we have that the following estimates hold for $t \in [t_b, t_1]$

$$\left.\begin{aligned} e(t) &= \bar{e}(t) + O(\epsilon) \\ x(t) &= \bar{x}(t) + O(\epsilon) \\ \hat{x}(t) &= \hat{x}(t) + O(\epsilon) \end{aligned}\right\} \quad (105)$$

It follows then from the continuity properties of the functions $V(\cdot)$, $l(\cdot)$, $u(\cdot)$, $R(\cdot)$ that for $t \in [t_b, t_1]$

$$V(e) \to V(\bar{e}), \quad l(e) \to l(\bar{e}), \quad u(\hat{x}) \to u(\bar{x}), \quad R(x) \to R(\bar{x}) \qquad \text{as } \epsilon \to 0 \quad (106)$$

and hence

$$\lim_{t\to T_f} V(e(t)) + \int_{t_b}^{T_f} (l(e) + u(\hat{x})R(x)u(\hat{x}))\,\mathrm{d}t \to \lim_{t\to T_f} V(\bar{e}(t)) + \int_{t_b}^{T_f} (l(\bar{e}) + u(\bar{x})R(\bar{x})u(\bar{x}))\,\mathrm{d}t \quad \text{as } \epsilon \to 0 \quad (107)$$

From the boundedness of the trajectories of the closed-loop system established in Part 2, there exists a positive real number M that bounds the integrand of equation (31). Using the fact that $t_b = O(\epsilon)$, we have that

$$\int_0^{t_b} (l(e) + u(\hat{x})R(x)u(\hat{x}))\, \mathrm{d}t \le \int_0^{t_b} M\, \mathrm{d}t \le M\epsilon = O(\epsilon) \qquad (108)$$

Similarly, from the stability of the reduced closed-loop system under state feedback established in Part 2 and the fact that $t_b = O(\epsilon)$, there exists a positive real number M' such that

$$\int_0^{t_b} (l(\bar{e}) + u(\bar{x})R(\bar{x})u(\bar{x}))\, \mathrm{d}t \le \int_0^{t_b} M'\, \mathrm{d}t \le M'\epsilon = O(\epsilon) \qquad (109)$$

Combining equations (107)–(109), we obtain

$$\lim_{t \to T_f} V(e(t)) + \int_0^{T_f} (l(e) + u(\hat{x})R(x)u(\hat{x}))\, \mathrm{d}t \to \lim_{t \to T_f} V(\bar{e}(t)) + \int_0^{T_f} (l(\bar{e}) + u(\bar{x})R(\bar{x})u(\bar{x}))\, \mathrm{d}t \quad \text{as } \epsilon \to 0 \qquad (110)$$

and from equation (103), we finally have

$$J = \lim_{t \to T_f} V(e(t)) + \int_0^{T_f} (l(e) + u(\hat{x})R(x)u(\hat{x}))\, \mathrm{d}t \to V(\bar{e}(0)) \quad \text{as } \epsilon \to 0 \qquad (111)$$

This completes the proof of the theorem. □

References

BASAR, T., and BERNHARD, P., 1990, *H^∞-Optimal Control and Related Minimax Design Problems* (Boston: Birkhäuser).

CHOW, J. H., and KOKOTOVIC, P. V., 1978, Near-optimal feedback stabilization of a class of nonlinear singularly perturbed systems. *SIAM Journal of Control Optimization*, **16**, 756–770.

CHRISTOFIDES, P. D., 2000, Robust output feedback control of nonlinear singularly perturbed systems. *Automatica*, **36**, 45–52.

CHRISTOFIDES, P. D., and DAOUTIDIS, P., 1996, Feedback of two-time-scale nonlinear systems. *International Journal of Control*, **63**, 965–994.

CHRISTOFIDES, P. D., TEEL, A. R., and DAOUTIDIS, P., 1996, Robust semi-global output tracking for nonlinear singularly perturbed systems. *International Journal of Control*, **65**, 639–666.

CORLESS, M., 1993, Control of uncertain nonlinear systems. *J. Dyn. Sys., Meas. & Contr.*, **115**, 362–372.

EL-FARRA, N. H., and CHRISTOFIDES, P. D., 1999, Robust optimal control of nonlinear systems. *Systems and Control Letters*, submitted. A version of this paper was presented at 1999 American Control Conference, pp. 124–128, San Diego, CA.

FREEMAN, R. A., and KOKOTOVIC, P. V., 1996, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques* (Boston: Birkhäuser).

FRIDMAN, E., 1996, Near-optimal H^∞ control of linear singularly perturbed systems. *IEEE Transactions on Automatic Control*, **41**, 236–240.

FRIDMAN, E., 1998, Exact slow-fast decomposition of the Hamilton–Jacobi equation of singularly perturbed systems. *Proceedings of American Control Conference*, Philadelphia, PA, pp. 1503–1507.

ISIDORI, A., 1994, A necessary condition for nonlinear H_∞ control via measurement feedback. *Systems and Control Letters*, **23**, 169–177.

ISIDORI, A., 1999, Stabilization of nonlinear systems using output feedback. In G. Picci and D. S. Gilliam (Eds) *Dynamical Systems, Control, Coding, Computer Vision: New Trends, Interfaces, and Interplay*, Vol. 25 (Boston: Birkhäuser), pp. 111–133.

ISIDORI, A., and ASTOLFI, A., 1992, Disturbance attenuation and H_∞ control via measurement feedback in a nonlinear system. *IEEE Transactions on Automatic Control*, **37**, 1283–1293.

ISIDORI, A., and KANG, W., 1995, H_∞ control via measurement feedback for general nonlinear systems. *IEEE Transactions on Automatic Control*, **40**, 466–472.

JIANG, Z. P., and MAREELS, I., 1997, A small gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Transactions on Automatic Control*, **42**, 292–308.

JIANG, Z. P., TEEL, A. R., and PRALY, L., 1995, Small-gain theorem for ISS systems and applications. *Mathematics of Control Signals Systems*, **7**, 95–120.

KALMAN, R. E., 1964, When is a control system optimal? *Journal of Basic Engineering*, **86**, 51–60.

KANELLAKOPOULOS, I., KOKOTOVIC, P. V., and MARINO, R., 1991, An extended direct scheme for robust adaptive nonlinear control. *Automatica*, **27**, 247–255.

KHALIL, H., 1994, Robust servomechanism output feedback controller for feedback linearizable systems. *Automatica*, **30**, 1587–1599.

KHALIL, H. K., 1996, *Nonlinear Systems* (New York: Macmillan Publishing Company).

KHALIL, H. K., and ESFANDIARI, F., 1993, Semiglobal stabilization of a class of nonlinear systems using feedback. *IEEE Transactions on Automatic Control*, **38**, 1412–1415.

KRSTIC, M., KANELLAKOPOULOS, I., and KOKOTOVIC, P., 1995, *Nonlinear and Adaptive Control Design* (New York: Wiley).

KRSTIC, M., and LI, Z. H., 1997, Inverse optimal design of input-to-sate stabilizing nonlinear controllers. *IEEE Transactions on Automatic Control*, **43**, 336–350.

LEITMANN, G., 1993, On one approach to the control of uncertain systems. *Journal of Dynamic Systems, Measurement and Control*, **115**, 373–380.

LIN, Y., SONTAG, E. D., and WANG, Y., 1995, Input to state stabilizability of parametrized family of systems. *International Journal of Robust and Nonlinear Control*, **5**, 187–205.

MAHMOUD, N. A., and KHALIL, H. K., 1996, Asymptotic regulation of minimum phase nonlinear systems using output feedback. *IEEE Transactions on Automatic Control*, **41**, 1402–1412.

MARINO, R., and TOMEI, P., 1993a, Global adaptive output feedback control of nonlinear systems, part 2: nonlinear parameterization. *IEEE Transactions on Automatic Control*, **38**, 33-48.

MARINO, R., and TOMEI, P., 1994b, Global adaptive output feedback control of nonlinear systems, part 1: Linear parameterization. *IEEE Transactions on Automatic Control*, **38**, 17–32.

MARINO, R., and TOMEI, P., 1993 c, Robust stabilization of feedback linearizable time-varying uncertain nonlinear systems. *Automatica*, **29**, 181–189.

MOYLAN, P. J., and ANDERSON, B. D. O., 1973, Nonlinear regulator and an inverse optimal control problem. *IEEE Transactions on Automatic Control*, **18**, 460–465.

PAN, Z., and BASAR, T., 1993, Robustness of H^{∞} controllers to nonlinear perturbations. *Proceedings of 32nd IEEE Conference on Decision and Control*, San Antonio, TX, pp. 1638–1643.

QU, Z., 1993, Robust control of nonlinear uncertain systems under generalized matching conditions. *Automatica*, **29**, 985–998.

SASTRY, S. S., and ISIDORI, A., 1989, Adaptive control of linearizable systems. *IEEE Transactions on Automatic Control*, **34**, 1123–1131.

SEPULCHRE, R., JANKOVIC, M., and KOKOTOVIC, P., 1997, *Constructive Nonlinear Control* (Berlin–Heidelberg: Springer-Verlag).

SONTAG, E. D., 1989 a, Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, **34**, 435–443.

SONTAG, E. D., 1989 b, A 'universal' construction of Artstein's theorem on nonlinear stabilization. *Systems and Control Letters*, **13**, 117–123.

TEEL, A., KADIYALA, R., KOKOTOVIC, P. V., and SASTRY, S. S. 1991, Indirect techniques for adaptive input–output linearization of nonlinear systems. *International Journal of Control*, **53**, 193–222.

TEEL, A., and PRALY, L., 1994, Global stabilizability and observability imply semi-global stabilizability by output feedback. *Systems and Control Letters*, **22**, 313–325.

TEEL, A., and PRALY, L., 1995, Tools for semi-global stabilization by partial state and output feedback. *SIAM Journal of Control and Optimization*, **33**, 1443–1488.

THAU, F. E., 1967, On the inverse optimal control problem for a class of nonlinear autonomous systems. *IEEE Transactions on Automatic Control*, **12**, 674–681.

VAN DER SCHAFT, A. J., 1992, L_2-gain analysis of nonlinar systems and nonlinear state feedback H_{∞} control. *IEEE Transactions on Automatic Control*, **37**, 770–783.