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Robust inverse optimal control laws for nonlinear systems

Nael H. El-Farra and Panagiotis D. Christofides*,†

Department of Chemical Engineering, University of California, Los Angeles, CA 90095-1592, U.S.A.

SUMMARY

This work proposes a robust inverse optimal controller design for a class of nonlinear systems with bounded, time-varying uncertain variables. The basic idea is that of re-shaping the scalar nonlinear gain of an L_gV controller, based on Sontag's formula, so as to guarantee certain uncertainty attenuation properties in the closed-loop system. The proposed gain re-shaping is shown to yield a control law that enforces global boundedness of the closed-loop trajectories, robust asymptotic output tracking with an arbitrary degree of attenuation of the effect of uncertainty on the output, and inverse optimality with respect to a meaningful cost that penalizes the tracking error and the control action. The performance of the control law is illustrated through a simulation example and compared with other controller designs.

KEY WORDS: model uncertainty; Lyapunov's direct method; Sontag's formula; inverse optimality

1. INTRODUCTION

All practical control systems have to be robust with respect to model uncertainty (e.g. unknown or partially known time-varying process parameters and exogenous disturbances) and utilize reasonable control action to achieve the desired performance and robustness specifications. Model uncertainty, if not taken into account in the controller design, may cause severe deterioration in the nominal closed-loop performance or even lead to closed-loop instability. On the other hand, unnecessarily large control action may also cause poor closed-loop behaviour owing to the presence of constraints on the capacity of control actuators.

The problem of designing controllers for nonlinear systems with uncertain variables, that enforce closed-loop stability and output tracking, has received significant attention in the past. For feedback linearizable nonlinear systems with constant uncertain variables, adaptive techniques have been employed to design controllers that enforce asymptotic stability and output tracking (e.g. see References [1, 2]). On the other hand, for feedback linearizable nonlinear systems with time-varying uncertain variables that satisfy the so-called matching condition, Lyapunov's direct method has been used to design robust state feedback controllers

*Correspondence to: Professor P.D. Christofides, Department of Chemical Engineering, University of California, Los Angeles, School of Engineering and Applied Science, Box 951592, Los Angeles, CA 90095-1592, U.S.A
†E-mail: pdc@seas.ucla.edu

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that enforce boundedness and arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output (e.g. see References [3, 4]). More recently, robust output feedback controllers have also been designed through combination of robust state feedback controllers with high-gain observers [5, 6]. Additional results on the robust stabilization of feedback linearizable nonlinear systems can be found in References [7, 8].

Even though the above works provide systematic methods for adaptive and robust controller design, they do not lead in general to controllers that are optimal with respect to a meaningful cost. An approach to address the design of robust optimal controllers is within the nonlinear H_∞ control framework (e.g. [9]). However, the explicit construction of controllers in this approach requires the analytic solution of the steady-state Hamilton–Jacobi–Isaacs (HJI) partial differential equation which is not a feasible task in general. An alternative approach which does not require solving the HJI equation is the inverse optimal control approach. Inverse optimal control problems have a long history in control theory (e.g. see References [10–15]). The central idea of the inverse optimal approach is to compute a robust stabilizing control law together with the appropriate penalties that render the cost functional well defined and meaningful in some sense. This approach provides a convenient route for robust optimal controller design and is well motivated by the fact that the closed-loop robustness achieved as a result of controller optimality is largely independent of the specific choice of the cost functional so long as this cost functional is meaningful [16]. The appealing features of the inverse optimal approach have motivated its use for the design of robust optimal controllers in References [17–19]. These controllers, however, do not necessarily lead to an arbitrary degree of attenuation of the effect of uncertainty on the closed-loop output. This feature is particularly desirable in the case of non-vanishing uncertainty that can change the nominal equilibrium point of the system.

In this work, we propose a robust inverse optimal controller design that enforces an arbitrary degree of attenuation of the effect of uncertainty on the output of the closed-loop system. The controller design is obtained by re-shaping the scalar nonlinear gain of a Lyapunov-based controller based on Sontag's formula, in a way that guarantees the desired uncertainty attenuation properties in the closed-loop system. The rest of the paper is organized as follows. In Section 2, we introduce the class of uncertain nonlinear systems considered and review some preliminaries on inverse optimal controller design. Then in Section 3 we formulate the robust inverse optimal control problem and present the proposed gain re-shaping, and the resulting control law. We discuss some of the advantages of the proposed design, relative to other designs that have been introduced in the literature. In contrast to previous works, the uncertain variables considered here do not obey state-dependent bounds and can change the nominal equilibrium point. Finally, in Section 4 we present a simulation example to illustrate the performance of the controller and compare it with other possible controller designs.

2. PRELIMINARIES

We consider single-input single-output, uncertain nonlinear systems with the following state–space description:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + \sum_{k=1}^{q} w_k(x)\theta_k(t) \\ y &= h(x) \end{aligned} \qquad (1)$$

where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}$ denotes the manipulated input, $\theta_k(t) \in \mathcal{W} \subset \mathbb{R}$ denotes the kth uncertain (possibly time varying) but bounded variable taking values in a non-empty compact convex subset $\mathcal{W}$ of $\mathbb{R}$, and $y \in \mathbb{R}$ denotes the output to be controlled. Without loss of generality, we assume that the origin is an equilibrium point of the nominal system ($u(t) = \theta_k(t) \equiv 0$) of Equation (1). The uncertain variable $\theta_k(t)$ may describe time-varying parametric uncertainty and/or exogenous disturbances. The vector functions $f(x)$, $w_k(x)$ and $g(x)$, and the scalar function $h(x)$ are assumed to be sufficiently smooth. Throughout the paper, the notation $\|\theta\|$ is used to denote ess.sup. $|\theta(t)|$, $t \geqslant 0$ where the function θ is measurable (with respect to the Lebesgue measure). For simplicity, we will suppress the time dependence in the notation of the uncertain variable $\theta_k(t)$.

Preparatory for its use as a tool for robust optimal controller design, we begin by reviewing the concept of inverse optimality in the context of robust stabilization (see also Reference [18]). To this end, consider the system of Equation (1) with $q = 1$ and $w(0) = 0$. Also, let $l(x)$ and $R(x)$ be two continuous scalar functions such that $l(x) \geqslant 0$ and $R(x) > 0 \ \forall \ x \in \mathbb{R}^n$ and consider the problem of finding a feedback control law $u(x)$ for the system of Equation (1) that achieves asymptotic stability of the origin and minimizes the infinite time cost functional

$$J = \int_0^\infty (l(x) + uR(x)u)\,\mathrm{d}t \tag{2}$$

The steady-state HJI equation associated with the system of Equation (1) and the cost of Equation (2) is

$$0 \equiv \inf_{u \in \mathbb{R}} \sup_{\theta \in \mathcal{W}} (l(x) + uR(x)u + L_f V + L_g V u + L_w V \theta) \tag{3}$$

where the value function V is the unknown. A smooth positive definite solution V to this equation will lead to a continuous state feedback $u(x)$ of the form

$$u = -p(x) = -\tfrac{1}{2} R^{-1}(x) L_g V \tag{4}$$

which provides stability, optimality and robustness with respect to the disturbance θ. However, such a smooth solution may not exist or may be extremely difficult to compute. Suppose, instead, that we were able to find a positive definite radially unbounded C^1 scalar function V such that

$$\inf_{u \in \mathbb{R}} \sup_{\theta \in \mathcal{W}} (L_f V + L_g V u + L_w V \theta) < 0 \quad \forall \ x \neq 0 \tag{5}$$

Now, if we can find a meaningful cost functional (i.e. $l(x) \geqslant 0$, $R(x) > 0$) such that the given V is the corresponding value function, then we will have indirectly obtained a solution to the HJI equation and can therefore compute the optimal control law of Equation (4). Such reasoning motivates following the inverse path. In the inverse approach, a stabilizing feedback control law is designed first and then shown to be optimal with respect to a well defined and meaningful cost functional of the form of Equation (2). The problem is inverse because the weights $l(x)$ and $R(x)$ in the cost functional are *a posteriori* computed from the chosen stabilizing feedback control law, rather than *a priori* specified by the designer.

A stabilizing control law $u(x)$ is said to be inverse optimal for the system of Equation (1) if it can be expressed in the form of Equation (4) where the negative definiteness of $\dot{V}$ is achieved with the control $u^* = -\frac{1}{2}p(x)$, that is

$$\sup_{\theta \in \mathscr{W}} \dot{V} = \sup_{\theta \in \mathscr{W}} (L_f V + L_g V u^* + L_w V \theta)$$
$$= L_f V - \tfrac{1}{2} L_g V p(x) + |L_w V|\theta_b < 0 \quad \forall\ x \neq 0 \tag{6}$$

where the worst-case uncertainty (i.e. one that maximizes $\dot{V}$) is given by $\theta = \mathrm{sgn}\,[L_w V(x)]\theta_b$ where $\theta_b = \|\theta\|$ (which is clearly an admissible uncertainty since it is both measurable and bounded). When the function $l(x)$ is set equal to $l(x) = -\sup_{\theta \in \mathscr{W}} \dot{V}$, then V is a solution to the following steady-state HJI equation:

$$0 \equiv l(x) + L_f V - \tfrac{1}{4} L_g V R^{-1}(x) L_g V + |L_w V|\theta_b \tag{7}$$

and the optimal (minimal) value of the cost J is $V(x(0))$. Finally, we recall the definition of input-to-state stability (ISS) for a system of the form of Equation (1).

Definition 1 (Sontag [20])
The system in Equation (1) (with $u \equiv 0$) is said to be ISS with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_0 \in \mathbb{R}^n$ and for each measurable, essentially bounded input $\theta(\cdot)$ on $[0, \infty)$ the solution of Equation (1) with $x(0) = x_0$ exists for each $t \geqslant 0$ and satisfies

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|\theta\|), \quad \forall\ t \geqslant 0 \tag{8}$$

3. ROBUST INVERSE OPTIMAL CONTROL OF NONLINEAR SYSTEMS

3.1. Control problem formulation

Referring to the uncertain nonlinear system of Equation (1), we consider the general case when the uncertain variables $w_k(x)\theta_k$ are non-vanishing, (i.e. $w_k(0)\theta \neq 0$). Under this assumption, the origin is no longer an equilibrium point for the uncertain system. Our objective is to synthesize a robust nonlinear state feedback controller of the general form

$$u = \mathscr{P}(x, \bar{v}) \tag{9}$$

where $\mathscr{P}(x, \bar{v})$ is a scalar function and $\bar{v} = [v\ v^{(1)}\ \cdots\ v^{(r)}]^{\mathrm{T}}$ is a generalized reference input ($v^{(k)}$ denotes the kth time derivative of the reference input v which is assumed to be a sufficiently smooth function of time), that guarantees global boundedness of the closed-loop trajectories, enforces the discrepancy between the output and the reference input to be asymptotically arbitrarily small, and minimizes a meaningful cost functional defined over a finite time-interval. To address this problem, the controller is designed using combination of geometric and Lyapunov techniques. In particular, the design is carried out by combining a Sontag-like control law with a robust control design proposed in Reference [4]. The analysis of the closed-loop system is performed by utilizing the concept of ISS and nonlinear small gain theorem-type arguments, and optimality is established using the inverse optimal control approach. Throughout the paper, we focus on the problem of unconstrained controller design. For results on constrained robust inverse optimal control of nonlinear systems, the reader is referred to Reference [21].

3.2. Controller synthesis

In order to proceed with the synthesis of the controller, we will impose the following assumptions on the system of Equation (1). The first assumption allows transforming the system into a partially linear form and is motivated by the requirement of output tracking.

Assumption 1

There exists an integer r and a set of co-ordinates:

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ T_1(x) \\ \vdots \\ T_{n-r}(x) \end{bmatrix} \tag{10}$$

where $T_1(x), \ldots, T_{n-r}(x)$ are scalar functions such that the system of Equation (1) takes the form

$$\begin{aligned} \dot{\zeta} &= A\zeta + b\left[L_f^r h(T^{-1}(\zeta,\eta)) + L_g L_f^{r-1} h(T^{-1}(\zeta,\eta))u + \sum_{k=1}^{q} L_{wk} L_f^{r-1} h(T^{-1}(\zeta,\eta))\theta_k\right] \\ \dot{\eta} &= \Psi(\zeta,\eta,\theta) \end{aligned} \tag{11}$$

where

$$A_{r\times r} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b_{r\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{12}$$

and $L_g L_f^{r-1} h(x) \neq 0$ for all $x \in \mathbb{R}^n$. Moreover, for each $\theta \in \mathcal{W}^q$, the states ζ, η are bounded if and only if x is bounded.

Assumption 1 is satisfied by SISO nonlinear systems that have a well-defined relative degree (i.e. input/output linearizable systems, see Reference [22] for further details) and, in addition, satisfy the matching condition (the relative order of the output with respect to the uncertainty, θ, is equal to the relative order with respect to the control input, u,) which ensures that the uncertain variables cannot have a stronger effect on the controlled output than the manipulated input. We note that the change of variables of Equation (10) is independent of θ (because the vector field $g(x)$ is independent of θ), and is invertible, since, for every x, the variables ζ, η are uniquely determined by Equation (10).

Following [4], the requirement of ISS of the η subsystem in Equation (11) is imposed to allow the synthesis of a robust state feedback controller that enforces the requested properties in the closed-loop system globally. Assumption 2 that follows states this requirement.

Assumption 2
The dynamical system $\dot{\eta} = \Psi(\zeta, \eta, \theta)$ is ISS with respect to ζ, θ.

In order to achieve attenuation of the effect of the uncertain variables on the output, we assume the existence of known bounds that capture the size of the uncertain variables for all times. In practice, information of this kind can be obtained from physical considerations, preliminary simulations, experimental data, etc.

Assumption 3
There exist known positive constants θ_{bk} such that $\|\theta_k(t)\| = \theta_{bk}$.

Before we state the main result of this section, we need to introduce the following notation: $e = [e_1 \; e_2 \; \cdots \; e_r]^{\mathrm{T}}$, $e_i = \zeta_i - v^{(i-1)}$ where $i = 1, \ldots, r$. With this notation, we define

$$\begin{aligned} \bar{f}(e, \eta, \bar{v}) &= Ae + b(L_f^r h(T^{-1}(e, \eta, \bar{v})) - v^{(r)}) \\ \bar{g}(e, \eta, \bar{v}) &= bL_g L_f^{r-1} h(T^{-1}(e, \eta, \bar{v})), \;\; \bar{w}_k(e, \eta, \bar{v}) = bL_{wk} L_f^{r-1} h(T^{-1}(e, \eta, \bar{v})) \end{aligned} \tag{13}$$

Theorem 1 below provides a formula of the robust state feedback controller and states precise conditions under which the proposed controller enforces the desired properties in the closed-loop system. The proof of this theorem is given in Appendix A.

Theorem 1
Consider the uncertain nonlinear system of Equation (1), for which Assumptions 1, 2 and 3 hold, under the state feedback law:

$$\begin{aligned} u =& -[c_0 + r_s(x) + r_d(x)] L_{\bar{g}} V \\ =& -\left(c_0 + \frac{L_{\bar{f}} V + \sqrt{(L_{\bar{f}} V)^2 + (L_{\bar{g}} V)^4}}{(L_{\bar{g}} V)^2} + \frac{(\rho + \chi \sum_{k=1}^{q} \theta_{bk} |L_{wk} L_f^{r-1} h(x)|)}{(L_g L_f^{r-1} h(x))^2 \left(\dfrac{|L_{\bar{g}} V|}{|L_g L_f^{r-1} h(x)|} + \phi \right)} \right) L_{\bar{g}} V \end{aligned} \tag{14}$$

where $V = e^{\mathrm{T}} P e$, P is a positive definite matrix that satisfies $A^{\mathrm{T}} P + PA - Pbb^{\mathrm{T}} P < 0$, and c_0, ρ, χ, and ϕ are adjustable parameters that satisfy $c_0 > 0$, $\rho > 0$, $\chi > 2$, and $\phi > 0$. Furthermore, assume that the uncertain variables are non-vanishing in the sense that there exists positive real numbers δ_k, μ_k such that $|\bar{w}_k(e, \eta)| \leqslant \delta_k |2b^{\mathrm{T}} Pe| + \mu_k \;\; \forall \; e \in D$, where $D = \{e \in \mathbb{R}^r : |2b^{\mathrm{T}} Pe| \leqslant \phi(\frac{1}{2} \chi - 1)^{-1}\}$. Then for any initial condition, and for every positive real number d, there exists $\phi^*(d) > 0$ such that if $\phi \in (0, \phi^*(d)]$, the following holds:

(1) The trajectories of the closed-loop system are bounded.

(2) The output of the closed-loop system satisfies a relation of the form

$$\limsup_{t\to\infty} |y(t) - v(t)| \leqslant d \tag{15}$$

(3) The control law of Equation (14) minimizes the cost functional

$$J = V(e(T_f)) + \int_0^{T_f} (l(e(t)) + u(t)R(x(t))u(t))\,\mathrm{d}t \tag{16}$$

where $R(x) = \frac{1}{2}[c_0 + r_s(x) + r_d(x)]^{-1} > 0$ for all $x \in \mathbb{R}^n$ and $l(e) = -L_{\bar{f}}V + \frac{1}{4}L_{\bar{g}}VR^{-1}(x)L_{\bar{g}}V - \sum_{k=1}^{q} |L_{wk}V|\theta_{bk} > 0 \ \forall \ t \in [0, T_f]$, where $T_f = \inf\{T \geqslant 0 : e(t) \in \Gamma \ \forall \ t \geqslant T\}$, $\Gamma = \{e \in \mathbb{R}^r : e^{\mathrm{T}}Pe \leqslant \lambda_{\max}(P)\varepsilon^2\}$, $\varepsilon = \max\{\alpha_1^{-1}(2\bar{\phi}^*), \alpha_2^{-1}(2\bar{\phi}^*)\}$, where $\bar{\phi}^* = \mu\theta_b\phi^*$, and $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions that satisfy, respectively, $\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} \geqslant \alpha_1(|e|)$ and $\frac{1}{2}[-L_{\bar{f}}V + \sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}] \geqslant \alpha_2(|e|)$.

Remark 1

Referring to the controller of Equation (14), one can easily observe that it is comprised of two components. The first component, $u_s = -[c_0 + r_s(x)]L_{\bar{g}}V$, is responsible for achieving stabilization and reference-input tracking in the nominal (i.e. $\theta(t) \equiv 0$) closed-loop system. This component, with $c_0 = 0$, is the so-called Sontag's formula proposed in Reference [23] and is continuous everywhere, since V satisfies the small control property. The second component of the controller, $u_d = -r_d(x)L_{\bar{g}}V$, enforces, on the other hand, output tracking with an arbitrary degree of asymptotic attenuation of the effect of θ on y. This component is also continuous everywhere since $L_gL_f^{r-1}h(x) \neq 0$ for all x (from Assumption 1) and $\phi > 0$. A key feature of the uncertainty compensator, $r_d(x)$, is the presence of a scaling term of the form $|x|/(|x| + \phi)$ which allows us, not only to tune the degree of uncertainty attenuation (by adjusting ϕ), but also to use smaller and smaller control effort to cancel the uncertainties as we get closer and closer to the equilibrium point. The full weight (gain) of the compensator is used only when the state is far from the equilibrium point (since ϕ is small).

Remark 2

Owing to the persistent nature of the uncertainty ($\bar{w}_k(0, \eta)\theta \neq 0$) and the fact that asymptotic convergence to the equilibrium point of the nominal system is not possible, J cannot achieve a finite value over the infinite time interval. Therefore, the cost functional of Equation (16) is defined only over a finite time interval $[0, T_f]$. The size of this interval is determined by the time required for the tracking error trajectory to reach and enter a terminal set, Γ, centred around the origin, without ever leaving again. In the proof of Theorem 1, we show that $\dot{V} < 0$ on and outside the boundary of Γ. This, together with the fact that the boundary of Γ is a level set of V, implies that, once inside Γ, the closed-loop trajectory cannot escape. The size of Γ scales with ϕ and, depending on the desired degree of attenuation of the effect of the uncertainty on the output, can be made arbitrarily small by adjusting the controller tuning parameter ϕ. Therefore, the discrepancy between the closed-loop output and the reference input can be made arbitrarily small. In addition to the terminal time T_f, the cost functional of Equation (16) imposes a terminal penalty given by the term $V(e(T_f))$ to penalize the tracking error trajectory for its inability to converge to the origin for $t > T_f$.

Remark 3

Regarding the properties of the cost functional of Equation (16), we observe that J is well defined according to the formulation of the optimal control problem for nonlinear systems [16]. In particular, since the error trajectory of the closed-loop system converges to the terminal set Γ as time tends to T_f, the cost functional of Equation (16) is meaningful in the sense that the weights $l(e)$ and $R(x)$ satisfy the following properties for all $t \in [0, T_f]$. The function $l(e)$ is continuous, positive definite, and bounded from below by a class $\mathcal{K}$ function of the norm of e (see proof of Theorem 1). The function $R(x)$ is continuous, strictly positive and the term $uR(x)u$ has a unique global minimum at $u = 0$ implying a larger control penalty for u farther away from zero.

Remark 4

The linear growth bound on the functions $\bar{w}_k(e, \eta)$ is imposed to ensure that the results of Theorem 1 hold globally. Note, however, that this condition is required to hold only over the set D (containing the origin) whose size can be made arbitrarily small by appropriate selection of the tuning parameters ϕ and χ. This bound is consistent with the non-vanishing nature of the uncertain variables considered, since it implies that the functions $\bar{w}_k(e, \eta)$ do not vanish when $e = 0$.

Remark 5

The re-shaping of the scalar gain of the nominal $L_g V$ controller, $u_s = -r_s(x) L_{\bar{g}} V$, via the addition of the uncertainty compensator, $r_d(x)$, allows us to achieve attenuation of the effect of θ on y without using unreasonably large (high-gain) control action. The use of a high-gain controller of the form $u = (1/\varepsilon)u_s$ where ε is a small positive parameter could also lead to uncertainty attenuation at the expense of employing unnecessarily large control action (see the example in Section 4 for an illustration of this fact).

Remark 6

It is important to compare the controller design proposed in Theorem 1 with the following robust nonlinear controller proposed in Reference [4]:

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \sum_{i=0}^{r} \frac{\beta_i}{\beta_r} (v^{(i)} - L_f^i h(x)) + u_d \tag{17}$$

where $u_d = -r_d(x) L_{\bar{g}} V$, which is continuous (since $L_g L_f^{r-1} h(x) \neq 0$ and $\phi > 0$) and guarantees boundedness of the trajectories of the closed-loop system and output tracking with arbitrary degree of asymptotic attenuation of the effect of θ on y. The controller of Equation (17) consists of a term that cancels the nonlinearities of the system of Equation (11), with $\theta(t) \equiv 0$, that can be cancelled by feedback, and a term that compensates for the effect of θ on y. While the control law of Equation (17) can be shown to be optimal with respect to some performance index, it remains to be seen whether the index is meaningful or not. In particular, we note that the controller of Equation (17) may generate unnecessarily large control action to cancel beneficial nonlinearities and, therefore, may not in general be optimal with respect to a meaningful performance index.

Remark 7
An alternative to the gain-reshaping procedure underlying the controller design in Theorem 1 is the one proposed in Reference [19] which involves incorporating the uncertainty compensator within (rather than adding it to) the 'nominal' gain of Sontag's formula. The resulting L_gV controller in this case has a scalar gain that is structurally similar to that of Sontag's formula, except that the term L_fV appears with the uncertainty compensator added to it. In particular, when the compensator in Theorem 1 is used, this gain-reshaping approach yields the following control law:

$$u = -\left(\frac{L_{\bar{f}}^{*}V + \sqrt{(L_{\bar{f}}^{*}V)^2 + (L_{\bar{g}}V)^4}}{(L_{\bar{g}}V)^2}\right) L_{\bar{g}}V \tag{18}$$

where

$$L_{\bar{f}}^{*}V = L_{\bar{f}}V + \chi \sum_{k=1}^{q} \theta_{bk} |L_{\bar{w}k}V| \left(\frac{|2b^{\mathrm{T}}Pe|}{|2b^{\mathrm{T}}Pe| + \phi}\right)$$

Using calculations similar to those in Appendix A, one can show that the above control law is also globally robustly stabilizing and inverse optimal with respect to a meaningful cost. However, a potential drawback of this design is the fact that it can limit the ability of the controller to fully 'cancel out' the uncertainty, especially when $L_{\bar{f}}^{*}V$ is negative. This, in turn, can lead either to a level of asymptotic uncertainty attenuation smaller than that achieved by the controller of Equation (14), or to larger control action in order to enforce the same attenuation level. To illustrate this point, consider the following system:

$$\dot{x} = u + \theta \tag{19}$$

Since the system is scalar, we take $V = \frac{1}{2}x^2$ and get $L_gV = L_wV = x$ and . For this choice, the control law based on Equation (14) is

$$u = -\left(1 + \frac{\chi\theta_b}{|x| + \phi}\right)x \tag{20}$$

while the control law based on Equation (18) is

$$u = -\left(\frac{\chi\theta_b}{|x| + \phi} + \sqrt{1 + \left(\frac{\chi\theta_b}{|x| + \phi}\right)^2}\right)x \tag{21}$$

It is straightforward to verify that, for the same choice of tuning parameters ϕ and χ, both controllers enforce the same level of asymptotic uncertainty attenuation, with an ultimate bound on the state of $|x| \leqslant d$ where $d = \phi(\chi - 1)^{-1}$. However, it is also clear that the controller gain in Equation (21) is larger, pointwise, than that of the control law in Equation (20). The discrepancy between the two gains reflects the cost of incorporating the uncertainty compensator within a Sontag-type gain, as done in Equation (21), as opposed to adding it to the nominal gain of Sontag's formula, as done in Equation (20). Note that to reduce the controller gain in Equation (21) requires an increase in ϕ and/or reduction in χ which, in either case, implies a lesser degree of uncertainty attenuation (larger residual set).

Remark 8
For the special case when the uncertain variables are vanishing (i.e. $\bar{w}_k(0,\eta)\theta = 0$), the origin is an equilibrium point for the uncertain system. In this case, one can show, using calculations similar to those in Appendix A, that the control law of Equation (14) globally asymptotically stabilizes the origin and forces the output tracking error to converge, asymptotically, to zero ($d = 0$). Furthermore, inverse optimality can be established over the infinite time interval. In particular, one can show that the control law of Equation (14) minimizes a cost functional of the form of Equation (16) with $T_f = \infty$ and zero terminal penalty.

4. ILLUSTRATIVE EXAMPLE

In order to illustrate the performance of the developed control law, we consider a continuous stirred tank reactor where an irreversible first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The inlet stream consists of pure A at flow rate $F = 100$ L/min, concentration $C_{A0} = 1.0$ mol/L and temperature $T_{A0} = 290$ K. Under standard modelling assumptions, the mathematical model for the process takes the form

$$\begin{aligned}\frac{\mathrm{d}C_A}{\mathrm{d}t} &= \frac{F}{V}(C_{A0} - C_A) - k_0 \exp\left(\frac{-E}{RT}\right) C_A \\ \frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{F}{V}(T_{A0} - T) + \frac{(-\Delta H)}{\rho_m c_p} k_0 \exp\left(\frac{-E}{RT}\right) C_A + \frac{Q}{\rho_m c_p V}\end{aligned} \tag{22}$$

where C_A denotes the reactant concentration, T denotes the reactor temperature, Q denotes the rate of heat input to the reactor, $V = 100$ L is the volume of the reactor, $k_0 = 7.39\ \mathrm{min}^{-1}$, $E = 27.4$ KJ/mol, $\Delta H = 718.31$ KJ/mol are the pre-exponential constant, the activation energy and the enthalpy of the reaction, $c_p = 0.239$ J/g K and $\rho_m = 1000$ g/L, are the heat capacity and fluid density in the reactor. For these values of process parameters, the process (with $Q = 0$) has an unstable equilibrium point at $(C_{As}, T_s) = (0.731\ \mathrm{mol/L}, 1098.5\ \mathrm{K})$. The control objective is the regulation of the reactant concentration at the (open-loop) unstable equilibrium point, in the presence of time varying, persistent disturbances in the feed temperature and uncertainty in the heat of reaction, by manipulating the rate of heat input Q.

Defining the variables $x_1 = (C_A - C_{As})/C_{As}$, $x_2 = (T - T_s)/T_s$, $u = Q/(\rho_m c_p V T_s)$, $\theta_1(t) = T_{A0} - T_{A0s}$, $\theta_2(t) = \Delta H - \Delta H_{\mathrm{nom}}$, $y = x_1$, where the subscript s denotes the steady-state values, it is straightforward to verify that the process model of Equation (2) can be cast in the form of Equation (1) with

$$f(x) = \begin{bmatrix} 0.368 - x_1 - \delta(x_1, x_2) \\ -0.736 - x_2 + 2\delta(x_1, x_2) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w_1(x) = \begin{bmatrix} 0 \\ \dfrac{1}{T_s} \end{bmatrix}$$

$$w_2(x) = \begin{bmatrix} 0 \\ \dfrac{C_{As}}{\rho_m c_p T_s} \delta(x_1, x_2) \end{bmatrix} \tag{23}$$

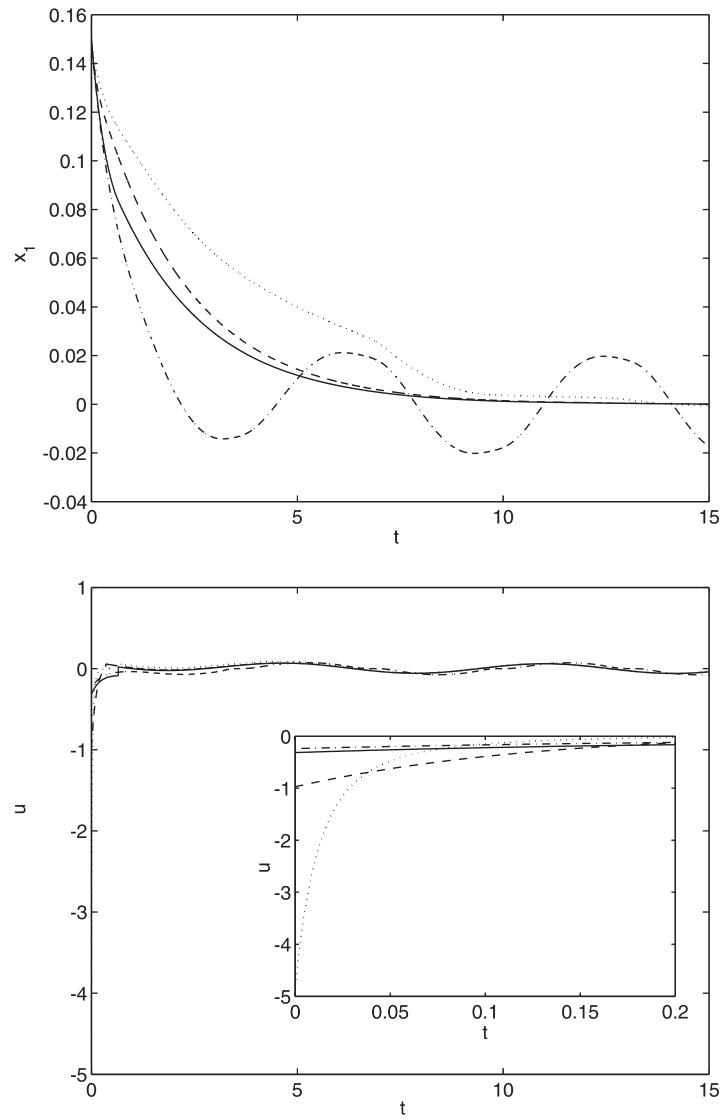

Figure 1. Closed-loop output and manipulated input profiles under the controller of Equation (14) (solid lines), the controller of Equation (14) with $\chi = 0$ (dashed–dotted lines), the controller of Equation (17) (dashed lines), and the high-gain controller (dotted lines).

where

$$\delta(x_1, x_2) = (1 + x_1) \exp\left(2 - \frac{3}{x_2 + 1}\right)$$

and has a relative degree $r = 2$. To simulate the effect of the uncertainty, we consider the time-varying functions $\theta_k(t) = \theta_{bk} \sin(t)$, $k = 1, 2$ with upper bounds $\theta_{b1} = 30$ K and

$\theta_{b2} = 28.7$ KJ/mol, respectively. To proceed with the design of the controller, we initially use the co-ordinate transformation $e_1 = x_1$, $e_2 = 0.368 - x_1 - \delta(x_1, x_2)$ to cast the input/output dynamics of the system of Equation (22) in the following form:

$$\dot{e} = \bar{f}(e) + \bar{g}(e)u + \bar{w}_1(e)\theta_1 + \bar{w}_2(e)\theta_2 \tag{24}$$

where the functions $\bar{f}$, $\bar{g}$, $\bar{w}_1$, $\bar{w}_2$ can be computed directly from Equation (13). We use the result of Theorem 1 to design the necessary controller which takes the form of Equation (14) with $q = 2$, $r = 2$, and

$$P = \begin{bmatrix} 1 & 0.45 \\ 0.45 & 1 \end{bmatrix}$$

Moreover, the following values are used for the tuning parameters of the controller: $\phi = 0.0001$, $\chi = 1.6$ and $c_0 = 0.0001$ to guarantee that the output of the closed-loop system satisfies a relation of the form $\limsup_{t\to\infty} |y - v| \leqslant 0.005$.

Closed-loop simulations were performed to evaluate the robustness and optimality properties of the controller. The results are shown by the solid lines in Figure 1, which depict the controlled output (top) and manipulated input (bottom) profiles. One can immediately see that the robust optimal controller drives the closed-loop output close to the desired steady state, while attenuating the effect of the uncertainty on the output. Included also in Figure 1 is the closed-loop output profile (top plot, dashed–dotted line) and corresponding input profile (bottom plot, dashed–dotted line) when the controller of Equation (14) is implemented without the uncertainty compensation component ($\chi = 0$.) It is clear from the result that the effect of uncertainty is significant and leads to poor transient performance and offset if unaccounted for.

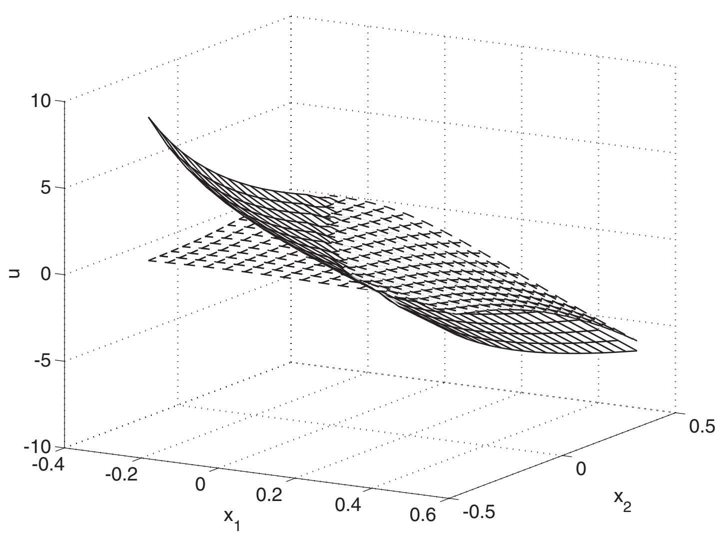

Figure 2. Control actions computed by the controller of Equation (14) (dashed lines) and the controller of Equation (17) (solid lines).

For the sake of comparison, we also considered the robust nonlinear controller of Equation (17) (see Remark 6 above) as well as the nonlinear high-gain controller of the form $u = (1/\varepsilon)[c_0 + r_s(x)]L_{\bar{g}}V$ where no uncertainty compensation is included and ε is a small positive number. The results for these two cases (starting from the same initial condition) are shown in Figure 1 by the dashed and dotted lines, respectively, for a choice of the tuning parameters: $\beta_0 = 4,\ \beta_1 = 9,\ \beta_2 = 1,\ \varepsilon = 0.05$. It is clear from the input profiles that, compared with these two controllers, the controller of Equation (14) starts by requesting significantly smaller control action to achieve a relatively faster closed-loop response and enforce, asymptotically, the same level of attenuation of the effect of uncertainty on x_1. Finally, in order to establish the fact that the robust optimal controller of Equation (14) does not use unnecessary control action compared with the controller of Equation (17), we compared the control actions generated by the two controllers for a broad range of x_1 and x_2. The results are shown in Figure 2 and clearly indicate that the controller of Equation (14) (dashed lines) uses smaller control action, pointwise, than the controller of Equation (17) (solid lines).

APPENDIX A

Proof of Theorem 1

Part 1: In this part, we establish that the controller of Equation (14) enforces global boundedness of the closed-loop trajectories and that the closed-loop output satisfies Equation (15). To simplify the notation, we consider only the case of a single uncertain variable, i.e. $q = 1$ (extension to the case when $q > 1$ is conceptually straightforward). Consider the representation of the closed-loop system in the transformed co-ordinates:

$$\begin{aligned}\dot{e} &= \bar{f}(e, \eta, \bar{v}) + \bar{g}(e, \eta, \bar{v})u + \bar{w}(e, \eta, \bar{v})\theta \\ \dot{\eta} &= \Psi(e, \eta, \bar{v}, \theta) \\ y &= e_1 + v\end{aligned} \tag{A1}$$

where the functions $\bar{f}, \bar{g}, \bar{w}$ are defined in Equation (13). We proceed in two steps. Initially, we use a Lyapunov argument to show that, starting from any initial condition, the states of the closed-loop e-subsystem are bounded and converge, asymptotically, to an arbitrarily small neighbourhood, centred around the origin. We derive bounds that capture the evolution of the states of the e and η subsystems. Then we invoke a small gain argument to show that the trajectories of the e,η interconnected system remain bounded for all times and that the output satisfies the relation of Equation (15).

Step 1: Computing the time derivative of V along the trajectories of the e-subsystem in Equation (A1), we get

$$\begin{aligned}\dot{V} &= L_{\bar{f}}V + L_{\bar{g}}Vu + L_{\bar{w}}V\theta \\ &= -c_0(L_{\bar{g}}V)^2 - \sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} \\ &\quad - \left(\frac{\rho + \chi\theta_b|L_w L_f^{r-1}h(x)|}{(L_g L_f^{r-1}h(x))^2\left(\dfrac{|L_{\bar{g}}V|}{|L_g L_f^{r-1}h(x)|} + \phi\right)}\right)(L_{\bar{g}}V)^2 + L_{\bar{w}}V\theta\end{aligned} \tag{A2}$$

Substituting the expressions $L_{\bar{g}}V = (2b^{\mathrm{T}}Pe)L_gL_f^{r-1}h(x)$ and $L_{\bar{w}}V = (2b^{\mathrm{T}}Pe)L_wL_f^{r-1}h(x)$ into the above equation, we obtain

$$\begin{aligned}\dot{V} \leqslant & -\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} \\ & -\left(\frac{\rho + \chi\theta_b|L_wL_f^{r-1}h(x)|}{(|2b^{\mathrm{T}}Pe| + \phi)}\right)|2b^{\mathrm{T}}Pe|^2 + |2b^{\mathrm{T}}Pe||L_wL_f^{r-1}h(x)|\theta_b \\ = & -\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} - \frac{\rho|2b^{\mathrm{T}}Pe|^2}{|2b^{\mathrm{T}}Pe| + \phi} \\ & - \frac{\theta_b|L_wL_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|((\chi-1)|2b^{\mathrm{T}}Pe| - \phi)}{|2b^{\mathrm{T}}Pe| + \phi}\end{aligned} \tag{A3}$$

From the last inequality and the fact that $\rho > 0$ and $\chi > 2$, it is clear that whenever $|2b^{\mathrm{T}}Pe| \geqslant \phi(\chi-1)^{-1}$, the time derivative of the Lyapunov function satisfies

$$\dot{V} \leqslant -\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} \tag{A4}$$

which is negative definite since, for all $e \neq 0$, $L_{\bar{g}}V = 0 \Rightarrow L_{\bar{f}}V < 0$. To analyse the sign of $\dot{V}$ when $|2b^{\mathrm{T}}Pe| < \phi(\chi-1)^{-1}$, we use the bound $|L_wL_f^{r-1}h(x)| \leqslant \delta|2b^{\mathrm{T}}Pe| + \mu$ and the fact that $\phi - (\chi-1)|2b^{\mathrm{T}}Pe| < \phi$ to obtain the following estimates:

$$\begin{aligned}& \theta_b|L_wL_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|(\phi - (\chi-1)|2b^{\mathrm{T}}Pe|) \\ & \leqslant \theta_b|L_wL_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|\phi \\ & \leqslant \theta_b\delta\phi|2b^{\mathrm{T}}Pe|^2 + \theta_b\mu\phi|2b^{\mathrm{T}}Pe|3 \quad \forall\ |2b^{\mathrm{T}}Pe| < \phi(\chi-1)^{-1}\end{aligned} \tag{A5}$$

Substituting the estimates of Equation (A5) directly into Equation (A3) yields

$$\dot{V} \leqslant -\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} + \frac{(-\rho + \delta\theta_b\phi)|2b^{\mathrm{T}}Pe|^2}{|2b^{\mathrm{T}}Pe| + \phi} + \frac{\theta_b\mu\phi|2b^{\mathrm{T}}Pe|}{|2b^{\mathrm{T}}Pe| + \phi} \tag{A6}$$

It is clear from the above equation, and the fact that $|2b^{\mathrm{T}}Pe|/(|2b^{\mathrm{T}}Pe| + \phi) \leqslant 1$, that if $\phi \leqslant \rho/\delta\theta_b := \phi_1^*$, $\dot{V}$ satisfies

$$\dot{V} \leqslant -\sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4} + \bar{\phi} \tag{A7}$$

irrespective of the value of $|2b^{\mathrm{T}}Pe|$, where $\bar{\phi} = \theta_b\mu\phi$. Consequently, there exists a function $\alpha_1(\cdot)$ of class $\mathcal{K}$ such that

$$\dot{V} \leqslant -\alpha_1(|e|) + \bar{\phi} \tag{A8}$$

Choosing ϕ small enough such that $\bar{\phi} \leqslant \frac{1}{2}\alpha_1(|e|)$, we have

$$\dot{V} \leqslant -\tfrac{1}{2}\alpha_1(|e|) < 0 \quad \forall\ |e| \geqslant \alpha_1^{-1}(2\bar{\phi}) \tag{A9}$$

This inequality shows that $\dot{V}$ is negative outside the set, $B_1 := \{e \in \mathbb{R}^r : |e| \leqslant \alpha_1^{-1}(2\bar{\phi})\}$. A direct application then of the result of Theorem 5.1 and its corollaries in Reference [24] allows us to conclude that starting from any initial condition, the following ISS inequality holds for the e states of the system of Equation (A1)

$$|e(t)| \leqslant \bar{\beta}_e(|e(0)|, t) + \bar{\gamma}_e(\phi) \quad \forall\ t \geqslant 0 \tag{A10}$$

where $\bar{\beta}_e$ is a class $\mathcal{KL}$ function and $\bar{\gamma}_e$ is a class $\mathcal{K}_\infty$ function. From Assumption 2, we have that the η states of the system of Equation (A1) possess an ISS property with respect to e and θ

$$|\eta(t)| \leqslant \bar{\beta}_\eta(|\eta(0)|, t) + \bar{\gamma}_\eta(\|[e^{\mathrm{T}} \;\; \theta]^{\mathrm{T}}\|) \leqslant \bar{\beta}_\eta(|\eta(0)|, t) + \bar{\gamma}_{\eta_1}(\|e\|) + \bar{\gamma}_{\eta_2}(\|\theta\|) \tag{A11}$$

uniformly in $\bar{v}$, where $\bar{\gamma}_{\eta_1}, \bar{\gamma}_{\eta_2}$ are class $\mathcal{K}$ functions defined as $\bar{\gamma}_{\eta_1}(s) = \bar{\gamma}_{\eta_2}(s) = \bar{\gamma}_\eta(2s)$.

Step 2: We now analyse the behaviour of the interconnected dynamical system comprised of the states e, η in Equation (A1) for which the inequalities of Equations (A10) and (A11) hold. We first define the following positive real numbers: $\delta_e = D^{\bar{e}} + \phi^*$, $\delta_\eta = D^{\bar{\eta}} + \bar{\gamma}_{\eta_1}(\delta_e) + \phi^*$, $D^{\bar{e}} = \bar{\beta}_e(\bar{\delta}_e, 0) + \bar{\gamma}_e(\phi^*)$, $D^{\bar{\eta}} = \bar{\beta}_\eta(\bar{\delta}_\eta, 0) + \bar{\gamma}_{\eta_2}(\theta_b)$, $\phi^* = \min\{\frac{1}{2}\phi_1^*, \bar{\gamma}_e^{-1}(d)\}$ where $d > 0$ is arbitrary and $\bar{\delta}_e, \bar{\delta}_\eta$ are any positive real numbers. Then, using a contradiction argument similar to the one used in References [4, 25], one can show that if $\phi \in (0, \phi^*]$, the evolution of the states e, η, starting from any initial states that satisfy $|e(0)| \leqslant \bar{\delta}_e$, $|\eta(0)| \leqslant \bar{\delta}_\eta$, satisfies the inequalities $|e(t)| \leqslant \delta_e$, $|\eta(t)| \leqslant \delta_\eta$ for all times. Finally, for $\phi \in (0, \phi^*]$ and for any initial states, taking the lim sup of both sides of Equation (A10) as $t \to \infty$, we have

$$\begin{aligned}\limsup_{t\to\infty} |y(t) - v(t)| &\leqslant \limsup_{t\to\infty} |e(t)| \\ &\leqslant \limsup_{t\to\infty} (\bar{\beta}_e(|e(0)|, t) + \bar{\gamma}_e(\phi)) \leqslant \bar{\gamma}_e(\phi^*) \leqslant d\end{aligned} \tag{A12}$$

Part 2: In this part, we prove that the control law of Equation (14) is optimal with respect to a meaningful cost functional of the form of Equation (16). We proceed in two steps. In the first step, we show that the cost functional defined in Equation (16) is a meaningful one. In the second step, we show that the stabilizing control law of Equation (14) minimizes this cost functional.

Step 1: From its definition in Theorem 1, $l(e)$ is given by

$$l(e) = -L_{\bar{f}}V + \tfrac{1}{4} L_{\bar{g}} V R^{-1}(x) L_{\bar{g}} V - |L_{\bar{w}} V|\theta_b \tag{A13}$$

Direct substitution of the expression for $R^{-1}(x)$ given in Theorem 1 in the above equation yields

$$\begin{aligned} l(e) &= \frac{1}{2} c_0 (L_{\bar{g}}V)^2 + \frac{1}{2}[-L_{\bar{f}}V + \sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}] \\ &\quad + \frac{1}{2} \frac{(\rho + \chi\theta_b |L_w L_f^{r-1} h(x)|)|2b^{\mathrm{T}}Pe|^2}{|2b^{\mathrm{T}}Pe| + \phi} - |L_{\bar{w}}V|\theta_b \\ &\geqslant \frac{1}{2}[-L_{\bar{f}}V + \sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}] + \frac{1}{2}\left(\frac{\rho|2b^{\mathrm{T}}Pe|^2}{|2b^{\mathrm{T}}Pe| + \phi}\right) \\ &\quad + \frac{\theta_b |2b^{\mathrm{T}}Pe| |L_w L_f^{r-1} h(x)| ((\frac{1}{2}\chi - 1)|2b^{\mathrm{T}}Pe| - \phi)}{|2b^{\mathrm{T}}Pe| + \phi} \end{aligned} \tag{A14}$$

where we used the expression for $L_{\bar{w}}V$ and the fact that $c_0 > 0$ to derive the above inequality. From this inequality and the fact that $\rho > 0$ and $\chi > 2$, it is clear that whenever $|2b^{\mathrm{T}}Pe| \geqslant \phi(\frac{1}{2}\chi - 1)^{-1}$, we have

$$l(e) \geqslant \tfrac{1}{2}[-L_{\bar{f}}V + \sqrt{(L_{\bar{f}}V)^2 + (L_{\bar{g}}V)^4}] \tag{A15}$$

which is positive definite. To analyse the sign of $l(e)$ when $|2b^{\mathrm{T}}Pe| < \phi(\frac{1}{2}\chi - 1)^{-1}$, we use the bound $|L_w L_f^{r-1} h(x)| \leqslant \delta |2b^{\mathrm{T}}Pe| + \mu$ and the fact that $(\frac{1}{2}\chi - 1)|2b^{\mathrm{T}}Pe| - \phi > -\phi$ to obtain the following estimates:

$$\begin{aligned}&\theta_b|L_w L_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|(\phi-(\tfrac{1}{2}\chi-1)|2b^{\mathrm{T}}Pe|)\\&\quad\geqslant-\theta_b|L_w L_f^{r-1}h(x)||2b^{\mathrm{T}}Pe|\phi\\&\quad\geqslant-\theta_b\delta\phi|2b^{\mathrm{T}}Pe|^2-\theta_b\mu\phi|2b^{\mathrm{T}}Pe|\quad\forall\ |2b^{\mathrm{T}}Pe|<\phi(\tfrac{1}{2}\chi-1)^{-1}\end{aligned}\tag{A16}$$

Substituting the estimates of Equation (A16) directly into Equation (A14) yields

$$l(e)\geqslant\frac{1}{2}[-L_{\bar f}V+\sqrt{(L_{\bar f}V)^2+(L_{\bar g}V)^4}]+\frac{((1/2)\rho-\delta\theta_b\phi)|2b^{\mathrm{T}}Pe|^2}{|2b^{\mathrm{T}}Pe|+\phi}-\frac{\theta_b\mu\phi|2b^{\mathrm{T}}Pe|}{|2b^{\mathrm{T}}Pe|+\phi}\tag{A17}$$

From the above equation, it is clear that if $\phi\leqslant P/2\delta\theta_b:=\phi_2^*$, $l(e)$ satisfies

$$l(e)\geqslant\tfrac{1}{2}[-L_{\bar f}V+\sqrt{(L_{\bar f}V)^2+(L_{\bar g}V)^4}]-\bar\phi\tag{A18}$$

irrespective of the value of $|2b^{\mathrm{T}}Pe|$. Therefore, there exists a function $\alpha_2(\cdot)$ of class $\mathscr{K}$ such that

$$\begin{aligned}l(e)&\geqslant\alpha_2(|e|)-\bar\phi\\&\geqslant\tfrac{1}{2}\alpha_2(|e|)>0\quad\forall\ |e|\geqslant\alpha_2^{-1}(2\bar\phi)\end{aligned}\tag{A19}$$

The last inequality implies that $l(e)$ is positive outside the set $B_2:=\{e\in\mathbb{R}^r:|e|\leqslant\alpha_2^{-1}(2\bar\phi)\}$. Note that this set is completely contained within the set Γ since $|e|\leqslant\alpha_2^{-1}(2\bar\phi)\leqslant\alpha_2^{-1}(2\bar\phi^*)\leqslant\varepsilon\Rightarrow e^{\mathrm{T}}Pe\leqslant\lambda_{\max}(P)|e|^2\leqslant\lambda_{\max}(P)\varepsilon^2$ where $\varepsilon:=\max\{\alpha_1^{-1}(2\bar\phi^*),\alpha_2^{-1}(2\bar\phi^*)\}$ and $\lambda_{\max}(P)>0$ is the maximum eigenvalue of the matrix P. Similarly, the set B_1, defined right after Equation (A9), is also contained in Γ, and therefore $\dot V<0$ on and outside the boundary of Γ. From its definition in Theorem 1, T_f is the minimum time for the trajectories of the closed-loop system to reach and enter Γ without ever leaving again (note that the boundary of Γ is a level set of V). Therefore, we have that $|e(t)|\geqslant\alpha_2^{-1}(2\bar\phi)\ \forall\ t\in[0,T_f]$. Hence, $l(e)>0\ \forall\ t\in[0,T_f]$. Note also that $R(x)>0$ for all x. Therefore, the cost functional of Equation (16) is a meaningful one.

Step 2: In this step, we prove that control law of Equation (14) minimizes the cost functional of Equation (16). Substituting

$$\alpha=u+\tfrac{1}{2}R^{-1}(x)L_{\bar g}V\tag{A20}$$

into Equation (16), we get the following chain of equalities:

$$\begin{aligned}J&=V(e(T_f))+\int_0^{T_f}(l(e(t))+u(t)R(x(t))u(t))\,\mathrm{d}t\\&=V(e(T_f))+\int_0^{T_f}(-L_{\bar f}V+\tfrac{1}{2}L_{\bar g}VR^{-1}(x)L_{\bar g}V-|L_{\bar w}V|\theta_b-L_{\bar g}V\alpha)\,\mathrm{d}t\\&\quad+\int_0^{T_f}\alpha R(x)\alpha\,\mathrm{d}t\\&=V(e(T_f))-\int_0^{T_f}\sup_{\theta\in\mathscr{W}}(\dot V)\,\mathrm{d}t+\int_0^{T_f}\alpha R(x)\alpha\,\mathrm{d}t\end{aligned}\tag{A21}$$

Note that the term $J^*=V(e(T_f))-\int_0^{T_f}\sup_{\theta\in\mathscr{W}}(\dot V)\,\mathrm{d}t$ is bounded from above since

$$J^*=V(e(T_f))-\int_0^{T_f}\sup_{\theta\in\mathscr{W}}(\dot V)\,\mathrm{d}t\leqslant V(e(T_f))-\int_0^{T_f}\dot V\,\mathrm{d}t=V(e(0))\tag{A22}$$

Since $R(x) > 0$ for all $x \in \mathbb{R}^n$, it is clear that the minimum of J is J^*. This minimum is achieved when $\alpha(t) \equiv 0$ which proves that the controller of Equation (14) minimizes the cost defined in Equation (16). To complete the proof of optimality, we need to show that $J^* = V(e(0))$. To this end, consider the uncertain variable $\theta_\Delta \in \mathcal{W}$ where for every $e_\Delta(0) \in \mathbb{R}^r$, every $u \in \mathbb{R}$, and every $\Delta > 0$, we have

$$\int_0^{T_f} \dot{V}(e_\Delta, u, \theta_\Delta)\,\mathrm{d}t \geqslant \int_0^{T_f} \sup_{\theta_\Delta \in \mathcal{W}} \dot{V}(e_\Delta, u, \theta_\Delta)\,\mathrm{d}t - \Delta \tag{A23}$$

(The existence of θ_Δ follows from the properties of $\theta(t)$.) From the inequality of Equation (A23), we obtain

$$V(e(0)) = V(e(T_f)) - \int_0^{T_f} \dot{V}\,\mathrm{d}t \leqslant V(e(T_f)) - \int_0^{T_f} \sup_{\theta_\Delta \in \mathcal{W}} \dot{V}(e_\Delta, u, \theta_\Delta)\,\mathrm{d}t + \Delta \tag{A24}$$

Combining the inequalities of Equations (A24) and (A22), we get

$$V(e(0)) - \Delta \leqslant J^* \leqslant V(e(0)) \tag{A25}$$

for arbitrarily small Δ. Clearly, if $J^* = V(e(0))$, the proof of optimality is complete. Otherwise, there exists $\mu > 0$ such that $J^* + \mu = V(e(0))$. Since Δ is arbitrary, we can choose it to be sufficiently small such that $\Delta < \mu$ and the inequality of Equation (A25) is violated. Thus, the only way to satisfy the inequality of Equation (A25) for arbitrary Δ is to set $J^* = V(e(0))$. This completes the proof of the theorem. □

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