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Bounded robust control of constrained multivariable nonlinear processes ☆

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Abstract

This work focuses on control of multi-input multi-output (MIMO) nonlinear processes with uncertain dynamics and actuator constraints. A Lyapunov-based nonlinear controller design approach that accounts explicitly and simultaneously for process nonlinearities, plant-model mismatch, and input constraints, is proposed. Under the assumption that all process states are accessible for measurement, the approach leads to the explicit synthesis of bounded robust multivariable nonlinear state feedback controllers with well-characterized stability and performance properties. The controllers enforce stability and robust asymptotic reference-input tracking in the constrained uncertain closed-loop system and provide, at the same time, an explicit characterization of the region of guaranteed closed-loop stability. When full state measurements are not available, a combination of the state feedback controllers with high-gain state observes and appropriate saturation filters, is employed to synthesize bounded robust multivariable output feedback controllers that require only measurements of the outputs for practical implementation. The resulting output feedback design is shown to inherit the same closed-loop stability and performance properties of the state feedback controllers and, in addition, recover the closed-loop stability region obtained under state feedback, provided that the observer gain is sufficiently large. The developed state and output feedback controllers are applied successfully to non-isothermal chemical reactor examples with uncertainty, input constraints, and incomplete state measurements. Finally, we conclude the paper with a discussion that attempts to put in perspective the proposed Lyapunov-based control approach with respect to the nonlinear model predictive control (MPC) approach and discuss the implications of our results for the practical implementation of MPC, in control of uncertain nonlinear processes with input constraints.

1. Introduction

As modern-day chemical processes are continuously faced with the requirements of becoming safer, more reliable, and more economical in operation, the need for a rigorous, yet practical, approach for the design of effective chemical process control systems that can meet these demands, becomes increasingly apparent. The development of such an approach, however, can be quite a challenging undertaking given the host of fundamental and practical problems that arise in process control systems and transcend the boundaries of specific applications. Although they may vary from one application to another and have different levels of significance, these issues remain generic in their relationship to the control design objectives. Central to these issues is the requirement that the control system provide satisfactory performance in the presence of strong process nonlinearities, model uncertainty, process variations, and actuator constraints. Process nonlinearities, plant-model mismatch, and actuator constraints represent some of the more salient features whose frequently encountered co-presence in many chemical processes can lead to severe performance deterioration and even closed-loop instability, if not appropriately accounted for in the controller design. This realization has consequently fostered a large and growing body of research work on these problems, towards the development of a variety of model-based nonlinear process control approaches, including both analytical and optimization-based methods. The area of nonlinear process control, for example, has been the subject of significant research activity over the past two decades, and the literature on this subject is quite extensive (e.g., see Bequette (1991), Kravaris and Arkun (1991) and

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Allgöwer and Doyle (1997) and the references therein for comprehensive reviews).

In addition to nonlinearity, the problem of designing robust controllers for nonlinear processes with model uncertainty, such as unknown process parameters and exogenous disturbance inputs, has received significant attention in the past (e.g., Arkun & Calvet, 1992; Christofides, Teel, & Daoutidis, 1996; Christofides, 2000; Kolavennu, Palanki, & Cockburn, 2000). Similarly, the detrimental effects of actuator constraints on the stability and performance of process control systems have motivated several studies on the analysis and control of processes with input constraints. Important contributions in this area include results within the model predictive control (MPC) framework (e.g., Rawlings, 1999; Schwarm & Nikolaou, 1999; Kurtz, Zhu, & Henson, 2000), constrained quadratic-optimal control (e.g. Chmielewski & Manousiouthakis, 1998) anti-windup designs (e.g., Kothare, Campo, Morari, & Nett, 1994; Kendi & Doyle, 1997; Teel, 1999; Kapoor & Daoutidis, 1999a), the synthesis of directionality compensation schemes (e.g., Soroush & Valluri, 1999; Mhatre & Brosilow, 2000) for constrained multivariable processes, the study of the nonlinear bounded control problem for a class of chemical reactors (Alvarez, Alvarez, & Suarez, 1991), and some general results on the dynamics of constrained nonlinear systems (Kapoor & Daoutidis, 1999b).

Despite notable progress in these areas, however, a number of important limitations continue to impact on the practical implementation of existing control approaches. One such limitation is the lack of a unified framework for dealing with the combined problems of nonlinearity, uncertainty, and constraints; which can place unacceptable limitations on the performance and stability of the closed-loop system. The lack of a unified control approach is due, in part, to the well-known computational difficulties associated with some of the available methods (e.g., nonlinear MPC) when dealing with the problems of nonlinearity, uncertainty, and constraints combined. In addition to computational complexity, there is also the lack of any theoretically well-established guarantees of closed-loop stability. In particular, the issue of identifying, a priori, the set of admissible initial conditions, starting from where closed-loop stability is guaranteed under uncertainty and constraints, remains to be adequately addressed within existing nonlinear control approaches, both analytical and optimization based. This issue has important implications for the practical implementation of a given control strategy since, absent any a priori guarantees of closed-loop stability, the control engineer is faced with the task of searching, via extensive and time-consuming closed-loop simulations, over the whole set of possible initial conditions.

Motivated by these problems, we have recently developed in El-Farra and Christofides (2001a) a unified framework for model-based control of single-input single-output (SISO) nonlinear processes with model uncertainty and input constraints. The proposed framework adopts a Lyapunov-based approach and leads to the explicit synthesis of bounded robust optimal nonlinear state feedback controllers that integrate the following desirable properties in the controller design, including: (1) robustness against significant (constant and time-varying) plant-model mismatch, (2) optimality with respect to meaningful performance criteria and subsequent use of reasonable control action to achieve the desired closed-loop objectives such as stability and robust asymptotic set-point tracking, and (3) explicit constraint-handling capabilities. In addition, the proposed approach provides an explicit characterization of the region of guaranteed closed-loop stability. Therefore, given an initial condition one can ascertain a priori, i.e. before controller implementation and without the need to run closed-loop simulations, whether closed-loop stability can be guaranteed under uncertainty and constraints. The proposed control strategy was shown to possess clear advantages over the traditional two-step approaches in which a feedback controller is designed first in the absence of constraints and then modified to account for the presence of constraints through some anti-windup modification scheme. In addition to this work, other advances on bounded, Lyapunov-based, state feedback control of nonlinear systems with disturbances include the results in Liberzon (1999) on disturbance attenuation within the input-to-state stability (ISS) framework, and the results in Suarez, Solis-Daun, and Alvarez-Ramirez (2002) on global robust stabilization of multi-input systems.

Even though the work in El-Farra and Christofides (2001a) provides a systematic approach for state feedback nonlinear controller design for SISO nonlinear processes with model uncertainty and input constraints, it does not address the additional important problems of multivariable interactions and lack of complete process state measurements, which are frequently encountered in chemical process control systems. Virtually all practical process control systems are characterized by multivariable interactions where the control problem involves the regulation of more than one output by manipulating more than one input. When input constraints are not dealt with explicitly in the controller design, multivariable interactions can cause further deterioration in the process performance due to problems such as process directionality. In addition, the lack of complete state measurements renders the state feedback controller designs not directly suitable for practical implementation and necessitates the design of appropriate state estimators.

Motivated by these considerations, we focus in this work on control of multi-input multi-output (MIMO) nonlinear processes with model uncertainty and input constraints. A unified Lyapunov-based control approach, that accounts explicitly for process nonlinearities, model uncertainty, input constraints, multivariable interactions, and the lack of full state measurements, is proposed. When measurements of the full state are available, the approach leads to the explicit synthesis of bounded robust multivariable nonlinear state feedback controllers that enforce stability and robust asymptotic set-point tracking in the constrained uncertain closed-loop system and provide, at the same time, an explicit characterization of the region of guaranteed closed-loop stability. When complete state measurements are not available, a combination of the state feedback controllers with high-gain state observes and appropriate saturation filters, is employed to synthesize bounded robust multivariable output feedback controllers that require only measurements of the outputs for practical implementation. The resulting output feedback design enforces the same closed-loop stability and performance properties of the state feedback controllers and, in addition, practically preserves the region of guaranteed closed-loop stability obtained under state feedback. The developed state and output feedback controllers are applied successfully to non-isothermal multivariable chemical reactor examples with uncertainty, input constraints, and incomplete state measurements. Finally, we conclude the paper with a discussion that attempts to put in perspective the proposed Lyapunov-based control approach with respect to the nonlinear MPC approach, and discuss the implications of our results for the practical implementation of MPC in control of uncertain nonlinear processes with input constraints.

2. Preliminaries

We consider MIMO nonlinear processes with uncertain variables and input constraints with the following state-space description:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i + \sum_{k=1}^{q} w_k(x)\theta_k(t), \tag{1}$$

$$y_i = h_i(x), \quad i = 1, \ldots, m,$$

where $x \in \mathbb{R}^n$ denotes the vector of process state variables, u_i denotes the ith constrained manipulated input taking values in a nonempty convex subset U_i of $\mathbb{R}$ (i.e., $u_i \in U_i = [u_{i,\min}, u_{i,\max}] \subset \mathbb{R}$), $\theta_k(t) \in \mathcal{W} \subset \mathbb{R}$ denotes the kth uncertain (possibly time-varying) but bounded variable taking values in a nonempty compact convex subset $\mathcal{W}$ of $\mathbb{R}$, $y_i \in \mathbb{R}$ denotes the ith output to be controlled. The uncertain variable $\theta_k(t)$ may describe time-varying parametric uncertainty and/or exogenous disturbances. It is assumed that the origin is the equilibrium point of the nominal (i.e., $u(t)=\theta_k(t) \equiv 0$) system of Eq. (1). The vector functions $f(x)$, $g_i(x)$, $w_k(x)$, and the scalar functions $h_i(x)$ are assumed to be sufficiently smooth on their domains of definition. Throughout the paper, the Lie derivative notation will used. In particular, $L_f\bar{h}$ denotes the standard Lie derivative of a scalar function $\bar{h}(x)$ with respect to the vector function $f(x)$, $L_f^k\bar{h}$ denotes the kth order Lie derivative and $L_{g_i}L_f^{k-1}\bar{h}$ denotes the mixed Lie derivative where $g_i(x)$ is a vector function. We will also need the definition of a class KL function. A function $\beta(s,t)$ is said to be of class KL if, for each fixed t, the function $\beta(s,\cdot)$ is continuous, increasing, and zero at zero and, for each fixed s, the function $\beta(\cdot,t)$ is non-increasing and tends to zero at infinity.

Referring to the system of Eq. (1), we define the relative order of the output y_i with respect to the vector of manipulated inputs $u = [u_1 u_2 \cdots u_m]^{\mathrm{T}}$ as the smallest integer r_i for which

$$[L_{g_1}L_f^{r_i-1}h_i(x) \ \cdots \ L_{g_m}L_f^{r_i-1}h_i(x)] \not\equiv [0 \ \cdots \ 0] \tag{2}$$

or $r_i = \infty$ if such an integer does not exist. We also define the characteristic matrix (Isidori, 1989)

$$C(x) = \begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & \cdots & L_{g_m}L_f^{r_1-1}h_1(x) \\ \vdots & \cdots & \vdots \\ L_{g_1}L_f^{r_m-1}h_m(x) & \cdots & L_{g_m}L_f^{r_m-1}h_m(x) \end{bmatrix}. \tag{3}$$

Finally, we recall the definition of ISS for a system of the form of Eq. (1).

Definition 1 (Sontag (1989a)). The system in Eq. (1) (with $u \equiv 0$) is said to be ISS with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_\circ \in \mathbb{R}^n$ and for each measurable, essentially bounded input $\theta(\cdot)$ on $[0,\infty)$ the solution of Eq. (1) with $x(0) = x_\circ$ exists for each $t \geqslant 0$ and satisfies

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|\theta\|) \quad \forall t \geqslant 0. \tag{4}$$

3. Bounded robust Lyapunov-based control: state feedback

In this section, we focus on the state feedback control problem, where the full process state is assumed to be available for measurement. The output feedback control problem will be discussed in Section 5. We begin, in Section 3.1, with the control problem formulation and then present the controller synthesis results in Section 3.2.

3.1. Control problem formulation

In order to formulate our control problem, we consider the MIMO nonlinear process of Eq. (1) and assume, initially, that the uncertain variable terms $w_k(x)\theta_k$ are vanishing (in the sense that $w_k(0)\theta_k = 0$ for any $\theta_k \in \mathcal{W}$-note that this does not require the variable θ_k itself to vanish in time). Under this assumption, the origin, which is an equilibrium point for the nominal process, continues to be an equilibrium point for the uncertain process. The results for the case when the uncertain variables are non-vanishing (i.e., perturb the nominal equilibrium point) are discussed later (see Remark 7 below). For the control problem at hand, our objectives are two fold. The first is to synthesize, via Lyapunov-based control methods, a bounded robust multivariable nonlinear state feedback control law of the general form

$$u = \mathscr{P}(x, \bar{v}), \tag{5}$$

that enforces the following closed-loop properties in the presence of uncertainty and input constraints, including: (a) asymptotic stability, and (b) robust asymptotic reference-input tracking with an arbitrary degree of attenuation of the effect of the uncertainty on the output of the closed-loop system. In Eq. (5), $\mathscr{P}(x,\bar{v})$ is an m-dimensional bounded vector function (i.e., $|u| \leqslant u_{\max}$, where $|\cdot|$ denotes the Euclidean norm and $u_{\max}$ is the maximum magnitude of the Euclidean norm of the vector of manipulated inputs allowed by the constraints), $\bar{v}$ is a generalized reference input which is assumed to be a sufficiently smooth function of time. Our second objective is to explicitly characterize the region of guaranteed closed-loop stability associated with the controller, namely the set of admissible initial states starting from where, stability and robust output tracking can be guaranteed in the constrained uncertain closed-loop system.

3.2. Controller synthesis

To proceed with the controller synthesis task, we will impose the following three assumptions on the process of Eq. (1). We initially assume that there exists a coordinate transformation that renders the system of Eq. (1) partially linear. This assumption is motivated by the requirement of robust output tracking and is formulated precisely below:

Assumption 1. There exist a set of integers $(r_1, r_2, \ldots, r_m)$ and a coordinate transformation $(\zeta,\eta) = T(x)$ such that the representation of the system of Eq. (1), in the (ζ,η) coordinates, takes the form:

$$
\begin{aligned}
\dot{\zeta}_1^{(i)} &= \zeta_2^{(i)},\\
&\vdots\\
\dot{\zeta}_{r_i-1}^{(i)} &= \zeta_{r_i}^{(i)},\\
\dot{\zeta}_{r_i}^{(i)} &= L_f^{r_i}h_i(x) + \sum_{j=1}^{m} L_{g_j}L_f^{r_i-1}h_i(x)u_j + \sum_{k=1}^{q} L_{w_k}L_f^{r_i-1}h_i(x)\theta_k,\\
\dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta),\\
&\vdots\\
\dot{\eta}_{n-\sum_i r_i} &= \Psi_{n-\sum_i r_i}(\zeta,\eta,\theta),\\
y_i &= \zeta_i^{(i)}, \quad i = 1,\ldots,m,
\end{aligned}
\tag{6}
$$

where $x = T^{-1}(\zeta,\eta)$, $\zeta = [\zeta^{(1)^{\mathrm{T}}} \cdots \zeta^{(m)^{\mathrm{T}}}]^{\mathrm{T}}$, $\eta = [\eta_1 \cdots \eta_{n-\sum_i r_i}]^{\mathrm{T}}$, $\theta = [\theta_1 \cdots \theta_q]^{\mathrm{T}}$.

Assumption 1 includes the matching condition of our robust control methodology. In particular, we consider systems of the form Eq. (1) for which the time derivatives of the outputs y_i, up to order $r_i - 1$, are independent of the uncertain variables θ_k. Notice that this condition is different from the standard one which restricts the uncertainty to enter the system of Eq. (1) in the same equation with the manipulated input u. Defining the tracking error variables $e_k^{(i)} = \zeta_k^{(i)} - v_i^{(k-1)}$ and introducing the vector notation $e^{(i)} = [e_1^{(i)}\ e_2^{(i)} \cdots\ e_{r_i}^{(i)}]^{\mathrm{T}}$, $e = [e^{(1)^{\mathrm{T}}} e^{(2)^{\mathrm{T}}} \cdots e^{(m)^{\mathrm{T}}}]^{\mathrm{T}}$, where $i = 1,\ldots,m$, $k = 1,\ldots,r_i$, the ζ-subsystem of Eq. (6) can be re-written in the following more compact form:

$$
\dot{e} = Ae + r(e,\eta,\bar{v}) + B[C_1(x)u + C_2(x)\theta], \tag{7}
$$

where A, B are constant matrices of dimensions $(\sum_{i=1}^{m} r_i) \times (\sum_{i=1}^{m} r_i)$ and $(\sum_{i=1}^{m} r_i) \times m$, respectively, $r(e,\eta,\bar{v})$ is a $(\sum_{i=1}^{m} r_i) \times 1$ continuous nonlinear vector function, and $\bar{v} = [\bar{v}_1^{\mathrm{T}}\ \bar{v}_2^{\mathrm{T}} \cdots \bar{v}_m^{\mathrm{T}}]^{\mathrm{T}}$ where $\bar{v}_i = [v_i\ v_i^{(1)} \cdots v_i^{(r_i)}]^{\mathrm{T}}$ is a smooth vector function, and $v_i^{(k)}$ is the kth time derivative of the external reference input v_i (which is assumed to be a smooth function of time.) The specific forms of these functions are omitted for brevity. The $m \times m$ matrix $C_1(x)$ is the characteristic matrix of the system of Eq. (1) defined in Eq. (3) while $C_2(x)$ is an $m \times q$ matrix that is structurally similar to $C_1(x)$, except that the vector fields g_i are replaced by w_k. In order to simplify the presentation of our results, the matrix $C_1(x)$ will be assumed non-singular uniformly in x. This assumption can be readily relaxed if robust dynamic state feedback, instead of robust static state feedback, is used to solve the control problem (see Isidori (1989) for details). Finally, we define the function $\bar{f}(e,\eta,\bar{v}) = Ae + r(e,\eta,\bar{v})$, denote by $\bar{g}_i(e,\eta,\bar{v})$ the ith column of the matrix function $g(x) = BC_1(x)$, $i = 1,\ldots,m$, and denote by $\bar{w}_k(e,\eta,\bar{v})$ the kth column of the matrix function $W(x) = BC_2(x)$, $k = 1,\ldots,q$.

Following El-Farra and Christofides (2001a), the requirement of ISS is imposed on the η-subsystem of Eq. (6) to ensure bounded stability of the internal dynamics and allow the synthesis of the desired controller on the basis of the e-subsystem. This assumption, stated below, is standard in the process control literature concerned with enforcing output tracking and is satisfied by many practical systems, including the chemical reactor models studied in this paper which do not exhibit non-minimum phase behavior (see also Christofides et al. (1996) and El-Farra and Christofides (2001a) for additional examples).

Assumption 2. The dynamical system

$$
\begin{aligned}
\dot{\eta}_1 &= \Psi_1(e,\eta,\theta,\bar{v})\\
&\vdots\\
\dot{\eta}_{n-\sum_i r_i} &= \Psi_{n-\sum_i r_i}(e,\eta,\theta,\bar{v})
\end{aligned}
\tag{8}
$$

is ISS with respect to e uniformly in θ, $\bar{v}$.

In order to achieve attenuation of the effect of the uncertain variables on the outputs, we assume the existence of known (but not necessarily small) bounds that capture the size of the uncertain variables for all times.

Assumption 3. There exist known constants θ_{bk} such that $\|\theta_k\| = \theta_{bk}$ where $\|\theta_k\|$ denotes ess.sup $|\theta_k(t)|^n$, $t \geqslant 0$.

Theorem 1 that follows provides the explicit synthesis formula for the desired bounded robust multivariable state feedback control law and states precise conditions that guarantee closed-loop stability and robust asymptotic output tracking in the presence of uncertainty and input constraints. The key idea in the proposed controller design is that of bounding a robust Lyapunov-based controller, and is inspired by the results on bounded control in Lin and Sontag (1991) for systems without uncertainty. Analysis of the closed-loop system utilizes combination of standard Lyapunov techniques together with the concept of ISS and nonlinear small-gain type arguments. The proof of the theorem can be found in the appendix.

Theorem 1. *Consider the constrained uncertain nonlinear process of Eq.* (1), *for which Assumptions* 1–3 *hold*, *under the static state feedback law*:

$$u = -\tfrac{1}{2} R^{-1}(x)(L_{\bar{g}}V)^{\mathrm{T}} \tag{9}$$

where

$$\frac{1}{2} R^{-1}(x) = \frac{L_{\bar{f}}^{*}V + \sqrt{\left(L_{\bar{f}}^{**}V\right)^2 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^4}}{|(L_{\bar{g}}V)^{\mathrm{T}}|^2\left[1 + \sqrt{1 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}\right]} \tag{10}$$

and

$$L_{\bar{f}}^{*}V = L_{\bar{f}}V + \left(\rho|2Pe| + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar{w}k}V|\right)\left(\frac{|2Pe|}{|2Pe|+\phi}\right),$$

$$L_{\bar{f}}^{**}V = L_{\bar{f}}V + \rho|2Pe| + \chi\sum_{k=1}^{q}\theta_{bk}|L_{\bar{w}k}V|. \tag{11}$$

$V = e^{\mathrm{T}}Pe$, *P is a positive definite matrix that satisfies the Riccati matrix inequality $A^{\mathrm{T}}P + PA - \mathrm{PBB}^{\mathrm{T}}P < 0$, $L_{\bar{g}}V = [L_{\bar{g}_1}V \ \cdots \ L_{\bar{g}_m}V]$ is a row vector, and ρ, χ and ϕ are adjustable parameters that satisfy $\rho > 0$, $\chi > 1$ and $\phi > 0$. Let Π be the set defined by*

$$\Pi(\theta_b, u_{\max}) = \{x \in \mathbb{R}^n : L_{\bar{f}}^{**}V \leqslant u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|\}. \tag{12}$$

Then, *if the origin is contained in the interior of Π*, *given any initial condition $x_0 \in \Omega$, where $\Omega = \{x \in \mathbb{R}^n : |x| \leqslant \delta_s\}$ is an invariant subset of Π*, *there exists $\phi^* > 0$ such that if $\phi \in (0, \phi^*]$*:

(1) *The constrained closed-loop system is asymptotically stable* (*i.e. there exists a function β of class KL such that $|x(t)| \leqslant \beta(|x_0|, t), \forall t \geqslant 0$*).

(2) *The outputs of the closed-loop system satisfy a relation of the form*

$$\limsup_{t\to\infty} |y_i(t) - v_i(t)| = 0, \quad i = 1, \ldots, m. \tag{13}$$

Remark 1. In addition to the explicit controller synthesis formula, Theorem 1 provides an explicit characterization of the region in the state space where the desired closed-loop stability and set-point tracking properties are guaranteed under the proposed control law. This characterization is obtained from the inequality of Eq. (12) which describes a closed state-space region where the time derivative of the Lyapunov function is guaranteed to be negative definite, along the trajectories of the constrained uncertain closed-loop system, and the control action satisfies the input constraints. Provided that the origin is contained in the interior of the region described by the set Π, any closed-loop trajectory that evolves within this region is guaranteed to converge to the origin with the available control action. In fact, it is not difficult to see how this inequality is closely linked to the classical Lyapunov stability condition

$$\dot{V} = L_{\bar{f}}V + L_{\bar{g}}Vu + \sum_{k=1}^{q} L_{\bar{w}_k}V\theta_k \leqslant 0 \tag{14}$$

when the additional requirements that $|u| \leqslant u_{\max}$ and $|\theta_k| \leqslant \theta_{bk}$ are imposed on Eq. (14). The key idea here is that by taking the constraints directly into account (i.e., bounding the controller), we automatically obtain information about the region where both stability is guaranteed and the input constraints are respected. The inequality of Eq. (12) can therefore be used to identify the set of admissible initial states starting from where closed-loop stability is guaranteed under uncertainty and constraints (region of closed-loop stability). A reasonable estimate of the stability region can be obtained by computing an invariant subset, Ω, (preferably the largest) within the set Π (see El-Farra and Christofides (2001a) for further details). This invariant subset can then be used to check a priori (i.e., before controller implementation) whether closed-loop stability is guaranteed starting from a given initial condition. It is important to note here that the representation of the invariant subset by a Euclidean ball of size δ_s in Theorem 1 is done only for notational convenience; the stability results of the theorem hold for any invariant subset (which can be of arbitrary shape) that can be constructed within Π and contains the origin in its interior. Note that, since the set Π is closed, this requires that the origin lie in the interior of Π as assumed in Theorem 1. This assumption assures the existence of an invariant subset and can be checked prior to the practical implementation of the results of the theorem. However, it can be shown, by means of local Lipschitz arguments, that, if the vector fields are sufficiently smooth on their domains of definition and the adjustable parameters in Eq. (12) chosen properly, this assumption is automatically satisfied.

Remark 2. By inspection of the inequality in Eq. (12), it is clear that the size of this set is governed, as expected, by the size of the constraints and the size of the uncertainty. Therefore, under extremely tight constraints ($u_{\max}$ close to zero) and/or large plant-model mismatch, this set may contain only the origin (particularly for open-loop unstable processes where $L_{\bar{f}}V > 0$). In this case, the origin would be the only admissible initial condition for which closed-loop stability can be guaranteed. In most applications of practical interest, however, the stability region contains the origin in its interior and is not empty (see the chemical reactor examples studied in Sections 4 and 6 for a demonstration of this fact). As discussed in Section 7, another feature of the inequality of Eq. (12) is that it suggests a general way for estimating the stability region, which can be used in conjunction with other approaches (such as optimization-based control approaches) to provide the necessary a priori closed-loop stability guarantees. Although the inequality of Eq. (12) pertains specifically to the bounded robust controller of Eqs. (9)–(11), similar ideas can be used to identify the feasible initial conditions for other control approaches.

Remark 3. Regarding the continuity properties of the controller of Eqs. (9)–(11), it is important to note that this controller is smooth, away from the origin, because the functions $L_{\bar{f}}^{*}V$, $L_{\bar{f}}^{**}V$, $L_{\bar{g}}V$ are smooth on their domains of definition. In addition, the controller is continuous at the origin because the Lyapunov function, $V = e^{\mathrm{T}}Pe$, satisfies the small control property for the system of Eq. (7) (see Sontag (1989b) as well as Freeman and Kokotovic (1996, Chapter 4) for further details and illustrations of this point). These continuity properties can be proven by following the same arguments used in the proof of Theorem 1 in Sontag (1989b) to prove continuity of the nominal (i.e. with $\theta = 0$) unbounded version of the controller (see also the proof of Proposition 3.43 in Sepulchre, Jankovic, and Kokotovic (1997) for a similar proof). The same arguments can be extended to the bounded robust controller of Eqs. (9)–(11).

Remark 4. An important problem, frequently encountered in control of multivariable processes with input constraints, is the problem of directionality. This is a performance deterioration problem that arises when a multivariable controller, designed on the basis of the unconstrained MIMO process, is "clipped" to achieve a feasible plant input. Unlike the SISO case, clipping an unconstrained controller output to achieve a feasible plant input may not lead in MIMO systems to a plant output that is closest to the unconstrained plant output, thus steering the plant in the wrong directions and leading to performance deterioration. Controllers that do not account explicitly for input constraints suffer from this problem and often require the design of directionality compensators to prevent the resulting excessive performance deterioration (e.g., Soroush & Valluri, 1999). It is important to note that the Lyapunov-based control approach proposed in Theorem 1 avoids the directionality problem by taking input constraints directly into account in the controller design. In particular, note that the controller of Eqs. (9) and (10) inherently respects the constraints within the region described by Eq. (12) and, therefore, starting from any initial condition in Ω, there is never a mismatch between the controller output and actual plant input at any time.

Remark 5. The bounded nonlinear controller of Eqs. (9)–(11) possesses certain optimality properties characteristic of its ability to use reasonable control action to accomplish the desired closed-loop objectives. Using techniques from the inverse optimal control approach (e.g., see Freeman & Kokotovic, 1996; Sepulchre et al., 1997; El-Farra & Christofides, 2001a), one can rigorously prove that, within a well-defined subset of the set Π, this controller is optimal with respect to an infinite-time meaningful cost functional of the form

$$J = \int_0^{\infty} (l(e) + u^{\mathrm{T}}R(x)u)\,\mathrm{d}t, \tag{15}$$

where $l(e)$ is a positive definite penalty on the tracking error (and its time derivatives), that is bounded below by a quadratic function of the norm of the tracking error, and $R(x) > 0$ is a positive definite penalty weight on the control action, and thus the above cost functional imposes sensible penalties on both the tracking error and control action. The inverse optimal approach provides a rigorous framework for associating meaningful optimality (i.e., meaningful performance criteria) with certain stabilizing controllers (such as those of Eqs. (9)–(11)) and therefore helps explain the basis for their optimality properties. An advantage of this approach is that it allows the design of optimal feedback controllers without recourse to the unwieldy task of solving the Hamilton–Jacobi–Isaacs equation which is the optimality condition for the robust stabilization problem. Inverse optimal control design is motivated by the fact that both the optimality benefits (e.g., avoiding wasteful cancellation of beneficial nonlinearities) and the robustness properties that the controller possesses as a consequence of optimality (e.g., stability margins) are largely independent of the specific choice of the cost functional as long as it is a meaningful one. Finally, one can easily show that the minimum cost achieved by the state feedback controller is $V(e(0))$.

Remark 6. The inequality of Eq. (12) captures the classical tradeoff between stability and performance. To see this, note that the size of the region of guaranteed closed-loop stability (obtained from Eq. (12)) can be enlarged by using small values for the controller tuning parameters χ and ρ. This enlargement of the stability region, however, comes at the expense of the controller's robust performance, since large values for the tuning parameters χ and ρ are typically required to achieve a significant degree of attenuation of the effect of disturbances and plant-model mismatch on the outputs of the closed-loop system. Therefore, in selecting the controller tuning parameters, one must strike a balance between the need to stabilize the process from a given initial condition and the requirement of achieving a satisfactory level of uncertainty attenuation.

Remark 7. The result of Theorem 1 can be extended to the case when the uncertain variables in Eq. (1) are non-vanishing. In this case, starting from any initial state within the invariant set Ω, the bounded robust controller of Eqs. (9)–(11) enforces boundedness of the process states and robust asymptotic output tracking with an arbitrary degree of attenuation of the effect of uncertainty on the outputs of the closed-loop system. Owing to the persistent nature of the uncertainty, asymptotic convergence to the origin in this case is not possible. Instead, the controller sends the closed-loop trajectory into a small neighborhood of the origin, the size of which (i.e. the asymptotic tracking error) can be made arbitrarily small by selecting the tuning parameters ϕ to be sufficiently small and/or selecting the tuning parameter χ to be sufficiently large. The problem of non-vanishing uncertainty has been addressed in detail in El-Farra and Christofides (2001a) for SISO nonlinear processes. It is worth noting that the use of a Lyapunov-based control approach allows the synthesis of a robust controller that can effectively attenuate the effect of both constant and time-varying persistent uncertainty on the closed-loop outputs, which cannot be achieved using classical uncertainty compensation techniques, including integral action and parameter adaptation in the controller. For nonlinear controller design, Lyapunov methods have provided useful and systematic synthesis tools (e.g., see Kazantzis & Kravaris, 1999).

Remark 8. Referring to the practical implementation of the proposed control approach, we initially verify whether Assumptions 1 and 2 hold for the nonlinear process under consideration, and determine the available bounds on the uncertain variables. Next, given the constraints on the manipulated inputs, the inequality of Eq. (12) is used to compute the estimate of the region of guaranteed closed-loop stability, Ω, and check whether a given initial condition is admissible. Then, the synthesis formula of Eqs. (9)–(11) is used to design the controller which is then implemented on the process. Note that the actual controller synthesis requires only off-line computations, including differentiation and algebraic calculations to compute the various terms in the analytical controller formula of Eqs. (9)–(11). This task can be easily coded into a computer using available software for symbolic manipulations (e.g., MATHEMATICA). Furthermore, the approach provides explicit guidelines for the selection of the tuning parameters that guarantee closed-loop stability and the desired robust performance. For example, the tuning parameters χ and ϕ are responsible for achieving the desired degree of uncertainty attenuation. A large value for χ (greater than one) and a small value for ϕ (close to zero) lead to significant attenuation of the effect of uncertainty on the closed-loop outputs.

Table 1
Process parameters and steady-state values for the reactor of Eq. (16)

$V = 100.0$ l
$R = 8.314$ J/mol K
$C_{A0s} = 1.0$ mol/l
$T_{A0s} = 310.0$ K
$\Delta H = -4.78 \times 10^4$ J/mol
$k_0 = 7.20 \times 10^{10}$ min^{-1}
$E = 8.31 \times 10^4$ J/mol
$c_p = 0.239$ J/g K
$\rho = 1000.0$ g/l
$F = 100.0$ l/min
$C_{As} = 0.577$ mol/l
$T_s = 395.3$ K

4. Application to a multivariable exothermic chemical reactor

Consider a well-mixed continuous stirred tank reactor where an irreversible elementary exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The feed to the reactor consists of pure A at flow rate F, molar concentration C_{A0} and temperature T_{A0}. Under standard modeling assumptions, the process model takes the following form:

$$V\frac{\mathrm{d}C_A}{\mathrm{d}t} = F(C_{A0} - C_A) - k_0 \mathrm{e}^{-(E/RT)} C_A V,$$

$$V\frac{\mathrm{d}T}{\mathrm{d}t} = F(T_{A0} - T) + \frac{(-\Delta H)}{\rho c_p} k_0 \mathrm{e}^{(-E/RT)} C_A V + \frac{Q}{\rho c_p}, \tag{16}$$

where C_A denotes the concentration of species A, T denotes the temperature of the reactor, Q denotes the rate of heat input to the reactor, V denotes the volume of the reactor, k_0, E, ΔH denote the pre-exponential constant, the activation energy, and the enthalpy of the reaction, c_p and ρ, denote the heat capacity and density of the fluid in the reactor. The steady-state values and process parameters are given in Table 1. For these parameters, it was verified that the given equilibrium point is an unstable one (the system also possesses two locally asymptotically stable equilibrium points).

The control objective is to regulate both the reactor temperature and reactant concentration at the (open-loop) unstable equilibrium point by manipulating both the rate of heat input/removal and the inlet reactant concentration. The control objective is to be accomplished in the presence of: (1) time-varying persistent disturbances in the feed stream temperature, (2) parametric uncertainty in value of the heat of reaction, and (3) hard constraints on the manipulated inputs. Defining $x_1 = C_A$, $x_2 = T$, $u_1 = C_{A0} - C_{A0s}$, $u_2 = Q$, $\theta_1(t) = T_{A0} - T_{A0s}$, $\theta_2(t) = \Delta H - \Delta H_{\mathrm{nom}}$, $y_1 = C_A$, $y_2 = T$, where the subscript s denotes the steady-state value and ΔH_{nom} denotes the nominal value of the heat of reaction, the process model of Eq. (16) can be cast in the form of Eq. (1) with

$$f(x)=\begin{bmatrix}\frac{F}{V}(C_{A0s}-C_A)-k_0\mathrm{e}^{-E/RT}C_A\\ \frac{F}{V}(T_{A0s}-T)+\frac{(-\Delta H_{\mathrm{nom}})}{\rho c_p}k_0\mathrm{e}^{-E/RT}C_A\end{bmatrix},$$

$$g_1(x)=\begin{bmatrix}\frac{F}{V}\\ 0\end{bmatrix},\quad g_2(x)=\begin{bmatrix}0\\ \dfrac{1}{\rho c_p V}\end{bmatrix},$$

$$w_1(x)=\begin{bmatrix}0\\ \frac{F}{V}\end{bmatrix},\quad w_2(x)=\begin{bmatrix}0\\ k_0\mathrm{e}^{-E/RT}C_A\end{bmatrix}, \tag{17}$$

$h_1(x)=x_1,\ h_2(x)=x_2$. In all simulation runs, the following time-varying function was considered to simulate the effect of exogenous disturbances in the feed temperature

$$\theta_1(t)=\theta_0\sin(3t), \tag{18}$$

where $\theta_0 = 0.08T_{A0s}$. In addition, a parametric uncertainty of 50% in the heat of reaction was considered, i.e. $\theta_2(t) = 0.5(-\Delta H_{\mathrm{nom}})$. The upper bounds on the uncertain variables were therefore taken to be $\theta_{b1} = 0.08T_{A0s}$, $\theta_{b2} = 0.5|(-\Delta H_{\mathrm{nom}})|$. Also, the following constraints were imposed on the manipulated inputs: $|u_1| \leqslant 1.0$ mol/l, and $|u_2| \leqslant 92$ KJ/s.

A quadratic Lyapunov function of the form $V=(C_A-C_{As})^2+(T-T_s)^2$ was used to design the multivariable bounded robust controller of Eqs. (9)–(11) and to compute the associated region of guaranteed closed-loop stability, with the aid of Eq. (12). Since the uncertainty considered is non-vanishing (see Remark 7), the following values were used for the tuning parameters: $\chi=8.0$, $\phi=0.0001$, and $\rho=0.01$ to guarantee that the outputs of the closed-loop system satisfy a relation of the form $\limsup_{t\to\infty}|y_i-v_i|\leqslant 0.0005$, $i=1,2$.

Several closed-loop simulation runs were performed to evaluate the robustness and constraint-handling capabilities of the multi-variable controller. Fig. 1 depicts the startup reactor temperature and reactant concentration profiles, for an initial condition that lies within the stability region computed from Eq. (12). The solid lines represent the process response when the bounded robust controller is tuned properly and implemented on the process. The dashed lines on the other hand correspond to the response when the controller is designed and implemented without the robust uncertainty compensation component (i.e., $\chi=0$). Finally the dotted lines represent the open-loop response. When compared with the open-loop profiles, it is clear that, starting from the given admissible initial condition, the properly tuned bounded robust controller successfully drives both outputs to the desired steady state, in the presence of input constraints, while simultaneously attenuating the effect of the uncertain variables on the outputs. It is noteworthy that this conclusion was reached a priori (before controller implementation), based on the characterization of the stability region given in Theorem 1. From the dashed profiles we see that the uncertain variables have a significant effect on the process outputs and that failure to explicitly account for them in the controller design leads to instability and poor transient performance. Fig. 2 shows the corresponding profiles for the manipulated inputs. Note that since the initial condition is chosen within the region of guaranteed stability, the well-tuned multivariable controller generates, as expected, control action (solid lines) that respects the constraints imposed.

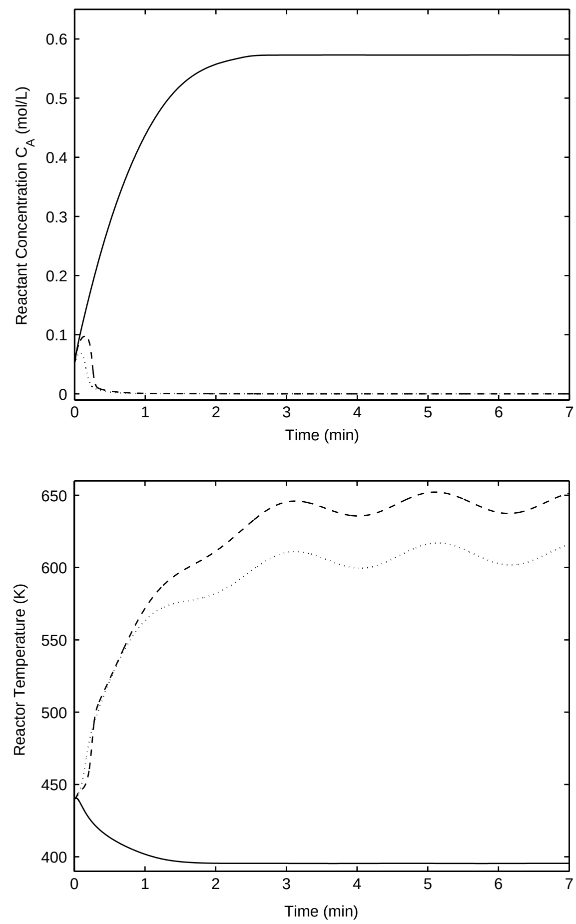

Fig. 1. Controlled outputs: reactant concentration (top) and reactor temperature (bottom) profiles under the bounded multivariable nonlinear state feedback controller with uncertainty compensation and initial condition inside the stability region (solid), without uncertainty compensation (dashed) and under open-loop conditions (dotted).

In Fig. 3, we tested the ability of the controller to robustly stabilize the process at the desired steady state, starting from initial conditions that lie outside the region of guaranteed closed-loop stability. In this case, no a priori guarantees can be made regarding closed-loop stability. In fact, from the process response denoted by the dashed lines in Fig. 3, we see that, starting from the given initial condition, the controller is unable to successfully stabilize the process or attenuate the effect of uncertainty on the process outputs. The reason for this can be seen from the corresponding manipulated inputs' profiles in Fig. 4 (dashed lines) which show

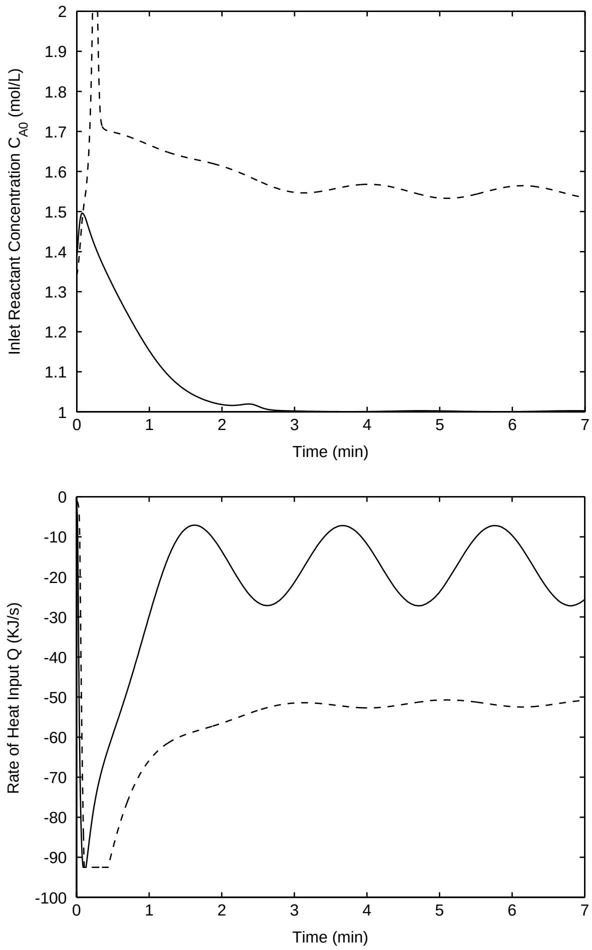

Fig. 2. Manipulated inputs: inlet reactant concentration (top) and rate of heat input (bottom) profiles under the bounded multivariable nonlinear state feedback controller with uncertainty compensation and initial condition inside the stability region (solid), without uncertainty compensation (dashed).

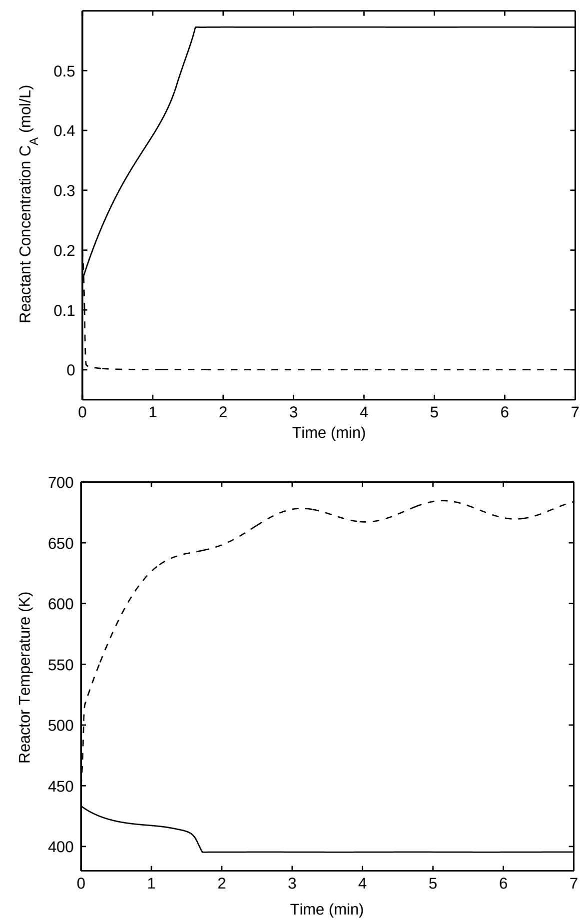

Fig. 3. Controlled outputs: reactant concentration (top) and reactor temperature (bottom) profiles under the bounded robust multivariable nonlinear state feedback controller for initial conditions outside the region described by Eq. (12).

that both the inlet reactant concentration and rate of heat input stay saturated for all times, indicating that stabilizing the process from the given initial condition requires significantly larger control action, which the controller is unable to provide due to the constraints. The combination of insufficient reactant material in the incoming feed stream and insufficient cooling of the reactor prompt an increase in the reaction rate and, subsequently, an increase in the reactor temperature and depletion of the reactant material in the reactor. Note that since the region of guaranteed closed-loop stability given in Theorem 1 does not necessarily capture all the admissible initial conditions (this remains an resolved problem in control), it is possible for the controller to robustly stabilize the process starting from some initial conditions outside this region. An example of this is shown by the controlled outputs and manipulated inputs' profiles denoted by solid lines in Figs. 3 and 4, respectively.

In addition to robust stabilization, we also tested the robust set-point tracking capabilities of the controller under uncertainty and constraints. The results for this case are shown in Figs. 5 and 6 which depict the profiles of the controlled outputs and manipulated inputs, respectively, in response to a 0.4 mol/l increase in the reactant concentration set-point and 40 K decrease in the reactor temperature set-point. It is clear from the figures that the controller successfully achieves the requested tracking while simultaneously attenuating the effect of uncertainty and generating control action that respects the constraints imposed. Finally, the state-space region, where the bounded robust multivariable controller satisfies the input constraints, was computed using Eq. (12) and is depicted in Fig. 7 (top plot). For the sake of comparison, we included in Fig. 7 (bottom plot) the corresponding region for a multivariable input–output linearizing controller that cancels process nonlinearities (the stabilizing linear terms were not included since the associated stabilization cost of these terms would yield an even smaller region). Evolution of the closed-loop trajectories within either region (which is insured by constructing the largest invariant subregion within) guarantees closed-loop stability. It is clear from the comparison that the bounded robust multivariable controller satisfies the constraints for a wider range of process

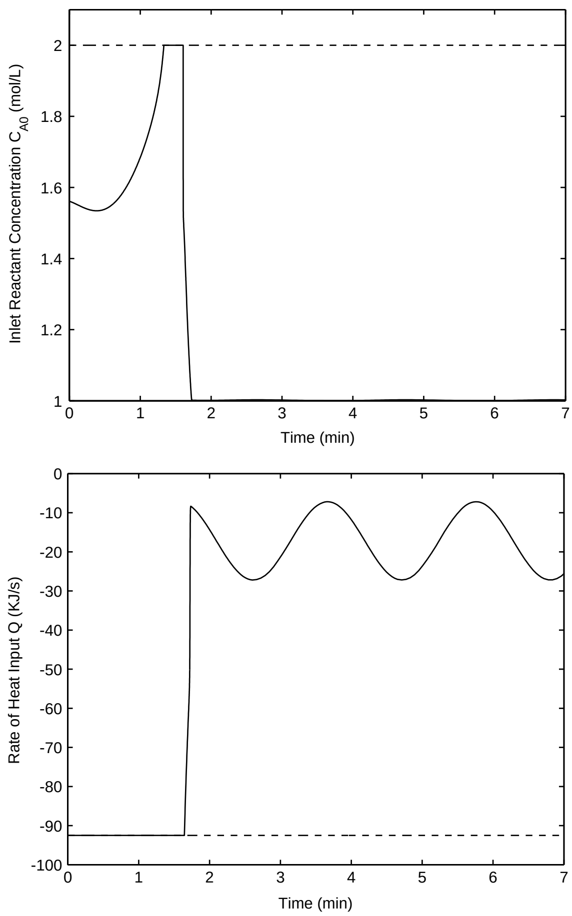

Fig. 4. Manipulated inputs: inlet reactant concentration (top) and rate of heat input (bottom) profiles under the bounded multivariable nonlinear state feedback controller for initial conditions outside the region described by Eq. (12).

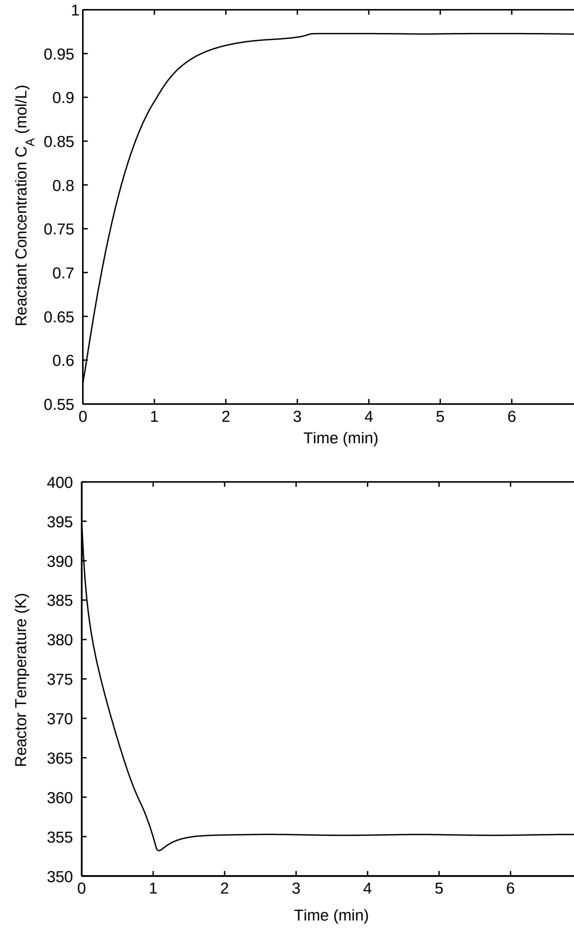

Fig. 5. Controlled outputs: reactant concentration (top) and reactor temperature (bottom) profiles under the bounded robust multivariable nonlinear state feedback controller for a set-point increase of 0.4 mol/l in the reactant concentration and a set-point decrease of 40 K in the reactor temperature.

operating conditions. This is a consequence of the fact that, unlike the input–output linearizing controller, the bounded multivariable controller accounts explicitly for constraints. Furthermore, the cost of cancelling process nonlinearities by the linearizing controller renders many initial conditions, far from the equilibrium point, infeasible. In contrast, the bounded controller avoids the unnecessary cancellation of nonlinearities and employs reasonably smaller control action to stabilize the process. This, in turn, results in a larger set of operating conditions that satisfy the constraints.

5. State estimation and output feedback control

The feedback controller of Eqs. (9)–(11) was designed under the assumption of accessibility of all process states for measurement. In chemical process control, however, more so than in other control areas, the complete state often cannot be measured. For example, the concentration of certain intermediates in chemical reactors may not be accessible for online measurements and therefore cannot be used directly for feedback purposes. In the past few years, significant advances have been made in the direction of output feedback controller design for the purpose of robustly stabilizing nonlinear systems. In this section, we address the problem of synthesizing bounded robust output feedback controllers for the class of constrained uncertain multivariable nonlinear processes in Eq. (1). Owing to the explicit presence of time-varying uncertain variables in the process of Eq. (1), the approach that is followed for output feedback controller design is based on combination of high-gain observers and state feedback controllers (see also Khalil (1994), Teel and Praly (1994), Mahmoud and Khalil (1996), and Christofides (2000) for results on output feedback control for unconstrained nonlinear systems). To this end, we initially formulate the control problem in Section 5.1 and then present its solution in Section 5.2.

5.1. Control problem formulation

Referring to the system of Eq. (1), our objective is to synthesize a bounded robust nonlinear dynamic output feedback

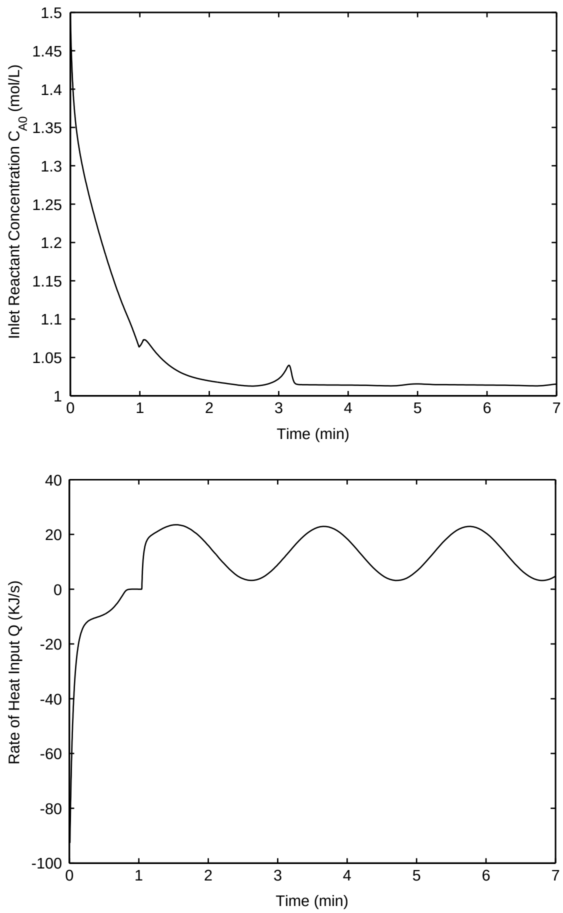

Fig. 6. Manipulated inputs: inlet reactant concentration (top) and rate of heat input (bottom) profiles under the bounded multivariable nonlinear state feedback controller for a set-point increase of 0.4 mol/l in the reactant concentration and a set-point decrease of 40 K in the reactor temperature.

controller of the form:

$$\dot{\omega} = \mathscr{F}(\omega, y, \bar{v}),$$

$$u = \mathscr{P}(\omega, y, \bar{v}), \tag{19}$$

where $\omega \in \mathbb{R}^s$ is an observer state, $\mathscr{F}(\omega, y, \bar{v})$ is a vector function, $\mathscr{P}(\omega, y, \bar{v})$ is a *bounded* vector function, $\bar{v}$ is a generalized reference input, that: (a) enforces, in the presence of actuator constraints, exponential stability and asymptotic robust output tracking with arbitrary degree of attenuation of the effect of the uncertainty on the output, and (b) provides an explicit characterization of the region where the aforementioned properties are guaranteed.

The design of the dynamic controller is carried out using combination of a high-gain state observer and the bounded robust state feedback controller proposed in Section 3. In particular, the system $\dot{\omega} = \mathscr{F}(\omega, y, \bar{v})$ in Eq. (19) is constructed to provide estimates of the process state variables from the measured outputs, while the bounded static component $\mathscr{P}(\omega, y, \bar{v})$ is synthesized to enforce the requested properties in the closed-loop system and at the same time provide the necessary explicit characterization of the region of guaranteed closed-loop stability. The stability analysis of the closed-loop system employs singular perturbation techniques (due to the high-gain nature of the observer) and utilizes the concept of ISS and nonlinear small gain theorem-type arguments.

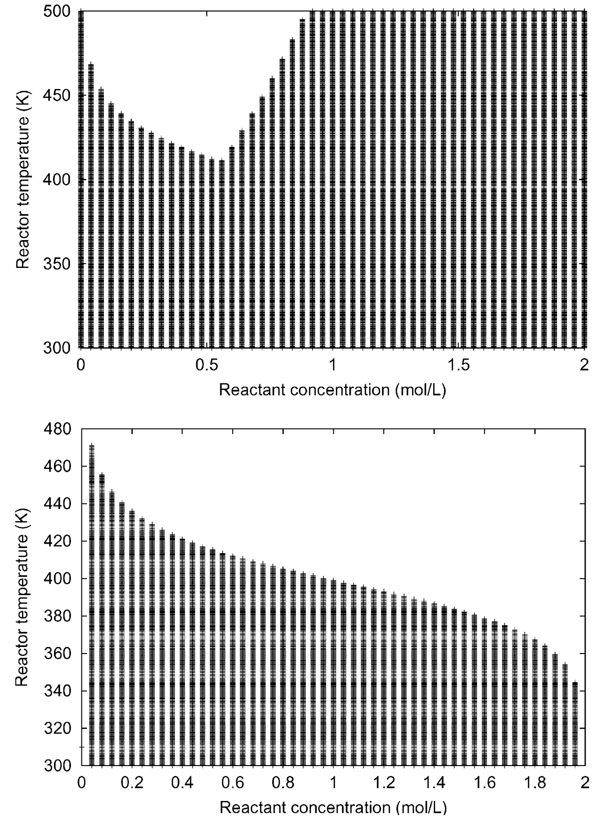

Fig. 7. Comparison between the state space regions where the bounded robust multivariable controller (top) and a multivariable input–output linearizing controller (bottom) satisfy the input constraints. The regions are used to provide estimates of the corresponding stability regions.

5.2. Controller synthesis

In order to proceed with the controller synthesis task, we need to slightly modify Assumption 1 to the following one:

Assumption 4. There exist a set of integers $(r_1, r_2, \ldots, r_m)$ and a coordinate transformation $(\zeta, \eta) = T(x)$ such that the representation of the system of Eq. (1), in the (ζ, η) coordinates takes the form of Eq. (6) with

$$\dot{\eta}_1 = \Psi_1(\zeta, \eta)$$

$$\vdots$$

$$\dot{\eta}_{n-\sum_i r_i} = \Psi_{n-\sum_i r_i}(\zeta, \eta). \tag{20}$$

We note that the change of variables of Eqs. (6) and (20) is independent of θ and is invertible, since, for every x, the variables ζ, η are uniquely determined by the transformation $(\zeta, \eta) = T(x)$. This implies that if we can estimate the values of ζ, η for all times, using appropriate state observers, then we automatically obtain estimates of x for all times. This property will be exploited later to synthesize a state estimator for the system of Eq. (1) on the basis of the system of Eq. (6). We also note that Assumption 1 includes the matching condition of our robust control method. In particular, we consider systems of the form Eq. (1) for which the uncertain variables enter the system in the same equation with the manipulated input. This assumption is motivated by our requirement to eliminate the presence of θ in the η subsystem. This requirement and the stability requirement of Assumption 5 below will allow including in the controller a replica of the η subsystem of Eq. (20) which provide estimates of the η states.

Assumption 5. The system of Eq. (20) is ISS with respect to ζ with $\beta_\eta(|\eta(0)|, t) = K_\eta|\eta(0)|\mathrm{e}^{-at}$ where K_η, and a are positive real numbers and $K_\eta \geqslant 1$.

Assumption 5 is satisfied by many chemical processes of practical interest (e.g., see El-Farra and Christofides (2001a) and the chemical reactor example in Section 6). We are now ready to state the main result of this section. Theorem 2 that follows provides the explicit synthesis formula for the desired bounded robust output feedback controller and states precise conditions that guarantee closed-loop stability and robust asymptotic output tracking in the presence of uncertainty and input constraints. The proof of the theorem can be found in the appendix.

Theorem 2. *Consider the constrained uncertain nonlinear process of Eq.* (1), *for which Assumptions* 3–5 *hold*, *under the output feedback controller*

$$\dot{\tilde{y}}^{(i)} = A\tilde{y}^{(i)} + F_i(y_i - y_0^{(i)}),$$

$$\dot{\omega}_1 = \Psi_1(\mathrm{sat}(\tilde{y}), \omega),$$

$$\vdots$$

$$\dot{\omega}_{n-\sum r_i} = \Psi_{n-\sum r_i}(\mathrm{sat}(\tilde{y}), \omega),$$

$$u = -\tfrac{1}{2}R^{-1}(\hat{x})(\widehat{L_{\bar{G}}V})^{\mathrm{T}}, \tag{21}$$

where $\hat{x} = T^{-1}(\mathrm{sat}(\tilde{y}), \omega)$, $\tilde{y} = [\tilde{y}^{(1)^{\mathrm{T}}} \ \cdots \ \tilde{y}^{(m)^{\mathrm{T}}}]^{\mathrm{T}}$, $\omega = [\omega_1 \ \cdots \ \omega_{n-\sum_i r_i}]^{\mathrm{T}}$, $F_i = [L_i a_1^{(i)} \ L_i^2 a_2^{(i)} \ \cdots \ L_i^{r_i} a_{r_i}^{(i)}]^{\mathrm{T}}$, $i = 1, \ldots, m$. *Let* $\bar{\varepsilon} = \max\{1/L_i\}$. *Then for each pair of positive real numbers* (δ_b, d) *such that* $\beta(\delta_b, 0) + d \leqslant \delta_s$, *where* $\beta(\cdot,\cdot)$ *and* δ_s *were defined in Theorem* 1, *and for each pair of positive real numbers* $(\delta_\theta, \delta_{\bar{v}})$, *there exists* $\phi^* > 0$, *and for each* $\phi \in (0, \phi^*]$, *there exists* $\bar{\varepsilon}^*(\phi) > 0$, *such that if* $\phi \in (0, \phi^*]$, $\bar{\varepsilon} \in (0, \bar{\varepsilon}^*(\phi)]$, $\mathrm{sat}(\cdot) = \min\{1, \zeta_{\max}/|\cdot|\}(\cdot)$ *with* $\zeta_{\max}$ *being the maximum value of the vector* ζ *for* $|\zeta| \leqslant \beta_\zeta(\delta_\zeta, 0)$ *where* β_ζ *is a class KL function and* δ_ζ *is the maximum value of the vector* $[l_1^{\mathrm{T}}(x) \ l_2^{\mathrm{T}}(x) \ \cdots \ l_m^{\mathrm{T}}(x)]^{\mathrm{T}}$ *for* $|x| \leqslant \delta_b$, *where* $l_i(x) = [h_i(x) \ L_f h_i(x) \ \cdots \ L_f^{r_i-1}h_i(x)]^{\mathrm{T}}$, $|x(0)| \leqslant \delta_b$, $\|\theta\| \leqslant \delta_\theta$, $\|\bar{v}\| \leqslant \delta_{\bar{v}}$, $|\tilde{y}(0)| \leqslant \delta_\zeta$, $\omega(0) = \eta(0) + O(\varepsilon)$, *the following holds in the presence of constraints*:

(1) *The closed-loop system is asymptotically* (*and locally exponentially*) *stable*.

(2) *The outputs of the closed-loop system satisfy a relation of the form*

$$\limsup_{t\to\infty} |y_i(t) - v_i(t)| = 0, \quad i = 1, \ldots, m. \tag{22}$$

Remark 9. The robust output feedback controller of Eq. (21) consists of m high-gain observers, each of which provides estimates of the derivatives of one of the m controlled outputs, y_i, up to order $r_i - 1$, and thus estimates of the variables $\zeta_1^{(i)}, \ldots, \zeta_{r_i}^{(i)}$, an observer that simulates the inverse dynamics of the system of Eq. (20), and a bounded robust static feedback controller (see Theorem 1) that uses measurements of the outputs and estimates of the states to attenuate the effect of the uncertain variables on the process outputs and enforce reference input tracking. The use of high-gain observers allows us to achieve a certain degree of separation in the output feedback controller design, where the observer design is carried out independently of the state feedback design. This is possible because of the disturbance rejection properties of high-gain observers that allow asymptotic recovery of the performance achieved under state feedback, where "asymptotic" here refers to the behavior of the system as the poles of the observer approach infinity. It is important to note here that while such "separation principle" holds for the class of feedback linearizable nonlinear systems considered in this paper, one should not expect the task of observer design to be independent from the state feedback design for more general nonlinear systems. In fact, even in linear control design, when model uncertainties are taken into consideration, the design of the observer cannot be separated from the design of the state feedback control.

Remark 10. In designing the output feedback controller of Eq. (21), we use a standard saturation function, sat, to eliminate the peaking phenomenon typically exhibited by high-gain observers in their transient behavior. The origin of this phenomenon owes to the fact that as the observer poles approach infinity, its exponential modes will decay to zero arbitrarily fast, but the amplitude of these modes will approach infinity, thus producing impulsive-like behavior. To eliminate observer peaking, we use the saturation filter in conjunction with the high-gain observers, to eliminate (or filter out) wrong estimates of the process output derivatives provided by the observer for short times. The idea here is to exploit our knowledge of an estimate of the stability region obtained under state feedback (Ω), where the process states evolve, to derive bounds on the actual values of the outputs' derivatives, and then use these bounds to design the saturation filter. Therefore, during the short transient period when the estimates of the high-gain observers exhibit peaking, the saturation filter eliminates those estimates which exceed the state feedback bounds, thus preventing peaking from being transmitted to the plant. Over the same period, the estimation error decays to small values, while the state of the plant remains close to its initial value. The validity of this idea is justified via asymptotic analysis from singular perturbation theory (see the proof of Theorem 2 in the appendix).

Remark 11. An important consequence of the combined use of high-gain observers and saturation filters is that the region of closed-loop stability obtained under state feedback remains practically preserved under output feedback. Specifically, starting from any compact subset of initial conditions (whose size is fixed by δ_b) within the state feedback region, there always exist observer gains, such that the dynamic *output* feedback controller of Eq. (21) continues to enforce asymptotic stability and reference-input tracking in the constrained uncertain closed-loop system, when the observer gains are selected to be sufficiently large. Instrumental in deriving this result is the use of the saturation filter, which allows us to use arbitrarily large observer gains, in order to recover the state feedback region, without suffering the detrimental effects of observer peaking. Note that the size of the output feedback region (δ_b) can be made close to that of the state feedback region (δ_s) by selecting d to be sufficiently small which, in turn, can be done by making $\bar{\varepsilon}$ sufficiently small. Therefore, although combination of the bounded state feedback controller with the observer results in some loss (given by d) in the size of the region of guaranteed closed-loop stability, this loss can be made small by selecting $\bar{\varepsilon}$ to be sufficiently small. As expected, the nature of this result is consistent with the semi-global result obtained in El-Farra and Christofides (2001b) for the unconstrained case.

Remark 12. In addition to preserving the stability region, the controller–observer combination of Eq. (21) practically preserves the optimality properties of the state feedback controller explained in Remark 4. The output feedback controller design is near-optimal in the sense that the cost incurred by implementing this controller tends to the optimal cost achieved by implementing the bounded state feedback controller when the observer gains are taken to be sufficiently large. Using standard singular perturbation arguments, one can show that cost associated with the output feedback controller is $O(\bar{\varepsilon})$ close to the optimal cost associated with the state feedback controller (i.e. $J_{\min} = V(e(0)) + O(\bar{\varepsilon})$). The basic reason for near-optimality is the fact that by choosing $\bar{\varepsilon}$ to be sufficiently small, the observer states can be made to converge quickly to the process states. This fact can be exploited to make the performance of the output feedback controller arbitrarily close to that of the optimal state feedback controller (see El-Farra and Christofides (2001b) for an analogous result for the unconstrained case).

Remark 13. Owing to the presence of the fast (high-gain) observer in the dynamical system of Eq. (21), the closed-loop system can be cast as a singularly perturbed system, where $\bar{\varepsilon} = \max\{1/L_i\}$ is the singular perturbation parameter. Within this system, the states of the high-gain observers, which provide estimates of the outputs and their derivatives, constitute the fast states, while the ω states of the observer and the states of the original system of Eq. (1) under state feedback represent the slow states. Owing to the dependence of the controller on both the slow and fast states, the control action computed by the static component in Eq. (21) is not $O(\bar{\varepsilon})$ close to that computed by the state feedback controller for all times. After the decay of the boundary layer term (fast transients of the high-gain observers), however, the static component in Eq. (21) approximates the state feedback controller to within $O(\bar{\varepsilon})$.

Remark 14. It is important to note that the asymptotic stability results of Theorem 1 (Theorem 2) are regional (semi-regional) and non-local. By definition, a local result is one that is valid provided that the initial conditions are sufficiently small. In this regard, neither the result of Theorem 1 nor that of Theorem 2 requires the initial conditions to be sufficiently small in order for stability to be guaranteed. The apparent limitations imposed on the size of the initial conditions that guarantee closed-loop stability follow from the fundamental limitations imposed by the input constraints and uncertainty, as can be seen from Eq. (12). In the absence of constraints, for example, one can establish asymptotic stability globally for the state feedback problem and semi-globally for the output feedback case. In fact, one of the valuable features of the results in Theorem 1 is that they provide an explicit procedure for constructing reasonably large estimates of the stability region, that depend only on the magnitude of the constraints and size of the uncertainty, which therefore allows one to start from initial conditions farther away (from the equilibrium point) than would be possible using existing approaches where no explicit characterization of the stability region is available and consequently one often has to restrict the initial conditions to be sufficiently close to the origin. In this light, our results allow control designers to break away from local and unnecessarily conservative results available in the literature.

6. Robust stabilization of a chemical reactor via output feedback control

To illustrate an application of the output feedback controller design presented in the previous section, we consider in this section a well-mixed continuous stirred tank reactor where three parallel irreversible elementary exothermic reactions of the form $A \xrightarrow{k_1} D$, $A \xrightarrow{k_2} U$ and $A \xrightarrow{k_3} R$ take place, where A is the reactant species, D is the desired product and U, R are undesired byproducts. The feed to the reactor consists of pure A at flow rate F, molar concentration C_{A0}

Table 2
Process parameters and steady-state values for the reactor of Eq. (23)

$V = 1000.0$ l
$R = 8.314$ J/mol K
$C_{A0s} = 4.0$ mol/l
$T_{A0s} = 300.0$ K
$\Delta H_1 = -5.0 \times 10^4$ J/mol
$\Delta H_2 = -5.2 \times 10^4$ J/mol
$\Delta H_3 = -5.4 \times 10^4$ J/mol
$k_{10} = 5.0 \times 10^4$ min^{-1}
$k_{20} = 5.0 \times 10^3$ min^{-1}
$k_{30} = 5.0 \times 10^3$ min^{-1}
$E_1 = 5.0 \times 10^4$ J/mol
$E_2 = 7.53 \times 10^4$ J/mol
$E_3 = 7.53 \times 10^4$ J/mol
$c_p = 0.231$ J/g K
$\rho = 1000.0$ g/l
$F = 83.3$ l/min
$T_s = 390.97$ K
$C_{As} = 3.58$ mol/l
$C_{Ds} = 0.42$ mol/l

and temperature T_{A0}. Due to the non-isothermal nature of the reactions, a jacket is used to remove/provide heat to the reactor. Under standard modeling assumptions, a mathematical model of the process can be derived from material and energy balances and takes the following form:

$$V\frac{\mathrm{d}T}{\mathrm{d}t} = F(T_{A0}-T)+\sum_{i=1}^{3}\frac{(-\Delta H_i)}{\rho c_p}k_{i0}\mathrm{e}^{(-E_i/RT)}C_A V+\frac{Q}{\rho c_p},$$

$$V\frac{\mathrm{d}C_A}{\mathrm{d}t} = F(C_{A0}-C_A)-\sum_{i=1}^{3}k_{i0}\mathrm{e}^{(-E_i/RT)}C_A V,$$

$$V\frac{\mathrm{d}C_D}{\mathrm{d}t} = -FC_D+k_{10}\mathrm{e}^{(-E_1/RT)}C_A V, \tag{23}$$

where C_A and C_D denote the concentrations of the species A and D, T denotes the temperature of the reactor, Q denotes rate of heat input/removal from the reactor, V denotes the volume of the reactor, ΔH_i, k_i, E_i, $i=1,2,3$, denote the enthalpies, pre-exponential constants and activation energies of the three reactions, respectively, c_p and ρ denote the heat capacity and density of the reactor. The values of the process parameters and the corresponding steady-state values are given in Table 2. It was verified that these conditions correspond to an unstable equilibrium point of the process of Eq. (23).

The control problem is formulated as the one of regulating both the concentration of the desired product C_D and the reactor temperature T at the unstable steady-state by manipulating the inlet reactant concentration C_{A0} and rate of heat input Q provided by the jacket. The control objective is to be accomplished in the presence of: (1) exogenous time-varying disturbances in the feed stream temperature, (2) parametric uncertainty in the enthalpy of the three reaction, and (3) hard constraints on the manipulated inputs. Defining $x_1=T$, $x_2=C_A$, $x_3=C_D$, $u_1=Q$, $u_2=C_{A0}-C_{A0s}$, $\theta_i=\Delta H_i-\Delta H_{i0}$, $i=1,2,3$, $\theta_4=T_{A0}-T_{A0s}$, $y_1=x_1$, $y_2=x_3$, where the subscript s denotes the steady-state values and ΔH_{i0} are the nominal values for the enthalpies, the process model of Eq. (23) can be written in the form of Eq. (1) with

$$f(x) = \begin{bmatrix} \frac{F}{V}(T_{A0s}-T)+\sum_{i=1}^{3}\frac{(-\Delta H_{i0})}{\rho c_p}k_{i0}\mathrm{e}^{(-E_i/RT)}C_A \\ \frac{F}{V}(C_{A0s}-C_A)-\sum_{i=1}^{3}k_{i0}\mathrm{e}^{(-E_i/RT)}C_A \\ -\frac{F}{V}C_D+k_{10}\mathrm{e}^{(-E_1/RT)}C_A \end{bmatrix},$$

$$g_1(x) = \begin{bmatrix} \frac{1}{\rho c_p V} \\ 0 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ \frac{F}{V} \\ 0 \end{bmatrix},$$

$$w_i(x) = \begin{bmatrix} k_{i0}\mathrm{e}^{(-E_i/RT)}C_A \\ 0 \\ 0 \end{bmatrix}, \quad i=1,2,3, \quad w_4(x) = \begin{bmatrix} \frac{F}{V} \\ 0 \\ 0 \end{bmatrix},$$

$h_1(x)=x_1$, and $h_2(x)=x_3$. To simulate the effect of uncertainty on the process outputs, we consider a time varying function of the form of Eq. (18), with $\theta_0=0.03T_{A0s}$, to simulate the effect of external disturbances in the feed temperature. We also consider a parametric uncertainty of 50% in the values of the enthalpies. Therefore, the bounds on the uncertain variables are taken to be $\theta_{bk}=0.5|(-\Delta H_{k0})|$, $k=1,2,3$, $\theta_{b4}=0.03T_{A0s}$. Also, the following constraints are imposed on the manipulated inputs: $|u_1| \leqslant 25$ KJ/s and $|u_2| \leqslant 4.0$ mol/l.

For this process, the relative degrees of the process outputs, with respect to the vector of manipulated inputs, are $r_1=1$, and $r_2=2$, respectively. Using Eq. (3), it can be verified that the decoupling matrix $C(x)$ is non-singular and that the assumptions of Theorem 2 are satisfied. In order to proceed with controller synthesis, we initially use the following coordinate transformation (in error variables form):

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \zeta_1^{(1)}-v_1 \\ \zeta_1^{(2)}-v_2 \\ \zeta_2^{(2)}-v_2^{(1)} \end{bmatrix} = \begin{bmatrix} h_1(x)-v_1 \\ h_2(x)-v_2 \\ L_f h_2(x) \end{bmatrix} = \begin{bmatrix} x_1-v_1 \\ x_3-v_2 \\ -\frac{F}{V}C_D+k_{30}\mathrm{e}^{(-E_1/RT)}C_A \end{bmatrix} \tag{24}$$

to cast the process model in its input–output form of Eq. (7), which is used directly for output feedback controller design. The necessary controller, whose practical implementation requires measurements of T and C_D only, consists of a combination of two high-gain observers (equipped with appropriate saturation filters) that provide estimates of the process states and a bounded robust static feedback component. The static component is designed, with the aid of the explicit synthesis formula in Eqs. (9)–(11), using a quadratic Lyapunov function of the form $V = e^{\mathrm{T}}Pe$, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 1 \\ 0 & 1 & \sqrt{3} \end{bmatrix}, \tag{25}$$

which is also used to compute an estimate of the region of guaranteed closed-loop stability, using Eq. (12). The high-gain observers are designed using Eq. (21) and consist of a replica of the linear part of the input–output dynamics of Eq. (7) plus a linear gain multiplying the discrepancy between the actual and the estimated values of the outputs. The output feedback controller takes the following form:

$$\begin{aligned} \dot{\tilde{y}}_1^{(1)} &= L_1 a_1^{(1)}(y_1 - \tilde{y}_1^{(1)}), \\ \dot{\tilde{y}}_1^{(2)} &= \tilde{y}_2^{(2)} + L_2 a_1^{(2)}(y_2 - \tilde{y}_1^{(2)}), \\ \dot{\tilde{y}}_2^{(2)} &= L_2^2 a_2^{(2)}(y_2 - \tilde{y}_1^{(2)}), \\ u &= -\tfrac{1}{2}R^{-1}(\mathrm{sat}(\tilde{y}))(\widehat{L_{\bar{g}}V})^{\mathrm{T}}, \end{aligned} \tag{26}$$

where $\tilde{y}_1^{(1)}$, $\tilde{y}_1^{(2)}$, $\tilde{y}_2^{(2)}$ are the estimates of the reactor temperature, reactant concentration, and desired product concentration, respectively, and the saturation filter is defined as

$$\mathrm{sat}(\tilde{y}_i^{(j)}) = \left\{ \begin{array}{ll} a_{m,i}^{(j)}, & \tilde{y}_i^{(j)} > a_m^{(j)} \\ \tilde{y}_i^{(j)}, & -a_{m,i}^{(j)} \leqslant \tilde{y}_i^{(j)} \leqslant a_{m,i}^{(j)}, \\ -a_{m,i}^{(j)}, & \tilde{y}_i^{(j)} \leqslant -\, a_{m,i}^{(j)} \end{array} \right\}. \tag{27}$$

The following values were used for the controller and observer parameters: $\chi = 3.5$, $\phi = 0.0001$, $\rho = 0.001$, $L_1 = 100$, $a_1^{(1)} = 10$, $L_2 = 400$, $a_1^{(2)} = 40$, and $a_2^{(2)} = 400$, $a_{m,1}^{(1)} = 395$, $a_{m,1}^{(2)} = 4$ and $a_{m,2}^{(2)} = 0.5$ to ensure that the process outputs satisfy a relation of the form $\limsup_{t\to\infty} |y_i - v_i| \leqslant 0.0005$, $i = 1, 2$.

Closed-loop simulations were performed to evaluate the robust stabilization capabilities of the output feedback controller, starting from an initial condition inside the stability region, and compare its performance with that of the state feedback controller and the open-loop response. Figs. 8 and 9 depict the controlled outputs (desired product concentration and reactor temperature) and manipulated inputs (inlet reactant concentration and rate of heat input) profiles, respectively, under output feedback control (solid lines), under state feedback control (dashed lines) assuming all process states are available for measurement, and under no control (dotted lines). It is clear, from the comparison with the open-loop profiles, that the output feedback controller successfully drives both controlled outputs to the desired steady state while simultaneously attenuating the effect of disturbances and model uncertainty on the process outputs and generating control action that respects the constraints imposed. Note that the controlled outputs and manipulated inputs' profiles obtained under the output feedback controller (solid lines) are very close to the profiles obtained under the state feedback controller (dashed lines). This closeness, starting from the same initial condition, illustrates two important features of the output feedback design. The first is the fact that, by selecting the observer gains to be sufficiently large, the performance of the output feedback controller approaches (or recovers) that of the state feedback controller. Since the process response under state feedback control can be shown to be optimal (in the inverse sense) with respect to a meaningful cost (see Remark 5), the performance of the output feedback

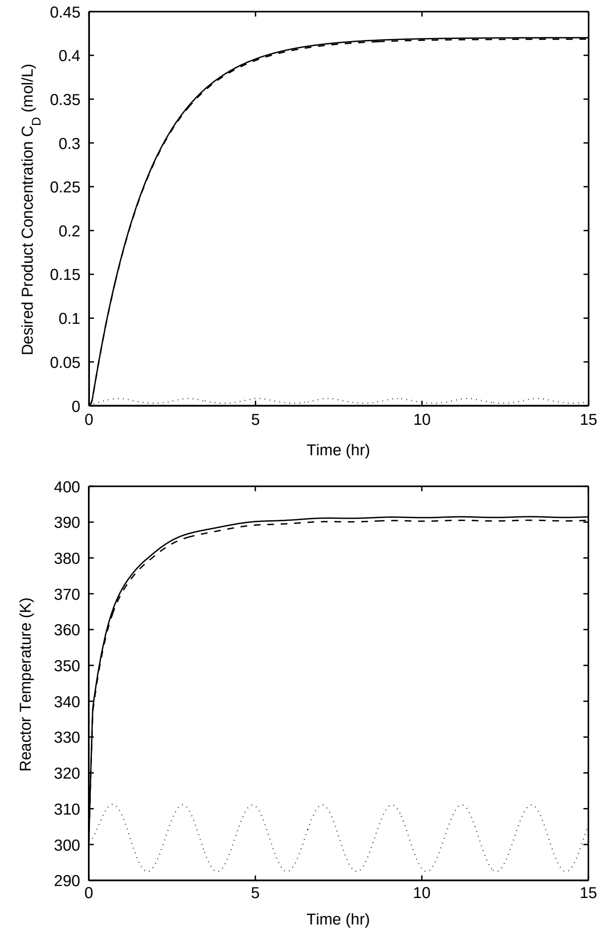

Fig. 8. Controlled outputs: desired product concentration (top) and reactor temperature (bottom) profiles under the bounded robust multivariable output feedback controller (solid), under the corresponding state feedback controller (dashed), for initial condition within the region of guaranteed stability, and under open-loop conditions (dotted).

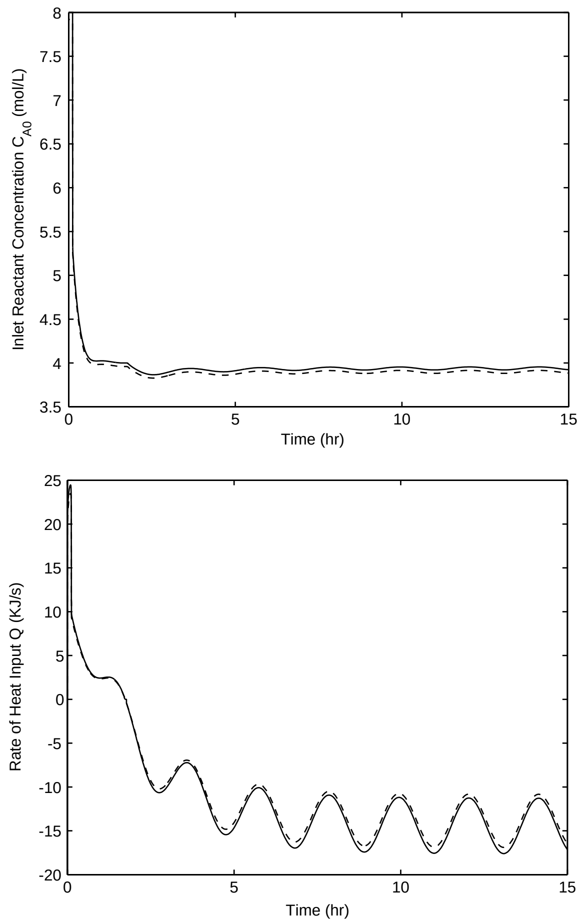

Fig. 9. Manipulated inputs: inlet reactant concentration (top) and rate of heat input (bottom) profiles under the bounded robust multivariable output feedback controller (solid), and under the corresponding state feedback controller (dashed), for an initial condition within the region of guaranteed stability.

controller then is near-optimal (with respect to the same cost). The second feature is that the set of admissible initial conditions, starting from where stability of the constrained closed-loop system is guaranteed under state feedback, remains practically preserved when the observer gain is sufficiently large.

7. Connections with MPC

The purpose of this section is to put the bounded robust Lyapunov-based control approach, proposed in this paper, in perspective with respect to the nonlinear model predictive control approach, for control of uncertain nonlinear processes with input constraints. Currently, in the Chemical Process Industry, MPC is the most established advanced control technique, whose success in several commercial applications is well-documented in the literature. The popularity of this approach stems largely from its ability to handle, among other issues, multivariable interactions, constraints, and optimization requirements, all in a consistent, systematic manner. This success notwithstanding, a number of important theoretical and practical problems, particularly within the nonlinear context, continue to be active areas of research. Our objective here is to highlight some of the implications of the proposed Lyapunov-based control approach for dealing with some of these problems and assisting the capabilities of the MPC approach. In particular, we illustrate how the same conceptual tools, put forth by the Lyapunov-based approach, can be used to address the issue of providing a priori closed-loop stability guarantees for the MPC algorithm. Towards this end, we will initially highlight, briefly, how the control problem is formulated within the MPC framework and what are the main tools required to solve it, for constrained uncertain nonlinear processes. For a more in-depth discussion of the theoretical and practical issues within model predictive control, the reader is referred to the following extensive reviews (Morari & Lee, 1991; Lee & Cooley, 1997; Mayne, 1997; Rawlings, 1999) and the references therein.

To simplify our discussion, let us consider first the nominal case where no modeling errors or disturbances are present ($\theta_k \equiv 0$ in Eq. (1)). Also since the MPC algorithm is typically implemented with digital computers, we will use the discrete version of Eq. (1)

$$x_{k+1} = f(x_k) + g(x_k)u_k. \tag{28}$$

For this case, MPC is conventionally formulated as solving on-line a finite horizon optimal control problem of the form

$$\min_{\mathscr{U}_k} \quad \sum_{i=1}^{p} x_{k+i|k}^{\mathrm{T}} Q x_{k+i|k} + \sum_{i=0}^{m-1} u_{k+i|k}^{\mathrm{T}} R u_{k+i|k}, \tag{29}$$

$$x_{k+i|k} \in \mathscr{X}, \quad i = 1, \ldots, p, \tag{30}$$

$$u_{k+i|k} \in \mathscr{V}, \quad i = 0, \ldots, m-1, \tag{31}$$

where $x_{k+i|k}$ stands for the prediction of x_{k+i} based on $x_{k|k}$, and the vector $\mathscr{U}_k = [u_{k|k}^{\mathrm{T}}, \cdots, u_{k+m-1|k}^{\mathrm{T}}]^{\mathrm{T}}$ represents the input trajectory to be computed. Q and R are positive definite matrices, p is the prediction horizon, m is the number of control moves, and Eqs. (30) and (31) represent the hard constraints on the future states and manipulated inputs. At each kth sample time, information is received about the current state of the process and based on this information together with the process model, the dependence of future states on the future manipulated inputs can be predicted. The future input trajectory is then determined according to the given optimality criterion and implemented until the next sample time, when the entire process is repeated based on new information. What results therefore is a discrete feedback control law $u_k = \phi(x_{k|k})$ that is given *implicitly* through the optimization problem, which must be solved at each time step. Note that since the future input trajectory is computed solely on the basis of the current information, MPC is an open-loop optimal control policy. The open-loop formulation allows for significant reduction in the complexity of the problem, by avoiding the intractable dynamic programming problem arising from optimal feedback control, but can also lead to a worse performance than what is achievable with feedback control (Lee & Cooley, 1997). A conceptual parallel to be noted here is that the Lyapunov-based control approach proposed in this work also avoids the intractable Hamilton–Jacobi–Bellman equation (HJB) (or the HJI equation in the case with uncertainty) which is the optimality condition for the stabilization problem, though through inverse, rather than open-loop, optimal control. What results in this case therefore is an explicit feedback law (see Theorem 1) that is optimal with respect to a meaningful (though not specified a priori) cost functional. Note also that, in both approaches, performance deterioration problems, such as process directionality in constrained multivariable processes, do not arise since the constraints are explicitly accounted for in the controller design.

When the process of Eq. (28) is linear, the cost quadratic, the sets $\mathscr{X}$ and $\mathscr{V}$ convex, the MPC optimization problem reduces to a quadratic program for which efficient software exists and, consequently, a number of control-relevant issues have been explored, including issues of closed-loop stability, performance, implementability, and constraint satisfaction. However, when the system is non-linear, the optimization problem will be, in general, non-convex, even if the cost functional and constraints are convex, and therefore a host of difficulties impacting on the implementability of the model predictive controller, ensue (see Mayne (1997) for a detailed discussion of these issues). In addition to the computational difficulties of solving a non-linear optimization problem at each time step, the implementation of MPC in this case comes without any well-established closed-loop properties, even nominal ones (Rawlings, 1999). Unless a non-linear MPC controller is exhaustively tested by simulation over the whole range of potential initial states, doubt will always remain as to whether or not a state will be encountered for which an acceptable solution to the finite horizon problem can be found. Note, in contrast, that the bounded nonlinear feedback controller of Eqs. (9)–(11), comes with well-characterized closed-loop stability and performance properties (both nominal and in the presence of uncertainty) through the inequality of Eq. (12), which is suited for the practical computation of the set of feasible initial states starting from where the constrained closed-loop system is guaranteed to be stable (see also Remark 1). We will return to this point shortly in our discussion to illustrate how similar ideas for characterizing the stability region can be used to provide a priori stability guarantees for the MPC approach.

Let us now turn our attention to the problem of controlling nonlinear systems with both model uncertainty and input constraints. Addressing the problem within an MPC framework in this case usually leads to a min–max formulation. For example, consider the case of time-varying uncertainty and let us assume that θ_k is a time sequence and $\theta_k \in \Theta\ \forall k$ where Θ is a compact set. The following min–max MPC formulation can be derived from game theory (Lee & Cooley, 1997):

$$\min_{\mathscr{U}_k} \max_{\theta_{k|k},\ldots,\theta_{k+p-1|k}} \sum_{i=1}^{p} x_{k+i|k}^{\mathrm{T}} Q x_{k+i|k} + \sum_{i=0}^{m-1} u_{k+i|k}^{\mathrm{T}} R u_{k+i|k}, \quad (32)$$

$$x_{k+i|k} \in \mathscr{X}, \quad i = 1, \ldots, p, \quad (33)$$

$$u_{k+i|k} \in \mathscr{V}, \quad i = 0, \ldots, m-1, \quad (34)$$

$$\theta_{k+i|k} \in \Theta, \quad i = 0, \ldots, p-1. \quad (35)$$

Solution to the above program computes an open-loop input trajectory that minimizes the "worst-case" cost for the chosen prediction horizon. It is to be noted that the state constraint of Eq. (33) must be satisfied for all possible sequences of $\theta_{k+i|k}$. The above optimization, which is non-convex in general (even for linear systems), must be solved on-line at each time step. In addition to the concomitant computational difficulties associated with this optimization, the above approach does not provide any explicit guarantees for closed-loop stability. Note that although the Lyapunov-based control approach proposed in Theorem 1 accounts explicitly for nonlinearities, uncertainty, and input constraints, it does not account explicitly for state constraints. To deal with this problem, one may formulate the constrained states as additional outputs to be controlled. Obviously, this may require the use of more control inputs and is therefore restricted to cases where the number of state constraints is not too large. It is to be noted also that the MPC formulation allows for incorporating state constraints directly as part of the constrained optimization problem (see Eq. (33)), though an actual solution to the nonlinear min–max optimization problem is, in general, difficult to obtain.

In addition to providing a solution to the control problem for uncertain nonlinear systems with input constraints, the Lyapunov-based approach also suggests a systematic way of obtaining closed-loop stability guarantees for the MPC algorithm. The idea here is to incorporate the Lyapunov stability condition of Eq. (14) (using a pre-determined control Lyapunov function as is done, for example, in contractive MPC (Yang & Polak, 1993) and CLF-based Receding Horizon Control (Primbs, Nevistic, & Doyle, 2000; Rawls & Freeman, 2000) as an additional inequality constraint on the optimization problem. This condition represents a constraint on the control input and applies only to the initial control action, and not the entire manipulated input trajectory computed, since it is only the first control move in that sequence that is actually implemented. Satisfaction of this constraint guarantees closed-loop stability. Then, using techniques similar to those used in deriving Eq. (12), one can obtain an estimate of the closed-loop stability region associated with the MPC controller. For example, when $\theta_k \equiv 0$, it is not difficult to see that a condition of the form

$$L_f V \leqslant -u_{\max}|(L_g V)^{\mathrm{T}}| \quad (36)$$

is sufficient to guarantee that the Lyapunov function decreases monotonically along the trajectories of the closed-loop system (i.e., $\dot{V} < 0$) regardless of the control input, so long as it satisfies the constraints (i.e., for any $|u| \leqslant u_{\max}$). Therefore, if the set described by Eq. (36) contains the origin in its interior, one can obtain a set of feasible initial conditions (by computing the largest invariant subset within) starting from where the constrained finite horizon optimization problem is guaranteed to have a solution for every point in the set. Note that the condition derived in Eq. (36) is more conservative than the one in Eq. (12). This owes to the fact that the estimate of Eq. (12) applies specifically to the bounded robust controller of Eqs. (9)–(11) while the estimate of Eq. (36) is independent of the control law. A similar expression to that of Eq. (36) can be derived to estimate the stability region and provide stability guarantees when uncertainty is present. More research work is needed in this direction to establish, rigorously, these possible links with model predictive control (see El-Farra, Mhasker, & Christofides (2002) for recent results in this direction) and address computational issues that arise in the practical implementation of nonlinear MPC.

8. Conclusions

A Lyapunov-based nonlinear controller design methodology, for MIMO nonlinear processes with uncertain dynamics, actuator constraints, and incomplete state measurements, was developed. Under the assumption that all process states are accessible for measurement, the approach led to the explicit synthesis of bounded robust multivariable nonlinear feedback controllers that enforce stability and robust asymptotic reference-input tracking in the constrained uncertain closed-loop system and provide, at the same time, an explicit characterization of the region of guaranteed closed-loop stability. When complete state measurements are not available, a combination of the bounded robust state feedback controllers with high-gain state observes and appropriate saturation filters, was employed to synthesize bounded robust multivariable output feedback controllers that require only measurements of the outputs for practical implementation. The resulting output feedback design was shown to inherit the same closed-loop stability and performance properties of the state feedback controller and, in addition, practically preserve the region of guaranteed closed-loop stability obtained under state feedback. The developed state and output feedback controllers were applied successfully to non-isothermal chemical reactor examples with uncertainty, input constraints, and incomplete state measurements. Finally, a discussion was provided to put the proposed Lyapunov-based approach in perspective with respect to the nonlinear model predictive control approach, and the implications of our results in the practical implementation of MPC were highlighted.

Appendix A.

Proof of Theorem 1. Consider the representation of the closed-loop system in terms of the transformed coordinates (e, η) introduced in Eqs. (6) and (7):

$$
\begin{aligned}
\dot{e} &= \bar{f}(e,\eta,\bar{v}) - \frac{1}{2}\sum_{i=1}^{m} \bar{g}_i(e,\eta,\bar{v}) R^{-1}(x) L_{\bar{g}_i} V \\
&\quad + \sum_{k=1}^{q} \bar{w}_k(e,\eta,\bar{v})\theta_k, \\
\dot{\eta} &= \Psi(e,\eta,\bar{v},\theta), \\
y_i &= e_1^{(i)} + v_i, \quad i = 1,\ldots,m,
\end{aligned} \tag{A.1}
$$

where $\Psi(\cdot) = [\Psi_1(\cdot) \ \cdots \ \Psi_{n-\sum_i r_i}(\cdot)]^{\mathrm{T}}$. We now follow a procedure to establish both asymptotic stability and reference-input tracking in the closed-loop system of Eq. (A.1). Initially, we show that the controller of Eqs. (9)–(11) satisfies the constraints within the region described by the set $\Pi(\theta_b, u_{\max})$ in Eq. (12). Then, using a Lyapunov argument we show that, starting from any initial condition that belongs to any invariant subset of $\Pi \subset \Omega$, the state feedback controller of Eqs. (9)–(11) asymptotically stabilizes the closed-loop e-subsystem and derive bounds that capture the evolution of the states of the e and η subsystems. Next, a small gain argument is invoked to show that the trajectories of the e–η interconnected closed-loop system remain bounded for all times. Finally, we show that the states of the full closed-loop system of Eq. (A.1) converge to the origin and that the outputs satisfy the relation of Eq. (13).

Step 1: To prove that the control law of Eqs. (9)–(11) satisfies the constraints within the region described by the set $\Pi(\theta_b, u_{\max})$, we have from Eqs. (9) and (10) that

$$
\begin{aligned}
|u(x)| &\leqslant \left|\frac{1}{2}R^{-1}(x)\right| |(L_{\bar{g}}V)^{\mathrm{T}}| \\
&\leqslant \frac{|L_{\bar{f}}^{*}V + \sqrt{(L_{\bar{f}}^{**}V)^2 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^4}|}{|(L_{\bar{g}}V)^{\mathrm{T}}|[1 + \sqrt{1 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]}.
\end{aligned} \tag{A.2}
$$

From the definitions of $L_{\bar{f}}^{*}V$ and $L_{\bar{f}}^{**}V$ in Eq. (11) and the fact that $\rho > 0$, it is clear that if $L_{\bar{f}}^{**}V \leqslant u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|$, then we also have $L_{\bar{f}}^{*}V \leqslant u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|$. Therefore, for any $x \in \Pi$, the following estimates hold:

$$
\begin{aligned}
&(L_{\bar{f}}^{**}V)^2 \leqslant (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2, \\
&L_{\bar{f}}^{*}V \leqslant u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|.
\end{aligned} \tag{A.3}
$$

Substituting the above estimates into Eq. (A.2) yields

$$
\begin{aligned}
|u(x)| &\leqslant \frac{u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|[1 + \sqrt{1 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]}{|(L_{\bar{g}}V)^{\mathrm{T}}|[1 + \sqrt{1 + (u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]}. \\
&= u_{\max}.
\end{aligned} \tag{A.4}
$$

Step 2: Consider the smooth, positive-definite radially unbounded function, $V:\mathbb{R}^{\sum_i r_i}\to R_{\geqslant 0}$, $V=e^{\mathrm{T}}Pe$ as a Lyapunov function candidate for the e-subsystem of Eq. (A.1). Computing the time-derivative of V along the trajectories of the closed-loop e-subsystem, we get

$$\begin{aligned}\dot V &= L_{\bar f}V + L_{\bar g}Vu + \sum_{k=1}^{q} L_{\bar w_k}V\theta_k\\
&= L_{\bar f}V - \left(\frac{L_{\bar f}^{*}V + \sqrt{(L_{\bar f}^{**}V)^2 + (u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^4}}{[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]}\right)\\
&\quad + \sum_{k=1}^{q} L_{\bar w_k}V\theta_k\\
&\leqslant \left(L_{\bar f}V + \chi\sum_{k=1}^{q}|L_{\bar w_k}V|\theta_{bk}\right)\\
&\quad - \left(\frac{L_{\bar f}^{*}V + \sqrt{(L_{\bar f}^{**}V)^2 + (u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^4}}{[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]}\right). \qquad \text{(A.5)}\end{aligned}$$

After performing some algebraic manipulations, the above inequality can be re-written as

$$\begin{aligned}\dot V &\leqslant \alpha(e,\eta,\bar v)\\
&\quad + \left(\frac{\sum_{k=1}^{q}\theta_{bk}|L_{\bar w_k}V|(\phi-(\chi-1)|2Pe|/|2Pe|+\phi) - \rho(|2Pe|^2/|2Pe|+\phi)}{[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]}\right), \qquad \text{(A.6)}\end{aligned}$$

where

$$\alpha(e,\eta,\bar v) = \left(\frac{(L_{\bar f}V + \chi\sum_{k=1}^{q}\theta_{kb}|L_{\bar w_k}V|)\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2} - \sqrt{(L_{\bar f}^{**}V)^2+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^4}}{[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]}\right). \qquad \text{(A.7)}$$

To analyze the sign of $\dot V$ in Eq. (A.6), we will initially study the sign of the term $\alpha(e,\eta,\bar v)$ on the right-hand side. It is clear that the sign of this term depends on the sign of the term $L_{\bar f}V+\chi\sum_{k=1}^{q}\theta_{kb}|L_{\bar w_k}V|$. To this end, we consider the following two cases:

Case 1: $L_{\bar f}^{**}V\leqslant 0$.

Since $L_{\bar f}^{**}V = L_{\bar f}V + \rho|2Pe| + \chi\sum_{k=1}^{q}|L_{\bar w_k}V|\theta_{bk}$ and ρ is a positive real number, the fact that $L_{\bar f}^{**}V\leqslant 0$ implies that $L_{\bar f}V+\chi\sum_{k=1}^{q}|L_{\bar w_k}V|\theta_{bk}\leqslant 0$. As a result, we have that $\alpha(e,\eta,\bar v)\leqslant 0$ and the time-derivative of V in this case satisfies the following bound:

$$\dot V \leqslant \left(\frac{\sum_{k=1}^{q}\theta_{bk}|L_{\bar w_k}V|((\phi-(\chi-1)|2Pe|)/(|2Pe|+\phi)) - \rho(|2Pe|^2/(|2Pe|+\phi))}{[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]}\right) \equiv \beta(e,\eta,\bar v). \qquad \text{(A.8)}$$

Case 2: $0 < L_{\bar f}^{**}V \leqslant u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|$.

In this case, we have

$$(L_{\bar f}^{**}V)^2 \leqslant (u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2 \qquad \text{(A.9)}$$

and therefore

$$\begin{aligned}&-\sqrt{(L_{\bar f}^{**}V)^2 + (u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^4}\\
&\quad = -\sqrt{(L_{\bar f}^{**}V)^2 + (u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}\\
&\quad \leqslant -(L_{\bar f}^{**}V)\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}. \qquad \text{(A.10)}\end{aligned}$$

Substituting the estimate of Eq. (A.10) in the expression for $\dot V$ in Eqs. (A.6) and (A.7) yields

$$\begin{aligned}\dot V &= \frac{-\rho|2Pe|\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}}{[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]} + \beta(e,\eta,\bar v)\\
&\leqslant \beta(e,\eta,\bar v). \qquad \text{(A.11)}\end{aligned}$$

From the above analysis, it is clear that whenever $L_{\bar f}^{**}V\leqslant u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|$, the inequality of Eq. (A.8) holds. To guarantee that the state of the closed-loop system satisfies $L_{\bar f}^{**}V\leqslant u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|$ for all time, we confine the initial conditions within an invariant subset Ω embedded within the region described by Eq. (12). Therefore, starting from any $x(0)\in\Omega$, the inequality of Eq. (A.8) holds. Referring to this inequality, note that since $\chi>1$ and $\rho>0$, it is clear that, whenever $|2Pe|>\phi/(\chi-1)$, the first term on the right-hand side is strictly negative, and therefore $\dot V$ satisfies

$$\begin{aligned}\dot V &\leqslant -\frac{\rho|2Pe|^2}{(|2Pe|+\phi)[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]}\\
&\leqslant -\frac{k_1|e|^2}{(|2Pe|+\phi)[1+\sqrt{1+(u_{\max}|(L_{\bar g}V)^{\mathrm{T}}|)^2}]}, \qquad \text{(A.12)}\end{aligned}$$

where $k_1 = 4\rho\lambda_{\min}(P^2) > 0$. To study the behavior of $\dot{V}$ when $|2Pe| \leqslant \phi/(\chi-1)$, we first note that since the functions $\bar{w}_k(e,\eta,\bar{v})$ are smooth and vanish when $e=0$, then there exists positive real constants ϕ_1^*, δ_k, $k=1,\ldots,q$, such that if $\phi \leqslant \phi_1^*$, the bound $|\bar{w}_k(e,\eta,\bar{v})| \leqslant \delta_k|e|$, holds for $|2Pe| \leqslant \phi/(\chi-1)$. Using this bound, we obtain the following estimates:

$$\begin{aligned}
&\sum_{k=1}^{q}\theta_{bk}|L_{\bar{w}_k}V|(\phi-(\chi-1)|2Pe|)\\
&\quad\leqslant \sum_{k=1}^{q}\theta_{bk}|L_{\bar{w}_k}V|\phi\\
&\quad=\sum_{k=1}^{q}\theta_{bk}|\bar{w}_k||2Pe|\phi\\
&\quad\leqslant k_2\phi\sum_{k=1}^{q}\theta_{bk}\delta_k|e|^2 \quad \forall|2Pe| \leqslant \frac{\phi}{\chi-1},
\end{aligned} \tag{A.13}$$

where $k_2 = 2\sqrt{\lambda_{\max}(P^2)} > 0$. Substituting the estimate of Eq. (A.13) directly into Eq. (A.8), we get

$$\begin{aligned}
\dot{V} &\leqslant \left(\frac{(k_2\phi\sum_{k=1}^{q}\theta_{bk}\delta_k - k_1)|e|^2}{(|2Pe|+\phi)[1+\sqrt{1+(u_{\max}|(L_{\bar{g}}V)^{\mathrm{T}}|)^2}]}\right)\\
&\qquad \forall|2Pe| \leqslant \frac{\phi}{\chi-1}.
\end{aligned} \tag{A.14}$$

If ϕ is sufficiently small to satisfy the bound $\phi \leqslant k_1/k_2\sum_{k=1}^{q}\delta_k\theta_{bk} \equiv \phi_2^*$, then it is clear from Eqs. (A.12)–(A.14) that the last inequality in Eq. (A.12) is satisfied, irrespective of the value of $|2Pe|$. In summary, we have that for any initial condition in the invariant set Ω (where Eq. (12) holds $\forall t \geqslant 0$), there exists $\phi^* \equiv \min\{\phi_1^*,\phi_2^*\}$ such that if $\phi \leqslant \phi^*$, $\dot{V}$ satisfies

$$\begin{aligned}
\dot{V} &\leqslant \frac{-k_1|e|^2}{(|2Pe|+\phi)[1+\sqrt{1+(u_{\max}L_{\bar{g}}V)^2}]}\\
&< 0 \quad \forall e \neq 0.
\end{aligned} \tag{A.15}$$

Consequently, there exists a function β_e of class KL (see Khalil, 1996 for details) such that the following ISS inequality holds for the e states of the system of Eq. (A.1)

$$|e(t)| \leqslant \beta_e(|e(0)|,t) \quad \forall t \geqslant 0 \tag{A.16}$$

and the origin of the e-subsystem is asymptotically stable. From Assumption 2, we have that the η subsystem of Eq. (A.1) possesses an ISS property with respect to e which implies that there exists a function β_η of class KL and a function γ_η of class K such that the following ISS inequality holds:

$$|\eta(t)| \leqslant \beta_\eta(|\eta(0)|,t)+\gamma_\eta(\|e\|) \quad \forall t \geqslant 0 \tag{A.17}$$

uniformly in θ, $\bar{v}$. Using the inequalities of Eqs. (A.16) and (A.17), it can be shown by means of a small gain argument, similar to that used in Christofides and Teel (1996) and Christofides et al. (1996), that the full closed-loop system is asymptotically stable for all initial conditions inside the invariant set Ω. The asymptotic output tracking result can finally be obtained by simply taking the limsup of both sides of Eq. (A.16) which yields

$$\limsup_{t\to\infty}|e(t)| = 0 \tag{A.18}$$

and, hence,

$$\limsup_{t\to\infty}|e_1^{(i)}(t)| = \limsup_{t\to\infty}|y_i(t)-v_i(t)| = 0, \quad i=1,\ldots,m. \tag{A.19}$$

This completes the proof of the theorem. □

Proof of Theorem 2. The proof of this theorem consists of three parts. In the first part, we use a singular perturbation formulation to represent the closed-loop system and show that the resulting fast subsystem is globally exponentially stable. In the second part, we focus on the closed-loop reduced (slow) system and derive bounds for its states. Then, in the third part, we use a technical lemma proved in Christofides (2000), to establish that these bounds continue to hold up to an arbitrarily small offset, for arbitrarily large compact subsets of the stability region obtained under state feedback. The resulting bounds are then analyzed to establish asymptotic and local exponential stability of the full closed-loop system, which is then used to establish Eq. (22), provided that ϕ and $\bar{\varepsilon}$ are sufficiently small.

Part 1: Defining the auxiliary error variables $\hat{e}_j^{(i)} = L_i^{r_i-j}(y_i^{(j-1)} - \tilde{y}_j^{(i)})$, $j=1,\ldots,r_i$, the vectors $e_o^{(i)} = [\hat{e}_1^{(i)}\hat{e}_2^{(i)}\ \cdots\ \hat{e}_{r_i}^{(i)}]^{\mathrm{T}}$, $e_o = [e_o^{(1)^{\mathrm{T}}}\ e_o^{(2)^{\mathrm{T}}}\ \cdots\ e_o^{(m)^{\mathrm{T}}}]^{\mathrm{T}}$, the parameters $\varepsilon_i = 1/L_i$, the matrices $\tilde{A}_i$ and the vector $\tilde{b}$:

$$\tilde{A}_i = \begin{bmatrix} -a_1^{(i)} & 1 & 0 & \cdots & 0\\ -a_2^{(i)} & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -a_{r_i-1}^{(i)} & 0 & 0 & \cdots & 1\\ -a_{r_i}^{(i)} & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1 \end{bmatrix}, \tag{A.20}$$

where $i=1,\ldots,m$, the system of Eq. (1) under the controller of Eq. (21) takes the following form:

$$\begin{aligned}
&\varepsilon_i\dot{e}_o^{(i)} = \tilde{A}_i e_o^{(i)} + \varepsilon_i\tilde{b}\psi_i(x,\hat{x},\theta,\phi,\bar{v}), \quad i=1,\ldots,m,\\
&\dot{\omega} = \Psi(\mathrm{sat}(\tilde{y}),\omega),\\
&\dot{x} = f(x) - \frac{1}{2}\sum_{i=1}^{m}g_i(x)R^{-1}(\hat{x})\widehat{L_{\bar{g}_i}V} + \sum_{k=1}^{q}w_k(x)\theta_k(t)
\end{aligned} \tag{A.21}$$

where $\psi_i(x, \hat{x}, \theta, \phi, \bar{v})$ is a Lipschitz function of its argument. Owing to the presence of the small parameters ε_i that multiply the time derivatives $\dot{e}_o^{(i)}$, the system of Eq. (A.21) can be, in general, a multiple-time-scale system. Therefore, the results proved in Khalil (1987) will be used to establish exponential stability of the fast dynamics. Defining $\bar{\varepsilon} = \max\{\varepsilon_i\}$, $i = 1, \dots, m$, multiplying each $e_o^{(i)}$-subsystem by $\bar{\varepsilon}/\varepsilon$, introducing the fast time-scale $\bar{\tau} = t/\bar{\varepsilon}$ and setting $\bar{\varepsilon} = 0$, the closed-loop fast subsystem takes the form:

$$\frac{de_o}{d\bar{\tau}} = \frac{\bar{\varepsilon}}{\varepsilon}\tilde{A}_i e_o, \quad i = 1, \dots, m. \tag{A.22}$$

The above system possesses a triangular (cascaded) structure, with the constant matrices $\tilde{A}_i$ being Hurwitz. Therefore, it satisfies Assumption 3 in Khalil (1987) and, since $\bar{\varepsilon}/\varepsilon \geqslant 1$, the same approach, as in Khalil (1987), can be used to deduce that the origin of this system is globally exponentially stable, i.e. there exists real numbers $k_3 \geqslant 1$, $a_3 > 0$ such that

$$|e_o(\bar{\tau})| \leqslant k_3 |e_o(0)| \mathrm{e}^{-a_3 \bar{\tau}} \quad \forall \bar{\tau} \geqslant 0. \tag{A.23}$$

Part 2: Using the coordinate change of Eq. (6) and (20), together with the notation introduced thereafter, we can re-write the closed-loop system of Eq. (A.21) in the following form:

$$\varepsilon_i \dot{e}_o = \tilde{A}_i e_o + \varepsilon_i \tilde{b} \psi_i(e, \eta, \hat{x}, \theta, \phi, \bar{v}),$$

$$\dot{\omega} = \Psi(\mathrm{sat}(\tilde{y}), \omega),$$

$$\dot{e} = \bar{f}(e, \eta, \bar{v}) - \frac{1}{2}\sum_{i=1}^{m} \bar{g}_i(e, \eta, \bar{v}) R^{-1}(\hat{x}) \widehat{L_{g_i}} V$$

$$+ \sum_{k=1}^{q} \bar{w}_k(e, \eta, \bar{v}) \theta_k,$$

$$\dot{\eta} = \Psi(e, \bar{v}, \eta),$$

$$y_i = e_1^{(i)} + v_i, \quad i = 1, \dots, m. \tag{A.24}$$

Consider the above system with $\bar{\varepsilon} = \max\{\varepsilon_i\} = 0$. Using the result of Theorem 1 in Khalil and Esfandiari (1993), it can be shown that $\eta(0) = \omega(0) + O(\bar{\varepsilon})$ implies $\eta(t) = \omega(t) + O(\bar{\varepsilon})$, $\forall\, t \geqslant 0$ and therefore $\eta(t) = \omega(t)$ $\forall t \geqslant 0$, when $\bar{\varepsilon} = 0$. The closed-loop reduced (slow) system of Eq. (A.24) therefore reduces to the one studied in the proof of Theorem 1 (see Eq. (A.1)) under state feedback, where we have already shown that given any initial condition such that $|x(0)| \leqslant \delta_s$, there exists $\phi^* > 0$ such that if $\phi \leqslant \phi^*$ and $|x(0)| \leqslant \delta_s$, the closed-loop system is asymptotically stable. Consequently, the denominator expression in Eq. (A.15) is bounded and there exist real numbers $k_e > 0$, $k_1 \geqslant 1$, $a_1 > 0$ such that if $|e(0)| \leqslant \delta_e$, $|\eta(0)| \leqslant \delta_\eta$, $\|\bar{v}\| \leqslant \delta_{\bar{v}}$, where $\delta_s = T^{-1}(\delta_e, \delta_\eta, \delta_{\bar{v}})$, we have that $\dot{V}$ satisfies $\dot{V} \leqslant -k_e |e|^2$ and the e states of the closed-loop system satisfy

$$|e(t)| \leqslant k_1 |e(0)| \mathrm{e}^{-a_1 t} \quad \forall t \geqslant 0 \tag{A.25}$$

which shows that the origin of the e-subsystem is exponentially stable. From this result and the fact that the η-subsystem (with $e = 0$) is locally exponentially stable, we have that the e, η interconnected closed-loop reduced (slow) system is also locally exponentially stable (see Khalil (1996) for details). Therefore, given the set of positive real numbers $(\delta_e, \delta_\eta, \delta_{\bar{v}})$, there exists $b > 0$ such that if $|e(0)| \leqslant \delta_e$, $|\eta(0)| \leqslant \delta_\eta$, $\|\bar{v}\| \leqslant \delta_{\bar{v}}$, the following bound holds

$$|\pi(t)| \leqslant k_2 |\pi(0)| \mathrm{e}^{-a_2 t} \quad \forall t \geqslant 0 \tag{A.26}$$

for all $|\pi(t)| \leqslant b$, where $\pi(t) = [e^{\mathrm{T}}(t)\ \eta^{\mathrm{T}}(t)]^{\mathrm{T}}$, for some $k_2 \geqslant 1$, $a_2 > 0$. Note that since the static component of the controller of Eq. (21) with $\hat{x} = x$ enforces asymptotic stability in the closed-loop slow system, the state ζ of the closed-loop slow system satisfies a bound of the following form $\forall t \geqslant 0$:

$$|\zeta(t)| \leqslant \beta_\zeta(\delta_\zeta, t), \tag{A.27}$$

where β_ζ is a class KL function and δ_ζ is the maximum value of the vector $[l_1^{\mathrm{T}}(x)\ l_2^{\mathrm{T}}(x)\ \cdots\ l_m^{\mathrm{T}}(x)]^{\mathrm{T}}$ for $|x| \leqslant \delta_b$, *where* $l_i(x) = [h_i(x)\ L_f h_i(x)\ \cdots\ L_f^{r_i - 1} h_i(x)]^{\mathrm{T}}$.

Based on the above bound and following the results of Khalil and Esfandiari (1993) and Teel and Praly (1995), we disregard estimates of $\tilde{y}$, obtained from the high-gain observer, with norm $|\tilde{y}| > \beta_\zeta(\delta_\zeta, 0)$. Hence, we set $\mathrm{sat}(\cdot) = \min\{1, \zeta_{\max}/|\cdot|\}(\cdot)$ where $\zeta_{\max}$ is the maximum value of the vector $[\zeta_1\ \zeta_2\ \cdots\ \zeta_r]$ for $|\zeta| \leqslant \beta_\zeta(\delta_\zeta, 0)$.

Part 3: Having analyzed the stability properties of both the fast and slow closed-loop systems in parts 1 and 2, respectively, it can be shown, with the aid of calculations similar to those performed in Christofides and Teel (1996) and El-Farra and Christofides (2001b), that the inequalities of Eq. (A.23) (derived for the fast system), and Eqs. (A.25)–(A.27) (derived for the slow system) continue to hold, for the states of the full closed-loop system, up to an arbitrarily small offset d for initial conditions in large compact subsets ($\Omega_b \subset \Omega$) where $\Omega_b = \{x \in \mathbb{R}^m : |x| \leqslant \delta_b\}$ and $\beta(\delta_b, 0) + d \leqslant \delta_s$, provided that the singular perturbation parameter $\bar{\varepsilon}$ is sufficiently small. The requirement that $\beta(\delta_b, 0) + d \leqslant \delta_s$ guarantees that during the initial boundary layer (when the fast states have not decayed yet), the "slow" states of the closed-loop system remain within the invariant region Ω. Therefore, given the pair (δ_b, d), the set (δ_θ, $\delta_{\bar{v}}$, δ_ζ), and with $\phi \in (0, \phi^*]$, there exists $\bar{\varepsilon}^{(1)} > 0$ such that if $\bar{\varepsilon} \in (0, \bar{\varepsilon}^{(1)}]$, $|x(0)| \leqslant \delta_b$, $|\tilde{y}| \leqslant \delta_\zeta$, $\|\theta_k\| \leqslant \delta_\theta$, $\|\bar{v}\| \leqslant \delta_{\bar{v}}$, then, for all $t \geqslant 0$, the states of the closed-loop singularly perturbed system satisfy

$$|e(t)| \leqslant k_1 |e(0)| \mathrm{e}^{-a_1 t} + d,$$

$$|\eta(t)| \leqslant K_\eta |\eta(0)| \mathrm{e}^{-a_2 t} + \gamma_\eta(\|e\|) + d,$$

$$|e_o(t)| \leqslant k_3 |e_o(0)| \mathrm{e}^{-a_3 (t/\varepsilon)} + d. \tag{A.28}$$

The above inequalities imply that the trajectories of the closed-loop singularly perturbed system will be bounded. Furthermore, as t increases, they will be ultimately bounded with an ultimate bound that depends on d. Since d is arbitrary, we can choose it small enough such that after a sufficiently large time, say $\tilde{t}$, the trajectories of the closed-loop system are confined within a small compact neighborhood of the origin of the closed-loop system. Obviously, $\tilde{t}$ depends on both the initial conditions and the desired size of the neighborhood, but is independent of $\bar{\varepsilon}$. For reasons that will become obvious shortly, we choose $d = b/4$ and let $\tilde{t}$ be the smallest time such that $\max\{k_1|e(0)|e^{-a_1\tilde{t}}, K_\eta|\eta(0)|e^{-a_2\tilde{t}} + \gamma_\eta(\|e\|),\ k_3|e_o(0)|e^{-a_3\tilde{t}/\varepsilon}\} \leqslant d$. Then it can be easily verified that

$$|\pi(t)| \leqslant b, \quad |e_o(t)| \leqslant b \quad \forall t \geqslant \tilde{t}. \tag{A.29}$$

Recall from Eqs. (A.23) and (A.26) that both the fast and slow subsystems are exponentially stable within the ball of Eq. (A.29). Then, a direct application of the result of Theorem 9.3 in Khalil (1996) can be performed to show that there exists $\bar{\varepsilon}^{(2)}$ such that if $\bar{\varepsilon} \leqslant \bar{\varepsilon}^{(2)}$, the singularly perturbed closed-loop system is locally exponentially stable and, therefore, once inside the ball of Eq. (A.29), the closed-loop trajectories converge to the origin as $t \to \infty$.

To summarize, we have that given the pair of positive real numbers (δ_b, d) such that $\beta(\delta_b, 0) + d \leqslant \delta_s$, given the set of positive real numbers $(\delta_\theta,\ \delta_{\bar{v}},\ \delta_\zeta)$, and with $\phi \in (0.\phi^*]$, there exists $\bar{\varepsilon}^* \equiv \min\{\bar{\varepsilon}^{(1)}, \bar{\varepsilon}^{(2)}\}$ such that if $|x(0)| \leqslant \delta_b$, $|\tilde{y}(0)| \leqslant \delta_\zeta$, $\|\theta_k\| \leqslant \delta_\theta$, $\|\bar{v}\| \leqslant \delta_{\bar{v}}$, and $\bar{\varepsilon} \in (0, \bar{\varepsilon}^*]$, the closed-loop trajectories are bounded and converge to the origin as time tends to infinity, i.e. the closed-loop system is asymptotically stable. The asymptotic output tracking result can then be established by noting that from Eq. (A.26) we have

$$\limsup_{t\to\infty} |e(t)| = 0 \tag{A.30}$$

and therefore

$$\limsup_{t\to\infty} |e_1^{(i)}(t)| = \limsup_{t\to\infty} |y_i(t) - v_i(t)| = 0, \quad i = 1, \ldots, m \tag{A.31}$$

This completes the proof of the theorem. □

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