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Brief paper

Uniting bounded control and MPC for stabilization of constrained linear systems ☆

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Abstract

In this work, a hybrid control scheme, uniting bounded control with model predictive control (MPC), is proposed for the stabilization of linear time-invariant systems with input constraints. The scheme is predicated upon the idea of switching between a model predictive controller, that minimizes a given performance objective subject to constraints, and a bounded controller, for which the region of constrained closed-loop stability is explicitly characterized. Switching laws, implemented by a logic-based supervisor that constantly monitors the plant, are derived to orchestrate the transition between the two controllers in a way that safeguards against any possible instability or infeasibility under MPC, reconciles the stability and optimality properties of both controllers, and guarantees asymptotic closed-loop stability for all initial conditions within the stability region of the bounded controller. The hybrid control scheme is shown to provide, irrespective of the chosen MPC formulation, a safety net for the practical implementation of MPC, for open-loop unstable plants, by providing a priori knowledge, through off-line computations, of a large set of initial conditions for which closed-loop stability is guaranteed. The implementation of the proposed approach is illustrated, through numerical simulations, for an exponentially unstable linear system.
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1. Introduction

Virtually all practical control systems are subject to hard constraints on their manipulated inputs. One of the key limitations imposed by input constraints is the restriction on the set of initial states of the closed-loop system that can be steered to the origin with the available control action. For a control policy to be effective in dealing with the problem of input constraints, it needs to provide not only the stabilizing feedback control law, but also an explicit characterization of the set of initial conditions starting from where constrained closed-loop stability is guaranteed. The absence of an a priori explicit characterization of this set can make an impact on the practical implementation of the given control policy by requiring extensive closed-loop simulations over the whole set of possible initial conditions, to check for stability, or by limiting the operation within an unnecessarily small and conservative neighborhood of the operating point. These considerations have motivated significant work on the design of stabilizing bounded control laws that guarantee explicitly defined, large regions of attraction for the closed-loop system (e.g., Lin & Sontag, 1991; Teel, 1992). Recently, in El-Farra and Christofides (2001, 2003), a class of bounded robust Lyapunov-based nonlinear controllers, inspired in part by the results on bounded control originally presented in Lin and Sontag (1991), was developed. The controllers enforce robust stability in the closed-loop system and provide, at the same time, an explicit characterization of the region of guaranteed closed-loop stability. Despite their well-characterized stability and constraint-handling

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properties, the above controllers are not necessarily optimal with respect to an arbitrary performance criterion (in El-Farra & Christofides, 2001, it is shown that these controllers are inverse optimal with respect to meaningful costs).

Currently, model predictive control (MPC), also known as receding horizon control (RHC), is one of the few control methods for handling constraints within an optimal control setting. Numerous research investigations into the stability properties of MPC have led to a plethora of MPC formulations that focus on closed-loop stability (see, for example, Allgower & Chen, 1998; Morari & Lee, 1999; Mayne, Rawlings, Rao, & Scokaert, 2000) for extensive surveys of these developments). For open-loop stable and integrating linear systems with bounded inputs, infinite horizon formulations have been developed that are globally stabilizing (e.g., Rawlings & Muske, 1993; Zheng & Morari, 1995). This progress notwithstanding, the issue of obtaining, a priori (i.e. before controller implementation), an analytic characterization of the region of constrained closed-loop stability for MPC controllers remains to be adequately addressed, particularly for open-loop (exponentially) unstable plants where, unlike stable plants, stabilization is the overriding requirement. Part of the difficulty in this regard owes to the fact that, unlike in analytical bounded control, the stability of MPC feedback loops depends on a complex interplay between several factors such as the stabilizability of the initial condition, the penalties in the performance index, and the choice of the control horizon. A priori knowledge of the stability region of MPC requires an explicit characterization of these interplays. This difficulty can have an impact on the implementation of MPC on unstable plants by requiring extensive closed-loop simulations over the whole set of possible initial conditions, to check for stability.

Motivated by the above, we propose in this paper a controller switching strategy that merges the bounded control approach with MPC in a way that allows both approaches to complement the stability and optimality properties of each other. The guiding principle in realizing this strategy is the idea of decoupling optimality from the characterization of the region of constrained closed-loop stability. Specifically, by relaxing the optimality requirement, an explicit bounded feedback control law is designed and an explicit large estimate of the region of constrained closed-loop stability, which is not unnecessarily conservative, is computed. An optimal MPC controller that minimizes a given cost functional subject to the same constraints is then designed and implemented within the stability region of the bounded controller. Switching laws, that place appropriate restrictions on the evolution of the closed-loop trajectory under MPC within the stability region are then constructed to orchestrate the transition between the two controllers in a way that guarantees closed-loop stability for all initial conditions within the stability region. The switching scheme is shown to provide a safe mechanism for the practical implementation of MPC, especially for open-loop unstable systems, by providing, through off-line computations, a priori knowledge of a large set of initial conditions for which closed-loop stability is guaranteed.

The idea of switching, between different controllers (or models), for the purpose of achieving some objective that either cannot be achieved or is more difficult to achieve using a single controller has been widely used in the literature, and in a variety of contexts (e.g., Rugh & Shamma, 2000; Hespanha et al., 2001; Bemporad & Morari, 1999; Banerjee & Arkun, 1998; Aufderheide, Prasad, & Bequette, 2001). In this work, switching is employed between two structurally different, though complementary, control approaches as a tool for reconciling the objectives of optimal stabilization of the constrained closed-loop system (through MPC) and the a priori (off-line) determination of set of initial conditions for which closed-loop stability is guaranteed (through bounded control). The rest of the paper is organized as follows. In Section 2, we present the class of systems considered and review briefly how the constrained control problem is addressed in both bounded control and MPC. We then proceed in Section 3 to formulate the controller switching problem and propose a number of switching schemes that, with varying degrees of flexibility, address the problem. Finally, the implementation of the proposed switching schemes is demonstrated through numerical simulations.

2. Preliminaries

In this work, we consider continuous-time linear time-invariant (LTI) systems with input constraints, with the following form:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in \mathcal{U} \subset \mathbb{R}^m, \tag{1}$$

where $x = [x_1 \cdots x_n]' \in \mathbb{R}^n$ denotes the vector of state variables, $u = [u_1 \cdots u_m]'$ is the vector of manipulated inputs, taking values in a compact and convex subset, $\mathcal{U} := \{u \in \mathbb{R}^m : \|u\| \leqslant u_{\max}\}$, where $u_{\max} > 0$ is the magnitude of input constraints. The matrices A and B are constant $n \times n$ and $n \times m$ matrices, respectively. The pair (A, B) is assumed to be controllable. Throughout the paper, the notation $\|\cdot\|$ is used to denote the standard Euclidean norm of a vector, while the notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x'Qx$ for all $x \in \mathbb{R}^n$, where Q is a positive definite symmetric matrix and x' denotes the transpose of x. Furthermore, the notation $x(T^-)$ is used to denote the limit of $x(t)$ as T is approached from the left, i.e. $x(T^-) = \lim_{t \to T^-} x(t)$. We assume that measurements of the entire state, $x(t)$, are available for all t.

2.1. Bounded Lyapunov-based control

Consider the Lyapunov function candidate $V = x'Px$, where P is a positive-definite symmetric matrix that satisfies the Riccati equation

$$A'P + PA - PBB'P = -\bar{Q} \tag{2}$$

for some positive-definite matrix $\bar{Q}$. Using this Lyapunov function, we can construct, using Sontag's formula for bounded controls proposed in Lin and Sontag (1991) (see also El-Farra and Christofides, 2001), the following bounded nonlinear controller:

$$u(x) = -2k(x)B'Px := b(x), \tag{3}$$

where

$$k(x) = \left(\frac{L_f V + \sqrt{(L_f V)^2 + (u_{\max}\|(L_g V)'\|)^4}}{\|(L_g V)'\|^2 \left[1 + \sqrt{1 + (u_{\max}\|(L_g V)'\|)^2}\right]} \right), \tag{4}$$

$L_f V = x'(A'P + PA)x$ and $(L_g V)' = 2B'Px$. This control law is continuous everywhere in the state space and smooth away from the origin. For the above controller, one can show, using a Lyapunov argument, that whenever the closed-loop state trajectory evolves within the state space region described by the set

$$\Phi(u_{\max}) = \{x \in \mathbb{R}^n\colon L_f V < u_{\max}\|(L_g V)'\|\} \tag{5}$$

the resulting control action respects the constraints (i.e. $\|u\| \leqslant u_{\max}$) and enforces, simultaneously, the negative-definiteness of the time-derivative of the Lyapunov function, $\dot{V} < 0$, along the trajectories of the closed-loop system. Note that the size of the set Φ depends on the magnitude of the constraints in a way such that, the tighter the constraints, the smaller the region described by this set. Starting from any initial state within $\Phi(u_{\max})$, asymptotic stability of the constrained closed-loop system can be guaranteed, provided that the closed-loop trajectory remains within the region described by $\Phi(u_{\max})$. To ensure this, we consider initial conditions that belong to an invariant subset (preferably the largest) which we denote by $\Omega(u_{\max})$. This idea was also used in El-Farra and Christofides (2001) in the context of bounded robust control of constrained nonlinear systems. A common way of constructing such a subset is using the level sets of V, i.e.

$$\Omega(u_{\max}) = \{x \in \mathbb{R}^n\colon x'Px \leqslant c_{\max}\} \tag{6}$$

where $c_{\max} > 0$ is the largest number for which all nonzero elements of $\Omega(u_{\max})$ are contained within $\Phi(u_{\max})$. The invariant region, described by the set $\Omega(u_{\max})$, provides an estimate of the stability region, starting from where the origin of the constrained closed-loop system under the control law of Eqs. (3)–(4) is guaranteed to be asymptotically stable.

Remark 1. The bounded control law of Eqs. (3)–(4) will be used in Section 3 to illustrate the basic idea of the proposed hybrid control scheme. Our choice of using this particular design is motivated by its explicit structure and well-defined region of stability. However, our results are not restricted to this particular design. Any other analytical bounded control law, with an explicit structure and well-defined region of stability, can be used including, for example the bounded controls developed in Teel (1992) for certain classes of constrained linear systems.

Remark 2. The exact computation of the null controllable region, for general constrained linear systems, remains an open problem in control (e.g., see Hu, Lin, & Qiu, 2002 for some recent results in this area). Most research has therefore focused on obtaining an estimate of the domain of attraction under a given stabilizing feedback law. A common and computationally convenient way of providing this estimate is through Lyapunov-based invariant regions (level sets). While these regions do not necessarily capture the entire domain of attraction, we will use them throughout the paper only for a concrete illustration of the basic ideas of the results. It is possible to obtain better (i.e. less conservative) estimates by using, for example, combination of several Lyapunov functions.

2.2. Model predictive control

We consider model predictive control of the system described by Eq. (1). For this case, MPC at state x and time t is conventionally obtained by solving, on-line, a finite horizon optimal control problem of the form

$$P(x,t)\colon \ \min\{J(x,t,u) \mid u \in S\}, \tag{7}$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into $\mathscr{U}, T$ is the specified horizon. A control $u(\cdot)$ in S is characterized by the sequence $\{u[k]\}$ where $u[k] := u(k\Delta)$. A control $u(\cdot)$ in S satisfies $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. The performance index is given by

$$J(x,t,u(\cdot)) = \int_t^{t+T} [\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2]\,\mathrm{d}s + F(x(t+T)), \tag{8}$$

where R and Q are strictly positive definite, symmetric matrices, $x^u(s;x,t)$ denotes the solution of Eq. (1), due to control u, with initial state x at time t, and $F(\cdot)$ denotes the terminal penalty. In addition to penalties on the state and control action, the objective function may also include penalties on the rate of input change, reflecting limitations on actuator speed (e.g., a large valve requiring few seconds to change position). The minimizing control $u^0(\cdot) \in S$ is then applied to the plant over the interval $[k\Delta, (k+1)\Delta)$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M(x) := u^0(t;x,t). \tag{9}$$

It is well known that the control law defined by Eqs. (7)–(9) is not necessarily stabilizing. To achieve closed-loop stability, early versions of MPC focused on tuning the horizon length, T, and/or increasing the terminal penalty (see Bitmead, Gevers, & Wertz, 1990, for a critique of these approaches), while more recent formulations focused on imposing stability constraints on the optimization (see Allgower & Chen, 1998; Mayne et al., 2000 for surveys of different constraints proposed in the literature and the concomitant theoretical issues). Obviously, even when stability constraints are imposed on the optimization, any guarantee of closed-loop stability remains contingent upon making the appropriate choice of the initial condition, which must belong to the stability region of the MPC controller which, in turn, is a complex function of the constraints, the performance objective, and the horizon length. However, the implicit nature of the MPC control law, obtained through repeated on-line optimization, limits our ability to obtain, a priori, an explicit characterization of the admissible initial conditions starting from where the given MPC controller (with fixed performance index and horizon length) is guaranteed to enforce asymptotic closed-loop stability. Therefore, the initial conditions are tested usually through closed-loop simulations which can add to the computational burden prior to the implementation of MPC.

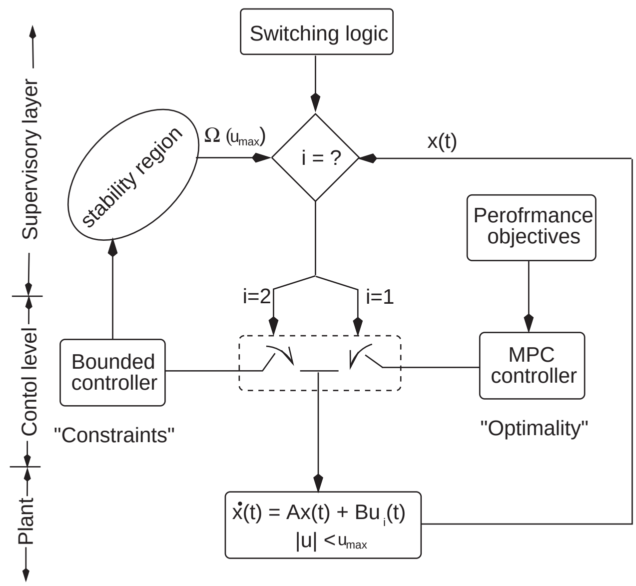

Fig. 1. A schematic representation of the hierarchical hybrid control structure merging the bounded and MPC controllers.

3. Uniting bounded control and MPC: a hybrid control strategy

By comparing the bounded and MPC controller designs reviewed in the previous section, some tradeoffs with respect to their stability and optimality properties are observed. For example, while the bounded controller possesses a well-defined region of admissible initial conditions that guarantee closed-loop stability in the presence of constraints, the performance of this controller is not guaranteed to be optimal with respect to an arbitrary performance criterion (in El-Farra & Christofides, 2001 we show that a more general class of bounded robust controllers are inverse optimal with respect to a meaningful, though not specified a priori, cost functional that imposes penalty on the state and control action). On the other hand, the MPC controller is well-suited for handling constraints within an optimal control setting; however, the analytical characterization of its set of admissible initial conditions is a more difficult task than it is through bounded control. In this section, we show how to reconcile the two approaches by means of a switching scheme that combines the desirable properties of both approaches.

3.1. Problem formulation and overview of solution

Consider the linear time-invariant system of Eq. (1), subject to input constraints, $\|u\| \leqslant u_{\max}$, for which the bounded controller of Eqs. (3)–(4) and MPC controller of Eqs. (7)–(9) have been designed. We formulate the controller switching problem as the one of designing a set of switching laws that orchestrate the transition between the MPC controller and the bounded controller in a way that: (1) respects input constraints, (2) guarantees asymptotic stability of the origin of the closed-loop system starting from any initial condition in the set $\Omega(u_{\max})$ defined in Eq. (6) and (3) guarantees recovery of the MPC optimal performance whenever the pertinent stability criteria are met. For a precise statement of the problem, we first cast the system of Eq. (1) as a switched system of the form

$$\dot{x} = Ax + Bu_{i(t)}, \quad \|u_i\| \leqslant u_{\max}, \quad i(t) \in \{1,2\}, \tag{10}$$

where $i : [0,\infty) \to \{1,2\}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, implying that only a finite number of switches between the two controllers is allowed on any finite interval of time. The index $i(t)$, which takes values in the finite set $\{1,2\}$, represents a discrete state that indexes the control input, u, with the understanding that $i(t)=1$ if and only if $u_i(x(t)) = M(x(t))$ (i.e. MPC is used) and $i(t)=2$ if and only if $u_i(x(t)) = b(x(t))$ (i.e. bounded control is used). The value of $i(t)$ is determined by a higher-level supervisor responsible for executing the transition between the two controllers. Our goal is to construct a switching law of the form $i(t) = \psi(x(t),t)$ that provides the supervisor with the switching times that ensure stabilizing transitions between the two controllers. This, in turn, determines the time-course of the discrete state $i(t)$. A schematic representation of the proposed hybrid control structure is depicted in Fig. 1.

In the remainder of this section, we present two switching schemes that address this problem. The first scheme is formalized in Theorem 1 (Section 3.2) and focuses primarily on closed-loop stability using classical MPC formulations, while the second scheme deals with feasibility-based

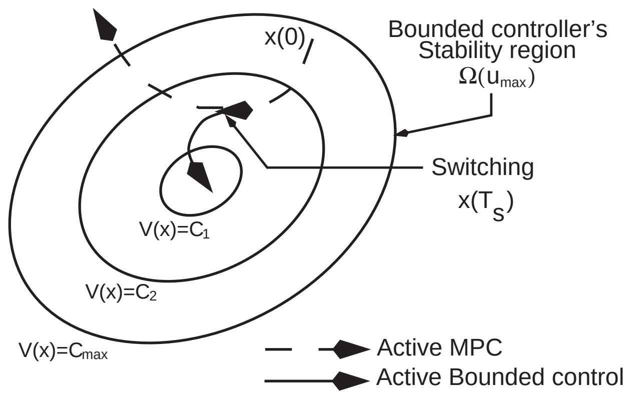

Fig. 2. A schematic representation of the implementation of the stability-based switching scheme of Theorem 1.

switching and shows how more advanced MPC formulations can also be accommodated by adjusting the switching logic appropriately. The proof of Theorem 1 is given in the appendix.

3.2. Stability-based controller switching

Theorem 1. *Consider the constrained LTI system of Eq.* (10), *with any initial condition* $x(0) = x_0 \in \Omega(u_{\max})$, *where* $\Omega(u_{\max})$ *was defined in Eq.* (6), *under the model predictive controller of Eqs.* (7)–(9). *Also let* $T_s > 0$ *be the earliest time for which the closed-loop state under MPC satisfies*

$$-\|x(T_s^-)\|_Q^2 + \|B'Px(T_s^-)\|^2 + 2x'(T_s^-)PBu(T_s^-) \geqslant 0 \tag{11}$$

then the switching rule

$$i(t) = \begin{cases} 1 & 0 \leqslant t < T_s, \\ 2 & t \geqslant T_s, \end{cases} \tag{12}$$

where $i(t) = 1 \Leftrightarrow u_i(x(t)) = M(x(t))$ *and* $i(t) = 2 \Leftrightarrow u_i(x(t)) = b(x(t))$, *guarantees that the origin of the constrained closed-loop system is asymptotically stable.*

Remark 3. Theorem 1 describes a stability-based switching strategy for control of linear systems with input constraints. The three main components of this strategy include the bounded controller, the MPC controller, and a higher-level supervisor that orchestrates the switching between the two controllers. The implementation of this hybrid strategy is best understood through the following stepwise procedure (see Fig. 2):

- Given the system model of Eq. (1) and the constraints on the input, design the bounded controller using Eqs. (3)–(4). Given the performance objective and a choice of the horizon length, design the MPC controller.
- Compute the stability region estimate for the bounded controller, $\Omega(u_{\max})$, using Eqs. (5)–(6).
- Initialize the closed-loop system, using the MPC controller, at any initial condition x_0 within $\Omega(u_{\max})$.
- Monitor the evolution of the closed-loop trajectory (by checking Eq. (11) at each time) until the earliest time, T_s, that Eq. (11) is met.
- At the earliest time that Eq. (11) is met, discontinue MPC implementation, switch to the bounded controller and implement it for all future times.

Remark 4. The relations of Eqs. (11)–(12) represent the switching rule that the supervisor observes when deciding if a switch between the two controllers is needed. The implementation of this rule requires only the evaluation of the algebraic expression on the left-hand side in Eq. (11), which is the rate at which the Lyapunov function (used in designing the bounded controller and characterizing its stability region) grows or decays along the trajectories of the closed-loop system. By observing this rule, the supervisor is essentially tracking the temporal evolution of V under MPC, so that whenever an increase in V is detected after its initial implementation, MPC is disengaged from the closed-loop and the bounded controller is switched in, thus steering the closed-loop trajectory to the origin asymptotically. This logic guarantees that, under MPC, the closed-loop trajectory never escapes $\Omega(u_{\max})$ before the bounded controller can be switched in. The idea of designing the switching logic based on monitoring the state's temporal evolution with respect to the stability region was introduced in El-Farra and Christofides (2002) for control of switched (multi-modal) nonlinear systems with input constraints.

Remark 5. In the case when Eq. (11) is never fulfilled, i.e. the Lyapunov function continues to decay monotonically along the closed-loop trajectories under MPC, the switching rule of Eq. (12) ensures that only MPC is implemented for all times (no switching occurs) since it is asymptotically stabilizing. In this case, the optimal performance of MPC is fully recovered.

Remark 6. Note that the proposed approach does not turn $\Omega(u_{\max})$ into a stability region for MPC. What the approach does, however, is turn $\Omega(u_{\max})$ into a stability region for the switched closed-loop system. The value of this can be understood in light of the difficulty in obtaining, a priori, an analytical characterization of the set of admissible initial conditions that the MPC controller can steer to the origin in the presence of input constraints. By using the bounded controller as a fall-back controller, the switching scheme allows us to safely initialize the closed-loop system anywhere within $\Omega(u_{\max})$ using MPC, with the guarantee that the bounded controller can always intervene (through switching) to "rescue" closed-loop stability in case the closed-loop starts to become unstable under MPC (due, e.g., to a poor choice of the initial condition or improper tuning of MPC). This safety feature distinguishes the bounded controller from other fall-back controllers that could be used, such as PID controllers, which do not provide a priori knowledge of the constrained stability region and, therefore, do not guarantee a safe transition in the case of unstable plants. For these controllers, a safe transition is critically dependent on issues such as whether or not the operator is able to properly tune the controller online, which, if not achieved, can possibly result in instability and/or performance degradation. The transition from the MPC to the bounded controller, on the other hand, is not fraught with such uncertainty and is always safe because of the a priori knowledge of the stability region. This aspect helps make plant operation safer and smoother.

Remark 7. In contrast to Lyapunov-based MPC approaches (e.g. contractive MPC (Polak & Yang, 1993), CLF-based RHC, (Primbs, Nevistic, & Doyle, 2000), the Lyapunov-based stability condition is enforced at the supervisory level, via continuous monitoring of $V(t)$, rather than being incorporated as an inequality constraint in the optimization problem. This fact may help reduce the complexity of the optimization problem, without loss of stability, by reducing the number of constraints involved. The underlying idea here is that of decoupling the stability requirement from optimality. In the proposed hybrid scheme, it is switching to the bounded controller, with its well-defined stability region, that safeguards against potential closed-loop instability arising as a consequence of implementing MPC, for which the stability region is not known a priori. On the other hand, MPC provides, by design, the desired optimal performance under constraints. The switching logic then ensures that any such optimal solution is implemented only when it is asymptotically stabilizing.

3.3. Switching using advanced MPC formulations

In Theorem 1, a classical MPC formulation (with no stability constraints) was used as an example to illustrate the basic idea of controller switching and how it can be used to aid MPC implementation. The value of the hybrid control structure (see Fig. 1), however, is not restricted to classical MPC formulations, and extends to any MPC formulation, for which a priori knowledge of the set of admissible initial conditions is lacking (or computationally expensive to obtain). The structure can be adapted to more advanced versions of MPC by appropriate design of the switching logic. To illustrate this point, consider the case when advanced MPC algorithms, that employ stability constraints in the optimization formulation, are used to compute the control action. An example of such formulations is the following:

$$\begin{aligned} &\min_{u(\cdot)} \int_t^{t+T} (x'(s)Qx(s) + u'(s)Ru(s))\,\mathrm{d}s, \\ &\text{s.t.} \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ &\qquad u(\cdot) \in S, \quad x(t+T) = 0 \end{aligned} \tag{13}$$

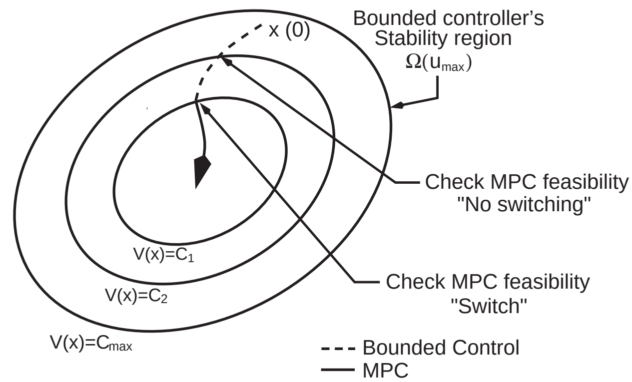

Fig. 3. A schematic representation of feasibility-based switching using a stability-handling MPC formulation.

which uses a terminal equality constraint (other types of stability constraints, such as control Lyapunov function (CLF)-based inequality constraints of the form $V(x(t+T)) < V(x(t))$ can be treated similarly). By incorporating stability conditions as part of the constrained optimization problem, asymptotic stability under MPC can be guaranteed provided that the initial optimization yields a feasible solution. However, inclusion of the stability constraints does not, by itself, provide any a priori explicit knowledge of the feasible initial conditions, which must then be identified through simulations. This observation suggests that the hybrid control structure can be used here to safeguard closed-loop stability in the event of MPC infeasibility. Specifically, in lieu of monitoring the growth of V, a different switching logic, based on feasibility of the optimization in Eq. (13), needs to be implemented. The switching algorithm in this case can be summarized as follows (see also Fig. 3 for a schematic representation):

- Given any initial condition x_0 within $\Omega(u_{\max})$, check, off-line, whether the constrained optimization in Eq. (13) yields a feasible solution. If a solution exists, no switching is needed and MPC can be implemented for all time.
- If an initial feasible solution is not found, implement the bounded controller instead, i.e. set $u(0) = b(x_0)$.
- While the bounded controller is in the closed-loop, continue to check, off-line, and as frequently as desired, the feasibility of the optimization in Eq. (13).
- At the earliest time that a feasible solution is found, switch to the MPC controller, else the bounded controller remains active.

The above discussion underscores an important attribute of the proposed hybrid control structure, and that is the fact that it is not proposed as a substitute for stability constraints (even though it guarantees stability when the MPC formulation is not stabilizing), but as a reliable fall-back mechanism from which any MPC formulation can benefit.

Remark 8. An important issue in the practical implementation of MPC is the selection of the horizon length. It is well known that this selection can have a profound effect on nominal closed-loop stability, and results are available in the literature that establish, under certain assumptions, the existence of “sufficiently large” finite horizons that ensure stability (e.g., Rawlings & Muske, 1993; Chmielewski & Manousiouthakis, 1996). However, a priori knowledge (without closed-loop simulations) of the minimum horizon length that guarantees feasibility or closed-loop stability, from an arbitrary initial condition, is currently not available. Therefore, in practice the horizon length is typically chosen based on heuristic criteria, tested through closed-loop simulations, and varied, if necessary, to achieve stability. Note that in all the switching schemes proposed in Section 3, closed-loop stability is maintained independent of the horizon length, and therefore selection of the horizon length can be made solely on the basis of what is computationally practical for the size of the optimization problem, without increasing, unnecessarily, the horizon length (and consequently the computational load) out of concern for stability.

Remark 9. The supervisory checks required in the implementation of the switching scheme do not incur any additional computational costs beyond that involved in MPC implementation alone. In the event that MPC is infeasible from a given initial condition (e.g., due to insufficient number of control moves), the supervisor, checks as frequently as desired, the initial feasibility of the MPC by actually solving the optimization problem at each time step and implementing the resulting solution (if found). If a solution is not found at a given time step, the supervisor implements the bounded controller (an inexpensive algebraic computation) and continues, in the mean time, to solve the optimization until it finds a feasible solution to implement.

Remark 10. The switching schemes proposed in this work differ, both in their objective and implementation, from other MPC formulations involving switching which have appeared earlier in the literature. For example, in dual mode MPC (Michalska & Mayne, 1993) the strategy includes switching from MPC to a locally stabilizing controller once the state is brought near the origin by MPC. The purpose of switching in this approach is to relax the terminal equality constraint whose implementation is computationally burdensome for nonlinear systems. However, the set of initial conditions for which MPC is guaranteed to steer the state close to the origin is not explicitly known a priori. In contrast, switching from MPC to the bounded controller is used in our work only to prevent any potential closed-loop instability arising from implementing MPC without the a priori knowledge of the admissible initial conditions. So depending on the stability properties of the chosen MPC, switching may or may not occur, and if it occurs, it can take place near or far from the origin. Finally, we note that the notion of “switching” from a predictive to an unconstrained LQR controller, for times beyond a finite control horizon, has been used in Sznaier and Damborg (1987), Chmielewski and Manousiouthakis (1996), and Scokaert and Rawlings (1998), in the context of determining a finite horizon that solves the infinite horizon constrained LQR problem.

4. Simulation example

Consider the following linear system

$$\dot{x} = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \tag{14}$$

where both inputs u_1, u_2 are constrained in the interval $[-5, 5]$ and the open-loop system has an unstable equilibrium point at the origin (A has two positive eigenvalues). We initially use Eqs. (3)–(4) to design the bounded controller and construct its stability region via Eqs. (5)–(6). The matrix P is chosen to be

$$P = \begin{bmatrix} 1.1362 & 0.8102 \\ 0.8102 & 1.8658 \end{bmatrix}.$$

For the MPC controller, the parameters in the objective function of Eq. (8) are chosen as penalty on the states, $Q = qI$, with $q = 1$, penalty on the control inputs, $R = rI$, with $r = 3$ and a horizon length of $T = 1$. The resulting quadratic program is solved using the MATLAB subroutine QuadProg, and the set of ODEs are integrated using the MATLAB solver ODE45.

As shown by the solid curve in Fig. 4, applying the MPC controller from the initial condition $x_0 = [6 \ \ -2]'$ (which belongs to the stability region of the bounded controller, Ω, represented by the ellipse in Fig. 4) leads to closed-loop instability. The corresponding input and output profiles are shown by the solid lines in Fig. 5. Using the switching scheme of Theorem 1, however, we find that the supervisor detects an increase in V at $t = 0.425$ and therefore immediately switches at this time from the MPC controller to the bounded controller in order to preserve closed-loop stability. As expected, the bounded controller asymptotically stabilizes the plant (see dotted lines in Figs. 4 and 5). It is important to note that one could try to tune the horizon length further in order to achieve stability using the MPC controller. For example, we see from the dashed lines in Figs. 4 and 5 that stability can be achieved by increasing the horizon length to $T = 1.5$. However, this conclusion could not be reached a priori, i.e. before running the closed-loop simulation in its entirety to check whether the choice $T = 1.5$ is appropriate. In contrast, closed-loop stability starting from the given initial condition, is guaranteed, a priori, under the switching scheme of Theorem 1.

In the following set of simulation runs, we demonstrate an application of the feasibility-based switching scheme,

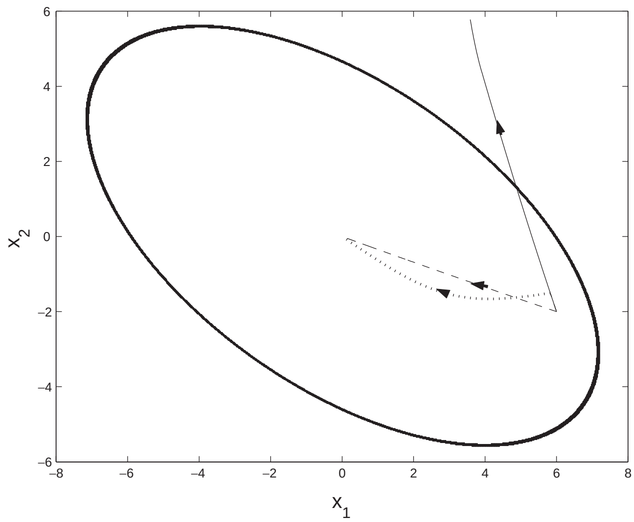

Fig. 4. Closed-loop state trajectory under MPC with $T=1$ (solid), under the switching scheme of Theorem 1 with $T=1$ (dotted), and under MPC with $T=1.5$ (dashed).

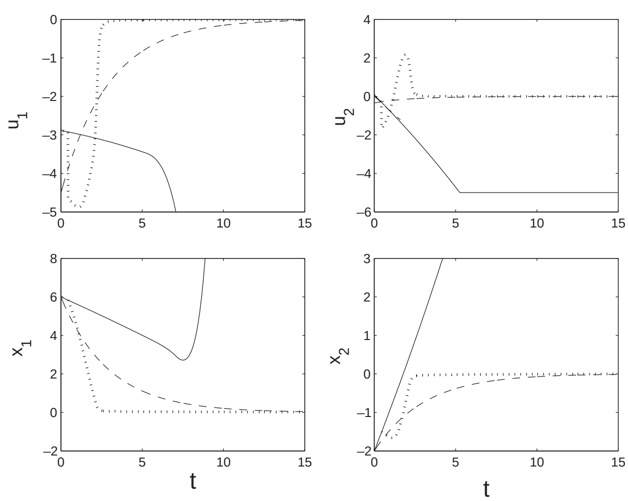

Fig. 5. Input and closed-loop state profiles under MPC with $T=1$ (solid), under the switching scheme of Theorem 1 with $T=1$ (dotted), and under MPC with $T=1.5$ (dashed).

proposed in Section 3.3, which uses a stable version of MPC, specifically, one employing a terminal equality constraint. For the MPC controller, the parameters in the objective function (Eq. (13)) are chosen as $q=1$ and $r=1$. The horizon length is chosen to be $T=1.2$. As shown by the solid lines in Fig. 6, starting from an initial condition $x_0=[2\ -2]'$, MPC is found to be feasible and is therefore implemented and kept in the closed-loop for all times driving the trajectory asymptotically to the origin. The corresponding input and output profiles are shown by the solid lines in Fig. 7. The scenario, therefore, represents a reasonably well-tuned MPC for initial conditions around the origin. Now suppose that, after settling at the origin, a temporary disturbance drives the closed-loop trajectory to the point $x_i=[5\ -5]'$ (which is still within the

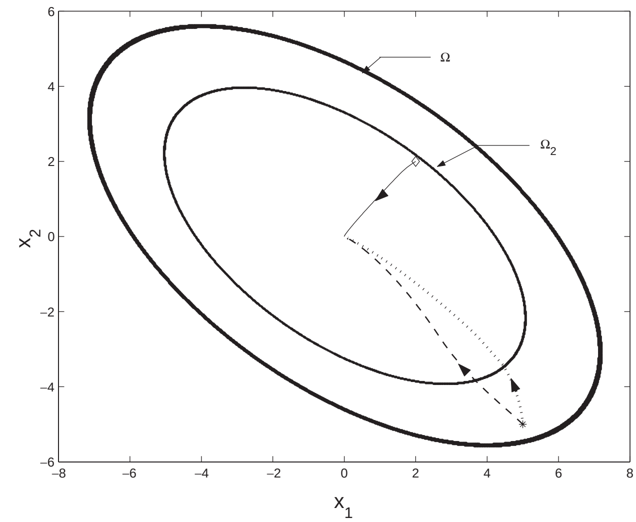

Fig. 6. Closed-loop state trajectory starting close to the origin using MPC with $T=1.2$ (solid) and from a point farther away in state space using the switching scheme with a horizon length $T=1.2$ (dotted) and using only MPC with $T=1.5$ (dashed).

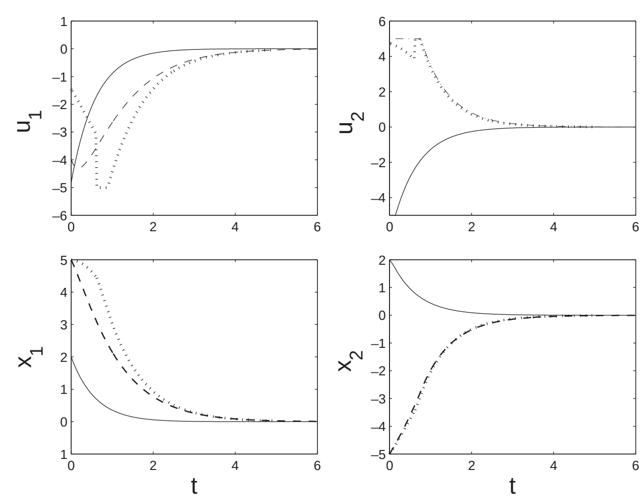

Fig. 7. Input and closed-loop state profiles starting close to the origin using MPC with $T=1.2$ (solid) and from a point farther away in state space using the switching scheme with a horizon length $T=1.2$ (dotted) and using only MPC with $T=1.5$ (dashed).

stability region of the bounded controller). After the disturbance disappears, MPC starting from this initial condition is found to be infeasible and, therefore, the bounded controller is employed instead. Implementation of the bounded controller proceeds until the closed-loop trajectory crosses the level set denoted by Ω_2 in Fig. 6, which occurs at $t=0.6$. At this point, feasibility of MPC is checked again and the MPC controller is found to be feasible. Consequently, the bounded controller is terminated immediately and the MPC controller is employed in the closed-loop for the remaining time (inside Ω_2). This scenario is shown by the dotted lines in Figs. 6 and 7. Once again, note that using a higher value of the horizon length, $T=1.5$ can lead to MPC being feasible from the initial condition $x_i = [5\ -5]'$ (dashed lines in Figs. 6 and 7). This implies that, once the closed-loop state is driven by the disturbance to x_i, a re-tuning of the horizon length can make MPC feasible. However, a priori knowledge of the value of the horizon length that is guaranteed to work could not be obtained, short of testing feasibility at that condition. The prospect of disturbances throwing the system around would therefore necessitate running extensive closed loop simulations to determine such a value. The feasibility-based switching scheme, on the other hand, clearly guarantees closed-loop stability for any value of the horizon length. Following the disturbance, it drives the closed-loop trajectory to a point at which the original MPC controller (with $T = 1.2$) becomes feasible again and can therefore be implemented. In this particular example, feasibility was checked only at one level set. The frequency of feasibility checking could be made larger by introducing more level sets, or by checking feasibility at predetermined times.

Appendix.

Proof of Theorem 1. *Step* 1: Consider the system of Eq. (1), under the control law of Eqs. (3)–(4). Evaluating the time-derivative of the Lyapunov function along the closed-loop trajectories

$$\dot{V} = L_f V + L_g V u$$
$$= \frac{L_f V\sqrt{1+(u_{\max}\|(L_g V)'\|)^2} - \sqrt{(L_f V)^2+(u_{\max}\|(L_g V)'\|)^4}}{\left[1+\sqrt{1+(u_{\max}\|(L_g V)'\|)^2}\right]}. \tag{15}$$

It is clear from the last equality above that when $L_f V < 0$, we have $\dot{V} < 0$. Furthermore, when $0 \leqslant L_f V < u_{\max}\|(L_g V)'\|$, we have $(L_f V)^2 < (u_{\max}\|(L_g V)'\|)^2$ and, therefore,

$$-\sqrt{(L_f V)^2+(u_{\max}\|L_g V\|)^4} < L_f V\sqrt{1+(u_{\max}\|(L_g V)'\|)^2}. \tag{16}$$

Substituting the above estimate into the expression for $\dot{V}$ in Eq. (15), we have that $\dot{V} < 0$. To summarize: whenever $L_f V < u_{\max}\|(L_g V)'\|$, we have $\dot{V} < 0$. Since $\Omega(u_{\max})$ is taken to be the largest invariant set, where this inequality holds for all $x \neq 0$, then starting from any initial state $x(0) \in \Omega(u_{\max})$, the inequality of Eq. (5) holds for all times, and consequently, $\dot{V} < 0,\ \forall\ 0 \neq x \in \Omega(u_{\max})$, which implies that the closed-loop system, under the control law of Eqs. (3)–(4) is asymptotically stable.

Step 2: Consider the switched system of Eq. (10), subject to the switching rule of Eqs. (11)–(12), with any initial state $x(0) \in \Omega(u_{\max})$. From the definition of T_s given in Theorem 1, it is clear that if T_s is a finite number, then $\dot{V}(x^M(t)) < 0\ \forall\ 0 \leqslant t < T_s$, where the notation $x^M(t)$ denotes the closed-loop state under MPC at time t, which implies that $x(t) \in \Omega(u_{\max})\ \forall\ 0 \leqslant t < T_s$ (or that $x(T^-) \in \Omega(u_{\max})$). This fact, together with the continuity of the solution of the switched system, $x(t)$, (which follows from the fact that the right-hand side of Eq. (10) is continuous in x and piecewise continuous in t since only finite number of switches is allowed over any finite time interval) implies that, upon switching (instantaneously) to the bounded controller at $t = T_s$, we have $x(T_s) \in \Omega(u_{\max})$ and $u(t) = b(x(t))$ for all $t \geqslant T_s$. Therefore, from our analysis in step 1 we conclude that $\dot{V}(x^b(t)) < 0\ \forall\ t \geqslant T_s$. In summary, the switching rule of Eqs. (11)–(12) guarantees that, starting from any $x(0) \in \Omega(u_{\max})$, $\dot{V}(x(t)) < 0\ \forall\ 0 \neq x \in \Omega(u_{\max})$, $\forall\ t \geqslant 0$, which implies that the switched closed-loop system is asymptotically stable. Note that if no such T_s exists, then we simply have $\dot{V}(x^M(t)) < 0\ \forall\ t \geqslant 0$ and the closed-loop system is, again, asymptotically stable. This completes the proof of the theorem.

References

Allgower, F., & Chen, H. (1998). Nonlinear model predictive control schemes with guaranteed stability. In R. Berber, & C. Kravaris (Eds.), *NATO ASI on nonlinear model based process control* (pp. 465–494). Dordrecht: Kluwer Academic Publishers.

Aufderheide, B., Prasad, V., & Bequette, B. W. (2001). A comparison of fundamental model-based and multiple model predictive control. *Proceedings of 40th IEEE Conference on Decision and Control*, Orlando, FL (pp. 4863–4868).

Banerjee, A., & Arkun, Y. (1998). Model predictive control of plant transitions using a new identification technique for interpolating nonlinear models. *Journal of Process Control*, *8*, 441–457.

Bemporad, A., & Morari, M. (1999). Control of systems integrating logic, dynamics and constraints. *Automatica*, *35*, 407–427.

Bitmead, R. R., Gevers, M., & Wertz, V. (1990). *Adaptive optimal control-the thinking man's GPC*. Englewood Cliffs, NJ: Prentice-Hall.

Chmielewski, D., & Manousiouthakis, V. (1996). On constrained infinite-time linear quadratic optimal control. *Systems & Control Letters*, *29*, 121–129.

El-Farra, N. H., & Christofides, P. D. (2001). Integrating robustness optimality and constraints in control of nonlinear processes. *Chemical Engineering Science*, *56*, 1841–1869.

El-Farra, N. H., & Christofides, P. D. (2002). Switching and feedback laws for control of constrained switched nonlinear systems. In C. J. Tomlin & M. R. Greenstreet (Eds.), *Lecture Notes in Computer Science*, Vol. 2289 (pp. 164–178.)

El-Farra, N. H., & Christofides, P. D. (2003). Bounded robust control of constrained multivariable nonlinear processes. *Chemical Engineering Science*, *58*, 3025–3047.

Hespanha, J., Liberzon, D., Morse, A. S., Anderson, B. D. O., Brinsmead, T. S., & De Bruyne, F. (2001). Multiple model adaptive control. Part 2: Switching. *International Journal of Robust & Nonlinear Control*, *11*, 479–496.

Hu, T., Lin, Z., & Qiu, L. (2002). An explicit description of null controllable regions of linear systems with saturating actuators. *Systems & Control Letters*, *47*, 65–78.

Lin, Y., & Sontag, E. D. (1991). A universal formula for stabilization with bounded controls. *Systems & Control Letters*, *16*, 393–397.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. M. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, *36*, 789–814.

Michalska, H., & Mayne, D. Q. (1993). Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, *38*, 1623–1633.

Morari, M., & Lee, J. H. (1999). Model predictive control: Past, present and future. *Computers & Chemical Engineering*, *23*, 667–682.

Polak, E., & Yang, T. H. (1993). Moving horizon control of linear systems with input saturation and plant uncertainty—Part 1: Robustness. *International Journal of Control*, *58*, 613–638.

Primbs, J. A., Nevistic, V., & Doyle, J. C. (2000). A receding horizon generalization of pointwise min-norm controllers. *IEEE Transactions on Automatic Control*, *45*, 898–909.

Rawlings, J. B., & Muske, K. R. (1993). The stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, *38*, 1512–1516.

Rugh, W. J., & Shamma, J. S. (2000). Research on gain scheduling. *Automatica*, *36*, 1401–1425.

Scokaert, P. O. M., & Rawlings, J. B. (1998). Constrained linear quadratic regulation. *IEEE Transactions on Automatic Control*, *43*, 1163–1169.

Sznaier, M., & Damborg, M. J. (1987). Suboptimal control of linear systems with state and control inequality constrains. *Proceedings of 26th IEEE Conference on Decision and Control*, Los Angeles, CA (pp. 761–762).

Teel, A. (1992). Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, *18*, 165–171.

Zheng, A., & Morari, M. (1995). Stability of model predictive control with mixed constraints. *IEEE Transactions on Automatic Control*, *40*, 1818–1823.

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