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Hybrid predictive control of nonlinear systems: method and applications to chemical processes

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SUMMARY

A hybrid control structure that unites bounded control with model predictive control (MPC) is proposed for the constrained stabilization of nonlinear systems. The structure consists of: (1) a finite-horizon model predictive controller, which can be linear or nonlinear, and with or without stability constraints, (2) a family of bounded nonlinear controllers for which the regions of constrained closed-loop stability are explicitly characterized and (3) a high-level supervisor that orchestrates switching between MPC and the bounded controllers. The central idea is to embed the implementation of MPC within the stability regions of the bounded controllers and employ these controllers as fall-back in the event that MPC is unable to achieve closed-loop stability (due, for example, to infeasibility of a given initial condition and/or horizon length, or due to computational difficulties in solving the nonlinear optimization). Switching laws, that monitor the closed-loop state evolution under MPC, are derived to orchestrate the transitions in a way that guarantees asymptotic closed-loop stability for all initial conditions within the union of the stability regions of the bounded controllers. The proposed hybrid control scheme is shown to provide a paradigm for the safe implementation of predictive control algorithms to nonlinear systems, with guaranteed stability regions. The efficacy of the proposed approach is demonstrated through applications to chemical reactor and crystallization process examples. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: nonlinear systems; input constraints; model predictive control; Lyapunov-based bounded control; controller switching; closed-loop stability region; process control

1. INTRODUCTION

The stabilization of dynamical systems using constrained control is an important problem that has been the subject of significant research work in control theory. Model predictive control (MPC) is a popular method for handling constraints within an optimal control setting. In MPC, the control action is obtained by solving repeatedly, on-line, a finite-horizon constrained open-loop optimal control problem. When the system is linear, the cost quadratic and the constraints

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convex, the MPC optimization problem reduces to a quadratic programme for which efficient software exists and, consequently, a number of control-relevant issues have been explored, including issues of closed-loop stability, performance, implementation and constraint satisfaction (e.g. see the tutorial paper [1]).

Many systems of practical interest, however, exhibit highly nonlinear behaviour that must be accounted for when designing the controller. In the literature, several nonlinear model predictive control (NMPC) schemes have been developed (e.g. see References [2–8]) that focus on the issues of stability, constraint satisfaction and performance optimization for nonlinear systems. A common idea of these approaches is to enforce stability by means of suitable penalties and constraints on the state at the end of the finite optimization horizon. Because of the system nonlinearities, however, the resulting optimization problem is non-convex and, therefore, much harder to solve, even if the cost functional and constraints are convex. The computational burden is more pronounced for NMPC algorithms that employ terminal stability constraints (e.g. equality constraints) whose enforcement requires intensive computations that typically cannot be performed within a limited time window.

In addition to the computational difficulties of solving a nonlinear optimization problem at each time step, one of the key challenges that impact on the practical implementation of NMPC is the inherent difficulty of characterizing, *a priori*, the set of initial conditions starting from where a given NMPC controller is guaranteed to stabilize the closed-loop system. For finite-horizon MPC, an adequate characterization of the stability region requires an explicit characterization of the complex interplay between several factors, such as the initial condition, the size of the constraints, the horizon length, the penalty weights, etc. Use of conservatively large horizon lengths to address stability only increases the size and complexity of the nonlinear optimization problem and could make it intractable. Furthermore, since feasibility of NMPC is determined through on-line optimization, unless an NMPC controller is exhaustively tested by simulation over the whole range of potential initial states, doubt will always remain as to whether or not a state will be encountered for which an acceptable solution to the finite-horizon problem can be found.

The above host of theoretical and computational issues continue to motivate research efforts in this area. Most available predictive control formulations for nonlinear systems, however, either do not explicitly characterize the stability region, or provide estimates of this region based on linearized models, used as part of some scheduling scheme between a set of local predictive controllers. The idea of scheduling of a set of local controllers to enlarge the operating region was proposed earlier in the context of analytic control of nonlinear systems (e.g. see References [9, 10]) and requires an estimate of the region of stability for the local controller designed at each scheduling point. Then by designing a set of local controllers with their estimated regions of stability overlapping each other, supervisory scheduling of the local controllers can move the state through the intersections of the estimated regions of stability of different controllers to the desired operating point. Similar ideas were used in References [11, 12] for scheduled predictive control of nonlinear systems. All of these approaches require the existence of an equilibrium surface that connects the scheduling points, and the resulting stability region estimate is the union of the local stability regions, which typically forms an envelope around the equilibrium surface. Stability region estimates based on linearization, however, are inherently conservative.

In contrast to this approach, progress within the area of analytic nonlinear controller design has made available bounded controllers with well-characterized stability properties (e.g. References [13–15]). The class of Lyapunov-based bounded controllers in Reference [13] was extended in References [16, 17] to handle input constraints and model uncertainty within an inverse optimal control setting, and provide at the same time an explicit characterization of the constrained stability region. Since the analytic controllers are designed on the basis of the nonlinear model, and also account explicitly for the constraints in the design of the controller, they yield a substantially larger region of stability than controllers based on linearized models. However, the resulting closed-loop performance is not necessarily optimal with respect to arbitrary cost functionals.

In a previous work [18], we developed a hybrid control scheme, uniting bounded control with MPC, for the stabilization of linear systems with input constraints. The scheme was predicated upon the idea of switching between a bounded controller, for which the region of constrained closed-loop stability is explicitly characterized, and a model predictive controller that minimizes a given performance objective subject to constraints. Switching laws were derived to orchestrate the transition between the two controllers in a way that reconciles the trade-offs between their respective stability and optimality properties, and guarantees asymptotic closed-loop stability for all initial conditions within the stability region of the bounded controller. The hybrid scheme was shown to provide a safety net for the implementation of MPC under state feedback.

In this work, we propose a hybrid control structure for the stabilization of nonlinear systems with input constraints. The central idea is to use a family of bounded nonlinear controllers, each with an explicitly characterized stability region, as fall-back controllers, and embed the operation of MPC within the union of these regions. In the event that the given predictive controller (which can be linear, nonlinear, or even scheduled) is unable to stabilize the closed-loop system (e.g., due to failure of the optimization algorithm, poor choice of the initial condition, insufficient horizon length, etc.), supervisory switching from MPC to any of the bounded controllers, whose stability region contains the state trajectory, guarantees closed-loop stability. The rest of the paper is organized as follows. In Section 2, some mathematical preliminaries are presented to describe the class of systems under consideration and briefly review how the constrained control problem is addressed within the bounded control and MPC frameworks. In Section 3, the controller switching problem is formulated first and then its solution is presented in the form of a hybrid control structure that employs switching between MPC and bounded control. Two representative switching schemes, that address (with varying degrees of flexibility) stability and performance objectives, are described to highlight the theoretical underpinnings and practical implications of the proposed hybrid control structure. Possible extensions of the supervisory switching logic, to address a variety of practical implementation issues, are also discussed. Finally in Section 4, the efficacy of the proposed approach is demonstrated through applications to chemical reactor and crystallization process examples.

2. PRELIMINARIES

In this work, we consider the problem of asymptotic stabilization of continuous-time nonlinear systems with input constraints, with the following state–space description

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1}$$

$$\|u\| \leqslant u_{\max} \tag{2}$$

where $x = [x_1 \cdots x_n]' \in \mathbb{R}^n$ denotes the vector of state variables, $u = [u_1 \cdots u_m]'$ is the vector of manipulated inputs, $u_{\max} \geqslant 0$ denotes the bound on the manipulated inputs, $f(\cdot)$ is a sufficiently smooth $n \times 1$ nonlinear vector function, and $g(\cdot)$ is a sufficiently smooth $n \times m$ nonlinear matrix function. Without loss of generality, it is assumed that the origin is the equilibrium point of the unforced system (i.e. $f(0) = 0$). Throughout the paper, the notation $\|\cdot\|$ will be used to denote the standard Euclidean norm of a vector, while the notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x'Qx$ for all $x \in \mathbb{R}^n$, where Q is a positive-definite symmetric matrix and x' denotes the transpose of x. In order to provide the necessary background for our main results in Section 3, we will briefly review in the remainder of this section the design procedure for, and the stability properties of, both the bounded and model predictive controllers, which constitute the basic components of our hybrid control scheme. For clarity of presentation, we will focus only on the state feedback problem where measurements of $x(t)$ are assumed to be available for all t (see Remark 15 below for a discussion on the issue of measurement sampling and how it can be handled).

2.1. Bounded Lyapunov-based control

Consider the system of Equations (1)–(2), for which a family of control Lyapunov functions (CLFs), $V_k(x)$, $k \in \mathcal{K} \equiv \{1, \ldots, p\}$ has been found (see Remark 1 below for a discussion on the construction of CLFs). Using each control Lyapunov function, we construct, using the results in Reference [13] (see also References [16, 17]), the following continuous bounded control law

$$u_k(x) = -k_k(x)(L_g V_k)'(x) \equiv b_k(x) \tag{3}$$

where

$$k_k(x) = \frac{L_f V_k(x) + \sqrt{(L_f V_k(x))^2 + (u_{\max}\|(L_g V_k)'(x)\|)^4}}{\|(L_g V_k)'(x)\|^2 \left[1 + \sqrt{1 + (u_{\max}\|(L_g V_k)'(x)\|)^2}\right]} \tag{4}$$

$L_f V_k(x) = (\partial V_k(x)/\partial x) f(x)$, $L_g V_k(x) = [L_{g_1} V_k(x) \cdots L_{g_m} V_k(x)]'$ and $g_i(x)$ is the ith column of the matrix $g(x)$. For the above controller, it can be shown, using standard Lyapunov arguments, that whenever the closed-loop state trajectory, x, evolves within the state–space region described by the set

$$\Phi_k(u_{\max}) = \{x \in \mathbb{R}^n : L_f V_k(x) < u_{\max}\|(L_g V_k)'(x)\|\} \tag{5}$$

then the controller satisfies the constraints, and the time derivative of the Lyapunov function is negative-definite. Therefore, starting from any initial state within the set $\Phi_k(u_{\max})$, asymptotic stability of the origin of the constrained closed-loop system can be guaranteed, provided that the state trajectory remains within region described by $\Phi_k(u_{\max})$ whenever $x \neq 0$. To ensure this, we consider initial conditions that belong to an invariant subset (preferably the largest), $\Omega_k(u_{\max})$ (this idea was also used in References [16, 17] in the context of bounded robust inverse optimal control of constrained nonlinear systems). One way to construct such a subset is using the level sets of V_k, i.e.

$$\Omega_k(u_{\max}) = \{x \in \mathbb{R}^n : V_k(x) \leqslant c_k^{\max}\} \tag{6}$$

where $c_k^{\max} > 0$ is the largest number for which $\Phi_k(u_{\max}) \supset \Omega_k(u_{\max}) \setminus \{0\}$. The union of the invariant regions described by the set

$$\Omega(u_{\max}) = \bigcup_{k=1}^{p} \Omega_k(u_{\max}) \tag{7}$$

then provides an estimate of the stability region, starting from where the origin of the constrained closed-loop system, under the appropriate control law from the family of Equations (3)–(4), is guaranteed to be asymptotically stable.

Remark 1

CLF-based stabilization of nonlinear systems has been studied extensively in the nonlinear control literature (e.g. see References [13, 19–21]). The construction of constrained CLFs (i.e. CLFs that take the constraints into account) remains a difficult problem (especially for nonlinear systems) that is the subject of ongoing research. For several classes of nonlinear systems that arise commonly in the modelling of practical systems, systematic and computationally feasible methods are available for constructing unconstrained CLFs (CLFs for the unconstrained system) by exploiting the system structure. Examples include the use of quadratic functions for feedback linearizable systems and the use of back-stepping techniques to construct CLFs for systems in strict feedback form. In this work, the bounded controllers in Equations (3)–(4) are designed using unconstrained CLFs, which are also used to explicitly characterize the associated regions of stability via Equations (5)–(6). While the resulting estimates do not necessarily capture the entire domain of attraction, they will be used throughout the paper only for a concrete illustration of the basic ideas of the results. It is possible to obtain substantially improved (i.e. less conservative) estimates by using, for example, a larger family of CLFs (see Section 4 for examples).

2.2. Model predictive control

In this section, we consider model predictive control of the system described by Equation (1), subject to the control constraints of Equation (2). In the literature, several MPC formulations are currently available, each with its own merits and limitations. While the results of this work are not restricted to any particular MPC formulation (see Remark 6 below), we will briefly describe here the 'traditional' formulation for the purpose of highlighting some of the theoretical and computational issues involved in the nonlinear setting. For this case, MPC at state x and time t is conventionally obtained by solving, on-line, a finite-horizon optimal control problem [22] of the form

$$P(x,t): \ \min\{J(x,t,u(\cdot)) | u(\cdot) \in S\} \tag{8}$$

$$\text{s.t.} \quad \dot{x} = f(x) + g(x)u \tag{9}$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into $\mathcal{U} := \{u \in \mathbb{R}^m : \|u\| \leqslant u_{\max}\}$ and T is the specified horizon. Equation (9) is a nonlinear model describing the time evolution of the states x.

A control $u(\cdot)$ in S is characterized by the sequence $\{u[k]\}$ where $u[k] := u(k\Delta)$. A control $u(\cdot)$ in S satisfies $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. The performance index is given by

$$J(x, t, u(\cdot)) = \int_t^{t+T} \left[\|x^u(s; x, t)\|_Q^2 + \|u(s)\|_R^2 \right] \mathrm{d}s + F(x(t+T)) \tag{10}$$

where R and Q are strictly positive-definite, symmetric matrices and $x^u(s; x, t)$ denotes the solution of Equation (1), due to control u, with initial state x at time t and $F(\cdot)$ denotes the terminal penalty. In addition to penalties on the state and control action, the objective function may also include penalties on the rate of input change reflecting limitations on actuator speed (e.g. a large valve requiring few seconds to change position). The minimizing control $u^0(\cdot) \in S$ is then applied to the plant over the interval $[k\Delta, (k+1)\Delta)$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M(x) = \mathrm{argmin}(J(x, t, u(\cdot))) = u^0(t; x, t) \tag{11}$$

While the use of a nonlinear model as part of the optimization problem is desirable to account for the system's nonlinear behaviour, it also raises a number of well-known theoretical and computational issues [22] that impact on the practical implementation of MPC. For example, in the nonlinear setting, the optimization problem is non-convex and, therefore, harder to solve than in the linear case. Furthermore, the issue of closed-loop stability is typically addressed by introducing penalties and constraints on the state at the end of the finite optimization horizon (see Reference [23] for a survey of different constraints proposed in the literature). Imposing constraints adds to the computational complexity of the nonlinear optimization problem which must be solved at each time instance.

Even if the optimization problem could be solved in a reasonable time, any guarantee of closed-loop stability remains critically dependent upon making the appropriate choice of the initial condition, which must belong to the predictive controller's region of stability (or feasibility), which, in turn, is a complex function of the constraints, the performance objective, and the horizon length. However, the implicit nature of the nonlinear MPC law, obtained through repeated on-line optimization, limits our ability to obtain, *a priori* (i.e. before controller implementation), an explicit characterization of the admissible initial conditions starting from where a given MPC controller (with a fixed performance index and horizon length) is guaranteed to asymptotically stabilize the nonlinear closed-loop system. Therefore, the initial conditions and the horizon lengths are usually tested through closed-loop simulations, which can add to the computational burden prior to the implementation of MPC.

3. THE HYBRID CONTROL STRATEGY: UNITING BOUNDED CONTROL AND MPC

By comparing the bounded controller and MPC designs presented in the previous section, some trade-offs with respect to their stability and optimality properties are evident. The bounded controller, for example, possesses a well-defined region of admissible initial conditions that guarantee constrained closed-loop stability. However, its performance may not be optimal with respect to an arbitrary performance criterion. MPC, on the other hand, provides the desired optimality requirement, but poses implementation difficulties and lacks an explicit characterization of the stability region. In this section, we show how to reconcile the two approaches by means of a switching scheme that provides a safety net for the implementation of MPC to nonlinear systems.

3.1. Formulation of the switching problem

Consider the constrained nonlinear system of Equations (1)–(2), for which the bounded controllers of Equations (3)–(4) and predictive controller of Equations (8)–(11) have been designed. The control problem is formulated as the one of designing a set of switching laws that orchestrate the transition between MPC and the bounded controllers in a way that guarantees asymptotic stability of the origin of the closed-loop system starting from any initial condition in the set $\Omega(u_{\max})$ defined in Equation (7), respects input constraints, and accommodates the optimality requirements whenever possible. For a precise statement of the problem, the system of Equation (1) is first cast as a switched system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u_{i(t)} \\ \|u_i\| &\leqslant u_{\max} \\ i(t) &\in \{1, 2\} \end{aligned} \tag{12}$$

where $i : [0, \infty) \to \{1, 2\}$ is the switching signal, which is assumed to be a piecewise continuous (from the right) function of time, implying that only a finite number of switches, between the predictive and bounded controllers, is allowed on any finite interval of time. The index, $i(t)$, which takes values in the set $\{1, 2\}$, represents a discrete state that indexes the control input $u(\cdot)$, with the understanding that $i(t) = 1$ if and only if $u_i(x(t)) = M(x(t))$ and $i(t) = 2$ if and only if $u_i(x(t)) = b_k(x(t))$ for some $k \in \mathcal{K}$. Our goal is to construct a switching law

$$i(t) = \psi(x(t), t) \tag{13}$$

that provides the set of switching times that ensure stabilizing transitions between the predictive and bounded controllers, in the event that the predictive controller is unable to enforce closed-loop stability. This in turn determines the time course of the discrete state $i(t)$.

In the remainder of this section, two switching schemes that address the above problem are presented. The first scheme is given in Theorem 1 (Section 3.2) and focuses primarily on the issue of closed-loop stability, while the second scheme, given in Theorem 2 (Section 3.3), provides more flexible switching rules that guarantee closed-loop stability and, simultaneously, enhance the overall closed-loop performance beyond that obtained from the first scheme. The proofs of both theorems are given in Appendix A.

3.2. Stability-based controller switching

Theorem 1

Consider the constrained nonlinear system of Equation (12), with any initial condition $x(0) \equiv x_0 \in \Omega_k(u_{\max})$, for some $k \in \mathcal{K} \equiv \{1, \ldots, p\}$, where Ω_k was defined in Equation (6), under the model predictive controller of Equations (8)–(11). Also let $\bar{T} \geqslant 0$ be the earliest time for which either the closed-loop state, under MPC, satisfies

$$L_f V_k(x(\bar{T})) + L_g V_k(x(\bar{T}))M(x(\bar{T})) \geqslant 0 \tag{14}$$

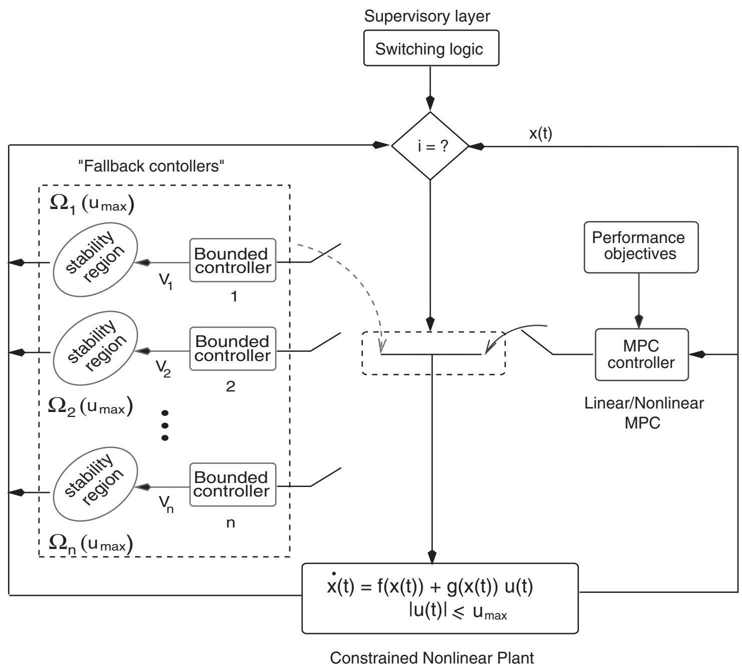

Figure 1. Schematic representation of the hybrid control structure merging MPC and a family of fall-back bounded controllers with their stability regions.

or the MPC algorithm fails to prescribe any control move. Then, the switching rule given by

$$i(t) = \begin{Bmatrix} 1, & 0 \leqslant t < \bar{T} \\ 2, & t \geqslant \bar{T} \end{Bmatrix} \tag{15}$$

where $i(t) = 1 \Leftrightarrow u_i(x(t)) = M(x(t))$ and $i(t) = 2 \Leftrightarrow u_i(x(t)) = b_k(x(t))$, guarantees that the origin of the switched closed-loop system is asymptotically stable.

Remark 2

Theorem 1 describes a stability-based switching strategy for control of nonlinear systems with input constraints. The main components of this strategy include the predictive controller, a family of bounded nonlinear controllers, with their estimated regions of constrained stability, and a high-level supervisor that orchestrates the switching between the controllers. A schematic representation of the hybrid control structure is shown in Figure 1. The implementation procedure of this hybrid control strategy is outlined below:

- Given the system model of Equation (1), the constraints on the input and the family of CLFs, design the bounded controllers using Equations (3)–(4). Given the performance objective, set up the MPC optimization problem.
- Compute the stability region estimate for each of the bounded controllers, $\Omega_k(u_{\max})$, using Equations (5)–(6), for $k = 1, \ldots, p$, and $\Omega(u_{\max}) = \bigcup_{k=1}^{p} \Omega_k(u_{\max})$.

- Initialize the closed-loop system under MPC at any initial condition, x_0, within Ω, and identify a CLF, $V_k(x)$, for which the initial condition is within the corresponding stability region estimate, Ω_k.
- Monitor the temporal evolution of the closed-loop trajectory (by checking Equation (14) at each time) until the earliest time that either Equation (14) holds or the MPC algorithm prescribes no solution, $\bar{T}$.
- If such a $\bar{T}$ exists, discontinue MPC implementation, switch to the kth bounded controller (whose stability region contains x_0) and implement it for all future times.

Remark 3

The use of multiple CLFs to design a family of bounded controllers, with their estimated regions of stability, allows us to initialize the closed-loop system from a larger set of initial conditions than in the case when a single CLF is used. Note, however, that once the initial condition is fixed, this determines both the region where MPC operation will be confined (and monitored) and the corresponding fall-back bounded controller to be used in the event of MPC failure. If the initial condition falls within the intersection of several stability regions, then any of the corresponding bounded controllers can be used as the fall-back controller.

Remark 4

The relation of Equations (14)–(15) represents the switching rule that the supervisor observes when contemplating whether a switch, between MPC and any of the bounded controllers at a given time, is needed. The left-hand side of Equation (14) is the rate at which the Lyapunov function grows or decays along the trajectories of the closed-loop system, under MPC, at time $\bar{T}$. By observing this rule, the supervisor tracks the temporal evolution of V_k, under MPC, such that whenever an increase in V_k is detected after the initial implementation of MPC (or if the MPC algorithm fails to prescribe any control move, e.g. due to optimization failure), the predictive controller is disengaged from the closed-loop system, and the appropriate bounded controller is switched in, thus steering the closed-loop trajectory to the origin asymptotically. This switching rule, together with the choice of the initial condition, guarantee that the closed-loop trajectory, under MPC, never escapes $\Omega_k(u_{\max})$ before the corresponding bounded controller can be activated. The idea of designing the switching logic based on monitoring the state's temporal evolution with respect to the stability region was introduced in Reference [24] in the context of control of switched (multi-modal) nonlinear systems with input constraints.

Remark 5

In the case when the condition of Equation (14) is never fulfilled (i.e. MPC continues to be feasible for all times and the Lyapunov function continues to decay monotonically for all times ($\bar{T} = \infty$)), the switching rule of Equation (15) ensures that only MPC is implemented for all times and that no switching to the fall-back controllers takes place. In this case, MPC is stabilizing and its optimal performance is fully recovered by the hybrid control structure. This particular feature underscores the central objective of the hybrid control structure, which is not to replace or subsume MPC but, instead, to provide a safe environment for the implementation of any optimal predictive control policy for which *a priori* guarantees of stability are not available. Note also that, to the extent that stability under MPC is captured by the given Lyapunov function, the notion of switching, as described in Theorem 1, does not result in loss of performance, since the transition to the bounded controller takes place only if MPC is infeasible or destabilizing. Clearly, under these circumstances the issue of optimality is not very meaningful for the predictive controller.

Remark 6

The fact that closed-loop stability is guaranteed, for all $x_0 \in \Omega$ (through supervisory switching), independently of whether MPC itself is stabilizing or not, allows us to use any desired MPC formulation within the switching scheme (and not just the one mentioned in Theorem 1), whether linear or nonlinear, and with or without terminal stability constraints or terminal penalties, without concern for loss of stability. This flexibility of using any desired MPC formulation has important practical implications for reducing the computational complexities that arise in implementing predictive control algorithms to nonlinear systems by allowing, for example, the use of less-computationally demanding NMPC formulations, or even the use of linear MPC algorithms instead (based on linearized models of the plant), with guaranteed stability (see Sections 4.1 and 4.2 for examples). In all cases, by embedding the operation of the chosen MPC algorithm within the large and well-defined stability regions of the bounded controllers, the switching scheme provides a safe fall-back mechanism that can be called upon at any moment to preserve closed-loop stability should MPC become unable to achieve closed-loop stability. Finally, we note that the Lyapunov functions used by the bounded controllers can also be used as a guide to design and tune the MPC in case a Lyapunov-based constraint is used at the end of the prediction horizon.

Remark 7

Note that no assumption is made regarding how fast the MPC optimization needs to be solved since, even if this time is relatively significant and the plant dynamics are unstable, the implementation of the switching rule in Theorem 1 guarantees an 'instantaneous' switch to the appropriate stabilizing bounded controller before such delays can adversely affect closed-loop stability (the times needed to check Equations (14)–(15) and compute the control action of the bounded controller are insignificant as they involve only algebraic computations). This feature is valuable when computationally intensive NMPC formulations fail, in the course of the on-line optimization, to provide an acceptable solution in a reasonable time. In this case, switching to the bounded controller allows us to safely abort the optimization without loss of stability.

Remark 8

One of the important issues in the practical implementation of finite-horizon MPC is the selection of the horizon length. It is well known that this selection can have a profound effect on nominal closed-loop stability as well as the size and complexity of the optimization problem. For NMPC, however, *a priori* knowledge of the shortest horizon length that guarantees closed-loop stability, from a given set of initial conditions (alternatively, the set of feasible initial conditions for a given horizon length) is not available. Therefore, in practice the horizon length is typically chosen using *ad hoc* selection criteria, tested through extensive closed-loop simulations, and varied, if necessary, to achieve stability. In the switching scheme of Theorem 1, closed-loop stability is maintained independently of the horizon length. This allows the predictive control designer to choose the horizon length solely on the basis of what is computationally practical for the size of the optimization problem, and without increasing, unnecessarily, the horizon length (and consequently the computational load) out of concern for stability. Furthermore, even if a conservative estimate of the necessary horizon length were to be ascertained for a small set of initial conditions before MPC implementation (say through extensive closed-loop simulations and testing), then if some disturbance were to drive the state outside of this set during the on-line implementation of MPC, this estimate may not be sufficient to stabilize the closed-loop system from the new state. Clearly, in this case, running extensive closed-loop simulations on-line to try to find the new horizon length needed is not a feasible option, considering the computational difficulties of NMPC as well as the significant delays that would be introduced until (and if) the new horizon could be determined. In contrast to the on-line re-tuning of MPC, stability can be preserved by switching to the fall-back controllers (provided that the disturbed state still lies within Ω).

Remark 9

When compared with other MPC-based approaches, the proposed hybrid control scheme is conceptually aligned with Lyapunov-based approaches in the sense that it, too, employs a Lyapunov stability condition to guarantee asymptotic closed-loop stability. However, this condition is enforced at the supervisory level, via continuous monitoring of the temporal evolution of V_k and explicitly switching between two controllers, rather than being incorporated in the optimization problem, as is customarily done in Lyapunov-based MPC approaches, whether as a terminal inequality constraint (e.g., contractive MPC [5, 25], CLF-based RHC for unconstrained nonlinear systems [26]) or through a CLF-based terminal penalty (e.g. References [6, 8]). The methods proposed in these works do not provide an explicit characterization of the set of states starting from where feasibility and/or stability is guaranteed *a priori*. Furthermore, the idea of switching to a fall-back controller with a well-characterized stability region, in the event that the MPC controller does not yield a feasible solution, is not considered in these approaches.

Remark 10

The idea of switching between different controllers for the purpose of achieving some objective that either cannot be achieved or is more difficult to achieve using a single controller has been widely used in the literature, and in a variety of contexts (see, for example, References [10, 27–29]). In most of these works, however, switching takes place between multiple structurally similar controllers. In this work, switching is employed between two structurally different, though complementary, control approaches as a tool for reconciling the objectives of optimal stabilization of the constrained closed-loop system (through MPC) and the *a priori* (off-line) determination of set of initial conditions for which closed-loop stability is guaranteed (through bounded control). The proposed switching schemes also differ, both in their objective and implementation, from other MPC formulations that involve switching. For example, in dual mode MPC [3] the strategy includes switching from MPC to a locally stabilizing controller once the state is brought near the origin by MPC. The purpose of switching in this approach is to relax the terminal equality constraint whose implementation is computationally burdensome for nonlinear systems. However, the set of initial conditions for which MPC is guaranteed to steer the state close to the origin is not explicitly known *a priori*. In contrast, switching from MPC to the bounded controller is used in our work only to prevent any potential closed-loop instability arising from implementing MPC without the *a priori* knowledge of the admissible initial conditions. Therefore, depending on the stability properties of the chosen MPC, switching may or may not occur, and if it occurs, it can take place near or far from the origin.

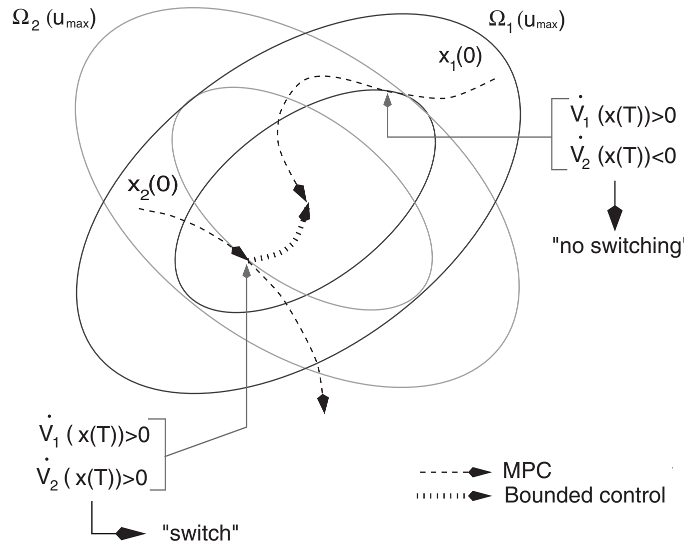

Figure 2. Schematic representation illustrating the main idea of the controller switching scheme proposed in Theorem 2.

3.3. Enhancing closed-loop performance

The switching rule in Theorem 1 requires monitoring only one of the CLFs (any one for which $x_0 \in \Omega_k$) and does not permit any transient increase in this CLF under MPC (by switching immediately to the appropriate bounded controller). While this condition is sufficient to guarantee closed-loop stability, it does not take full advantage of the behaviour of other CLFs at the switching time. For example, even though a given CLF, say V_1, may start increasing at time $\bar{T}$ under MPC, another CLF, say V_2, (for which $x(\bar{T})$ lies inside the corresponding stability region) may still be decreasing (see Figure 2). In this case, it would be desirable to keep MPC in the closed-loop (rather than switch to the bounded controller) and start monitoring the growth of V_2 instead of V_1, because if V_2 continues to decay, then MPC can be kept active for all times and its optimal performance fully recovered. To allow for such flexibility, we extend in this section the switching strategy of Theorem 1 by relaxing the switching rule. This is formalized in the following theorem. The proof is given in Appendix A.

Theorem 2

Consider the constrained nonlinear system of Equation (12), with any initial condition $x(0) \equiv x_0 \in \Omega(u_{\max})$, where $\Omega(u_{\max})$ was defined in Equation (7), under the model predictive controller of Equations (8)–(11). Let $T_k \geqslant 0$ be the earliest time for which

$$\begin{aligned} V_k(x(T_k))) &\leqslant c_k^{\max} \\ L_f V_k(x(T_k)) + L_g V_k(x(T_k)) M(x(T_k)) &\geqslant 0, \;\; k \in \mathcal{K} \end{aligned} \tag{16}$$

and define

$$T_k^*(t) = \left\{ \begin{array}{ll} T_k, & 0 \leqslant t \leqslant T_k \\ 0, & t > T_k \end{array} \right\} \tag{17}$$

Then, the switching rule given by

$$i(t) = \left\{ \begin{array}{ll} 1, & 0 \leqslant t < T^* \\ 2, & t \geqslant T^* \end{array} \right\} \tag{18}$$

where $T^* = \min\{T^i, T^f\}$, $T^i \geqslant 0$ is the earliest time for which the MPC algorithm fails to prescribe any control move, and $T^f = \max_j \{T_j^*\}$ for all j such that $x(t) \in \Omega_j(u_{\max})$, guarantees that the origin of the closed-loop system is asymptotically stable.

Remark 11

The implementation of the switching scheme proposed in Theorem 2 can be understood as follows:

- Initialize the closed-loop system using MPC, at any initial condition x_0 within Ω, and start monitoring the growth of all the Lyapunov functions whose corresponding stability regions contain the state.
- The supervisor disregards (i.e. permanently stops monitoring) any CLF whose value ceases to decay (i.e. any CLF for which the condition of Equation (16) holds) by setting the corresponding $T_k^*(t)$ to zero for all future times.
- As the closed-loop trajectory continues to evolve, the supervisor includes in the pool of Lyapunov functions it monitors any CLF for which the state enters the corresponding stability region, provided that this CLF has not been discarded at some earlier time.
- Continue implementing MPC as long as at least one of the CLFs being monitored is decreasing and the MPC algorithm yields a solution.
- If at any time, there is no active CLF (this corresponds to T^f), or if the MPC algorithm fails to prescribe any control move (this corresponds to T^i), switch to the bounded controller whose stability region contains the state at this time, else the MPC controller stays in the closed-loop system.

Remark 12

The purpose of permanently disregarding a given CLF, once its value begins to increase, is to avoid (or 'break') a potential perpetual cycle in which the value of such a CLF increases (at times when other CLFs are decreasing) and then decreases (at times when other CLFs are increasing) and keeps repeating this pattern without decaying to zero. The inclusion of such a CLF among the pool of active CLFs used to decide whether MPC should be kept active, would result in MPC staying in the closed-loop indefinitely, but without actually converging to the origin (i.e. only boundedness of the closed-loop state can be established but not asymptotic stability). Instead of disregarding, for all future times, a CLF that begins to increase at some point; however, an alternative strategy is to consider such a CLF in the supervisory decision making only if its value, at the current state, falls below the value from which it began to increase. This idea is similar to the one used in the stability analysis of switched systems using multiple Lyapunov functions (MLFs) (e.g. see References [24, 30]). The greater CLF-monitoring flexibility resulting from this policy can increase the likelihood of MPC implementation and enhanced closed-loop performance, while still guaranteeing asymptotic stability.

Remark 13
The switching schemes proposed in Theorems 1 and 2 can be further generalized to allow for multiple switchings between MPC and the bounded controllers. For example, if a given MPC is initially found infeasible (e.g. due to some terminal equality constraints), the bounded controller can be activated initially, but need not stay in the closed-loop system for all future times. Instead, it could be employed only until it brings the closed-loop trajectory to a point where MPC becomes feasible, at which time MPC can take over (see Section 4 for illustrations of this scenario). This scheme offers the possibility of further enhancement in the closed-loop performance by implementing MPC for all the times that it is feasible (instead of using the bounded controller). If MPC runs into any feasibility or stability problems, then the bounded controller can be re-activated. Chattering problems, due to back and forth switching, are avoided by allowing only a finite number of switches over any finite time interval.

Remark 14
Note that the implementation of the switching schemes proposed in Theorems 1 and 2 can be easily adapted to the case when actuator dynamics are not negligible and, consequently, a sudden change in the control action (resulting from switching between controllers) may not be achieved instantaneously. A new estimate of the stability region under the bounded controllers can be generated by sufficiently 'stepping back' from the boundary of the original stability region estimate, in order to prevent the escape of the closed-loop trajectory as a result of the implementation of possibly incorrect control action for some time (due to the delay introduced by the actuator dynamics). The switching schemes can then be applied as described in Theorems 1 and 2, using the revised estimate of the stability region for the family of bounded controllers.

Remark 15
When the proposed hybrid control schemes are applied to a process on-line, state measurements are typically available only at discrete sampling instants (and not continuously). In this case, the restricted access that the supervisor has to the state evolution between sampling times can lead to the possibility that the closed-loop state trajectory under MPC may leave the stability region without being detected, particularly if the initial condition is close to the boundary of the stability region and/or the sampling period is too large. In such an event, switching to the bounded controller at the next sampling time may be too late to recover from the instability of MPC. To guard against this possibility, the switching rules in Theorems 1 and 2 can be modified by restricting the implementation of MPC within a subset of the stability region, computed such that, starting from this subset, the system trajectory is guaranteed to remain within the stability region after one sampling period. Explicit estimates of this subset, which is parameterized by the sampling period, can be readily obtained off-line by computing (or estimating) the time derivative of the Lyapunov function under the maximum allowable control action and then integrating both sides of the resulting inequality over one sampling period. Note that this computation of a 'worst-case' estimate does not require knowledge of the solution of the closed-loop system.

Remark 16
The hybrid control schemes proposed in this paper can be extended to deal with the case when both input and state constraints are present. In one possible extension, state constraints would

be incorporated directly as part of the constrained optimization problem that yields the model predictive control law. In addition, estimates of the stability regions for the bounded controllers would be obtained by intersecting the region described by Equation (5) with the region described by the state constraints, and computing the largest invariant subset within the intersection. Using these estimates (which now account for both input and state constraints), implementation of the switching schemes can proceed following the same logic outlined for each case.

4. APPLICATION TO CHEMICAL PROCESS EXAMPLES

In this section, we present two simulation studies of chemical process examples to demonstrate the implementation of the proposed hybrid predictive control structure and evaluate its effectiveness.

4.1. Application to a chemical reactor example

We consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The inlet stream consists of pure A at flow rate F, concentration C_{A0} and temperature T_{A0}. Under standard modelling assumptions, the mathematical model for the process takes the form

$$\begin{aligned} \dot{C}_A &= \frac{F}{V}(C_{A0} - C_A) - k_0 \mathrm{e}^{-E/RT_\mathrm{R}} C_A \\ \dot{T}_\mathrm{R} &= \frac{F}{V}(T_{A0} - T_\mathrm{R}) + \frac{(-\Delta H)}{\rho c_p} k_0 \mathrm{e}^{-E/RT_\mathrm{R}} C_A + \frac{UA}{\rho c_p V}(T_\mathrm{c} - T) \end{aligned} \tag{19}$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, T_c is the temperature of the coolant in the surrounding jacket, U is the heat-transfer coefficient, A is the jacket area, V is the volume of the reactor, k_0, E, ΔH are the pre-exponential constant, the activation energy, and the enthalpy of the reaction, c_p and ρ, are the heat capacity and fluid density in the reactor. The values of all process parameters can be found in Table I. At the

Table I. Process parameters and steady-state values

V	$= 100.0$ L
E/R	$= 8000$ K
C_{A0}	$= 1.0$ mol/l
T_{A0}	$= 400.0$ K
ΔH	$= -2.0 \times 10^5$ J/mol
k_0	$= 4.71 \times 10^8$ min^{-1}
c_p	$= 1.0$ J/g K
ρ	$= 1000.0$ g/L
UA	$= 1.0 \times 10^5$ J/min K
F	$= 100.0$ L/min
C_A^s	$= 0.52$ mol/L
T_R^s	$= 398.97$ K
$T_\mathrm{c}^\mathrm{nom}$	$= 302$ K

nominal operating condition of $T_c^{nom} = 302$ K, the system has three equilibrium points, one of which is unstable. The control objective is to stabilize the reactor at the unstable equilibrium point $(C_A^s,\ T_R^s) = (0.52, 398.9)$ using the coolant temperature, T_c, as the manipulated input with constraints: $275\ \text{K} \leqslant T_c \leqslant 370\ \text{K}$.

Defining $x = [x_1\ x_2]' = [(C_A - C_A^s)\ (T_R - T_R^s)]'$ and $u = T_c - T_c^{nom}$, the process model of Equation (19) can be written in the form of Equation (1). Defining an auxiliary output, $y = h(x) = x_1$ (for the purpose of designing the controller), and using the invertible co-ordinate transformation: $\xi = [\xi_1\ \xi_2]' = T(x) = [x_1\ f_1(x)]'$, where $f_1(x) = \dot{x}_1$, the system of Equation (19) can be transformed into the following partially linear form:

$$\dot{\xi} = A\xi + bl(\xi) + b\alpha(\xi)u \tag{20}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = [0\ \ 1]', \quad l(\xi) = L_f^2 h(T^{-1}(\xi))$$

$L_f^2 h(T^{-1}(\xi))$ is the second-order Lie derivative of $h(\cdot)$ along the vector field $f(\cdot)$, $\alpha(\xi) = L_g L_f h \times (T^{-1}(\xi))$ is the mixed-order Lie derivative. The system of Equation (20) will be used to design the bounded controllers and compute their estimated regions of stability. A common choice of CLFs for this system is quadratic functions of the form, $V_k = \xi' P_k \xi$, where the positive-definite matrix P_k is chosen to satisfy the Riccati matrix inequality: $A'P_k + P_k A - P_k bb' P_k < 0$. The following matrices

$$P_1 = \begin{bmatrix} 1.45 & 1.0 \\ 1.0 & 1.45 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.55 & 0.1 \\ 0.1 & 0.55 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 8.02 & 3.16 \\ 3.16 & 2.53 \end{bmatrix} \tag{21}$$

were used to construct a family of three CLFs and three bounded controllers, and compute their stability region estimates, Ω_k', $k = 1, 2, 3$, in the ξ-co-ordinate system. The corresponding stability regions in the (C_A, T_R) co-ordinate system are then computed using the transformation $\xi = T(x)$ defined earlier. The union of these regions, Ω', is shown in Figure 3.

For the design of the predictive controller, a linear MPC formulation (based on the linearization of the process model around the unstable equilibrium point) with terminal equality constraints, $x(t + T) = 0$, is chosen for the sake of illustration (other MPC formulations that use terminal penalties, instead of terminal equality constraints, could also be used). The parameters in the objective function of Equation (10) are chosen as $Q = qI$, with $q = 1$, $R = rI$, with $r = 1.0$, and $F = 0$. We also choose a horizon length of $T = 0.25$ in implementing the MPC controller. The resulting quadratic programme is solved using the MATLAB subroutine QuadProg, and the set of nonlinear ODEs is integrated using the MATLAB solver ODE45.

As shown by the solid trajectory in Figure 3, starting from the initial condition $[C_A(0)\ \ T_R(0)]' = [0.46\ \ 407.0]'$, MPC using a horizon length of $T = 0.25$ yields a feasible solution, and when implemented in the closed-loop, stabilizes the nonlinear closed-loop system. The corresponding state and input profiles are shown in Figures 4(a)–4(c). Starting from the initial condition $[C_A(0)\ \ T_R(0)]' = [0.45\ \ 385.0]'$ (dashed lines in Figures 3–4), however, the linear MPC controller is infeasible. Recognizing that the initial condition is within the stability region estimate, Ω_3', the supervisor implements the third bounded controller, while continuously checking feasibility of MPC. At $t = 0.4$, the predictive controller becomes feasible and, therefore, the supervisor

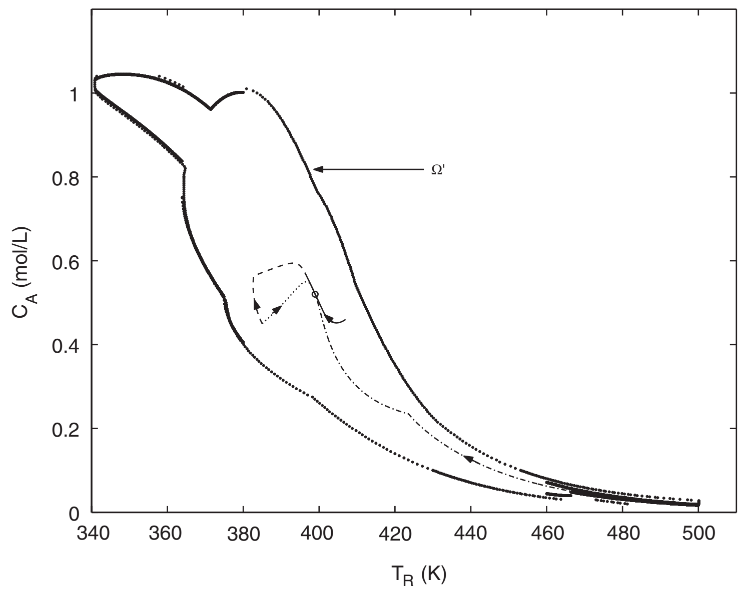

Figure 3. Implementation of the proposed hybrid control structure: Closed-loop state trajectory under MPC with $T = 0.25$ (solid trajectory), under the switched MPC/bounded controller(3) with $T = 0.25$ (dashed trajectory), under MPC with $T = 0.5$ (dotted trajectory), and under the switched bounded controller(2)/MPC with $T = 0.5$ (dash–dotted line).

switches to MPC and keeps monitoring the evolution of V_3. In this case, the value of V_3 keeps decreasing and MPC stays in the closed-loop for all future times, thus asymptotically stabilizing the nonlinear plant. Note that, from the same initial condition, if the horizon length in the MPC is increased to $T = 0.5$, MPC yields a feasible solution and, when implemented, asymptotically stabilizes the closed-loop system (dotted lines in Figures 3–4). However, the initial feasibility, based on the linearized model, does not necessarily imply that the predictive controller will be stabilizing, or for that matter even feasible at future times. Furthermore, this value of the horizon length (which yields a feasible solution) could not be determined *a priori* (without actually solving the optimization problem with the given initial condition), and stability of the closed-loop could not be ascertained, *a priori*, without running the closed-loop simulation in its entirety.

While our particular choice of implementing linear MPC (using the linearized model) facilitates implementation by making the optimization problem more easily solvable, the stabilizability of a given initial condition is limited, not only by the possibility of insufficient horizon length, but also by the linearization procedure itself. To demonstrate this, we consider an initial condition, $[C_A(0)\ T_R(0)]' = [0.028\ 484.0]'$, belonging to the stability region Ω', (dash–dotted lines in Figures 3–4). For this initial condition, linear MPC is found infeasible, no matter how large T is chosen to be, suggesting that this initial condition is outside the feasibility region based on the linear model. Therefore, using the Lyapunov function, V_2 (since the initial condition belongs to Ω_2'), the supervisor activates the second bounded controller which brings the state trajectory closer to the desired equilibrium point, while continuously checking

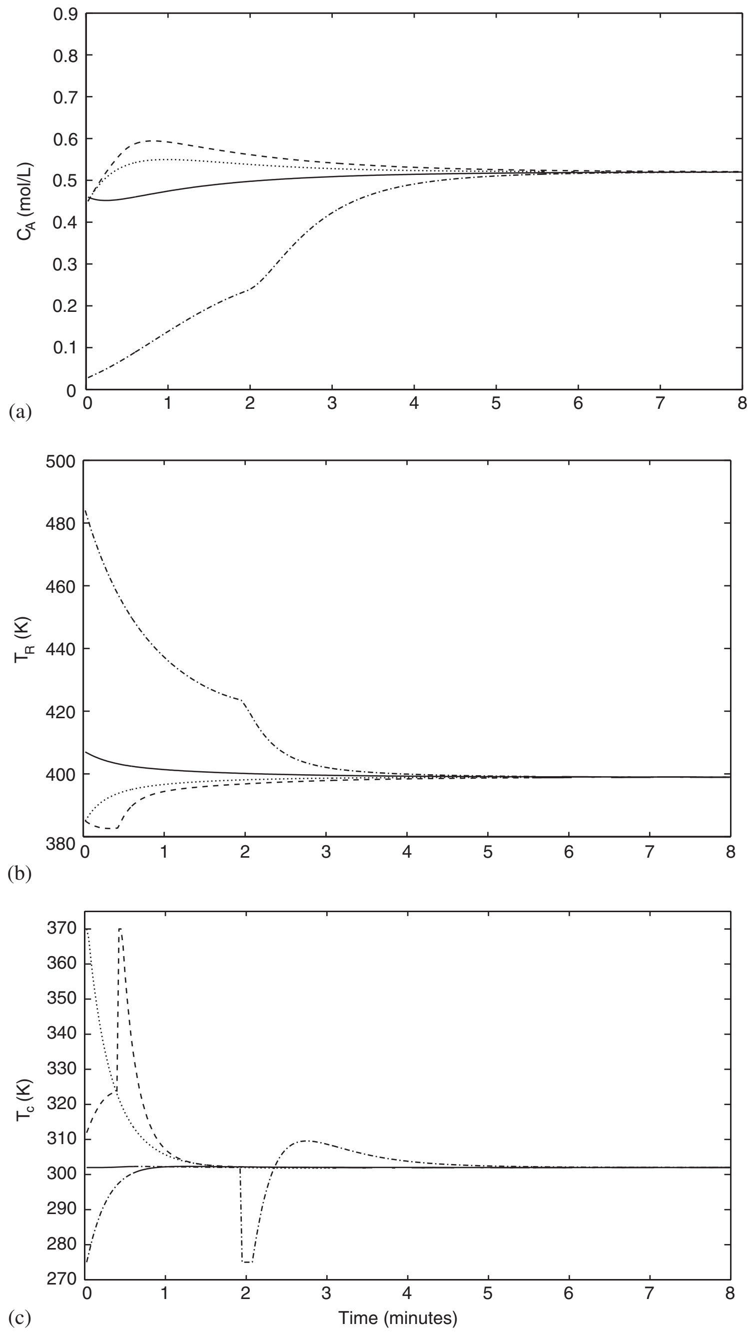

Figure 4. Closed-loop reactant concentration profile (a), reactor temperature profile (b), and coolant temperature profile (c) under MPC with $T = 0.25$ (solid trajectory), under the switched MPC/bounded controller(3) with $T = 0.25$ (dashed trajectory), under MPC with $T = 0.5$ (dotted trajectory), and under the switched bounded controller(2)/MPC with $T = 0.5$ (dash–dotted line).

feasibility of linear MPC. At $t = 1.925$, the predictive controller with $T = 0.5$ is found to be feasible and is, therefore, employed to asymptotically stabilize the closed-loop system.

To demonstrate some of the performance benefits of using the more flexible switching rules in Theorem 2, we consider the same control problem, described above, with relaxed constraints on the manipulated input: $250\ \mathrm{K} \leqslant T_c \leqslant 500\ \mathrm{K}$. Using these constraints, a new set of four bounded controllers are designed using a family of four CLFs of the form $V_k = \xi' P_k \xi,\ k = 1, 2, 3, 4$, where

$$P_1 = \begin{bmatrix} 1.03 & 0.32 \\ 0.32 & 3.26 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.4 & 0.32 \\ 0.32 & 1.28 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1.45 & 1.0 \\ 1.0 & 1.45 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 4.78 & 2.24 \\ 2.24 & 2.14 \end{bmatrix} \tag{22}$$

The stability region estimates of the controllers, $\Omega'_k,\ k = 1, 2, 3, 4$, are depicted in Figure 5. The relaxed input constraints are also incorporated in the design of the predictive controller, using the same MPC formulation employed in the preceding simulations. Starting from the initial condition $[C_A(0)\ \ T_R(0)]' = [0.75\ \ 361.0]'$ (Figures 6(a)–6(c)), the predictive controller, with $T = 0.1$, does not yield a feasible solution and, therefore, the supervisor implements the first bounded controller, using V_1, instead. At $t = 0.95$, however, MPC yields a feasible solution and, therefore, the supervisor switches to MPC. Even though $\dot{V}_1 > 0$ at this time, recognizing the fact that the state at this time ($[0.84\ \ 363.3]'$, denoted by Δ in Figure 5) belongs to Ω_2' and that $\dot{V}_2 < 0$, the supervisor continues to implement MPC while monitoring $\dot{V}_2$ (instead of V_1). At $t = 1.1$, the supervisor detects that $\dot{V}_2 > 0$. However, the state at this time ($[0.84\ \ 378.0]'$, denoted by $\diamondsuit$ in Figure 5) is within $\Omega_3{}'$ where $\dot{V}_3 < 0$. Therefore, the supervisor continues the implementation of

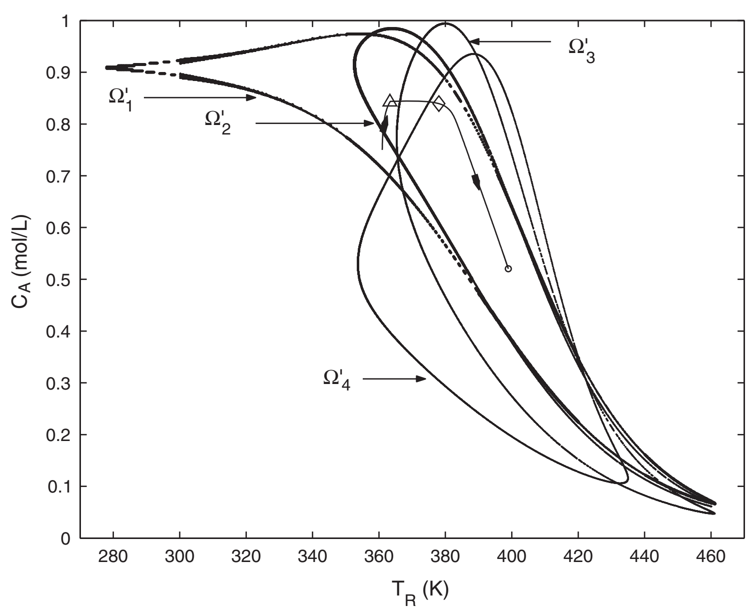

Figure 5. Closed-loop state trajectory under the switching rules of Theorem 2 with $T = 0.1$ for MPC.

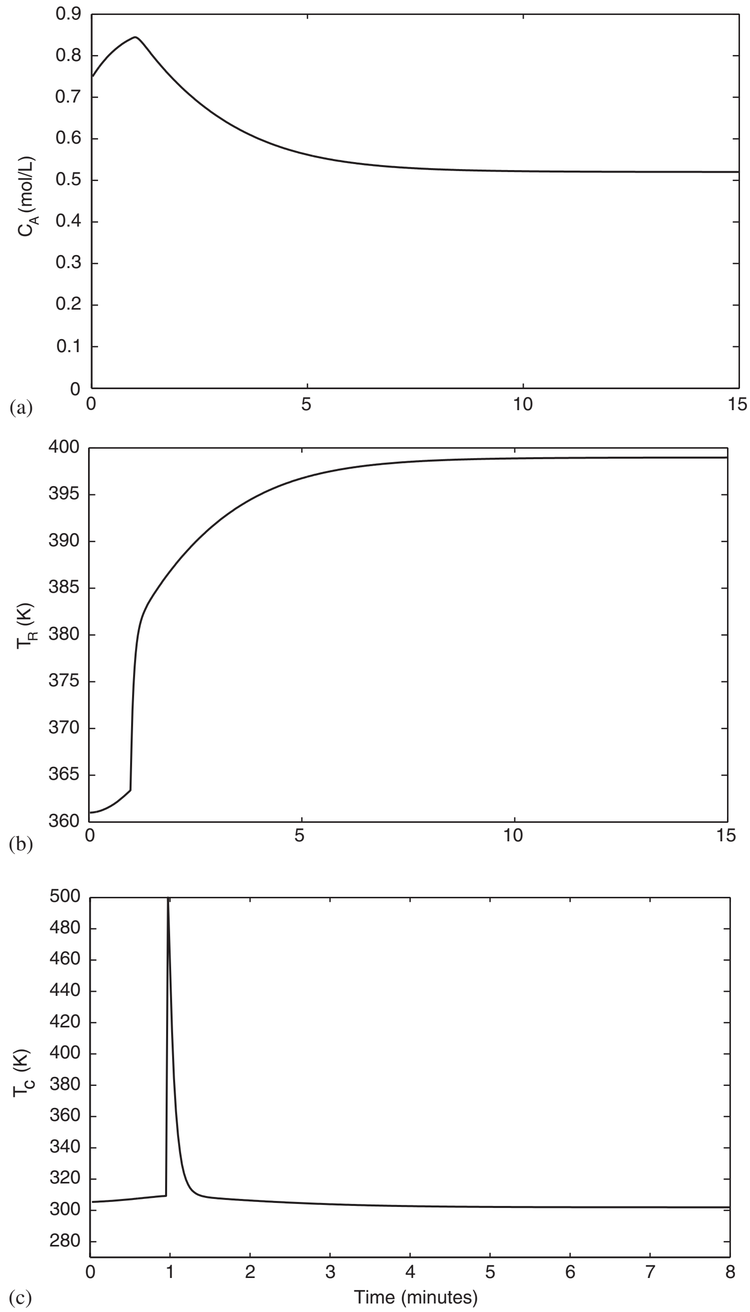

Figure 6. Closed-loop reactant concentration profile (a), reactor temperature profile (b), and coolant temperature profile (c) under the switching scheme of Theorem 2 with $T = 0.1$ for MPC.

MPC while monitoring V_3. From this point onwards, V_3 continues to decay monotonically and MPC is implemented in the closed-loop system for all future times to achieve asymptotic stability. Note that the switching scheme of Theorem 1 would have dictated a switch back to the first bounded controller at $t = 0.95$ and would not have allowed for MPC to be implemented in closed-loop, leading to a total cost of $J = 1.81 \times 10^6$ for the objective function of Equation (10). The switching rules in Theorem 2, on the other hand, allow the implementation of MPC for all the times that it is feasible, leading to a lower total cost of $J = 1.64 \times 10^5$, while guaranteeing, at the same time, closed-loop stability.

4.2. Application to a continuous crystallizer example

We consider a continuous crystallizer described by a fifth-order moment model of the following form:

$$\begin{aligned}
\dot{x}_0 &= -x_0 + (1 - x_3)\,\mathrm{Da}\exp\left(\frac{-F}{y^2}\right) \\
\dot{x}_1 &= -x_1 + yx_0 \\
\dot{x}_2 &= -x_2 + yx_1 \\
\dot{x}_3 &= -x_3 + yx_2 \\
\dot{y} &= \frac{1 - y - (\alpha - y)yx_2}{1 - x_3} + \frac{u}{1 - x_3}
\end{aligned} \tag{23}$$

where x_i, $i = 0, 1, 2, 3$, are dimensionless moments of the crystal size distribution, y is a dimensionless concentration of the solute in the crystallizer, and u is a dimensionless concentration of the solute in the feed (the reader may refer to References [31, 32] for a detailed process description, population balance modelling of the crystal size distribution and derivation of the moments model, and to Reference [33] for further results and references in this area). The values of the dimensionless process parameters are chosen to be: $F = 3.0$, $\alpha = 40.0$ and $\mathrm{Da} = 200.0$. For these values, and at the nominal operating condition of $u^{\mathrm{nom}} = 0$, the above system has an unstable equilibrium point, surrounded by a stable limit cycle. The control objective is to stabilize the system at the unstable equilibrium point, $x^s = [x_0^s\ x_1^s\ x_2^s\ x_3^s\ y^s]' = [0.0471, 0.0283, 0.0169, 0.0102, 0.5996]'$, where the superscript s denotes the desired steady state, by manipulating the dimensionless solute feed concentration, u, subject to the constraints: $-1 \leqslant u \leqslant 1$.

To facilitate the design of the bounded controller, we initially transform the system of Equation (23) into the normal form. To this end, we define the auxiliary output variable, $\bar{y} = h(x) = x_0$, and introduce the invertible co-ordinate transformation: $[\xi'\ \eta']' = T(x) = [x_0\ f_1(x)\ x_1\ x_2\ x_3]'$, where $\xi = [\xi_1\ \xi_2]' = [x_0\ f_1(x)]'$, $\bar{y} = \xi_1$, $f_1(x) = -x_0 + (1 - x_3)\,\mathrm{Da}\exp(-F/y^2)$, and $\eta = [\eta_1\ \eta_2\ \eta_3]' = [x_1\ x_2\ x_3]'$. The state–space description of the system in the transformed co-ordinates takes the form

$$\begin{aligned}
\dot{\xi} &= A\xi + bl(\xi, \eta) + b\alpha(\xi, \eta)u \\
\dot{\eta} &= \Psi(\eta, \xi)
\end{aligned} \tag{24}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = [0\ 1]', \quad l(\xi,\eta) = L_f^2 h(T^{-1}(\xi,\eta))$$

$L_f^2 h(T^{-1}(\xi,\eta))$ is the second-order Lie derivative of the scalar function, $h(\cdot)$, along the vector field $f(\cdot)$, and $\alpha(\xi,\eta) = L_g L_f h(T^{-1}(\xi,\eta))$ is the mixed Lie derivative. The forms of $f(\cdot)$ and $g(\cdot)$ can be obtained by re-writing the system of Equation (23) in the form of Equation (1), and are omitted for brevity.

The partially linear ξ-subsystem in Equation (24) is used to design a bounded controller that stabilizes the full interconnected system of Equation (24) and, consequently, the original system of Equation (23). For this purpose, a quadratic function of the form, $V_\xi = \xi' P \xi$, is used as a CLF in the controller synthesis formula of Equation (3), where the positive-definite matrix, P, is chosen to satisfy the Riccati matrix equality: $A'P + PA - Pbb'P = -Q$ where Q is a positive-definite matrix. An estimate of the region of constrained closed-loop stability for the full system is obtained by defining a composite Lyapunov function of the form $V_c = V_\xi + V_\eta$, where $V_\eta = \eta' P_\eta \eta$ and P_η is a positive-definite matrix, and choosing a level set of V_c, Ω_c, for which $\dot{V}_c < 0$ for all x in Ω_c. The two-dimensional projections of the stability region are shown in Figure 7 for all possible combinations of the system states.

In designing the predictive controller, a linear MPC formulation, with a terminal equality constraint of the form $x(t+T) = 0$, is chosen (based on the linearization of the process model of Equation (23) around the unstable equilibrium point). The parameters in the objective function of Equation (10) are taken to be: $Q = qI$, with $q = 1$, $R = rI$, with $r = 1.0$, and $F = 0$. We also choose a horizon length of $T = 0.5$ in implementing the predictive controller. The resulting quadratic programme is solved using the MATLAB subroutine QuadProg, and the nonlinear closed-loop system is integrated using the MATLAB solver ODE45.

In the first set of simulation runs, we test the ability of the predictive controller to stabilize the closed-loop system starting from the initial condition, $x(0) = [0.046\ 0.0277\ 0.0166\ 0.01\ 0.58]'$. The result is shown by the solid lines in Figure 8(a)–8(e) where it is seen that the predictive controller, with a horizon length of $T = 0.5$, is able to stabilize the closed-loop system at the desired equilibrium point. Starting from the initial condition $x(0) = [0.032\ 0.035\ 0.010\ 0.009\ 0.58]'$, however, the predictive controller yields no feasible solution. If the terminal equality constraint is removed, to make MPC yield a feasible solution, we see from the dashed lines in Figure 8(a)–8(e) that the resulting control action cannot stabilize the closed-loop system and sends the system states into a limit cycle. On the other hand, when the switching scheme of Theorem 1 is employed, the supervisor immediately switches to the bounded controller which in turn stabilizes the closed-loop system at the desired equilibrium point. This is depicted by the dotted lines in Figure 8(a)–8(e). The manipulated input profiles for the three scenarios are shown in Figure 8(f).

5. CONCLUDING REMARKS

In this work, a hybrid control structure, uniting bounded control with MPC, was proposed for the stabilization of nonlinear systems with input constraints. The structure consists of: (1) a high-performance model predictive controller, (2) a family of fall-back Lyapunov-based

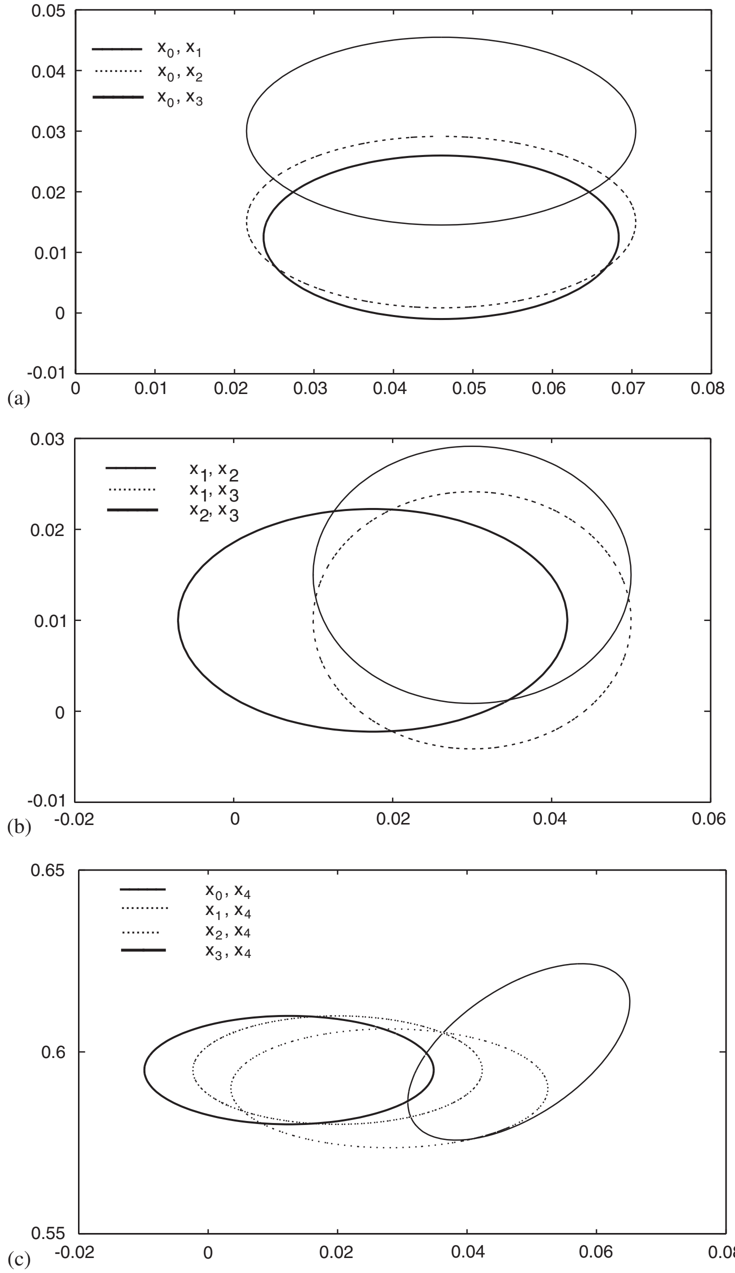

Figure 7. Implementation of the proposed hybrid control structure to a continuous crystallizer: two-dimensional projections of the stability region for the ten distinct combinations of the process states.

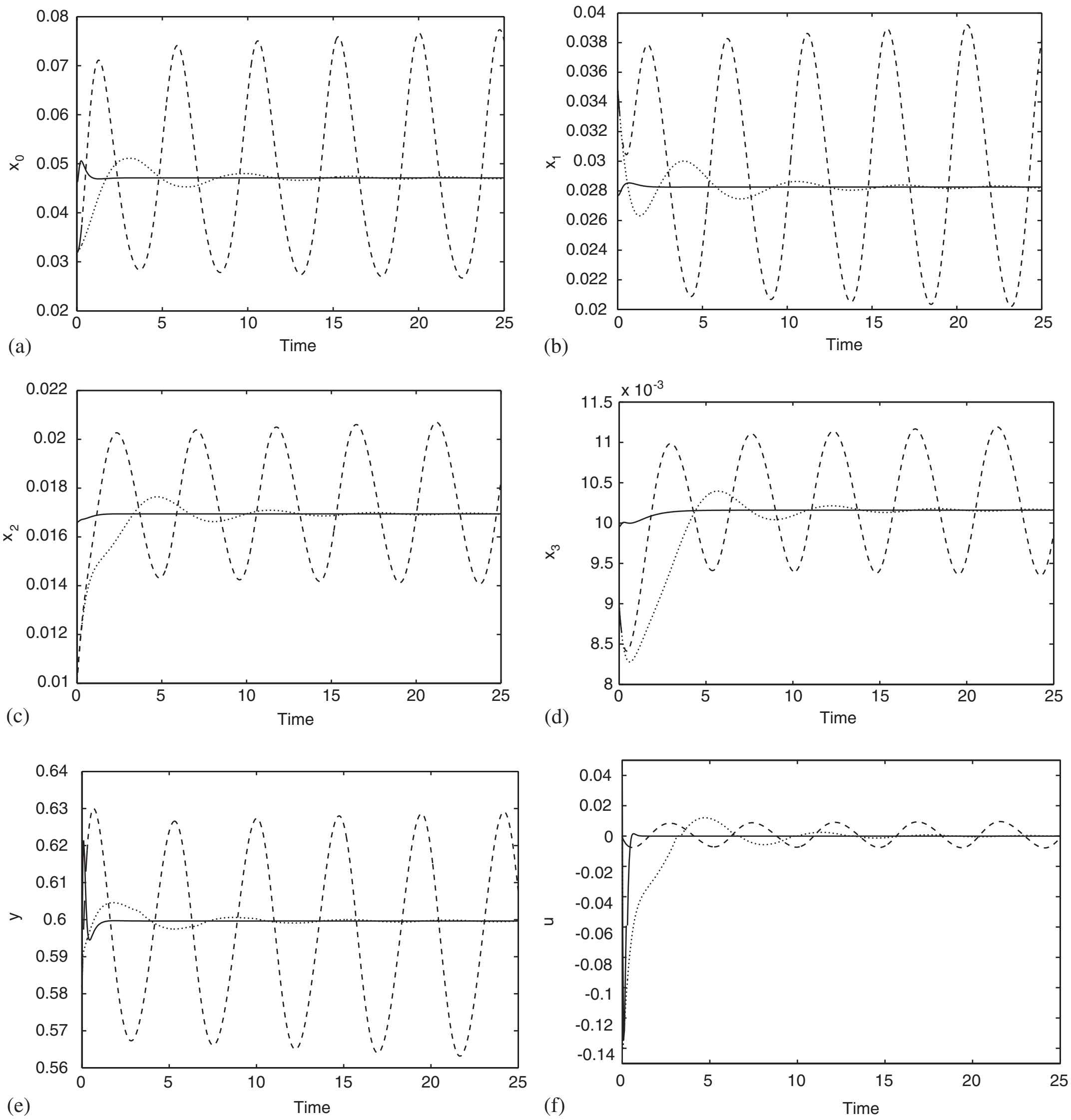

Figure 8. Closed-loop profiles of the dimensionless crystallizer moments (a)–(d), the solute concentration in the crystallizer (e), and the manipulated input (f) under MPC with stability constraints (solid line), under MPC without terminal constraints (dashed line), and using the switching scheme of Theorem 1 (dotted line).

bounded nonlinear controllers, each with a well-defined stability region and (3) a high-level supervisor that orchestrates switching between MPC and the bounded controllers in a way that safeguards closed-loop stability in the event of MPC instability or infeasibility. The basic idea was to embed the implementation of MPC within the stability regions of the bounded controllers, and derive a set of supervisory switching rules that monitor the evolution of the closed-loop trajectory and place appropriate restrictions on the growth of the Lyapunov functions in a way that guarantees asymptotic stability for all initial conditions within the union of all stability regions of the bounded controllers. By tailoring the switching logic appropriately, the hybrid control structure was shown to provide, irrespective of the chosen MPC formulation, a safety net for the implementation of predictive control algorithms to constrained nonlinear systems. Finally, the implementation of the switching schemes was demonstrated through applications to chemical reactor and crystallization process examples.

APPENDIX A

Proof of Theorem 1

Step 1: Substituting the control law of Equations (3)–(4) into the system of Equation (1) and evaluating the time derivative of the Lyapunov function along the trajectories of the closed-loop system, it can be shown that $\dot{V}_k < 0$ for all $x \in \Phi_k(u_{\max})$ (where $\Phi_k(u_{\max})$ was defined in Equation (5). Since $\Omega_k(u_{\max})$ (defined in Equation (6)) is an invariant subset of $\Phi_k(u_{\max}) \cup \{0\}$, it follows that for any $x(0) \in \Omega_k(u_{\max})$, the origin of the closed-loop system, under the control law of Equations (3)–(4), is asymptotically stable.

Step 2: Consider the switched closed-loop system of Equation (12), subject to the switching rule of Equations (14)–(15), with any initial state $x(0) \in \Omega_k(u_{\max})$. From the definition of $\bar{T}$ given in Theorem 1, it is clear that if $\bar{T}$ is a finite number, then $\dot{V}_k(x^M(t)) < 0 \ \forall \ 0 \leqslant t < \bar{T}$, where the notation $x^M(t)$ denotes the closed-loop state under MPC at time t, which implies that $x(t) \in \Omega_k(u_{\max}) \ \forall 0 \leqslant t < \bar{T}$ (or that $x(\bar{T}^-) \in \Omega_k(u_{\max})$). This fact, together with the continuity of the solution of the switched system, $x(t)$, (following from the fact that the right-hand side of Equation (1) is continuous in x and piecewise continuous in time) implies that, upon switching (instantaneously) to the bounded controller at $t = \bar{T}$, we have $x(\bar{T}) \in \Omega_k(u_{\max})$ and $u(t) = b_k(x(t))$ for all $t \geqslant \bar{T}$. Therefore, from our analysis in step 1, we conclude that $\dot{V}_k(x^{b_k}(t)) < 0 \ \forall \ t \geqslant \bar{T}$. In summary, the switching rule of Equations (14)–(15) guarantees that, starting from any $x(0) \in \Omega_k(u_{\max})$, $\dot{V}_k(x(t)) < 0 \ \forall x \neq 0$, $x \in \Omega_k(u_{\max})$, $\forall \ t \geqslant 0$, which implies that the origin of the switched closed-loop system is asymptotically stable. Note that if no such $\bar{T}$ exists, then we simply have from Equations (14)–(15) that $\dot{V}_k(x^M(t)) < 0 \ \forall \ t \geqslant 0$, and the origin of the closed-loop system is also asymptotically stable. This completes the proof of the theorem. □

Proof of Theorem 2

The proof of this theorem, for the case when only one bounded controller is used as the fall-back controller (i.e. $p = 1$), is same as that of Theorem 1. To simplify the proof, we prove the result only for the case when the family of fall-back controllers consists of two controllers (i.e. $p = 2$). Generalization of the proof to the case of a family of p controllers, where $2 < p < \infty$, is conceptually straightforward. Also, without loss of generality, we consider the case when the optimization problem in MPC is feasible for all times (i.e. $T^i = \infty$), since if it is not (i.e. $T^i < \infty$), then the switching time is simply taken to be the minimum of $\{T^i, T^f\}$, as stated in Theorem 2.

Without loss of generality, let the closed-loop system be initialized within the stability region of the first bounded controller, $\Omega_1(u_{\max})$, under MPC. Then one of the following scenarios will take place:

Case 1: $\dot{V}_1(x^M(t)) < 0$ for all $t \geqslant 0$. The switching law of Equations (16)–(18) dictates in this case that MPC be implemented for all times. Since $x(0) \in \Omega_1(u_{\max})$, where Ω_1 is a level set of V_1, and $\dot{V}_1 < 0$, then the state of the closed-loop system is bounded and converges to the origin as $t \to \infty$.

Case 2: $\dot{V}_1(x^M(T_1)) \geqslant 0$ and $x^M(T_1) \in \Omega_1(u_{\max})$, for some finite $T_1 > 0$. In this case, one of the following scenarios will occur:

(a) If $x(T_1)$ is outside of $\Omega_2(u_{\max})$, then it follows from Equations (16)–(18) that the supervisor will set $T^* = T_f = T_1$ and switch to the first bounded controller at $t = T_1$, using V_1 as the CLF, which enforces asymptotic stability as discussed in step 2 of the proof of Theorem 1.

(b) If $x(T_1) \in \Omega_2(u_{\max})$ and $\dot{V}_2(x(T_1)) < 0$ (i.e. $0 < T_1 < T_2$), then the supervisor will keep MPC in the closed-loop system at T_1. If T_2 is a finite number, then the supervisor will set $T^* = T_f = T_2$ (since $T_1^* = 0$ for all $t > T_1$ from Equation (17)) at which time it will switch to the second bounded controller, using V_2 as the CLF. Since $\dot{V}_2(x^M(t)) < 0$ for all $T_1 \leqslant t < T_2$, and $x(T_1) \in \Omega_2(u_{\max})$, then $x(T^{*-}) \in \Omega_2(u_{\max})$. By continuity of the solution of the closed-loop system, it follows that $x(T^*) \in \Omega_2(u_{\max})$; and since $\Omega_2(u_{\max})$ is the stability region corresponding to V_2, this implies that upon implementation of the corresponding bounded controller for all future times, asymptotic stability is achieved. Note that if T_2 does not exist (or $T_2 = \infty$), then we simply have $x(T_1) \in \Omega_2(u_{\max})$ and $\dot{V}_2(x(T_1)) < 0$ for all $t \geqslant T_1$, which implies that the origin of the closed-loop system is again asymptotically stable.

(c) If $0 < T_2 < T_1$ (i.e. $x(T_2) \in \Omega_2(u_{\max})$, $V_1(x(T_2)) < 0$ and $V_2(x(T_2)) \geqslant 0$), then it follows from Equations (10)–(18) that the supervisor will set $T^* = T_f = T_1$ (since $T_2^* = 0$ for all $t > T_2$ from Equation (17)) and switch to the first bounded controller, using V_1 as the CLF. Since $\dot{V}_1(x^M(t)) < 0$ for all $t < T_1$, and $x(0) \in \Omega_1(u_{\max})$, then $x(T^{*-}) \in \Omega_1(u_{\max})$. By continuity of the solution of the closed-loop system, we have $x(T^*) \in \Omega_1(u_{\max})$, and since $\Omega_1(u_{\max})$ is the stability region corresponding to V_1, this implies that upon implementation of the first bounded controller for the remaining time, asymptotic closed-loop stability is achieved. This completes the proof of the theorem. □

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