Available online at www.sciencedirect.com
SCIENCE DIRECT®
Systems & Control Letters 54 (2005) 1163–1182
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Output feedback control of switched nonlinear systems using multiple Lyapunov functions☆

Nael H. El-Farra[1], Prashant Mhaskar, Panagiotis D. Christofides*

Department of Chemical Engineering, University of California, Los Angeles, CA 90095-1592, USA

Received 13 January 2003; received in revised form 17 June 2003; accepted 2 April 2005
Available online 26 May 2005

Abstract

This work presents a hybrid nonlinear control methodology for a broad class of switched nonlinear systems with input constraints. The key feature of the proposed methodology is the integrated synthesis, via multiple Lyapunov functions, of "lower-level" bounded nonlinear feedback controllers together with "upper-level" switching laws that orchestrate the transitions between the constituent modes and their respective controllers. Both the state and output feedback control problems are addressed. Under the assumption of availability of full state measurements, a family of bounded nonlinear state feedback controllers are initially designed to enforce asymptotic stability for the individual closed-loop modes and provide an explicit characterization of the corresponding stability region for each mode. A set of switching laws are then designed to track the evolution of the state and orchestrate switching between the stability regions of the constituent modes in a way that guarantees asymptotic stability of the overall switched closed-loop system. When complete state measurements are unavailable, a family of output feedback controllers are synthesized, using a combination of bounded state feedback controllers, high-gain observers and appropriate saturation filters to enforce asymptotic stability for the individual closed-loop modes and provide an explicit characterization of the corresponding output feedback stability regions in terms of the input constraints and the observer gain. A different set of switching rules, based on the evolution of the state estimates generated by the observers, is designed to orchestrate stabilizing transitions between the output feedback stability regions of the constituent modes. The differences between the state and output feedback switching strategies, and their implications for the switching logic, are discussed and a chemical process example is used to demonstrate the proposed approach.

Keywords: Switched systems; Multiple Lyapunov functions; Input constraints; Output feedback control; Bounded nonlinear control; Hybrid chemical processes

☆ Financial support from NSF, CTS-0129571, is gratefully acknowledged.
*Corresponding author. Tel.: +1 310 794 1015; fax: +1 310 206 4107.
E-mail addresses: nhelfarra@ucdavis.edu (N.H. El-Farra), mhaskar@seas.ucla.edu (P. Mhaskar), pdc@seas.ucla.edu (P.D. Christofides).
[1]Current address: Department of Chemical Engineering & Materials Science, University of California, Davis, One Shields Avenue, Davis, CA 95616-5294.

0167-6911/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2005.04.005

1. Introduction

The study of hybrid systems in control is motivated by the fundamentally hybrid nature of many modern-day control systems, which are characterized by the interaction of lower-level continuous dynamics and upper-level discrete or logical components. In many of these systems, the continuous dynamics arise from the underlying physical laws such as mass, momentum, and energy conservation, and are usually modelled by continuous-time differential equations. Discrete events, on the other hand, can arise from a variety of sources, including inherent physico-chemical discontinuities in the continuous dynamics (e.g., phase changes, flow reversals), controlled transitions between different operating regimes, the use of discrete actuators and sensors in the control system (e.g., on/off valves, binary sensors), and the use of logic-based switching for supervisory and safety control. It is well understood at this stage that the interaction of discrete events with even simple continuous dynamical systems can lead to complex dynamics and, possibly, to undesirable outcomes if not appropriately accounted for in the control system design.

Even though theory for the analysis and control of purely continuous-time systems exists and, to a large extent, is well-developed, similar techniques for combined discrete-continuous systems are limited at present, primarily due to the difficulty of extending the available concepts and tools to treat the hybrid nature of these systems and their changing dynamics. Motivated by this and the abundance of situations where hybrid systems arise in practice, significant research work has focused on hybrid systems over the last decade, covering a broad range of problems including, for example, modeling [37,2], simulation [2,15,14], optimization [16], and control [26,22,5,3,29].

A class of hybrid systems that has attracted significant attention, because it can model several practical control problems that involve the integration of supervisory logic-based control schemes and feedback control algorithms, is the class of switched (or multi-modal) systems. For this class, results have been developed for stability analysis using the tools of multiple Lyapunov functions for linear [30] and nonlinear systems [31,4,38], and the concept of dwell-time [17]; the reader may refer to [24,7] for a survey of results in this area. These results have motivated the development of methods for control of various classes of switched systems (e.g., [36,39,18,8]). Despite much progress, significant research remains to be done in the direction of nonlinear and constrained control of switched systems, especially since the majority of practical switched systems exhibit inherently nonlinear dynamics and the control action is often subject to hard actuator constraints. In [10], a hybrid control methodology for a class of switched nonlinear systems with input constraints, that accounts for the interactions between the "lower-level" constrained continuous dynamics and the "upper-level" discrete or logical components, was developed. The key idea was the integrated synthesis, via multiple Lyapunov functions (MLFs), of bounded nonlinear continuous controllers and switching laws that orchestrate the transitions between the constituent control modes to guarantee stability of the switched closed-loop system. The proposed methodology was extended to switched systems with both input constraints and model uncertainty, and applied to the problem of fault-tolerant control of chemical processes in [12].

In addition to input constraints, another important issue that must be accounted for in the control system design is the lack of complete state measurements. For switched systems, this issue affects both the design and implementation of the lower-level controllers and the upper-level switching laws which both have to be based on state estimates. Motivated by these considerations, we present in this work a nonlinear output feedback control method for a class of switched nonlinear systems with input constraints. The key idea is the coupling between the switching logic and the stability regions arising from the limitations imposed by both input constraints and the lack of full state measurements on the dynamics of the constituent modes of the switched system. Using MLFs, the proposed method involves the integration of output feedback control and switching based on state estimates. A family of output feedback controllers are synthesized, using a combination of bounded state feedback controllers, high-gain observers and appropriate saturation filters to enforce asymptotic stability for the individual closed-loop modes and provide an explicit characterization of the corresponding output feedback stability regions in terms of the input constraints and the observer gain. A set of switching rules that track the evolution of the state estimates generated by the observers are then designed to orchestrate stabilizing transitions between the output feedback stability regions of the constituent modes.

The remainder of the manuscript is organized as follows. In Section 2, we present the class of switched systems considered and briefly review MLF stability analysis. In Section 3, we address the state feedback control problem to provide the necessary background for the output feedback control problem which is addressed in Section 4. The differences between the state and output feedback switching strategies, and their implications for the design and implementation of the switching logic, are discussed. Finally, in Section 5, the proposed methodology is demonstrated using a chemical process example.

2. Preliminaries

2.1. Class of systems

We consider the class of switched nonlinear systems represented by the following state-space description:

$$\begin{aligned}
&\dot{x}(t) = f_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))u_{\sigma(t)} \\
&y_m = h_m(x) \\
&\|u_\sigma\| \leqslant u_\sigma^{\max}, \quad \sigma(t) \in \mathscr{I} = \{1, \ldots, N\},
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of continuous-time state variables, $u_\sigma(t) = [u_\sigma^1(t) \cdots u_\sigma^m(t)]^{\mathrm{T}}$ denotes the vector of manipulated inputs taking values in the nonempty compact subset $\mathscr{U} := \{u_\sigma \in \mathbb{R}^m : \|u_\sigma\| \leqslant u_\sigma^{\max}\}$, $y_m \in \mathbb{R}^m$ denotes the vector of measured variables, $h_m(x)$ is a sufficiently smooth function on $\mathbb{R}^m$, $\sigma : [0, \infty) \rightarrow \mathscr{I}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, i.e., $\sigma(t_k) = \lim_{t \rightarrow t_k^+} \sigma(t)$ for all k, implying that only a finite number of switches is allowed on any finite interval of time. The variable, $\sigma(t)$, which takes values in the finite index set, $\mathscr{I}$, represents a discrete state that indexes the vector field $f(\cdot)$, the matrix $G(\cdot)$, and the control input $u(\cdot)$, which altogether determine $\dot{x}$. For each value that σ assumes in $\mathscr{I}$, the temporal evolution of the continuous state, x, is governed by a different set of differential equations. Systems of the form of Eq. (1) are therefore referred to as multi-modal, or of variable structure. They consist of a finite family of N continuous-time nonlinear subsystems (or modes) and some rules for switching between them. These rules define a switching sequence that describes the temporal evolution of the discrete state. Throughout the paper, we use the notations t_{i_k} and $t_{i'_k}$ to denote, the kth times that the ith subsystem is switched in and out, respectively, i.e., $\sigma(t_{i_k}^+) = \sigma(t_{i'_k}^-) = i$, for all $k \in Z_+$. With this notation, it is understood that the continuous state evolves according to $\dot{x} = f_i(x) + G_i(x)u_i$ for $t_{i_k} \leqslant t < t_{i'_k}$.

It is assumed that all entries of the vector functions $f_i(x)$ and the $n \times m$ matrices $G_i(x)$, are sufficiently smooth on $\mathbb{R}$. Without loss of generality, we assume that $f_i(0) = 0$ for all $i \in \mathscr{I}$. We also assume that the state x does not jump at the switching instants, i.e., the solution $x(\cdot)$ is everywhere continuous. Note that changes in the discrete state $\sigma(t)$ (i.e., transitions between the continuous dynamical modes) may, in general, be a function of time, state or both. When changes in $\sigma(t)$ depend only on inherent process characteristics, the switching is referred to as autonomous. However, when $\sigma(t)$ is chosen by some higher process such as a controller or human operator, then the switching is controlled. In this paper, we focus on controlled switching where mode transitions are decided and executed by some higher-level supervisor. This class of systems arises naturally in the context of coordinated supervisory and feedback control of chemical process systems (see Section 5 for an example).

2.2. Stability analysis via multiple Lyapunov functions

Preparatory for its use in control, we will briefly review in this section the main idea of MLFs as a tool for stability analysis of switched systems. To this end, consider the switched system of Eq. (1), with $u_i(t) \equiv 0, i \in \mathscr{I}$, and suppose that we can find a family of Lyapunov-like functions $\{V_i : i \in \mathscr{I}\}$, each associated with the vector field $f_i(x)$. A Lyapunov-like function for the system $\dot{x} = f_i(x)$, with equilibrium point $x_{\mathrm{eq}} = 0 \in \Omega_i \subset \mathbb{R}^n$, is a real-valued function $V_i(x)$, with continuous partial derivatives, defined over the region Ω_i, satisfying the conditions: (1)

$V_i(0) = 0$ and $V_i(x) > 0$ for $x \neq 0$, $x \in \Omega_i$, and (2) $\dot{V}_i = [\partial V_i(x)/\partial x] f_i(x) \leqslant 0$, for $x \in \Omega_i$. The following theorem provides sufficient conditions for stability.

Theorem 1 (*Decarlo et al., 2000 [7], see also Branicky [4]*). *Given the N-switched nonlinear system of Eq.* (1), *with $u_i(t) \equiv 0$, $i \in \mathscr{I}$, suppose that each vector field f_i has an associated Lyapunov-like function V_i in the region Ω_i, each with equilibrium point $x_{\mathrm{eq}}=0$, and suppose $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $\sigma(t)$ be a given switching sequence such that $\sigma(t)$ can take on the value of i only if $x(t) \in \Omega_i$, and in addition*

$$V_i(x(t_{i_k})) \leqslant V_i(x(t_{i_{k-1}})), \tag{2}$$

where t_{i_k} denotes the kth time that the vector field f_i is switched in, i.e., $\sigma(t_{i_k}^-) \neq \sigma(t_{i_k}^+) = i$. Then, the equilibrium point, $x_{\mathrm{eq}} = 0$, of the system of Eq. (1), *with $u_i(t) \equiv 0$, $i \in \mathscr{I}$, is Lyapunov stable.*

Remark 1. From the definition of a Lyapunov-like function, V_i is monotonically non-increasing on every time interval where the ith subsystem is active, and the set Ω_i represents the part of the state space where $\dot{V}_i \leqslant 0$. The idea of Theorem 1 above is that even if there exists such a Lyapunov function for each subsystem, f_i, individually (i.e., each mode is stable), restrictions must be placed on the switching scheme to guarantee stability of the overall switched system. In fact, it is possible to construct examples of globally asymptotically stable systems and a switching rule that sends all trajectories to infinity (see [4] for some examples). A sufficient condition to guarantee Lyapunov stability is to require, as in Eq. (2), that for every mode i, the value of V_i at the beginning of each interval on which the ith subsystem is active not exceed the value at the beginning of the previous such interval. Some variations and generalizations of this result are discussed in [31,38]. A stronger condition than the one proposed in Eq. (2) will be used in Theorem 2 to enforce asymptotic stability (see Section 3.2).

Having reviewed how multiple Lyapunov functions can be used to analyze the stability of switched nonlinear systems without control inputs, we proceed in the next two sections to use the MLF framework as a tool for control of switched nonlinear systems with input constraints. For a clear presentation of the main results of this paper, we will start in Section 3 by reviewing the state feedback control problem (i.e., with $y_m = x$) which will provides the necessary foundation for formulating and solving the output feedback control problem in Section 4.

3. State feedback control of switched nonlinear systems

Referring to the system of Eq. (1), we first consider the state feedback control problem, i.e., $h_m(x) = x$. For purely continuous-time systems, the idea of using a candidate Lyapunov function for designing feedback controllers has been used extensively, though made explicit only with the introduction of the concept of a control Lyapunov function.

Definition 1 (*Sontag [33]*). A control Lyapunov function (CLF) for a nonlinear control system of the form $\dot{x} = f(x) + G(x)u$ is a smooth, proper, and positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ with the property that for every fixed $x \neq 0$, there exists admissible values $u^1, \ldots, u^m$ for the controls such that

$$\inf_{u \in \mathscr{U}} \{L_f V + L_{g_1} V u^1 + \cdots + L_{g_m} V u^m\} < 0, \tag{3}$$

where $L_f V = [\partial V / \partial x] f(x)$, g_i is the ith column of the matrix G.

Just as the existence of a Lyapunov function is necessary and sufficient for the stability of a system without inputs, the existence of a CLF is necessary and sufficient for the stabilizability of a system with a control input [1,33]. Note from Eq. (3) that, for a CLF, $(L_G V)^{\mathrm{T}}(x) = 0 \implies L_f V(x) < 0$ for all $x \neq 0$, where $L_G V = [L_{g_1} V \cdots L_{g_m} V]$. By definition, any Lyapunov function whose time-derivative can be rendered negative-definite via control is a CLF. The importance of the CLF concept is that, when a CLF is known, a stabilizing control law can be selected from a choice of explicit expressions (such as those in [25]). In the context of control of switched systems, a generalization of this concept, similar to the MLF idea, is that of multiple control Lyapunov functions (MCLFs). The idea is to use a family of control Lyapunov functions, one for each subsystem, to: (a) design a family of nonlinear feedback controllers that stabilize the individual subsystems, and (b) design a set of stabilizing switching laws that orchestrate the transition between the constituent modes and their respective controllers. In many ways, the connections between MLFs and MCLFs for switched systems conceptually parallel the connections between classical Lyapunov functions and CLFs for continuous systems.

3.1. Control problem formulation

Consider the switched nonlinear system of Eq. (1). Given that switching is controlled by some higher-level supervisor, the problem we focus on is how to orchestrate switching between the various subsystems in a way that respects the constraints and guarantees asymptotic closed-loop stability. To this end, we formulate the following two control objectives. The first is to synthesize a family of N bounded nonlinear state feedback controllers of the form

$$u_i = -k_i(x)(L_{G_i}V_i)^{\mathrm{T}}, \quad i = 1, \ldots, N, \tag{4}$$

where V_i is a CLF for the ith mode and $L_{G_i}V_i$ is a row vector of the form $[L_{g_i^1}V_i \cdots L_{g_i^m}V_i]$, that: (1) satisfy the constraints, (2) enforce asymptotic stability for the individual closed-loop subsystems, and (3) provide an explicit characterization of the set of admissible initial conditions starting from where each mode is guaranteed to be stable. The second objective is to identify a set of switching laws, $\sigma(t) = \psi(x)$, that orchestrate the transition between the constituent modes and their respective controllers in a way that respects the input constraints and guarantees asymptotic stability of the constrained switched closed-loop system.

In order to proceed with the design of the controllers, we need to impose the following assumption on the system of Eq. (1).

Assumption 1. For every $i \in \mathscr{I}$, *a CLF*, V_i, exists for the system $\dot{x} = f_i(x) + G_i(x)u_i$.

CLF-based stabilization of nonlinear systems has been studied extensively in the nonlinear control literature (e.g., see [1,13,32]). For several classes of nonlinear systems that arise commonly in the modeling of practical systems, systematic methods are available for constructing CLFs by exploiting the system structure (see Section 3.3 for a discussion of feedback linearizable systems and Section 5 for an example).

3.2. Switching rules under state feedback

Theorem 2 below provides a formula for the bounded nonlinear state feedback controllers used to stabilize the constituent subsystems and states switching conditions that guarantee the desired properties in the constrained switched closed-loop system. The proof of this theorem is given in the appendix.

Theorem 2. *Consider the switched nonlinear system of Eq.* (1), *for which a family of CLFs* V_i, $i=1, \ldots, N$ *exist*, *under the following family of bounded nonlinear feedback controllers*:

$$u_i = -k_i(x, u_i^{\max})(L_{G_i}V_i(x))^{\mathrm{T}}, \quad i = 1, \ldots, N, \tag{5}$$

where

$$k(x, u_i^{\max}) = \begin{cases} \dfrac{\alpha_i(x) + \sqrt{\alpha_i^2(x) + (u_i^{\max}\|b_i^{\mathrm{T}}(x)\|)^4}}{\|b_i^{\mathrm{T}}(x)\|^2[1+\sqrt{1+(u_i^{\max}\|b_i^{\mathrm{T}}(x)\|)^2}]}, & b_i^{\mathrm{T}}(x) \neq 0, \\ 0, & b_i^{\mathrm{T}}(x) = 0, \end{cases} \tag{6}$$

with $\alpha_i(x) = L_{f_i}V_i(x) + \rho_i V_i(x)$, $\rho_i > 0$ *and* $b_i(x) = L_{G_i}V_i(x)$. *Let* $\Phi_i(u_i^{\max})$ *be the largest set of x, containing the origin, such that*

$$L_{f_i}V_i(x) + \rho_i V_i(x) \leqslant u_i^{\max}\|(L_{G_i}V_i(x))^{\mathrm{T}}\|. \tag{7}$$

Also, let $\Omega_i^*(u_i^{\max}) := \{x \in \mathbb{R}^n : V_i(x) \leqslant \delta_{x,i}\}$ *be a level set of* V_i, *completely contained in* Φ_i, *for some* $\delta_{x,i} > 0$, *and assume, without loss of generality, that* $x(0) \in \Omega_i^*(u_i^{\max})$ *for some* $i \in \mathscr{I}$. *If, at any given time*, T, *the following conditions hold*:

$$x(T) \in \Omega_j^*(u_j^{\max}), \tag{8}$$

$$V_j(x(T)) < V_j(x(t_{j*})) \tag{9}$$

for some $j \in \mathscr{I}$, $j \neq i$, *where* $t_{j*} < T$ *is the time when the jth subsystem was last switched in, i.e.*, $\sigma(t_{j*}^-) \neq \sigma(t_{j*}^+) = j$, *then setting* $\sigma(t) = j$, *for* $t \geqslant T^+$, *guarantees that the origin of the switched closed-loop system is asymptotically stable.*

Remark 2. Note that asymptotic stability of each mode of the closed-loop system implies that there exists a family of class $\mathscr{KL}$ functions β_i, $i = 1, \ldots, N$ such that a bound of the following form holds for each closed-loop mode:

$$\|x(t)\| \leqslant \beta_i(\|x(0)\|, t). \tag{10}$$

This property will be used later in the design of the output feedback controllers.

Remark 3. Referring to the family of N nonlinear feedback controllers given in Eqs. (5) and (6), we note that they are synthesized, via multiple control Lyapunov functions, by reshaping the nonlinear gain of the bounded $L_G V$ controller proposed in [25] (see also [23,27,9,11] for examples of other extensions of the universal formula in [25]). The term $-\rho_i V_i$ is added to enforce (local) exponential stability which will be needed in designing the appropriate output feedback controllers. The continuity properties of the controllers can be analyzed using arguments similar to those presented in [32]. In particular, consider separately the open set, $\Pi = \{x | b_i(x)^{\mathrm{T}} \neq 0 \textit{ or } \alpha_i(x) < 0\}$, and its complement, $\Pi^{\mathrm{c}} = \mathbb{R}^n \backslash \Pi$. Inside Π, the control law of Eqs. (5) and (6) is a smooth function of x if $\alpha_i(\cdot)$ and $b_i^{\mathrm{T}}(\cdot)$ are smooth, since the top expression in Eq. (6), as a function of $\alpha_i \in \mathbb{R}$ and $b_i^{\mathrm{T}} \in \mathbb{R}^m$, is analytic when $b_i(x)^{\mathrm{T}} \neq 0$ or $\alpha_i(x) < 0$. Since V_i is a CLF, the set Π is the whole state space except for the origin (because of the strict inequality in Eq. (3)). Then the set Π^{c} is just the origin, $x = 0$. The control law of Eqs. (5) and (6) is continuous at the origin if and only if the CLF satisfies the small control property, which is a mild assumption on V_i since it is required to hold only at the origin. If V_i is not a CLF, the set Π^{c} may include points other than the origin in which case the continuity of the control law of Eqs. (5) and (6) would require the small control property to hold at every point of Π^{c}, which is a more restrictive assumption.

Remark 4. The use of CLF-based controllers of the form of Eqs. (5) and (6) is motivated by the fact that this class of controllers account explicitly for input constraints and provide an explicit characterization of the constrained stability region. Specifically, the ith inequality in Eq. (7) describes a state-space region, $\Phi_i(u_i^{\max})$, where the ith control law satisfies the constraints and forces V_i to decrease monotonically along the trajectories of the ith closed-loop subsystem. Note that the inequality captures the dependence of the size of this region on the size of the constraints (tighter constraints yield a smaller region). However, since $\Phi_i(u_i^{\max})$ is not necessarily invariant, there is no guarantee that a trajectory starting in $\Phi_i(u_i^{\max})$ will remain forever in this region and, consequently, no guarantee that $\dot{V}_i$ will stay negative. To guarantee that $\dot{V}_i$ remains negative for all the times that the ith mode is active, we compute an invariant set, $\Omega_i^*(u_i^{\max})$, (preferably the largest) within $\Phi_i(u_i^{\max})$ (see [21] for details on how to construct this set). This set represents an estimate of the stability region associated with each mode. Note that, unlike the sets Ω_i introduced in Theorem 1 in the context of autonomous switching, the sets $\Omega_i^*(u_i^{\max})$: (1) are a function of the control constraints, and (2) do not partition the state-space into neighboring regions where each mode is confined; rather, the regions described by these sets can be overlapping implying that a given mode is allowed to remain active even if x crosses into the stability region of another mode.

Remark 5. The switching rules of Eqs. (8) and (9) determine, implicitly, the times when switching from mode i to mode j is permissible. The first rule, which tracks the temporal evolution of the continuous state, x, requires that, at the desired switching time, the continuous state reside within the stability region of the subsystem to be activated. This ensures that once this subsystem is activated its constraints are satisfied and its Lyapunov function continues to decay for as long as that mode remains active. Note that this condition applies at every time that the supervisor considers switching from one mode to another. In contrast, the second switching rule, which tracks the evolution of the Lyapunov functions, applies only when the target mode j has been previously activated. In this case, Eq. (9) requires that V_j at the current "switch in" be less than its value at the previous "switch in." This requirement is less conservative than the one proposed in Theorem 2 in [10] and allows switching to take place earlier. Note that, unlike the condition of Eq. (2) in Theorem 1, where only Lyapunov-stability can be concluded, the condition of Eq. (9) requires a strict inequality in order to enforce asymptotic stability of the origin of the switched closed-loop system. When only a finite number of switches is considered over the infinite time-interval, the condition of Eq. (9) can be relaxed to allow for finite increases in V_j (see [10,12] for the reasoning behind this as well as a discussion of other possible extensions). In this case, switching based on Eq. (8) alone is sufficient to enforce asymptotic closed-loop stability (see the simulation example in Section 5).

Remark 6. Even though the switching conditions require knowledge of the temporal evolution of the closed-loop state, $x(t)$, the a priori knowledge of the solution of the constrained closed-loop nonlinear system (which is difficult to obtain in general) is not needed for the practical implementation of the proposed approach. Instead, the supervisor can monitor (on-line) how x evolves in time to determine if and when the switching conditions are satisfied. If the conditions are satisfied for the desired target mode at some time, then switching can take place safely. Otherwise, the current mode is kept active. Note that, absent any failures in the control system, maintaining closed-loop stability does not require switching since the closed-loop system is always initialized within the stability region of at least one of the constituent modes and, therefore, absent switching, the closed-loop trajectory will simply remain in this invariant set and stabilize at the desired equilibrium point. In many practical situations, however, the ability of the control system to deal with failure situations requires consideration of multiple control configurations and switching between them to preserve closed-loop stability in the event that the active control configuration fails. For such cases, the switching rules proposed in Theorem 2 (based on the stability regions) can be used to explicitly identify the appropriate fall-back control actuator configuration that should be activated to preserve closed-loop stability (see [12] for some examples). Clearly, if failure occurs when the state is outside the stability regions of all the available configurations, then closed-loop stability cannot be preserved because of the fundamental limitations imposed by the constraints on the stability regions as well as the number of backup configurations that are available. However, our approach in this case provides a useful way for analyzing when the control system can or cannot tolerate failures. For example, by analyzing the overlap of the stability regions of the given configurations, one can decide the time periods during which the control system (under a given configuration) cannot tolerate failure (which are the times that the trajectory spends outside the stability region of the other configurations). By relaxing the constraints (i.e., enlarging the stability regions) and/or increasing the available control configurations (this is ultimately limited by system design considerations) one can reduce the possibility of failures taking place outside of the stability regions of all configurations.

Remark 7. In addition to fault-tolerant control, another reason to consider switching, which could be tied to the performance of the process, is the need to cope with changes in the operating conditions which can arise, for example, due to changes in raw materials, changes in energy sources or the desire to enhance the overall process yield (e.g., via the addition/removal of certain streams; see the simulation example in Section 5). In such instances, Theorem 2 provides the conditions that can be checked on-line to determine the feasibility of switching.

Remark 8. Note that it is possible for more than one subsystem to satisfy the switching rules given in Eqs. (8) and (9) at a given time T. This occurs when x lies within the intersection of the stability regions of several modes. In this case, Theorem 2 guarantees that a switch from the current mode to any of these modes is feasible, but does not specify which one to choose. The supervisor's decision to activate a particular mode can be made on the basis of a higher-level switching objective.

Remark 9. The assumption that $f_i(0) = 0$ is not necessary for proving stability of the switched closed-loop system; however, this condition does simplify the statement of Theorem 1 (note that Theorem 1 refers to the open-loop system). Since closed-loop stability in Theorem 2 is proved using the result of Theorem 1 (see the proof in the appendix), we therefore need to have that, under the designed control laws, the origin is the equilibrium point of the various modes of the switched closed-loop system. However, no assumption about the origin being the equilibrium point of the open-loop modes need be made when control is considered, since it is the closed-loop system whose stability is of interest and not the open-loop system. Furthermore, the assumption that the various closed-loop modes share the same equilibrium point is only needed when an infinite sequence of switching times is considered since, otherwise, asymptotic stability cannot be achieved. However, when only a finite number of switches is considered over the infinite time-interval, the closed-loop modes need not share the same equilibrium point, since the closed-loop system will eventually settle in a final mode where the state will converge to its equilibrium point achieving asymptotic stability (see the simulation example in Section 5). In this case, switching is carried out based on the stability regions condition of Eq. (8) only (the condition of Eq. (9) can be relaxed).

3.3. Application to input/output linearizable systems

An important class of nonlinear systems that has been studied extensively within control theory is that of input/output feedback linearizable systems. This class arises frequently in practical problems where the objective is to force the controlled output to follow some reference-input trajectory (rather than stabilize the full state at some nominal equilibrium point). In this section, we illustrate how the feedback and switching methodology proposed in Theorem 2 can be applied when the individual subsystems of the switched system are input/output linearizable. For simplicity, we limit our attention to the single-input single-output case. Consider the system:

$$\begin{aligned}
&\dot{x}(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t))u_{\sigma(t)},\\
&y_c = h_c(x),\\
&\sigma(t) \in \mathscr{I} = \{1, \ldots, N\},
\end{aligned} \tag{11}$$

where $y_c \in \mathbb{R}$ is the controlled output and $h_c(x)$ is a sufficiently smooth scalar function. Suppose that, for all $i \in \mathscr{I}$, there exists an integer r (this assumption is made only to simplify notation and can be readily relaxed to allow a different relative degree r_i for each subsystem) and a set of coordinates (see [19] for a detailed treatment of feedback linearizable nonlinear systems)

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = \mathscr{X}(x) = \begin{bmatrix} h_c(x) \\ L_{f_i} h_c(x) \\ \vdots \\ L_{f_i}^{r-1} h_c(x) \\ \chi_{1,i}(x) \\ \vdots \\ \chi_{n-r,i}(x) \end{bmatrix}, \tag{12}$$

where $\chi_1(x), \ldots, \chi_{n-r}(x)$ are nonlinear scalar functions of x, such that the system of Eq. (11) takes the form:

$$\begin{aligned}
&\dot{\zeta}_1 = \zeta_2,\\
&\vdots\\
&\dot{\zeta}_{r-1} = \zeta_r,\\
&\dot{\zeta}_r = L_{f_i}^r h_c(\mathscr{X}^{-1}(\zeta,\eta)) + L_{g_i} L_{f_i}^{r-1} h_c(\mathscr{X}^{-1}(\zeta,\eta))u_i,\\
&\dot{\eta}_1 = \Psi_{1,i}(\zeta,\eta),\\
&\vdots\\
&\dot{\eta}_{n-r} = \Psi_{n-r,i}(\zeta,\eta),\\
&y = \zeta_1,
\end{aligned} \tag{13}$$

where $L_{g_i} L_{f_i}^{r-1} h_c(x) \neq 0$ for all $x \in \mathbb{R}^n$, $i \in \mathscr{I}$ and $\Psi_{1,i} \cdots \Psi_{n-r,i}$ are nonlinear functions of their arguments describing the evolution of the inverse dynamics of the ith mode. Under the assumption that the η-subsystem is input-to-state stable (ISS) (see [34] for details) with respect to ζ for each $i \in \mathscr{I}$, i.e., there exists a function β of class $\mathscr{KL}$ and a function γ of class $\mathscr{K}$ such that for each $\eta(0) = \eta_0 \in \mathbb{R}^{n-r}$, $\|\eta(t)\| \leqslant \beta(\|\eta_0\|, t) + \gamma(\|e\|^s)$, $\forall t \geqslant 0$ where $\|e\|^s$ denotes ess.sup.$\|e(t)\|$, $t \geqslant 0$ (given a measurable function $f : \mathscr{T} \to \mathbb{R}$, where $\mathscr{T}$ is a measure space with measure μ, the essential supremum is defined as $ess.sup.f(t) = \inf\{M : \mu\{t : f(t) > M\} = 0\}$, i.e., it is the smallest positive integer M such that $\|f\|$ is bounded by M almost everywhere), the controller synthesis task for each mode can be addressed on the basis of the partially linear ζ-subsystem. To this end, upon introducing the notation $e_k = \zeta_k - v^{(k-1)}$, $e = [e_1 e_2 \cdots e_r]^{\mathrm{T}}$, $\bar{v} = [v v^{(1)} \cdots v^{(r-1)}]^{\mathrm{T}}$, where $v^{(k)}$ is the kth time derivative of the reference input v which is assumed to be a smooth function of time, the ζ-subsystem of Eq. (13) can be further transformed into the following more compact form:

$$\dot{e} = \bar{f}_i(e,\eta,\bar{v}) + \bar{g}_i(e,\eta,\bar{v})u_i, \quad i = 1, \ldots, N, \tag{14}$$

where $\bar{f}_i(e,\eta,\bar{v}) = Ae + bL_{f_i}^r h_c(\mathscr{X}^{-1}(e,\eta,\bar{v}))$, $\bar{g}_i(e,\eta,\bar{v}) = bL_{g_i} L_{f_i}^{r-1} h_c(\mathscr{X}^{-1}(e,\eta,\bar{v}))$ are $r \times 1$

vector functions, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 1 \end{bmatrix} \tag{15}$$

are an $r \times r$ matrix and $r \times 1$ vector, respectively. For systems of the form of Eq. (14), a simple choice for a CLF is a quadratic function (see, for example, [32])

$$\bar{V}_i = e^{\mathrm{T}} P_i e, \tag{16}$$

where the positive-definite matrix P_i is chosen to satisfy the following Ricatti inequality:

$$A^{\mathrm{T}} P_i + P_i A - P_i b b^{\mathrm{T}} P_i < 0. \tag{17}$$

Using these quadratic CLFs, a controller can be designed for each mode using Eqs. (5) and (6) applied to the system of Eq. (14). Using a standard Lyapunov argument, it can be shown that each controller asymptotically stabilizes the e states in each mode. This result together with the ISS assumption on the η states can then be used to show, via a small gain argument, that the full closed-loop e-η interconnection, for each individual mode, is asymptotically stable.

Remark 10. Note that the Lyapunov functions, $\bar{V}_i$, used in designing the controllers, are in general different from the Lyapunov functions, V_i, used in implementing the switching rules. Owing to the ISS property of the η-subsystem of each mode, only a Lyapunov function for the e subsystem, namely $\bar{V}_i$, is needed and used to design a controller that stabilizes the full e–η interconnection for each mode. However, when implementing the switching rules (constructing Ω_i^* and verifying Eq. (9)), we need to track the evolution of x (and hence the evolution of both e and η). Therefore, the Lyapunov functions used in verifying the switching conditions at any given time, V_i, are based on x. From the asymptotic stability of each mode, the existence of these Lyapunov functions is guaranteed by converse Lyapunov theorems. For systems with relative degree $r = n$, the choice $\bar{V}_i = V_i$ is sufficient.

Remark 11. Note that, unless the open-loop modes of the switched system of Eq. (11) share the same equilibrium point, only the closed-loop e-subsystems of Eq. (14) can be made to share the same equilibrium point (essentially through the controllers enforcing the same set-point for all the modes), while the uncontrolled η-subsystems may have different equilibrium points, thus causing the different modes of the full closed-loop system to possess different equilibrium points. However, as we noted earlier in Remark 9, when applying the results of Theorem 2 with a finite number of switches, asymptotic closed-loop stability is guaranteed by switching on the basis of the stability regions (Eq. (8)) and the fact that the closed-loop system will eventually settle in a final mode where the state will converge to the equilibrium point of that mode (see the simulation example in Section 5).

4. Output feedback control of switched nonlinear systems

The feedback controllers and switching rules presented in Section 3 were designed under the assumption of accessibility of all the process states for measurement. In this section, we consider the case when some of the states of the system of Eq. (1) are not available for measurement.

4.1. Control problem formulation

Referring to the switched nonlinear system of Eq. (1), our objectives for the output feedback control problem include: (a) the synthesis a family of N bounded nonlinear dynamic output feedback controllers of the general form:

$$\begin{aligned} &\dot{\omega} = \mathscr{F}_i(\omega, y_m), \\ &u_i = -p_i(\omega, y_m, u_i^{\max}), \quad i = 1, \ldots, N, \end{aligned} \tag{18}$$

where $\omega \in \mathbb{R}^n$ is a state, $\mathscr{F}_i(\cdot)$ is a vector function, $p_i(\cdot)$ is a bounded nonlinear function, that enforce asymptotic (and local exponential) stability for the individual closed-loop subsystems, and provide, for each mode, an explicit characterization of the constrained stability region under output feedback, and (b) the design a set of switching laws, $\sigma(t) = \psi'(\omega, y_m)$, that orchestrate, based on the state estimates, stabilizing transitions between the constituent closed-loop modes.

In the remainder of this section, we first review an output feedback controller design, based on a combination of high-gain observers, saturation filters and the state feedback controllers of Eqs. (5) and (6), and characterize the stability properties of the closed-loop system under output feedback control (see also [20,35,6,11] for results on output feedback control of nonlinear systems). We then present switching laws based on available state estimates that guarantee closed-loop stability for the switched closed-loop system.

4.2. Output feedback controller synthesis

In order to synthesize an output feedback controller that enforces the requested closed-loop properties for each mode, we will need to impose the following assumption on the system of Eq. (1). To simplify the notation, we will focus on the case of a single measured output. The results, however, can be readily generalized to the case of multiple measured outputs.

Assumption 2. For each $i \in \mathscr{I}$, there exists a set of coordinates:

$$[\xi_i] = \begin{bmatrix} \xi_i^{(1)} \\ \xi_i^{(2)} \\ \vdots \\ \xi_i^{(n)} \end{bmatrix} = \chi_i(x) = \begin{bmatrix} h_m(x) \\ L_{f_i} h_m(x) \\ \vdots \\ L_{f_i}^{n-1} h_m(x) \end{bmatrix}, \tag{19}$$

such that the system of Eq. (1) takes the form:

$$\begin{aligned} \dot{\xi}_i^{(1)} &= \xi_i^{(2)}, \\ &\vdots \\ \dot{\xi}_i^{(n-1)} &= \xi_i^{(n)}, \\ \dot{\xi}_i^{(n)} &= L_{f_i}^n h_m(\chi_i^{-1}(\xi_i)) + L_{g_i} L_{f_i}^{n-1} h_m(\chi_i^{-1}(\xi_i)) u_i, \end{aligned} \tag{20}$$

where $L_{g_i} L_{f_i}^{n-1} h_m(x) \neq 0$ for all $x \in \mathbb{R}^n$. Also, $\xi_i \longrightarrow 0$ if and only if $x \longrightarrow 0$.

We note that the change of variables is invertible, since for every x, the variable ξ_i is uniquely determined by the transformation $\xi_i = \chi_i(x)$. This implies that if one can estimate the values of ξ_i for all times, using an appropriate state observer, then we automatically obtain estimates of x for all times, which can be used to implement the state feedback controller. The existence of such a transformation will facilitate the design of the high-gain observers which will be instrumental in preserving the same closed-loop stability properties achieved under full state feedback.

Proposition 1 below presents the output feedback controller used for each mode and characterizes its stability properties. The proof of the proposition, which invokes singular perturbation arguments, is a special case of the proof of Theorem 2 in [11], and is omitted for brevity. To simplify the statement of the proposition, we first introduce the following notation. We define $\bar{\alpha}_i(\cdot)$ as a class $\mathscr{K}$ function that satisfies $\bar{\alpha}_i(\|x\|) \leqslant V_i(x)$. We also define the set $\Omega_{b,i} := \{x \in \mathbb{R}^n : V_i(x) \leqslant \delta_{b,i}\}$, where $\delta_{b,i} < \delta_{x,i}$ is chosen such that $\beta_i(\bar{\alpha}_i^{-1}(\delta_{b,i}), 0) < \bar{\alpha}_i^{-1}(\delta_{x,i})$, where $\beta_i(\cdot,\cdot)$ is a class $\mathscr{KL}$ function defined in Eq. (10) and $\delta_{x,i}$ is a positive real number defined in Theorem 1.

Proposition 1. *Consider the nonlinear system of Eq.* (1), *for a fixed mode*, $\sigma(t) = i$, *under the output feedback controller*:

$$\begin{aligned} \dot{\tilde{y}} &= \begin{bmatrix} -L_i a_1^{(i)} & 1 & 0 & \cdots & 0 \\ -L_i^2 a_2^{(i)} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -L_i^n a_r^{(i)} & 0 & 0 & \cdots & 0 \end{bmatrix} \tilde{y} \\ &\quad + \begin{bmatrix} L_i a_1^{(i)} \\ L_i^2 a_2^{(i)} \\ \vdots \\ L_i^n a_n^{(i)} \end{bmatrix} y_m, \end{aligned} \tag{21}$$

$$u_i = -k_i(\hat{x}, u_i^{\max})(L_{G_i} V_i(\hat{x}))^{\mathrm{T}},$$

where the parameters, $a_1^{(i)}, \ldots, a_n^{(i)}$ *are chosen such that the polynomial* $s^n + a_1^{(i)} s^{n-1} + a_2^{(i)} s^{n-2} + \cdots + a_n^{(i)} = 0$ *is Hurwitz*, $\hat{x} = \chi_i^{-1}(sat(\tilde{y}))$, $sat(\cdot) = \min\{1, \zeta_{\max,i}/\|\cdot\|\}(\cdot)$, *with* $\zeta_{\max,i} = \beta_\zeta(\delta_{\zeta,i}, 0)$, *where* β_ζ *is a class* $\mathscr{KL}$ *function and* $\delta_{\zeta,i}$ *is the maximum value of* $\|[h_m(x) L_{f_i} h_m(x) \cdots L_{f_i}^{n-1} h_m(x)]^{\mathrm{T}}\|$ *for* $V_i(x) \leqslant \delta_{b,i}$ *and let* $\varepsilon_i = 1/L_i$. *Then, given* $\Omega_{b,i}$, *there exists* $\varepsilon_i^* > 0$, *such that if* $\varepsilon_i \in (0, \varepsilon_i^*]$, $x(0) \in \Omega_{b,i}$, *and* $\|\tilde{y}(0)\| \leqslant \delta_{\zeta,i}$, *the origin of the closed-loop system is asymptotically* (*and locally exponentially*) *stable. Furthermore, given* $\varepsilon_i \in (0, \varepsilon_i^*]$ *and some real number* $e_{m,i} > 0$, *there exists a real number* $T_i^b > 0$ *such that* $\|x(t) - \hat{x}(t)\| \leqslant e_{m,i}$ *for all* $t \geqslant T_i^b$.

Remark 12. The output feedback controller of Eq. (21) consists of a high-gain observer which provides estimates of the derivatives of the outputy_m up to order $n-1$, denoted as $\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_{n-1}$, and thus estimates of the variables $\xi_i^{(1)}, \ldots, \xi_i^{(n)}$ (note from Assumption 1 that $\xi_i^{(k)} = d^{k-1}y_m/\mathrm{d}t^{k-1}, k = 1, \ldots, n$),and a static state feedback controller that enforces closed-loopstability. To eliminate the peaking phenomenon associated with the high-gain observer, we use a standard saturation function, *sat*, to eliminate wrong estimates of the output derivatives for short times. The use of a high-gain observer (together with the saturation filter) allows us to practically preserve the closed-loop stability region obtained under state feedback. Specifically, starting from any compact subset of initial conditions within the state feedback region ($\Omega_{b,i} \subset \Omega_i^*$), the output feedback controller of Eq. (21) continues to enforce asymptotic stability in the closed-loop system provided that the observer gain is chosen sufficiently large. As expected, the nature of this semi-regional result is consistent with the semi-global result obtained for the unconstrained case [6,40]. It should be noted, however, that while the output feedback stability region can, in principle, be chosen as close as desired to its state feedback counterpart by increasing the observer gain L_i, it is well known that large observer gains can amplify measurement noise and induce poor performance. This points to a fundamental trade-off that cannot be resolved by simply changing the estimation scheme. For example, although one could replace the high-gain observer design with other observer designs (for example, a moving horizon estimator) to gain a better handle on measurement noise, it is difficult in such schemes to obtain an explicit relationship between the observer tuning parameters and the output feedback stability region.

Remark 13. Owing to the presence of the fast (high-gain) observer in the dynamical system of Eq. (21), the closed-loop system for the *i*th mode can be cast as a two time-scale system and, therefore, represented in the following singularly perturbed form, where $\varepsilon_i = 1/L_i$ is the singular perturbation parameter:

$$\begin{aligned} \varepsilon_i \dot{e}_{\mathrm{o}} &= A e_{\mathrm{o}} + \varepsilon_i b \Delta_i(x, \hat{x}), \\ \dot{x} &= f_i(x) - g_i(x) p_i(\hat{x}, u_i^{\max}), \end{aligned} \tag{22}$$

where e_{o} is a vector of the auxiliary error variables $\hat{e}_i = L^{n-i}(y^{(i-1)} - \tilde{y}_i)$, A is an $n \times n$ matrix, $b = [0 \ \cdots \ 0 \ 1]^{\mathrm{T}}$ is a $n \times 1$ vector, and Δ_i is a Lipschitz function of its argument. It is clear from the above representation that, within the singular perturbation formulation, the observer error states, e_{o}, which are directly related to the estimates of the output and its derivatives up to order $n-1$, constitute the fast states of the singularly perturbed system of Eq. (22), while the states of the original system of Eq. (1) under the static component of the controller represent the slow states.

Remark 14. Note that asymptotic stability of each mode of the closed-loop system under the output feedback controller of Eq. (21) implies that there exists a Lyapunov function, V_i^{c}, for each mode of the closed-loop system, $i = 1, \ldots, N$, such that $\dot{V}_i^{\mathrm{c}}(x_f) < 0$, where $x_f = [x^{\mathrm{T}} \ e_{\mathrm{o}}^{\mathrm{T}}]^{\mathrm{T}}$ is the state vector of the full closed-loop system of Eq. (22). The existence of $\dot{V}_i^{\mathrm{c}}$ can be ascertained using a standard converse Lyapunov theorem argument (see, for example, Theorem 3.14 in [21]). We will use these Lyapunov functions in the next section to design the appropriate switching rules under output feedback.

4.3. Switching logic under output feedback

Owing to the lack of full state measurements, the supervisor can rely only on the available state estimates to decide whether switching at any given time is permissible. This necessitates that the supervisor be able to make reliable inferences regarding the position of the states based upon the available state estimates. Proposition 2 below establishes the existence of a set, $\Omega_{s,i}$, such that once the state estimation error has fallen below a certain value (note that the decay rate can be controlled by adjusting L_i), the presence of the state within the output feedback stability region, $\Omega_{b,i}$, can be guaranteed by verifying the presence of the state estimates in the set $\Omega_{s,i}$. A similar notion was used in [28] in the context of hybrid predictive control of linear systems under output feedback. The proof of Proposition 2 follows from the continuity of the function $V_i(\cdot)$, and relies on the fact that given a positive real number, $\delta_{b,i}$, (i.e., given a desired output feedback stability region), one can find positive real numbers $e^*_{m,i}$ and $\delta_{s,i}$ such that if the estimation error is below $e^*_{m,i}$ (i.e., $\|x-\hat{x}\| \leqslant e^*_{m,i}$) and the estimate is within $\Omega_{s,i}$ (i.e., $V_i(\hat{x}) \leqslant \delta_{s,i}$ or $\hat{x} \in \Omega_{s,i}$), then the state itself must be within $\Omega_{b,i}$, i.e., $V_i(x) \leqslant \delta_{b,i}$ (see [28] for a proof in the linear case).

Proposition 2. *Given any positive real number $\delta_{b,i}$, there exists a positive real number $e^*_{m,i}$, and a set $\Omega_{s,i} := \{x \in \mathbb{R}^n : V_i(x) \leqslant \delta_{s,i}\}$ such that if $\|x-\hat{x}\| \leqslant e_{m,i}$, where $e_{m,i} \in (0, e^*_{m,i}]$ then $\hat{x} \in \Omega_{s,i} \implies x \in \Omega_{b,i}$.*

We are now ready to proceed with the design of the switching logic. To this end, consider the switched nonlinear system of Eq. (1) for which Assumption 2 holds and, for each mode, an output feedback controller of the form of Eq. (21) has been designed and a Lyapunov function V_i^{c} has been determined. Given the desired output feedback stability regions $\Omega_{b,i} \subset \Omega_i^*$, $i=1,\ldots,N$, we choose, for simplicity, $\varepsilon_1=\varepsilon_2=\cdots=\varepsilon_n \leqslant \min\{\varepsilon_i^*\}$ (i.e., the same observer gain is used for all modes). Also assume that, for each mode, and for choices of $e_{m,i} \leqslant e^*_{m,i}$, the sets $\Omega_{s,i}$ (see Proposition 2) and the times $T_{b,i}$ (see Proposition 1) have been determined, and let $T_b^{\max} = \max\{T_{b,i}\}$, $i=1,\ldots,N$. Theorem 3 below presents the output feedback switching rules. The proof is given in the appendix.

Theorem 3. *Assume, without loss of generality, that $x(0) \in \Omega_{b,i}(u_i^{\max})$ for some $i \in \mathscr{I}$ and choose $\tilde{y}(0)$ such that $\|\tilde{y}(0)\| \leqslant \delta_{\zeta,i}$, where $\delta_{\zeta,i}$ was defined in Proposition 1. Let T_{old} be the time that the current mode was last switched in. Let M_i be such that $\|z_1 - z_2\| \leqslant e_{m,i} \implies |V_i^{\mathrm{c}}(z_1) - V_i^{\mathrm{c}}(z_2)| \leqslant M_i$. If, at any given time, T, the following conditions hold*:

$$T \geqslant T_{\mathrm{old}} + T_b^{\max}, \quad \hat{x}(T) \in \Omega_{s,j}(u_j^{\max}) \tag{23}$$

for some $j \in \mathscr{I}$, $j \neq i$, and

$$V_i^{\mathrm{c}}(\hat{x}_f(T)) + 2M_i < V_i^{\mathrm{c}}(\hat{x}_f(t_{i*})), \tag{24}$$

where $\hat{x}_f = [\hat{x}^{\mathrm{T}} \;\; e_{\mathrm{o}}^{\mathrm{T}}]^{\mathrm{T}}$, $e_{\mathrm{o}} = [\hat{e}_1 \;\; \hat{e}_2 \;\; \cdots \;\; \hat{e}_n]^{\mathrm{T}}$, $\hat{e}_k = L^{n-k}(y^{(k-1)} - \tilde{y}_k)$ and $t_{i*} < T$ is the time when the ith subsystem was last switched out, i.e., $\sigma(t_{i*}^+) \neq \sigma(t_{i*}^-)=i$, then setting $\sigma(T^+)=j$ and

$$\tilde{y}(T^+) = \left\{ \begin{array}{c} \tilde{y}(T) \text{ if } \|\tilde{y}(T)\| \leqslant \delta_{\zeta,j}, \\ \tilde{y}(T)\frac{\delta_{\zeta,j}}{\|\tilde{y}(T)\|} \text{ if } \|\tilde{y}(T)\| > \delta_{\zeta,j}, \end{array} \right\} \tag{25}$$

guarantees that the origin of the switched closed-loop system of Eqs. (1), (21), (23) *and* (24) *is asymptotically stable.*

Remark 15. The fact that the switching rules are based on the state estimates has several important implications that distinguish the output feedback switching logic from its state feedback counterpart. First, note that the switching rules dictate that there be a time interval of at least $T_b^{\max}$ between two consecutive switches. This is done to ensure that for the given choice of the observer gain, and once a given mode is switched in, the estimation error has enough time to decrease to a sufficiently small value such that, from that point in time onwards, the position of the state can be inferred by looking at the state estimate. Recall from Proposition 2 that the relation $\hat{x} \in \Omega_{s,j} \implies x \in \Omega_{b,j}$ holds only when the estimation error is sufficiently small. Second, the decision to switch is not based on $\hat{x}$ entering $\Omega_{b,j}$ (under state feedback it was based on x entering Ω_j^*); rather it is based on $\hat{x}$ entering $\Omega_{s,j}$. The inference that $\hat{x} \in \Omega_{s,j} \implies x \in \Omega_{b,j}$, however, can be made only once the error has dropped sufficiently, and this is guaranteed to happen after the closed-loop system has evolved in mode i for a time $T_b^{\max} \geqslant T_{b,i}$. Therefore, a switch is not executed before an interval of length $T_b^{\max}$ elapses (from the last switching instance) even if $\hat{x}$ enters $\Omega_{s,j}$ at some earlier time. Furthermore, in contrast to the switching rules under state feedback, the MLF condition of Eq. (24) is checked for switch-out rather than switch-in times (once again, this is to ensure that the error has decreased sufficiently). Also, in contrast to Theorem 2, the MLF condition of Eq. (24) is checked using the Lyapunov function, $V_i^{\mathrm{c}}(x_f)$, for the full closed-loop system of Eq. (22), instead of the CLF, $V_i(x)$, used in the controller design. Finally, since the MLF condition is checked using the state estimate, Eq. (24) requires that the value of V_i^{c}, based on $\hat{x}_f$, decrease by a margin large enough to guarantee the decay of the value of V_i^c based on x_f.

Remark 16. Note that the values of the observer state, $\tilde{y}$, are re-initialized after switching using Eq. (25) to ensure that $\|\tilde{y}(T^+)\| \leqslant \delta_{\zeta,j}$ which, from Proposition 1, is necessary for $\Omega_{b,j}$ to continue to be the output feedback stability region for mode j.

5. Application to a chemical process example

Consider a continuous stirred tank reactor where three parallel, irreversible, first-order exothermic reactions of the form $A \xrightarrow{k_1} D$, $A \xrightarrow{k_2} U$ and $A \xrightarrow{k_3} R$ take place, where A is the reactant species and D, U, R denote three product species. The reactor has two inlet streams: the first continuously feeds pure A at flow rate $F = 83.33\,\mathrm{L/min}$, concentration $C_{A0} = 4.0\,\mathrm{mol/L}$ and temperature $T_{A0} = 300\,\mathrm{K}$, while the second can be turned on or off (by means of an on/off valve) during reactor operation. When turned on, the second stream feeds pure A at flow rate $F^* = 200\,\mathrm{L/min}$, concentration $C_{A0}^* = 5.0\,\mathrm{mol/L}$ and temperature $T_{A0}^* = 500\,\mathrm{K}$. Under standard modeling assumptions, the mathematical model for the process takes the form:

$$\begin{aligned}
\dot{C}_A &= \frac{F}{V}(C_{A0} - C_A) + (\sigma(t) - 1)\frac{F^*}{V}(C_{A0}^* - C_A) \\
&\quad - \sum_{i=1}^{3} R_i(C_A, T), \\
\dot{T} &= \frac{F}{V}(T_{A0} - T) + (\sigma(t) - 1)\frac{F^*}{V}(T_{A0}^* - T) \\
&\quad + \sum_{i=1}^{3} G_i(C_A, T) + \frac{Q}{\rho_m c_{pm} V},
\end{aligned} \tag{26}$$

where $R_i(C_A, T) = k_{0i}\exp(-E_i/RT)C_A$, $G_i(C_A, T) = [(\Delta H_i)/\rho_m c_{pm}]R_i(C_A, T)$, C_A denotes the concentration of species A, T denotes the temperature of the reactor, Q denotes the rate of heat input to the reactor, V denotes the volume of the reactor, k_{0i}, E_i, ΔH_i denote the pre-exponential constants, the activation energies, and the enthalpies of the three reactions, respectively, c_{pm} and ρ_m, denote the heat capacity and density of the fluid in the reactor. The values of these process parameters can be found in Table 1. $\sigma(t)$ is a discrete variable that takes a value of 1 when the second

Table 1
Process parameters and steady-state values

$F = 83.33\,\mathrm{L/min}$
$F^* = 200.0\,\mathrm{L/min}$
$T_0 = 300.0\,\mathrm{K}$
$T_0^* = 500.0\,\mathrm{K}$
$C_{A0} = 4.0\,\mathrm{mol/L}$
$C_{A0}^* = 5.0\,\mathrm{mol/L}$
$V = 1000.0\,\mathrm{L}$
$R = 8.314\,\mathrm{J/mol\,K}$
$\Delta H_1 = -5.0 \times 10^4\,\mathrm{J/mol}$
$\Delta H_2 = -5.2 \times 10^4\,\mathrm{J/mol}$
$\Delta H_3 = -5.4 \times 10^4\,\mathrm{J/mol}$
$k_{01} = 5.0 \times 10^4\,\mathrm{min}^{-1}$
$k_{02} = 5.0 \times 10^3\,\mathrm{min}^{-1}$
$k_{03} = 5.0 \times 10^3\,\mathrm{min}^{-1}$
$E_1 = 5.0 \times 10^4\,\mathrm{J/mol}$
$E_2 = 7.53 \times 10^4\,\mathrm{J/mol}$
$E_3 = 7.53 \times 10^4\,\mathrm{J/mol}$
$\rho_m = 1000.0\,\mathrm{g/L}$
$c_{pm} = 0.231\,\mathrm{J/g\,K}$
$T_s(\sigma = 1, 2) = 388.58\,\mathrm{K}$
$C_{As}(\sigma = 1) = 3.59\,\mathrm{mol/L}$
$C_{As}(\sigma = 2) = 4.55\,\mathrm{mol/L}$

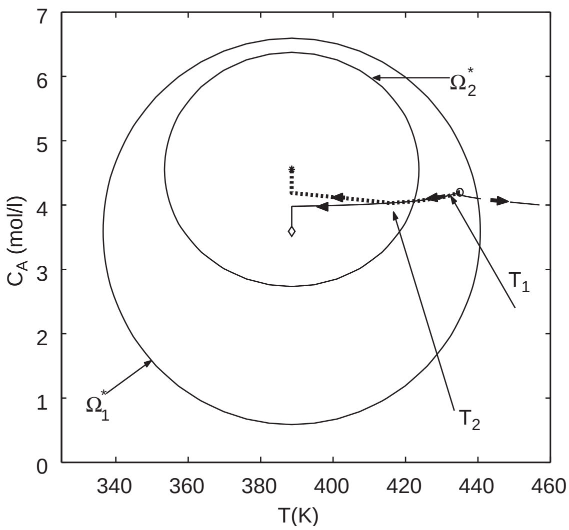

Fig. 1. A phase plot showing the stability region estimates Ω_1^*, Ω_2^*.

inlet stream is turned off and a value of 2 when it is turned on. Initially, it is assumed that $\sigma = 1$. During reactor operation, however, it is desired to enhance the product concentration by feeding more reactant material through the second inlet stream ($\sigma = 2$). This requirement gives rise to two distinct modes between which switching is desired. For the parameters given in Table 1, it was verified that for mode 1, the open-loop system (with $Q = 0$) has three equilibrium points, one of which is unstable (see Table 1).

The control objectives are to: (1) stabilize the reactor temperature at the open-loop unstable steady-state of mode 1 ($T_s = 388.58\,\mathrm{K}$), and (2) maintain the temperature at this steady-state when the reactor switches to mode 2. Note that, with this requirement, both closed-loop modes share the same steady-state temperature but have different steady-state reactant concentrations (see Remark 11 and Fig. 1 for the different equilibrium points). The control objective is to be accomplished by manipulating Q, subject to the constraint $|Q| \leqslant 1 \times 10^4\,\mathrm{KJ/min}$. Defining $x_1 = T$ and $x_2 = C_A$, the process model of Eq. (26) can be cast in the form of Eq. (11) with $y_c = x_1$. For this feedback linearizable process, the controlled output has a relative degree of $r = 1$ and, therefore, using a coordinate transformation of the form of Eq. (12), with $e = T - T_s$ and $\eta = C_A$, a scalar system of the form of Eq. (14), describing the input/output dynamics, can be obtained for controller design:

$$\dot{e} = \bar{f}_i(e, \eta, \bar{v}) + \bar{g}_i(e, \eta, \bar{v})u_i, \quad i = 1, 2, \tag{27}$$

where $\bar{v} = T_s$, $\bar{f}_1(\cdot) = \frac{F}{V}(C_{A0} - C_A) - \sum_{i=1}^{3} k_{0i} \exp(-E_i/RT)C_A$, $f_2(\cdot) = \frac{F}{V}(C_{A0} - C_A) + \frac{F^*}{V}(C^*_{A0} - C_A) - \sum_{i=1}^{3} k_{0i} \exp(-E_i/RT)C_A$, $\bar{g}_1(\cdot) = \bar{g}_2(\cdot) = 1/\rho_m c_{pm} V$. Two quadratic, positive-definite functions of the form $\bar{V}_1 = \bar{V}_2 = \frac{1}{2}c_\delta e^2$, where $c_\delta = 1/T_s^2$, are then used to synthesize, on the basis of the e-subsystems, two bounded nonlinear controllers (one for each mode) of the form:

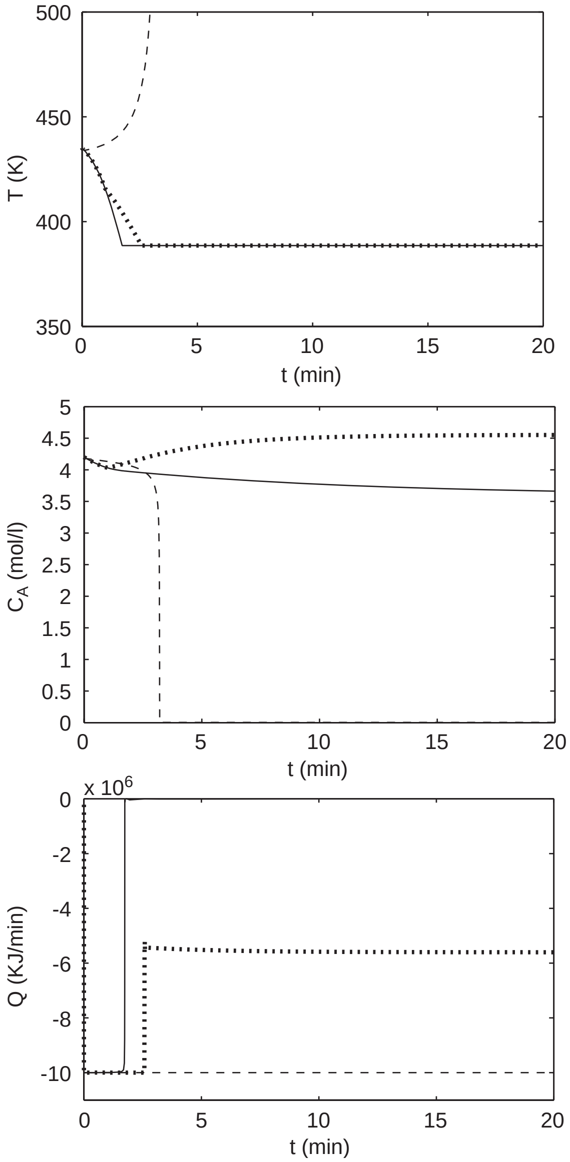

Fig. 2. Closed-loop state and input profiles when the reactor is initialized within Ω_1^* and operated in mode 1 for all times (solid), when the reactor switches to mode 2 at $t = 0.1$ min (dashed), and when the switch is executed at $t = 1.0$ min (dotted).

$$u_i = -\left(\frac{L_{\bar{f}_i}\bar{V}_i + \rho\bar{V}_i + \sqrt{(L_{\bar{f}_i}\bar{V}_i + \rho\bar{V}_i)^2 + (u_i^{\max} L_{\bar{g}_i}\bar{V}_i)^4}}{(L_{\bar{g}_i}\bar{V}_i)^2[1 + \sqrt{1 + (u_i^{\max} L_{\bar{g}_i}\bar{V}_i)^2}]} \right) L_{\bar{g}_i}\bar{V}_i, \quad i = 1, 2, \tag{28}$$

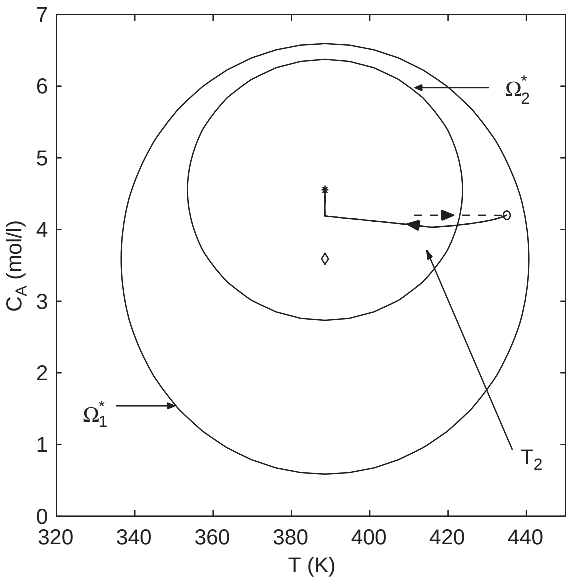

Fig. 3. Closed-loop state trajectories under state (dotted) and output (solid) feedback, and the state estimate trajectory (dashed) under output feedback are shown for the case when the reactor is initialized within Ω_1^* and the switch is executed at $t = 1.0$ min $:= T_2$.

where $L_{\bar{f}_i}\bar{V}_i = [\partial \bar{V}_i/\partial e]\bar{f}_i$ and $L_{\bar{g}_i}\bar{V}_i = [\partial \bar{V}_i/\partial e]\bar{g}_i$, for $i = 1, 2$ (note that the $\bar{V}_i$'s are CLFs for the e-subsystem only and not for the full system of Eq. (26)—see Remark 10). To estimate the stability region for each mode, we use the following Lyapunov functions based on the full system: for mode 1, we use $V_1 = \frac{1}{2}c_1((T - T_s)/T_s)^2 + \frac{1}{2}c_2((C_A - C_{As})/C_{As})^2$, where $c_1 = 38.8$, $c_2 = 1.0$, and for mode 2 we use $V_2 = \frac{1}{2}c_3((T - T_s)/T_s)^2 + \frac{1}{2}c_4((C_A - C_{As})/C_{As})^2$ where $c_3 = 19.4$, $c_4 = 1.0$. The invariant regions, denoted in Fig. 1 by Ω_1^*, Ω_2^*, respectively, represent estimates of the stability regions for each mode.

We first present the case when both T and C_A are available for measurement. The solid lines in Figs. 1 and 2 depict the temperature, concentration and heat input profiles when the reactor is initialized at $x_0 = [435\,\mathrm{K}, 4.2\,\mathrm{mol/L}]^{\mathrm{T}} \in \Omega_1^*$ and operated in mode 1 for all times (with no switching). We observe that the controller for this mode successfully stabilizes the reactor temperature at the desired steady-state. The dashed lines in Figs. 1 and 2 depict the result when the reactor (initialized at x_0 within Ω_1^*) switches to

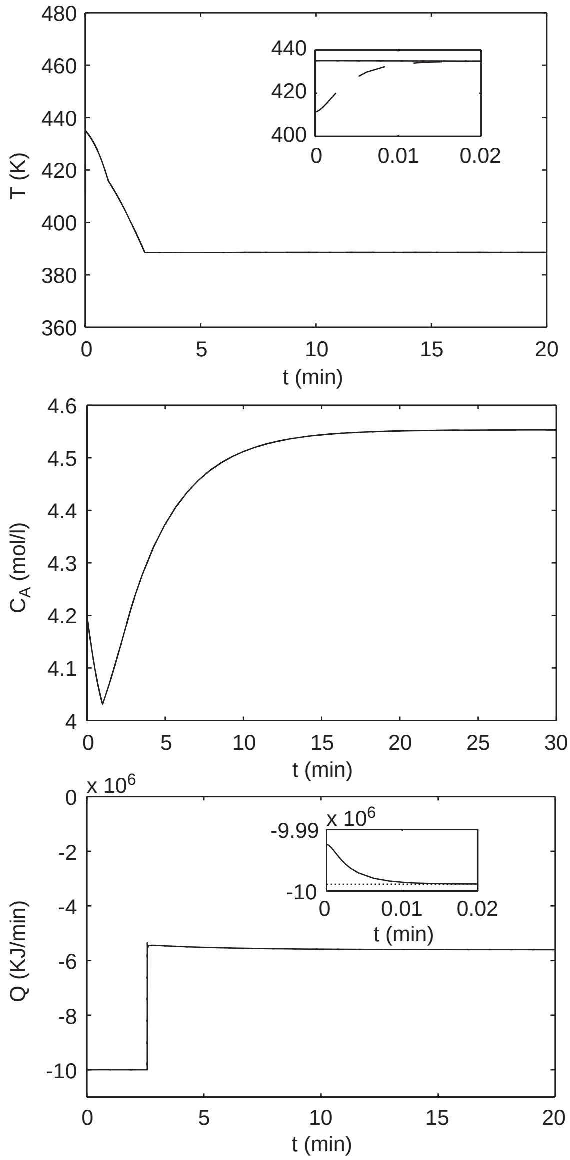

Fig. 4. Closed-loop state profiles under state (dotted) and output (solid) feedback; the state estimate profiles under output feedback (dashed), and the manipulated input profiles are shown for the case when the reactor is initialized within Ω_1^* and the switch is executed at $t = 1.0$ min.

mode 2 (with its corresponding controller) at a randomly chosen time of $t = T_1 = 0.1$ min. It is clear that in this case the controller is unable to stabilize the temperature at the desired steady-state. The reason is the fact that at $t = 0.1$ min, the state of the system lies

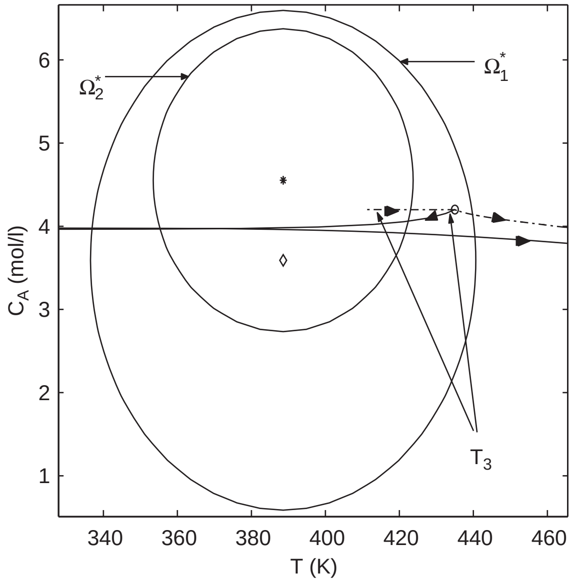

Fig. 5. The solid line shows the closed-loop state trajectory when the reactor is initialized within Ω_1^* and operated in mode 1 for all times with $L = 0.1$. The dotted and dash–dotted lines show, respectively, the closed-loop state and state estimate trajectories for $L = 100.0$ when the reactor switches to mode 2 at $t = 0.002$ min $:= T_3$.

outside the stability region of mode 2 and, therefore, the available control action is insufficient to achieve stabilization as can be seen from the input profile in Fig. 2 (dashed lines). To avoid this instability, we use the switching scheme proposed in Theorem 2. To this end, the reactor is initialized in mode 1 at x_0 and the closed-loop state is monitored (dotted trajectory in Fig. 1). At $t = T_2 = 1.0$ min, the state is observed to belong to Ω_2^* (i.e., the condition in Eq. (8) is satisfied) and, consequently, the supervisor switches to mode 2 (note that the condition of Eq. (9) is not needed here since mode 1 is never reactivated). The temperature, concentration and heat input profiles for this case are given in Fig. 2 (dotted lines) which show that the controllers successfully drive the reactor temperature to the desired steady-state and maintain it there with the available control action (note that the concentration settles at a higher steady-state value than that of mode 1 thus achieving our switching objective of enhancing reactant concentration).

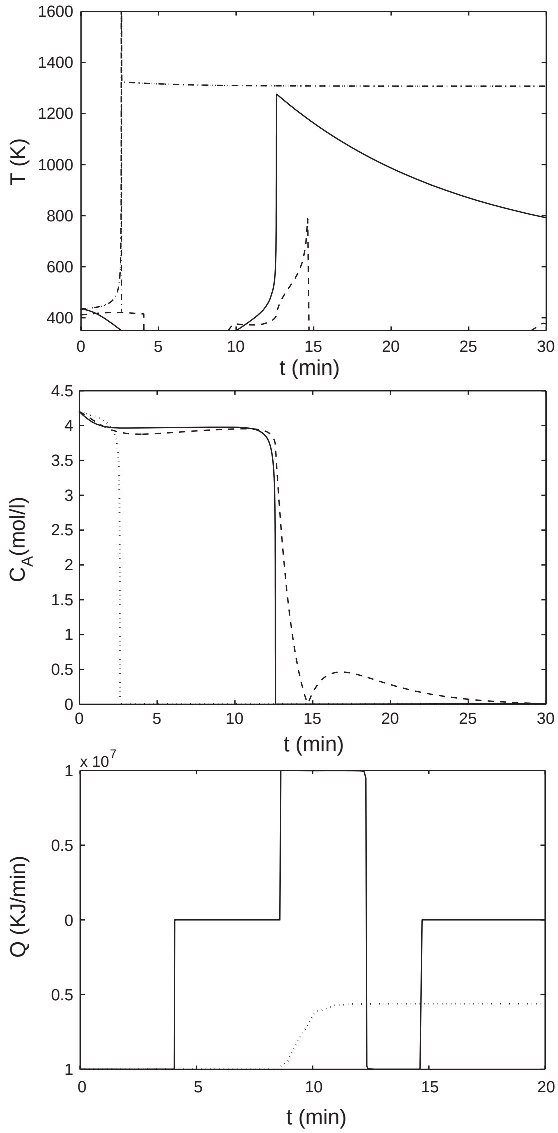

Fig. 6. Closed-loop state and input profiles: the solid (state) and dashed (state estimates) lines show the case when the reactor is initialized within Ω_1^* and operated in mode 1 for all times with $L = 0.1$. The dotted (state) and dash–dotted (state estimates) lines shows the case with $L = 100.0$ when the reactor switches to mode 2 at $t = 0.002$ min $:= T_3$.

We now consider the case when only C_A is measured. For this choice of the measured output, Assumption 2 is satisfied and, therefore, an output feedback controller of the form of Eq. (21) is designed for each mode. The values of the observer parameters in the state estimator design of Eq. (21) are chosen as

$L_1 = L_2 = 100$, $a_1^{(1)} = a_1^{(2)} = 10$ and $a_2^{(1)} = a_2^{(2)} = 20$. We first demonstrate the implementation of the switching rule of Eq. (23). As shown in Fig. 3, starting from the same initial condition considered under state feedback, $x(0) = [435\,\mathrm{K}, 4.2\,\mathrm{mol/L}]^{\mathrm{T}}$, and with $\hat{x}(0) = [411.3\,\mathrm{K}, 4.2\,\mathrm{mol}/L]^{\mathrm{T}}$ (the state and state estimate trajectories are shown by the solid and dashed lines, respectively), implementing the output feedback controller of Eq. (21) for mode 1 and switching to mode 2 (and its associated output feedback controller) at $t = 1.0$ (upon observing that the state estimates are within Ω_2^*) results in closed-loop stability. Included in Figs. 3 and 4 also are the corresponding state (dotted lines in Fig. 4) and manipulated input (dotted lines in Fig. 4) trajectories under state feedback control. Notice that the manipulated input profile under output feedback control differs from that under state feedback control for a brief period of time, due to the initial error in the state estimates, but converges to that under state feedback very quickly as the estimates converge to the true state values.

To demonstrate the importance of using an observer gain consistent with the choice of the output feedback stability region, we present in Figs. 5 and 6 (solid lines) a scenario, where starting from the same initial conditions in Ω_1^*, the reactor is operated in mode 1 for all times but with a lower value of the observer gain, $L = 0.1$. In this case, the error in the control action, resulting from the error in the value of the state estimates, leads to instability, demonstrating the need for appropriate choice of the observer parameters.

Finally, we illustrate the importance of waiting for a small period of time before a decision regarding switching is made even if a sufficiently large value of the observer gain is being used (i.e., a value for which Ω_1^* and Ω_2^* closely estimate the output feedback stability region). To this end, we use $L_1 = L_2 = 100$, as in the first scenario. Starting from the same initial conditions, it is observed that at $t = 0.002$ min $:= T_3$ the state estimates reside in Ω_2^* (see dotted (which coincides with the solid line) and dash–dotted lines in Fig. 5). Notice that while the state estimates belong to Ω_2^*, the true states are outside of Ω_2^* at this time. If the switch is executed immediately (as done under state feedback; see Theorem 2), then the closed-loop system becomes unstable. In contrast, if the decision regarding a switch is made after waiting for a sufficiently long period of time (as required by the switching rule of Eq. (23)) after which the state estimates have converged to their true values, closed-loop stability is achieved (see solid lines in Figs. 3 and 4).

Appendix

Proof of Theorem 2. To prove this theorem, we proceed in two steps. In the first step we show, for each individual mode (without switching), that starting from any initial condition within the set $\Omega_i^*(u_i^{\max})$ the corresponding feedback control law asymptotically stabilizes the closed-loop subsystem. In the second step, we use this fact together with the MLF stability result of Theorem 1 to show that the switching laws of Eqs. (8) and (9) enforce asymptotic stability in the switched closed-loop system starting from any initial condition that belongs to any of the sets $\Omega_i^*(u_i^{\max})$.

Step 1: Consider the ith subsystem of the switched nonlinear system of Eq. (1) for which Assumption 1 holds. Substituting the control law of Eqs. (5) and (6), evaluating the time-derivative of the Lyapunov function along the closed-loop trajectories, and using the fact that $\|(L_{G_i}V_i)^{\mathrm{T}}\|^2 = (L_{G_i}V_i)(L_{G_i}V_i)^{\mathrm{T}}$, we obtain

$$\begin{aligned}
\dot{V}_i &= L_{f_i}V_i + L_{G_i}V_i u_i \\
&= L_{f_i}V_i - L_{G_i}V_i \left(\frac{L_{f_i}V_i + \rho_i V_i + \sqrt{(L_{f_i}V_i + \rho_i V_i)^2 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^4}}{\|(L_{G_i}V_i)^{\mathrm{T}}\|^2[1 + \sqrt{1 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^2}]} \right) (L_{G_i}V_i)^{\mathrm{T}} \\
&= \frac{L_{f_i}V_i\sqrt{1 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^2} - \sqrt{(L_{f_i}V_i + \rho_i V_i)^2 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^4} - \rho_i V_i}{[1 + \sqrt{1 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^2}]}. \qquad (29)
\end{aligned}$$

It is clear from the last equality above that when $L_{f_i}V_i + \rho_i V_i < 0$, we have $\dot{V}_i < 0$. Furthermore, when $0 < L_{f_i}V_i + \rho V_i \leqslant u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|$, we have

$(L_{f_i}V_i + \rho_i V_i)^2 < (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^2$ and, therefore,

$$\begin{aligned}
&-\sqrt{(L_{f_i}V_i + \rho_i V_i)^2 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^4} \\
&\quad = -\sqrt{(L_{f_i}V_i + \rho_i V_i)^2 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^2(u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^2} \\
&\quad < -(L_{f_i}V_i + \rho_i V_i)\sqrt{1 + (u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|)^2}.
\end{aligned} \tag{30}$$

Substituting the above estimate into the expression for $\dot{V}_i$ in Eq. (29), we have that $\dot{V}_i < 0$. To summarize, we see that whenever $L_{f_i}V_i + \rho V_i \leqslant u_i^{\max}\|(L_{G_i}V_i)^{\mathrm{T}}\|$, we have $\dot{V}_i < 0$. Since $\Omega_i^*(u_i^{\max})$ is the largest invariant set where this inequality holds for all nonzero x, then starting from any initial state $x(0) \in \Omega_i^*(u_i^{\max})$, we have that

$$\dot{V}_i < 0 \ \forall x \neq 0, \quad i = 1, \ldots, N, \tag{31}$$

which implies that the individual closed-loop subsystems are asymptotically stable.

Step 2: Consider now the switched closed-loop system and, without loss of generality, suppose that $x(0) \in \Omega_i^*(u_i^{\max})$ for some $i \in \mathscr{I}$. Then it follows from Eq. (31) above and the invariance of $\Omega_i^*(u_i^{\max})$ that the Lyapunov function for this mode, V_i, decays monotonically along the trajectories of the closed-loop system for as long as mode i is to remain active, i.e., for all times such that $\sigma(t) = i$. If at any time T such that $x(T) \in \Omega_j^*(u_j^{\max})$ for some $j \in \mathscr{I}$, $j \neq i$, we set $\sigma(t) = j$ for $t \geqslant T^+$ (i.e., activate mode j and its respective controller), then using the same argument, it is clear that the corresponding Lyapunov function for this mode, V_j, will also decay monotonically for as long as we keep $\sigma(t) = j$. By tracking the closed-loop trajectory in this manner, we conclude that, starting from any $x(0) \in \Omega_i^*(u_i^{\max})$ for any $i \in \mathscr{I}$ and as long as the ith mode (and its controller) is activated only at a time when $x(t) \in \Omega_i^*(u_i^{\max})$, we have that for all $i_k \in \mathscr{I}$, $k \in Z_+$

$$\dot{V}_{\sigma(t_{i_k})} < 0 \quad \forall t \in [t_{i_k}, t_{i'_k}), \tag{32}$$

where t_{i_k} and $t_{i'_k}$ refer, respectively, to the times that the ith mode is switched in and out for the kth time, by the supervisor. Furthermore, from Eq. (9), we have that for any admissible switching time t_{i_k}

$$V_i(x(t_{i_k})) < V_i(x(t_{i_{k-1}})). \tag{33}$$

Using Eqs. (32) and (33), a direct application of the MLF result of Theorem 2.3 in [4] can be performed to conclude that the origin of the switched closed-loop system, under the switching laws of Theorem 2, is Lyapunov stable. To prove asymptotic stability, we note from the strict inequality in Eq. (33) that for every (infinite) sequence of switching times $t_{i_1}, t_{i_2}, \ldots$ such that $\sigma(t_{i_k}^+) = i$, the sequence $V_{\sigma(t_{i_1})}, V_{\sigma(t_{i_2})}, \ldots$ is decreasing and positive, and therefore has a limit $L \geqslant 0$. We have

$$\begin{aligned}
0 = L - L &= \lim_{k\to\infty} V_{\sigma(t_{i_{k+1}}^+)}(x(t_{i_{k+1}})) \\
&\quad - \lim_{k\to\infty} V_{\sigma(t_{i_k}^+)}(x(t_{i_k})) \\
&= \lim_{k\to\infty} [V_i(x(t_{i_{k+1}})) - V_i(x(t_{i_k}))].
\end{aligned} \tag{34}$$

Note that the term in brackets in the above equation is strictly negative for all nonzero x and zero only when $x = 0$. Therefore, there exists a function α of class $\mathscr{K}$ (i.e., continuous, increasing, and zero at zero) such that

$$[V_i(x(t_{i_{k+1}})) - V_i(x(t_{i_k}))] \leqslant -\alpha(\|x(t_{i_k})\|). \tag{35}$$

Substituting the above estimate into Eq. (34), we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} [V_i(x(t_{i_{k+1}})) - V_i(x(t_{i_k}))] \\
&\leqslant \lim_{k\to\infty} [-\alpha(\|x(t_{i_k})\|)] \leqslant 0,
\end{aligned} \tag{36}$$

which implies that $x(t)$ converges to the origin, which together with Lyapunov stability, implies that the origin of the switched closed-loop system is asymptotically stable. This completes the proof of Theorem 2. □

Proof of Theorem 3. *Step* 1: Consider the switched closed-loop system where $x(0) \in \Omega_{b,i}$, $\|\tilde{y}(0)\| \leqslant \delta_{\zeta,i}$ and $\varepsilon_i \in (0, \varepsilon_i^*]$, for some $i \in \mathscr{I}$. Then it follows from the result of Proposition 1 that the Lyapunov function for this mode, V_i^{c}, decays monotonically, along the trajectories of the closed-loop system, for as long as mode i is to remain active, i.e., for all times such that $\sigma(t) = i$. Consider now the scenario where at some given time T, we have $\hat{x}(T) \in \Omega_{s,j}$ and the system switches from mode i to mode j. Since we set $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n \leqslant \min\{\varepsilon_i^*\}$, $i = 1, \ldots, N$, we have $\varepsilon_j \leqslant \varepsilon_j^*$. Since the closed-loop system has been evolving in mode i for a time greater than or equal to $T_b^{\max} > T_{b,i}$ (from Eq. (23)), it follows from Proposition 1 that $\|x(T) - \hat{x}(T)\| \leqslant e_{m,i}$. This, together with the fact that $\hat{x}(T) \in \Omega_{s,j}$, implies (using Proposition 2) that$x(T) \in \Omega_{b,j}$. Also, once mode j is switched in, the switching rule of Eq. (25) sets $\|\tilde{y}(T^+)\| \leqslant \delta_{\zeta,j}$. All the conditions in Proposition 1 are, therefore, satisfied ($x(T^+) \in \Omega_{b,j}$, $\|\tilde{y}(T^+)\| \leqslant \delta_{\zeta,j}$ and $\varepsilon_j \leqslant \varepsilon_j^*$), which implies that the corresponding Lyapunov function for this mode, V_j^{c}, will also decay monotonically for $t > T$, and as long as we keep $\sigma(t) = j$. In this manner, we have that for all $j_k \in \mathscr{I}$, $k \in Z_+$:

$$\dot{V}^{\mathrm{c}}_{\sigma(t_{j_k})} < 0 \quad \forall t \in [t_{j_k}, t_{j'_k}), \tag{37}$$

where t_{j_k} and $t_{j'_k}$ refer, respectively, to the times that the jth mode is switched in and out for the kth time, by the supervisor.

Step 2: The first part of Eq. (23) ensures that the system has stayed in mode i for at least a time $T_b^{\max} \geqslant T_{b,i}$ before switching to mode j. It follows from Proposition 1, therefore, that $\|x(T) - \hat{x}(T)\| \leqslant e_{m,i}$. From the continuity of the function $V_i^{\mathrm{c}}(\cdot)$, we get that for a given $e_{m,i}$, there exists a positive real number $M_i(e_{m,i})$ such that if $\|x_f - \hat{x}_f\| \leqslant e_{m,i}$, $|V_i^{\mathrm{c}}(x_f) - V_i^{\mathrm{c}}(\hat{x}_f)| \leqslant M_i$. Therefore, we can write, for $x_f(t_{i'_{k-1}})$ and $x_f(t_{i'_k})$:

$$V_i^{\mathrm{c}}(\hat{x}_f(t_{i'_{k-1}})) - M_i \leqslant V_i^{\mathrm{c}}(x_f(t_{i'_{k-1}})), \tag{38}$$

$$V_i^{\mathrm{c}}(\hat{x}_f(t_{i'_k})) + M_i \geqslant V_i^{\mathrm{c}}(x_f(t_{i'_k})). \tag{39}$$

From Eq. (24), we have for any admissible switching time $T = t_{i'_k}$

$$V_i^{\mathrm{c}}(\hat{x}_f(t_{i'_{k-1}})) - M_i > V_i^{\mathrm{c}}(\hat{x}_f(t_{i'_k})) + M_i, \tag{40}$$

which, together with Eqs. (38) and (39), implies

$$V_i^{\mathrm{c}}(x_f(t_{i'_k})) < V_i^{\mathrm{c}}(x_f(t_{i'_{k-1}})). \tag{41}$$

Using Eqs. (37)–(41), one can finally show, with calculations similar to those used in the proof of Theorem 2 (see Eqs. (32)–(36)), that the origin of the switched closed-loop system, under the switching laws of Eqs. (23) and (24), is asymptotically stable. Note that the switching law enforces that if any of the conditions are not satisfied, then the closed-loop system continues to evolve in mode i for all times; in this case closed-loop stability can be shown using the result of Proposition 1. This completes the proof of Theorem 3. □

References

[1] Z. Artstein, Stabilization with relaxed controls, Nonlinear Anal. 7 (1983) 1163–1173.

[2] P.I. Barton, C.C. Pantelides, Modeling of combined discrete/continuous processes, A.I.Ch.E. J. 40 (1994) 966–979.

[3] A. Bemporad, M. Morari, Control of systems integrating logic, dynamics and constraints, Automatica 35 (1999) 407–427.

[4] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, IEEE Trans. Automat. Control 43 (1998) 475–482.

[5] M.S. Branicky, V.S. Borkar, S.K. Mitter, A unified framework for hybrid control: model and optimal control theory, IEEE Trans. Automat. Control 43 (1998) 31–45.

[6] P.D. Christofides, Robust output feedback control of nonlinear singularly perturbed systems, Automatica 36 (2000) 45–52.

[7] R.A. Decarlo, M.S. Branicky, S. Petterson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems, Proc. IEEE 88 (2000) 1069–1082.

[8] M. Demetriou, N. Kazantzis, A new actuator activation policy for performance enhancement of controlled diffusion processes, Automatica 40 (2004) 415–421.

[9] N.H. El-Farra, P.D. Christofides, Integrating robustness, optimality, and constraints in control of nonlinear processes, Chem. Eng. Sci. 56 (2001) 1841–1868.

[10] N.H. El-Farra, P.D. Christofides, Switching and feedback laws for control of constrained switched nonlinear systems, in: C.J. Tomlin, M.R. Greenstreet (Eds.), Lecture Notes in Computer Science Series, vol. 2289, Springer, Berlin, 2002, pp. 164–178.

[11] N.H. El-Farra, P.D. Christofides, Bounded robust control of constrained multivariable nonlinear processes, Chem. Eng. Sci. 58 (2003) 3025–3047.

[12] N.H. El-Farra, P.D. Christofides, Coordinating feedback and switching for control of hybrid nonlinear processes, A.I.Ch.E. J. 49 (2003) 2079–2098.

[13] R.A. Freeman, P.V. Kokotovic, Robust Nonlinear Control Design: State-Space and Lyapunov Techniques, Birkhauser, Boston, 1996.

[14] V. Garcia-Onorio, B.E. Ydstie, Distributed, asynchronous and hybrid simulation of process networks using recording controllers, Internat. J. Robust Nonlinear Control 14 (2004) 227–248.

[15] R.L. Grossman, A. Nerode, A.P. Ravn, H. Rischel, Hybrid systems, Lecture Notes in Computer Science, vol. 736, Springer, New York, 1993.

[16] I.E. Grossmann, S.A. van den Heever, I. Harjukoski, Discrete optimization methods and their role in the integration

of planning and scheduling, in: Proceedings of Sixth International Conference on Chemical Process Control, Tucson, AZ, 2001, pp. 124–152.
[17] J.P. Hespanha, A.S. Morse, Stability of switched systems with average dwell time, in: Proceedings of 38th IEEE Conference on Decision and Control, Phoenix, AZ, 1999, pp. 2655–2660.
[18] B. Hu, X. Xu, P.J. Antsaklis, A.N. Michel, Robust stabilizing control law for a class of second-order switched systems, Systems Control Lett. 38 (1999) 197–207.
[19] A. Isidori, Nonlinear Control Systems: An Introduction, third ed., Springer, Berlin-Heidelberg, 1995.
[20] H.K. Khalil, Robust servomechanism output feedback controller for feedback linearizable systems, Automatica 30 (1994) 1587–1599.
[21] H.K. Khalil, Nonlinear Systems, second ed., Macmillan Publishing Company, New York, 1996.
[22] M.D. Lemmon, P.J. Antsaklis, Timed automata and robust control: can we now control complex dynamical systems?, in: Proceedings of 36th IEEE Conference on Decision and Control, San Diego, CA, 1997, pp. 108–113.
[23] D. Liberzon, ISS and integral-ISS disturbance attenuation with bounded controls, in: Proceedings of 38th IEEE Conference on Decision and Control, Phoenix, AZ, 1999, pp. 2501–2506.
[24] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Systems Mag. 19 (1999) 59–70.
[25] Y. Lin, E.D. Sontag, A universal formula for stabilization with bounded controls, Systems Control Lett. 16 (1991) 393–397.
[26] J. Lygeros, D.N. Godbole, S.S. Sastry, A game theoretic approach to hybrid system design, in: R. Alur, T. Henzinger (Eds.), Lecture Notes in Computer Science, vol. 1066, Springer, Berlin, 1996, pp. 1–12.
[27] M. Malisoff, E.D. Sontag, Universal formulas for feedback stabilization with respect to Minkowski balls, Systems Control Lett. 40 (2000) 247–260.
[28] P. Mhaskar, N.H. El-Farra, P.D. Christofides, Hybrid predictive control of process systems, A.I.Ch.E. J. 50 (2004) 1242–1259.
[29] D. Mignone, A. Bemporad, M. Morari, A framework for control, fault detection, state estimation, and verification of hybrid systems, in: Proceedings of American Control Conference, San Diego, CA, 1999, pp. 134–139.
[30] P. Peleties, R. DeCarlo, Asymptotic stability of *m*-switched systems using Lyapunov-like functions, in: Proceedings of American Control Conference, Boston, MA, 1991, pp. 1679–1684.
[31] S. Pettersson, B. Lennartson, Stability and robustness for hybrid systems, in: Proceedings of 35th IEEE Conference on Decision and Control, Kobe, Japan, 1996, pp. 1202–1207.
[32] R. Sepulchre, M. Jankovic, P. Kokotovic, Constructive Nonlinear Control, Springer, Berlin-Heidelberg, 1997.
[33] E.D. Sontag, A Lyapunov-like characterization of asymptotic controllability, SIAM J. Control Opt. 21 (1983) 462–471.
[34] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Trans. Automat. Control 34 (1989) 435–443.
[35] A. Teel, L. Praly, Global stabilizability and observability imply semi-global stabilizability by output feedback, Systems Control Lett. 22 (1994) 313–325.
[36] M.A. Wicks, P. Peleties, R.A. DeCarlo, Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems, European J. Control 4 (1998) 140–147.
[37] E.C. Yamalidou, J. Kantor, Modeling and optimal control of discrete-event chemical processes using Petri nets, Comp. Chem. Eng. 15 (1990) 503–519.
[38] H. Ye, A.N. Michel, L. Hou, Stability theory for hybrid dynamical systems, IEEE Trans. Automat. Control 43 (1998) 461–474.
[39] M. Zefran, J.W. Burdick, Design of switching controllers for systems with changing dynamics, in: Proceedings of 37th IEEE Conference on Decision and Control, Tampa, FL, 1998, pp. 2113–2118.
[40] P.D. Christofides, A.R. Teel, Singular perturbations and input-to-state stability, IEEE Trans. Automat. Control 41 (1996) 1645–1650.