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Global stabilization of the Kuramoto–Sivashinsky equation via distributed output feedback control

Panagiotis D. Christofides *, Antonios Armaou

Department of Chemical Engineering, School of Engineering & Applied Sciences, University of California, Los Angeles, 405 Hilgard Avenue, Box 951592, CA 90095, USA

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Abstract

This work addresses the problem of global exponential stabilization of the Kuramoto–Sivashinsky equation (KSE) subject to periodic boundary conditions via distributed static output feedback control. Under the assumption that the number of measurements is equal to the total number of unstable and critically stable eigenvalues of the KSE and a necessary and sufficient stability condition is satisfied, linear static output feedback controllers are designed that globally exponentially stabilize the zero solution of the KSE. The controllers are designed on the basis of finite-dimensional approximations of the KSE which are obtained through Galerkin's method. The theoretical results are confirmed by computer simulations of the closed-loop system.

Keywords: Kuramoto–Sivashinsky equation; Distributed control; Global stabilization; Wave suppression

1. Introduction

The KSE is a nonlinear dissipative fourth-order partial differential equation (PDE) of the form

$$\frac{\partial x}{\partial t} = -v\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z}, \tag{1}$$

where $v > 0$ is the so-called instability parameter, which describes incipient instabilities in a variety of physical and chemical systems. Examples include falling liquid films [8], unstable flame fronts [21,19,22], Belouzov–Zabotinskii reaction patterns [16,17] and interfacial instabilities between two viscous fluids [14]. Analytical and numerical studies of the dynamics of Eq. (1) with periodic boundary conditions (e.g., [23,8,13,15]) have revealed the existence of steady and periodic wave solutions, as well as chaotic behavior for very small values of v.

In addition to the existence of complex solution patterns, the above studies have revealed that the dominant dynamics of the KSE can be adequately characterized by a small number of degrees of freedom (e.g., [23]). Motivated by this, we recently addressed [2,1] the design of linear/nonlinear finite-dimensional output feedback

* Corresponding author. Tel.: 000-310-794-1015; fax: 000-310-206-4107.
E-mail address: pdc@seas.ucla.edu (P.D. Christofides)

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controllers for stabilization of the zero solution of the KSE on the basis of ordinary differential equation (ODE) approximations, obtained through linear/nonlinear Galerkin's method, that accurately describe the dominant dynamics. However, even though these control algorithms achieve stabilization of the zero solution of the KSE for any value of the instability parameter v, their application is limited to local (i.e., for sufficiently small initial conditions) stabilization. This limitation motivates the study of the problem of global stabilization of the KSE. In this direction, a nonlinear boundary feedback controller was proposed in [18] that enhances the rate of convergence to the spatially uniform steady state of the KSE for values of v for which this steady state is naturally stable.

In this paper, we consider the problem of global exponential stabilization of the zero solution, $x(z,t)=0$, of the KSE with periodic boundary conditions, for any value of the instability parameter v, via distributed static output feedback control. Under the assumptions that the number of measurements is equal to the total number of unstable and critically stable eigenvalues of the KSE and a necessary and sufficient stability condition is satisfied, linear static output feedback controllers are designed that achieve global (i.e., for every initial condition) stabilization of the $x(z,t)=0$ solution of Eq. (1). The proposed output feedback controllers are designed on the basis of ODE approximations of the KSE obtained through Galerkin's method. Numerical simulations of the closed-loop system, for different values of the instability parameter, confirm the theoretical results.

2. Preliminaries

We consider the integrated form of the *controlled* Kuramoto–Sivashinsky equation

$$\frac{\partial x}{\partial t} = -v\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z} + \sum_{i=1}^{m} b_i u_i(t),$$

$$y_\kappa^m = \int_{-\pi}^{\pi} s_\kappa(z) x \,\mathrm{d}z, \quad \kappa = 1,\ldots,p, \tag{2}$$

subject to the periodic boundary conditions

$$\frac{\partial^j x}{\partial z^j}(-\pi,t) = \frac{\partial^j x}{\partial z^j}(+\pi,t), \quad j=0,\ldots,3, \tag{3}$$

and the initial condition

$$x(z,0) = x_0(z), \tag{4}$$

where $x \in L_p^2(-\pi,\pi)$ is the state of the system, $L_p^2(-\pi,\pi)$ denotes the space of square integrable functions that satisfy the boundary conditions of Eq. (3) (i.e., $L_p^2(-\pi,\pi) = \{x \in L^2(-\pi,\pi)$: $(\partial^j x/\partial z^j)(-\pi,t) = (\partial^j x/\partial z^j)(+\pi,t),\ j=0,\ldots,3\}$), z is the spatial coordinate, v is the instability parameter which is assumed to be known, t is the time and 2π is the length of the spatial domain, $x_0(z) \in L_p^2([-\pi,\pi])$ is the initial condition, m is the number of manipulated inputs (i.e., variables that can be manipulated externally to modify the dynamics of Eq. (2) in a desired fashion), $u_i(t)$ is the ith manipulated input, $b_i(z)$ is the actuator distribution function (i.e., $b_i(z)$ determines how the control action computed by the ith control actuator, $u_i(t)$, is distributed (e.g., point or distributed actuation) in the spatial interval $[-\pi,\pi]$), $y_\kappa^m \in \mathbb{R}$ denotes a measured output, and $s_\kappa(z)$ is a known smooth function of z which is determined by the location and type of the measurement sensors (e.g., point/distributed sensing). We note that in the case of point actuation (sensing) which influences (measures) the system at z_0 (i.e., $b_i(z)$ or $s_\kappa(z)$ is equal to $\delta(z-z_0)$ where $\delta(\cdot)$ is the standard Dirac function), we approximate the function $\delta(z-z_0)$ by the finite value $1/2\varepsilon$ in the interval $[z_0-\varepsilon, z_0+\varepsilon]$ (where ε is a small positive real number) and zero elsewhere in $[-\pi,\pi]$. Finally, in $L_p^2([-\pi,\pi])$, we define the inner product and norm: $(\omega_1,\omega_2) = \int_{-\pi}^{\pi} \omega_1(z)\omega_2(z)\,\mathrm{d}z$, $\|\omega_1\|_2 = (\omega_1,\omega_1)^{1/2}$ where ω_1,ω_2 are two elements of $L_p^2([-\pi,\pi])$.

The dynamics of Eq. (2) depend heavily on the value of v. To make this point clear, we compute the linearization of the system of Eq. (1) around $x(z,t)=0$, which takes the form

$$\frac{\partial x}{\partial t} = -v\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} \tag{5}$$

and consider the corresponding eigenvalue problem

$$\mathscr{A}\phi_n = -v\frac{\partial^4 \phi_n}{\partial z^4} - \frac{\partial^2 \phi_n}{\partial z^2} = \mu_n\phi_n, \quad n=1,\ldots,\infty, \tag{6}$$

subject to

$$\frac{\partial^j \phi_n}{\partial z^j}(-\pi,t) = \frac{\partial^j \phi_n}{\partial z^j}(+\pi,t), \quad j=0,\ldots,3, \tag{7}$$

where μ_n denotes an eigenvalue and ϕ_n denotes an eigenfunction. A direct computation of the solution of the above eigenvalue problem yields $\mu_0=0$ with $\psi_0(z)=1/\sqrt{2\pi}$, and $\mu_n=-vn^4+n^2$ (μ_n is an eigenvalue of multiplicity two) with eigenfunctions $\phi_n(z)=(1/\sqrt{\pi})\sin(nz)$ and $\psi_n(z)=(1/\sqrt{\pi})\cos(nz)$, $n=1,\ldots,\infty$. Clearly, a pair of eigenvalues of the system of Eq. (5) crosses the imaginary axis when

$$v=\frac{1}{n^2}, \quad n=1,\ldots,\infty. \tag{8}$$

Apparently, the smallest value of v, for which the $x(z,t)=0$ solution of the system of Eq. (5) is about to become unstable is $v=1$ and when $1/n^2 > v > 1/(n+1)^2$, the system of Eq. (5) has $2n$ positive eigenvalues.

The above stability analysis implies that the spatially uniform steady state, $x(z,t)=0$, of the nonlinear system of Eqs. (1)–(3) is locally unstable when $v<1$. Instead, there is a generation of *stable* spatially non-uniform stationary solutions as well as spatially non-uniform periodic solutions, while for very small values of v no stable solutions exist and the system of Eq. (2) exhibits chaotic behavior (the reader may refer to [8,15] for detailed characterizations of the solution patterns for various ranges of values of v).

3. Static output feedback control

In this section, we synthesize output feedback controllers that globally stabilize the system of Eqs. (2) and (3) at $x(z,t)=0$. We initially use Galerkin's method to derive ODE approximations of the system of Eq. (2) that capture the dynamics of the unstable modes. Expanding the solution of the system of Eq. (2) in an infinite series in terms of the eigenfunctions of the operator of Eq. (6), we obtain

$$x(z,t)=\sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z)+\sum_{n=0}^{\infty}\beta_n(t)\psi_n(z), \tag{9}$$

where $\alpha_n(t)$, $\beta_n(t)$ are time-varying coefficients and $\phi_n(z)=(1/\sqrt{\pi})\sin(nz)$, $\psi_0(z)=1/\sqrt{2\pi}$ and $\psi_n(z)=(1/\sqrt{\pi})\cos(nz)$. Substituting the above expansion for the solution, $x(z,t)$, into the system of Eq. (2) and taking the inner product in $L_p^2([-\pi,\pi])$ with the adjoint eigenfunctions, $\phi_n^*(z)=(1/\sqrt{\pi})\sin(nz)$, $\psi_0^*(z)=1/\sqrt{2\pi}$ and $\psi_n^*(z)=(1/\sqrt{\pi})\cos(nz)$ of the operator of Eq. (6) (note that the operator of Eq. (6) subject to the boundary condition of Eq. (7) is self-adjoint, i.e., $\phi_n(z)=\phi_n^*(z)$, $\psi_0(z)=\psi_0^*(z)$, $\psi_n(z)=\psi_n^*(z)$), the following infinite system of ODEs is obtained:

$$\dot{\alpha}_n=(-vn^4+n^2)\alpha_n - f_{n\alpha}+\sum_{i=1}^{m} b_{i\alpha}^n u_i(t), \quad n=1,\ldots,\infty,$$

$$\dot{\beta}_n = (-\nu n^4 + n^2)\beta_n - f_{n\beta} + \sum_{i=1}^{m} b_{i\beta}^n u_i(t), \quad n = 0,\ldots,\infty, \tag{10}$$

$$y_\kappa^m = \int_{-\pi}^{\pi} s_\kappa(z)\left(\sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z) + \sum_{n=0}^{\infty}\beta_n(t)\psi_n(z)\right)\mathrm{d}z, \quad \kappa = 1,\ldots,p,$$

where

$$f_{n\alpha} = \int_{-\pi}^{\pi}\phi_n(z)\left(\sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z) + \sum_{n=0}^{\infty}\beta_n(t)\psi_n(z)\right)\left(\sum_{n=1}^{\infty}\alpha_n(t)\frac{\mathrm{d}\phi_n}{\mathrm{d}z}(z) + \sum_{n=0}^{\infty}\beta_n(t)\frac{\mathrm{d}\psi_n}{\mathrm{d}z}(z)\right)\mathrm{d}z,$$

$$f_{n\beta} = \int_{-\pi}^{\pi}\psi_n(z)\left(\sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z) + \sum_{n=0}^{\infty}\beta_n(t)\psi_n(z)\right)\left(\sum_{n=1}^{\infty}\alpha_n(t)\frac{\mathrm{d}\phi_n}{\mathrm{d}z}(z) + \sum_{n=0}^{\infty}\beta_n(t)\frac{\mathrm{d}\psi_n}{\mathrm{d}z}(z)\right)\mathrm{d}z, \tag{11}$$

$$b_{i\alpha}^n = \int_{-\pi}^{\pi}\phi_n(z)b_i(z)\,\mathrm{d}z, \qquad b_{i\beta}^n = \int_{-\pi}^{\pi}\psi_n(z)b_i(z)\,\mathrm{d}z$$

and

$$\frac{\mathrm{d}\phi_n}{\mathrm{d}z}(z) = \frac{n}{\sqrt{\pi}}\cos(nz), \qquad \frac{\mathrm{d}\psi_n}{\mathrm{d}z}(z) = -\frac{n}{\sqrt{\pi}}\sin(nz).$$

Owing to its infinite-dimensional nature, the system of Eq. (10) cannot be directly used for the design of controllers that can be implemented in practice (i.e., the practical implementation of controllers which are designed on the basis of this system will require the computation of infinite sums which cannot be done by a computer). Instead, we will base the controller design on linear finite-dimensional approximations of this system. Moreover, the maximum number of eigenvalues of $\mathscr{A}$ which are identical, for any ν, is four, and thus, the use of five control actuators is sufficient for stabilizing any stabilizable finite-dimensional approximation of the system of Eq. (10) with linear state feedback (four control actuators are needed due to the presence of four identical unstable eigenvalues and one control actuator is needed to ensure that the conservation of mass condition, $\int_{-\pi}^{\pi} x(z,t)\,\mathrm{d}z = 0$, is satisfied at steady state). We note that even though the use of five control actuators is sufficient to stabilize any stabilizable finite-dimensional approximation of the system of Eq. (10) with linear state feedback, it is not necessary. For example, when $\nu = 0.4$, $\mathscr{A}$ possesses only two identical unstable eigenvalues, and thus, stabilization can be achieved with three control actuators; see simulation section for a numerical demonstration of this point.

Assuming that the number of unstable eigenvalues of the system of Eq. (5) is $2l$ (i.e., $1/l^2 > \nu > 1/(l+1)^2$) and using m control actuators (where $m \leqslant 5$), the infinite set of ODEs of Eq. (11) can be written in the following form:

$$\begin{aligned}
\dot{\alpha}_u &= A_u\alpha_u + B_u u - \tilde{f}_u,\\
\dot{\alpha}_s &= A_s\alpha_s + B_s u - \tilde{f}_s,\\
y^m &= S_u\alpha_u + S_s\alpha_s,
\end{aligned} \tag{12}$$

where $\alpha_u = [\beta_0\ \alpha_1\ \beta_1\ \alpha_2\ \beta_2\ \cdots\ \alpha_l\ \beta_l]^{\mathrm{T}}$, $\alpha_s = [\alpha_{l+1}\ \beta_{l+1}\ \cdots]^{\mathrm{T}}$, $\tilde{f}_u = [0\ f_{1\alpha}\ f_{1\beta}\ f_{2\alpha}\ f_{2\beta}\ \cdots\ f_{l\alpha}\ f_{l\beta}]^{\mathrm{T}}$, $\tilde{f}_s = [f_{l+1\alpha}\ f_{l+1\beta}\ f_{l+2\alpha}\ f_{l+2\beta}\ \cdots]^{\mathrm{T}}, y^m = [y_1^m\ y_2^m\ \cdots\ y_p^m]^{\mathrm{T}}$, $u \in \mathbb{R}^m$ and

$$A_u = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -v+1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -v+1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -l^4 v + l^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -l^4 v + l^2 \end{bmatrix}, \quad B_u = \begin{bmatrix} b_{1\beta}^0 & b_{2\beta}^0 & \cdots & b_{m\beta}^0 \\ b_{1\alpha}^1 & b_{2\alpha}^1 & \cdots & b_{m\alpha}^1 \\ b_{1\beta}^1 & b_{2\beta}^1 & \cdots & b_{m\beta}^1 \\ \vdots & \vdots & \cdots & \vdots \\ b_{1\alpha}^l & b_{2\alpha}^l & \cdots & b_{m\alpha}^l \\ b_{1\beta}^l & b_{2\beta}^l & \cdots & b_{m\beta}^l \end{bmatrix}, \tag{13}$$

$$A_s = \begin{bmatrix} -(l+1)^4 v + (l+1)^2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(l+1)^4 v + (l+1)^2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \end{bmatrix}, \tag{14}$$

$$B_s = \begin{bmatrix} b_{1\beta}^{l+1} & b_{2\beta}^{l+1} & \cdots & b_{m\beta}^{l+1} \\ b_{1\beta}^{l+2} & b_{2\beta}^{l+2} & \cdots & b_{m\beta}^{l+2} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}, \quad S_u = \begin{bmatrix} s_{1\beta}^0 & s_{1\alpha}^1 & s_{1\beta}^1 & \cdots & s_{1\alpha}^l & s_{1\beta}^l \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ s_{p\beta}^0 & s_{p\alpha}^1 & s_{p\beta}^1 & \cdots & s_{p\alpha}^l & s_{p\beta}^l \end{bmatrix}, \tag{15}$$

$$S_s = \begin{bmatrix} s_{1\alpha}^{l+1} & s_{1\beta}^{l+1} & \cdots \\ \vdots & \vdots & \cdots \\ s_{p\alpha}^{l+1} & s_{p\beta}^{l+1} & \cdots \end{bmatrix}. \tag{16}$$

The following assumption is needed in order to obtain estimates of the states α_u of the system of Eq. (12) from the measurements y_κ^m, $\kappa = 1, \ldots, p$.

Assumption 1. $p = 2l + 1$ (i.e., the number of measurements is equal to the number of eigenvalues which are in the closed right-half of the complex plane), and the inverse of the operator S_u exists so that $\hat{\alpha}_u = S_u^{-1} y^m$, where $\hat{\alpha}_u$ is an estimate of α_u.

Theorem 1 that follows provides a necessary and sufficient condition under which, for a given value of v, there exists a linear static output feedback controller that uses five control actuators and $2l+1$ measurements to globally exponentially stabilize the zero solution of the Kuramoto–Sivashinsky equation.

Theorem 1. *Let the number of unstable eigenvalues of the system of Eq.* (5) *be* $2l$ (*i.e.*, $1/l^2 > v > 1/(l+1)^2$), $m = 5$ *and Assumption* 1 *hold. Then, the existence of a matrix* K *so that the following infinite range matrix*:

$$A_{\mathrm{cl}} = \begin{bmatrix} A_u + B_u K & B_u K S_u^{-1} S_s \\ B_s K & A_s + B_s K S_u^{-1} S_s \end{bmatrix} \tag{17}$$

is Hurwitz (*i.e., it possesses eigenvalues with strictly negative real part*), *is necessary and sufficient for the following output feedback controller*:

$$u = K S_u^{-1} y^m, \tag{18}$$

to ensure that the $x(z,t) = 0$ *solution of the closed-loop system* (*Eqs.* (2)–(17)) *is exponentially stable, for any initial condition in* $L_p^2([-\pi,\pi])$ *and any* $1/l^2 > v > 1/(l+1)^2$.

Remark 1. The requirement on matrix A_{cl} depends on the structure of the matrices B_u, S_u, and therefore, on the shape of the actuator distribution functions b_i and the measurement sensor-shape functions s_κ which, for most practical applications, cannot be chosen by the control designer. Whenever the requirement on matrix A_{cl} is not satisfied, one can use more than five control actuators to ensure that the system of Eq. (12) is stabilizable via output feedback. Furthermore, the requirement of Assumption 1 that the inverse of the operator S_u exists is necessary for the solution of the output feedback control problem, and clearly depends on the shape and location of the measurement sensors. However, this requirement does not appear to be restrictive from a practical standpoint (see numerical results in the next section). Finally, we note that the requirement $m = 5$ in Theorem 1 is imposed to ensure stabilization of the closed-loop system even for values of v for which $\mathscr{A}$ possesses four identical unstable eigenvalues; clearly, it is possible to achieve stabilization of the KSE with a smaller number of control actuators as long as the matrix A_{cl} is stable (see Section 4 for such an example).

Remark 2. Even though the matrix A_{cl} of Eq. (17) has infinite range, the fact that the eigenvalues of the matrix A_s grow in a fourth-order fashion in the left-half of the complex plane implies that there exists an N sufficiently large so that the verification of the stability of A_{cl} can be done on the basis of a finite-dimensional approximation obtained by keeping the first $(2l + 1 + 2N)$ rows and columns of A_{cl} (this fact is numerically verified in Section 4). This means that the design of the gain matrix K which stabilizes the closed-loop system (Eqs. (2)–(18)) can be done on the basis of a $(2l + 1 + 2N)$-dimensional system utilizing standard pole-placement methods for static output feedback of finite-dimensional systems (see, for example, [4, Section 13.5] for details). This fact can be also rigorously established by using a singular perturbation formulation of the closed-loop system similar to the one employed in [9] to show stability of a parabolic PDE system under robust output feedback control. The details of this analysis are given in the appendix.

Remark 3. When the measurement sensor shape functions are chosen so that $S_u^{-1}S_s = 0$ (which implies that $\hat{\alpha}_u = \alpha_u$; state feedback case), the matrix A_{cl} of Eq. (17) simplifies to

$$A_{cl} = \begin{bmatrix} A_u + B_uK & 0 \\ B_sK & A_s \end{bmatrix} \tag{19}$$

and the infinite-dimensional stabilizability requirement of Theorem 1 reduces to the one of stability of the finite-dimensional matrix $A_u + B_uK$ (i.e., the pair $[A_u \;\; B_u]$ should be controllable).

Remark 4. On the other hand, when the distribution functions of the control actuators are chosen so that $B_s = 0$ (which implies that no spill-over [2] effect is present in the closed-loop system), the matrix A_{cl} of Eq. (17) simplifies to

$$A_{cl} = \begin{bmatrix} A_u + B_uK & B_uKS_u^{-1}S_s \\ 0 & A_s \end{bmatrix} \tag{20}$$

and the stabilizability requirement of Theorem 1 again reduces to the one of stability of $A_u + B_uK$.

Remark 5. Even though static output feedback control requires a larger number of measurements ($2l + 1$ versus m, with $m \leqslant 5$) for small values of v and is more sensitive to measurement noise than dynamic output feedback, we prefer to use static feedback of y^m in the controller of Eq. (18) because it allows us to globally exponentially stabilize the zero solution; this is not possible when linear finite-dimensional dynamic output feedback control is used [2].

Remark 6. The exponential stability of the closed-loop system ensures a certain degree of robustness with respect to sufficiently small disturbances and uncertainty in process parameters (i.e., v, $b_{i\alpha}^n, b_{i\beta}^n, s_\kappa$), as well as robustness of the closed-loop system with respect to fast and asymptotically stable unmodeled dynamics (for example, actuator and sensor dynamics). The problem of designing robust controllers that explicitly account and compensate for the presence of uncertainty will not be addressed in this paper; the reader may refer to [9] for results on robust control of parabolic PDE systems.

Remark 7. We note that the proposed approach can also be employed to compute necessary and sufficient conditions for the global exponential stabilization of the zero solution of the KSE subject to nonperiodic boundary conditions (such as the ones studied in [18]) via distributed static output feedback control.

Remark 8. Even though the proposed approach allows achieving *global* stabilization of the zero solution of the KSE, improved *local* convergence rates of the state of the closed-loop system to the zero solution can be achieved by utilizing nonlinear controllers constructed on the basis of ODE models obtained from combination of Galerkin's method with approximate inertial manifolds (see [1] for details and [5,6,20] for other applications of inertial manifold theory to control of nonlinear distributed parameter systems). Furthermore, it is important to note that the L_p^2 exponential convergence of $x(z,t)$ to the zero solution does not necessarily lead to pointwise convergence.

Remark 9. We note that the approach that we followed for controller synthesis based on finite-dimensional approximations of the Kuramoto–Sivashinsky equation obtained via Galerkin's method is motivated from the fact that the dominant dynamics of the equation are characterized by a finite (small) number of degrees of freedom. This approach has been successfully used to control parabolic PDEs arising in the context of transport-reaction processes (see, for example, [3,11,12,7]).

Proof of Theorem 1. *Part 1*: We initially show that if the condition of Eq. (17) holds, then the controller of Eq. (18) exponentially stabilizes the Kuramoto–Sivashinsky equation without the term $-x(\partial x/\partial z)$. Under the controller of Eq. (18), the closed-loop system without the term $-x(\partial x/\partial z)$ takes the form

$$\frac{\partial x}{\partial t} = -v\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} + bKS_u^{-1}y^m, \tag{21}$$

where $b = [b_1\ b_2\ \cdots\ b_m]$. Applying Galerkin's method to the above system, the following infinite set of ordinary differential equations is obtained:

$$\begin{aligned} \dot{\alpha}_u &= A_u\alpha_u + B_uKS_u^{-1}(S_u\alpha_u + S_s\alpha_s), \\ \dot{\alpha}_s &= A_s\alpha_s + B_sKS_u^{-1}(S_u\alpha_u + S_s\alpha_s) \end{aligned} \tag{22}$$

or

$$\begin{bmatrix} \dot{\alpha}_u \\ \dot{\alpha}_s \end{bmatrix} = \begin{bmatrix} A_u + B_uK & B_uKS_u^{-1}S_s \\ B_sK & A_s + B_sKS_u^{-1}S_s \end{bmatrix} \begin{bmatrix} \alpha_u \\ \alpha_s \end{bmatrix}. \tag{23}$$

From the above system, it is clear that the existence of a matrix K so that the matrix of Eq. (17) is Hurwitz is necessary and sufficient for the local exponential stability of the zero solution of the closed-loop system of Eq. (21) for $1/l^2 > v > 1/(l+1)^2$.

Part 2: In this part of the proof, we use a Lyapunov argument to show that a linear static output feedback controller of the form of Eq. (18) that exponentially stabilizes the closed-loop system of Eq. (21), ensures also global exponential stability of the zero solution of the Kuramoto–Sivashinsky equation with the term $-x(\partial x/\partial z)$. To this end, we first use the fact that exponential stability of the system of Eq. (23) implies that there exists a positive constant c such that the linear self-adjoint closed-loop operator

$$\bar{\mathscr{A}}x = -v\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} + bK \begin{bmatrix} \int_{-\pi}^{\pi} s_1(z)x(z,t)\,\mathrm{d}z \\ \int_{-\pi}^{\pi} s_2(z)x(z,t)\,\mathrm{d}z \\ \int_{-\pi}^{\pi} s_3(z)x(z,t)\,\mathrm{d}z \\ \cdots \\ \int_{-\pi}^{\pi} s_{2l}(z)x(z,t)\,\mathrm{d}z \\ \int_{-\pi}^{\pi} s_{2l+1}(z)x(z,t)\,\mathrm{d}z \end{bmatrix}, \tag{24}$$

satisfies $(x, \bar{\mathscr{A}}x) \leqslant -c\, \|x\|_2^2$. Now, multiplying the nonlinear closed-loop system

$$\frac{\partial x}{\partial t} = -v\frac{\partial^4 x}{\partial z^4} - \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z} + bKS_u^{-1}y^m, \tag{25}$$

by x and integrating from $[-\pi, \pi]$ and using the notation of Eq. (24), we obtain

$$\int_{-\pi}^{\pi} x\frac{\partial x}{\partial t}\,\mathrm{d}z = \int_{-\pi}^{\pi} x\bar{\mathscr{A}}x\,\mathrm{d}z - \int_{-\pi}^{\pi} x^2\frac{\partial x}{\partial z}\,\mathrm{d}z. \tag{26}$$

Using the boundary conditions of Eq. (3), one can show that (see [23] for details)

$$\int_{-\pi}^{\pi} x^2\frac{\partial x}{\partial z}\,\mathrm{d}z = 0, \tag{27}$$

which implies that Eq. (26) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|x\|_2^2 = (x, \bar{\mathscr{A}}x) \leqslant -c\,\|x\|_2^2\,. \tag{28}$$

From the above inequality, the exponential stability of the solution, $x(z,t) = 0$, of the closed-loop system (Eq. (21)), for any initial condition in $L_p^2([-\pi, \pi])$, follows. □

4. Numerical results

In this section, we evaluate, through computer simulations, the ability of the output feedback controller of Eq. (18) to stabilize the system of Eqs. (2) and (3) at the steady state $x(z,t) = 0$. Two simulation runs were performed for $v = 0.4$ and 0.2. For the numerical simulation of the KSE, we used a 51-order nonlinear ordinary differential equation model obtained from the application of Galerkin's method to the system of Eq. (2) (the use of higher-order Galerkin approximations in simulating the open-loop system of Eqs. (2) and (3) as well as the closed-loop system led to identical numerical results). In all the simulation runs, the system is assumed to be at a spatially nonuniform initial condition

$$x_0 = \frac{1.0}{\sqrt{2\pi}} + \frac{2.5}{\sqrt{\pi}}\sum_{n=1}^{5}(\sin(nz) + \cos(nz)) \in L_p^2([-\pi, \pi]).$$

In the first simulation run, we set $v = 0.4$ which implies that the linearized system of Eq. (5) possesses two identical eigenvalues in the right-half of the complex plane. Therefore, the use of three control actuators and three measurement sensors is necessary and sufficient for achieving stabilization of the closed-loop system. One distributed control actuator with $b_1(z) = 1/\sqrt{2\pi}$, two point control actuators placed at $z = -\pi/2$ and

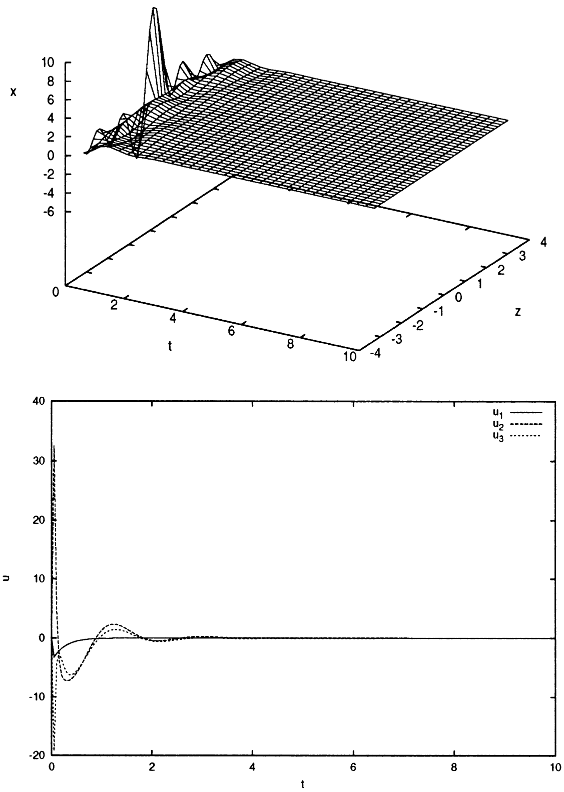

Fig. 1. Closed-loop spatio-temporal profile of $x(z,t)$ (top figure) and manipulated input profiles (bottom figure), for $v=0.4$.

$z=\pi/6$ (i.e., $b_2(z)=\delta(z+(\pi/2))$ and $b_3(z)=\delta(z-(\pi/6))$), one distributed sensor with $s_1(z)=1/\sqrt{2\pi}$ and two-point measurement sensors placed at $z=0.0$ and $z=-\pi/4$ (i.e., sensor shape functions $s_2(z)=\delta(z-0.0)$, $s_3(z)=\delta(z+(\pi/4))$) are used to stabilize the system of Eqs. (2) and (3) at $x(z,t)=0$. The control actions, $u_1(t),u_2(t),u_3(t)$, were computed from the formula of Eq. (18) with

$$K=\begin{bmatrix}-4.000 & -2.404 & 4.163\\ 0.000 & 6.026 & -3.478\\ 0.000 & 0.000 & -6.958\end{bmatrix} \tag{29}$$

to ensure that the matrix A_{cl} is stable. Fig. 1 shows the closed-loop spatio-temporal profile of $x(z,t)$ and the profiles of the three manipulated inputs. It is clear that the controller stabilizes the state of the system at $x(z,t)=0$, indicating that the use of three control actuators suffices to stabilize the system for $v=0.4$.

In the second simulation run, we used $v=0.2$ which implies that the linearized system of Eq. (5) possesses four identical unstable eigenvalues (i.e., $\mu_1=\mu_2=0.8$; both μ_1,μ_2 are eigenvalues of multiplicity 2), thus requiring five control actuators and five measurement sensors of $x(z,t)$ to stabilize the system at $x(z,t)=0$. One distributed control actuator with $b_1(z)=1/\sqrt{2\pi}$, four-point control actuators placed at $z=\pi/2$, $z=\pi/6$, $z=-\pi/4$ and $z=-\pi/2$, (i.e., $b_2(z)=\delta(z-(\pi/2))$, $b_3(z)=\delta(z-(\pi/6))$, $b_4(z)=\delta(z+(\pi/4))$ and $b_5(z)=\delta(z+(\pi/2))$), one distributed measurement sensor with $s_1(z)=1/\sqrt{2\pi}$ and four-point measurement sensors placed at $z=\pi/6$, $z=\pi/4$, $z=-\pi/4$ and $z=-\pi/2$, (i.e., $s_2(z)=\delta(z-(\pi/6))$, $s_3(z)=\delta(z-(\pi/4))$, $s_4(z)=\delta(z+(\pi/4))$ and $s_5(z)=\delta(z+(\pi/2))$) are used to stabilize the system at $x(z,t)=0$. The control actions, $u_1(t),u_2(t),u_3(t),u_4(t),u_5(t)$,

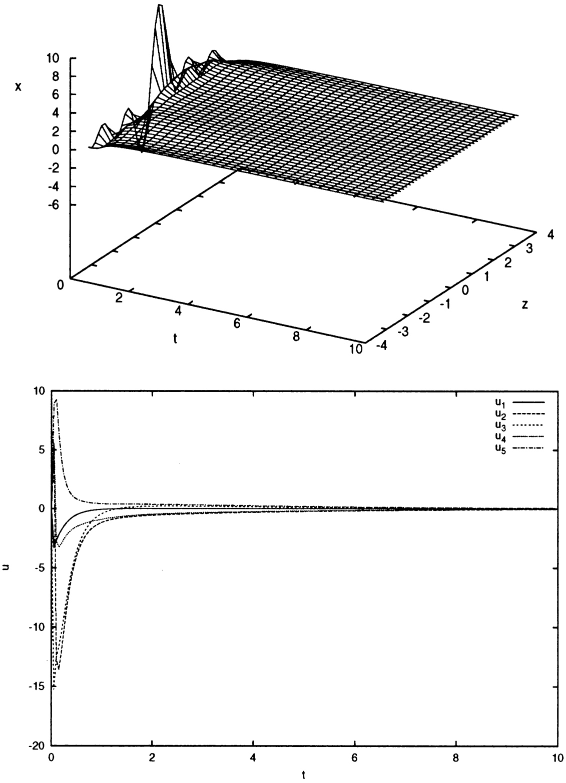

Fig. 2. Closed-loop spatio-temporal profile of $x(z,t)$ (top figure) and manipulated input profiles (bottom figure), for $v = 0.2$.

were computed from the formula of Eq. (18) with

$$K = \begin{bmatrix} -4.000 & 0.000 & 3.621 & 0.298 & -2.263 \\ 0.000 & -2.836 & -1.175 & 1.175 & 2.836 \\ 0.000 & 0.000 & -3.836 & -2.713 & 0.000 \\ 0.000 & 0.000 & -3.322 & 3.323 & 0.000 \\ 0.000 & 2.836 & -0.744 & -2.531 & 2.836 \end{bmatrix}, \tag{30}$$

to ensure that the matrix A_{cl} is stable. The closed-loop profile of $x(z,t)$ and the profiles of the five manipulated inputs are displayed in Fig. 2. The stabilization of the state of the system at $x(z,t)=0$, for $v=0.2$, has been accomplished.

From the results of the simulation study, it is evident that the proposed control algorithm achieves stabilization of the Kuramoto–Sivashinsky equation at $x(z,t) = 0$ for values of v for which the zero solution is unstable.

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Appendix A

We show that given the value of the instability parameter v, if there exists a gain matrix K so that the condition of Eq. (17) is satisfied for a $(2l+1+2N)$-dimensional approximation of the closed-loop system of Eq. (21) without the term $-x(\partial x/\partial z)$, then the infinite dimensional closed-loop system of Eqs. (2) and (18) without the term $-x(\partial x/\partial z)$ is also globally exponentially stable. Note that the nonlinear term $-x(\partial x/\partial z)$ is neglected because in the proof of theorem 1 we have shown that a linear static output feedback controller that exponentially stabilizes the Kuramoto–Sivashinsky equation without the term $-x(\partial x/\partial z)$, enforces also global exponential stability of the zero solution of the Kuramoto–Sivashinsky equation when this term is present.

Given v, we fix l, and apply Galerkin's method to the closed-loop system of Eq. (21) without the term $-x(\partial x/\partial z)$ to obtain the following infinite set of ODEs:

$$\begin{aligned}
\dot{\alpha}_{\mathrm{sl}} &= A_{\mathrm{sl}}\alpha_{\mathrm{sl}} + B_{\mathrm{sl}}KS_u^{-1}y^m,\\
\dot{\alpha}_{\mathrm{f}} &= A_{\mathrm{f}}\alpha_{\mathrm{f}} + B_{\mathrm{f}}KS_u^{-1}y^m,\\
y^m &= S_{\mathrm{sl}}\alpha_{\mathrm{sl}} + S_s\alpha_{\mathrm{f}},
\end{aligned} \tag{A.1}$$

where $\alpha_{\mathrm{sl}} = [\beta_0 \ \alpha_1 \ \beta_1 \ \alpha_2 \ \beta_2 \ \cdots \ \alpha_{l+N} \ \beta_{l+N}]^{\mathrm{T}}$, $\alpha_{\mathrm{f}} = [\alpha_{l+N+1} \ \beta_{l+N+1} \ \cdots]^{\mathrm{T}}$, and the explicit structure of the matrices $A_{\mathrm{sl}}, B_{\mathrm{sl}}, A_{\mathrm{f}}, B_{\mathrm{f}}, S_{\mathrm{sl}}, S_s$ is similar to the structure of the corresponding matrices in the system of Eq. (12) and will be omitted for brevity.

Now, defining the parameter $\varepsilon = |\mu_1|/|\mu_{l+N+1}| < 1$ and multiplying the α_{f}-subsystem of Eq. (A.1) by ε, we obtain

$$\begin{aligned}
\dot{\alpha}_{\mathrm{sl}} &= A_{\mathrm{sl}}\alpha_{\mathrm{sl}} + B_{\mathrm{sl}}KS_u^{-1}y^m,\\
\varepsilon\dot{\alpha}_{\mathrm{f}} &= A_{\mathrm{f}\varepsilon}\alpha_{\mathrm{f}} + \varepsilon B_{\mathrm{f}}KS_u^{-1}y^m,\\
y^m &= S_{\mathrm{sl}}\alpha_{\mathrm{sl}} + S_s\alpha_{\mathrm{f}},
\end{aligned} \tag{A.2}$$

where $A_{\mathrm{f}\varepsilon} = \varepsilon A_{\mathrm{f}}$. In the above system, the operators A_{sl} and $A_{\mathrm{f}\varepsilon}$ generate semi-groups with growth rates which are of the same order of magnitude, i.e., the solutions α_{sl} and α_{f} of the systems $\dot{\alpha}_{\mathrm{sl}} = A_{\mathrm{sl}}\alpha_{\mathrm{sl}}$ and $\dot{\alpha}_{\mathrm{f}} = A_{\mathrm{f}\varepsilon}\alpha_{\mathrm{f}}$, respectively, satisfy: $|\alpha_{\mathrm{sl}}(t)| \leqslant K|\alpha_{\mathrm{sl}}(0)|\mathrm{e}^{k_1 t}$, $|\alpha_{\mathrm{f}}(t)| \leqslant K|\alpha_{\mathrm{f}}(0)|\mathrm{e}^{k_2 t}$, where K, k_1, k_2 are real numbers, and $\mathrm{O}(k_1) = \mathrm{O}(k_2)$ ($\mathrm{O}(\cdot)$ denotes the standard order of magnitude notation). This means that the system of Eq. (A.2) is in the standard singularly perturbed form with α_{sl} being the slow states and α_{f} being the fast states. The stability properties of this system can be analyzed within the singular perturbation framework by decomposing it into reduced-order subsystems describing its behavior in the fast and slow time scales. Specifically, the slow subsystem can be obtained by setting $\varepsilon = 0$ in the system of Eq. (A.2) that yields $\alpha_{\mathrm{f}} = 0$ and

$$\dot{\alpha}_{\mathrm{sl}} = A_{\mathrm{sl}}\alpha_{\mathrm{sl}} + B_{\mathrm{sl}}KS_u^{-1}S_{\mathrm{sl}}\alpha_{\mathrm{sl}}. \tag{A.3}$$

Under the assumption that there exists a gain matrix K so that the condition of Eq. (17) is satisfied for a $(2l+1+2N)$-dimensional approximation of the closed-loop system, we have that the above finite-dimensional system is exponentially stable. Furthermore, rewriting the system of Eq. (A.2) in the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, the following system which describes the fast dynamics of the system of Eq. (A.2) is obtained:

$$\frac{\mathrm{d}\alpha_{\mathrm{f}}}{\mathrm{d}\tau} = A_{\mathrm{f}\varepsilon}\alpha_{\mathrm{f}}, \tag{A.4}$$

which is also exponentially stable. Finally, utilizing results from singular perturbation theory for systems of the form of Eq. (A.2) in which the fast subsystem is an infinite-dimensional system [10], we have that the exponential stability of the slow and fast subsystems implies that there exists an ε^* such that if $\varepsilon \in (0, \varepsilon^*]$, the system of Eq. (A.2) is exponentially stable. From the definition of ε, it follows directly that ε decreases as N increases, thereby implying that given $v > 0$, there exists an N^* sufficiently large (i.e., ε^* sufficiently small) so that the design of the gain matrix K which stabilizes the infinite dimensional closed-loop system (Eqs. (2)–(18)) can be done on the basis of a $(2l+1+2N)$-dimensional approximation of this system, provided $N \geqslant N^*$. Note that ε^* (and thus, N^*) depends on all the matrices of the system of Eq. (A.2), and thus, in turn, it depends on ν, and the distribution functions of the measurement sensors and control actuators.

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