Technical Notes and Correspondence

Singular Perturbations and Input-to-State Stability

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Abstract—**This paper establishes a type of *total stability* for the input-to-state stability property with respect to singular perturbations. In particular, if the boundary layer system is uniformly globally asymptotically stable and the reduced system is input-to-state stable with respect to disturbances, then these properties continue to hold, up to an arbitrarily small offset, for initial conditions, disturbances, and their derivatives in an arbitrarily large compact set as long as the singular perturbation parameter is sufficiently small.**

I. Introduction

Many physical and chemical systems involve dynamical phenomena occurring in different time scales. Representative examples of nonlinear multiple-time scale systems include chemical reaction networks [2], high-purity distillation columns [13], electromechanical networks [4], flexible mechanical systems [7], etc. Typically, such processes are modeled within the mathematical framework of singular perturbations [12]. These models are then used as a basis for the design of control systems. However, the majority of models are characterized by disturbances, unknown model parameters, etc., complicating further the stability analysis and controller design for such systems.

Motivated by the above, the problem of analyzing the stability properties of nonlinear singularly perturbed systems has received considerable attention in the literature. Early results focused on local results or imposed growth conditions on the nonlinearities of the system (see [9] and [8] for example). More recently, for systems without disturbances, Saberi and Khalil [16] developed a set of sufficient interconnection conditions, expressed in terms of Lyapunov functions and their derivatives, to guarantee asymptotic stability of a point. Da and Corless [7] followed the same approach to study the asymptotic stability of a class of nonlinear singularly perturbed systems whose fast dynamics are marginally stable. Other methods based on the so-called geometric approach for singularly perturbed systems have also been proposed [3]. For systems with disturbances, approaches based on Lyapunov's second method have been proposed (see [6] for example) to derive sufficient conditions that guarantee boundedness of the trajectories of the system.

In this note, we address the analysis of singularly perturbed nonlinear systems having a reduced system that is input-to-state stable (see [17]) with respect to disturbances. We will show that this property is robust, in a sense to be made precise in Theorem 1, to uniformly globally asymptotically stable singular perturbations. Our result is novel in that absolutely no interconnection or growth conditions are required, and the disturbances are not required to be small and/or slowly varying. The robustness we obtain is of a "total stability" type with a flavor similar to the total stability result reported on in [18]. To establish our result, we will generate lemmas regarding the input-to-state stability property which are natural generalizations of [18, Lemma 3.2 and Th. 2]. An alternative, Lyapunov-based approach, based on an idea in [1], can be found in [5].

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II. Notation and Definitions

- $|\cdot|$ denotes the standard Euclidean norm, and $I_{n\times n}$ denotes the identity matrix of dimension $n \times n$.
- For any measurable (with respect to the Lebesgue measure) function $\theta: \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $\|\theta\|$ denotes ess.sup.$|\theta(t)|$, $t \geq 0$. For any strictly positive real number ρ, $\|\theta^\rho\|$ denotes ess.sup. $|\theta(t)|$, $t \geq \rho$.
- For a signal $u(t)$ defined on $[0, T)$ and for each $\tau \in [0, T)$, u_τ is a signal defined on $[0, \infty)$ given by

$$u_\tau(t) = \begin{Bmatrix} u(t) & t \in [0, \tau] \\ 0 & t \in (\tau, \infty) \end{Bmatrix}. \tag{1}$$

- A function $\gamma: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class K if it is continuous, nondecreasing, and is zero at zero. It is of class K_∞ if, in addition, it is strictly increasing and unbounded.
- A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.

We will work with various forms of the "input-to-state" stability property for systems of the form

$$\dot{x} = \phi(x, u_1, u_2) \tag{2}$$

where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u_1 \in \mathbb{R}^s$, $u_2 \in \mathbb{R}^m$ denote vectors of input variables (disturbances), and ϕ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^m$.

Definition 1: Let γ_{u_1} and γ_{u_2} be two functions of class K. The system in (2) is said to be input-to-state stable (ISS) with Lyapunov gain $(\gamma_{u_1}, \gamma_{u_2})$ if there exist a smooth function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, three functions a_1, a_2, a_3 of class K_∞, and two functions $\tilde{\gamma}_{u_1}, \tilde{\gamma}_{u_2}$ of class K such that for all $x \in \mathbb{R}^n$, all $u_1 \in \mathbb{R}^s$, $u_2 \in \mathbb{R}^m$, and all $s \geq 0$:

1)

$$a_1(|x|) \leq V(x) \leq a_2(|x|)$$

2)

$$\begin{aligned} |x| \geq \max\{\tilde{\gamma}_{u_1}(|u_1|), \tilde{\gamma}_{u_2}(|u_2|)\} \\ \Longrightarrow \dot{V}(x) = \frac{\partial V}{\partial x}\phi(x, u_1, u_2) \leq -a_3(|x|) \end{aligned}$$

3)

$$\gamma_{u_1}(s) = a_1^{-1}(a_2(\tilde{\gamma}_{u_1}(s))), \gamma_{u_2}(s) = a_1^{-1}(a_2(\tilde{\gamma}_{u_2}(s))). \tag{3}$$

A relationship between the above property and a bound on the state x as a function of time (as proposed in [17]) is given below. See [19] for related converse theorems.

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Lemma 0: If system (2) is ISS with Lyapunov gain $(\gamma_{u_1}, \gamma_{u_2})$ then there exists a function β of class KL such that for each $x_0 \in \mathbb{R}^n$ and for each pair of measurable, essentially bounded inputs $u_1(\cdot), u_2(\cdot)$, the solution of (2) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|) + \gamma_{u_2}(\|u_2\|). \tag{4}$$

Proof: From the hypothesis of the lemma and the results of [17] or [11, Th. 4.10], there exists a function β of class KL such that for each $x_0 \in \mathbb{R}^n$ and for each pair of measurable, essentially bounded inputs $u_1(\cdot), u_2(\cdot)$, the solution of (2) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + a_1^{-1}(a_2(\max\{\tilde{\gamma}_{u_1}(\|u_1\|), \tilde{\gamma}_{u_2}(\|u_2\|)\})). \tag{5}$$

Then the lemma follows from the fact that for any pair of nonnegative real numbers (g_1, g_2)

$$\begin{aligned} a_1^{-1}(a_2(\max\{g_1, g_2\})) &\leq \max\left\{a_1^{-1}(a_2(g_1)), a_1^{-1}(a_2(g_2))\right\} \\ &\leq a_1^{-1}(a_2(g_1)) + a_1^{-1}(a_2(g_2)). \end{aligned} \tag{6}$$

III. Singularly Perturbed Systems

We will focus on singularly perturbed nonlinear systems with the following state-space description:

$$\begin{aligned} \dot{x} &= f(x, z, \theta(t), \epsilon) \\ \epsilon\dot{z} &= g(x, z, \theta(t), \epsilon) \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ denote vectors of state variables, $\theta \in \mathbb{R}^q$ denotes the vector of the disturbances, and ϵ is a small positive parameter. The functions f and g are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times [0, \bar{\epsilon})$ for some $\bar{\epsilon} > 0$. The input vector $\theta(t)$ in (7) may represent constant or time-varying (not necessarily slowly) parameters, tracking signals and/or exogenous disturbances. In what follows, for simplicity, we will suppress the time-dependence in the notation of the vector of input variables $\theta(t)$.

A standard procedure that is followed for the analysis of systems in the form of (7) is the decomposition of the original system into separate reduced-order systems, each one associated with a different time scale. This procedure is called two-time scale decomposition [12]. Formally, it can be done by setting $\epsilon = 0$ in which case the dynamics of the state vector z becomes instantaneous and (7) takes the form

$$\begin{aligned} \dot{x} &= f(x, z_s, \theta, 0) \\ g(x, z_s, \theta, 0) &= 0 \end{aligned} \tag{8}$$

where z_s denotes a quasisteady state for the fast state vector z. Assumption 1 states that the singularly perturbed system in (7) is in *standard form.*

Assumption 1: The algebraic equation $g(x, z_s, \theta, 0) = 0$ possesses a unique root

$$z_s = h(x, \theta) \tag{9}$$

with the properties that $h: \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p$ and its partial derivatives $(\frac{\partial h}{\partial x}), \frac{\partial h}{\partial \theta}$ are locally Lipschitz.

Substituting (9) into (8), the following locally Lipschitz system is obtained:

$$\dot{x} = f(x, h(x, \theta), \theta, 0). \tag{10}$$

The dynamical system in (10) is called the *reduced system* or *slow subsystem.* The inherent two-time scale behavior of the system of (7) can be analyzed by defining a fast time scale

$$\tau = \frac{t}{\epsilon} \tag{11}$$

and the new coordinate $y := z - h(x, \theta)$. In the (x, y) coordinates, and with respect to the τ time scale, the singularly perturbed system in (7) takes the form

$$\begin{aligned} \frac{dx}{d\tau} &= \epsilon f(x, h(x, \theta) + y, \theta, \epsilon) \\ \frac{dy}{d\tau} &= g(x, h(x, \theta) + y, \theta, \epsilon) \\ &\quad - \epsilon\left[\frac{\partial h}{\partial x} f(x, h(x, \theta) + y, \theta, \epsilon) + \frac{\partial h}{\partial \theta}\dot{\theta}\right]. \end{aligned} \tag{12}$$

Setting ϵ equal to zero, the following locally Lipschitz system is obtained:

$$\frac{dy}{d\tau} = g(x, h(x, \theta) + y, \theta, 0). \tag{13}$$

Here, x and θ are to be thought of as constant vectors. In what follows, we will refer to the dynamical system in (13) as the *fast subsystem* or the *boundary layer* system. The assumptions that follow state our stability requirements on the slow and fast subsystems.

Assumption 2: The reduced system in (10) is ISS with Lyapunov gain γ.

Assumption 3: The equilibrium $y = 0$ of the boundary layer system in (13) is globally asymptotically stable, uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.

The main result of this note is given in the following theorem (the proof is given in Section V).

Theorem 1: Consider the singularly perturbed system in (7) and suppose Assumptions 1–3 hold and that $\theta(t)$ is absolutely continuous. Define $y = z - h(x, \theta)$ and let γ be the function given by Assumption 2. Then there exist functions β_x, β_y of class KL, and for each pair of positive real numbers (δ, d), there is an $\epsilon^* > 0$ such that if $\max\{|x(0)|, |y(0)|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$ and $\epsilon \in (0, \epsilon^*]$, then, for all $t \geq 0$

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + d \tag{14}$$

$$|y(t)| \leq \beta_y\left(|y(0)|, \frac{t}{\epsilon}\right) + d. \tag{15}$$

Remark 1: When g does not depend on θ, $\theta(t)$ can simply be measurable and there is no requirement on $\|\dot{\theta}\|$ (if $\dot{\theta}$ exists).

Remark 2: We emphasize that the result of Theorem 1 can be applied to arbitrarily large initial conditions $(x(0), y(0))$, uncertainty $\theta(t)$, and rate of change of uncertainty $\dot{\theta}(t)$. Furthermore, this result is novel even in the undisturbed case (i.e., $\theta(t) \equiv 0$) in that no growth or interconnection conditions are imposed to establish boundedness of the trajectories (compare with the discussion in Section VI).

Remark 3: In principle, a value for ϵ^* can be extracted from the proof of the theorem. However, as is the case for most general singular perturbation results for nonlinear systems (e.g., [16] and [7]), this value will typically be quite conservative.

IV. Further Facts About Input-to-State Stability

In this section, we give three lemmas on the input-to-state stability property that will be used in the proof of Theorem 1. These lemmas are natural extensions of results given in [18] and should be useful as tools for other nonlinear stability results. Throughout, the lemmas will refer to (2). When applying these lemmas in the proof of Theorem 1, we may be considering systems with three groups of inputs. The results for these cases are completely straightforward extensions of the cases presented below.

The first lemma is reminiscent of [18, Lemma 3.2].

Lemma 1: Assume that (2) with $u_2(t) \equiv 0$ is input-to-state stable with Lyapunov gain γ_{u_1}. Then, there exist an $m \times m$ matrix $B(x, u_1)$ of smooth functions, invertible for all $x \in \mathbb{R}^n$, $u_1 \in \mathbb{R}^s$ that satisfies $B(x, u_1) \equiv I_{m\times m}$ in a neighborhood of the origin, and a function γ_{u_2} of class K_∞ such that the system

$$\dot{x} = \phi(x, u_1, B(x, u_1)u_2) \tag{16}$$

is input-to-state stable with Lyapunov gain $(\gamma_{u_1}, \gamma_{u_2})$. Furthermore, if γ_{u_1} is of class K_∞, then the matrix B can be chosen to be independent of u_1.

Proof: From the hypothesis of the lemma and Definition 1, we have that there exist a smooth function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class K_∞ functions a_1, a_2, a_3, and a class K function $\tilde{\gamma}_{u_1}$ such that part 1 of (3) holds and

$$|x| \geq \tilde{\gamma}_{u_1}(|u_1|) \Longrightarrow \dot{V} = \frac{\partial V}{\partial x}\phi(x, u_1, 0) \leq -a_3(|x|) \tag{17}$$

where $\tilde{\gamma}_{u_1}(s) = a_2^{-1}(a_1(\gamma_{u_1}(s)))$ is a function of class K. Defining $\tilde{u}_2 := B(x, u_1)u_2$ and computing the time-derivative of the function V along the trajectories of (16), we get

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x}\phi(x, u_1, B(x, u_1)u_2)\\ &= \frac{\partial V}{\partial x}\phi(x, u_1, 0) + \frac{\partial V}{\partial x}(\phi(x, u_1, \tilde{u}_2) - \phi(x, u_1, 0)).\end{aligned} \tag{18}$$

Since ϕ is locally Lipschitz and V is smooth, and using (17), we have that there exist a class K_∞ function Ψ_1 and also a positive real number $L \geq 1$ such that

$$\begin{aligned}&|x| \geq \tilde{\gamma}_{u_1}(|u_1|) \Longrightarrow\\ &\dot{V} \leq -a_3(|x|) + |\tilde{u}_2|(L + \Psi_1(\max\{|x|, |u_1|, |\tilde{u}_2|\})).\end{aligned} \tag{19}$$

Let us now define the class K_∞ function $\tilde{\gamma}_{u_2}(s) := \max\{s, a_3^{-1}(2(1+L)(s))\}$, and let $b(s)$ be a smooth function that satisfies $b(s) \equiv 1$ in a neighborhood of the origin (i.e., there exists a $\delta_1 > 0$ such that $b(s) \equiv 1$, $\forall\ s \in [0, \delta_1]$) and moreover is chosen so that the inequality

$$0 < b(s) \leq \min\left\{\frac{1}{0.5 + \Psi_1(s)}, 1\right\} \tag{20}$$

holds for all $s \in \mathbb{R}_{\geq 0}$. Then, defining $X = [x^T\ u_1^T]^T$, we claim that the matrix

$$B(x, u_1) := b(|X|)I_{m\times m} \tag{21}$$

and the function $\gamma_{u_2}(s) = a_1^{-1}(a_2(\tilde{\gamma}_{u_2}(s)))$ satisfy the conditions of the lemma, i.e.,

$$\begin{aligned}&|x| \geq \max\{\tilde{\gamma}_{u_1}(|u_1|), \tilde{\gamma}_{u_2}(|u_2|)\} \Longrightarrow\\ &\dot{V} = \frac{\partial V}{\partial x}\phi(x, u_1, B(x, u_1)u_2) \leq -\frac{a_3(|x|)}{2}.\end{aligned} \tag{22}$$

First, using the fact from (20) that $b(|X|) \leq 1$, we have

$$\begin{aligned}&|x| \geq \tilde{\gamma}_{u_1}(|u_1|) \Longrightarrow\\ &\dot{V} \leq -a_3(|x|) + b(|X|)|u_2|\\ &\quad \times (L + \Psi_1(\max\{|x|, |u_1|, b(|X|)|u_2|\}))\\ &\quad \leq -a_3(|x|) + b(|X|)|u_2|(L + \Psi_1(\max\{|x|, |u_1|, |u_2|\})).\end{aligned} \tag{23}$$

From this it follows that

$$\begin{aligned}&|x| \geq \max\{\tilde{\gamma}_{u_1}(|u_1|), |u_2|\} \Longrightarrow\\ &\dot{V} \leq -a_3(|x|) + b(|X|)\,|u_2|(L + \Psi_1(\max\{|x|, |u_1|\}))\\ &\quad \leq -a_3(|x|) + b(|X|)\,|u_2|(L + \Psi_1(|X|)).\end{aligned} \tag{24}$$

Now, using the facts from (20) that $b(|X|)\Psi_1(|X|) \leq 1$ and $b(|X|) \leq 1$, we get

$$\begin{aligned}&|x| \geq \max\{\tilde{\gamma}_{u_1}(|u_1|), |u_2|\} \Longrightarrow\\ &\dot{V} \leq -\frac{1}{2}a_3(|x|) - \frac{1}{2}a_3(|x|) + (L+1)|u_2|.\end{aligned} \tag{25}$$

Finally, we have

$$\begin{aligned}&|x| \geq \max\{\tilde{\gamma}_{u_1}(|u_1|), |u_2|, a_3^{-1}(2(1+L)(|u_2|))\} \Longrightarrow\\ &\dot{V} \leq -\frac{1}{2}a_3(|x|).\end{aligned} \tag{26}$$

But from the definition of $\tilde{\gamma}_{u_2}$ we have (22).

We finally note that whenever the function $\tilde{\gamma}_{u_1}$ is of class K_∞, the inequality $|u_1| \leq \tilde{\gamma}_{u_1}^{-1}(|x|)$ can be used in the second inequality of (24) to get a new K_∞ function $\tilde{\Psi}_1$ which depends only on $|x|$. In this case, by choosing $b(s)$ to satisfy (20) for this new $\tilde{\Psi}_1$, one can then choose $B(x, u_1) = b(|x|)I_{m\times m}$, i.e., independent of u_1. □

Lemma 2 that follows is reminiscent of [18, Th. 2].

Lemma 2: Assume that (2) with $u_2(t) \equiv 0$ is input-to-state stable with Lyapunov gain γ_{u_1}. Then, there exist a function β of class KL, a continuous nonincreasing function $\sigma: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a function γ_{u_2} of class K such that for each $x_0 \in \mathbb{R}^n$, each essentially bounded input $u_1(\cdot)$, and each essentially bounded input $u_2(\cdot)$ satisfying $\|u_2\| \leq \sigma(\max\{|x(0)|, \|u_1\|\})$, the solution of (2) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies (4).

Proof: Referring to (2), we note that a direct application of Lemma 1 yields that there exist an $m \times m$ matrix $B(x, u_1)$ of smooth functions, invertible for all $x \in \mathbb{R}^n$, $u_1 \in \mathbb{R}^s$, and a function $\gamma_{\tilde{u}_2}$ of class K_∞ such that the following system:

$$\dot{x} = \phi(x, u_1, B(x, u_1)\tilde{u}_2) \tag{27}$$

is input-to-state stable with Lyapunov gain $(\gamma_{u_1}, \gamma_{\tilde{u}_2})$. Using Lemma 0, there exists a function β of class KL such that for each $x_0 \in \mathbb{R}^n$ and for each essentially bounded pair of inputs $u_1(\cdot), \tilde{u}_2(\cdot)$, the solution of (27) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|) + \gamma_{\tilde{u}_2}(\|\tilde{u}_2\|). \tag{28}$$

In the case where $\tilde{u}_2$ is only locally essentially bounded on $[0, \bar{T})$, we have by causality that for each $t \in [0, \bar{T})$

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|) + \gamma_{\tilde{u}_2}(\|\tilde{u}_{2_t}\|). \tag{29}$$

Identifying $\tilde{u}_2 = B^{-1}(x, u_1)u_2$, and using that u_1 and u_2 are assumed to be essentially bounded, if $[0, T)$ represents the maximal interval of definition for (27), then for each $t \in [0, T)$

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_{u_1}(\|u_1\|) + \gamma_{\tilde{u}_2}(\|B_t^{-1}\|\ \|u_2\|^{\frac{1}{2}}\ \|u_2\|^{\frac{1}{2}}). \tag{30}$$

The proof of Lemma 1 gives that $B(x, u_1)$ has the form $b(|X|)I_{m\times m}$, where $b(s)$ is a smooth function that satisfies $0 < b(s) \leq 1$, for all $s \in \mathbb{R}$, and $b(s) \equiv 1$, for all $s \in [0, \delta_1]$, where δ_1 is some positive real number. Let $L \geq 1$ and let Ψ_2 be a function of class K_∞ such that

$$\begin{aligned}&\frac{1}{b(\beta(|x(0)|, 0) + \gamma_{u_1}(\|u_1\|) + \gamma_{\tilde{u}_2}(1) + \|u_1\|)}\\ &\quad \leq L + \Psi_2(\max\{|x(0)|, \|u_1\|\}).\end{aligned} \tag{31}$$

Define the continuous nonincreasing function $\sigma(\cdot)$ as $\sigma(s) := (\frac{1}{2}(L + \Psi_2(s)))^{-2}$. Note that since $L \geq 1$, $\sigma(s) \leq 1$ for all $s \in \mathbb{R}$. We will show that if $\|u_2\| \leq \sigma(\max\{|x(0)|, \|u_1\|\})$ then x is defined on $[0, \infty)$ and $\frac{\|u_2\|^{\frac{1}{2}}}{\|b(|X|)\|} \leq 1$. Let $T' \in (0, T)$ be the maximal time such that for all $t \in [0, T')$

$$\frac{\|u_2\|^{\frac{1}{2}}}{b(\|x_t\| + \|u_1\|)} < 1. \tag{32}$$

The existence of such a T' follows from continuity of x and the fact that

$$\begin{aligned}&\frac{\|u_2\|^{\frac{1}{2}}}{b(|x(0)|+\|u_1\|)}\\&\quad\le (L+\Psi_2(\max\{|x(0)|,\|u_1\|\}))\sigma^{\frac{1}{2}}(\max\{|x(0)|,\|u_1\|\})\\&\quad\le \frac{1}{2}.\end{aligned} \tag{33}$$

From (30) and the fact $\|u_2\|\le\sigma(\max\{|x(0)|,\|u_1\|\})$, it follows that for all $t\in[0,T')$

$$\begin{aligned}&\frac{\|u_2\|^{\frac{1}{2}}}{b(\|x_t\|+\|u_1\|)}\\&\quad\le\frac{\|u_2\|^{\frac{1}{2}}}{b(\beta(|x(0)|,0)+\gamma_{u_1}(\|u_1\|)+\gamma_{\tilde u_2}(1)+\|u_1\|)}\\&\quad\le (L+\Psi_2(\max\{|x(0)|,\|u_1\|\}))\sigma^{\frac{1}{2}}(\max\{|x(0)|,\|u_1\|\})\\&\quad\le\frac{1}{2}.\end{aligned} \tag{34}$$

Again using continuity of x and the fact that T' is maximal, we must have that $T'=T$. This implies that the quantity $\|u_2\|^{\frac{1}{2}}/b(\|x_t\|+\|u_1\|)$ (and hence $\|u_2\|^{\frac{1}{2}}/b(\|X_t\|)$) is bounded on $[0,T)$. So using (30), x must be bounded on $[0,T)$. From the definition of T, we must have $T=\infty$, and we also have that (4) holds with $\gamma_{u_2}(s)=\gamma_{\tilde u_2}(s^{\frac{1}{2}})$. □

Lemma 3: Referring to (2), assume that (4) holds for each $x_0\in\mathbb{R}^n$, each essentially bounded input $u_1(\cdot)$, and each essentially bounded input $u_2(\cdot)$ such that $\|u_2\|\le\sigma(\max\{|x(0)|,\|u_1\|\})$. Then, for each pair of positive real numbers $(\bar\delta,\bar d)$, there exists a positive real number ρ^* such that for each $\rho\in(0,\rho^*]$, if $\max\{|x(0)|,\|u_1\|\}\le\bar\delta$ and $\|u_2\|\le\sigma(\max\{|x(0)|+\bar d,\|u_1\|\})$, then the solution of (2) with $x(0)=x_0$ exists for each $t\ge 0$ and satisfies

$$|x(t)|\le\beta(|x(0)|,t)+\gamma_{u_1}(\|u_1^\rho\|)+\gamma_{u_2}(\|u_2^\rho\|)+\bar d. \tag{35}$$

In case where (4) holds for all essentially bounded inputs $u_2(\cdot)$, (35) holds for all $(x(0),u_1,u_2)$ that satisfy $\max\{|x(0)|,\|u_1\|,\|u_2\|\}\le\bar\delta$.

Proof: First note that since (4) holds for the system in (2) for each essentially bounded input $u_2(\cdot)$ such that $\|u_2\|\le\sigma(\max\{|x(0)|,\|u_1\|\})$, and since $\max\{|x(0)|,\|u_1\|\}\le\bar\delta$ and $\|u_2\|\le\sigma(\max\{|x(0)|+\bar d,\|u_1\|\})$, $x(t)$ is bounded by a number depending on $(\bar\delta,\bar d)$, and thus $|\dot x|$ is bounded by a number depending on $(\bar\delta,\bar d)$. (In the case where (4) holds for all essentially bounded inputs $u_2(\cdot)$, M depends only on $\bar\delta$.) Thus, there exists a positive real number M (depending on $(\bar\delta,\bar d)$) such that

$$|x(t)|\le|x(0)|+tM,\quad\forall t\ge 0. \tag{36}$$

Now, observe that for fixed $s\ge 0$ and $t\ge 0$, the term $\beta(s+\bar TM,t-\bar T)-\beta(s,t)$ is nondecreasing as $\bar T\le t$ increases, and for fixed $s\ge 0$ and $\bar T\ge 0$ it decreases to zero as $t\to\infty$. From these two facts, we have that for each pair of positive real numbers $(\bar\delta,\bar d)$, there exists a positive real number T sufficiently large (without loss of generality, let $T\ge 1$) such that

$$\beta(s+M,t-1)-\beta(s,t)\le\bar d,\quad\forall s\in[0,\bar\delta],\quad\forall t\ge T. \tag{37}$$

Define $\rho^*\le 1$ such that

$$\beta(s,0)-\beta(s,t)+tM\le\bar d,\quad\forall s\in[0,\bar\delta],\quad\forall t\in[0,\rho^*] \tag{38}$$

and

$$\beta(s+\rho^*M,t-\rho^*)-\beta(s,t)\le\bar d,\quad\forall s\in[0,\bar\delta],\quad\forall t\in[0,T]. \tag{39}$$

To establish the result of the lemma, we will show that the bound in (35) holds for each $\rho\in(0,\rho^*]$ and all $t\ge 0$ by examining the time intervals $[0,\rho^*]$, $[\rho^*,T]$ and $[T,\infty)$, separately. First, note that $s\le\beta(s,0)$ for all $s\ge 0$ (consider (4) with $\|u_1\|=\|u_2\|=0$). Then, from (36) and using (38), we get, for all $t\in[0,\rho^*]$

$$\begin{aligned}|x(t)|&\le\beta(|x(0)|,0)+tM\\&\le\beta(|x(0)|,t)+[\beta(|x(0)|,0)-\beta(|x(0)|,t)]+tM\\&=:\beta(|x(0)|,t)+\bar d.\end{aligned} \tag{40}$$

Thus, (35) holds for all $t\in[0,\rho^*]$. For the remaining two intervals, note that using time invariance and (36) and (38), we have that if $\|u_2\|\le\sigma(\max\{|x(0)|+\bar d,\|u_1\|\})\le\sigma(\max\{|x(\rho^*)|,\|u_1\|\})$, then for all $t\in[\rho^*,\infty)$

$$\begin{aligned}|x(t)|&\le\beta(|x(\rho^*)|,t-\rho^*)+\gamma_{u_1}(\|u_1^{\rho^*}\|)+\gamma_{u_2}(\|u_2^{\rho^*}\|)\\&\le\beta(|x(0)|,t)+[\beta(|x(\rho^*)|,t-\rho^*)-\beta(|x(0)|,t)]\\&\quad+\gamma_{u_1}(\|u_1^{\rho^*}\|)+\gamma_{u_2}(\|u_2^{\rho^*}\|).\end{aligned} \tag{41}$$

(Notice that if (4) holds for all essentially bounded inputs $u_2(\cdot)$, (41) holds for all $(x(0),u_1,u_2)$ that satisfy $\max\{|x(0)|,\|u_1\|,\|u_2\|\}\le\bar\delta$.) From (41), using (36) and (39), we have for all $t\in[\rho^*,T]$

$$\begin{aligned}|x(t)|&\le\beta(|x(0)|,t)+\gamma_{u_1}(\|u_1^{\rho^*}\|)+\gamma_{u_2}(\|u_2^{\rho^*}\|)+\bar d\\&\le\beta(|x(0)|,t)+\gamma_{u_1}(\|u_1^\rho\|)+\gamma_{u_2}(\|u_2^\rho\|)+\bar d.\end{aligned} \tag{42}$$

Moreover, from (41), using (36), the fact that $\beta(s+\rho^*M,t-\rho^*)-\beta(s,t)\le\beta(s+M,t-1)-\beta(s,t)$, and (37), we have, for all $t\in[T,\infty)$

$$\begin{aligned}|x(t)|&\le\beta(|x(0)|,t)+[\beta(|x(0)|+M,t-1)-\beta(|x(0)|,t)]\\&\quad+\gamma_{u_1}(\|u_1^{\rho^*}\|)+\gamma_{u_2}(\|u_2^{\rho^*}\|)\\&\le\beta(|x(0)|,t)+\gamma_{u_1}(\|u_1^\rho\|)+\gamma_{u_2}(\|u_2^\rho\|)+\bar d.\end{aligned} \tag{43}$$

□

V. Proof of Theorem 1

To simplify notation, we assume $\bar\epsilon=\infty$. The modifications are obvious for the case where $\bar\epsilon$ is finite. We analyze (7) in the (x,y) coordinates. In the time scale $\tau=t/\epsilon$, the y dynamics are governed by

$$\begin{aligned}\frac{dy}{d\tau}&=g(x,h(x,\theta)+y,\theta,\epsilon)\\&\quad-\epsilon\left[\frac{\partial h}{\partial x}f(x,h(x,\theta)+y,\theta,\epsilon)-\frac{\partial h}{\partial\theta}\dot\theta\right].\end{aligned} \tag{44}$$

Since, by assumption, when $\epsilon=0$ the point $y=0$ is globally asymptotically stable uniformly in $x\in\mathbb{R}^n$, $\theta\in\mathbb{R}^q$, there exist (see [14, ch. 2]) three functions b_1,b_2,b_3 of class K_∞, and a smooth function $W:\mathbb{R}^n\times\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}_{\ge 0}$, the following conditions hold:

$$\begin{aligned}b_1(|y|)\le W(x,y,\theta)&\le b_2(|y|)\\\frac{\partial W}{\partial y}g(x,h(x,\theta)+y,\theta,0)&\le -b_3(|y|).\end{aligned} \tag{45}$$

Note the similarity between this property and what would be required for the boundary layer system to be ISS [with respect to (x,θ)] with Lyapunov gains which are identically zero. Obtaining the time-derivative (in the τ time scale) of $W(x,y,\theta)$ along the trajectories of (44) and performing calculations similar to the ones in the proofs of Lemmas 1 and 2, it can be shown that there exist a function β_y of class KL, a continuous nonincreasing function $\sigma_y:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$, and a function γ_{ϵ_y} of class K such that for each $y_0\in\mathbb{R}^p$, each triple of essentially bounded inputs $\theta,\dot\theta,x$, and each ϵ satisfying

$\epsilon \leq \sigma_y(\max\{|y(0)|, \|\theta\|, \|\dot{\theta}\|, \|x\|\})$, the solution of (44) with $y(0) = y_0$ exists for each $t \geq 0$ and satisfies

$$|y(t)| \leq \beta_y\left(|y(0)|, \frac{t}{\epsilon}\right) + \gamma_{\epsilon_y}(\epsilon). \tag{46}$$

The dynamics for x are governed by

$$\dot{x} = f(x, h(x,\theta) + y, \theta, \epsilon). \tag{47}$$

Using Assumption 2 and Lemma 1, we have that there exists a $p \times p$ matrix $\tilde{B}(x,\theta)$ of smooth functions, invertible for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and a function $\gamma_{\tilde{y}}$ of class K such that the system

$$\dot{x} = f(x, h(x,\theta) + \tilde{B}(x,\theta)\tilde{y}, \theta, 0) \tag{48}$$

is ISS with respect to $\theta, \tilde{y}$ with Lyapunov gain $(\gamma, \gamma_{\tilde{y}})$. Next, using Lemma 2, we have that there exist a function β_x of class KL, a continuous nonincreasing function $\sigma_x: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a function γ_{ϵ_x} of class K such that for each $x_0 \in \mathbb{R}^n$, each pair of essentially bounded inputs $\theta, \tilde{y}$, and each ϵ satisfying $\epsilon \leq \sigma_x(\max\{|x(0)|, \|\theta\|, \|\tilde{y}\|\})$, the solution of (47) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + \gamma_{\tilde{y}}(\|\tilde{y}\|) + \gamma_{\epsilon_x}(\epsilon). \tag{49}$$

Since we do not know *a priori* that x and $\tilde{y}$ are essentially bounded, to use the above inequalities we must work with the truncation of signals and exploit causality. Let (δ, d) be as given in the theorem so that $\max\{|x(0)|, |y(0)|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$, and let δ_x be a positive real number satisfying

$$\delta_x > \beta_x(\delta, 0) + \gamma(\delta) + d. \tag{50}$$

Note that since $\beta_x(s, 0) \geq s$, for all $s \in \mathbb{R}$, $\delta_x > \delta$ and, using continuity with respect to initial conditions, we can define $[0, T)$ with $T > 0$ to be the maximal interval in which $\|x_t\| < \delta_x$ for all $t \in [0, T)$. To show by contradiction that $T = \infty$ for ϵ sufficiently small, suppose T is finite.

Using the fact that σ_y is a continuous, nonincreasing function, it follows from the definition of T and causality that if $\epsilon \leq \epsilon_1 := \sigma_y(\delta_x)$, then (46) holds for all $t \in [0, T)$. Then, since $\tilde{y} = \tilde{B}^{-1}(x,\theta)y$ and $\tilde{B}(x,\theta) = b(|X|)I_{p\times p}$, where $X(t) = [x^T(t)\ \theta^T(t)]^T$ (see the proof of Lemma 1) we have for all $t \in [0, T)$

$$\begin{aligned}\|\tilde{y}_t\| &= \|(\tilde{B}^{-1}(x,\theta)y)_t\| \leq \|(\tilde{B}^{-1}(x,\theta))_t\|\|y_t\| \\ &\leq \frac{1}{b(\|X_t\|)}\left[\beta_y(|y(0)|, t/\epsilon) + \gamma_{\epsilon_y}(\epsilon)\right].\end{aligned} \tag{51}$$

Define

$$\delta_{\tilde{y}} := \frac{1}{b(\delta_x + \delta)}[\beta_y(\delta, 0) + \gamma_{\epsilon_y}(\epsilon_0)] \tag{52}$$

and note that since $\beta_y(s, 0) \geq s$, b is nonincreasing and $b(s)$ satisfies $0 < b(s) \leq 1$ for all $s \in \mathbb{R}$ (see the proof of Lemma 1), it follows that $\delta \leq \delta_{\tilde{y}}$ and $\|\tilde{y}_t\| \leq \delta_{\tilde{y}}$ for all $t \in [0, T)$.

Now, from the conditions under which (49) holds, and using that σ_x is nonincreasing, it follows from Lemma 3 that there exists a positive real number $\rho < T$ such that if

$$\begin{aligned}\epsilon \leq \epsilon_2 &:= \sigma_x(\delta_{\tilde{y}} + d/2) \\ &\leq \sigma_x(\max\{|x(0)| + d/2, \|\theta\|, \|\tilde{y}_t\|\}) \quad \forall t \in [0, T)\end{aligned} \tag{53}$$

then the solution of (47) with $x(0) = x_0$ exists for all $t \in [0, T)$ and satisfies

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + \gamma_{\tilde{y}}(\|\tilde{y}_t^\rho\|) + \gamma_{\epsilon_x}(\epsilon) + \frac{d}{2}. \tag{54}$$

Combining (51) and (54), if $\epsilon \leq \min\{\epsilon_1, \epsilon_2\}$ then for all $t \in [0, T)$

$$\begin{aligned}|x(t)| &\leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + \frac{d}{2} \\ &\quad + \gamma_{\tilde{y}}\left(\frac{1}{b(\|X_t\|)}\left[\beta_y\left(|y(0)|, \frac{\rho}{\epsilon}\right) + \gamma_\epsilon(\epsilon)\right]\right) + \gamma_{\epsilon_x}(\epsilon) \\ &\leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + \frac{d}{2} \\ &\quad + \gamma_{\tilde{y}}\left(\frac{1}{b(\delta_x + \delta)}\left[\beta_y\left(\delta, \frac{\rho}{\epsilon}\right) + \gamma_{\epsilon_y}(\epsilon)\right]\right) + \gamma_{\epsilon_x}(\epsilon).\end{aligned} \tag{55}$$

Since the last two terms converge to zero as ϵ goes to zero, there exists an $\epsilon_3 > 0$ such that if $\epsilon \leq \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$, then for all $t \in [0, T)$

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + d. \tag{56}$$

From the definition of δ_x, the assumption that T is finite and has continuity of x, there must exist some positive real number k such that $\|x_t\| < \delta_x$, for all $t \in [0, T + k)$. This contradicts that T is maximal. Hence, $T = \infty$ and (14) holds for all $t \geq 0$. Finally, letting ϵ_4 be such that $\gamma_{\epsilon_y}(\epsilon) \leq d$ for all $\epsilon \in [0, \epsilon_4]$, it follows that both (14) and (15) hold for $\epsilon \in (0, \epsilon^*]$ where $\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$.

VI. Discussion of Local Properties

Note that even with $\theta(t) \equiv 0$, Theorem 1 does not provide any result concerning stability of or convergence to the origin. To guarantee such properties further assumptions are required. Indeed, consider the singularly perturbed system

$$\begin{aligned}\dot{x} &= z - x^3 \\ \epsilon\dot{z} &= -z + \epsilon x\end{aligned} \tag{57}$$

where $x \in \mathbb{R}$, $z \in \mathbb{R}$. One can easily see that the above system is in standard form, $h(x, \theta) = 0$, and its fast dynamics

$$\frac{dy}{d\tau} = -y \tag{58}$$

possess a globally exponentially stable equilibrium $(y = 0)$. Moreover, the reduced system takes the form

$$\dot{x} = -x^3 \tag{59}$$

which clearly possesses a globally asymptotically stable equilibrium $(x = 0)$. Thus, the assumptions of Theorem 1 are satisfied, and its result can be applied. However, the origin is not an asymptotically stable equilibrium for (57). This can be easily seen considering the linearization of (57) around the origin

$$\begin{aligned}\dot{x} &= z \\ \epsilon\dot{z} &= -z + \epsilon x\end{aligned} \tag{60}$$

which possesses an eigenvalue in the right half of the complex plane for all positive values of ϵ. It is clear from the above example that we cannot draw any conclusions about the stability properties of the equilibrium point of the full order system from knowledge of the stability properties of the reduced-order systems without some type of additional interconnection condition. There are several possible interconnection conditions that can be imposed to guarantee stability of and convergence to the origin. Most efficient conditions are essentially related to the small gain theorem (see [21] and [15]) in one way or another. Saberi and Khalil provide Lyapunov conditions in [16] which are closely related to small gain conditions in an L_2 setting. An L_∞ nonlinear small gain condition is given in [20] and [10]. Perhaps the most straightforward, although conservative, interconnection condition is the assumption that with $\theta(t) \equiv 0$, the origin of the slow and fast subsystems are locally exponentially stable. In this case, it can be shown [16] that there exists ϵ sufficiently small such that the origin of (7) is locally exponentially stable and that the basin of attraction does not shrink as ϵ becomes small. In general, when a local interconnection condition is given that has this "nonshrinking" property, the result of Theorem 1 can be used to show stability and convergence from an arbitrary large compact set under Assumptions 1–3. This is obtained by choosing d_x, d_y sufficiently small to guarantee convergence in finite time to the domain of attraction of the origin.

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On The Stability Domain Estimation via a Quadratic Lyapunov Function: Convexity and Optimality Properties for Polynomial Systems

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***Abstract*—The problem of estimating the stability domain of the origin of an n-order polynomial system is considered. Exploiting the structure of this class of systems it is shown that, for a given quadratic Lyapunov function, an estimate of the stability domain can be obtained by solving a suitable convex optimization problem. This estimate is shown to be optimal for an important subclass including both quadratic and cubic systems, and its accuracy in the general polynomial case is discussed via several examples.**

NOTATION

$e_i = [0 \cdots 1 \cdots 0]'$	ith basis vector.
I_n	Identity matrix of order n.
$T = T' > 0\ (\geq 0)$	Positive definite (semidefinite) matrix.
$T = T' < 0\ (\leq 0)$	Negative definite (semidefinite) matrix.
$T = -T'$	Skew-symmetric matrix.
T_f	Factor of $T = T' > 0$, i.e., $T = T_f' T_f$.
$\lVert T \rVert_2$	Induced l_2-norm of matrix T.
$\lambda_M[T]$	Maximum eigenvalue of T.
$T_1 \otimes T_2$	Kronecker product of T_1 and T_2.
$T(v) = T_0 + v_1 T_1 + \cdots v_l T_l$	Affine linear matrix on $v = [v_1, \cdots, v_l]' \in R^l$.

I. INTRODUCTION

The problem of estimating the stability domain [usually called domain of attraction (DA)] of an equilibrium point is well known in the area of nonlinear system analysis and control. In spite of a lot of work (see [1], [2], and references therein), there are no simple ways for finding reasonable guaranteed stability domains especially when high-order general nonlinear systems are considered. Roughly speaking, a typical algorithm for computing estimates of the stability domain essentially consists of two distinct parts (see, for instance, [3]–[7]): i) a part where a Lyapunov function is selected according to some specified rules and ii) a part where an estimate of the domain of attraction is computed for the selected Lyapunov function. While part i) strongly depends on the used algorithm, part ii) is common to all the algorithms and represents the crucial point of the procedure. More precisely, part ii) requires one to compute the region where the derivative of the chosen Lyapunov function is negative definite and to embed the contour surfaces of the Lyapunov function within this region. This problem can be cast as a nonconvex optimization problem which is in general difficult to solve for the presence of local extrema. Most of the proposed approaches are based on gridding techniques (see, e.g., [3] and [4]) which make them not very effective for higher dimensional systems. More recently, a more efficient

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