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Robust semi-global output tracking for nonlinear singularly perturbed systems

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In this work, a robust controller design methodology for a broad class of singularly perturbed nonlinear systems with time-varying uncertainties is developed. When the fast subsystem is stabilizable and the slow subsystem is input/output linearizable with input-to-state stable (ISS) inverse dynamics, the developed state feedback controller guarantees boundedness of the trajectories of the closed-loop system and robust output tracking with arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output, for initial conditions and uncertainties in an arbitrarily large compact set, so long as the singular perturbation parameter is sufficiently small. The controller is a continuous function of the state of the system and its construction requires the knowledge of bounding functions on the size of uncertainty. The developed method is applied to a non-isothermal continuous stirred tank reactor with fast jacket dynamics.

Notation

c_{pm}	heat capacity of the reacting mixture
c_{pj}	heat capacity of the fluid in the jacket
$d, d_{\bar{\varepsilon}_r}, d_{\bar{\varepsilon}}, d_\eta$	positive real numbers
E_1, E_2	activation energies
$F, \tilde{F}, \tilde{F}_{\text{nom}}, f_1, f_2$	vector functions
F_r, F_j	flow rates
$G, \tilde{G}, g_1, g_2$	vector functions associated with the input
h	output scalar function
k^{T}	covector function
$\bar{k}, \tilde{k}$	positive real numbers
k_{10}, k_{20}	pre-exponential constants
$L_f h$	Lie derivative of a scalar function h with respect to the vector function f
$L_f^k h$	kth order Lie derivative
$L_g L_f^{k-1} h$	mixed Lie derivative
$\mathbf{Q}_1$	matrix of dimension $n \times p$ associated with the slow state vector x
$\mathbf{Q}_2$	matrix of dimension $p \times p$ associated with the fast state vector z
$\mathbb{R}$	real line
$\mathbb{R}^i$	i-dimensional euclidean space
$r, \tilde{r}$	integers associated with the input $\tilde{u}$ in the reduced systems
$s, \tilde{s}$	integers associated with the variable θ in the reduced systems

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T	transpose
t	time
u	input
$\tilde{u}$	auxiliary input
V_j, V_r	volumes of the liquid holdup in the jacket and reactor
v	reference input
x	vector of the slow state variables
y	output
z	vector of the fast state variables
β_k	adjustable parameters
$\delta, \delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{e}_r}, \delta_{\bar{\eta}}, \delta_{\bar{e}}, \delta_\eta, \bar{\tilde{\delta}}_{\bar{e}_r}, \bar{\tilde{\delta}}_{\bar{\eta}}, \delta_{\bar{v}}$	positive real numbers
$\Delta H_{r_1}, \Delta H_{r_2}$	enthalpy of the reactions
$\varepsilon, \varepsilon^{\bar{e}_r}, \varepsilon^{\eta}, \varepsilon^{\bar{e}}$	singular perturbation parameters
ζ	state vector
η	state vector
ρ_j	density of the reacting mixture
ρ_m	density of the fluid in the jacket
ϕ	adjustable parameter
$\in$	belongs to

1. Introduction

Nonlinear multiple-time-scale systems arise naturally in a wide variety of engineering applications. Representative examples of multiple-time-scale nonlinear systems include biochemical (Christofides and Daoutidis 1996) and catalytic reactors (Chang and Aluko 1984), distillation columns (Levine and Rouchon 1991), fluidized catalytic crackers (Monge and Georgakis 1987), flexible mechanical systems (Corless *et al.* 1993), electromechanical networks (Chow 1982), etc. For these systems, it is well-established (Kokotovic *et al.* 1986, Christofides and Daoutidis 1996) that a direct application of nonlinear feedback control methods (see, for example, the books by Isidori 1989, Nijmeijer and van der Schaft 1990), without taking into account the interactions of slow and fast phenomena may lead to controller ill-conditioning and/or destabilization of the fast dynamics in the closed-loop system.

The mathematical framework of singular perturbations has proven to be a natural setting for the modelling and control of two-time-scale systems (Kokotovic *et al.* 1986). Singular perturbation methods take advantage of the intrinsic two-time-scale behaviour of the system and decompose the original model into two lower-order models representing the slow and fast dynamics. Then one can infer the asymptotic properties of the original model from knowledge of the behaviour of the lower-order models.

An additional difficulty encountered in the synthesis of control systems for actual processes is the inadequacy of the majority of process models to exactly capture the dynamic behaviour of the process under consideration. Model uncertainties typically arise from unknown or poorly known process parameters and/or unmeasured exogenous disturbance inputs. It is well-established that the presence of model uncertainties may cause serious deterioration of the nominal performance and in some instances closed-loop instability (Corless and Leitmann 1988, Dorato *et al.* 1993).

Motivated by the above considerations, significant research effort has focused on the synthesis of robust controllers for uncertain nonlinear systems. For this purpose, adaptive control schemes (Sastry and Isidori 1989, Teel *et al.* 1991) have been proposed. The main disadvantage of this approach is that the uncertainty is assumed to be time-invariant, which significantly restricts the practical applicability of these methods. An alternative approach for the treatment of uncertain systems is to address the controller design using Lyapunov functions (Corless and Leitmann 1981, 1988). In this approach, one utilizes the existence of bounding functions on the size of uncertainty and Lyapunov's direct method to design robust controllers that guarantee boundedness of the trajectories of the closed-loop system. This approach is applicable to systems with time-varying uncertainties.

For uncertain two-time-scale nonlinear systems with stable fast dynamics, an approach that utilizes a combination of singular perturbation methods and adaptive control schemes has been proposed by Taylor *et al.* (1989) and Kanellakopoulos *et al.* (1991). For the same class of systems, an alternative approach that utilizes a combination of singular pertubation theory and controller design via Lyapunov functions to derive robust controllers has been developed by Corless (1987) and Khorasani (1989). For linear singularly perturbed systems with time-varying uncertainties for which the fast dynamics may be unstable, a class of nonlinear composite controllers, synthesized using Lyapunov's direct method, was proposed by Corless *et al.* (1993). For linear systems, other available results concern the use of H^∞ control methods (Khalil and Chen 1992, Pan and Basar 1993) and stochastic control methods (see Kokotovic *et al.* 1986 and the references therein) for robust controller design.

In this work we consider a broad class of uncertain singularly perturbed nonlinear systems. The uncertainty is allowed to be time-varying. We assume that the fast subsystem is stabilizable and the slow subsystem is input/output linearizable with ISS inverse dynamics. For such systems, we synthesize continuous, possibly time-varying, state feedback controllers that guarantee boundedness of the trajectories of the closed-loop system and achieve an arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output, for initial conditions and uncertainties in an arbitrarily large compact set, so long as the singular perturbation parameter is sufficiently small. Our main result is based on a recent result concerning the robustness of the input-to-state stability property with respect to uniformly globally asymptotically stable singular perturbations, developed by Christofides and Teel (1995) (see also Christofides and Teel 1996) and uses calculations similar to those used in the derivation of the ISS nonlinear small gain theorem established by Jiang *et al.* (1995) (see also Teel 1996). The explicit construction of the controller requires knowledge of upper bounds on the size of uncertainty. The developed control methodology is applied to a typical chemical reactor example with time-scale multiplicity and uncertainties, and its performance and robustness characteristics are evaluated through simulations.

2. Notation and facts

$|\cdot|$ denotes the standard euclidean norm and $\mathrm{sgn}(\cdot)$ denotes the sign function.

For any measurable (with respect to the Lebesgue measure) function $\theta: \mathbb{R}_{\geqslant 0} \rightarrow \mathbb{R}^m$; $\|\theta\|$ denotes ess.sup $|\theta(t)|$, $t \geqslant 0$.

A function $W: \mathbb{R}^n \rightarrow \mathbb{R}_{\geqslant 0}$ is said to be positive definite if $W(x)$ is positive for all non-zero x and is zero at zero.

A function $W: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be proper if $W(x)$ tends to $+\infty$ as $|x|$ tends to $+\infty$.

A function $\gamma: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class K if it is continuous, increasing and is zero at zero. It is of class K_∞, if in addition, it is proper.

A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class KL if for each fixed t the function $\beta(\cdot, t)$ is of class K, and for each fixed s the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.

For any function γ of class K_∞, its inverse function is well defined and is again of class K_∞.

A matrix $A(x)$ of dimension $n \times n$ is said to be Hurwitz uniformly in $x \in \mathbb{R}^n$ if there exists a positive real number c such that $\mathrm{Re}[\lambda_i(A(x))] \leqslant -c, i = 1, \ldots, n$ for all $x \in \mathbb{R}^n$, where λ_i denotes the ith eigenvalue of the matrix.

Let A and B be two matrices of dimensions $n_1 \times n_2$ and $n_2 \times n_1$, respectively, and let $I_{n_1 \times n_1}, I_{n_2 \times n_2}$ also denote the identity matrices of dimensions $n_1 \times n_1, n_2 \times n_2$. Then the following identity holds:

$$\det(I_{n_1 \times n_1} + AB) = \det(I_{n_2 \times n_2} + BA) \tag{1}$$

3. Preliminaries

We will consider uncertain singularly perturbed nonlinear systems with the following state-space description:

$$\left.\begin{aligned} \dot{x} &= f_1(x, \theta(t)) + \mathbf{Q}_1(x, \theta(t))\, z + g_1(x, \theta(t))\, u \\ \varepsilon \dot{z} &= f_2(x, \theta(t)) + \mathbf{Q}_2(x, \theta(t))\, z + g_2(x, \theta(t))\, u \\ y &= h(x) \end{aligned}\right\} \tag{2}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ denote vectors of state variables, $u \in \mathbb{R}$ denotes the input, $\theta \in \mathbb{R}^q$ denotes the vector of the uncertain (possibly time-varying) variables, $y \in \mathbb{R}$ denotes the output (to be controlled), and ε is a small positive parameter which can be interpreted as the speed ratio of the slow versus the fast dynamical phenomena of the system. The vector of uncertain variables θ may include time-varying parametric uncertainties and/or unmeasured exogenous disturbances. The vector functions $f_1(x, \theta)$, $f_2(x, \theta)$, $g_1(x, \theta)$ and $g_2(x, \theta)$ are sufficiently smooth, $\mathbf{Q}_1(x, \theta)$ and $\mathbf{Q}_2(x, \theta)$ are sufficiently smooth matrices of appropriate dimensions, and $h(x)$ is a sufficiently smooth scalar function. Our assumptions will impose, among other things, that the output y and the input u in the system of (2) are in deviation variables, and the rate of change of the uncertainty, $\dot{\theta}$, is bounded. In the remainder of this paper, for simplicity, we will suppress the time-dependence in the notation of the vector of uncertain variables $\theta(t)$.

Models of the form of (2) arise naturally in a wide variety of engineering applications; they are characterized by an explicit separation of the slow and fast dynamics, owing to the presence of the small parameter ε that multiplies the derivative of the state vector z. They are also characterized by a linear structure with respect to the fast state vector z, consistent with the fact that the main nonlinearities in chemical processes (Christofides and Daoutidis 1996) are usually associated with the slow dynamics. Models of the form of (2) include as special cases the models considered by Khorasani (1989) and Corless *et al.* (1993).

A standard procedure which is followed for the analysis of systems of the form of (2) is the decomposition of the original system into separate reduced-order systems, each one associated with a different time-scale. This procedure is called two-time-scale decomposition (Kokotovic *et al.* 1986). To review this procedure, let us assume that the matrix $\mathbf{Q}_2(x,\theta)$ is invertible uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$ (this assumption will be removed later). Setting $\varepsilon = 0$, in which case the dynamics of the z state becomes instantaneous, the system of (2) takes the form

$$\dot{x} = f_1(x,\theta) + \mathbf{Q}_1(x,\theta)\, z_s + g_1(x,\theta)\, u \tag{3}$$

$$f_2(x,\theta) + \mathbf{Q}_2(x,\theta)\, z_s + g_2(x,\theta)\, u = 0 \tag{4}$$

where z_s denotes a quasi-steady-state for the fast state vector z. The fact that the matrix $\mathbf{Q}_2(x,\theta)$ is invertible implies that the algebraic equation (4) admits a unique solution of the form

$$z_s = -[\mathbf{Q}_2(x,\theta)]^{-1}[f_2(x,\theta) + g_2(x,\theta)\, u] \tag{5}$$

Note that the quasi-steady-state z_s is an explicit function of the uncertainty vector θ, which implies that the exact value of the vector z_s is not known. Substituting (5) into (3), the following dynamical system is obtained:

$$\left.\begin{aligned} \dot{x} &= F(x,\theta) + G(x,\theta)\, u \\ y &= h(x) \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} F(x,\theta) &= f_1(x,\theta) - \mathbf{Q}_1(x,\theta)[\mathbf{Q}_2(x,\theta)]^{-1} f_2(x,\theta) \\ G(x,\theta) &= g_1(x,\theta) - \mathbf{Q}_1(x,\theta)[\mathbf{Q}_2(x,\theta)]^{-1} g_2(x,\theta) \end{aligned}\right\} \tag{7}$$

The dynamical system of (6) is called the reduced system or slow subsystem. Note that the control u appears linearly in the above system which is an implication of the linearity in z in the original system.

The inherent two-time-scale behaviour of the system of (2) can be analysed by defining a fast time-scale:

$$\tau = \frac{t}{\varepsilon} \tag{8}$$

In this new time-scale the original system takes the form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \varepsilon[f_1(x,\theta) + \mathbf{Q}_1(x,\theta)\, z + g_1(x,\theta)\, u] \tag{9}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = f_2(x,\theta) + \mathbf{Q}_2(x,\theta)\, z + g_2(x,\theta)\, u \tag{10}$$

By setting ε equal to zero, the following system is obtained:

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = f_2(x,\theta) + \mathbf{Q}_2(x,\theta)\, z + g_2(x,\theta)\, u \tag{11}$$

where x can be considered equal to its initial value $x(0)$ and θ can be viewed as constant. In what follows, we will refer to the dynamical system of (11) as the fast subsystem.

In this work, we are dealing with systems of the form of (2) for which the matrix $\mathbf{Q}_2(x,\theta)$ may be singular or possess eigenvalues in the right-half of the complex plane for some $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$. The following assumption states our stabilizability requirement on the open-loop fast subsystem of (11).

Assumption 1: *The pair* $[Q_2(x,\theta)\ \ g_2(x,\theta)]$ *is stabilizable uniformly in* $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$, *in the sense that there exists a covector field* $k^T(x)$ *such that the matrix* $Q_2(x,\theta) + g_2(x,\theta)\, k^T(x)$ *is Hurwitz uniformly in* $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$.

In closing this section, we define what is meant by input-to-state stability for a system of the form of (6).

Definition 1 (Sontag 1989): The system in (6) (with $u \equiv 0$) is said to be ISS with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_0 \in \mathbb{R}^n$ and for each measurable and essentially bounded input $\theta(\cdot)$ on $[0, \infty)$ the solution of (6) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|\theta\|), \quad \forall t \geq 0 \tag{12}$$

4. Problem statement: approach

The direct application of inversion-based control algorithms to multiple-time-scale systems of the form of (2) may lead to controller ill-conditioning (i.e. the controller generates infinite control action in the limit as $\varepsilon \to 0$) and/or closed-loop instability owing to slightly non-minimum-phase behaviour (Christofides and Daoutidis 1996). Furthermore, the presence of time-varying uncertainties in the model, if it is not taken into account in the controller design may lead to serious degradation of the nominal performance and in some instances to closed-loop instability.

In this work, we address and solve the problem of synthesizing well-conditioned continuous, possibly time-varying, static state feedback controllers that preserve the two-time-scale nature of the original system, with the following objectives for the closed-loop system:

(*a*) boundedness of the trajectories;

(*b*) arbitrary degree of asymptotic attenuation of the effect of the uncertainty on the output;

(*c*) robust output tracking for changes in the reference input.

The above requirements will be obtained for sufficiently small ε.

The approach followed for the synthesis of the controller is essentially a two-step one. In the first step we take advantage of Assumption 1 to design a preliminary control law that transforms the original singularly perturbed system of (2) into a new singularly perturbed system in standard form with globally exponentially stable fast dynamics. In the second step we synthesize a control law that uses feedback of the slow state vector x only and utilizes the knowledge of upper bounds on the size of uncertainty to achieve robust output tracking with arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output. It is established that the resulting controller enforces the requested three properties in the closed-loop system, provided that the singular perturbation parameter ε is sufficiently small.

5. Feedback control of uncertain singularly perturbed nonlinear systems

In this section we will develop a solution to the state feedback control problem, specified in the preceding section, for uncertain singularly perturbed nonlinear systems of the form of (2) for which the matrix $Q_2(x,\theta)$ may be singular or possess eigenvalues which lie in the right-half of the complex plane, for some $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Motivated by the affine appearance of the fast variable z in the model of (2), we will initially consider control laws of the form

$$u = \tilde{u} + k^{\mathrm{T}}(x)\, z \tag{13}$$

where $k^{\mathrm{T}}(x)$ is a vector field on $\mathbb{R}^n$ and $\tilde{u}$ is an auxiliary input. The control law of (13) allows us to stabilize the fast dynamics of the system of (2) by choosing appropriately the gain $k^{\mathrm{T}}(x)$ (see Assumption 1).

Under a control law of the form of (13) the system of (2) takes the form

$$\left.\begin{aligned} \dot{x} &= f_1(x,\theta) + [\mathrm{Q}_1(x,\theta) + g_1(x,\theta)\, k^{\mathrm{T}}(x)]\, z + g_1(x,\theta)\,\tilde{u} \\ \varepsilon \dot{z} &= f_2(x,\theta) + [\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\, k^{\mathrm{T}}(x)]\, z + g_2(x,\theta)\,\tilde{u} \end{aligned}\right\} \tag{14}$$

On can immediately observe that a control law of the form of (13) preserves the two-time-scale nature of the original system, and the linearity with respect to the state z and the auxiliary input $\tilde{u}$. Furthermore, performing a standard two-time-scale decomposition, one can easily show that the fast subsystem is given by

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = f_2(x,\theta) + [\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\, k^{\mathrm{T}}(x)]\, z + g_2(x,\theta)\,\tilde{u} \tag{15}$$

Clearly, the control law of (13) guarantees the stability of the fast dynamics if the gain $k^{\mathrm{T}}(x)$ is chosen in such a manner so that the matrix $\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\, k^{\mathrm{T}}(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Note that this is always possible because of Assumption 1.

The reduced system corresponding to the singularly perturbed system of (14) takes the form

$$\left.\begin{aligned} \dot{x} &= \tilde{F}(x,\theta) + \tilde{G}(x,\theta)\,\tilde{u} \\ y &= h(x) \end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned} \tilde{F}(x,\theta) &= f_1(x,\theta) - [\mathrm{Q}_1(x,\theta) + g_1(x,\theta)\, k^{\mathrm{T}}(x)]\,[\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\, k^{\mathrm{T}}(x)]^{-1} f_2(x,\theta) \\ \tilde{G}(x,\theta) &= g_1(x,\theta) - [\mathrm{Q}_1(x,\theta) + g_1(x,\theta)\, k^{\mathrm{T}}(x)]\,[\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\, k^{\mathrm{T}}(x)]^{-1} g_2(x,\theta) \end{aligned}\right\} \tag{17}$$

The usual approach followed for the design of stabilizing controllers for uncertain nonlinear systems of the form of (16) involves two steps: first, the derivation of bounds (in terms of continuous functions) for the uncertainty; second, the design of a robust controller for the system of (16) via Lyapunov's direct method, utilizing the bounding functions for the uncertainty. This approach was originally proposed by Corless and Leitmann (1981) and was further developed by Corless and Leitmann (1988) and Khorasani (1989). However, there are certain disadvantages associated with this approach. First, the actual construction of the controller requires the existence and knowledge of a nonlinear Lyapunov function for the nominal system. Second, there are certain matching conditions that the uncertainty vector θ has to satisfy for the applicability of this method. From the aforementioned difficulties, it is clear that the first one poses fundamental limitations on the practical applicability of this method, and the second one may or may not pose restrictions, depending on the specific structure of the system under consideration. Motivated by the above considerations, research efforts have focused on the design of robust controllers using Lyapunov's direct method for uncertain systems of the form of (16) with weaker matching conditions (e.g. Qu 1993).

In what follows, we consider the synthesis of robust nonlinear control laws for systems of the form (16). The central differences with the above-mentioned robust controller design methods are that the synthesis of the controller does not require the existence of a nonlinear Lyapunov function for the nominal system, whereas the matching condition required for the application of this theory is different to the standard one employed in Corless and Leitmann (1981).

We now proceed with the design of the controller. Motivated by the requirement of output tracking with attenuation of the effect of the uncertainty on the output, we initially assume that there exists a coordinate transformation that renders the system of (16) in partially linear form. Our requirement is precisely formulated in Assumption 2.

Assumption 2: *The vector of uncertain variables $\theta(t)$ is absolutely continuous, and there exists an integer $\tilde{r}$ and a set of coordinates*

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{\tilde{r}} \\ \eta_1 \\ \vdots \\ \eta_{n-\tilde{r}} \end{bmatrix} = \chi(x, \theta) = \begin{bmatrix} h(x) \\ L_{\tilde{F}} h(x) \\ \vdots \\ L_{\tilde{F}}^{\tilde{r}-1} h(x) \\ \chi_1(x, \theta) \\ \vdots \\ \chi_{n-\tilde{r}}(x, \theta) \end{bmatrix} \tag{18}$$

where $\chi_1(x, \theta), \ldots, \chi_{n-\tilde{r}}(x, \theta)$ are scalar functions such that the reduced system of (16) *takes the form*

$$\left.\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{\tilde{r}-1} &= \zeta_{\tilde{r}} \\ \dot{\zeta}_{\tilde{r}} &= L_{\tilde{F}}^{\tilde{r}} h(\chi^{-1}(\zeta, \eta, \theta)) + L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(\chi^{-1}(\zeta, \eta, \theta))\, \tilde{u} \\ \dot{\eta}_1 &= \Psi_1(\zeta, \eta, \theta, \dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\tilde{r}} &= \Psi_{n-\tilde{r}}(\zeta, \eta, \theta, \dot{\theta}) \\ y &= \zeta_1 \end{aligned}\right\} \tag{19}$$

where $L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x) \neq 0$ *for all $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$. Moreover, for each $\theta \in \mathbb{R}^q$, the states ζ and η are bounded if and only if the state x is bounded.*

Remark 1: We note that Assumption 2 implicitly includes the matching condition of our theory. Note that this condition is different to the standard one which restricts the uncertainty vector θ to lie in the span of the vector field $\tilde{G}(x, \theta)$ (Taylor *et al.* 1989, Corless *et al.* 1993). Specifically, it allows one to handle a larger class of uncertain inputs than the standard one, but requires posing a stronger stability requirement on the inverse dynamics of the system of (19) (see Assumption 3 below). The motivation for considering this matching condition is given by the fact that it is satisfied by a large number of practical applications, including the chemical reactor example of §7, where the standard matching condition does not hold in the slow subsystem.

The following assumption poses our stability requirement on the inverse dynamics of the reduced system of (16).

Assumption 3: *The dynamical system*

$$\left.\begin{aligned}\dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta})\\ &\vdots\\ \dot{\eta}_{n-\tilde{r}} &= \Psi_{n-\tilde{r}}(\zeta,\eta,\theta,\dot{\theta})\end{aligned}\right\} \tag{20}$$

is ISS with respect to ζ, θ *and* $\dot{\theta}$.

Remark 2: We note that although, from a theoretical point of view, Assumptions 2 and 3 limit the class of systems for which our methodology is applicable, they are satisfied by the majority of chemical processes of practical interest (Christofides and Daoutidis 1996). Furthermore, we remark that even in the case where these assumptions hold around a reasonably large neighbourhood of the equilibrium point, but not necessarily globally, the application of our methodology is still possible (see the chemical reactor example of § 7).

Defining the variable $\tilde{\delta}(x,\theta)$ as

$$\tilde{\delta}(x,\theta) = [\tilde{F}(x,\theta) - \tilde{F}_{\text{nom}}(x)] \tag{21}$$

where $\tilde{F}_{\text{nom}}(x)$ denotes the vector field resulting from $\tilde{F}(x,\theta)$ by setting θ equal to its nominal value θ_0 and using the fact that $L_{\tilde{F}}^{k-1}h(x) = L_{\tilde{F}_{\text{nom}}}^{k-1}h(x), k = 1,\ldots,\tilde{r}$ (which follows from the structure of (19)), the system of (19) can be written as

$$\left.\begin{aligned}\dot{\zeta}_1 &= \zeta_2\\ &\vdots\\ \dot{\zeta}_{\tilde{r}-1} &= \zeta_{\tilde{r}}\\ \dot{\zeta}_{\tilde{r}} &= L_{\tilde{F}_{\text{nom}}}^{\tilde{r}} h(\chi^{-1}(\zeta,\eta,\theta)) + L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(\chi^{-1}(\zeta,\eta,\theta))\,\tilde{u} + L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(\chi^{-1}(\zeta,\eta,\theta))\\ \dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta})\\ &\vdots\\ \dot{\eta}_{n-\tilde{r}} &= \Psi_{n-\tilde{r}}(\zeta,\eta,\theta,\dot{\theta})\end{aligned}\right\} \tag{22}$$

where $\zeta_k = L_{\tilde{F}_{\text{nom}}}^{k-1} h(x), k = 1,\ldots,\tilde{r}$.

In most practical applications, there exists partial information about the uncertain terms of the process model. Information of this kind may result from physical considerations, preliminary simulations, experimental data, etc. In what follows, we will quantify possible knowledge about the uncertainty by assuming the existence of known state-dependent, possibly time-varying, bounds that capture the size of uncertainty for all times. Assumption 4 formalizes this requirement.

Assumption 4: *The term* $L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)$ *has known constant sign for all* $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, *and there exist known functions* $\tilde{c}_1(x,t)$, $\tilde{c}_2(x,t)$ *such that the following conditions hold:*

$$\begin{aligned}|L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(x)| &\leqslant \tilde{c}_1(x,t)\\ 0 < \tilde{c}_2(x,t) &\leqslant |L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)|\end{aligned} \tag{23}$$

for all $\theta \in \mathbb{R}^q$.

Remark 3: We can assume, without loss of generality, that (θ, x) restricted in compact sets implies that $\tilde{c}_1(x,t)$ is bounded and $\tilde{c}_2(x,t)$ is bounded away from zero.

We are now in a position to proceed with the design of the auxiliary input $\tilde{u}$ to achieve the asymptotic attenuation of the effect of the uncertainty on the output. In particular, we consider static feedback laws of the form

$$\tilde{u} = r_1(x, t)[p(x) + q(x)\bar{v} + r_2(x, t)] \tag{24}$$

where $p(x)$, $r_1(x, t)$ and $r_2(x, t)$ are scalar functions, $q(x)$ is a row vector function, with $r_1(x, t)q(x) \neq 0$ for all $x \in \mathbb{R}^n$, $t \in [0, \infty)$, and $\bar{v} = [v \;\; v^{(1)} \;\; \dots \;\; v^{(\bar{r})}]^{\mathrm{T}}$, where $v^{(k)}$ denotes the kth time derivative of the reference input v, which is assumed to be a sufficiently smooth function of time.

The control law of (24) consists of three components. The component $r_1(x, t)$ which is used to account for the effect of uncertainty that enters the system coupled with the input, the component $p(x) + q(x)\bar{v}$ which is responsible for the output tracking and stabilization of the nominal reduced system of (16), and the component $r_2(x, t)$ which is responsible for the attenuation of the effect of the uncertainty on the output in the closed-loop reduced system.

We are now in position to state the main result of this paper in the form of a theorem.

Theorem 1: *Consider the uncertain singularly perturbed nonlinear system of* (2), *for which Assumptions* 1–4 *hold, under the static state feedback law*

$$\begin{aligned} u = \operatorname{sgn}[L_{\bar{G}} L_{\bar{F}}^{\bar{r}-1} h(x)][\tilde{c}_2(x, t)]^{-1} \Bigg\{ & \sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}}(v^{(k)} - L_{\bar{F}_{\mathrm{nom}}}^{k} h(x)) \\ & + \sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}}(v^{(k-1)} - L_{\bar{F}_{\mathrm{nom}}}^{k-1} h(x)) \\ & - 2\left[\tilde{c}_1(x, t) + \left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}}(v^{(k)} - L_{\bar{F}_{\mathrm{nom}}}^{k} h(x))\right|\right] w(x, \phi) \Bigg\} + k^{\mathrm{T}}(x) z \end{aligned} \tag{25}$$

where the feedback gain $k^{\mathrm{T}}(x)$ *is such that the matrix* $\mathbf{Q}_2(x, \theta) + g_2(x, \theta)k^{\mathrm{T}}(x)$ *is Hurwitz uniformly in* $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q, \beta_k$ *are parameters chosen so that the polynomial* $s^{\bar{r}-1} + (\beta_{\bar{r}-1}/\beta_{\bar{r}})s^{\bar{r}-2} + \cdots + (\beta_2/\beta_{\bar{r}})s + (\beta_1/\beta_{\bar{r}}) = 0$ *is Hurwitz, and the scalar function* $w(x, \phi)$ *is given by*

$$w(x, \phi) = \frac{\displaystyle\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}}(L_{\bar{F}_{\mathrm{nom}}}^{k-1} h(x) - v^{(k-1)})}{\left|\displaystyle\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}}(L_{\bar{F}_{\mathrm{nom}}}^{k} h(x) - v^{(k-1)})\right| + \phi} \tag{26}$$

where ϕ *is an adjustable parameter. Then, for each set of positive real numbers* $\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}$ *and* d *there exists* $\phi^* > 0$, *and for each* $\phi \in (0, \phi^*]$ *there exists* $\varepsilon^*(\phi) > 0$, *such that if* $\phi \in (0, \phi^*]$, $\varepsilon \in (0, \varepsilon^*(\phi)]$ *and* $|x(0)| \leqslant \delta_x, |z(0)| \leqslant \delta_z, \|\theta\| \leqslant \delta_\theta, \|\dot{\theta}\| \leqslant \delta_{\dot{\theta}}, \|\bar{v}\| \leqslant \delta_{\bar{v}}$, *the output of the closed-loop system satisfies a relation of the form*

$$\limsup_{t \to \infty} |y(t) - v(t)| \leqslant d \tag{27}$$

Remark 4: The feedback gain $k^{\mathrm{T}}(x)$ can be designed using standard linear control methods, such as pole placement, optimal control, etc. (Kokotovic *et al.* 1986).

Remark 5: Referring to the controller of (25), one can easily observe that is composed of two pieces: the term $k^{\mathrm{T}}(x)\, z$, which is responsible for the stabilization of the fast dynamics of the closed-loop system, and the term

$$a(\boldsymbol{x}, \bar{\boldsymbol{v}}, t) := \operatorname{sgn}[L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(\boldsymbol{x})]\,[\tilde{c}_2(\boldsymbol{x}, t)]^{-1} \left\{ \sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} (v^{(k)} - L_{\tilde{F}_{\mathrm{nom}}}^{k} h(\boldsymbol{x})) + \sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} (v^{(k-1)} - L_{\tilde{F}_{\mathrm{nom}}}^{k-1} h(\boldsymbol{x})) - 2\left[\tilde{c}_1(\boldsymbol{x}, t) + \left| \sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} (v^{(k)} - L_{\tilde{F}_{\mathrm{nom}}}^{k} h(\boldsymbol{x})) \right| \right] w(\boldsymbol{x}, \phi) \right\} \quad (28)$$

which guarantees boundedness of the trajectories of the closed-loop reduced system and outout tracking with arbitrary degree of asymptotic attention of the effect of θ on y.

Remark 6: Note that the result of Theorem 1 holds for arbitrarily large initial conditions, uncertainties and their derivatives, and does not impose any kind of interconnection or growth conditions on the nonlinearities of the system.

Remark 7: In practical applications, the value of the singular perturbation parameter is typically fixed by the process, say ε_{p}, and thus there is a limit on how small the ultimate bound d can be chosen. For example, the proof of the theorem can be used to calculate ϕ^* from the desired $(\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d)$ and, in turn, the value ε^* for $\phi \leqslant \phi^*$. If this ε^* is less than ε_p, then d may need to be readjusted (increased) so that $\varepsilon^* \geqslant \varepsilon_{\mathrm{p}}$. Of course, if ε_{p} is too large, there may be no value of d that works. On the other hand, the value of ε^*, calculated from the proof of the theorem, is typically conservative, and so it can be useful to check the appropriateness of ϕ^* (and d) through computer simulations.

Remark 8: Theorem 1 provides a bound for the discrepancy between the output and the reference input that holds asymptotically. A characterization of the transient performance of the controller of (25) though a bound for the quantity $\|y(t) - v(t)\|$ in terms of the norm of the initial condition and the ultimate bound d is established in the proof of the theorem (consider (54) with $\|\tilde{e}_{\tilde{r}}\|$ derived from (48) as $|\tilde{e}_r(0)| + 2\phi$).

Proof: The proof of the theorem consists of three main parts. In the first part the global exponential stability of the closed-loop fast subsystem is established. In the second part the closed-loop reduced system is analysed using Lyapunov techniques to derive ISS inequalities that capture the evolution of the states. Then a direct application of the result of Theorem 2 developed by Christofides and Teel (1995) (which is briefly summarized in the Appendix) is made to establish that these ISS inequalities continue to hold up to an arbitrarily small offset, for arbitrarily large initial conditions. In the third part the resulting ISS inequalities are studied, using techniques similar to those used by Jiang *et al.* (1995) to show boundedness of the trajectories and establish the inequality of (27). All the above results will be obtained for sufficiently small values of ϕ and ε.

Part 1: in this part of the proof we will establish that the closed-loop fast subsystem is globally exponentially stable. Under the control law of (25) the closed-loop system takes the form

$$\dot{x} = \tilde{F}(x, \theta) + \tilde{G}(x, \theta)\, a(x, \bar{v}, t) + [\mathbf{Q}_1(x, \theta) + g_1(x, \theta)\, k^{\mathrm{T}}(x)]\,[z - C(x, \theta, \phi, \bar{v})] \quad (29)$$

$$\varepsilon \dot{z} = [\mathbf{Q}_2(x, \theta) + g_2(x, \theta)\, k^{\mathrm{T}}(x)]\,[z - C(x, \theta, \phi, \bar{v})] \quad (30)$$

where

$$C(x, \theta, \phi, \bar{v}) = -[\mathbf{Q}_2(x, \theta) + g_2(x, \theta)\, k^{\mathrm{T}}(x)]^{-1}\,[f_2(x, \theta) + g_2(x, \theta)\, a(x, \bar{v}, t)] \quad (31)$$

It is clear that the closed-loop fast subsystem

$$\frac{dz}{d\tau} = [\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\,k^{\mathrm{T}}(x)]\,[z - C(x,\theta,\phi,\bar{v})] \tag{32}$$

possesses a globally exponentially stable equilibrium manifold of the form of (31), since the matrix $\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\,k^{\mathrm{T}}(x)$ is Hurwitz by the appropriate choice of the feedback gain $k^{\mathrm{T}}(x)$.

Part 2: to proceed with the rest of the proof, we consider the representation of the closed-loop in terms of the ζ, η and z coordinates. For ease of notation, we set $e_z = (z - C(x,\theta,\phi,\bar{v}))$, $x = \chi^{-1}(\zeta,\eta,\theta)$. We then have

$$\left.\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 + e_z\,\bar{\Psi}_1(\zeta,\eta,\theta,\bar{v})\\
&\;\vdots\\
\dot{\zeta}_{\bar{r}-1} &= \zeta_{\bar{r}} + e_z\,\bar{\Psi}_{\bar{r}-1}(\zeta,\eta,\theta,\bar{v})\\
\dot{\zeta}_{\bar{r}} &= L^{\bar{r}}_{\bar{f}_{\mathrm{nom}}}\mathrm{h}(x) + L_{\tilde{g}}L^{\bar{r}-1}_{\bar{f}}h(x) + e_z\,\bar{\Psi}_{\bar{r}}(\zeta,\eta,\theta,\bar{v}) + L_{\bar{G}}L^{\bar{r}-1}_{\bar{f}}h(x)\,a(x,\bar{v},t)\\
\dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) + e_z\,\bar{\Psi}_{\bar{r}+1}(\zeta,\eta,\theta,\bar{v})\\
&\;\vdots\\
\dot{\eta}_{n-\bar{r}} &= \Psi_{n-\bar{r}}(\zeta,\eta,\theta,\dot{\theta}) + e_z\,\bar{\Psi}_n(\zeta,\eta,\theta,\bar{v})\\
\varepsilon\dot{z} &= [\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\,k^{\mathrm{T}}(x)]\,e_z
\end{aligned}\right\} \tag{33}$$

where $\bar{\Psi}_i$, $i = 1,\dots,n$ are Lipschitz functions of their arguments. By introducing the variables

$$e_i = \zeta_i - v^{(i-1)}, \quad i = 1,\dots,\bar{r}, \quad \tilde{e}_{\bar{r}} = e_{\bar{r}} + \sum_{k=1}^{\bar{r}-1}\frac{\beta_k}{\beta_{\bar{r}}}e_k$$

and the notation $\bar{v} = [v \;\; v^{(1)} \;\dots\; v^{(\bar{r}-1)}]^{\mathrm{T}}$, the system of (33) takes the form

$$\left.\begin{aligned}
\dot{e}_1 &= e_2 + e_z\,\bar{\Psi}_1(\bar{e},\tilde{e}_{\bar{r}},\bar{v},\eta,\theta)\\
&\;\vdots\\
\dot{e}_{\bar{r}-1} &= -\sum_{k=1}^{\bar{r}-1}\frac{\beta_k}{\beta_{\bar{r}}}e_k + \tilde{e}_{\bar{r}} + e_z\,\bar{\Psi}_{\bar{r}-1}(\bar{e},\tilde{e}_{\bar{r}},\bar{v},\eta,\theta)\\
\dot{\tilde{e}}_{\bar{r}} &= e_z\,\bar{\Psi}_{\bar{r}}(\bar{e},\tilde{e}_{\bar{r}},\bar{v},\eta,\theta) + L^{\bar{r}}_{\bar{f}_{\mathrm{nom}}}\mathrm{h}(x) - v^{(\bar{r})} + L_{\tilde{g}}L^{\bar{r}-1}_{\bar{f}}h(x)\\
&\quad + \sum_{k=1}^{\bar{r}-1}\frac{\beta_k}{\beta_{\bar{r}}}e_{k+1} + L_{\bar{G}}L^{\bar{r}-1}_{\bar{f}}h(x)\,a(x,\bar{v},t)\\
\dot{\eta}_1 &= \Psi_1(\bar{e},\tilde{e}_{\bar{r}},\bar{v},\eta,\theta,\dot{\theta}) + e_z\,\bar{\Psi}_{\bar{r}+1}(\bar{e},\tilde{e}_{\bar{r}},\bar{v},\eta,\theta)\\
&\;\vdots\\
\dot{\eta}_{n-\bar{r}} &= \Psi_{n-\bar{r}}(\bar{e},\tilde{e}_{\bar{r}},\bar{v},\eta,\theta,\dot{\theta}) + e_z\,\bar{\Psi}_n(\bar{e},\tilde{e}_{\bar{r}},\bar{v},\eta,\theta)\\
\varepsilon\dot{z} &= [\mathrm{Q}_2(x,\theta) + g_2(x,\theta)\,k^{\mathrm{T}}(x)]\,e_z
\end{aligned}\right\} \tag{34}$$

Set $\bar{e} = [e_1 \;\; e_2 \;\dots\; e_{\bar{r}-1}]^{\mathrm{T}}$, $\bar{\eta} = [\bar{e}^{\mathrm{T}} \;\; \eta^{\mathrm{T}}]^{\mathrm{T}}$. We now follow a two-step procedure to establish ISS inequalities for the states $\bar{e}$, $\tilde{e}_{\bar{r}}$ and $\bar{\eta}$ of the system of (34). Specifically, in the first step we derive these inequalities in the absence of singular perturbations, i.e. $e_z = 0$. In the second step we establish, using the result of Theorem A.1 given in the Appendix, that these properties also hold up to an arbitrarily small offset, for initial conditions and uncertainties in an arbitrarily large compact set, so long as ε is sufficiently small.

Step 1: initially, we note that a straightforward Lyapunov function argument can be used to show that there exist positive real numbers, k_1, a and $\gamma_{\tilde{e}_{\bar{r}}}$ such that the following ISS inequality holds for the reduced $\bar{e}$ subsystem:

$$|\bar{e}(t)| \leqslant k_1 e^{-at}|\bar{e}(0)| + \gamma_{\tilde{e}_{\bar{r}}}\|\tilde{e}_{\bar{r}}\| \tag{35}$$

In what follows, our objective is to show that the state $\tilde{e}_{\bar{r}}$ of the reduced system of (34) possesses an ISS property with respect to $\tilde{e}_{\bar{r}}$, $\bar{e}$, $\bar{v}$, η and θ, and moreover the gain function saturates at ϕ. Let us consider the system composed of the state $\tilde{e}_{\bar{r}}$ of the system of (34):

$$\dot{\tilde{e}}_{\bar{r}} = L^{\bar{r}}_{\tilde{F}_{\text{nom}}} h(x) - v^{(\bar{r})} + L_{\tilde{\delta}} L^{\bar{r}-1}_{\tilde{F}} h(x) + \sum_{k=1}^{\bar{r}-1} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1} + L_{\tilde{G}} L^{\bar{r}-1}_{\tilde{F}} h(x)\, a(x, \bar{v}, t) \tag{36}$$

To establish that the state $\tilde{e}_{\bar{r}}$ of the above system satisfies an ISS property with respect to $\bar{e}$, $\tilde{e}_{\bar{r}}$, $\bar{v}$, η and θ, we consider the following smooth function $V: \mathbb{R} \rightarrow \mathbb{R}_{\geqslant 0}$:

$$V = \tfrac{1}{2}\tilde{e}^2_{\bar{r}} \tag{37}$$

Defining $\tilde{e}_{\bar{r}+1} = L^{\bar{r}}_{\tilde{F}_{\text{nom}}} h(x) - v^{(\bar{r})}$, $M = |L_{\tilde{G}} L^{\bar{r}-1}_{\tilde{F}} h(x)|\,[\tilde{c}_2(x,t)]^{-1}$, and computing the time-derivative of V along the trajectory of the system of (36), we obtain

$$\dot{V} = \tilde{e}_{\bar{r}}\Bigg[\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1} + L_{\tilde{\delta}} L^{\bar{r}-1}_{\tilde{F}} h(x) + M\left\{-\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1} - \tilde{e}_{\bar{r}} - 2\left[\tilde{c}_1(x,t) + \left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1}\right|\right] w(x,\phi)\right\}\Bigg] \tag{38}$$

Furthermore, it is straightforward to show that the representation of the function $w(x, \phi)$ in terms of the variable $\tilde{e}_{\bar{r}}$ is given by

$$w(\tilde{e}_{\bar{r}}, \phi) = \frac{\tilde{e}_{\bar{r}}}{|\tilde{e}_{\bar{r}}| + \phi} \tag{39}$$

By substituting (39) into (38), we have

$$\left.\begin{aligned}
\dot{V} &\leqslant M\tilde{e}_{\bar{r}}\Bigg\{-\tilde{e}_{\bar{r}} - 2\left[\tilde{c}_1(x,t) + \left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1}\right|\right]\frac{\tilde{e}_{\bar{r}}}{|\tilde{e}_{\bar{r}}| + \phi} \\
&\qquad + (M^{-1} - 1)\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1} + L_{\tilde{\delta}} L^{\bar{r}-1}_{\tilde{F}} h(x)\Bigg\} \\
&\leqslant M\Bigg\{-\tilde{e}^2_{\bar{r}} - 2\left[\tilde{c}_1(x,t) + \left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1}\right|\right]\frac{\tilde{e}^2_{\bar{r}}}{|\tilde{e}_{\bar{r}}| + \phi} \\
&\qquad + |\tilde{e}_{\bar{r}}|\left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1}\right| + |\tilde{e}_{\bar{r}}|\,|L_{\tilde{\delta}} L^{\bar{r}-1}_{\tilde{F}} h(x)|\Bigg\} \\
&\leqslant M\Bigg\{-\tilde{e}^2_{\bar{r}} - \left[\tilde{c}_1(x,t) + \left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1}\right|\right]\frac{\tilde{e}^2_{\bar{r}}}{|\tilde{e}_{\bar{r}}| + \phi} \\
&\qquad + \phi\left[|L_{\tilde{\delta}} L^{\bar{r}-1}_{\tilde{F}} h(x)| + \left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1}\right|\right]\frac{|\tilde{e}_{\bar{r}}|}{|\tilde{e}_{\bar{r}}| + \phi}\Bigg\}
\end{aligned}\right\} \tag{40}$$

From the last inequality and the fact that $M \geqslant 1$ (this follows from the definition of M and Assumption 4), it follows from Assumption 3 that whenever $|\tilde{e}_{\bar{r}}| > \phi$, the time-derivative of the Lyapunov function satisfies $\dot{V} \leqslant -\tilde{e}_{\bar{r}}^2$. This fact implies that the ultimate bound on the state $\tilde{e}_{\bar{r}}$ of the system of (36) depends only on the parameter ϕ and is independent of the states $\bar{e}$ and η and the vector of uncertain variables θ.

Let us now study the time-derivative of V for $|\tilde{e}_{\bar{r}}| < \phi$. To simplify the notation, we set $\mathscr{U} = [\tilde{e}_{\bar{r}} \;\; \bar{\eta}^{\mathrm{T}} \;\; \theta^{\mathrm{T}} \;\; \bar{v}^{\mathrm{T}}]^{\mathrm{T}}$. Then (40) can be written as

$$\begin{aligned} \dot{V} &\leqslant M\left\{-\tilde{e}_{\bar{r}}^2 + |\tilde{e}_{\bar{r}}|\left[|L_{\bar{g}} L_{\bar{f}}^{\bar{r}-1} h(x)| + \left|\sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} e_{k+1}\right|\right]\right\} \\ &\leqslant -\tilde{e}_{\bar{r}}^2 + |\tilde{e}_{\bar{r}}|\,\rho(|\mathscr{U}|) \end{aligned} \tag{41}$$

where ρ is a class K_∞ function. Summarizing, we have that $\dot{V}$ satisfies the following properties:

$$\dot{V} \leqslant -\frac{|\tilde{e}_{\bar{r}}|^2}{2}, \quad |\tilde{e}_{\bar{r}}| \geqslant \min\{\phi, 2\rho(|\mathscr{U}|)\} =: \tilde{\gamma}_{\mathscr{U}}(|\mathscr{U}|) \tag{42}$$

From the above equation, a direct application of the result of Theorem 4.10 reported by Khalil (1992), can be performed to conclude that the following ISS inequality holds for the state $\tilde{e}_{\bar{r}}$ of the system of (36):

$$|\tilde{e}_{\bar{r}}(t)| \leqslant e^{-0.5t}|\tilde{e}_{\bar{r}}(0)| + \tilde{\gamma}_{\mathscr{U}}(\|\mathscr{U}\|) \tag{43}$$

Before we proceed with the rest of this step, we will assume that $\phi \in (0, \phi^*]$, where

$$\phi^* = \frac{d}{1 + 2\gamma_{\tilde{e}_{\bar{r}}}} \tag{44}$$

Consider now the reduced system

$$\left.\begin{aligned} \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_{\bar{r}-1} &= -\sum_{k=1}^{\bar{r}-1} \frac{\beta_k}{\beta_{\bar{r}}} e_k + \tilde{e}_{\bar{r}} \\ \dot{\eta}_1 &= \Psi_1(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta, \dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\bar{r}} &= \Psi_{n-\bar{r}}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta, \dot{\theta}) \end{aligned}\right\} \tag{45}$$

From Assumption 3, we have that the η state of the above system possesses an ISS property with respect to $\bar{e}$, $\tilde{e}_{\bar{r}}$, θ and $\dot{\theta}$. We also note that the system of (45) is a cascaded interconnection of two ISS subsystems, and thus the state $\bar{\eta} = [\bar{e}^{\mathrm{T}} \;\; \eta^{\mathrm{T}}]^{\mathrm{T}}$ possesses an ISS property with respect to $\tilde{e}_{\bar{r}}$, θ and $\dot{\theta}$ (Sontag 1989, Jiang *et al.* 1995). To apply the result of Theorem A.1, in step 2 of the proof, we need the existence of a converse function for the system in (45). The existence of a converse function for this system can be derived by utilizing the converse theorem developed by Sontag and Wang (1995). In particular, we have that there exists a converse function for the system in (45) and the existence of this function implies that there exist a function $\beta_{\bar{\eta}}$ of class KL and a function $\bar{\gamma}_{\bar{\eta}}$ of class K such that the following ISS inequality holds for the state $\bar{\eta}$:

$$\left. \begin{aligned} |\bar{\eta}(t)| &\leqslant \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\bar{\eta}}(\|[\tilde{e}_{\bar{r}}\,\theta^{\mathrm{T}}\,\dot{\theta}^{\mathrm{T}}]^{\mathrm{T}}\|) \\ &\leqslant \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(\|\tilde{e}_{\bar{r}}\|) + \bar{\gamma}_{\theta}(\|\theta\|) + \bar{\gamma}_{\dot{\theta}}(\|\dot{\theta}\|) \end{aligned} \right\} \quad (46)$$

where $\bar{\gamma}_{\tilde{e}_{\bar{r}}}$, $\bar{\gamma}_{\theta}$ and $\bar{\gamma}_{\dot{\theta}}$ are class K functions, respectively, defined as $\bar{\gamma}_{\tilde{e}_{\bar{r}}}(s) = \bar{\gamma}_{\theta}(s) = \bar{\gamma}_{\dot{\theta}}(s) = \bar{\gamma}_{\bar{\eta}}(3s)$.

Step 2: we will now utilize the result of Theorem A.1 reported in the Appendix to establish that the ISS inequalities of (43)–(46) continue to hold up to an arbitrarily small offset, for the states $\tilde{e}_{\bar{r}}$ and $\bar{\eta}$ of the singularly perturbed system of (34). In order to proceed with the application of this result, we define a set of positive real numbers $(\delta_{\tilde{e}_{\bar{r}}}, \delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, \tilde{\delta}_{\bar{\eta}}, \tilde{\delta}_{\tilde{e}_{\bar{r}}}, d_{\tilde{e}_{\bar{r}}}, d_{\bar{\eta}})$, where δ_z, δ_θ, $\delta_{\dot{\theta}}$ and $\delta_{\bar{v}}$ were specified in the statement of the theorem, $d_{\bar{\eta}}$ is an arbitrary positive real number

$$\left. \begin{aligned} \delta_{\tilde{e}_{\bar{r}}} &\geqslant \max_{|x| \leqslant \delta_x, |\bar{v}| \leqslant \delta_{\bar{v}}} \left\{ \left| \sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} (v^{(k-1)} - L_{\bar{f}_{\text{nom}}}^{k-1} h(x)) \right| \right\} \\ \delta_{\bar{\eta}} &\geqslant \max_{|x| \leqslant \delta_x, |\bar{v}| \leqslant \delta_{\bar{v}}, \theta| \leqslant \delta_\theta} \left\{ \sum_{k=1}^{\bar{r}-1} |(v^{(k-1)} - L_{\bar{f}_{\text{nom}}}^{k-1} h(x))| + \sum_{\nu=1}^{n-\bar{r}} |\chi_\nu(x, \theta)| \right\} \end{aligned} \right\} \quad (47)$$

where $\chi_\nu(x, \theta)$, $\nu = 1, \ldots, n-\bar{r}$, are scalar field defined in assumption 2, $\tilde{\delta}_{\tilde{e}_{\bar{r}}} > \delta_{\tilde{e}_{\bar{r}}} + 2\phi$, $\tilde{\delta}_{\bar{\eta}} > D^{\bar{\eta}} + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(\tilde{\delta}_{\tilde{e}_{\bar{r}}})$, with $D^{\bar{\eta}} := \beta_{\bar{\eta}}(\delta_{\bar{\eta}}, 0) + \bar{\gamma}_{\theta}(\delta_\theta) + \bar{\gamma}_{\dot{\theta}}(\delta_{\dot{\theta}}) + d_{\bar{\eta}}$, and $d_{\tilde{e}_{\bar{r}}} = \phi < d$ (without any loss of generality). The reasons for these choices will become clear subsequently.

Let us now consider the singularly perturbed system composed of the states $(\tilde{e}_{\bar{r}}, z)$ of the system of (34). For this system, the application of Theorem A.1 is possible because this system is in a standard form, possesses a uniformly globally exponentially stable fast subsystem, and the reduced system of (36) is input-to-state stable with respect to $\mathscr{U}$ (see (43)). Then there is an $\varepsilon^{\tilde{e}_{\bar{r}}}(\phi) > 0$ such that if $\varepsilon \in (0, \varepsilon^{\tilde{e}_{\bar{r}}}(\phi)]$ and $|\tilde{e}_{\bar{r}}(0)| \leqslant \delta_{\tilde{e}_{\bar{r}}}$, $|z(0)| \leqslant \delta_z$, $\|\theta\| \leqslant \delta_\theta$, $\|\bar{v}\| \leqslant \delta_{\bar{v}}$, $\|\bar{\eta}\| \leqslant \tilde{\delta}_{\bar{\eta}}$, $\|\tilde{e}_{\bar{r}}\| \leqslant \tilde{\delta}_{\tilde{e}_{\bar{r}}}$, then

$$|\tilde{e}_{\bar{r}}(t)| \leqslant e^{-0.5t}|\tilde{e}_{\bar{r}}(0)| + \bar{\gamma}_{\mathscr{U}}(\|\mathscr{U}\|) + \phi \quad (48)$$

Note also that the explicit dependence of the upper bound on the singular perturbation parameter $\varepsilon^{\tilde{e}_{\bar{r}}}$ on ϕ is due to the appearance of parameter ϕ in the right-hand side of the differential equation which describes the closed-loop fast dynamics.

It can be also verified that the result of Theorem A.1 reported in the Appendix is also applicable to the singularly perturbed system composed of the states $(\bar{e}, \eta, z)$ of the system (34), with the same converse function which exists for the system in (45) and its resulting $(\beta_{\bar{\eta}}, \bar{\gamma}_{\bar{\eta}})$ (see also the discussion at the end of step 1). So we have that if $\varepsilon \in (0, \varepsilon^{\bar{\eta}}(\phi)]$ and $|\bar{\eta}(0)| \leqslant \delta_{\bar{\eta}}$, $|z(0)| \leqslant \delta_z$, $\|\theta\| \leqslant \delta_\theta$, $\|\dot{\theta}\| \leqslant \delta_{\dot{\theta}}$, $\|\bar{v}\| \leqslant \delta_{\bar{v}}$, $\|\tilde{e}_{\bar{r}}\| \leqslant \tilde{\delta}_{\tilde{e}_{\bar{r}}}$, then

$$\left. \begin{aligned} |\bar{\eta}(t)| &\leqslant \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\bar{\eta}}(\|[\tilde{e}_{\bar{r}}\,\theta^{\mathrm{T}}\,\dot{\theta}^{\mathrm{T}}]^{\mathrm{T}}\|) + d_{\bar{\eta}} \\ &\leqslant \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(\|\tilde{e}_{\bar{r}}\|) + \bar{\gamma}_{\theta}(\|\theta\|) + \bar{\gamma}_{\dot{\theta}}(\|\dot{\theta}\|) + d_{\bar{\eta}} \end{aligned} \right\} \quad (49)$$

For the same reasons discussed above, the upper bound $\varepsilon^{\bar{\eta}}$ depends on ϕ.

Part 3: in this part of the proof we will show that given the set of positive real numbers $\delta_{\tilde{e}_{\bar{r}}}$, $\delta_{\bar{\eta}}$, δ_z, δ_θ, $\delta_{\dot{\theta}}$, $\delta_{\bar{v}}$ and d (already specified) and with ϕ^* defined as in (44) and

$\phi \in (0, \phi^*]$, there exists $\varepsilon^*(\phi) \in (0, \varepsilon^{\bar\eta}(\phi)]$, such that if $\varepsilon \in (0, \varepsilon^*(\phi)]$ and $|\tilde{e}_{\bar r}(0)| \leqslant \delta_{\tilde{e}_{\bar r}}, |\bar\eta(0)| \leqslant \delta_{\bar\eta}, |z(0)| \leqslant \delta_z, \|\theta\| \leqslant \delta_\theta, \|\dot\theta\| \leqslant \delta_{\dot\theta}, \|\bar v\| \leqslant \delta_{\bar v}$, the output of the closed-loop system of (30) satisfies the relation of (27).

To establish this result, we will analyse the behaviour of the dynamical system composed of the states $\tilde{e}_{\bar r}$ and $\bar\eta$ of the system of (34), for which the inequalities of (48) and (49) hold, using calculations similar to those used in Jiang *et al.* (1995). In the first step we will use a contradiction argument to show that if $\phi \in (0, \phi^*]$ the evolution of the states $\tilde{e}_{\bar r}$ and $\bar\eta$, starting from initial conditions that satisfy $|\tilde{e}_{\bar r}(0)| \leqslant \delta_{\tilde{e}_{\bar r}}, |\bar\eta(0)| \leqslant \delta_{\bar\eta}$, and for θ, $\dot\theta$ and $\bar v$ such that $\|\theta\| \leqslant \delta_\theta, \|\dot\theta\| \leqslant \delta_{\dot\theta}, \|v\| \leqslant \delta_{\bar v}$ satisfies the following inequalities:

$$|\tilde{e}_{\bar r}(t)| \leqslant \tilde{\delta}_{\tilde{e}_{\bar r}}, \quad |\bar\eta(t)| \leqslant \tilde{\delta}_{\bar\eta} \tag{50}$$

for all times. In the second step we will analyse the asymptotic behaviour of the system composed of the states $\bar e$ and $\tilde{e}_{\bar r}$ of (34) and establish that the inequality of (27) holds.

Step 1: we will proceed by contradiction. Let T be the smallest time such that there is a δ so that $t \in (T, T+\delta)$ implies either $|\tilde{e}_{\bar r}(t)| > \tilde{\delta}_{\tilde{e}_{\bar r}}$ or $|\bar\eta(t)| > \tilde{\delta}_{\bar\eta}$. Then, for each $t \in [0, T]$ the conditions of (50) hold. Consider the functions $\tilde{e}_{\bar r}^{\mathrm{T}}(t)$ and $\bar\eta^{\mathrm{T}}(t)$ defined as follows:

$$\tilde{e}_{\bar r}^{\mathrm{T}}(t) = \left\{\begin{matrix} \tilde{e}_{\bar r}(t), & t \in [0, T] \\ 0, & t \in (T, \infty) \end{matrix}\right\}, \quad \bar\eta^{\mathrm{T}}(t) = \left\{\begin{matrix} \bar\eta(t) & t \in [0, T] \\ 0, & t \in (T, \infty) \end{matrix}\right\} \tag{51}$$

From the facts that $\|\theta\| \leqslant \delta_\theta$ and $\|\dot\theta\| \leqslant \delta_{\dot\theta}$ we have that

$$\sup_{0 \leqslant t \leqslant T} (\beta_{\bar\eta}(|\bar\eta(0)|, t) + \bar\gamma_\theta(\|\theta\|) + \bar\gamma_{\dot\theta}(\|\dot\theta\|) + d_{\bar\eta}) \leqslant \beta_{\bar\eta}(\delta_{\bar\eta}, 0) + \bar\gamma_{\dot\theta}(\delta_{\dot\theta}) + \bar\gamma_\theta(\delta_\theta) + d_{\bar\eta} =: D^{\bar\eta} \tag{52}$$

$$\sup_{0 \leqslant t \leqslant T} (e^{-0\cdot 5t}|\tilde{e}_{\bar r}(0)| + \tilde\gamma_{\mathscr{U}}(\|\mathscr{U}_T\|) + \phi) \leqslant \delta_{\tilde{e}_{\bar r}} + \phi + \phi \leqslant \delta_{\tilde{e}_{\bar r}} + 2\phi$$

By combining the above inequalities with (48), and also combining (52) with (49), we have that for each $\phi \in (0, \phi^*], \varepsilon \in (0, \varepsilon^{\bar\eta}(\phi)]$ and for all $t \geqslant 0$:

$$\left.\begin{aligned} \|\tilde{e}_{\bar r}^{\mathrm{T}}\| &\leqslant \delta_{\tilde{e}_{\bar r}} + 2\phi < \tilde{\delta}_{\tilde{e}_{\bar r}} \\ \|\bar\eta^{\mathrm{T}}\| &\leqslant D^{\bar\eta} + \bar\gamma_{\tilde{e}_{\bar r}}(\|\tilde{e}_{\bar r}^{\mathrm{T}}\|) \leqslant D^{\bar\eta} + \bar\gamma_{\tilde{e}_{\bar r}}(\tilde{\delta}_{\tilde{e}_{\bar r}}) < \tilde{\delta}_{\bar\eta} \end{aligned}\right\} \tag{53}$$

By continuity, we have that there exists some positive real number $\bar k$ such that $\|\tilde{e}_{\bar r}^{\mathrm{T}+\bar k}(t)\| \leqslant \tilde{\delta}_{\tilde{e}_{\bar r}}$ and $\|\bar\eta^{\mathrm{T}+\bar k}(t)\| \leqslant \tilde{\delta}_{\bar\eta}, \forall t \in [0, T+\bar k]$. This contradicts the definition of T. Hence, (50) holds $\forall t \geqslant 0$.

Step 2: in this step, we will analyse the asymptotic properties of the interconnection consisting of the states $\bar e$ and $\tilde{e}_{\bar r}$ of the system of (34), in order to recover the inequality of (27). We initially proceed with a direct application of Theorem A.1 reported in the Appendix to the interconnection composed of the states $(\bar e, z)$ of the system of (34). To this end, we consider the set of positive real numbers $(\delta_{\bar e}, \delta_z, \delta_\theta, \delta_{\dot\theta}, \delta_{\bar v}, d_{\bar e})$, where:

$$\delta_{\bar e} \geqslant \max_{|x| \leqslant \delta_x, |\bar v| \leqslant \delta_{\bar v}} \left\{ \sum_{k=1}^{\bar r-1} |(v^{(k-1)} - L_{\bar F_{\mathrm{nom}}}^{k-1} h(x))| \right\},$$

$\delta_z, \delta_\theta, \delta_{\dot\theta}, \delta_{\bar v}$ were specified in the theorem, and $d_{\bar e} = \phi$, without any loss of generality. Then there is an $\varepsilon^{\bar e}(\phi) \in (0, \varepsilon^{\bar\eta}(\phi)]$ such that if $\varepsilon \in (0, \varepsilon^{\bar e}(\phi)]$ and $|\bar e(0)| \leqslant b, |z(0)| \leqslant \delta_z, \|\theta\| \leqslant \delta_\theta, \|\dot\theta\| \leqslant \delta_{\dot\theta}, \|\bar v\| \leqslant \delta_{\bar v}$, then

$$|\bar e(t)| \leqslant k_1 e^{-at}|\bar e(0)| + \gamma_{\tilde{e}_{\bar r}}\|\tilde{e}_{\bar r}\| + \phi \tag{54}$$

In step 1 we established that the trajectories of the closed-loop system are bounded for all time, for $\phi \in (0, \phi^*]$ and $\varepsilon \in (0, \varepsilon^{\varepsilon}(\phi)]$.

Claim: The following inequality holds:

$$\limsup_{t \to \infty} |\bar{e}(t)| \leqslant \gamma_{\tilde{e}_r}(\limsup_{t \to \infty} |\tilde{e}_r(t)|) + \phi \tag{55}$$

Proof: First we note that

$$\|\bar{e}(t)\| \leqslant k_1 \delta_{\bar{z}} + \gamma_{\tilde{e}_r} \tilde{\delta}_{\tilde{e}_r} + \phi =: b \tag{56}$$

The fact that the inequality of (54) holds for every initial time t_0 (Christofides and Teel 1995), yields that $\forall t \geqslant t_0$

$$|\bar{e}(t)| \leqslant k_1 e^{-a(t-t_0)} |\bar{e}(t_0)| + \gamma_{\tilde{e}_r} \sup_{\tau \geqslant t_0} |\tilde{e}_r(\tau)| + \phi \tag{57}$$

Pick $t_0 = t/2$. We then have

$$|\bar{e}(t)| \leqslant k_1 e^{-a(t/2)}(b) + \gamma_{\tilde{e}_r} \sup_{\tau \geqslant \frac{t}{2}} |\tilde{e}_r(\tau)| + \phi \tag{58}$$

By taking the lim sup of both sides, the result of the claim can be obtained. □

Furthermore we have that (see (48)):

$$\limsup_{t \to \infty} |\tilde{e}_r(t)| \leqslant \phi + \phi \leqslant 2\phi \tag{59}$$

By combining the inequalities of (55)–(59), the following bound can be written for $\bar{e}$:

$$\limsup_{t \to \infty} |\bar{e}(t)| \leqslant \gamma_{\tilde{e}_r}(2\phi) + \phi \leqslant d \tag{60}$$

We then have that for each $\phi \in (0, \phi^*]$ and $\varepsilon \in (0, \varepsilon^{\varepsilon}(\phi)]$

$$\limsup_{t \to \infty} |y(t) - v(t)| = \limsup_{t \to \infty} |\bar{e}_1(t)| \leqslant d \tag{61}$$

Summarizing, the above inequality and all the other results of the theorem are obtained for $\phi \in (0, \phi^*]$ and $\varepsilon^*(\phi) = \varepsilon^{\varepsilon}(\phi)$. The proof of the theorem is complete. □

6. A note on the matching condition

In our development we have considered singularly perturbed systems of the form of (2) for which the matrix $\mathbf{Q}_2(x, \theta)$ could be singular for some $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$. To develop a solution to the state feedback control problem we essentially followed a two-step procedure. In the first step we employed a preliminary control law of the form of (13) to derive a two-time-scale system in standard form with globally exponentially stable fast dynamics. In the second step we specified the explicit form of the auxiliary input $\tilde{u}$ to achieve arbitrary degree of asymptotic attenuation of the effect of the uncertainty on the output. A direct implication of this approach is that the coordinate change of Assumption 2 (which implicitly includes the matching condition of our methodology) depends explicitly on the feedback gain $k^{\mathrm{T}}(x)$ employed to stabilize the fast dynamics. Therefore, it may be possible, depending on the structure of the system under consideration, to choose the gain $k^{\mathrm{T}}(x)$ to stabilize the fast dynamics as well as to ensure that the matching condition of our theory is satisfied.

In this section our objective is to show that in the case of singularly perturbed systems of the form of (2) in standard form, i.e. systems for which the matrix $\mathbf{Q}_2(x,\theta)$ is invertible uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$, with possibly unstable fast dynamics, the matching condition of this methodology is independent of the selection of the gain $k^{\mathrm{T}}(x)$ employed to stabilize the fast dynamics, and therefore it can be verified on the basis of the open-loop reduced system (6).

In order to simplify the statement of our result, we make the following definitions. Referring to the system of (6), let r denote an integer for which $\partial y^{(k)}/\partial u \equiv 0$, if $k = 1, \ldots, r-1$, for all $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$, and $\partial y^{(r)}/\partial u \not\equiv 0$, for all $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$, and let s denote the smallest integer for which $\partial y^{(s)}/\partial \theta \not\equiv 0$, for some $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$. Similarly, referring to the system of (16), let $\tilde{r}$ denote an integer for which $\partial y^{(k)}/\partial u \equiv 0$, if $k = 1, \ldots, \tilde{r}-1$, for all $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$, and $\partial y^{(\tilde{r})}/\partial u \not\equiv 0$, for all $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$, and let $\tilde{s}$ denote the smallest integer for which $\partial y^{(\tilde{s})}/\partial \theta \not\equiv 0$, for some $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$. The main result of this section is summarized in Proposition 1.

Proposition 1: *Consider the uncertain singularly perturbed nonlinear system of* (2), *for which the matrix* $\mathbf{Q}_2(x,\theta)$ *is invertible uniformly in* $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$ *and Assumption* 1 *holds. Then, if* r *is well-defined and the condition* $r \leqslant s$ *holds for the system of* (6) *and the feedback gain* $k^{\mathrm{T}}(x)$ *is chosen so that the matrix* $\mathbf{Q}_2(x,\theta)+g_2(x,\theta)k^{\mathrm{T}}(x)$ *is Hurwitz uniformly in* $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q, \tilde{r}$ *is well-defined and the condition* $r = \tilde{r} \leqslant \tilde{s}$ *holds for the system of* (16).

Proof: Initially, a direct application of the identity of (1) yields

$$\begin{aligned}\det(\mathbf{Q}_2(x,\theta)+g_2(x,\theta)k^{\mathrm{T}}(x)) &= \det(\mathbf{Q}_2(x,\theta))\det(I_{p\times p}+\mathbf{Q}_2^{-1}(x,\theta)g_2(x,\theta)k^{\mathrm{T}}(x))\\ &= \det(\mathbf{Q}_2(x,\theta))\det(1+k^{\mathrm{T}}(x)\mathbf{Q}_2^{-1}(x,\theta)g_2(x,\theta))\end{aligned} \quad (62)$$

From the hypothesis of invertibility of the matrix $\mathbf{Q}_2(x,\theta)$ uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$ and the above calculation, it follows directly that the matrix $\mathbf{Q}_2(x,\theta)+g_2(x,\theta)k^{\mathrm{T}}(x)$ is invertible uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$ if and only if the scalar $1+k^{\mathrm{T}}(x)\mathbf{Q}_2^{-1}(x,\theta)g_2(x,\theta)$ is non-zero uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$.

It is now straightforward to show that the control law of (13), calculated on the quasi-steady-state of (5) takes the form

$$u = [1+k^{\mathrm{T}}(x)\mathbf{Q}_2^{-1}(x,\theta)g_2(x,\theta)]^{-1}[\tilde{u}-k^{\mathrm{T}}(x)\mathbf{Q}_2^{-1}(x,\theta)f_2(x,\theta)] \quad (63)$$

Substitution of the state feedback law of (63) into the open-loop reduced system of (6) yields the following reduced system:

$$\left.\begin{aligned}\dot{x} &= F(x,\theta)+G(x,\theta)[1+k^{\mathrm{T}}(x)\mathbf{Q}_2^{-1}(x,\theta)g_2(x,\theta)]^{-1}[\tilde{u}-k^{\mathrm{T}}(x)\mathbf{Q}_2^{-1}(x,\theta)f_2(x,\theta)]\\ y &= h(x)\end{aligned}\right\} \quad (64)$$

where the vector functions $F(x,\theta)$ and $G(x,\theta)$ are given by (6). Now, it can easily be verified that the reduced system of (64) is identical to the reduced system of (16). Note that these systems were derived by commuting the operations setting $\varepsilon = 0$ and closing the feedback loop (see also Kokotovic *et al.* 1986 and Corless *et al.* 1993 for an analogous result in the case of linear singularly perturbed systems).

On the other hand, it is clear from the structure of the system of (64), and the hypothesis $r \leqslant s$ that a direct differentiation of the output of the system of (64) up to order r yields the following expressions:

$$\left.\begin{aligned} y &= h(x) \\ y^{(1)} &= L_F h(x) \\ y^{(2)} &= L_F^2 h(x) \\ &\vdots \\ y^{(r-1)} &= L_F^{r-1} h(x) \\ y^{(r)} &= L_F^r h(x) + L_G L_F^{r-1} h(x)\,[1 + k^{\mathrm{T}}(x)\,\mathbf{Q}_2^{-1}(x,\theta)\,g_2(x,\theta)]^{-1} \\ &\quad \times [\tilde{u} - k^{\mathrm{T}}(x)\,\mathbf{Q}_2^{-1}(x,\theta) f_2(x,\theta)] \end{aligned}\right\} \tag{65}$$

On the basis of the above expressions, it is clear that for the reduced system of (64) (and therefore for the reduced system of (16)), $r = \tilde{r} \leqslant \tilde{s}$. This completes the proof of the proposition. □

From the result of Proposition 1, it is clear that for systems in standard form, the verification of the condition $r \leqslant s$ on the basis of the open-loop reduced system of (16) is sufficient for the satisfaction of the matching condition of the developed robust control methodology.

7. Application to a non-isothermal continuous stirred tank reactor

Two parallel irreversible reactions of the form

$$A \xrightarrow{k_1} B, \quad A \xrightarrow{k_2} C \tag{66}$$

take place in a continuous stirred tank reactor which is shown in Fig. 1. The desired product is the species B, whereas species C is the side product. The inlet stream consists of pure A at flow rate F, concentration C_{A0} and temperature T_{A0}. The reactions are endothermic and a heating jacket is used to heat the reactor. Fluid is added to the jacket at a flow rate F_j and an inlet temperature T_{j0}. The rates of the reactions are assumed to be of the following form:

$$r_1 = -k_{10} \exp\left(\frac{-E_1}{RT_r}\right) C_A$$

$$r_2 = -k_{20} \exp\left(\frac{-E_2}{RT_r}\right) C_A$$

Under the following assumptions: perfect mixing in the reactor, uniform temperature in the reactor, constant volume of the liquid in the reactor, constant density and heat capacity of the reacting liquid, constant density and heat capacity of the fluid in the jacket: the process model consists of the following set of material and energy balances.

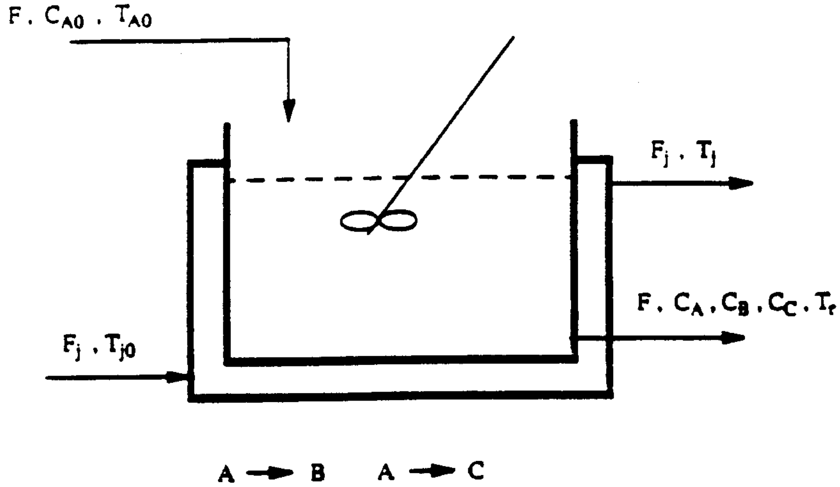

Figure 1. A continuous stirred tank reactor.

Reactor mass balance for the species A:

$$V_r \frac{\mathrm{d}C_A}{\mathrm{d}t} = F_r(C_{A0} - C_A) - k_{10} e^{(-E_1/RT_r)} C_A V_r - k_{20} e^{(-E_2/RT_r)} C_A V_r \tag{67}$$

Reactor mass balance for the species B:

$$V_r \frac{\mathrm{d}C_B}{\mathrm{d}t} = -F_r C_B + k_{10} e^{(-E_1/RT_r)} C_A V_r \tag{68}$$

Reactor energy balance:

$$V_r \frac{\mathrm{d}T_r}{\mathrm{d}t} = F_r(T_{A0} - T_r) + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{(-E_1/RT_r)} C_A V_r + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} e^{(-E_2/RT_r)} C_A V_r + \frac{UA_r}{\rho_m C_{pm}} (T_j - T_r) \tag{69}$$

Jacket energy balance:

$$V_j \frac{\mathrm{d}T_j}{\mathrm{d}t} = F_j T_{j0} - F_j T_j + \frac{UA_r}{\rho_j c_{pj}} (T_r - T_j) \tag{70}$$

where C_A and C_B denote the concentrations of the species A and B, T_r and T_j denote the temperatures of the reactor and the jacket, V_r and V_j denote the volumes of the reactor and the jacket, k_{10}, k_{20}, E_1, E_2, ΔH_1 and ΔH_2 denote the pre-exponential constants, the activation energies and the enthalpies of the two reactions, c_{pm} and c_{pj} and ρ_m and ρ_j denote the heat capacities and densities of the reactor and the jacket, respectively, and U denotes the heat transfer coefficient.

Typically, the volume of the reactor V_r is much larger than the volume of the jacket V_j; thus, the parameter ε can be defined as

$$\varepsilon = \frac{V_j}{V_r} \tag{71}$$

The control objective is the regulation of the concentration of the species B in the reactor, in order to maintain the production of B at the desired level, by manipulating the inlet temperature of the fluid in the jacket. The inlet concentration and temperature of the species A, C_{A0} and T_{A0}, respectively, are assumed to be the main uncertainties.

Defining

$$x_1 = C_A, \quad x_2 = C_B, \quad x_3 = T_r, \quad z_1 = T_j, \quad u = T_{j0} - T_{j0s} \tag{72}$$

$$\theta_1 = C_{AO} - C_{AOs}, \quad \theta_2 = T_{AO} - T_{AOs}, \quad y = C_B \tag{73}$$

where the subscript s denotes the steady-state values, one obtains the representation of the form of (2) with

$$f_1(x,\theta) = \begin{bmatrix} F_r(C_{A0s} - x_1) + F_r\theta_1 - k_{10}e^{(-E_1/Rx_3)}x_1V_r - k_{20}e^{(-E_2/Rx_3)}x_1V_r \\ -F_r x_2 + k_{10}e^{(-E_1/Rx_3)}x_1V_r \\ F_r(T_{A0s} - x_3) + F_r\theta_2 + \dfrac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10}e^{(-E_1/Rx_3)}x_1V_r \\ +\dfrac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20}e^{(-E_2/Rx_3)}x_1V_r - \dfrac{UA_r}{\rho_m c_{pm}} x_3 \end{bmatrix}$$

$$\mathbf{Q}_1(x,\theta) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{UA_r}{\rho_m c_{pm}} \end{bmatrix}, \quad g_1(\boldsymbol{x},\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad f_2(\boldsymbol{x},\theta) = \left[F_j T_{j0s} + \frac{UA_r}{\rho_j c_j} x_3\right]$$

$$\mathbf{Q}_2(x,\theta) = \left[-\left(F_j + \frac{UA_r}{\rho_j c_j}\right)\right], \quad g_2(\boldsymbol{x},\theta) = [F_j], \quad h(\boldsymbol{x}) = [x_2]$$

One can easily verify that the system is in standard form with globally exponentially stable fast dynamics. Setting $\varepsilon = 0$, the representation of the open-loop reduced system can easily be obtained with

$$F(\boldsymbol{x},\theta) = \begin{bmatrix} F_r(C_{A0s} - x_1) + F_r\theta_1 - k_{10}e^{(-E_1/Rx_3)}x_1V_r - k_{20}e^{(-E_2/Rx_3)}x_1V_r \\ -F_r x_2 + k_{10}e^{(-E_1/Rx_3)}x_1V_r \\ F_r(T_{A0s} - x_3) + F_r\theta_2 + \dfrac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10}e^{(-E_1/Rx_3)}x_1V_r + \dfrac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20}e^{(-E_2/Rx_3)}x_1V_r \\ +\dfrac{UA}{\rho_m c_{pm}} \dfrac{1}{F_j + \dfrac{UA}{\rho_j c_j}} \left[\dfrac{UA}{\rho_j c_j} T_r + F_j T_{j0s}\right] \end{bmatrix}$$

$$G(\boldsymbol{x},\theta) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{UA}{\rho_m c_{pm}} \dfrac{1}{F_j + \dfrac{UA}{\rho_j c_j}} F_j \end{bmatrix}, \quad h(\boldsymbol{x}) = [x_2]$$

It can easily be verified that a local version of Assumption 2 holds (note also that for this particular example we do not need to assume that θ_1 and θ_2 are absolutely continuous).

The values of the process parameters and the corresponding steady-state values

$V_r = 1{\cdot}00\ \mathrm{m^3}$
$V_j = 0{\cdot}08\ \mathrm{m^3}$
$A_r = 6{\cdot}0\ \mathrm{m^2}$
$U = 1000{\cdot}0\ \mathrm{kcal\ h^{-1}\ m^{-2}\ K^{-1}}$
$R = 1{\cdot}987\ \mathrm{kcal\ kmol^{-1}\ K^{-1}}$
$C_{A0s} = 3{\cdot}75\ \mathrm{kmol\ m^{-3}}$
$T_{A0s} = 310{\cdot}0\ \mathrm{K}$
$T_{j0s} = 357{\cdot}5\ \mathrm{K}$
$\Delta H_1 = 5{\cdot}4 \times 10^4\ \mathrm{kcal\ mol^{-1}}$
$\Delta H_2 = 50{\cdot}67 \times 10^4\ \mathrm{kcal\ mol^{-1}}$
$k_{10} = 3{\cdot}36 \times 10^{[illegible]}\ \mathrm{h^{-1}}$
$k_{20} = 7{\cdot}21 \times 10^6\ \mathrm{h^{-1}}$
$E_1 = 8{\cdot}0 \times 10^4\ \mathrm{kcal\ kg^{-1}}$
$E_2 = 9{\cdot}0 \times 10^4\ \mathrm{kcal\ kg^{-1}}$
$c_{pm} = 0{\cdot}231\ \mathrm{kcal\ kg^{-1}\ K^{-1}}$
$c_{pj} = 0{\cdot}2\ \mathrm{kcal\ kg^{-1}\ K^{-1}}$
$\rho_m = 900{\cdot}0\ \mathrm{kg\ m^{-3}}$
$\rho_j = 800{\cdot}0\ \mathrm{kg\ m^{-3}}$
$F = 3{\cdot}0\ \mathrm{m^3\ h^{-1}}$
$F_j = 20{\cdot}0\ \mathrm{m^3\ h^{-1}}$
$C_{As} = 1{\cdot}125\ \mathrm{kmol\ m^{-3}}$
$C_{Bs} = 1{\cdot}875\ \mathrm{kmol\ m^{-3}}$
$T_{rs} = 300{\cdot}0\ \mathrm{K}$
$T_{js} = 320{\cdot}0\ \mathrm{K}$

Process parameters.

of the process variables are given in the Table. It was verified that these conditions correspond to a stable equilibrium point. We will now verify that a local version of Assumption 3 also holds. To this end, consider the coordinate transformation

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} h(x) \\ L_F h(x) \\ \chi(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -F_r x_2 + k_{10} e^{(-E_1/Rx_3)} x_1 V_r \\ x_1 - C_{A0s} \end{bmatrix} \tag{74}$$

In the new coordinates the η_1 dynamics take the form

$$\dot{\eta}_1 = -F_r \eta_1 + F_r \theta_1 - (\zeta_2 + F_r \zeta_1) - k_{20} \left(\frac{\zeta_2 + F_r \zeta_1}{k_{10}(\eta_1 + C_{A0s}) V_r} \right)^{E_2/E_1} (\eta_1 + C_{A0s}) V_r \tag{75}$$

For a meaningful problem η_1 should always be greater than zero (note that η_1 represents the concentration of the species A in the reactor). By setting $\zeta_1 = \zeta_2 = \theta_1 = 0$, the above equation reduces to

$$\dot{\eta}_1 = -F_r \eta_1 \tag{76}$$

From the above, it follows that the system of (75) possesses a local input-to-state stability property with respect to θ and ζ, and thus Assumption 3 holds locally. Moreover, Assumption 4 takes the form

$$|L_{\tilde{g}} L_F h(x)| \leqslant \tilde{c}_1(x, t) = \left[k_{10} e^{(-E_1/Rx_3)} V_r F_r |\theta_1| + k_{10} e^{(-E_1/Rx_3)} \frac{E_1}{Rx_3^2} V_r F_r |\theta_2| \right] \tag{77}$$

where $|\theta_1|$ and $|\theta_2|$ denote the upper bounds on the size of uncertain variables.

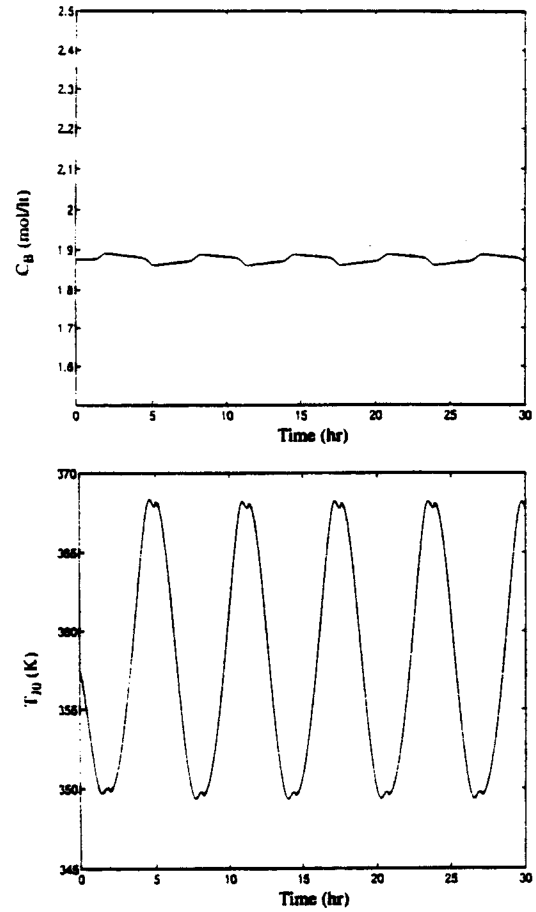

Figure 2. Open-loop (dotted) and closed-loop (solid) output profiles and input profile in the presence of uncertainty ($\phi = 0{\cdot}01, \varepsilon = 0{\cdot}08$).

We now proceed with the design of the controller. Motivated by physical arguments, we assume that the reference input is constant, and thus the controller of Theorem 1 takes the form

$$u = [L_{G_{\text{nom}}} L_{F_{\text{nom}}} h(\boldsymbol{x})]^{-1} \left\{ -\sum_{k=1}^{2} \frac{\beta_k}{\beta_2} L^k_{F_{\text{nom}}} h(\boldsymbol{x}) + \sum_{k=1}^{2} \frac{\beta_k}{\beta_2} (v^{(k-1)} - L^{k-1}_{F_{\text{nom}}} h(\boldsymbol{x})) \right.$$
$$\left. -2\left[\tilde{c}_1(\boldsymbol{x}, t) + \left| \sum_{k=1}^{2} \frac{\beta_k}{\beta_r} L^k_{\tilde{F}_{\text{nom}}} h(\boldsymbol{x}) \right| \right] w(\boldsymbol{x}, \phi) \right\} \tag{78}$$

where the scalar function $w(\boldsymbol{x}, \phi)$ is given by

$$w(\boldsymbol{x}, \phi) = \frac{\left[\dfrac{\beta_1}{\beta_2}(x_2 - v) + L_{F_{\text{nom}}} h(\boldsymbol{x}) \right]}{\left| \dfrac{\beta_1}{\beta_2}(x_2 - v) + L_{F_{\text{nom}}} h(\boldsymbol{x}) \right| + \phi} \tag{79}$$

By taking into account the time constants of the open-loop reduced system, the following values were chosen for the parameters β_k:

$$\beta_0 = 1{\cdot}0, \quad \beta_1 = 5{\cdot}0, \quad \beta_2 = 10{\cdot}0$$

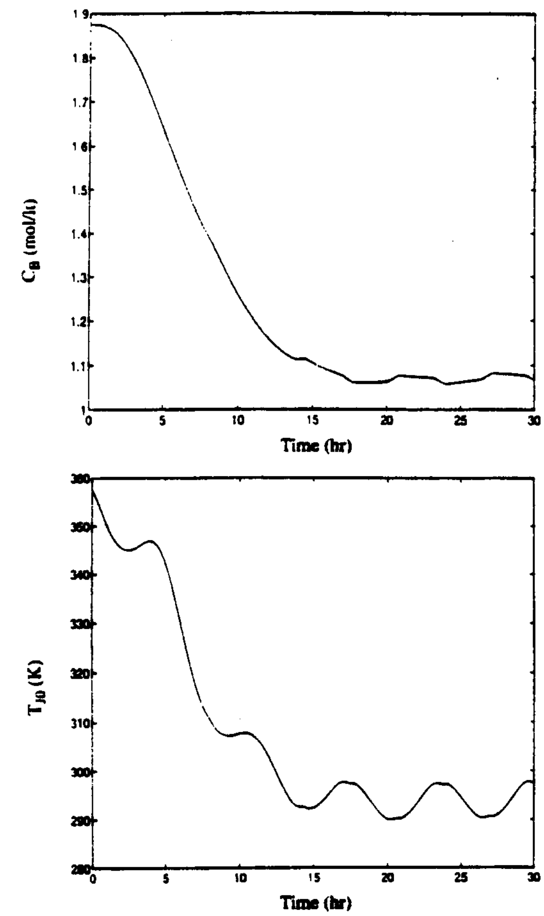

Figure 3. Output and input profiles for reference input tracking in the presence of uncertainty ($\phi = 0{\cdot}01, \varepsilon = 0{\cdot}08$).

The controller of (78) was implemented with $\phi = 0{\cdot}01$ to guarantee that the output of the system satisfies a relation of the form

$$\limsup_{t \to \infty} |y - v| \leq 0{\cdot}02 \tag{80}$$

for a value of the singular perturbation parameter ε equal to 0·08.

Simulations were performed to evaluate the performance and robustness properties of the controller. The following uncertainties were considered:

$$\theta_1 = \theta_{10} \sin(t), \quad \theta_2 = \theta_{20} \sin(t) \tag{81}$$

where $\theta_{10} = 0{\cdot}3$ mol/l and $\theta_{10} = 3{\cdot}0$ K. The upper bounds on the uncertainty were assumed to be $|\theta_1| = 0{\cdot}3, |\theta_2| = 3{\cdot}0$. In all simulation runs, it was verified that the states of the system stay bounded.

The first simulation run addressed the regulatory characteristics of the controller in the presence of uncertainties. Figure 2 shows the controlled output and the open-loop response of the process as well as the profile of the input. One can immediately observe that the effect of uncertainties on the output has been significantly reduced and the output of the process remains very close to the nominal value of the reference input, as predicted by the theory.

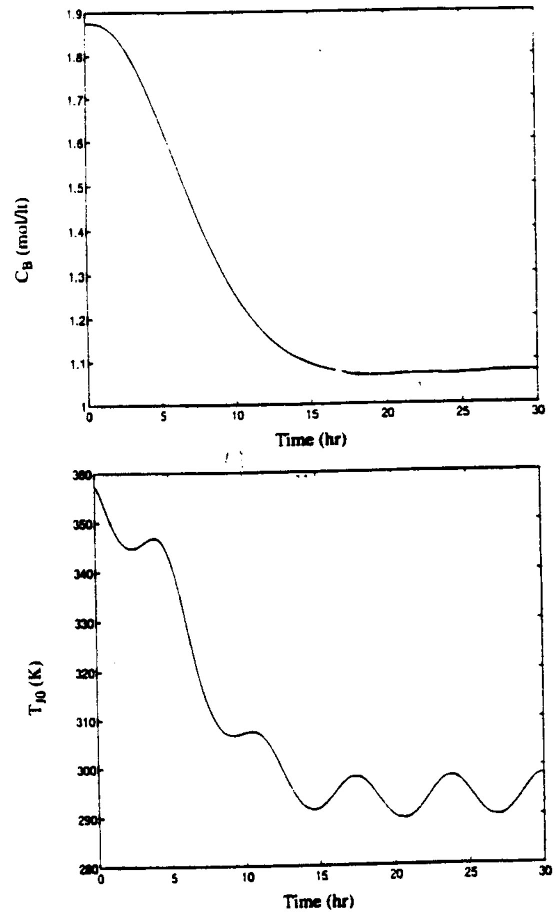

Figure 4. Output and input profiles for reference input tracking in the presence of uncertainty ($\phi = 0{\cdot}001, \varepsilon = 0{\cdot}08$).

The second simulation run addressed the reference input tracking capabilities of the controller in the presence of uncertainties. A 0·8 mol/l (almost 43 %) decrease in the reference input value ($v = 1{\cdot}075$) was imposed at time $t = 0$. The output and input profiles are shown in Fig. 3. It is clear that the controller achieves the requested degree of asymptotic attenuation of the effect of the uncertainty on the output and regulates the output to the new reference input value despite the presence of uncertainties.

In the final run we considered the performance of the controller for $\phi = 0{\cdot}001$ under the conditions used in the previous simulation. Figure 4 depicts the corresponding output and input profiles. As expected, the degree of attenuation of the uncertainty increases (compare with Fig. 3), while the output of the system is driven to the new reference input value.

8. Conclusions

In this work we considered a broad class of singularly perturbed systems with time-varying uncertainties, whose fast dynamics may be unstable or singular. The uncertainty is allowed to be time-varying. It is assumed that the boundary layer system is stabilizable and the reduced system is input/output linearizable with ISS inverse dynamics. For such systems, we proposed a class of well-conditioned control laws that guarantee boundedness of the trajectories of the closed-loop system and achieve arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output, for initial conditions and uncertainties in an arbitrarily large compact set, so long as the singular perturbation parameter is sufficiently small. Finally, the derived controller was successfully applied to a non-isothermal continuous stirred tank reactor, with fast jacket dynamics and uncertainties in the concentration and temperature of the inlet stream, modelled by a singularly perturbed system in standard form.

Appendix

In this appendix we recall a result developed by Christofides and Teel (1995) (see also Christofides and Teel 1996). Consider the singularly perturbed system

$$\left.\begin{aligned} \dot{x} &= f(x, z, \theta(t), \varepsilon) \\ \varepsilon\dot{z} &= g(x, z, \theta(t), \varepsilon) \end{aligned}\right\} \tag{A 1}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $\theta \in \mathbb{R}^q$. The functions f and g are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times [0, \bar{\varepsilon})$, for ε sufficiently small.

Assumption A.1: *The algebraic equation $g(x, z_s, \theta) = 0$ possesses a unique root*

$$z_s = h(x, \theta) \tag{A 2}$$

with the properties that $h: \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p$ and its partial derivatives $(\partial h/\partial x, \partial h/\partial \theta)$ are locally Lipschitz and $h(0, 0) = 0$.

Consider also the corresponding reduced and boundary layer system

$$\dot{x} = f(x, h(x, \theta), \theta, 0) \tag{A 3}$$

$$\frac{dy}{d\tau} = g(x, h(x, \theta) + y, \theta, 0) \tag{A 4}$$

where $y = z - h(x, \theta)$. The assumptions that follow state our stability requirements on the reduced and boundary layer system.

Assumption A.2: *The reduced system in* (A 3) *is input-to-state stable with the class-KL function β_x and the class-K function γ_x; more precisely, there exist a smooth function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class-K_∞ functions a_1, a_2, a_3 and a_4 such that*

$$a_1(|x|) \leq V(x) \leq a_2(|x|) \tag{A 5}$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x, h(x, \theta), \theta, 0) \leq a_4(|\theta|) - a_3(|x|) \tag{A 6}$$

for all $\theta \in \mathbb{R}^q$ and all $x \in \mathbb{R}^n$, hold and the existence of such a V implies that (12) *holds with $\beta = \beta_x$ and $\gamma = \gamma_x$.*

Assumption A.3: *The equilibrium $y = 0$ of the boundary layer system in* (A 4) *is globally asymptotically stable, uniformly in $x \in \mathbb{R}^n, \theta \in \mathbb{R}^q$.*

The result is as follows.

Theorem A.1: *Consider the singularly perturbed system in* (A 1) *for which Assumption* A.1 *holds and let $y = z - h(x, \theta)$. If $\theta(t)$ is absolutely continuous and Assumptions* A.2 *and* A.3 *hold, then there exists a function β_y of class KL such that, for each set of positive real* *numbers* $(\delta_x, \delta_y, \delta_\theta, \delta_{\dot\theta}, d_x, d_y)$, *there is an* $\varepsilon^* > 0$ *such that if* $\varepsilon \in (0, \varepsilon^*]$ *and* $|\mathbf{x}(0)| \leqslant \delta_x$, $|y(0)| \leqslant \delta_y$, $\|\theta\| \leqslant \delta_\theta$, $\|\dot\theta\| \leqslant \delta_{\dot\theta}$, *then*

$$|\mathbf{x}(t)| \leqslant \beta_x(|\mathbf{x}(0)|, t) + \gamma_x(\|\theta\|) + d_x \tag{A 7}$$

$$|y(\tau)| \leqslant \beta_y(|y(0)|, \tau) + d_y \tag{A 8}$$

where β_x *and* γ_x *are the functions defined in Assumption* A.2.

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