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Automatica 36 (2000) 45–52

www.elsevier.com/locate/automatica

Brief Paper

Robust output feedback control of nonlinear singularly perturbed systems☆

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Received 20 January 1997; revised 1 February 1999; received in final form 16 March 1999

Abstract

In this work, we consider nonlinear singularly perturbed systems with time-varying uncertain variables, for which the fast subsystem is asymptotically stable and the slow subsystem is input/output linearizable and possesses input-to-state stable (ISS) inverse dynamics. For these systems, we synthesize a robust output feedback controller that ensures boundedness of the state and enforces robust asymptotic output tracking with attenuation of the effect of the uncertain variables on the output of the closed-loop system. The controller is constructed through combination of a high-gain observer with a robust state feedback controller synthesized via Lyapunov's direct method. The proposed controller enforces the aforementioned properties in the closed-loop system, for initial conditions, uncertainty and rate of change of uncertainty in arbitrarily large compact sets, provided that the singular perturbation parameter is sufficiently small and the observer gain is sufficiently large.

Keywords: Singular perturbations; Input-to-state stability; Lyapunov's direct method; State estimation; Chemical processes

1. Introduction

Many industrial processes exhibit nonlinear behavior and involve physicochemical phenomena occurring in separate time-scales. It is well established that a direct application of standard control methods to multiple-time-scale processes, without accounting for the presence of time-scale multiplicity, may lead to controller ill-conditioning and/or closed-loop instability. To circumvent these problems, the control of multiple-time-scale processes is usually addressed within the framework of singular perturbations (e.g., Kokotovic, Khalil, & O'Reilly, 1986; Christofides & Daoutidis, 1996).

In addition to nonlinearities and time-scale multiplicity, many industrial processes involve unknown process parameters and external disturbances. Therefore, the problem of designing controllers for nonlinear systems with uncertain variables, that enforce output tracking with attenuation of the effect of the uncertain variables on the output, has received considerable attention. For feedback linearizable nonlinear systems with time-varying uncertain variables that satisfy the so-called matching condition, robust state feedback controllers have been designed via Lyapunov's direct method to solve this problem locally (the reader may refer to Corless (1993) for a review of results in this area). Recently, for a class of nonlinear systems with time-varying uncertain variables that admit a disturbance-strict-feedback form without zero dynamics, a robust output feedback controller was designed in Khalil (1994) that solves this problem for arbitrarily large initial conditions and uncertainty (semi-global result). This result was generalized in Mahmoud and Khalil (1996) to nonlinear systems with asymptotically stable zero dynamics. In Christofides, Teel and Daoutidis, (1996), robust *state* feedback controllers were synthesized for nonlinear singularly perturbed systems with time-varying uncertain variables.

In this work, we address the problem of synthesizing a robust *output* feedback controller for nonlinear singularly perturbed systems with uncertain variables, for which the fast subsystem is asymptotically stable and the slow subsystem is input/output linearizable and possesses input-to-state stable inverse dynamics. A dynamic controller is synthesized, through combination of a

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Y. Yamamoto under the direction of Editor R. Tempo.

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0005-1098/00/$ - see front matter
PII: S0005-1098(99)00105-3

high-gain observer with a robust state feedback controller synthesized via Lyapunov's direct method, that ensures boundedness of the state and achieves arbitrary degree of asymptotic attenuation of the effect of the uncertain variables on the output of the closed-loop system. The derived controller enforces the requested objectives in the closed-loop system, for initial conditions, uncertainty and rate of change of uncertainty in arbitrarily large compact sets, as long as the singular perturbation parameter is sufficiently small and the observer gain is sufficiently large. A successful application of the proposed control method to a chemical reactor example can be found in Christofides (1998).

2. Notation

- $|\cdot|$ denotes the standard Euclidean norm, $sgn(\cdot)$ denotes the sign function, and $sat(\cdot)$ denotes the saturation function, defined as

$$sat(s) = \begin{cases} a_m & \text{if } s \geq a_m, \\ s & \text{if } -a_m < s < a_m, \\ -a_m & \text{if } s \leq -a_m, \end{cases} \tag{1}$$

  where $s \in \mathbb{R}$ and a_m is a positive real number. For a vector $x \in \mathbb{R}^n$, $sat(x) = [sat(x_1)\ sat(x_2)\ \cdots\ sat(x_n)]^{\mathrm{T}}$. $O(\varepsilon)$ denotes the standard order of magnitude notation i.e., $\delta(\varepsilon) = O(\varepsilon)$ if there exist positive constants k and c such that $|\delta(\varepsilon)| \leq k|\varepsilon|$, $\forall |\varepsilon| < c$. $L_f h$ denotes the Lie derivative of a scalar field h with respect to the vector field f. $L_f^k h$ denotes the kth order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.
- For any measurable (with respect to the Lebesgue measure) function $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $\|\theta\|$ denotes $\operatorname{ess\,sup} |\theta(t)|$, $t \geq 0$. For any strictly positive real number ρ, $\|\theta^\rho\|$ denotes $\operatorname{ess\,sup} |\theta(t)|$, $t \geq \rho$.
- A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class K if it is continuous, increasing and is zero at zero. It is of class K_∞, if in addition, it is proper. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.

3. Preliminaries

We consider single-input single-output nonlinear singularly perturbed systems with uncertain variables of the form

$$\begin{aligned} \dot{x} &= f_1(x, \theta(t)) + Q_1(x, \theta(t))z + g_1(x, \theta(t))u, \\ \varepsilon \dot{z} &= f_2(x, \theta(t)) + Q_2(x, \theta(t))z + g_2(x, \theta(t))u, \\ y &= h(x), \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ denote vectors of state variables, $u \in \mathbb{R}$ denotes the manipulated input, $\theta \in \mathbb{R}^q$ denotes the vector of the uncertain time-varying variables, $y \in \mathbb{R}$ denotes the controlled output, and ε is a small positive parameter. $f_1(x, \theta(t))$, $f_2(x, \theta(t))$, $g_1(x, \theta(t))$ and $g_2(x, \theta(t))$ are sufficiently smooth vector functions, $Q_1(x, \theta(t))$ and $Q_2(x, \theta(t))$ are sufficiently smooth matrices, and $h(x)$ is a sufficiently smooth scalar function.

Setting $\varepsilon = 0$ in the system of Eq. (2) and assuming that $Q_2(x, \theta)$ is invertible uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, the following slow subsystem is obtained:

$$\begin{aligned} \dot{x} &= F(x, \theta) + G(x, \theta)u, \\ y &= h(x), \end{aligned} \tag{3}$$

where $F(x, \theta) = f_1(x, \theta) - Q_1(x, \theta)[Q_2(x, \theta)]^{-1} f_2(x, \theta)$ and $G(x, \theta) = g_1(x, \theta) - Q_1(x, \theta)[Q_2(x, \theta)]^{-1} g_2(x, \theta)$. Defining the fast time-scale $\tau = t/\varepsilon$, deriving the representation of the system of Eq. (2) in the τ time scale, and setting $\varepsilon = 0$, the following fast subsystem is obtained:

$$\frac{dz}{d\tau} = f_2(x, \theta) + Q_2(x, \theta)z + g_2(x, \theta)u, \tag{4}$$

where x can be considered equal to its initial value $x(0)$ and θ can be viewed as constant.

Assumption 1. The matrix $Q_2(x, \theta)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.

Definition [(Sontag, 1989)]. The system in Eq. (3) (with $u \equiv 0$) is said to be input-to-state stable (ISS) with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_0 \in \mathbb{R}^n$ and for each measurable, essentially bounded input $\theta(\cdot)$ on $[0, \infty)$ the solution of Eq. (3) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|\theta\|), \quad \forall t \geq 0. \tag{5}$$

4. Robust output feedback controller synthesis

Motivated by the assumption of global asymptotic stability of the fast dynamics of the system of Eq. (2), we will synthesize the requisite robust dynamic output feedback controller on the basis of the slow subsystem of Eq. (3). To this end, we will need to impose the following three assumptions on the slow subsystem of Eq. (3). The first assumption is motivated by the requirement of output tracking and states the existence of a coordinate change that renders the system of Eq. (3) partially linear.

Assumption 2. There exists an integer r and a set of coordinates

$$\begin{bmatrix}\zeta\\ \eta\end{bmatrix}=\begin{bmatrix}\zeta_1\\ \zeta_2\\ \vdots\\ \zeta_r\\ \eta_1\\ \vdots\\ \eta_{n-r}\end{bmatrix}=\mathscr{X}(x)=\begin{bmatrix}\theta(x)\\ L_Fh(x)\\ \vdots\\ L_F^{r-1}h(x)\\ \chi_1(x)\\ \vdots\\ \chi_{n-r}(x)\end{bmatrix}, \tag{6}$$

where $\chi_1(x), \ldots, \chi_{n-r}(x)$ are nonlinear scalar functions of x, such that the reduced system of Eq. (3) takes the form:

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2,\\ &\vdots\\ \dot{\zeta}_{r-1} &= \zeta_r,\\ \dot{\zeta}_r &= L_F^r h(\mathscr{X}^{-1}(\zeta,\eta),\theta) + L_G L_F^{r-1} h(\mathscr{X}^{-1}(\zeta,\eta),\theta)u,\\ \dot{\eta}_1 &= \Psi_1(\zeta,\eta),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta,\eta),\\ y &= \zeta_1,\end{aligned} \tag{7}$$

where $L_G L_F^{r-1} h(x,\theta) \neq 0$ for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.

We note that the change of variables of Eq. (6) is independent of θ and $\dot{\theta}$, and invertible, since, for every x, the variables ζ, η are uniquely determined by Eq. (6). This implies that if we can estimate the values of ζ, η for all times, using appropriate state observers, then we automatically obtain estimates of x for all times. This property will be exploited later to synthesize a state estimator for the system of Eq. (3) on the basis of the system of Eq. (7).

Assumption 3. The system

$$\begin{aligned}\dot{\eta}_1 &= \Psi_1(\zeta,\eta),\\ &\vdots\\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\zeta,\eta)\end{aligned} \tag{8}$$

is ISS with respect to ζ with $\beta_\eta(|\eta(0)|,t) = K_\eta|\eta(0)|e^{-at}$ where K_η, a are positive real numbers and $K_\eta \geq 1$.

Following Christofides et al. (1996), the requirement of input-to-state stability of the system of Eq. (8) with respect to ζ is imposed to allow the synthesis of a robust state feedback controller that enforces the requested properties in the closed-loop system for arbitrarily large initial conditions and uncertain variables. On the other hand, the requirement that $\beta_\eta(|\eta(0)|,t) = K_\eta e^{-at}$ implies that the inverse dynamics with $\zeta(t) \equiv 0$ are globally exponentially stable, which allows incorporating in the robust output feedback controller a dynamical system identical to the one of Eq. (8) that provides estimates of the variables η. Finally, we assume that there exist functions that capture the size of the uncertain term in the system of Eq. (7).

Assumption 4. The term $L_G L_F^{r-1} h(x,\theta)$ has known constant sign for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and there exist known functions $c_1(x,t)$, $c_2(x,t)$ such that the following conditions are satisfied:

$$|L_\delta L_F^{r-1} h(x,\theta)| \leq c_1(x,t), \tag{9}$$

$$0 < c_2(x,t) \leq |L_G L_F^{r-1} h(x,\theta)|, \tag{10}$$

where $\delta(x,\theta) = [F(x,\theta) - F_{\mathrm{nom}}(x)]$, for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.

Theorem 1 below provides the main result of this paper.

Theorem 1. *Consider the uncertain singularly perturbed nonlinear system of Eq.* (2), *for which Assumptions* 1–4 *hold, under the robust output feedback controller*:

$$\dot{\tilde{y}} = \begin{bmatrix} -La_1 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -L^{r-1}a_{r-1} & 0 & 0 & \cdots & 0 & 1\\ -L^r a_r & 0 & 0 & \cdots & 0 & 0\end{bmatrix}\tilde{y} + \begin{bmatrix} La_1\\ \vdots\\ L^{r-1}a_{r-1}\\ L^r a_r\end{bmatrix} y,$$

$$\begin{aligned}\dot{\omega}_1 &= \Psi_1(sat(\tilde{y}),\omega),\\ &\vdots\\ \dot{\omega}_{n-r} &= \Psi_{n-r}(sat(\tilde{y}),\omega),\end{aligned} \tag{11}$$

$$\begin{aligned} u = a(\hat{x},\bar{v},\phi,t) :={}& sgn[L_G L_F^{r-1} h(\hat{x})][\beta_r c_2(\hat{x},t)]^{-1}\\ &\times \Bigg\{\sum_{k=1}^{r} \beta_k (v^{(k)} - L_{F_{\mathrm{nom}}}^k h(\hat{x}))\\ &+ \sum_{k=1}^{r} \beta_k (v^{(k-1)} - L_{F_{\mathrm{nom}}}^{k-1} h(\hat{x}))\\ &- (1+\chi)\left[c_1(\hat{x},t) + \left|\sum_{k=1}^{r} \beta_k (v^{(k)} - L_{F_{\mathrm{nom}}}^k h(\hat{x}))\right|\right]\\ &\frac{\sum_{k=1}^{r} \beta_k (L_{F_{\mathrm{nom}}}^{k-1} h(\hat{x}) - v^{(k-1)})}{\left|\sum_{k=1}^{r} \beta_k (L_{F_{\mathrm{nom}}}^{k-1} h(\hat{x}) - v^{(k-1)})\right| + \phi}\Bigg\}.\end{aligned}$$

$\hat{x} = \mathscr{X}^{-1}(sat(\tilde{y}),\omega))$, a_k, β_k *are parameters chosen so that the polynomials* $s^r + a_1 s^{r-1} + \cdots + a_{r-1}s + a_r = 0$, $\beta_r s^{r-1} + \beta_{r-1}s^{r-2} + \cdots + \beta_2 s + \beta_1 = 0$ *are Hurwitz, and* ϕ, χ *are adjustable positive parameters. Then, for each set of positive real numbers* $\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d$, *there exists* $\phi^*(\chi) > 0$ *and for each* $\phi \in (0, \phi^*(\chi)]$, *there exists an* $\varepsilon^*(\phi) > 0$, *such that if* $\phi \in (0, \phi^*]$, $\bar{\varepsilon} = \max\{\varepsilon, 1/L\} \in (0, \varepsilon^*(\phi)]$, $sat(\cdot) = \min\{1, \zeta_{\max}/|\cdot|\}(\cdot)$ *with* $\zeta_{\max}$ *being the maximum value of the vector* $[\zeta_1 \zeta_2 \cdots \zeta_r]$ *for*

$|\zeta| \le \beta_\zeta(\delta_\zeta, 0) + d$ where β_ζ is a class KL function and δ_ζ is the maximum value of the vector $[h(x) L_F h(x) \cdots L_F^{r-1} h(x)]$ for $|x| \le \delta_x$, $|x(0)| \le \delta_x$, $|z(0)| \le \delta_z$, $\|\theta\| \le \delta_\theta$, $\|\dot{\theta}\| \le \delta_{\dot{\theta}}$, $\|\bar{v}\| \le \delta_{\bar{v}}$, $|\tilde{y}(0)| \le \delta_\zeta$, $\omega(0) = \eta(0) + O(\bar{\varepsilon})$, the output of the closed-loop system satisfies a relation of the form

$$\limsup_{t \to \infty} |y(t) - v(t)| \le d. \tag{12}$$

Remark 1. The robust output feedback controller of Eq. (4) consists of a linear gain observer (see, for example, Khalil, 1994) which provides estimates of the derivatives of the output y up to order $r-1$, denoted as $\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_{r-1}$, and thus estimates of the variables $\zeta_1, \ldots, \zeta_r$(note that from Assumption 2 it follows directly that $\zeta_k = \mathrm{d}^{k-1} y / \mathrm{d}t^{k-1}$, $k = 1, \ldots, r$), and observer that simulates the inverse dynamics of the system of Eq. (7), and a static state feedback controller, synthesized via Lyapunov's direct method (see Christofides et al., 1996 for details), that attenuates the effect of the uncertain variables on the output and enforces reference input tracking. To eliminate the peaking phenomenon associated with the high-gain observer, we use the saturation function, *sat*, to eliminate wrong estimates of the output derivatives for short times.

Proof of Theorem 1. *Part* 1: Defining the auxiliary error variables $\hat{e}_i = L^{r-i}(y^{(i-1)} - \tilde{y}_i)$, $i = 1, \ldots, r$, the vector $e_0 = [\hat{e}_1 \ \hat{e}_2 \ \cdots \ \hat{e}_r]^{\mathrm{T}}$, the parameter $\mu = 1/L$, the matrix A and the vector b:

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{r-1} & 0 & 0 & \cdots & 0 & 1 \\ -a_r & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{13}$$

the system of Eq. (2) under the controller of Eq. (4) takes the form

$$\begin{aligned} \mu \dot{e}_0 &= A e_0 + \mu b \Omega(x, \hat{x}, \theta, \phi, \bar{v}), \\ \dot{\omega}_1 &= \Psi_1(sat(\tilde{y}), \omega), \\ &\vdots \\ \dot{\omega}_{n-r} &= \Psi_{n-r}(sat(\tilde{y}), \omega), \\ \dot{x} &= F(x, \theta) + G(x, \theta) a(\hat{x}, \bar{v}, \phi, t) \\ &\quad + Q_1(x, \theta)[z - C(x, \hat{x}, \theta, \phi, \bar{v})], \\ \varepsilon \dot{z} &= Q_2(x, \theta)[z - C(x, \hat{x}, \theta, \phi, \bar{v})], \end{aligned} \tag{14}$$

where $\hat{x} = \mathscr{X}^{-1}(sat(y_d - \Delta(\mu) e_0), \omega)$ and $y_d = [y^{(0)} \ y^{(1)} \ \cdots \ y^{(r-1)}]^{\mathrm{T}}$, $\Delta(\mu)$ is a diagonal matrix whose ith diagonal element is μ^{r-i}, and $\Omega(x, \hat{x}, \theta, \phi, \bar{v})$ is a Lipschitz function of its argument and $C(x, \hat{x}, \theta, \phi, \bar{v}) = -[Q_2(x, \theta)]^{-1}[f_2(x, \theta) + g_2(x, \theta) a(\hat{x}, \bar{v}, \phi, t)]$. Owing to the presence of the small parameters μ and ε that multiply the time derivatives $\dot{e}$ and $\dot{z}$, respectively, the system of Eq. (14) can be, in general, a three-time-scale one. Therefore, the results proved in Khalil (1987), will be used to establish asymptotic stability of the fast dynamics. Defining $\bar{\varepsilon} = \max\{\mu, \varepsilon\}$, multiplying the z-subsystem of Eq. (14) with $\bar{\varepsilon}/\varepsilon$ and the e_0-subsystem with $\bar{\varepsilon}/\mu$, introducing the fast time-scale $\bar{\tau} = t/\bar{\varepsilon}$ and setting $\bar{\varepsilon} = 0$,

$$\begin{aligned} \frac{\mathrm{d}e_0}{\mathrm{d}\bar{\tau}} &= \frac{\bar{\varepsilon}}{\mu} A e_0, \\ \frac{\mathrm{d}z}{\mathrm{d}\bar{\tau}} &= \frac{\bar{\varepsilon}}{\varepsilon} Q_2(x, \theta)[z - C(x, \hat{x}, \theta, \phi, \bar{v})]. \end{aligned} \tag{15}$$

The above system possesses a triangular (cascaded structure), the matrix A is Hurwitz and the matrix $Q_2(x, \theta)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Therefore, the system of Eq. (15) satisfies the Assumption 3 in Khalil (1987) and since $\bar{\varepsilon}/\mu \ge 1$, $\bar{\varepsilon}/\varepsilon \ge 1$, the same approach as in Khalil (1987) can be used to show that it possesses a globally exponentially stable equilibrium manifold of the form $e_0 = 0$, $z_s = C(x, x, \theta, \phi, \bar{v})$, for all values of μ and ε.

Part 2: In this part of the proof, we initially derive ISS bounds, when $\bar{\varepsilon} = 0$, for the states of the system of Eq. (14), in appropriately transformed coordinates, and then use the result of the Lemma A.1 given in the appendix to show that these bounds hold up to an arbitrarily small offset, for initial conditions, uncertainty and rate of change of uncertainty in an arbitrarily large compact set, provided that $\bar{\varepsilon}$ is sufficiently small. Defining the variables $e_z = (z - C(x, x, \theta, \phi, \bar{v}))$, $x = \mathscr{X}^{-1}(\zeta, \eta)$, $e_i = \zeta_i - v^{(i-1)}$, $i = 1, \ldots, r$, $\tilde{e}_r = e_r + \sum_{k=1}^{r-1} (\beta_k/\beta_r) e_k$, and the vector $\tilde{v} = [v \ v^{(1)} \ \cdots \ v^{(r-1)}]^{\mathrm{T}}$, the closed-loop system can be written as

$$\begin{aligned} \mu \dot{e}_0 &= A e_0 + \mu b \Omega(\bar{e}, \tilde{e}_r, \tilde{v}, \eta, \hat{x}, \theta, \phi, \bar{v}), \\ \dot{\omega}_1 &= \Psi_1(sat(\tilde{y}), \omega), \\ &\vdots \\ \dot{\omega}_{n-r} &= \Psi_{n-r}(sat(\tilde{y}), \omega), \\ \dot{e}_1 &= e_2 + e_z \bar{\Psi}_1(\bar{e}, \tilde{e}_r, \tilde{v}, \eta, \theta), \\ &\vdots \\ \dot{e}_{r-1} &= -\sum_{k=1}^{r-1} \frac{\beta_k}{\beta_r} e_k + \tilde{e}_r + e_z \bar{\Psi}_{r-1}(\bar{e}, \tilde{e}_r, \tilde{v}, \eta, \theta), \\ \dot{\tilde{e}}_r &= e_z \bar{\Psi}_r(\bar{e}, \tilde{e}_r, \tilde{v}, \eta, \theta) + L_{F_{\mathrm{nom}}}^r h(x) - v^{(r)} \\ &\quad + L_\delta L_F^{r-1} h(x, \theta) + \sum_{k=1}^{r-1} \frac{\beta_k}{\beta_r} e_{k+1} \\ &\quad + L_G L_F^{r-1} h(x, \theta) a(\hat{x}, \bar{v}, \phi, t), \\ \dot{\eta}_1 &= \Psi_1(\bar{e}, \tilde{e}_r, \tilde{v}, \eta) + e_z \bar{\Psi}_{r+1}(\bar{e}, \tilde{e}_r, \tilde{v}, \eta, \theta), \\ &\vdots \\ \dot{\eta}_{n-r} &= \Psi_{n-r}(\bar{e}, \tilde{e}_r, \tilde{v}, \eta) + e_z \bar{\Psi}_n(\bar{e}, \tilde{e}_r, \tilde{v}, \eta, \theta), \\ \varepsilon \dot{z} &= Q_2(\bar{e}, \tilde{e}_r, \tilde{v}, \eta, \theta) e_z, \end{aligned} \tag{16}$$

where $\bar{\Psi}_i$, $i = 1, \dots, n$, are Lipschitz functions of their arguments.

Step 1: Consider the system of Eq. (16) with $\bar{\varepsilon} = \max\{\varepsilon, \mu\} = 0$ and define $\bar{e} = [e_1\ e_2\ \cdots\ e_{r-1}]^{\mathrm{T}}$ and $\bar{\eta} = [\bar{e}^{\mathrm{T}}\ \eta^{\mathrm{T}}]^{\mathrm{T}}$. Utilizing a standard Lyapunov function argument, it is straightforward to show that there exist positive real numbers, $k_1, a, \gamma_{\tilde{e}_r}$ such that the following ISS inequality holds for the reduced $\bar{e}$ subsystem:

$$|\bar{e}(t)| \le k_1 \mathrm{e}^{-at}|\bar{e}(0)| + \gamma_{\tilde{e}_r}\|\tilde{e}_r\|. \tag{17}$$

We now study the dynamics of the state $\tilde{e}_r$ of the system of Eq. (16) with $\bar{\varepsilon} = 0$ to show that it possesses an ISS property with respect to $\tilde{e}_r, \bar{e}, \bar{v}, \eta, \theta$, and moreover the magnitude of the gain function is bounded by ϕ. In order to prove this result, we initially need to show that for the system of Eq. (16) $\omega(0) = \eta(0) + O(\bar{\varepsilon})$ implies $\eta(t) = \omega(t) + O(\bar{\varepsilon})$, $\forall t \ge 0$. To this end, consider the singularly perturbed system comprised of the states e_0, ω, η, z of the system of Eq. (16). For this system, it is straightforward to verify that it satisfies the assumptions of Theorem 1 reported in Khalil and Esfandiari (1993). Applying this theorem, we have that there exists a positive real number ε_0, such that for any positive real number δ_ω satisfying $\delta_\omega \ge \max/|x| \le \delta_x \left\{ \sum_{v=1}^{n-r} |\chi_v(x)| \right\}$ where $\chi_v(x)$, $v = 1, \dots, n-r$ are the functions defined in Assumption 2, the states (η,ω) of this system, starting from any initial conditions that satisfy $\eta(0) = \omega(0) + O(\bar{\varepsilon})$ (with $\max\{\eta(0), \omega(0)\} \le \delta_\omega$), if $\bar{\varepsilon} \in (0, \varepsilon_0]$, satisfy $\eta(t) = \omega(t) + O(\bar{\varepsilon})$, $\forall t \ge 0$. Since $\eta(t) = \omega(t)$, $\forall t \ge 0$, when $\bar{\varepsilon} = 0$, the closed-loop slow system reduces to the one studied in Christofides et al. (1996). Thus, employing the same calculations as those in Christofides et al. (1996), one can show that the following bound holds for the state $\tilde{e}_r(t)$ of the system of Eq. (16) with $\bar{\varepsilon} = 0$, for $t \ge 0$:

$$|\tilde{e}_r(t)| \le \mathrm{e}^{-0.5t}|\tilde{e}_r(0)| + \tilde{\gamma}_{\mathscr{U}}(\|\mathscr{U}\|), \tag{18}$$

where $\tilde{\gamma}_{\mathscr{U}}(|\mathscr{U}|) = \min\{\phi/\chi, 2\phi\rho(|\mathscr{U}|)\}$, ρ is a class K_∞ function and $\mathscr{U} = [\tilde{e}_{\tilde{r}}\ \bar{e}^{\mathrm{T}}\ \eta^{\mathrm{T}}\ \theta^{\mathrm{T}}\ \bar{v}^{\mathrm{T}}]^{\mathrm{T}}$. Now, we consider the $(\bar{e},\eta)$-subsystem of the system of Eq. (16) with $e_z = 0$. For this system, it can be shown that the state $\bar{\eta} = [\bar{e}^{\mathrm{T}}\ \eta^{\mathrm{T}}]^{\mathrm{T}}$ possesses an ISS property with respect to $\tilde{e}_r$, and therefore, there exists a converse function, and the existence of this function implies that there exist a function $\beta_{\bar{\eta}}$ of class KL and a function $\bar{\gamma}_{\bar{\eta}}$ of class K such that the following ISS inequality holds for the state $\bar{\eta}$:

$$|\bar{\eta}(t)| \le \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\bar{\eta}}(\|\tilde{e}_r\|). \tag{19}$$

Finally, we note that the static component of the controller of Eq. (4) with $\hat{x} = x$, i.e., $a(x, \bar{v}, \phi, t)$, enforces global ultimate boundedness in the closed-loop slow system, and thus, the state ζ of the closed-loop slow system satisfies a bound of the following form $\forall t \ge 0$:

$$|\zeta(t)| \le \beta_\zeta(\delta_\zeta, t) + \bar{\gamma}_{\mathscr{U}}(\|\mathscr{U}\|), \tag{20}$$

where β_ζ is a class KL function, $\bar{\gamma}_{\mathscr{U}}(|\mathscr{U}|) = \min\{\phi/\chi, 2\phi\bar{\rho}(|\mathscr{U}|)\}$, $\bar{\rho}$ is a class K_∞ function and δ_ζ is the maximum value of the vector $[h(x)\ L_F h(x)\ \cdots\ L_F^{r-1}h(x)]$ for $|x| \le \delta_x$ (see Christofides et al., 1996 on how to compute explicitly the above bound). Based on the above bound and following the results of Khalil and Esfandiari (1993), and Teel and Praly (1995), we disregard estimates of $\tilde{y}$, obtained from the high-gain observer, with norm $|\tilde{y}| > \beta_\zeta(\delta_\zeta, 0) + d$, where $d > \bar{\gamma}_{\mathscr{U}}(\phi/\chi)$. Hence, we set $sat(\cdot) = \min\{1, \zeta_{\max}/|\cdot|\}(\cdot)$ where $\zeta_{\max}$ is the maximum value of the vector $[\zeta_1\ \zeta_2\ \cdots\ \zeta_r]$ for $|\zeta| \le \beta_\zeta(\delta_\zeta, 0) + d$.

Step 2: In order to apply the result of Lemma A.1 in the appendix, we need to define a set of positive real numbers $(\delta_{\tilde{e}_r}, \delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, \tilde{\delta}_{\bar{\eta}}, \tilde{\delta}_{\tilde{e}_r}, d_{\tilde{e}_r}, d_{\bar{\eta}})$,where $\delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}$ were specified in the statement of the theorem, $d_{\bar{\eta}}$ is an arbitrary positive real number

$$\delta_{\tilde{e}_r} \ge \frac{\max}{|x| \le \delta_x, |\bar{v}| \le \delta_{\bar{v}}} \left\{ \left| \sum_{k=1}^{r} \frac{\beta_k}{\beta_r} (v^{(k-1)} - L_{\tilde{F}_{\mathrm{nom}}}^{k-1} h(x)) \right| \right\},$$

$$\tilde{\delta}_{\tilde{e}_r} > \delta_{\tilde{e}_r} + 2\phi,$$

$$\delta_{\bar{\eta}} \ge \frac{\max}{|x| \le \delta_x, |\bar{v}| \le \delta_{\bar{v}}} \left\{ \sum_{k=1}^{r-1} |(v^{(k-1)} - L_{\tilde{F}_{\mathrm{nom}}}^{k-1} h(x))| + \sum_{v=1}^{n-r} |\chi_v(x)| \right\}$$

$\tilde{\delta}_{\bar{\eta}} > D^{\bar{\eta}} + \bar{\gamma}_{\tilde{e}_r}(\tilde{\delta}_{\tilde{e}_r})$, with $D^{\bar{\eta}} := \beta_{\bar{\eta}}(\delta_{\bar{\eta}}, 0) + d_{\bar{\eta}}$, and $d_{\tilde{e}_r} = \phi < d$ (without loss of generality).

First, consider the singularly perturbed system comprised of the states (ζ, z) of the closed-loop system. This system is in standard form, possesses a uniformly globally exponentially stable fast subsystem, and its corresponding reduced system is ISS with respect to $\mathscr{U}$. These properties allow a direct application of the result of Lemma A.1 with $\delta = \max\{\delta_\zeta, \delta_z, \delta_\theta, \delta_{\bar{v}}, \tilde{\delta}_{\bar{\eta}}, \tilde{\delta}_{\tilde{e}_r}\}$ and $d = \phi$, to obtain the existence of a positive real number $\varepsilon^\zeta(\phi)$ such that if $\bar{\varepsilon} \in (0, \varepsilon^\zeta(\phi)]$, and $|\zeta(0)| \le \delta_\zeta$, $|z(0)| \le \delta_z$, $\|\theta\| \le \delta_\theta$, $\|\bar{v}\| \le \delta_{\bar{v}}$, $\|\eta\| \le \delta_\omega$, then

$$\begin{aligned} |\zeta(t)| &\le \beta_\zeta(\delta_\zeta, t) + \bar{\gamma}_{\mathscr{U}}(\|\mathscr{U}\|) + \phi \\ &\le \beta_\zeta(\delta_\zeta, 0) + \bar{\gamma}_{\mathscr{U}}\left(\frac{\phi}{\chi}\right) + \phi. \end{aligned} \tag{21}$$

Let $\bar{\phi}$ be such that $\bar{\gamma}_{\mathscr{U}}(\bar{\phi}/\chi) + \bar{\phi} \le d$. Consider now the singularly perturbed system comprised of the states $(\tilde{e}_r, z)$ of the system of Eq. (16). This system is in standard form, possesses a uniformly globally exponentially stable fast subsystem, and its corresponding reduced system is ISS with respect to $\mathscr{U}$. These properties allow a direct application of the result of Lemma A.1 with $\delta = \max\{\delta_{\tilde{e}_r}, \delta_z, \delta_\theta, \delta_{\bar{v}}, \tilde{\delta}_{\bar{\eta}}, \tilde{\delta}_{\tilde{e}_r}\}$ and $d = \phi$, to obtain the existence of a positive real number $\varepsilon^{\tilde{e}_r}(\phi)$ such that if

$\bar{\varepsilon} \in (0, \varepsilon^{\tilde{e}_r}(\phi)]$, and $|\tilde{e}_r(0)| \leq \delta_{\tilde{e}_r}$, $|z(0)| \leq \delta_z$, $\|\theta\| \leq \delta_\theta$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, $\|\bar{\eta}\| \leq \tilde{\delta}_{\bar{\eta}}$, $\|\tilde{e}_r\| \leq \tilde{\delta}_{\tilde{e}_r}$, then

$$|\tilde{e}_r(t)| \leq \mathrm{e}^{-0.5t}|\tilde{e}_r(0)| + \tilde{\gamma}_{\mathscr{U}}(\|\mathscr{U}\|) + \phi. \tag{22}$$

Similarly, it can be shown that Lemma A.1, with $\delta = \max\{\delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, \tilde{\delta}_{\tilde{e}_r}\}$ and $d = d_{\bar{\eta}}$, can be applied to the system comprised of the states $(\bar{e}, \eta, z)$ of the system of Eq. (16), with the same converse function which exists for the $(\bar{e}, \eta)$-subsystem of the system of Eq. (16) with $e_z = 0$, and its resulting $(\beta_{\bar{\eta}}, \bar{\gamma}_{\bar{\eta}})$. Thus, we have that there exist positive real numbers $\varepsilon^{\bar{\eta}}(\phi)$ such that if $\bar{\varepsilon} \in (0, \varepsilon^{\bar{\eta}}(\phi)]$ and $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$, $|z(0)| \leq \delta_z$, $\|\theta\| \leq \delta_\theta$, $\|\dot{\theta}\| \leq \delta_{\dot{\theta}}$, $\|\bar{v}\| \leq \delta_{\bar{v}}$, $\|\tilde{e}_r\| \leq \tilde{\delta}_{\tilde{e}_r}$, then

$$|\bar{\eta}(t)| \leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\bar{\eta}}(\|\tilde{e}_r\|) + d_{\bar{\eta}}. \tag{23}$$

Part 3: Finally, the inequalities of Eqs. (18)–(23) can be manipulated using a small-gain theorem type argument similar to the one used in the proof of Theorem 1 in Christofides et al. (1996) to prove that states of the closed-loop system are bounded and its output satisfies the relation of Eq. (12). For brevity, this argument will not be repeated here. □

Acknowledgements

Financial support in part by UCLA through the SEAS Dean's Fund and the Petroleum Research Fund, administered by the ACS, is gratefully acknowledged.

Appendix

Definition A.1 (Sontag, 1989). Let γ_θ be a function of class K. The system of Eq. (3) with $u(t) = 0$ is said to be ISS with Lyapunov gain (γ_θ) if there exist a smooth function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, three functions a_1, a_2, a_3 of class K_∞, and a function $\tilde{\gamma}_\theta$ of class K such that for all $x \in \mathbb{R}^n$, all $\theta \in \mathbb{R}^q$ and all $s \geq 0$,

$$a_1(|x|) \leq V(x) \leq a_2(|x|),$$

$$|x| \geq \tilde{\gamma}_\theta(|\theta|) \Rightarrow \dot{V}(x) = \frac{\partial V}{\partial x} F(x, \theta) \leq -a_3(|x|),$$

$$\gamma_\theta(s) = a_1^{-1}(a_2(\tilde{\gamma}_\theta(s))). \tag{A.1}$$

Lemma A.1. *Consider the following singularly perturbed system*:

$$\dot{x} = \bar{f}(x, z_1, sat(z_2), \theta),$$

$$\varepsilon \dot{z}_1 = \bar{g}_1(x, z_1, sat(z_2), \theta),$$

$$\varepsilon \dot{z}_2 = A z_2 + \varepsilon \bar{g}_2(x, z_1, sat(z_2), \theta), \tag{A.2}$$

where $x \in \mathbb{R}^n$, $z_1 \in \mathbb{R}^{p_1}$, $z_2 \in \mathbb{R}^{p_2}$, $\theta \in \mathbb{R}^q$, $\bar{f}, \bar{g}_1, \bar{g}_2$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \times \mathbb{R}^q$, and A is a constant matrix. Suppose that the algebraic equation $\bar{g}(x, z_1, 0, \theta) = 0$ possesses a unique root $z_1 = h_1(x, \theta)$ and that the system

$$\dot{x} = \bar{f}(x, h_1(x, \theta), 0, \theta) \tag{A.3}$$

is ISS with Lyapunov gain γ. Also, define $y_1 = z_1 - h_1(x, \theta)$ and suppose that the fast subsystem

$$\frac{\mathrm{d}y_1}{\mathrm{d}\tau} = \bar{g}_1(x, h_1(x, \theta) + y_1, 0, \theta) \tag{A.4}$$

is globally asymptotically stable, uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Finally, suppose that A is Hurwitz and that $\theta(t)$ is absolutely continuous, and define $z_2(0) = \xi(0)/\varepsilon^{p_2}$, where $\xi \in \mathbb{R}^{p_2}$. Whenever the above assumptions are satisfied, there exist functions β_x, β_{y_1} of class KL and strictly positive constants K_1, a_1, and for each pair of positive real numbers (δ, d), there exists a positive real number ε^* such that if $\max\{|x(0)|, |y_1(0)|, |\xi(0)|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$, $\varepsilon \in (0, \varepsilon^*]$ *then, for all* $t \geq 0$,

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + d, \tag{A.5}$$

$$|y_1(t)| \leq \beta_{y_1}\left(|y_1(0)|, \frac{t}{\varepsilon}\right) + d, \tag{A.6}$$

$$|z_2(t)| \leq K_1 |z_2(0)| \mathrm{e}^{-a_1(t/\varepsilon)} + d. \tag{A.7}$$

Proof of Lemma A.1. To prove the lemma, we will initially assume that the states of the system of Eq. (A.2) are bounded and derive bounds that capture the evolution of the states for all times. Then, we will work with truncations and exploit causality to show that if these bounds hold for $t \in [0, T]$, where T is finite, then they will also hold for all $t \in [0, \infty)$, provided that ε is sufficiently small. We initially consider the system which describes the evolution of y_1 and z_2 in the fast time scale $\tau = t/\varepsilon$

$$\frac{\mathrm{d}y_1}{\mathrm{d}\tau} = \bar{g}_1(x, h_1(x, \theta) + y_1, sat(z_2), \theta) - \varepsilon\left[\frac{\partial h_1}{\partial x}\bar{f}(x, h_1(x, \theta) + y_1, sat(z_2), \theta) - \frac{\partial h_1}{\partial \theta}\dot{\theta}\right],$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}\tau} = A z_2 + \varepsilon \bar{g}_2(x, h_1(x, \theta) + y_1, sat(z_2), \theta). \tag{A.8}$$

From the assumption that the matrix A is Hurwitz, we have that the z_2-subsystem of the above system, with $\varepsilon = 0$, is globally exponentially stable. This implies that there exist positive real numbers $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and a smooth function $\bar{W}:\mathbb{R}^{p_2} \to \mathbb{R}_{\geq 0}$, such that the following conditions hold:

$$\gamma_1|z_2|^2 \leq \bar{W}(z_2) \leq \gamma_2|z_2|^2,$$
$$\frac{\partial \bar{W}}{\partial z_2}Az_2 \leq -\gamma_3|z_2|^2, \tag{A.9}$$
$$\left|\frac{\partial \bar{W}}{\partial z_2}\right| \leq \gamma_4|z_2|.$$

Computing the time derivative of $\bar{W}(z_2)$ along the trajectories of the z_2-subsystem of Eq. (A.8), we obtain

$$\frac{d\bar{W}}{dt} = \frac{\partial \bar{W}}{\partial z_2}\frac{dz_2}{dt} \leq -\frac{\gamma_3}{\varepsilon}|z_2|^2 + \gamma_4|z_2|\bar{g}_2(x, h_1(x,\theta) + y_1, sat(z_2), \theta). \tag{A.10}$$

Using that the *sat* function is globally bounded and assuming that x, y_1 and θ are essentially bounded, we have that there exists a positive real number γ_5 such that $|\bar{g}_2(x, h_1(x,\theta) + y_1, sat(z_2), \theta)| \leq \gamma_5$. So from Eq. (A.10), we have that

$$\frac{d\bar{W}}{dt} \leq -\frac{\gamma_3}{\varepsilon}|z_2|^2 + \gamma_4|z_2|\gamma_5,$$
$$\leq -\frac{\gamma_6}{\varepsilon}\bar{W}, \quad \text{for } \bar{W} \geq \varepsilon^2\gamma_7 \tag{A.11}$$

where $\gamma_6 = \gamma_3/2\gamma_1$ and $\gamma_7 = 4\gamma_4^2\gamma_5^2\gamma_1/\gamma_3^2$. The last inequality implies that there exist strictly positive constants K_1, a_1 and a function $\gamma_{\varepsilon_{z_2}}$ of class K, such that for each $\xi_0 \in \mathbb{R}^{p_2}$ satisfying $|\xi(0)| = |\xi_0| \leq \delta$, each essentially bounded inputs x, y_1, θ, and each ε satisfying $\varepsilon \in (0, \varepsilon_1]$ where $\varepsilon_1 := (K_1\delta\gamma_3/2\gamma_4\gamma_5)^{1/(p_2+1)}$, the solution of the z_2-subsystem of Eq. (A.8) with $z_2(0) = \xi(0)/\varepsilon^{p_2}$ exists for each $t \geq 0$, and satisfies

$$|z_2(t)| \leq K_1|z_2(0)|e^{-a_1(t/\varepsilon)} + \gamma_{\varepsilon_{z_2}}(\varepsilon). \tag{A.12}$$

We now study the system that describes the evolution of y_1

$$\frac{dy_1}{d\tau} = \bar{g}_1(x, h_1(x,\theta) + y_1, sat(z_2), \theta) - \varepsilon\left[\frac{\partial h_1}{\partial x}\bar{f}(x, h_1(x,\theta) + y_1, sat(z_2), \theta) - \frac{\partial h_1}{\partial \theta}\dot{\theta}\right]. \tag{A.13}$$

From the assumption that the equilibrium $y_1 = 0$ of the above system with $\varepsilon = 0$ and $z_2(t) \equiv 0$ is globally asymptotically stable uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, we have from Lemma 1 in Christofides and Teel (1996) that there exists a $p_1 \times p_1$ matrix $\tilde{B}_1(y_1)$ of smooth functions, invertible for all $y_1 \in \mathbb{R}^{p_1}$, and a function $\gamma_{\tilde{z}_2}$ of class K such that the system

$$\frac{dy_1}{d\tau} = \bar{g}_1(x, h_1(x,\theta) + y_1, \tilde{B}_1(y_1)\tilde{z}_2, \theta) \tag{A.14}$$

is ISS with respect to $\tilde{z}_2$ with Lyapunov gain $\gamma_{\tilde{z}_2}$. Now, using Lemma 2 in Christofides and Teel (1996), we have that there exists a function β_{y_1} of class KL, a continuous nonincreasing function $\bar{\sigma}_{y_1}:\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of the form $\bar{\sigma}_{y_1}(s) = (\frac{1}{2}(l + \Phi(s)))^{-2}$, where $l \geq 1$ and $\Phi(s)$ is a function of class K_∞, and a function $\bar{\gamma}_{\varepsilon_{y_1}}$ of class K, such that for each $y_{1_0} \in \mathbb{R}^p$, each triple of essentially bounded inputs $x, \theta, \dot{\theta}$, and each ε satisfying $\varepsilon \leq \bar{\sigma}_{y_1}(\max\{|y_1(0)|, \|\tilde{z}_2\|, \|\theta\|, \|\dot{\theta}\|, \|x\|\})$, the solution of the system:

$$\frac{dy_1}{d\tau} = \bar{g}_1(x, h_1(x,\theta) + y_1, \tilde{B}_1(y_1)\tilde{z}_2, \theta) - \varepsilon\left[\frac{\partial h_1}{\partial x}\bar{f}(x, h_1(x,\theta) + y_1, \tilde{B}_1(y_1)\tilde{z}_2, \theta) - \frac{\partial h_1}{\partial \theta}\dot{\theta}\right] \tag{A.15}$$

with $y_1(0) = y_{1_0}$ exists for each $t \geq 0$, and satisfies:

$$|y_1(t)| \leq \beta_{y_1}\left(|y_1(0)|, \frac{t}{\varepsilon}\right) + \gamma_{\tilde{z}_2}(\|\tilde{z}_2\|) + \bar{\gamma}_{\varepsilon_{y_1}}(\varepsilon). \tag{A.16}$$

We now return to the system that describes the evolution of the state x

$$\dot{x} = f(x, h_1(x,\theta) + y_1, sat(z_2), \theta). \tag{A.17}$$

Using that the above system with $y_1(t) \equiv 0$ and $z_2(t) \equiv 0$ is ISS with Lyapunov gain γ and Lemma 1 in Christofides and Teel, 1996, we have that there exists a $p_1 \times p_1$ matrix $\tilde{B}_2(x,\theta)$ of smooth functions, invertible for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and a function $\gamma_{\tilde{y}_1}$ of class K such that the system

$$\dot{x} = f(x, h_1(x,\theta) + \tilde{B}_2(x,\theta)\tilde{y}_1, 0, \theta) \tag{A.18}$$

is ISS with respect to $\theta, \tilde{y}_1$ with Lyapunov gain $(\gamma, \gamma_{\tilde{y}_1})$. Applying Lemma 1 in Christofides and Teel (1996) again, we have that there exists a $p_2 \times p_2$ matrix $\tilde{B}_3(x, \theta, \tilde{y}_1)$ of smooth functions, invertible for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, $\tilde{y}_1 \in \mathbb{R}^{p_1}$, and a function $\gamma_{\tilde{z}_2}$ of class K such that the solution of the system

$$\dot{x} = f(x, h_1(x,\theta) + \tilde{B}_2(x,\theta)\tilde{y}_1, \tilde{B}_3(x,\theta,\tilde{y}_1)\tilde{z}_2, \theta) \tag{A.19}$$

with $x(0) = x_0$ exists for each $t \geq 0$, and satisfies

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + \gamma_{\tilde{y}_1}(\|\tilde{y}_1\|) + \gamma_{\tilde{z}_2}(\|\tilde{z}_2\|). \tag{A.20}$$

The proof of the theorem can now be completed working with truncations of the bounds of Eqs. (A.12)–(A.20) (since x, y_1, z_2 are not a priori bounded) and using a contradiction argument similar to the one used in the proof of Theorem 1 in Christofides and Teel (1996). In particular, let (δ, d) be as given in the theorem, so that $\max\{|x(0)|, |y_1(0)|, |\xi(0)|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$, and $\delta_x > \beta_x(\delta, 0) + \gamma(\delta) + d$ and $\delta_{y_1} > \beta_{y_1}(\delta, 0) + d$. Note that since $\beta_x(s,0) \geq s$, for all $s \in \mathbb{R}$, $\delta_x > \delta$ and, using continuity with respect to the initial conditions, we can define $[0, T)$ with $T > 0$ to be the maximal interval in which $\|x_t\| < \delta_x$ and $\|y_{1_t}\| < \delta_{y_1}$ for all $t \in [0, T)$. To show by contradiction that $T = \infty$, for ε sufficiently small, suppose that T is finite.

Using the fact that $\bar{\sigma}_{y_1}$ is a continuous, nonincreasing function, it follows from the definition of T and causality that if $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$ where $\varepsilon_2 := \bar{\sigma}_{y_1}(\delta_x)$, then Eqs. (A.12)–(A.16) hold for all $t \in [0,T)$. Then, since $\tilde{z}_2 = \tilde{B}_1^{-1}(y_1)sat(z_2)$, $\tilde{y}_1 = \tilde{B}_2^{-1}(x,\theta)y_1$, $z_2^- = \tilde{B}_3^{-1}(x,\theta,\tilde{y}_1)sat(z_2)$, $\tilde{B}_1(y_1) = b_1(|y_1|)I_{p_1 \times p_1}$, and $\tilde{B}_2(x,\theta) = b_2(|X|)I_{p_1 \times p_1}$ where $X(t) = [x^T(t)\ \theta^T(t)]^T$, and $\tilde{B}_3(x,\theta,\tilde{y}_1) = b_3(|\bar{X}|)I_{p_2 \times p_2}$ where $\bar{X}(t) = [x^T(t)\ \theta^T(t)\ \tilde{y}_1^T]^T$ (see the proof of Lemma 1 in Christofides and Teel (1996)), we have for all $t \in [0, T)$:

$$\|\tilde{z}_{2_t}\| \le \frac{1}{b_1(\|y_{1_t}\|)}a_m, \qquad \|\bar{z}_{2_t}\| \le \frac{1}{b_3(\|\bar{X}\|_t\|)}a_m, \tag{A.21}$$

where a_m is the magnitude of the saturation function. Define $\delta_{\tilde{z}_2} := (1/b_1(\delta_{y_1}))a_m$, $\delta_{\bar{z}_2} := (1/b_3(\delta_x + \delta + \delta_{\tilde{y}_1}))a_m$, $\delta_{\tilde{y}_1} := (1/b_2(\delta_x + \delta))[\beta_{y_1}(\delta, 0) + d]$ and note that since $\|sat(z_{2_t})\| \le a_m$ and $0 < b_i(s) \le 1$ for all $s \in R$ (see the proof of Lemma 1 in Christofides and Teel (1996)), it follows that $\delta \le \delta_{\tilde{z}_2}$ and $\|\tilde{z}_{2_t}\| \le \delta_{\tilde{z}_2}$ and $\|\bar{z}_{2_t}\| \le \delta_{\bar{z}_2}$ for all $t \in [0, T)$.

Now from the conditions under which Eq. (A.16) holds and using that $\bar{\sigma}_1$ is nonincreasing, it follows from Lemma 3 in Christofides and Teel (1996) that there exists a positive real number $\rho_1 < T$ such that, if $\varepsilon \le \varepsilon_3 := \bar{\sigma}_1(\max\{\delta_{\tilde{z}_2}, \delta_x\} + \frac{d}{2}) \le \bar{\sigma}_1(\max\{|y_1(0)|, \|\tilde{z}_2\|, \|\theta\|, \|\dot{\theta}\|, \|x\|\})$, $\forall t \in [0, T)$, then the solution of Eq. (A.16) with $y_1(0) = y_{1_0}$ exists for all $t \in [0, T)$ and satisfies

$$|y_1(t)| \le \beta_{y_1}\left(|y_1(0)|, \frac{t}{\varepsilon}\right) + \gamma_{\tilde{z}_2}(\|\tilde{z}_2^{\rho_1}\|) + \bar{\gamma}_{\varepsilon_{y_1}}(\varepsilon) + \frac{d}{2}. \tag{A.22}$$

Combining Eqs. (A.21) and (A.22), if $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, then for all $t \in [0,T)$

$$|y_1(t)| \le \beta_{y_1}\left(|y_1(0)|, \frac{t}{\varepsilon}\right) + \frac{d}{2} + \gamma_{\tilde{z}_2}\left(\frac{1}{b_1(\delta_{y_1})} sat(K_1|z_2(0)|e^{-a_1(\rho_1/\varepsilon)} + \gamma_{\varepsilon_{z_2}}(\varepsilon))\right) + \bar{\gamma}_{\varepsilon_{y_1}}(\varepsilon). \tag{A.23}$$

Since the last two terms converge to zero as ε goes to zero, there exists an $\varepsilon_4 > 0$ such that if $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ then, for all $t \in [0,T)$, $|y_1(t)| \le \beta_{y_1}(|y_1(0)|, t/\varepsilon) + d < \delta_{y_1}$. Now from the conditions under which Eq. (A.20) holds it follows from Lemma 3 in Christofides and Teel (1996) that for $\bar{\delta} = \max\{\delta, \delta_{\tilde{z}_2}, \delta_{\tilde{y}_1}, \delta_{\bar{z}_2}\}$ there exists a positive real number $\rho_2 < T$ such that, if $\max\{|x(0)|, \|\theta\|, \|\tilde{y}_1\|, \|\bar{z}_2\| \le \bar{\delta}$, then the solution of the system of Eq. (A.19) with $x(0) = x_0$ exists for each $t \ge 0$, and satisfies

$$|x(t)| \le \beta_x(|x(0)|, t) + \gamma(\|\theta\|) + \gamma_{\tilde{y}_1}(\|\tilde{y}_1^{\rho_2}\|) + \gamma_{\bar{z}_2}(\|\bar{z}_2^{\rho_2}\|). \tag{A.24}$$

Developing the explicit expressions for $\|\tilde{y}_1^{\rho_2}\|$ and $\|\bar{z}_2^{\rho_2}\|$ and substituting them into Eq. (A.24), one can show that there exists an $\varepsilon_4 > 0$ such that if $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ then, for all $t \in [0,T)$, $|x(t)| \le \delta_x$. From the definition of δ_x, δ_{y_1}, the assumption that T is finite and continuity of x, there must exist some positive real number k such that $\|x_t\| < \delta_x$ and $\|y_{1_t}\| < \delta_{y_1}$, for all $t \in [0, T + k)$. This contradicts that T is maximal. Hence, $T = \infty$ and the inequalities of Eqs. (A.5)–(A.6) hold for all $t \in [0, \infty)$. Finally, letting ε_5 be such that $\gamma_{\varepsilon_{z_2}}(\varepsilon) \le d$ for all $\varepsilon \in [0, \varepsilon_5]$ it follows that the inequalities of Eqs. (A.5)–(A.7) hold for $\varepsilon \in (0, \varepsilon^*]$ where $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$. □

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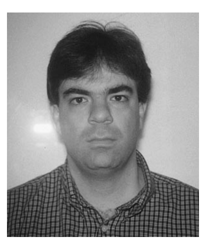

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