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Robust control of parabolic PDE systems

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Abstract—This article proposes a methodology for the synthesis of nonlinear robust feedback controllers for diffusion–convection–reaction processes with time-varying uncertain variables described by systems of quasi-linear parabolic partial differential equations (PDEs), for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional (possibly unstable) slow one and an infinite-dimensional stable fast complement. Combination of Galerkin's method with approximate inertial manifolds is used to derive ordinary differential equation (ODE) systems of dimension equal to the number of slow modes that accurately describe the dominant dynamics of the PDE system. These ODE systems are used for the synthesis of robust controllers that guarantee boundedness of the state and output tracking with arbitrary degree of asymptotic attenuation of the effect of the uncertain variables on the output of the closed-loop system. The robust controllers are synthesized via Lyapunov's direct method and utilize bounding functions on the magnitude of the uncertain terms. The developed methodology is successfully applied to a catalytic packed-bed reactor with unknown heat of reaction.

Keywords: Galerkin's method, approximate inertial manifolds, robust nonlinear control, diffusion–convection–reaction processes.

1. INTRODUCTION

Many chemical engineering processes are inherently nonlinear and are characterized by the presence of strong spatial variations due to the coupling of diffusive and convective mechanisms. Representative examples of industrially important diffusion–convection–reaction processes include fluidized-bed and packed-bed reactors (Ray, 1981, Georgakis *et al.*, 1977, Christofides and Daoutidis, 1996b) and chemical vapor deposition reactors (Economou *et al.*, 1989; Badgwell *et al.*, 1995). The mathematical models which describe the spatiotemporal behavior of these processes are typically obtained from the dynamic conservation equations and consist of systems of quasi-linear *parabolic* partial differential equations (PDEs).

The main feature of parabolic PDE systems is that the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement. This implies that the dynamic behavior of such systems can be approximately described by ordinary differential equation (ODE) systems. Therefore, the standard approach to the control of linear/quasi-linear parabolic PDE systems (see, for example, Georgakis *et al.*, 1977; Balas, 1979; Ray, 1981; Chen and Chang, 1992) involves the application of Galerkin's method to the PDE system to derive ODE systems that accurately describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for controller synthesis. The main advantage of this approach is that the exponential stability of the fast eigenmodes ensures that a controller which exponentially stabilizes the closed-loop ODE system, stabilizes also the closed-loop parabolic PDE system. On the other hand, the main disadvantage of this approach is that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large (Chakravarti *et al.*, 1995; Bangia *et al.*, 1997; Aling *et al.*, 1997), leading to complex controller design and high dimensionality of the resulting controllers.

Motivated by this, recent research efforts on control of parabolic PDE systems have focused on the problem of synthesizing low-dimensional output feedback controllers (Gay and Ray, 1995; Christofides and Daoutidis, 1997a; Sano and Kunimatsu, 1995). In Gay and Ray (1995), a method was proposed to address this problem for linear parabolic PDEs, that uses the singular functions of the differential operator instead of the eigenfunctions in the series expansion of the solution. For nonlinear parabolic PDE systems, a natural approach to address this problem is based on the concept of inertial manifold (IM) (see Temam, 1988 and the references therein). An IM is a positively invariant, finite-dimensional Lipschitz manifold, which attracts every trajectory exponentially. If an IM exists, the dynamics of the parabolic PDE system restricted on the inertial manifold is described by a set of ODEs called the inertial form. However, the explicit derivation of the inertial form requires the computation of the closed-form expression of the IM, which is a very difficult task in most practical applications. In order to overcome this problem, a novel procedure, based on singular perturbations, was proposed in Christofides and Daoutidis (1997a) for the construction of approximations of the inertial manifold [called approximate inertial manifolds (AIMs)]. The aims were to derive ODE systems of dimension equal to the number of slow modes, that yield solutions which are close, up to a desired accuracy, to the ones of the PDE system (see also Foias *et al.*, 1989a,b for alternative approaches for the construction of AIMs). These ODE systems were used as the basis for the synthesis of nonlinear output feedback controllers that guarantee stability and enforce the output of the closed-loop system to follow, up to a desired accuracy, a prespecified response. The concept of IM was also used in Sano and Kunimatsu (1995) to address the problem of stabilization of a parabolic PDE with boundary finite-dimensional feedback control.

A practically important controller design problem for diffusion–convection–reaction processes is the one of synthesizing nonlinear robust controllers that compensate for the effect of time-varying unknown process parameters and external disturbances on the process output. Despite its practical importance, the problem of synthesizing robust controllers for parabolic PDE systems has been only studied for linear systems. Specifically, the problem of robust stabilization was initially studied in Curtain and Glover (1986) and Gauthier and Xu (1989) for linear parabolic PDE systems in the frequency domain and H^{∞} control methods were developed. In Jacobson and Nett (1988), precise relations were derived between state-space and frequency-domain control theoretic concepts for a large class of linear infinite-dimensional systems, which led to the development of the state-space counterparts of the frequency-domain H^{∞} results (e.g., Keulen, 1993). In the area of robust control of quasi-linear PDE systems, Lyapunov-based controller design methods were recently developed for hyperbolic PDE systems (i.e. convection–reaction processes) in Alonso and Ydstie (1995) and Christofides and Daoutidis (1998). Despite the recent progress, there are no available methods for the synthesis of robust controllers for quasi-linear parabolic PDE systems that compensate for the effect of uncertain variables on the process output.

In this paper, we study the robust control problem for diffusion–convection–reaction processes with time-varying uncertain variables described by systems of quasi-linear parabolic PDEs, for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional (possibly unstable) slow one and an infinite-dimensional stable fast complement. The objective is to develop a general and practical methodology for the synthesis of nonlinear robust feedback controllers that guarantee boundedness of the state and output tracking with arbitrary degree of asymptotic attenuation of the effect of the uncertain variables on the output of the closed-loop system.

Specifically, the paper is structured as follows. Initially, Galerkin's method is used to derive an ODE system of dimension equal to the number of slow modes, which is subsequently used to synthesize robust controllers via Lyapunov's direct method. Singular perturbation methods are employed to establish that the degree of asymptotic attenuation of the effect of uncertain variables on the output, enforced by these controllers, is proportional to the degree of separation of the fast and slow modes of the spatial differential operator. Then, for processes for which such a degree of uncertainty attenuation is not sufficient, a sequential procedure, based on the concept of approximate inertial manifold, is developed for the synthesis of robust controllers which achieve *arbitrary* degree of asymptotic uncertainty attenuation in the closed-loop parabolic PDE system. The developed methodology is successfully applied to a catalytic packed-bed reactor with unknown heat of reaction and is shown to significantly outperform a nonlinear controller design which does not account for the presence of the uncertain variable.

2. PRELIMINARIES

2.1. *Description of parabolic PDEs with uncertain variables*

In this work, we consider quasi-linear parabolic PDE systems with uncertain variables, with a state-space description of the form

$$\frac{\partial \bar{x}}{\partial t} = A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2} + wb(z)u + f(\bar{x}) + W(\bar{x}, r(z)\theta(t))$$
$$y^i = \int_{\alpha}^{\beta} c^i(z)\,k\bar{x}dz, \quad i = 1, \ldots, l \tag{1}$$

subject to the boundary conditions:

$$C_1\bar{x}(\alpha, t) + D_1\frac{\partial \bar{x}}{\partial z}(\alpha, t) = R_1$$
$$C_2\bar{x}(\beta, t) + D_2\frac{\partial \bar{x}}{\partial z}(\beta, t) = R_2 \tag{2}$$

and the initial condition:

$$\bar{x}(z, 0) = \bar{x}_0(z) \tag{3}$$

where $\bar{x}(z, t) = [\bar{x}_1(z, t) \cdots \bar{x}_n(z, t)]^{\mathrm{T}}$ denotes the vector of state variables, $[\alpha, \beta] \subset \mathbb{R}$ is the domain of definition of the process, $z \in [\alpha, \beta)$ is the spatial coordinate, $t \in [0, \infty)$ is the time, $u = [u^1\ u^2 \cdots u^l]^{\mathrm{T}} \in \mathbb{R}^l$ denotes the vector of manipulated inputs, $\theta(t) = [\theta_1(t)\ \theta_2(t) \cdots \theta_q(t)] \in \mathbb{R}^q$ denotes the vector of uncertain variables, which may include uncertain process parameters or exogenous disturbances, and $y^i \in \mathbb{R}$ denotes the controlled output. $\partial\bar{x}/\partial z$, $\partial^2\bar{x}/\partial z^2$ denote the first- and second-order spatial derivatives of $\bar{x}$, $f(\bar{x})$,

$W(\bar{x}, r(z)\ \theta(t))$ are vector functions, w, k, R_1, R_2 are constant vectors, A, B, C_1, D_1, C_2, D_2 are constant matrices, and $\bar{x}_0(z)$ is the initial condition.

$b(z)$ is a known smooth vector function of z of the form $b(z) = [b^1(z)\ b^2(z) \cdots b^l(z)]$, where $b^i(z)$ describes how the control action $u^i(t)$ is distributed in the interval $[z_i, z_{i+1}] \subset [\alpha, \beta]$, $r(z) = [r_1(z) \cdots r_q(z)]$, where $r_k(z)$ is a known smooth function of z which specifies the position of action of the uncertain variable θ_k on $[\alpha, \beta]$, and $c^i(z)$ is a known smooth function of z which is determined by the specification of the ith controlled output in the interval $[\alpha, \beta]$. Whenever the control action enters the system at a single point z_0, with $z_0 \in [z_i, z_{i+1}]$ (i.e. point actuation), the function $b^i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_0 - \varepsilon, z_0 + \varepsilon]$, where ε is a small positive real number, and zero elsewhere in $[z_i, z_{i+1}]$. Figure 1 shows the location of the manipulated inputs and controlled outputs in the case of a prototype example. Throughout the paper, we will use the order of magnitude notation $O(\varepsilon)$. In particular, $\delta(\varepsilon) = O(\varepsilon)$ if there exist positive real numbers k_1 and k_2 such that: $|\delta(\varepsilon)| \leqslant k_1|\varepsilon|, \forall |\varepsilon| < k_2$.

Systems of the form of eq. (1) describe the majority of diffusion–convection–reaction processes for which the diffusion coefficient and the thermal conductivity are independent of temperature and concentrations, and thus, the corresponding spatial differential operators are linear. They are characterized by linear appearances of the manipulated inputs and the controlled outputs. These features are common in most practical applications where the wall temperature is usually chosen to be the manipulated input [the reader may refer to (Christofides and Daoutidis, 1996a) for a complete discussion on how the wall temperature can be indirectly manipulated in practice through manipulation of the jacket inlet flow rate], while the controlled outputs are usually some of the state variables of the process (e.g. temperature, concentrations). Moreover, the nonlinearities appear in an additive fashion, which is also typical in many chemical process applications (e.g. complex reaction rates, Arrhenius dependence of reaction rates on temperature). Finally, the nonlinear appearance of the vector of uncertain variables in eq. (1) allows accounting for all possible unknown variables, e.g. heat of reactions, pre-exponential constants, activation energies, temperature and concentration of lateral inlet streams, etc.

The following example illustrates modeling of a diffusion–reaction process in the form of eq. (1).

Illustrative example (Ray, 1981): Consider a long, thin rod, being heated in a furnace. The furnace is filled with species A and a catalytic reaction of the form $A \to B$ takes place on the rod. Under the assumptions of uniform reaction rate on the rod, constant density and heat capacity of the rod and excess of species A on the furnace, the mathematical model which describes the spatiotemporal evolution of the rod temperature consists of the following quasi-linear parabolic PDE:

$$\rho_r c_{pr} \frac{\partial T_r}{\partial \bar{t}} = k_t \frac{\partial^2 T_r}{\partial \bar{z}^2} + (-\Delta H_r)\, k_0 \mathrm{e}^{-E/RT_r} + q_h(\bar{t})$$

subject to the non-flux boundary conditions:

$$z = 0, \frac{\partial T_r}{\partial \bar{z}} = 0;\ z = l, \frac{\partial T_r}{\partial \bar{z}} = 0$$

and the initial condition:

$$T_r(z, 0) = T_0.$$

In the above model, T_r denotes the temperature of the reactor, ΔH_r denotes the enthalpy of the reaction, ρ_r, c_{pr}, k_t denote the density, heat capacity and thermal conductivity of the rod, k_0 E, denote the pre-exponential constant, and the activation energy of the reaction, l is the length of the rod and $q_h(\bar{t})$ denotes the heating rate which is assumed to be spatially uniform. The control objective is the regulation of the temperature profile in the rod through manipulation of the heating rate q_h, in the presence of time-varying uncertainty in the enthalpy of the reaction ΔH_r. For the control of the process, we assume that there is available one control actuator, with distribution function $b(z) = 1$.

Defining the dimensionless variables:

$$t = \frac{\bar{t} k_t}{\rho_r c_{pr} l^2}, \quad z = \frac{\bar{z}}{l}, \quad \bar{x}(z, t) = \frac{T_r}{T_0}, \quad \gamma = \frac{E}{RT_0},$$

$$\beta_{T0} = \frac{(-\Delta H_{r0})\, k_0 l^2 \mathrm{e}^{-\gamma}}{T_0}, \quad \theta(t) = \beta_{\mathrm{T}} - \beta_{T0} \tag{4}$$

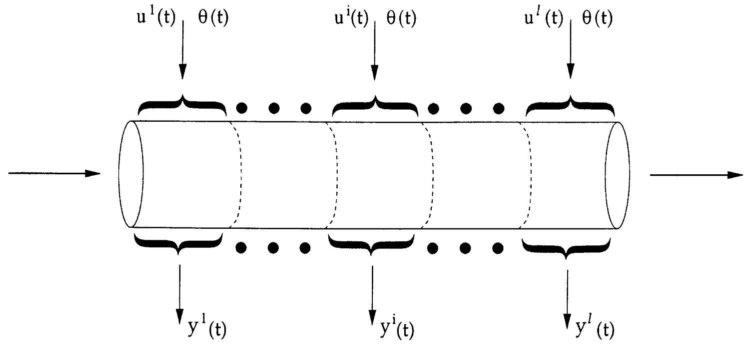

Fig. 1. Specification of the control problem in a prototype example.

and the manipulated input and controlled output as

$$u(t) = \frac{q_h l^2}{kT_0}, \quad y(t) = \int_0^1 c(z)\bar{x}(z,t)\,\mathrm{d}z \tag{5}$$

the original set of equations can be put in the form of eqs. (1)–(3):

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_{T0} \mathrm{e}^{\gamma \bar{x}/(1+x)} + u(t) + \mathrm{e}^{\gamma \bar{x}/(1+x)}\theta(t)$$

$$y(t) = \int_0^1 c(z)\bar{x}(z,t)\,\mathrm{d}z \tag{6}$$

$$z=0,\ \frac{\partial \bar{x}}{\partial z} = 0; \quad z=1,\ \frac{\partial \bar{x}}{\partial z} = 0$$

$$\bar{x}(z,0) = 1$$

2.2. *Formulation of the parabolic PDE system as an infinite-dimensional system — eigenvalue problem*

In this subsection, we precisely characterize the class of parabolic PDE systems of the form of eq. (1) which we consider in the manuscript. To this end, we formulate the parabolic PDE system of eq. (1) as an infinite-dimensional system in the Hilbert space $\mathscr{H}([\alpha,\beta],\mathbb{R}^n)$ [this will also simplify significantly the notation of the paper, since the boundary conditions of eq. (2) will be directly included in the formulation; see eq. (10) below], with $\mathscr{H}$ being the space of n-dimensional vector functions defined on $[\alpha,\beta]$ that satisfy the boundary condition of eq. (2), with inner product and norm:

$$(\omega_1,\omega_2) = \int_\alpha^\beta (\omega_1(z),\omega_2(z))_{\mathbb{R}^n}\,\mathrm{d}z$$
$$\|\omega_1\|_2 = (\omega_1,\omega_1)^{1/2} \tag{7}$$

where ω_1,ω_2 are two elements of $\mathscr{H}([\alpha,\beta];\mathbb{R}^n)$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function x on $\mathscr{H}([\alpha,\beta],\mathbb{R}^n)$ as

$$x(t) = \bar{x}(z,t), \quad t>0, \quad z\in[\alpha,\beta], \tag{8}$$

and the operators as

$$\mathscr{A}x = A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2}, \quad \mathscr{B}u = wbu,$$

$$\mathscr{W}(x,\theta) = W(\bar{x}, r\theta), \quad \mathscr{C}x = (c,kx) \tag{9}$$

$$x \in D(\mathscr{A}) = \left\{ x\in\mathscr{H}([\alpha,\beta];\mathbb{R}^n);\quad C_1 x(\alpha) + D_1\frac{\partial x}{\partial z}(\alpha) = R_1,\quad C_2 x(\beta) + D_2\frac{\partial x}{\partial z}(\beta) = R_2 \right\}$$

where $c = [c^1\ c^2 \cdots c^l]$, the system of eqs. (1)–(3) takes the form

$$\dot{x} = \mathscr{A}x + \mathscr{B}u + f(x) + \mathscr{W}(x,\theta), \quad x(0)=x_0$$
$$y = \mathscr{C}x \tag{10}$$

where $f(x(t)) = f(\bar{x}(z,t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear terms $f(x)$, $\mathscr{W}(x,\theta)$ are locally Lipschitz with respect to their arguments and satisfy $f(0)=0$, $\mathscr{W}(0,0)=0$.

For $\mathscr{A}$, the eigenvalue problem is defined as:

$$\mathscr{A}\phi_j = \lambda_j\phi_j, \quad j=1,\ldots,\infty \tag{11}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction; the eigenspectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, is defined as the set of all eigenvalues of $\mathscr{A}$, i.e. $\sigma(\mathscr{A}) = \{\lambda_1,\lambda_2,\ldots,\}$. Assumption 1 that follows states that the eigenspectrum of $\mathscr{A}$ can be partitioned into a finite-dimensional part consisting of m slow eigenvalues and a stable infinite-dimensional complement containing the remaining fast eigenvalues, and that the separation between slow and fast eigenvalues of $\mathscr{A}$ is large.

Assumption 1 (Christofides and Daoutidis, 1997a):

1. $\mathrm{Re}\{\lambda_1\} \geqslant \mathrm{Re}\{\lambda_2\} \geqslant \cdots \geqslant \mathrm{Re}\{\lambda_j\} \geqslant \cdots$, *where* $\mathrm{Re}\{\lambda_j\}$ *denotes the real part of* λ_j.
2. $\sigma(\mathscr{A})$ *can be partitioned as* $\sigma(\mathscr{A}) = \sigma_1(\mathscr{A}) + \sigma_2(\mathscr{A})$, *where* $\sigma_1(\mathscr{A})$ *consists of the first* m (*with* m *finite*) *eigenvalues*, i.e. $\sigma_1(\mathscr{A}) = \{\lambda_1,\ldots,\lambda_m\}$, *and* $|\mathrm{Re}\{\lambda_1\}|/|\mathrm{Re}\{\lambda_m\}| = \mathrm{O}(1)$.
3. $\mathrm{Re}\,\lambda_{m+1} < 0$ *and* $|\mathrm{Re}\{\lambda_m\}|/|\mathrm{Re}\{\lambda_{m+1}\}| = \mathrm{O}(\varepsilon)$ *where* $\varepsilon := |\mathrm{Re}\,\lambda_1|/|\mathrm{Re}\,\lambda_{m+1}| < 1$ *is a small positive number.*

The assumption of finite number of unstable eigenvalues is always satisfied for parabolic PDE systems (Friedman, 1976), while the assumption of discrete eigenspectrum and the assumption of existence of only a few dominant modes that describe the dynamics of the parabolic PDE system are usually satisfied by the majority of diffusion-convection-reaction processes (see the heating rod example below and the catalytic packed-bed reactor example at section 4).

Whenever assumption 1 holds $\mathscr{A}$ generates a strongly continuous semigroup of bounded linear operators $U(t)$ (this is the analogue of the concept of state transition matrix used for ODE systems in the case of infinite-dimensional systems) which implies that the generalized solution of the system of eq. (10) is given by (Friedman, 1976):

$$x(t) = U(t)x_0 + \int_0^t U(t-s)\,(\mathscr{B}u(s) + f(x(s)) + \mathscr{W}(x(s),\theta(s)))\,\mathrm{d}s \tag{12}$$

$U(t)$ satisfies the following growth property:

$$\|U(t)\|_2 \leqslant K_1 \mathrm{e}^{a_1 t}, \quad \forall\ t\geqslant 0 \tag{13}$$

where K_1, a_1 are positive real numbers, with $K_1 \geqslant 1$ and $a_1 \geqslant \mathrm{Re}\,\lambda_1$. If a_1 is strictly negative, we will say that $\mathscr{A}$ generates an exponentially stable semigroup $U(t)$.

Illustrative example (cont'd): For the heating rod example, the Hilbert space is $\mathscr{H}([0,1],\mathbb{R})$, the state function on $\mathscr{H}([0,1],\mathbb{R})$ is defined as $x(t)=\bar{x}(z,t)$, and the operator $\mathscr{A}$ takes the form

$$\mathscr{A}x=\frac{\partial^2\bar{x}}{\partial z^2}$$
$$x\in D(\mathscr{A})=\left\{x\in\mathscr{H}([0,1];\mathbb{R});\ z=0,\ \frac{\partial x}{\partial z}=0;\ z=1,\ \frac{\partial x}{\partial z}=0\right\}. \tag{14}$$

Furthermore, the manipulated input, uncertain variable and controlled output and measured output operators as

$$\mathscr{B}u=u,\ \mathscr{W}(x,\theta)=\mathrm{e}^{\gamma x/(1+x)}\theta(t),\ \mathscr{C}x=(c,x) \tag{15}$$

and the nonlinear term $f(x)$ as

$$f(x)=e^{\gamma\bar{x}/(1+\bar{x})}\beta_{T0} \tag{16}$$

The eigenvalue problem for $\mathscr{A}$ can be solved analytically and the solution is of the form

$$\lambda_j=-j^2\pi^2,\ j=0,\ldots,\infty$$
$$\phi_0=1,\ \phi_j=\sqrt{2}\cos(j\,\pi\,z),\ j=1,\ldots,\infty \tag{17}$$

From the values of λ_j, it is clear that the first eigenvalue ($\lambda_0=0$) is the dominant one, which implies that assumption 1 is satisfied for this example.

3. ROBUST CONTROL OF PARABOLIC PDE SYSTEMS

3.1. *Problem formulation*

The objective of this section is to synthesize nonlinear robust controllers for the infinite-dimensional system of eq. (19) on the basis of appropriate finite-dimensional systems, which enforce the following properties in the closed-loop system: (a) boundedness of the state, (b) output tracking for changes in the reference input, and (c) asymptotic attenuation of the effect of uncertain variables on the output.

To develop a solution to the above problem, we will initially transform the system of eq. (10) into an equivalent set of infinite ordinary differential equations. Letting $\mathscr{H}_s$, $\mathscr{H}_f$ be two subspaces of $\mathscr{H}$, defined as $\mathscr{H}_s=\operatorname{span}\{\phi_1,\phi_2,\ldots,\phi_m\}$ and $\mathscr{H}_f=\operatorname{span}\{\phi_{m+1},\phi_{m+2},\ldots,\}$, and defining the orthogonal projection operators P_s and P_f such that $x_s=P_sx$, $x_f=P_fx$, the state x of the system of eq. (10) can be decomposed as

$$x=x_s+x_f=P_sx+P_fx=\sum_{j=1}^{m}\alpha_j(t)\phi_j+\sum_{j=m+1}^{\infty}\alpha_j(t)\phi_j \tag{18}$$

where $\alpha_j(t)$ is the amplitude of the jth mode. Applying P_s and P_f to the system of eq. (10) and using the above decomposition for x, the system of eq. (10) can be equivalently written in the following form:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t}=\mathscr{A}_sx_s+\mathscr{B}_su+f_s(x_s,x_f)+\mathscr{W}_s(x_s,x_f,\theta)$$
$$\frac{\partial x_f}{\partial t}=\mathscr{A}_fx_f+\mathscr{B}_fu+f_f(x_s,x_f)+\mathscr{W}_f(x_s,x_f,\theta) \tag{19}$$
$$y=\mathscr{C}x_s+\mathscr{C}x_f$$
$$x_s(0)=P_sx(0)=P_sx_0,\ x_f(0)=P_fx(0)=P_fx_0$$

where $\mathscr{A}_s=P_s\mathscr{A}P_s$, $\mathscr{B}_s=P_s\mathscr{B}$, $f_s=P_sf$, $\mathscr{W}_s=P_s\mathscr{W}$, $\mathscr{A}_f=P_f\mathscr{A}P_f$, $\mathscr{B}_f=P_f\mathscr{B}$, $f_f=P_ff$, $\mathscr{W}_f=P_f\mathscr{W}$, and the notation $\partial x_f/\partial t$ is used to denote that the state x_f belongs in an infinite-dimensional space. In the above system, $\mathscr{A}_s$ is a diagonal matrix of dimension $m\times m$ of the form $\mathscr{A}_s=\operatorname{diag}\{\lambda_j\}$, $f_s(x_s,x_f)$, $f_f(x_s,x_f)$ $\mathscr{W}_s(x_s,x_f,\theta)$, $\mathscr{W}_f(x_s,x_f,\theta)$ are Lipschitz vector functions, and $\mathscr{A}_f$ is an unbounded differential operator which generates a strongly continuous exponentially stable semigroup (following from part 3 of assumption 1 and the selection of $\mathscr{H}_s$ and $\mathscr{H}_f$). In the remainder of the paper, we will assume, in order to simplify the presentation of our results, that the performance specification functions $c^i(z)$ are chosen such that $\mathscr{C}x_f\equiv 0$.

In order to derive precise conditions that guarantee that the proposed controllers enforce the desired properties in the infinite-dimensional closed-loop system and precisely characterize the degree of asymptotic attenuation of the uncertain variables on the output, we will formulate the system of eq. (19) within the framework of singular perturbations. Such a formulation is motivated by the fact that the system of eq. (19) exhibits two-time-scale behavior (which is a consequence of part 3 of assumption 1). Using that $\varepsilon=|\mathrm{Re}\,\lambda_1|/|\mathrm{Re}\,\lambda_{m+1}|$, the system of eq. (19) can be written in the following form:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t}=\mathscr{A}_sx_s+\mathscr{B}_su+f_s(x_s,x_f)+\mathscr{W}_s(x_s,x_f,\theta)$$
$$\varepsilon\frac{\partial x_f}{\partial t}=\mathscr{A}_{f\varepsilon}x_f+\varepsilon\mathscr{B}_fu+\varepsilon f_f(x_s,x_f)+\varepsilon\mathscr{W}_f(x_s,x_f,\theta) \tag{20}$$
$$y=\mathscr{C}x_s$$

where $\mathscr{A}_{f\varepsilon}$ is an unbounded differential operator defined as $\mathscr{A}_{f\varepsilon}=\varepsilon\mathscr{A}_f$. Since $\varepsilon\ll 1$ (following from assumption 1, part 3) and the operators $\mathscr{A}_s$, $\mathscr{A}_{f\varepsilon}$ generate semigroups with growth rates which are of the same order of magnitude, the system of eq. (20) is in the standard singularly perturbed form [see Kokotovic *et al.*, (1986) for a precise definition of standard form], with x_s being the slow states and x_f being the fast states.

Introducing the fast time-scale $\tau=t/\varepsilon$ and setting $\varepsilon=0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq. (20):

$$\frac{\partial x_f}{\partial \tau} = \mathscr{A}_{f\varepsilon} x_f. \tag{21}$$

From the fact that Re $\lambda_{m+1} < 0$ and the definition of ε, we have that the above system is globally exponentially stable. Setting $\varepsilon = 0$ in the system of eq. (20), we have that $x_f = 0$ and thus, the finite-dimensional slow system takes the form

$$\begin{aligned}\frac{dx_s}{dt} &= \mathscr{A}_s x_s + f_s(x_s, 0) + \mathscr{B}_s u + \mathscr{W}_s(x_s, 0, \theta)\\ &=: F_0(x_s) + \sum_{i=1}^{l} \mathrm{B}_0^i u_0^i + \mathrm{W}_0(x_s, 0, \theta)\\ y_{si} &= \mathscr{C}^i x_s =: h_{i0}(x_s)\end{aligned} \tag{22}$$

where the subscript s in y_{si} denotes that this output is associated with an ODE system and the subscript 0 in $(F_0, \mathrm{B}_0^i, u_0^i, \mathrm{W}_0, h_{i0})$ denotes that they are elements of the $O(\varepsilon)$ approximation of the x_s-subsystem. In the above system, a new notation was introduced to facilitate the presentation of the robust controller synthesis results in the next subsections. We finally note that the above system is identical to the one obtained by applying the standard Galerkin's method (e.g. Ray, 1981) to the system of eq. (10), keeping the first m ODEs and completely neglecting the x_f-subsystem.

3.2. *Robust controller synthesis*

In this subsection, we synthesize robust controllers for the system of eq. (1), on the basis of the finite-dimensional system of eq. (22), using Lyapunov's direct method and precisely characterize the ultimate uncertainty attenuation level. Motivated by the requirement of output tracking with attenuation of the effect of the uncertainty on the output and the fact that the system of eq. (22) includes only uncertain variables that appear in an additive fashion, we consider the synthesis of robust control laws of the form:

$$u_0 = p_0(x_s) + Q_0(x_s)\bar{v} + r_0(x_s, t) \tag{23}$$

where $p_0(x_s)$, $r_0(x_s, t)$ are vector functions, $Q_0(x_s)$ is a matrix, and $\bar{v}$ is a vector of the form $\bar{v} = [v \; v^{(1)} \cdots \; v^{(r)}]^{\mathrm{T}}$, where $v^{(k)}$ denotes the kth time derivative of the external reference input v, which is assumed to be a sufficiently smooth function of time. The control law of eq. (23) comprises of the component $p_0(x_s) + Q_0(x_s)\bar{v}$, which is responsible for the output tracking and stabilization of the closed-loop slow system, and the component $r_0(x_s, t)$ which is responsible for the asymptotic attenuation of the effect of the uncertain variables on the outputs of the closed-loop slow system.

In order to derive an explicit formula of the control law of eq. (23), we will impose the following three assumptions on the system of eq. (22). We initially assume that there exists a coordinate transformation that renders the system of eq. (22) partially linear and that the time-derivatives of the output y_i up to order $r_i - 1$ are independent of the vector of uncertain variables θ. Assumption 2 that follows states precisely this requirement:

Assumption 2: *Referring to the system of eq.* (20), *there exist a set of integers* $(r_1, r_2, \ldots, r_l)$ *and a coordinate transformation* $(\zeta, \eta) = T(x_s, \theta)$ *such that the representation of the system, in the coordinates* (ζ, η), *takes the form*

$$\begin{aligned}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)}\\
&\vdots\\
\dot{\zeta}_{r_1-1}^{(1)} &= \zeta_{r_1}^{(1)}\\
\dot{\zeta}_{r_1}^{(1)} &= L_{F_0}^{r_1} h_{10}(T^{-1}(\zeta, \eta, \theta))\\
&\quad + \sum_{i=1}^{l} L_{B_0^i} L_{F_0}^{r_1-1} h_{10}(T^{-1}(\zeta, \eta, \theta)) u_0^i\\
&\quad + L_{W_0} L_{F_0}^{r_1-1} h_{10}(T^{-1}(\zeta, \eta, \theta))\\
&\vdots\\
\dot{\zeta}_1^{(l)} &= \zeta_2^{(l)}\\
&\vdots\\
\dot{\zeta}_{r_l-1}^{(l)} &= \zeta_{r_l}^{(l)}\\
\dot{\zeta}_{r_l}^{(l)} &= L_{F_0}^{r_l} h_{l0}(T^{-1}(\zeta, \eta, \theta))\\
&\quad + \sum_{i=1}^{l} L_{B_0^i} L_{F_0}^{r_l-1} h_{l0}(T^{-1}(\eta, \zeta, \theta)) u_0^i\\
&\quad + L_{W_0} L_{F_0}^{r_l-1} h_{l0}(T^{-1}(\eta, \zeta, \theta))\\
\dot{\eta}_1 &= \Psi_1(\zeta, \eta, \theta, \dot{\theta})\\
&\vdots\\
\dot{\eta}_{m-\sum_i r_i} &= \Psi_{m-\sum_i r_i}(\zeta, \eta, \theta, \dot{\theta})\\
y_{si} &= \zeta_1^{(i)}, \; i = 1, \ldots, l
\end{aligned} \tag{24}$$

where

$$x_s = T^{-1}(\zeta, \eta, \theta), \; \zeta = [\zeta^{(1)} \cdots \zeta^{(l)}]^T, \; \eta = [\eta_1 \cdots \eta_{n-\sum_i r_i}]^{\mathrm{T}}.$$

We note that the above assumption is always satisfied for systems for which $r_i = 1$, for all $i = 1, \ldots, l$. In most practical applications, this requirement can be easily achieved by selecting the form of the actuator distribution functions $b_i(z)$ to be different than the form of the eigenfunctions ϕ_j for $j = 2, \ldots, \infty$ (i.e. pick $b_i(z)$ so that $b_i(z) \neq \phi_j$, for $i = 1, \ldots, l$, $j = 2, \ldots, \infty$). Finally, we note that we do not require $\theta(t)$ to enter the system of eq. (20) in the same differential equations as u which is a standard assumption in many robust controller design methods (e.g. Corless and Leitmann, 1981).

Referring to the system of eq. (24), we will assume, for simplicity, that the matrix:

$$C_0(x_s) = \begin{bmatrix} L_{B_0^1} L_{F_0}^{r_1-1} h_{10}(x_s) & \cdots & L_{B_0^l} L_{F_0}^{r_1-1} h_{10}(x_s) \\ \vdots & \cdots & \vdots \\ L_{B_0^1} L_{F_0}^{r_l-1} h_{l0}(x_s) & \cdots & L_{B_0^l} L_{F_0}^{r_l-1} h_{l0}(x_s) \end{bmatrix} \tag{25}$$

is nonsingular uniformly in $x_s \in \mathscr{H}^n$.

Assumption 3 that follows states that the system which describes the unforced (i.e. $\eta(t) = \theta(t) = \dot{\theta}(t) \equiv 0$) inverse dynamics of the system of eq. (24) is locally exponentially stable (in other words the system of eq. (24) is assumed to be minimum phase). This assumption is standard in most nonlinear control methods for ODE systems (e.g. Kravaris and Arkun, 1991) and is satisfied by many practical applications. This assumption is needed to establish that the state of the closed-loop slow system is locally bounded.

Assumption 3: *The dynamical system*:

$$\begin{aligned}\dot{\eta}_1 &= \Psi_1(\zeta, 0, 0, 0)\\ &\vdots\\ \dot{\eta}_{m-\sum_i r_i} &= \Psi_{m-\sum_i ri}(\zeta, 0, 0, 0)\end{aligned} \tag{26}$$

is locally exponentially stable.

Assumption 4 that follows requires the existence of a nonlinear time-varying bounding function that captures the size of the uncertain terms in the system of eq. (22). Such a bounding function is typically obtained from physical considerations, preliminary simulations or experimental data. The requirement of existence of a bounding function is standard in all Lyapunov-based robust control methods (see e.g. Corless and Leitmann, 1981; Christofides *et al.*, 1996; Christofides and Daoutidis, 1997b).

Assumption 4: *There exists a known function $c_0(x_s, t)$ such that the following condition holds*:

$$\begin{aligned}|[L_{\mathscr{W}_0} L_{F_0}^{r_1-1} h_{10}(T^{-1}(\zeta, \eta)) \cdots\\ L_{W_0} L_{F_0}^{r_l-1} h_{l0}(T^{-1}(\eta, \zeta))]^{\mathrm{T}}| \leqslant c_0(x_s, t)\end{aligned} \tag{27}$$

for all $x_s \in \mathscr{H}^m$, $\theta \in \mathbb{R}^q$, $t \geqslant 0$.

Whenever assumptions 2–4 are satisfied, it is possible to synthesize a robust controller on the basis of the system of eq. (22) using Lyapunov's direct method [the reader may refer to (Christofides and Daoutidis, 1997b) for details on robust controller design for nonlinear multi-input multi-output ODE systems]. Theorem 1 that follows provides an explicit formula of the robust controller, conditions that ensure boundedness of the state, and a precise characterization of the ultimate uncertainty attenuation level. The proof of the theorem is given in the appendix. To simplify the statement of the theorem, we set $\bar{v}_i = [v_i\, v_i^{(1)} \cdots v_i^{(r_i)}]^{\mathrm{T}}$ and $\bar{v} = [\bar{v}_1^{\mathrm{T}}\, \bar{v}_2^{\mathrm{T}} \cdots \bar{v}_m^{\mathrm{T}}]^{\mathrm{T}}$.

Theorem 1: *Consider the parabolic PDE system of eq.* (10) *for which assumption* 1 *holds, and the finite-dimensional system of eq.* (22), *for which assumptions* 2–4 *hold, under the robust controller*:

$$\begin{aligned}u_0 = [C_0(x_s)]^{-1}\Bigg\{ &\sum_{i=1}^{l}\sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (v_i^{(k)} - L_{F_0}^k h_{i0}(x_s))\\ &+ \sum_{i=1}^{l}\sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (v_i^{(k-1)} - L_{F_0}^{k-1} h_{i0}(x_s))\\ &- 2[c_0(x_s, t)]\\ &\times \frac{\displaystyle\sum_{i=1}^{l}\sum_{k=1}^{r_i-1} \frac{\beta_{ik}}{\beta_{ir_i}} (L_{F_0}^{k-1} h_{i0}(x_s) - v_i^{(k-1)})}{\displaystyle\left|\sum_{i=1}^{l}\sum_{k=1}^{r_i-1}\right| \frac{\beta_{ik}}{\beta_{ir_i}} (L_{F_0}^{k-1} h_{i0}(x_s) - v_i^{(k-1)})| + \phi}\Bigg\}\end{aligned} \tag{28}$$

where

$$\frac{\beta_{ik}}{\beta_{ir_i}} = \left[\frac{\beta_{ik}^1}{\beta_{ir_i}^1} \cdots \frac{\beta_{ik}^l}{\beta_{ir_i}^l}\right]^{\mathrm{T}}$$

are column vectors of parameters chosen so that the roots of the equation det$[\mathrm{B(s)}] = 0$, *where* B(s) *is an* 1×1 *matrix, whose* (i, j)*th element is of the form*

$$\sum_{k=1}^{r_i} \frac{\beta_{jk}^i}{\beta_{jr_i}^i} s^{k-1},$$

lie in the open left-half of the complex plane, and ϕ is an adjustable parameter. Then, there exist positive real numbers (δ, ϕ^*) such that for each $\phi \leqslant \phi^*$, there exists $\varepsilon^*(\phi)$, such that if $\phi \leqslant \phi^*$, $\varepsilon \leqslant \varepsilon^*(\phi)$ and $\max\{|x_s(0)|, |x_f(0)|, \|\theta\|, \|\dot{\theta}\|, \|\bar{v}\|\} \leqslant \delta$,

(a) *the state of the infinite-dimensional closed-loop system is bounded, and*

(b) *the outputs of the infinite-dimensional closed-loop system satisfy*:

$$\lim_{t\to\infty}|y_i - v_i| \leqslant d_0,\ i = 1, \ldots, l \tag{29}$$

where $\mathrm{d}_0 = \mathrm{O}(\phi) + \mathrm{O}(\varepsilon)$ *is a positive real number.*

Remark 1: Theorem 1 establishes that the ultimate uncertainty attenuation level in the case of synthesizing a robust controller on the basis of the system of eq. (22) is $d_0 = O(\phi) + O(\varepsilon)$. This result is intuitively expected because the controller of eq. (28) ensures that the outputs of the $O(\varepsilon)$ approximation of the closed-loop parabolic PDE system satisfy $\lim_{t\to\infty}|y_{s_i} - v_i| \leqslant d_\phi = O(\phi)$, $i = 1, \ldots, l$ [the fact that $d_\phi = O(\phi)$ is rigorously established in the proof of the theorem; see eqs. (62)–(70)].

Remark 2: Regarding the practical application of theorem 1, one has to initially verify assumption 1 (separation of eigenvalues of $\mathscr{A}$ into slow and fast ones), and then verify assumptions 2–4 on the basis of the system of eq. (22). Then, the synthesis formula of eq. (28) can be directly used to derive the explicit form of the controller (see section 4 for an application of this procedure to a catalytic reactor example).

Remark 3: We note that the validity of the approach which we followed here to synthesize the nonlinear robust controller of eq. (28) relies on the large separation of slow and fast modes of the spatial differential operator of the parabolic PDE system. This approach is not applicable to hyperbolic PDE systems (i.e.

convection–reaction processes) where the eigenmodes cluster along vertical or nearly vertical asymptotes in the complex plane and thus, the controller synthesis problem has to be addressed directly on the basis of the hyperbolic PDE system [see (Christofides and Daoutidis, 1998) for a comprehensive solution to the robust controller synthesis problem for hyperbolic PDE systems].

3.3. *Improving uncertainty attenuation using approximate inertial manifolds*

The controller of eq. (28), synthesized on the basis of the slow system of eq. (22), enforces an ultimate uncertainty attenuation level $d_0 = O(\phi) + O(\varepsilon)$. Even though, this degree of attenuation may be sufficient for several practical applications where the degree of separation of slow and fast modes is large (i.e. ε is very small), it may not be sufficient for diffusion–convection–reaction processes for which ε is close to one. The objective of this section is to propose a procedure for the synthesis of robust controllers that achieve *arbitrary* degree of asymptotic attenuation of the effect of the uncertain variables on the outputs. The proposed procedure is based on the construction of higher-order (higher than $O(\varepsilon)$) m-dimensional approximations of the x_s-subsystem of eq. (19) (which will be used for controller design) by utilizing a geometric framework based on the concept of inertial manifold for systems of the form of eq. (10). The inertial manifold is an appropriate tool for improving uncertainty attenuation because if the trajectories of the infinite dimensional system of eq. (10) are on the manifold, then this system is exactly described by an m-dimensional slow system. We note that the concept of inertial manifold used in this work is a direct generalization, of the one introduced by Temam for quasi-linear parabolic PDE systems without time-varying inputs (see for example Temam, 1988), to systems with time-varying inputs. A concept of inertial manifold for Navier–Stokes equations with time-dependent inputs, similar to the one used here, has been introduced in (Jones and Titi, 1994).

An inertial manifold $\mathscr{M}$ for the system of eq. (10) is defined as a subset of $\mathscr{H}$, which satisfies the following properties: (i) $\mathscr{M}$ is a finite-dimensional Lipschitz manifold, (ii) $\mathscr{M}$ is a graph of a Lipschitz function $\Sigma(x_s, u, \theta, \varepsilon)$ mapping $\mathscr{H}_s \times \mathbb{R}^l \times \mathbb{R}^q \times (0, \varepsilon^*]$ into $\mathscr{H}_f$ and for every solution $x_s(t)$, $x_f(t)$ of eq. (20) with $x_f(0) = \Sigma(x_s(0), u(t), \theta(t), \varepsilon)$, then

$$x_f(t) = \Sigma(x_s(t), u(t), \theta(t), \varepsilon), \quad \forall\ t \geqslant 0 \tag{30}$$

and (iii) $\mathscr{M}$ attracts every trajectory exponentially. Owing to the second and third properties of the IM, the dynamics of the system of eq. (10) restricted to $\mathscr{M}$ are exactly described by the following m-dimensional system (called inertial form):

$$\begin{aligned}\frac{dx_s}{dt} &= \mathscr{A}_s x_s + \mathscr{B}_s u + f_s(x_s, \Sigma(x_s, u, \theta, \varepsilon)) \\ &\quad + \mathscr{W}_s(x_s, \Sigma(x_s, u, \theta, \varepsilon), \theta) \\ y &= \mathscr{C} x_s\end{aligned} \tag{31}$$

where $\Sigma(x_s, u, \theta, \varepsilon)$ is the solution of the following partial differential equation [which was obtained by differentiating eq. (30) and utilizing eq. (20)]:

$$\begin{aligned}&\varepsilon \frac{\partial \Sigma}{\partial x_s} [\mathscr{A}_s x_s + \mathscr{B}_s u + f_s(x_s, x_f) \\ &\quad + \mathscr{W}_s(x_s, \Sigma(x_s, u, \theta, \varepsilon), \theta)] + \varepsilon \frac{\partial \Sigma}{\partial u} \dot{u} + \varepsilon \frac{\partial \Sigma}{\partial \theta} \dot{\theta} \\ &\quad = \mathscr{A}_{f\varepsilon} x_f + \varepsilon \mathscr{B}_f u + \varepsilon f_f(x_s, x_f) + \varepsilon \mathscr{W}_f(x_s, x_f, \theta)).\end{aligned} \tag{32}$$

Since the inertial form of eq. (31) exactly describes the long-term dynamics of the system of eq. (10), it follows that a robust feedback controller synthesized on the basis of the inertial form, using the methodology described in the previous subsection, will ensure that the outputs of the closed-loop parabolic PDE system satisfy $\lim_{t\to\infty} |y_i - v_i| \leqslant d = O(\phi)$ (i.e. the ultimate uncertainty attenuation level in the closed-loop system is independent of ε), provided that ε is sufficiently small. However, because of the complexity present in computing the exact form of $\Sigma(x_s, u, \theta, \varepsilon)$ from eq. (32), it is impossible to directly utilize the inertial form of eq. (31) for controller synthesis. In order to overcome the problems associated with the computation of $\Sigma(x_s, u, \theta, \varepsilon)$, we will use a procedure which involves expansion of $\Sigma(x_s, u, \theta, \varepsilon)$ and u in a power series in ε, to compute approximations of $\Sigma(x_s, u, \theta, \varepsilon)$ (approximate inertial manifolds) and approximations of the inertial form, of desired accuracy [see also (Christofides and Daoutidis, 1997a) for a similar procedure for the construction of AIMs for systems of parabolic PDEs without uncertain variables].

Specifically, we consider an expansion of $\Sigma(x_s, u, \theta, \varepsilon)$ and u in a power series in ε:

$$\begin{aligned}u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots + \varepsilon^k u_k + O(\varepsilon^{k+1}) \\ \Sigma(x_s, u, \theta, \varepsilon) &= \Sigma_0(x_s, u, \theta) + \varepsilon \Sigma_1(x_s, u, \theta) \\ &\quad + \varepsilon^2 \Sigma_2(x_s, u, \theta) + \cdots + \varepsilon^k \Sigma_k(x_s, u, \theta) + O(\varepsilon^{k+1})\end{aligned} \tag{33}$$

where u_k, Σ_k are smooth functions. Substituting the expressions of eq. (33) into eq. (32), and equating terms of the same power in ε, one can obtain approximations of $\Sigma(x_s, u, \theta, \varepsilon)$ up to a desired order. Substituting the expansion for $\Sigma(x_s, u, \theta, \varepsilon)$ and u up to order k into eq. (31), the following approximation of the inertial form is obtained:

$$\begin{aligned}\frac{dx_s}{dt} &= \mathscr{A}_s x_s + \mathscr{B}_s(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots + \varepsilon^k u_k) \\ &\quad + f_s(x_s, \Sigma_0(x_s, u, \theta) + \varepsilon \Sigma_1(x_s, u, \theta) \\ &\quad + \varepsilon^2 \Sigma_2(x_s, u, \theta) + \cdots + \varepsilon^k \Sigma_k(x_s, u, \theta)) \\ &\quad + \mathscr{W}_s(x_s, \Sigma_0(x_s, u, \theta) + \varepsilon \Sigma_1(x_s, u, \theta) \\ &\quad + \varepsilon^2 \Sigma_2(x_s, u, \theta) + \cdots + \varepsilon^k \Sigma_k(x_s, u, \theta), \theta) \\ y &= \mathscr{C} x_s.\end{aligned} \tag{34}$$

The above approximation procedure is motivated and validated from the fact that the inertial form of eq. (31) reduces to the system of eq. (22) as $\varepsilon \to 0$, which ensures that the inertial form is well-posed with respect to ε. The expansion of u in a power series in ε in eq. (33) is motivated by our intention to appropriately modify the synthesis of the controller such that the outputs of the $O(\varepsilon^{k+1})$ approximation of the closed-loop inertial form satisfy $\lim_{t\to\infty}|y_{s_i} - v_i| \leqslant d$, where $d = d_\phi$. The construction of the robust control law of eq. 33 to achieve this objective can be performed following a sequential procedure. Specifically, the component $u_0 = p_0(x_s) + Q_0(x_s)\ v + r_0(x_s, t)$ can be initially synthesized on the basis of the $O(\varepsilon)$ approximation of the inertial form [eq. (22)]; then, the component $u_1 = p_1(x_s) + Q_1(x_s)\ v + r_1(x_s, t)$ can be synthesized on the basis of the $O(\varepsilon^2)$ approximation of the inertial form. In general, at the kth step, the component $u_k = p_k(x_s) + Q_k(x_s)\ v + r_k(x_s, t)$ can be synthesized on the basis of the $O(\varepsilon^k)$ approximation of the inertial form [eq. (34)]. The synthesis of $[p_v(x_s), Q_v(x_s), r_v(x_s, t)]$, $v = 0, \ldots, k$, can be performed, at each step, utilizing the methodology presented in the previous section for the synthesis of the component $u_0 = p_0(x_s) + Q_0(x_s)\ v + r_0(x_s, t)$.

Theorem 2 that follows provides conditions that ensure boundedness of the state, and a precise characterization of the ultimate uncertainty attenuation level in the case of using robust control laws of the form of eq. (33) which are synthesized by following the above procedure (the proof of the theorem is similar to the one of theorem 1 and thus, will be omitted for brevity).

Theorem 2: *Consider the parabolic PDE system of eq.* (10) *for which assumption* 1 *holds, under the robust control law:*

$$u = p_0(x_s) + Q_0(x_s)v + r_0(x_s, t) + \varepsilon(p_1(x_s) + Q_1(x_s)v + r_1(x_s, t)) + \cdots + \varepsilon^k(p_k(x_s) + Q_k(x_s)v + r_k(x_s, t)) \tag{35}$$

where $[p_v(x_s), Q_v(x_s), r_v(x_s, t)]$, $v = 0, \ldots, k$ *are synthesized, under the assumptions 2–4, following the aforementioned procedure. Then, there exist positive real numbers* (δ, ϕ^*) *such that for each* $\phi \leqslant \phi^*$, *there exists* $\varepsilon^*(\phi)$, *such that if* $\phi \leqslant \phi^*$, $\varepsilon \leqslant \varepsilon^*(\phi)$ *and* $\max\{|x_s(0)|, \|x_f(0)\|, \|\theta\|, \|\dot{\theta}\|, \|\bar{v}\|\} \leqslant \delta$,

(a) *the state of the infinite-dimensional closed-loop system is bounded, and*

(b) *the outputs of the infinite-dimensional closed-loop system satisfy*:

$$\lim_{t\to\infty}|y_i - v_i| \leqslant d_k, \quad i = 1, \ldots, l \tag{36}$$

where $d_k = O(\phi) + O(\varepsilon^{k+1})$ *is a positive real number.*

Remark 4: The procedure employed above for the construction of AIMs led to the construction of time-dependent deterministic AIMs which explicitly depend on $u(t)$ and $\theta(t)$ [see eq. (33)]. For these AIMs, theorem 2 establishes that, if the the initial conditions, the magnitude of the uncertain variables and the rate of change of the uncertain variables are sufficiently small, then the solution of the parabolic PDE system of eq. (10) converges exponentially fast to a neighborhood of the AIM, independently of k [order of AIM used in the approximate inertial form of eq. (34)]. The restriction on the rate of variation of $\theta(t)$ is expected since the presence of a fast varying time-dependent variable may induce additional fast dynamics in the parabolic PDE system of eq. (10), thereby rendering the proposed procedure for the construction of AIMs inappropriate.

Remark 5: Since $\varepsilon < 1$ and thus, $\varepsilon^2 \ll \varepsilon$, an uncertainty attenuation level $d_1 = O(\phi) + O(\varepsilon^2)$ should be satisfactory for most practical applications. In such a case, an application of the proposed approximation procedure yields that the $O(\varepsilon^2)$ approximation of $\Sigma(x_s, u, \theta, \varepsilon)$ is of the form

$$\Sigma(x_s, u, \theta, \varepsilon) = (\mathscr{A}_{f\varepsilon})^{-1}[-\varepsilon f_f(x_s) - \varepsilon\mathscr{B}_f u_0 - \varepsilon\mathscr{W}_f(x_s, \theta)] \tag{37}$$

and the corresponding $O(\varepsilon^2)$ approximation of the inertial form is given by

$$\begin{aligned}\frac{dx_s}{dt} &= \mathscr{A}_s x_s + \mathscr{B}_s u_0 + \varepsilon\mathscr{B}_s u_1 + f_s(x_s, \mathscr{A}_{f\varepsilon})^{-1} \\ &\quad \times [-\varepsilon f_f(x_s) - \varepsilon\mathscr{B}_f u_0 - \varepsilon\mathscr{W}_f(x_s, \theta)]) \\ &\quad + \mathscr{W}_s(x_s, \mathscr{A}_{f\varepsilon})^{-1}[-\varepsilon f_f(x_s) - \varepsilon\mathscr{B}_f u_0 \\ &\quad - \varepsilon\mathscr{W}_f(x_s, \theta)], \theta) \\ &=: F_1(x_s, \varepsilon) + \sum_{i=1}^{l}\mathscr{B}_1^i(\varepsilon)\, u_1^i + \mathscr{W}_1(x_s, \theta, \varepsilon) \\ y_{si} &= \mathscr{C}^i x_s =: h_{i1}(x_s).\end{aligned} \tag{38}$$

The necessary robust controller which ensures that $\lim_{t\to\infty}|y_i - v_i| \leqslant d_1 = O(\phi) + O(\varepsilon^2)$, $i = 1, \ldots, l$ takes then the form:

$$\begin{aligned}u_0 = [C_0(x_s)]^{-1}\Bigg\{ &\sum_{i=1}^{l}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k)} - L_{F_0}^k h_{i0}(x_s)) \\ &+ \sum_{i=1}^{l}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k-1)} - L_{F_0}^{k-1}h_{i0}(x_s)) \\ &- 2[c_0(x_s, t)] \\ &\times \frac{\sum_{i=1}^{l}\sum_{k=1}^{r_i-1}\frac{\beta_{ik}}{\beta_{ir_i}}(L_{F_0}^{k-1}h_{i0}(x_s) - v_i^{(k-1)})}{\left|\sum_{i=1}^{l}\sum_{k=1}^{r_i-1}\frac{\beta_{ik}}{\beta_{ir_i}}(L_{F_0}^{k-1}h_{i0}(x_s) - v_i^{(k-1)})\right| + \phi}\Bigg\}\end{aligned}$$

$$+\,\varepsilon\left\{[C_1(x_s)]^{-1}\left\{\sum_{i=1}^{l}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k)}-L_{F_1}^{k}h_{i1}(x_s))\right.\right.$$

$$+\sum_{i=1}^{l}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k-1)}-L_{F_1}^{k-1}h_{i1}(x_s))$$

$$-\,2[c_1(x_s,t)]$$

$$\left.\left.\times\frac{\sum_{i=1}^{l}\sum_{k=1}^{r_i-1}\frac{\beta_{ik}}{\beta_{ir_i}}(L_{F_1}^{k-1}h_{i1}(x_s)-v_i^{(k-1)})}{\left|\sum_{i=1}^{l}\sum_{k=1}^{r_i-1}\frac{\beta_{ik}}{\beta_{ir_i}}(L_{F_1}^{k-1}h_{i1}(x_s)-v_i^{(k-1)})\right|+\phi}\right\}\right\}$$

where the explicit form of $C_1(x_s)$, $c_1(x_s, t)$ is omitted for brevity.

Remark 6: Following the proposed approximation procedure, it can be shown that the $O(\varepsilon)$ approximation of $\Sigma(x_s, 0, \varepsilon)$ is $\Sigma_0(x_s, u, \theta) = 0$ and the corresponding approximate inertial form is identical to the system of eq. (22) (obtained via Galerkin's method) with $u(t) \equiv 0$. This system does not utilize any information about the structure of the fast subsystem, thus yielding an ultimate degree of attenuation for the closed-loop parabolic PDE system $d_0 = O(\phi) + O(\varepsilon)$ (theorem 1). On the other hand, the $O(\varepsilon^2)$ approximation of $\Sigma(x_s, u, \varepsilon, \theta)$ is of the form of eq. (37) and the corresponding open-loop approximate inertial form does utilize information about the structure of the fast subsystem, and thus allows to obtain an ultimate degree of attenuation for the closed-loop parabolic PDE system $d_1 = O(\phi) + O(\varepsilon^2)$ (theorem 2).

Remark 7: The on-line implementation of the controllers of eqs. (28)–(35) requires that the values of the state variables x_s. The values of the variables $x_{s_1}, x_{s_2}, \ldots, x_{s_m}$ can be readily obtained from the equations $x_{s_j} = \int_\alpha^\beta x(z, t)\phi_j(z)\,\mathrm{d}z$, $j = 1, \ldots, m$. Of course, the state $x(z, t)$ cannot be exactly known at all positions and times, but it will be known only at a finite number of positions. This implies that there will be small error on the computation of $x_{s_j}(t)$, which decreases as the number of measurements of $x(z, t)$ along the length of the process increases.

Remark 8: The implementation of the controller of eq. (35) requires to explicitly compute the vector function $\Sigma_k(x_s, u, \theta)$. However, $\Sigma_k(x_s, u, \theta)$ has an infinite-dimensional range and therefore cannot be implemented in practice. Instead a finite-dimensional approximation of $\Sigma_k(x_s, u, \theta)$, say $\Sigma_k(x_s, u, \theta)$, can be derived by keeping the first $\bar{m}$ elements of $\Sigma_k(x_s, u, \theta)$ and neglecting the remaining infinite ones. Clearly, as $\bar{m} \to \infty$, $\Sigma_k(x_s, u, \theta)$ approaches $\Sigma_k(x_s, u, \theta)$. This implies that by picking $\bar{m}$ to be sufficiently large, the controller of eq. (39) with $\Sigma_k(x_s, u, \theta)$ instead of $\Sigma_k(x_s, u, \theta)$ guarantees stability and enforces the requirement of eq. (36) in the closed-loop infinite-dimensional system.

Remark 9: The robust controllers of eqs. (28)–(35) possess a robustness property with respect to fast and asymptotically stable unmodeled dynamics (i.e. the controllers enforce boundedness, output tracking and uncertainty attenuation in the closed-loop system, despite the presence of additional dynamics in the process model, as long as they are stable and sufficiently fast). This property of the controllers can be rigorously established by analyzing the closed-loop system with the unmodeled dynamics using singular perturbations [see (Christofides and Daoutidis, 1998) for a similar analysis in the case of robust control of hyperbolic PDE systems]. This robustness property of the controllers is of particular importance for many practical applications where unmodeled dynamics often occur due to actuator and sensor dynamics, fast process dynamics, etc. (see, for example, the application in the next section).

4. APPLICATION TO A PACKED-BED REACTOR

Consider the non-isothermal catalytic packed-bed reactor shown in Fig. 2, where a reaction of the form $A \to B$ takes place on the catalyst (Ray, 1981). The reaction is endothermic and a jacket is used to heat the reactor. Under the assumptions of negligible diffusive phenomena for the gas phase, constant density and heat capacity of the catalyst and the gas, and excess of species A in the reactor, the dynamic model of the process consists of the following set of energy balances:

- Energy balance for the gas phase.

$$\rho_f c_{pf}\frac{\partial T_g}{\partial \bar{t}} = -\,\rho_f c_{pf} v_l \frac{\partial T_g}{\partial \bar{z}} + h_c S_c (T - T_g) - h_g S_g (T_g - T_j). \tag{40}$$

Boundary condition:

$$\bar{z} = 0, \quad T_g = T_f. \tag{41}$$

- Energy balance for the catalyst.

$$\rho_s c_{ps}\frac{\partial T_c}{\partial \bar{t}} = k_e \frac{\partial^2 T_c}{\partial \bar{z}^2} + (-\Delta H)\,k_0 e^{-E/RT} - h_c S_c (T - T_g) - h_p S_h (T_c - T_j). \tag{42}$$

Boundary conditions:

$$\bar{z} = 0\ \frac{\partial T_c}{\partial \bar{z}} = 0, \quad \bar{z} = l\ \frac{\partial T_c}{\partial \bar{z}} = 0. \tag{43}$$

In the above model, ρ_f, c_{pf} and $\rho_s c_{ps}$ denote densities and heat capacities of the gas and the catalyst, respectively, v_l denotes the velocity of the gas, l denotes the length of the reactor, T_g, T_c, T_j denote the temperatures of the gas, the catalyst and the jacket, respectively, h_c, h_g, h_p denote heat transfer coefficients, S_c, S_g denote the pellet surface area and the wall heat transport area per unit volume, and ΔH, k_0, E denote the enthalpy, pre-exponential factor, and the activation energy of the reaction.

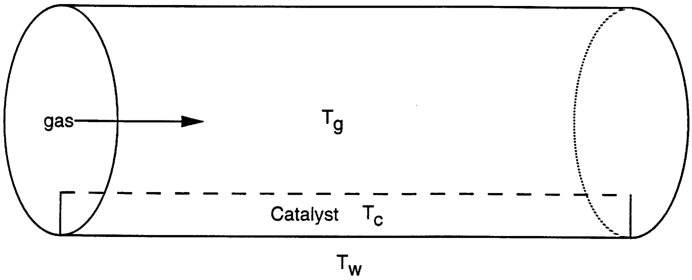

Fig. 2. A catalytic packed-bed reactor.

The main feature of the process is that the heat capacitance of the catalytic phase is much larger than the heat capacitance of the gas phase i.e. $\rho_s c_{ps} \gg \rho_f c_{pf}$. Therefore, the system of eqs. (40)–(42) possesses an inherent two-time-scale property, i.e., the dynamics of the gas temperature, T_g, are much faster than the dynamics of the catalyst temperature, T_c. The control problem considered is the one of controlling the temperature of the catalyst throughout the reactor (i.e. enforce a constant temperature on the catalyst) in order to maintain a desired degree of reaction rate, by manipulating the jacket temperature. The selection of T_c as the control variable is also motivated by the fact that T_c essentially determines the dynamics of the process. Notice that T_j is chosen to be the manipulated input with the understanding that in practice its manipulation is achieved indirectly through manipulation of the jacket inlet flow rate. The enthalpy of the reaction is considered to be the main uncertain variable.

In order to explicitly account in the controller design the presence of time-scale multiplicity in the process, we define a small positive parameter ε_p as the ratio of the heat capacitance of the gas phase vs the heat capacitance of the catalytic phase, i.e.

$$\varepsilon_p = \frac{\rho_f c_{pf} k_e}{\rho_s c_{ps} l^2} \tag{44}$$

and the dimensionless variables:

$$x_g = \frac{T_g - T_f}{T_f}, \quad \bar{x} = \frac{T_c - T_f}{T_f}, \quad u = \frac{T_j}{T_f},$$
$$\theta(t) = B - B_0, \quad t = \frac{\bar{t} k_e}{\rho_s c_{ps} l^2}, \quad z = \frac{\bar{z}}{l}, \quad \gamma = \frac{E}{RT_f} \tag{45}$$
$$\beta_c = \frac{h_c S_c l^2}{k_e}, \quad \beta_p = \frac{h_p S_c l^2}{k_e}, \quad B = \frac{(-\Delta H)\, l^2 k_0 \exp^{-\gamma}}{k_e T_f},$$
$$\alpha_c = \frac{h_c S_c l}{v_l \rho_f c_{pf}}, \quad \alpha_g = \frac{h_g S_h l}{v_l \rho_f c_{pf}}.$$

Using the above definitions, the process model of eqs. (40)–(43) can be written as

$$\varepsilon_p \frac{\partial x_g}{\partial t} = -\frac{\partial x_g}{\partial z} + \alpha_c(\bar{x} - x_g) - \alpha_g(\bar{x}_g - u)$$

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + B_0 exp^{\gamma\bar{x}/(1+\bar{x})} + \theta(t)\exp^{\gamma\bar{x}/(1+\bar{x})}$$
$$- \beta_c(\bar{x} - x_g) - \beta_p(\bar{x} - b(z)u) \tag{46}$$
$$y = \int_0^1 \bar{x}(z, t)\, dz$$

subject to the boundary conditions:

$$z = 0, \quad x_g = 0, \quad \frac{\partial \bar{x}}{\partial z} = 0, \quad z = 1, \quad \frac{\partial \bar{x}}{\partial z} = 0. \tag{47}$$

The values of the process parameters are given below:

$$\varepsilon_p = 0.01, \quad \gamma = 21.14, \quad \beta_c = 1.0, \quad \beta_p = 15.62,$$
$$B_0 = -0.003, \quad \alpha_c = 0.5, \quad \alpha_g = 0.5, \quad b(z) = 1. \tag{48}$$

For the above parameters, it was verified via simulations that the spatially uniforms steady state, $x_g(z, t) = 0$, $\bar{x}(z, t) = 0$, is a stable one.

In order to proceed with the controller design task, we initially exploit the fact that ε_p is small and set $\varepsilon_p = 0$ in the system of eq. (46), which yields the following system consisting of a first-order ODE in space and a parabolic PDE:

$$\frac{dx_g}{dz} = \alpha_c(\bar{x} - x_g) - \alpha_g(\bar{x}_g - u) \tag{49}$$

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + B_0 \exp^{\gamma\bar{x}/(1+\bar{x})} + \theta(t)\exp^{\gamma\bar{x}/(1+\bar{x})}$$
$$- \beta_c(\bar{x} - x_g) - \beta_p(\bar{x} - u) \tag{50}$$
$$y = \int_0^1 \bar{x}(z, t)\, dz.$$

Since the control objective is the regulation of the catalyst temperature, the parabolic PDE system of eq. (50) will be used for controller design. The state variable x_g which is present in the system of eq. (50) will be computed by integrating the ODE of eq. (49). For the system of eq. (50), the operator $\mathscr{A}$ takes the form

$$\mathscr{A}x = \frac{\partial^2 \bar{x}}{\partial z^2}$$

$$x \in D(\mathscr{A}) = \left\{x \in \mathscr{H}([0,1];\mathbb{R});\ z=0, \frac{\partial x}{\partial z} = 0;\right.$$

$$\left. z=1, \frac{\partial x}{\partial z} = 0\right\}. \quad (51)$$

Furthermore, the manipulated input, uncertain variable and controlled output operators as

$$\mathscr{B}u = \beta_p u,\ \mathscr{W}(x,\theta) = e^{\gamma\,\bar{x}/(1+\bar{x})}\theta(t),\ \mathscr{C}x = (c, x) \quad (52)$$

and the nonlinear term $f(x)$ as

$$f(x) = B_0 exp^{\gamma\bar{x}/(1+\bar{x})} - \beta_c(\bar{x} - x_g) - \beta_p\bar{x}. \quad (53)$$

The solution to the eigenvalue problem for $\mathscr{A}$ is

$$\lambda_j = -j^2\pi^2, \quad j = 0, \ldots, \infty,$$

$$\phi_0 = 1, \quad \phi_j = \sqrt{2}\cos(j\,\pi\,z), \quad j = 1, \ldots, \infty. \quad (54)$$

The values of λ_j indicate that the eigenspectrum of $\mathscr{A}$ can be partitioned into a finite-dimensional slow one (λ_0) and an infinite-dimensional fast one ($\lambda_1, \lambda_2, \lambda_3, \ldots,$). Furthermore, since $\lambda_0 = 0$ and thus, $\varepsilon = \lambda_0/\lambda_1 = 0$, it is reasonable to define $\mathscr{H}_s = \mathrm{span}\{\phi_0\}$. Applying Galerkin's method to the system of eq. (46), a one- dimensional ODE system is derived of the form

$$\frac{dx_s}{dt} = \int_0^1 (B_0 \exp^{\gamma x_s/(1+x_s)} - \beta_c(x_s - x_g) - \beta_p x_s)\,dz + \mathscr{B}_s u + \theta(t)\int_0^1 \exp^{\gamma x_s/(1+x_s)}\,dz \quad (55)$$

$$y_s = \mathscr{C}x_s.$$

One the basis of the above system, one can easily verify that assumptions 2 and 3 are trivially satisfied, while assumption 4 is satisfied with:

$$c_0(x,t) = \theta_m \int_0^1 \exp^{\gamma x_s/(1+x_s)}\,dz \quad (56)$$

where θ_m is a positive real number that satisfies $\theta_m \geqslant |\theta(t)\ |$. Therefore, the synthesis formula of eq. (28) was used to design a nonlinear robust controller on the basis of the system of eq. (55) of the form

$$u(t) = \frac{v - y_s - \int_0^1 (B_0 \exp^{\gamma x_s/(1+x_s)} - \beta_c(x_s - x_g) - \beta_p x_s)\,dz - \theta_m \frac{y_s - v}{|y_s - v| + \phi}\int_0^1 \exp^{\gamma x_s/(1+x_s)}\,dz}{\mathscr{B}_s} \quad (57)$$

where v is the reference input.

Simulation runs were performed to evaluate the performance and robustness properties of the controller, in the presence of the time-varying uncertain variable $B(t) = 10.0 B_0$ sint. The controller was implemented with $\beta_1 = 0.2$, $\beta_0 = 1.0$, $\phi = 0.05$ and $\theta_m = 10.0 B_0$. A detailed finite-difference discretization (100 discretization points) of the process model of eqs. (40)–(43) was used in the simulations. Initially, the process is assumed to be at the steady state $x_g(z,0) = 0$, $\bar{x}(z,0) = 0$, and a 0.05 increase in the

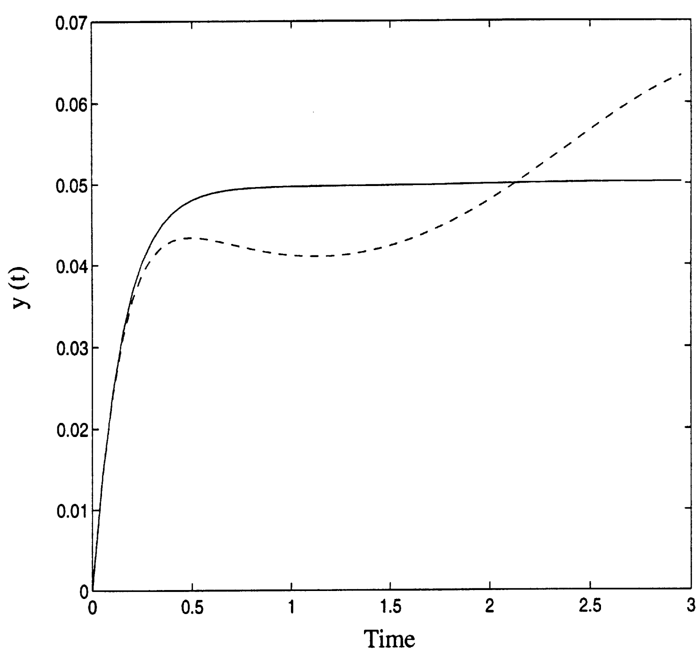

Fig. 3. (Solid line) Closed-loop output profile for reference input tracking. (Dashed line) Closed-loop output profile for reference input tracking (no uncertainty compensation).

value of the reference input is imposed at $t = 0$. The profile of the controlled output is shown in Fig. 3 (solid line), while the corresponding manipulated input profile is shown in Fig. 4. It is clear that the controller regulates the output at the new reference input value, attenuating the effect of the uncertain variable. The manipulated input changes smoothly with time in order to compensate for the effect of the uncertain variable. For the same simulation run, Fig. 5 displays the evolution of the catalyst temperature at all positions and times. The controller achieves excellent performance, regulating the catalyst temperature at each point in the reactor to the new reference input value. For the sake of comparison, we also implemented on the process the same controller without the term which compensates for

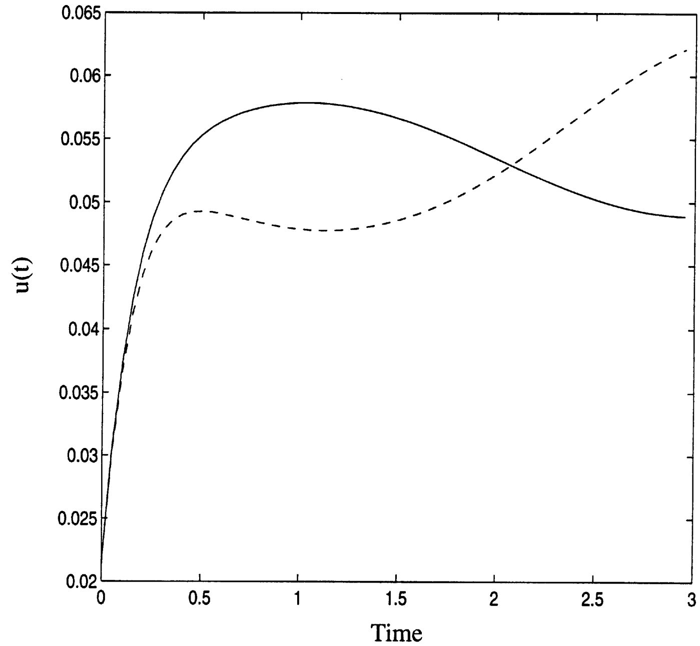

Fig. 4. (Solid line) Closed-loop manipulated input profile for reference input tracking. (Dashed line) Closed-loop manipulated input profile for reference input tracking (no uncertainty compensation).

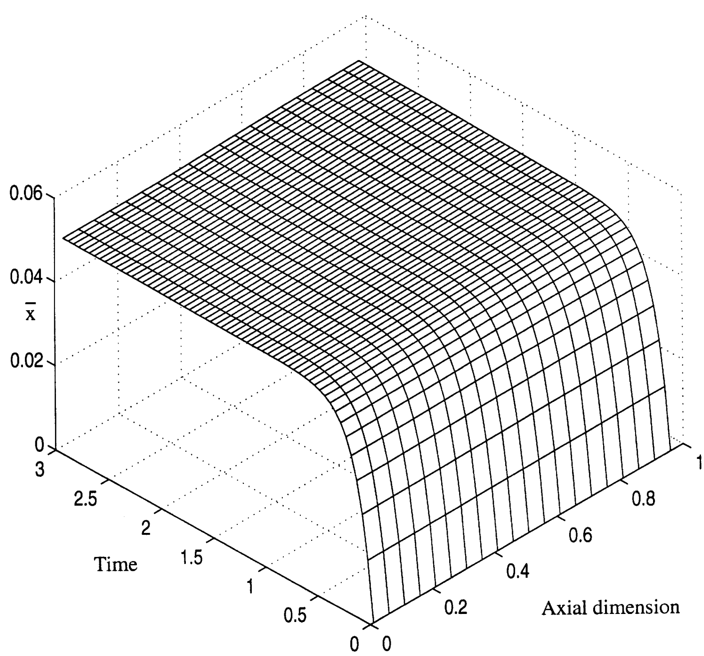

Fig. 5. Profile of evolution of catalyst temperature for reference input tracking.

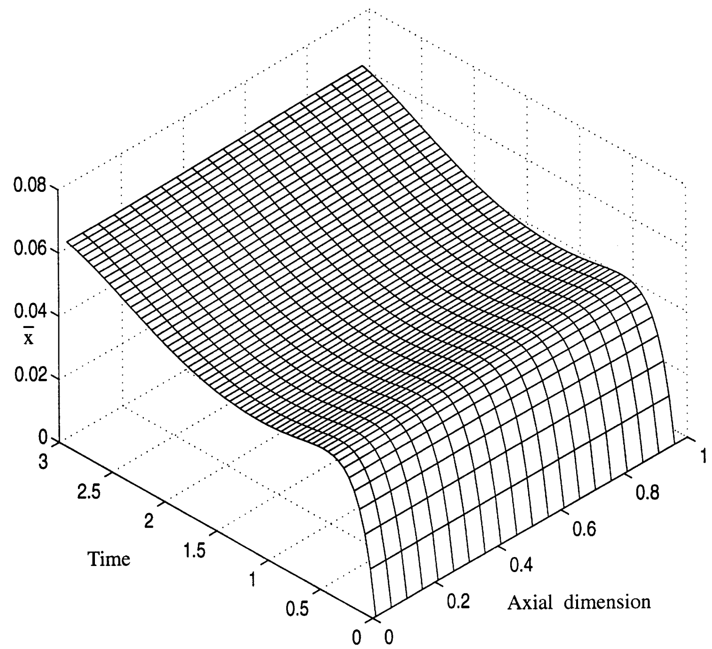

Fig. 6. Profile of evolution of catalyst temperature for reference input tracking (no uncertainty compensation).

the effect of the uncertain variable. The profile of the controlled output for this simulation run is given in Fig. 3 (dashed line), while the profile of the corresponding manipulated input and the spatiotemporal evolution of the catalyst temperature are given in Fig. 4 (dashed line) and Fig. 6, respectively. It is obvious that this controller cannot compensate for the effect of the uncertain variable, leading to poor performance. From the results of the simulation study, it is evident that the proposed methodology is a powerful tool for the synthesis of robust controllers which attenuate the effect of uncertain variables on the output for quasi-linear parabolic PDE systems.

Remark 10: We finally note that the one-dimensional model of eq. (55), which was used for the design of the controller of eq. (57), was obtained from standard Galerkin's method, and no improvement of its accuracy was pursued by using the concept of approximate inertial manifold. The reason is that the closed-loop performance and robustness properties of the controller of eq. (57) are clearly excellent (see the closed-loop output and state profiles in Fig. 3 and 5, respectively), thereby leaving no room for further improvement of the uncertainty attenuation properties of the controller by using approximate inertial manifolds. The fact that one mode suffices to design a nonlinear controller which yields an excellent closed-loop performance is expected since the control objective is to stabilize the process at a spatially uniform steady state, the control action is uniformly distributed along the length of the process and the first mode is spatially uniform (i.e. $\phi_0 = 1$).

5. CONCLUSIONS

In this paper we studied the control problem for diffusion–convection–reaction processes with time-varying uncertain variables described by systems of quasi-linear parabolic PDEs, for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional (possibly unstable) slow one and an infinite-dimensional stable fast complement. For such processes, we developed a methodology for the synthesis of nonlinear robust feedback controllers that guarantee boundedness of the state and output tracking with arbitrary degree of asymptotic attenuation of the effect of the uncertain variables on the output of the closed-loop system. Initially, Galerkin's method was used to derive an ODE system of dimension equal to the number of slow modes, which was subsequently used to synthesize robust controllers via Lyapunov's direct method. Singular perturbation methods were employed to establish that the degree of asymptotic uncertainty attenuation enforced by these controllers is proportional to the degree of separation of the fast and slow modes of the spatial differential operator. Then, for processes for which such a degree of uncertainty attenuation is not sufficient, a sequential procedure based on the concept of approximate inertial manifold was developed for the synthesis of robust controllers which enforce an arbitrarily small ultimate uncertainty attenuation level in the closed-loop parabolic PDE system. Finally, the developed methodology was successfully applied to a catalytic packed-bed reactor with unknown heat of reaction and was shown to significantly outperform a nonlinear controller design which does not account for the presence of the uncertain variable.

The solution to the robust control problem for parabolic PDE systems presented in this paper complements the solution to the robust control problem for hyperbolic PDE systems (Christofides and Daoutidis, 1998), leading to a general and practical framework for robust controller synthesis for diffusion–convection–reaction and convection–reaction processes.

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REFERENCES

Aling, H., Benerjee, S. Bangia, A. K., Cole, V., Ebert, J., Emani-Naeini, A., Jensen, K. F., Kevrekidis, I. G. and Shvartsman, S. (1997) Nonlinear model reduction for simulation and control of rapid thermal processing. In *Proceedings of American Control Conference*, 2233–2238, 1997, Albuquerque, NM.

Alonso, A. A. and Ydstie, E. B. (1995) Nonlinear control, passivity and the second law of thermodynamics, A.I.Ch.E. Annual Meeting, paper 181i, Miami Beach, FL.

Badgwell, T., Breedijk, T., Bushman, S. G., Butler, S. W., Chatterjee, S., Edgar, T. F., Toprac, A. J. and Trachtenberg, I. (1995) Modeling and control of microelectronics materials processing, *Comput. Chem. Engng* **19**, 1–41.

Balas, M. J. (1979) Feedback Control of Linear Diffusion Processes, *Int. J. Control* **29**, 523–533.

Bangia, A. K. Batcho, P. F., Kevrekidis, I. G. and Karniadakis, G. E. (1997) Unsteady 2-D flows in complex geometries: Comparative bifurcation studies with global eigenfunction expansion. *SIAM J. Sci. Comput.* **18**, 775–805.

Chakravarti, S., Marek, M. and Ray, W. H. (1995) Reaction-Diffusion system with Brusselator kinetics: control of a quasi-periodic route to chaos, *Phys. Rev. E.* **52**, 2407–2423.

Chen, C. C. and Chang, H. C. (1992) Accelerated disturbance damping of an unknown distributed system by nonlinear feedback, *A.I.Ch.E. J.* **38**, 1461–1476.

Christofides, P. D. and Daoutidis, P. (1996a) Feedback control of hyperbolic PDE systems, *A.I.Ch.E. J.* **42**, 3063–3086.

Christofides, P. D. and Daoutidis, P. (1996b) Nonlinear control of diffusion-convection-reaction processes, *Comput Chem. Engng* **20**(s) 1071–1076.

Christofides, P. D. and Daoutidis, P. (1997a) Finite-dimensional control of parabolic PDE systems using approximate inertial manifolds, *J. Math. Anal. Appl.* **216**, 398–420.

Christofides, P. D. and Daoutidis, P. (1997b) Robust control of multivariable two-time-scale nonlinear Systems, *J. Proc. Control* **7**, 313–328.

Christofides, P. D. and Daoutidis, P. (1988) Robust control of hyperbolic PDE systems, *Chem. Engng Sci.* **53**, 85–105.

Christofides, P. D. and Teel, A. R., and Daoutidis, P. (1996) Robust semi-global output tracking for nonlinear singularly perturbed systems, *Int. J. Control* **65**, 639–666.

Corless, M. and Leitmann, G. (1981) Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems. *IEEE Trans. Automat. Control* **26**, 1139–1144.

Curtain, R. F. and Glover, K. (1986) Robust stabilization of infinite dimensional systems by finite dimensional controllers, *Systems Control Lett.* **7**, 41–47.

Economou, D. J., Park, S.-K. and Williams, G. D. (1989) Uniformity of etching in parallel plate plasma reactors, *J. Electrochem. Soc.* **136**, 188–198.

Foias, C., Jolly, M., Kevrekidis, I., Sell, G. and Titi, E. (1989a) On the computation of inertial manifolds, *Phys. Lett. A* **131**, 433–437.

Foias, C., Sell, G. and Titi, E. (1989b) Exponential tracking and approximation of inertial manifolds for dissipative equations, *J. Dynamics Differential Equations* **1**, 199–244.

Friedman, A. (1976) *Partial Differential Equations*, Holt, Rinehart and Winston, New York.

Gauthier, J. P. and Xu, C. Z. (1989) H^{∞}-control of a distributed parameter system with non- minimum phase, *Int. J. Control* **53**, 45–79.

Gay, D. H. and Ray, W. H. (1995) Identification and control of distributed parameter systems by means of the singular value decomposition, *Chem. Engng Sci.* **50**, 1519–1539.

Georgakis, G. Aris, R. and Amundson, N. R. (1977) Studies in the control of tubular reactors: Part I & II & III, *Chem. Engng Sci.* **32**, 1359–1387.

Jacobson, C. A. and Nett, C. N. (1988) Linear state-space systems in infinite-dimensional space: The role and characterization of joint stabilizability/detectability, *IEEE Trans. Autom. Control* **33**, 541–551.

Jiang, Z. P., Teel, A. R. and Praly, L. (1995) Small-gain theorem for ISS systems and applications, *Math. Control Signals Systems* **7**, 95–120.

Jones, D. A. and Titi, E. S. (1994) A remark on quasi-stationary approximate inertial manifolds for the Navier–Stokes equations, *SIAM J. Math. Anal.* **25**, 894–914.

Keulen, B. (1993) *H_{∞}-Control for Distributed Parameter Systems: A State-Space Approach*, Birkhauser, Boston.

Khalil, H. K. (1992) *Nonlinear Systems*, Macmillan, New York.

Kokotovic, P. V., Khalil, H. K. and O'Reilly, J. (1986) *Singular Perturbations in Control: Analysis and Design*, Academic Press, London.

Kravaris, C. and Arkun, Y. (1991) Geometric nonlinear control – an overview. In *Proceedings of 4th International Conference on Chemical Process Control*, eds. Arkun, Y. and Ray, W. H. Padre Island, TX.

Ray, W. H. (1981) *Advanced Process Control*, McGraw-Hill, New York.

Sano, H. and Kunimatsu, N. (1995) An application of inertial manifold theory to boundary stabilization of semilinear diffusion systems, *J. Math. Anal. Appl.* **196**, 18–42.

Temam, R. (1988) *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York.

APPENDIX

Definitions

- For any measurable (with respect to the Lebesgue measure) function θ: $\mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$, $\|\theta\|$ denotes ess.sup. $|\theta(t)|$, $t \geqslant 0$.
- A function $\gamma: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class K if it is continuous, increasing and is zero at zero. It is of class K_∞, if in addition, $\gamma(s)$ tends to $+\infty$ as s tends to $+\infty$.
- A function $\beta: \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.

Definition (Khalil, 1992): *The system in eq.* (22) (*with* $u \equiv 0$) *is said to be locally input-to-state stable* (*ISS*) *with respect to* θ *if there exist a function* β *of class* KL, *a function* γ *of class* K *and a positive real number such that for each* $x_{s_0} \in \mathbb{R}^n$ *and for each measurable, essentially bounded input* $\theta(\cdot)$ *on* $[0, \infty)$ *that satisfy* $\max\{x_{s_0}, \|\theta\|\} \leqslant \hat{\delta}$, *the solution of eq.* (22) *with* $x_s(0) = x_{s_0}$ *exists for each* $t \geqslant 0$ *and satisfies*

$$|x_s(t)| \leqslant \beta(|x_s(0)|, t) + \gamma(\|\theta\|), \quad \forall\ t \geqslant 0. \tag{A.1}$$

Proof of Theorem 1: Under the control law of eq. (28) the closed-loop system takes the form

$$\frac{dx_s}{dt} = \mathscr{A}_s x_s + \mathscr{B}_s a_0(x_s, \bar{v}, t) + f_s(x_s, 0) + \mathscr{W}_s(x_s, 0, \theta) + \mathscr{R}_s(x_s, x_f, \theta)$$

$$\varepsilon \frac{\partial x_f}{\partial t} = \mathscr{A}_{f\varepsilon} x_f + \varepsilon \mathscr{B}_f a_0(x_s, \bar{v}, t) + \varepsilon f_f(x_s, 0) + \varepsilon \mathscr{W}_f(x_s, 0, \theta) + \varepsilon \mathscr{R}_f(x_s, x_f, \theta) \tag{A.2}$$

$$y = \mathscr{C} x_s$$

where $\mathscr{R}_s(x_s, x_f, \theta) = f_s(x_s, x_f) + \mathscr{W}_s(x_s, x_f, \theta) - f_s(x_s, 0) - \mathscr{W}_s(x_s, 0, \theta)$, $\mathscr{R}_f(x_s, x_f, \theta) = f_f(x_s, x_f) + \mathscr{W}_f(x_s, x_f, \theta) - f_f(x_s, 0) - \mathscr{W}_f(x_s, 0, \theta)$ with $\mathscr{R}_s(x_s, 0, \theta) = \mathscr{R}_f(x_s, 0, \theta) = 0$, and

$$a_0(x_s, \bar{v}, t) := [C_0(x_s)]^{-1} \left\{ \sum_{i=1}^{l} \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (v_i^{(k)} - L_{F_0}^k h_{i0}(x_s)) + \sum_{i=1}^{l} \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (v_i^{(k-1)} - L_{F_0}^{k-1} h_{i0}(x_s)) - 2[c_1(x_s, t)] \times \frac{\sum_{i=1}^{l} \sum_{k=1}^{r_i - 1} \frac{\beta_{ik}}{\beta_{ir_i}} (L_{F_0}^{k-1} h_{i0}(x_s) - v_i^{(k-1)})}{\left| \sum_{i=1}^{l} \sum_{k=1}^{r_i - 1} \frac{\beta_{ik}}{\beta_{ir_i}} (L_{F_0}^{k-1} h_{i0}(x_s) - v_i^{(k-1)}) \right| + \phi} \right\}. \tag{A3}$$

On the basis of the system of eq. (A.2), it is clear that since the operator $\mathscr{A}_{f\varepsilon}$ generates an exponentially stable semigroup, the infinite-dimensional closed-loop fast subsystem is exponentially stable.

Observing the similarity in the structure of the system of eq. (A.2) and the x_s-subsystem of eq. (22), using assumption 2, and introducing the variables

$$e_k^{(i)} = \zeta_k^{(i)} - v_i^{(k-1)}, \; i = 1, \ldots, l, \; k = 1, \ldots, r_i,$$

$$\tilde{e}_{r_i}^{(i)} = \sum_{i=1}^{l} e_{r_i}^{(i)} + \sum_{i=1}^{l} \sum_{k=1}^{r_i - 1} \beta_{ik}/\beta_{ir_i} e_k^{(i)}$$

and the notation $\bar{e}^{(i)} = [e_1^{(i)}\ e_2^{(i)} \cdots e_{r_i-1}^{(i)}]^T$, $\bar{e} = [e^{(1)T}\ e^{(2)T} \cdots e^{(l)T}]^T$, $\bar{\eta} = [\bar{e}^T\ \eta^T]$, $\tilde{e}_r = [\tilde{e}_{r_i}^{(1)}\ \tilde{e}_{r_i}^{(2)} \cdots \tilde{e}_{r_i}^{(l)}]^T$, $v_i = [v_i^{(0)}\ v_i^{(1)} \cdots v_i^{(r_i-1)}]^T$, $\bar{v} = [v_1^T\ v_2^T \cdots v_l^T]^T$, the representation of the closed-loop system of eq. (A.2):

$$\begin{aligned}
\dot{e}_1^{(1)} &= e_2^{(1)} + \bar{\Psi}_1^{(1)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, x_f) \\
&\ \vdots \\
\dot{e}_{r_1-1}^{(1)} &= -\sum_{i=1}^{l} \sum_{k=1}^{r_i-1} \frac{\beta_{ik}^1}{\beta_{ir_i}^1} e_k^{(i)} + \sum_{i=1}^{l} \tilde{e}_{r_1}^{(1)} + \bar{\Psi}_{r_1}^{(1)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, x_f) \\
\dot{\tilde{e}}_{r_1}^{(1)} &= \bar{\Psi}_{r_1}^{(1)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, x_f) + L_{W_0} L_{F_0}^{r_1-1} h_{10}(T^{-1}(\bar{e}, \tilde{e}_r, \bar{v}, \eta)) \\
&\quad + \sum_{i=1}^{l} L_{B_0^i} L_{F_0}^{r_1-1} h_{10}(T^{-1}(\bar{e}, \tilde{e}_r, \bar{v}, \eta)\ a^i(\bar{e}, \tilde{e}_r, \bar{v}, \bar{v}, \eta, t) \\
&\ \vdots \\
\dot{e}_1^{(l)} &= e_2^{(l)} + \bar{\Psi}_1^{(l)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta) \\
&\ \vdots \\
\dot{e}_{r_l-1}^{(l)} &= e_{r_l}^{(l)} + \bar{\Psi}_{r_l-1}^{(l)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, x_f) \\
\dot{e}_{r_l}^{(m)} &= \bar{\Psi}_{r_l}^{(l)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, x_f) + L_{W_0} L_{F_0}^{r_l-1} h_{l0}(T^{-1}(\bar{e}, \tilde{e}_r, \bar{v}, \eta)) \\
&\quad + \sum_{i=1}^{l} L_{B_0^i} L_{F_0}^{r_l-1} h_{l0}(T^{-1}(\bar{e}, \tilde{e}_r, \bar{v}, \zeta))\ a^i(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \bar{v}, t) \\
\dot{\eta}_1 &= \Psi_1(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, \dot{\theta}) + \bar{\Psi}_{\sum_i r_i + 1}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, x_f) \\
&\ \vdots \\
\dot{\eta}_{m - \sum_i r_i} &= \Psi_{m - \sum_i r_i}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, \dot{\theta}) + \bar{\Psi}_m(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta, x_f) \\
\varepsilon \frac{\partial x_f}{\partial t} &= \mathscr{A}_{f\varepsilon} x_f + \varepsilon \mathscr{B}_f a_0(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \bar{v}, t) + \varepsilon f_f(x_s, 0) \\
&\quad + \varepsilon \mathscr{W}_f(\bar{e}, \tilde{e}_r, \bar{v}, \eta, 0, \theta) + \varepsilon \mathscr{R}_f(\bar{e}, \tilde{e}_r, \bar{v}, \eta, x_f, \theta) \\
y_i &= \zeta_1^{(i)}
\end{aligned} \tag{A.4}$$

where $\bar{\Psi}_k^{(i)}$, $i = 1, \ldots, l$, $k = 1, \ldots, r_i$, and $\bar{\Psi}_{m - \sum_i r_i}, \ldots, \bar{\Psi}_m$, are Lipschitz functions of their arguments. In what follows, we derive bounds that capture the evolution of the states $\bar{e}, \tilde{e}_r, \bar{\eta}$ of the above system. We initially obtain these bounds, when $\varepsilon = 0$, and then we show that these bounds continue to hold up to an arbitrarily small offset for $\varepsilon > 0$.

First, note that the linear structure of $\bar{e}$ subsystem of the reduced system of eq. (A.4) and the fact that it is exponentially stable when $\tilde{e}_r \equiv 0$ allows using a direct Lyapunov function argument to show that there exist positive real numbers, k_1, a, γ_{e_r} such that the following ISS bound holds for the state $\bar{e}$ of the slow subsystem:

$$|\bar{e}(t)| \leqslant k_1 e^{-at} |\bar{e}(0)| + \gamma_{e_r} \|\tilde{e}_r\| \tag{A.5}$$

We will now show that the controller of eq. (28) ensures that the state vector $\tilde{e}_r = [\tilde{e}_{r_1}^{(1)}\ \tilde{e}_{r_2}^{(2)} \cdots\ \tilde{e}_{r_m}^{(m)}]^T$ of the system of eq. (A.4) with $\varepsilon = 0$ possesses an ISS property with respect to $\bar{e}, \bar{v}, \eta, \theta$, and moreover the gain function saturates at ϕ. To this end, we consider the $\varepsilon -$ independent system:

$$\tilde{e}_r = L_{W_0} L_{F_0}^{r_i-1} h_0(T^{-1}(\eta, \zeta)) + \sum_{i=1}^{l} \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} e_{k+1}^{(i)} + \left\{ -\sum_{i=1}^{l} \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} e_{k+1}^{(i)} - \tilde{e}_r - 2[c_0((\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta), t)] w(T^{-1}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta)), \phi) \right\} \tag{A.6}$$

where $\bar{\Psi}_r(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta) = [\bar{\Psi}_{r_1}^{(1)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta) \cdots \bar{\Psi}_{r_l}^{(l)}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta)]^T$, $v^{(r)} = [v_1^{(r_1)} \cdots v_m^{(r_m)}]^T$, $L_\delta L_F^{r-1} h(T^{-1}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta)) = [L_\delta L_F^{r_1-1} h_1(T^{-1}(\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta)) \cdots L_\delta L_F^{r_l-1} h_l(T^{-1}(\bar{e}, \tilde{e}_{r_1}, \bar{v}, \eta, \theta))]^T$. To establish that the above system is ISS with respect to $\bar{e}, \tilde{e}_r, \bar{v}, \eta, \theta$, we use the following smooth function $V: \mathscr{H}^m \rightarrow \mathbb{R}_{\geqslant 0}$:

$$V = \tfrac{1}{2}\tilde{e}_r^2. \tag{A.7}$$

Calculating the time-derivative of V along the trajectory of the system of eq. (A6), we have

$$\begin{aligned}\dot{V} = \tilde{e}_r^{\mathrm{T}}\Bigg[& L_{W_0}L_{F_0}^{r_i-1}h_0(T^{-1}(\eta,\zeta)) + \sum_{i=1}^{l}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}e_{k+1}^{(i)} \\ & + \Bigg\{ -\sum_{i=1}^{l}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}e_{k+1}^{(i)} - \tilde{e}_r \\ & - 2[c_0((\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta),t)]w(T^{-1}(\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta)),\phi)\Bigg\}\Bigg]. \end{aligned} \tag{A.8}$$

Furthermore, it is straightforward to show that the representation of the vector function $w(x, \phi)$ in terms of the vector $\tilde{e}_r$ is given by

$$w(\tilde{e}_r, \phi) = \frac{\tilde{e}_r}{|\tilde{e}_r| + \phi} \tag{A.9}$$

Substituting eqs. (A.9), we have

$$\begin{aligned}\dot{V} \leqslant \tilde{e}_r^{\mathrm{T}}\Bigg\{ & -\tilde{e}_r + L_{W_0}L_{F_0}^{r_i-1}h^0(T^{-1}(\eta,\zeta)) \\ & - 2[c_0(T^{-1}(\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta),t)]\frac{\tilde{e}_r}{|\tilde{e}_r|+\phi}\Bigg\} \\ \leqslant \Bigg\{ & -\tilde{e}_r^2 - 2[c_0(T^{-1}(\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta),t)]\frac{\tilde{e}_r^2}{|\tilde{e}_r|+\phi} \\ & + |\tilde{e}_r|c_0(T^{-1}(\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta),t)\Bigg\} \\ \leqslant \Bigg\{ & -\tilde{e}_r^2 - c_0(T^{-1}(\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta),t)\frac{\tilde{e}_r^2}{|\tilde{e}_r|+\phi} \\ & + \phi[c_0(T^{-1}(\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta),t)]\frac{\tilde{e}_r}{|\tilde{e}_r|+\phi}\Bigg\}. \end{aligned} \tag{A.10}$$

From the last inequality, it follows directly that if $|\tilde{e}_r| \geqslant \phi$, the time-derivative of the Lyapunov function satisfies $\dot{V} \leqslant -\tilde{e}_r^2$. This fact implies that the ultimate bound on the state $\tilde{e}_r$ of the system of eq. (A.6) depends only on the parameter ϕ and is independent of the states $\bar{e}, \eta$.

We will now analyze the time-derivative of V for $|\tilde{e}_r| < \phi$. For ease of notation, we set $\mathscr{U} = [\tilde{e}_r^{\mathrm{T}}\ \theta^{\mathrm{T}}\ \tilde{v}^{\mathrm{T}}]^{\mathrm{T}}$. Then, eq. (A.10) can be written as:

$$\begin{aligned}\dot{\mathrm{V}} &\leqslant \{-\tilde{e}_r^2 + |\tilde{e}_r|[c_0(T^{-1}(\bar{e},\tilde{e}_r,\tilde{v},\eta,\theta),t)]\} \\ &\leqslant -\tilde{e}_r^2 + |\tilde{e}_r|\rho(|\mathscr{U}|)\end{aligned} \tag{A.11}$$

where ρ is a class K_∞ function. Summarizing, we have that $\dot{V}$ satisfies the following properties:

$$\dot{V} \leqslant -\frac{|\tilde{e}_r|^2}{2}, \quad |\tilde{e}_r| \geqslant \min\{\phi, (2\rho(|\mathscr{U}|))\} =: \tilde{\gamma}_{\mathscr{U}}(|\mathscr{U}|). \tag{A.12}$$

Using the result of theorem 4.10 in Khalil (1992), we get that the following ISS bound holds for the state $\tilde{e}_r$ of the system of eq. (A.6):

$$\begin{aligned}|\tilde{e}_r(t)| &\leqslant \mathrm{e}^{-0.5t}|\tilde{e}_r(0)| + \tilde{\gamma}_{\mathscr{U}}(\|\mathscr{U}\|) \\ &\leqslant e^{-0.5t}|\tilde{e}_r(0)| + \phi.\end{aligned} \tag{A.13}$$

Referring to the singularly perturbed system comprised of the states $(\bar{e}, \eta, x_f)$ of the system of eq. (A.4), we have that its fast dynamics are globally exponentially stable and the slow system ($\bar{\eta} = [\bar{e}^{\mathrm{T}}\ \eta^{\mathrm{T}}]^{\mathrm{T}}$) is locally ISS with respect to $\tilde{e}_r, \theta, \dot{\theta}$ (Khalil, 1992). This implies that there exist a function $\beta_{\bar{\eta}}$ of class KL and functions $\bar{\gamma}_{e_r}, \bar{\gamma}_\theta, \bar{\gamma}_{\dot{\theta}}$ of class K such that the following ISS inequality holds for the state $\bar{\eta}$:

$$|\bar{\eta}(t)| \leqslant \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_r}(\|\tilde{e}_r\|) + \bar{\gamma}_\theta(\|\theta\|) + \bar{\gamma}_{\dot{\theta}}(\|\dot{\theta}\|). \tag{A.14}$$

We will now utilize the a result developed in Christofides and Daoutidis, (1998) which establishes robustness of the ISS property with respect to infinite-dimensional fast dynamics provided that they are stable and sufficiently fast, to show that the ISS inequalities of eqs. (A.13) and (A.14) continue to hold up to an arbitrarily small offset, for the states $\tilde{e}_r, \bar{\eta}$ of the singularly perturbed system of eq. (A.4). Following Christofides and Daoutidis, (1998), it can be shown that there exist positive real numbers $(\bar{\delta}, \bar{d}_0, \varepsilon^{\tilde{e}_r}(\phi))$ with $\bar{d}_0 = O(\varepsilon)$ such that if $\varepsilon \in (0, \varepsilon^{\tilde{e}_r}(\phi)]$ and $\max\{|\tilde{e}_r(0)|, \|x_f(0)\|, \|\theta\|, \|\bar{v}\|, \|\bar{\eta}\|\} \leqslant \bar{d}_0$, then

$$|\tilde{e}_r(t)| \leqslant \mathrm{e}^{-0.5t}|\tilde{e}_r(0)| + \phi + \bar{d}_0. \tag{A.15}$$

Furthermore, it can be also shown using the result from Christofides and Daoutidis (1998) that the singularly perturbed system comprised of $\bar{e}, \eta, z$, with the same converse function which exists for the reduced system comprised of the states $\bar{e}, \eta$ and its resulting $(\beta_{\bar{\eta}}, \bar{\gamma}_s)$. Thus, we have that there exist positive real numbers $(\tilde{\delta}, d_{\bar{\eta}}, \varepsilon^{\bar{\eta}}(\phi))$ such that if $\varepsilon \in (0, \varepsilon^{\bar{\eta}}(\phi)$ and $\max\{|\bar{\eta}(0)|, \|x_f(0)\|, \|\theta\|, \|\dot{\theta}\|, \|\bar{v}\|, \|\tilde{e}_r\|\} \leqslant \tilde{\delta}$, then

$$|\bar{\eta}(t) \leqslant \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{e_r}(\|\tilde{e}_r\|) + \bar{\gamma}_\theta(\|\theta\|) + \bar{\gamma}_{\dot{\theta}}(\|\dot{\theta}\|) + d_{\bar{\eta}}. \tag{A.16}$$

The proof of the theorem can be completed by: (a) analyzing the behavior of the dynamical system comprised of the states $\tilde{e}_r, \bar{\eta}$ of the system of eq. (A.4), for which the inequalities of eqs. (A.15) and (A.16) hold, using small-gain theorem type arguments [see also (Jiang *et al.*, 1995, Christofides *et al.*, 1996) for similar results on finite-dimensional systems] to establish boundedness of the state of the closed-loop system, and (b) combining the inequalities of eqs. (A.5)–(A.15) and using a claim proved in Christofides *et al.* (1996) to show that the output of the closed-loop system of eq. (A.2) satisfies the relation of eq. (29), for each $\phi \in (0, \phi^*]$, $\varepsilon^*(\phi) \in (0, \varepsilon^{\bar{\eta}}(\phi)$, and $\max\{|x_s(0)|, \|x_f(0)\|, \|\theta\|, \|\dot{\theta}\|, \|\bar{v}\|\} \leqslant \delta$.