Output Feedback Control of Nonlinear Two-Time-Scale Processes

Panagiotis D. Christofides†

Department of Chemical Engineering, University of California at Los Angeles, Los Angeles, California 90095-1592

This paper focuses on dynamic output feedback control of a broad class of nonlinear two-time-scale processes modeled within the mathematical framework of singular perturbations. A sequential procedure is employed to synthesize a nonlinear well-conditioned two-time-scale output feedback controller, which guarantees stability and enforces output tracking in the closed-loop system, provided that the separation of the slow and fast dynamics of the two-time-scale process is sufficiently large. The proposed controller is successfully applied to two-time-scale chemical processes, a series of two chemical reactors and a fluidized catalytic cracker, and is shown to outperform output feedback controller design methods that do not account for the presence of time-scale multiplicity.

1. Introduction

The majority of industrial processes are inherently nonlinear and involve unmeasured state variables. The presence of unmeasured state variables restricts significantly the implementation of nonlinear model-based control algorithms on industrial processes, and thus, may limit the achievable control quality. This has motivated extensive research activity on the design of nonlinear-state observers (i.e., dynamical systems that utilize on-line measurements of process outputs to produce estimates of all state variables (e.g., Kantor, 1989; Ciccarella et al., 1993; Kazantzis and Kravaris, 1995; Soroush, 1997)). The combination of the nonlinear-state observers and nonlinear-state feedback controllers, in order to derive nonlinear output feedback controllers, has also been studied extensively (see, for example, Limqueco and Kantor, 1990; Kravaris and Arkun, 1991; Bequette, 1991; Quintero-Marmol et al., 1991; Kravaris et al., 1994; Daoutidis and Christofides, 1995; Soroush, 1995; Kurtz and Henson, 1997 and the references therein).

In addition to nonlinearities and unmeasured state variables, many industrial processes involve physicochemical phenomena occurring in separate time scales. Examples of two-time-scale processes include fluidized catalytic crackers (the residence time in the reactor is significantly smaller than the one in the regenerator (Denn, 1986)), catalytic reactors (the thermal capacitance of the catalytic phase is much larger than the one of the gas phase (Denn, 1986)), and chemical vapor deposition reactors (coupling of fast and slow reactions), to name a few. It is well-established that a direct application of standard control methods (including the aforementioned ones) to two-time-scale processes, without accounting for the presence of time-scale multiplicity, may lead to controller ill-conditioning (i.e., the controller generates very large control actions in the presence of small modeling/measurement errors) and stiffness (i.e., the accurate numerical solution of the state observer equations requires a very small step of integration) and closed-loop instability due to slightly nonminimum phase behavior of the process (i.e., the zero dynamics of the process is a two-time-scale system with unstable fast dynamics) (Kokotovic et al., 1986; Christofides and Daoutidis, 1996).

In order to circumvent the above problems, the control of two-time-scale processes is usually addressed within the singular perturbation framework (Kokotovic et al., 1986). Within this framework, a two-time-scale process is initially modeled in the standard singularly perturbed form, where the fast and slow variables are explicitly separated due to the presence of a small parameter ϵ (called singular perturbation parameter) that multiplies the time derivative of the fast-state vector. Then, the singularly perturbed system is decomposed into separate well-conditioned reduced-order systems that describe the fast and slow dynamics of the original system. These reduced-order systems are used to synthesize well-conditioned nonlinear controllers, and singular perturbation techniques are employed to infer the asymptotic (i.e., for ϵ sufficiently small) properties of the two-time-scale closed-loop system from the knowledge of the behavior of the reduced-order closed-loop systems. Following this approach, two-time-scale *output* feedback controllers have been designed for *linear* singularly perturbed systems (e.g., O'Reilly, 1980; Kokotovic et al., 1986; Khalil, 1987; Calise et al., 1990; Wang et al., 1993), while optimal- (Kokotovic et al., 1986), geometric- (Christofides and Daoutidis, 1996), and Lyapunov-based (Christofides et al., 1996) *state* feedback controllers have been designed for *nonlinear* singularly perturbed systems. On the problem of state estimation of nonlinear two-time-scale systems, a method was recently proposed in Shouse and Taylor (1995) for the design of nonlinear two-time-scale discrete-time state observers.

In this paper, we study the output feedback control of a broad class of nonlinear two-time-scale processes, modeled within the mathematical framework of singular

† Tel.: (310)794-1015. Fax: (310)206-4107. E-mail: pdc@seas.ucla.edu.

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perturbations. We initially employ a sequential procedure to derive a nonlinear two-time-scale output feedback controller. The controller, which does not suffer from ill-conditioning and stiffness, globally asymptotically stabilizes the fast subsystem and enforces global asymptotic stability with asymptotic output tracking in the closed-loop slow subsystem. It is established, using a general stability theorem proved in Christofides and Teel (1996), that the controller enforces boundedness of the states and approximate asymptotic (i.e., for large times) output tracking in the closed-loop two-time-scale system for arbitrarily large initial conditions, provided that the separation of the slow and fast dynamical phenomena of the process is sufficiently large. Then, an explicit easy-to-use realization of the controller is derived that enforces local exponential stability and approximate output tracking, for all times, in the closed-loop two-time-scale system. Fundamental differences, on output feedback control of linear and nonlinear two-time-scale processes, are identified and discussed. Finally, the proposed controller is successfully applied to two-time-scale chemical processes, a series of two chemical reactors and a fluidized catalytic cracker, and is shown to outperform output feedback controller design methods that do not account for the presence of time-scale multiplicity.

2. Preliminaries

We will focus on two-time-scale nonlinear processes, modeled in singularly perturbed form, with the following state-space description:

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x})\mathbf{z} + \mathbf{g}_1(\mathbf{x})u$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})u \qquad (1)$$
$$y_1 = h_1(\mathbf{x}), \quad y_2 = \mathbf{p}(\mathbf{x})\mathbf{z}$$

where $\mathbf{x} \in \mathrm{IR}^n$ denotes the vector of *slow*-state variables and $\mathbf{z} \in \mathrm{IR}^p$ denotes vector of *fast*-state variables, $u \in \mathrm{IR}$ denotes the manipulated input, $y_1 \in \mathrm{IR}$ denotes the controlled output which is also assumed to be measurable, $y_2 \in \mathrm{IR}$ denotes the fast measured output, and ϵ is a small positive parameter which quantifies the speed ratio of the slow versus the fast dynamical phenomena of the process. $\mathbf{f}_1(\mathbf{x})$, $\mathbf{f}_2(\mathbf{x})$, $\mathbf{p}(\mathbf{x})$, $\mathbf{g}_1(\mathbf{x})$, and $\mathbf{g}_2(\mathbf{x})$ are sufficiently smooth vector functions, $\mathbf{Q}_1(\mathbf{x})$ and $\mathbf{Q}_2(\mathbf{x})$ are sufficiently smooth matrices of appropriate dimensions, and $h_1(\mathbf{x})$ is a sufficiently smooth scalar function. Throughout the paper, the standard order of magnitude notation, $O(\epsilon)$, will be used (i.e., $\delta(\epsilon) = O(\epsilon)$ if there exist positive constants k_1 and k_2 such that $|\delta(\epsilon)| \le k_1|\epsilon|$, $\forall$ $|\epsilon| < k_2$).

Referring to the system of eq 1, several remarks are in order: (a) the separation of slow and fast variables is explicit, owing to the presence of the small parameter ϵ that multiplies $\dot{\mathbf{z}}$, (b) the fast state variables $\mathbf{z}$ appear linearly (this is consistent with the fact that the main nonlinearities in chemical processes are usually associated with the slow dynamics), (c) the parameter ϵ appears only in the left-hand side (multiplying $\dot{\mathbf{z}}$) (this assumption is made in order to simplify the notation of the paper (the results of the paper can be directly applied to a more general class of singularly perturbed systems which includes ϵ in the right-hand side; see remark 10), and (d) there exists an additional measured output, y_2, which depends on the fast variable z (this is necessary in order to design a dynamic output feedback controller to stabilize the fast dynamics of the process (see subsection 3.2)).

The derivation of a singularly perturbed representation of a nonlinear two-time-scale process is, in general, a highly nontrivial task. The natural approach to address this problem involves defining the singular perturbation parameter ϵ, taking into account the physicochemical characteristics of the process, so that in the resulting singularly perturbed representation the separation of the fast and slow variables is consistent with the process dynamic behavior. This approach works for the majority of two-time-scale processes (see, for example, the applications considered in Kokotovic et al. (1986), Christofides and Daoutidis (1996), and the chemical process examples of section 4). Whenever this approach does not work, alternative approaches that utilize explicit coordinate changes (e.g., Marino and Kokotovic, 1988; Dochain and Bouaziz, 1993; Breusegem and Bastin, 1991; Kumar et al., 1998) or computational techniques (e.g., Lam and Goussis, 1994; Duchene and Rouchon, 1996) may be employed to derive a singularly perturbed representation of a two-time-scale process. We finally note that singular perturbation modeling of *linear* two-time-scale processes can be performed by using modal decomposition techniques (e.g., Kokotovic et al., 1986; Denn, 1986; Monge and Georgakis, 1987).

The explicit separation of the slow and fast variables in the system of eq 1 allows us to decompose it into separate reduced-order systems evolving in separate time scales. Specifically, setting $\epsilon = 0$ in the system of eq 1 and assuming that $\mathbf{Q}_2(\mathbf{x})$ is invertible uniformly in $\mathbf{x} \in \mathrm{IR}^n$ (this assumption will be removed later), the following slow subsystem is obtained:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \mathbf{G}(\mathbf{x})u$$
$$y_1^s = h_1(\mathbf{x}) \qquad (2)$$

where the superscript s in y_1^s denotes that the output is associated with the slow subsystem and

$$\mathbf{F}(\mathbf{x}) = \mathbf{f}_1(\mathbf{x}) - \mathbf{Q}_1(\mathbf{x})[\mathbf{Q}_2(\mathbf{x})]^{-1}\mathbf{f}_2(\mathbf{x})$$
$$\mathbf{G}(\mathbf{x}) = \mathbf{g}_1(\mathbf{x}) - \mathbf{Q}_1(\mathbf{x})[\mathbf{Q}_2(\mathbf{x})]^{-1}\mathbf{g}_2(\mathbf{x}) \qquad (3)$$

The system of eq 2 describes the slow dynamics of the two-time-scale system of eq 1.

Defining the fast time scale $\tau = t/\epsilon$, deriving the representation of the system of eq 1 in the τ time scale, and setting $\epsilon = 0$, the following fast subsystem is obtained:

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})u$$
$$y_2 = \mathbf{p}(\mathbf{x})\mathbf{z} \qquad (4)$$

where $\mathbf{x}$ can be considered equal to its initial value $\mathbf{x}(0)$. The system of eq 2 captures the fast dynamics of the two-time-scale system of eq 1.

In this work, we consider systems of the form of eq 1, for which the corresponding fast subsystem of the form of eq 4 is observable and stabilizables uniformly in $\mathbf{x} \in \mathrm{IR}^n$. This assumption will be exploited later (see section 3) in the design of a dynamic output feedback controller which operates in the fast time scale to stabilize the fast subsystem and is precisely formulated below.

Assumption 1. The pairs ($\mathbf{Q}_2(\mathbf{x})$, $\mathbf{g}_2(\mathbf{x})$) and ($\mathbf{p}(\mathbf{x})$, $\mathbf{Q}_2(\mathbf{x})$) are stabilizable and observable uniformly in $\mathbf{x} \in \mathrm{IR}^n$, respectively, in the sense that there exist sufficiently smooth vectors $\mathbf{k}^{\mathrm{T}}(\mathbf{x})$ and $\mathbf{l}(\mathbf{x})$ such that the matrices $\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})$ and $\mathbf{Q}_2(\mathbf{x}) - \mathbf{l}(\mathbf{x})\mathbf{p}(\mathbf{x})$ are Hurwitz uniformly in $\mathbf{x} \in \mathrm{IR}^n$.

Remark 1. In practice, the design of the gains $\mathbf{k}^{\mathrm{T}}(\mathbf{x})$ and $\mathbf{l}(\mathbf{x})$ can be performed by utilizing standard optimal control methods (Kokotovic et al., 1986) to ensure that the matrices $\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})$ and $\mathbf{Q}_2(\mathbf{x}) - \mathbf{l}(\mathbf{x})\mathbf{p}(\mathbf{x})$ are Hurwitz uniformly in $\mathbf{x} \in \mathrm{IR}^n$.

Finally, in order to point out differences on output feedback control of linear and nonlinear two-time-scale systems, we set $\mathbf{f}_1(\mathbf{x}) = \mathbf{A}_1\mathbf{x}$, $\mathbf{f}_2(\mathbf{x}) = \mathbf{A}_2\mathbf{x}$, $\mathbf{Q}_1(\mathbf{x}) = \mathbf{Q}_1$, $\mathbf{Q}_2(\mathbf{x}) = \mathbf{Q}_2$, $\mathbf{g}_1(\mathbf{x}) = \mathbf{g}_1$, $\mathbf{g}_2(\mathbf{x}) = \mathbf{g}_2$, $\mathbf{h}_1(\mathbf{x}) = \mathbf{c}\mathbf{x}$, and $\mathbf{p}(\mathbf{x}) = \mathbf{p}$, where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ are matrices and $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{p}$, and $\mathbf{c}$ are vectors of appropriate dimensions, to the system of eq 1, and we derive the following linear two-time-scale system:

$$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{Q}_1\mathbf{z} + \mathbf{g}_1 u$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{A}_2\mathbf{x} + \mathbf{Q}_2\mathbf{z} + \mathbf{g}_2 u \quad (5)$$
$$y_1 = \mathbf{c}\mathbf{x}, \quad y_2 = \mathbf{p}\mathbf{z}$$

which will be used to synthesize a linear two-time-scale output feedback controller (see subsection 3.4).

3. Output Feedback Control

3.1. Output Feedback Control Problem Formulation. Referring to the system of eq 1, the output feedback control problem is the one of synthesizing well-conditioned two-time-scale dynamic output feedback controllers of the following form:

$$\epsilon\dot{\boldsymbol{\omega}} = \mathcal{F}(\boldsymbol{\omega},\boldsymbol{\eta},y_1,y_2,v)$$
$$\dot{\boldsymbol{\eta}} = \mathcal{G}(\boldsymbol{\omega},\boldsymbol{\eta},y_1,v) \quad (6)$$
$$u = \mathcal{P}_1(\boldsymbol{\eta},v) + \mathcal{P}_2(\boldsymbol{\eta})\boldsymbol{\omega}$$

where $\boldsymbol{\omega} \in \mathrm{IR}^n$ and $\boldsymbol{\eta} \in \mathrm{IR}^p$ are the controller states, $\mathcal{G}(\boldsymbol{\omega},\boldsymbol{\eta},y_1,v)$ and $\mathcal{F}(\boldsymbol{\omega},\boldsymbol{\eta},y_1,y_2,v)$ are nonlinear vector functions, $\mathcal{P}_1(\boldsymbol{\eta},v)$ is a nonlinear scalar function, $\mathcal{P}_2(\boldsymbol{\eta})$ is a nonlinear vector function, and v is the setpoint, that enforce stability and output tracking in the closed-loop system, provided that ϵ is sufficiently small.

To address the above problem, we will initially characterize the properties enforced in the two-time-scale closed-loop system by a nonlinear two-time-scale output feedback controller that globally asymptotically stabilizes the fast subsystem and enforces global asymptotic stability (see Appendix for the definition of asymptotic stability of an equilibrium point) with asymptotic output tracking in the slow subsystem. To address this problem, we use a new stability theorem for nonlinear two-time-scale systems proved in Christofides and Teel (1996) and reviewed, for completeness, in the Appendix of the present paper. Then, an explicit easy-to-use realization of the controller will be derived that enforces local exponential stability (see Appendix for the definition of exponential stability of an equilibrium point) and approximate output tracking, for all times, in the closed-loop two-time-scale system. Finally, the counterpart of our result in the case of linear two-time-scale systems of the form of eq 5 will be derived, and differences in the nature of the solution of the output feedback control problem between linear and nonlinear two-time-scale systems will be pointed out.

3.2. A General Result. We initially assume that $\mathbf{x}$ is known (this assumption will be removed below) and consider dynamic output feedback laws of the following form:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\boldsymbol{\omega} + \mathbf{g}_2(\mathbf{x})u + \mathbf{l}(\mathbf{x})(y_2 - \mathbf{p}(\mathbf{x})\boldsymbol{\omega})$$
$$u = \tilde{u} + \mathbf{k}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\omega} \quad (7)$$

where $\tilde{u}$ is an auxiliary input, to stabilize the fast dynamics. The linear dependence of the controller of eq 7 in $\boldsymbol{\omega}$ is motivated from the linear appearance of the fast variable, $\mathbf{z}$, in the system of eq 1. Under a control law of the form of eq 7, the system of eq 1 takes the following form:

$$\epsilon\dot{\boldsymbol{\omega}} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\boldsymbol{\omega} + \mathbf{g}_2(\mathbf{x})u + \mathbf{l}(\mathbf{x})(y_2 - \mathbf{p}(\mathbf{x})\boldsymbol{\omega})$$
$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x})\mathbf{z} + \mathbf{g}_1(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\omega} + \mathbf{g}_1(\mathbf{x})\tilde{u} \quad (8)$$
$$\epsilon\dot{\mathbf{z}} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\omega} + \mathbf{g}_2(\mathbf{x})\tilde{u}$$

The fast dynamics of the above system are described by the following modified fast subsystem:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\boldsymbol{\omega} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\omega} + \mathbf{g}_2(\mathbf{x})\tilde{u} + \mathbf{l}(\mathbf{x})(y_2 - \mathbf{p}(\mathbf{x})\boldsymbol{\omega})$$
$$\frac{d\mathbf{z}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\omega} + \mathbf{g}_2(\mathbf{x})\tilde{u} \quad (9)$$

Defining the error vector $\mathbf{e}_\mathbf{z} = \boldsymbol{\omega} - \mathbf{z}$, the above system can be written as

$$\frac{d\mathbf{e}_\mathbf{z}}{d\tau} = [\mathbf{Q}_2(\mathbf{x}) - \mathbf{l}(\mathbf{x})\mathbf{p}(\mathbf{x})]\mathbf{e}_\mathbf{z}$$
$$\frac{d\mathbf{z}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + [\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})]\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\mathbf{e}_\mathbf{z} + \mathbf{g}_2(\mathbf{x})\tilde{u} \quad (10)$$

From the cascaded structure of the above system and assumption 1, we have that the modified fast subsystem of eq 9 can be made globally asymptotically stable uniformly in $\mathbf{x} \in \mathrm{IR}^n$ through appropriate selection of the controller and observer gains $\mathbf{l}(\mathbf{x})$ and $\mathbf{k}^{\mathrm{T}}(\mathbf{x})$, respectively.

Setting $\epsilon = 0$ in the system of eq 8, the following modified slow subsystem is obtained:

$$\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x}) + \tilde{\mathbf{G}}(\mathbf{x})\tilde{u}$$
$$y_1^s = h_1(\mathbf{x}) \quad (11)$$

where

$$\tilde{\mathbf{F}}(\mathbf{x}) = \mathbf{f}_1(\mathbf{x}) - [\mathbf{Q}_1(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})\mathbf{k}^T(\mathbf{x})][\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^T(\mathbf{x})]^{-1}\mathbf{f}_2(\mathbf{x})$$

$$\tilde{\mathbf{G}}(\mathbf{x}) = \mathbf{g}_1(\mathbf{x}) - [\mathbf{Q}_1(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})\mathbf{k}^T(\mathbf{x})][\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^T(\mathbf{x})]^{-1}\mathbf{g}_2(\mathbf{x}) \quad (12)$$

Referring to the system of eq 11, assumption 2 below states that there exists a dynamic output feedback controller that globally asymptotically stabilizes the closed-loop slow subsystem. Even though the design of such a dynamic output feedback controller is an unresolved, in general, task, this assumption will allow us to provide a general solution (see theorem 1) to the output feedback control problem for nonlinear two-time-scale processes and point out fundamental differences on the nature of the solution of the problem between linear and nonlinear systems (see remark 12). We note that the problem of synthesizing an easy-to-implement nonlinear two-time-scale output feedback controller that enforces local exponential stability and approximate output tracking in the closed-loop system is addressed in subsection 3.3.

Assumption 2. There exists a dynamic output feedback controller of the following form:

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\eta) + \tilde{\mathbf{G}}(\eta)\tilde{u} + \mathbf{s}(\eta)(y_1 - h_1(\eta))$$
$$\tilde{u} = \bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v \quad (13)$$

where $\mathbf{s}(\boldsymbol{\eta})$ is a sufficiently smooth vector function (observer gain) $\bar{p}(\boldsymbol{\eta})$ and $q(\boldsymbol{\eta})$ are sufficiently smooth scalar functions, and v is the setpoint, such that the closed-loop slow system

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})[\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v] + \mathbf{s}(\boldsymbol{\eta})(y_1 - h_1(\boldsymbol{\eta}))$$
$$\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x}) + \tilde{\mathbf{G}}(\mathbf{x})[\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v] \quad (14)$$
$$y_1^s = h_1(\mathbf{x})$$

is globally asymptotically stable and $\lim_{t\to\infty}(y_1^s - v) = 0$.

Combining the *fast* output feedback controller of eq 7 with the *slow* output feedback controller of eq 14, the following two-time-scale output feedback controller is obtained:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\boldsymbol{\eta})\boldsymbol{\omega} + \mathbf{l}(\boldsymbol{\eta})(y_2 - \mathbf{p}(\boldsymbol{\eta})\boldsymbol{\omega}) + g_2(\boldsymbol{\eta})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v + \mathbf{k}^T(\boldsymbol{\eta})\boldsymbol{\omega})$$
$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v) + \mathbf{s}(\boldsymbol{\eta})(y_1 - h_1(\boldsymbol{\eta})) \quad (15)$$
$$u = \bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v + \mathbf{k}^T(\boldsymbol{\eta})\boldsymbol{\omega}$$

which does not suffer from ill-conditioning and stiffness problems because the static component, $u = \bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v + \mathbf{k}^T(\boldsymbol{\eta})\boldsymbol{\omega}$, and the observer gains, $\mathbf{l}(\boldsymbol{\eta})$ and $\mathbf{s}(\boldsymbol{\eta})$, are independent of ϵ. Theorem 1 below establishes stability and output tracking results for the closed-loop system under the controller of eq 15, for arbitrarily large initial conditions.

Theorem 1. Consider the nonlinear two-time-scale system of eq 1, for which assumptions 1 and 2 hold, under the dynamic output feedback controller of eq 15. Then, for each pair of positive real numbers, δ and d, there exists an $\epsilon^* > 0$ such that if $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, $\max\{|\mathbf{x}(0)|, |\mathbf{z}(0)|, |\boldsymbol{\omega}(0)|, |\boldsymbol{\eta}(0)|, v\} \leq \delta$, and $\epsilon \in (0,\epsilon^*)$, the states of the closed-loop system are bounded and the output satisfies

$$\lim_{t\to\infty} |y_1(t) - v| \leq d \quad (16)$$

Remark 2. Theorem 1 establishes that the two-time-scale output feedback controller of eq 15 guarantees boundedness of the states and enforces asymptotic output tracking, up to an arbitrarily small offset, in the closed-loop system for arbitrarily large initial conditions, provided that ϵ is sufficiently small. We note that theorem 1 provides no local exponential stability result of the origin. In order to derive such a result, we need to assume (in addition to assumptions 1 and 2) that the closed-loop fast subsystem of eq 9 and the closed-loop slow subsystem of eq 14 are locally exponentially stable (see Theorem 2 below for such a result).

Remark 3. Notice that if the matrix $\mathbf{Q}_2(\mathbf{x})$ and the vectors $\mathbf{f}_2(\mathbf{x})$, $\mathbf{g}_2(\mathbf{x})$, and $\mathbf{p}(\mathbf{x})$ in the system of eq 1 are independent of $\mathbf{x}$, then the initialization requirement on the states of the slow observer, $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, is not required, since $\mathbf{k}^T(\boldsymbol{\eta})$ and $\mathbf{l}(\boldsymbol{\eta})$ can be chosen to be independent of $\boldsymbol{\eta}$.

Remark 4. The main advantages of the use of dynamic output feedback, over the use of static output feedback (Pan and Basar, 1994), to stabilize the fast dynamics, are (a) there is no need to assume the restrictive requirement that the triple ($\mathbf{Q}_2(\mathbf{x})$, $\mathbf{g}_2(\mathbf{x})$, $\mathbf{p}(\mathbf{x})$) is uniformly stabilizable/detectable and (b) dynamic output feedback control is more robust to output measurement noise than static output feedback control.

3.3. Controller Synthesis. The design of a nonlinear dynamic output feedback controller of the form of eq 14 to globally asymptotically stabilize the closed-loop slow subsystem is an unresolved, in general, task. For example, at this stage, there exists no general procedure for the design of the observer gain, $\mathbf{s}(\boldsymbol{\eta})$, that yields a slow state observer with globally asymptotically stable error (between the actual and the estimated state) dynamics. Furthermore, even if it is possible to design such an $\mathbf{s}(\boldsymbol{\eta})$ and also synthesize the component $\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v$ to globally asymptotically stabilize the closed-loop slow subsystem with state feedback, there is no guarantee that the resulting output feedback controller will globally asymptotically stabilize the closed-loop slow subsystem. Motivated by this, theorem 2 that follows provides an explicit formula of a dynamic output feedback controller of the form of eq 15 that enforces local exponential stability and approximate output tracking up to $O(\epsilon)$, for all times, in the closed-loop two-time-scale system.

Theorem 2. Consider the nonlinear two-time-scale system of eq 1 for which assumption 1 holds, under a dynamic output feedback controller of the following form:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^T(\boldsymbol{\eta})]\boldsymbol{\omega} + \mathbf{l}(\boldsymbol{\eta})(y_2 - p(\boldsymbol{\eta})\boldsymbol{\omega}) + \mathbf{g}_2(\boldsymbol{\eta})([\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^k h_1(\boldsymbol{\eta})\})$$

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})([\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^k h_1(\boldsymbol{\eta})\}) + \mathbf{S}(y_1 - h_1(\boldsymbol{\eta}))$$

$$u = [\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^k h_1(\boldsymbol{\eta})\} + \mathbf{k}^T(\boldsymbol{\eta})\boldsymbol{\omega} \quad (17)$$

where $\mathbf{S}$ is a column vector chosen so that the matrix $\partial\tilde{\mathbf{F}}/\partial\boldsymbol{\eta}(\boldsymbol{\eta}_s) - \mathbf{S}\ \partial h_1/\partial\boldsymbol{\eta}(\boldsymbol{\eta}_s)$ is Hurwitz and $\boldsymbol{\eta}_s$ is the operating steady state, $\tilde{r}$ is the relative order of the system of eq 11 (see definition 1 in the Appendix), and β_k are parameters chosen so that the roots of the polynomial $\beta_0 + \beta_1 s + \cdots + \beta_{\tilde{r}} s^{\tilde{r}} = 0$ have negative real part. Suppose also that the zero dynamics of the system of eq 11 is locally exponentially stable. Then, there exist positive real numbers, $\bar{\delta}$ and $\bar{\epsilon}$, such that if $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, $\max\{|\mathbf{x}(0)|, |\mathbf{z}(0)|, |\boldsymbol{\omega}(0)|, |\boldsymbol{\eta}(0)|, v\} \leq \bar{\delta}$, and $\epsilon \in (0,\bar{\epsilon}]$, the closed-loop system is exponentially stable and its output satisfies

$$y_1(t) = y_1^s(t) + O(\epsilon), \qquad t \geq 0 \quad (18)$$

where y_1^s is the solution of

$$\sum_{k=0}^{\tilde{r}} \beta_k \frac{d^k y_1^s}{dt^k} = v$$

Remark 5. Regarding the practical application of the result of theorem 2 to a two-time-scale process, one has to initially obtain a representation of the original process model in the standard singularly perturbed form and then use this representation to verify that assumption 1 holds and that the modified slow subsystem is minimum phase. If these assumptions are satisfied, then the synthesis formula of eq 17 can be directly used to derive the explicit form of the controller (see section 4 for applications of this procedure to two chemical process examples).

Remark 6. Regarding the controller of eq 17, we note that (a) it does not suffer from ill-conditioning and stiffness problems because the static component and the observer gains are independent of ϵ, (b) the static component of the slow controller,

$$[\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^k h_1(\boldsymbol{\eta})\}$$

which enforces exponential stability and output tracking in the closed-loop slow subsystem, was synthesized by using geometric control methods, and (c) the use of a constant gain, $\mathbf{S}$, in the slow observer ($\boldsymbol{\eta}$-subsystem) is not necessary; one could also use nonlinear gains (designed, for example, by using the procedure proposed in Kazantzis and Kravaris (1995)) and prove the same properties for the closed-loop system; such a choice, however, would probably lead to a more complex controller design.

Remark 7. When the open-loop fast subsystem of eq 1 is exponentially stable uniformly in $\mathbf{x} \in \mathrm{IR}^n$, there is no need to use fast dynamic output feedback to stabilize the fast dynamics (this is motivated by the fact that the fast dynamics of the process die out after a short time interval), and the controller of eq 21 can be simplified to

$$\dot{\boldsymbol{\eta}} = \mathbf{F}(\boldsymbol{\eta}) + \mathbf{G}(\boldsymbol{\eta})([\beta_r L_{\mathbf{G}} L_{\mathbf{F}}^{r-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{r} \beta_k L_{\mathbf{F}}^k h_1(\boldsymbol{\eta})\}) + \mathbf{S}(y_1 - h_1(\boldsymbol{\eta}))$$

$$u = [\beta_r L_{\mathbf{G}} L_{\mathbf{F}}^{r-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{r} \beta_k L_{\mathbf{F}}^k h_1(\boldsymbol{\eta})\} \quad (19)$$

where r is the relative order of the system of eq 2. The above controller can be directly synthesized by utilizing information about the open-loop slow subsystem of eq 2. In this case, the result of theorem 2 establishes a robustness property of the controller of eq 19, with respect to uniformly exponentially stable unmodeled dynamics (e.g., sensor and actuator dynamics which are neglected in the controller synthesis), provided that they are sufficiently fast.

Remark 8. For open-loop stable two-time-scale processes, the gain of the slow observer, $\mathbf{S}$, of the controller of eq 19 can also be set equal to zero; this is motivated by the fact that the open-loop stability of the slow subsystem guarantees the convergence of the estimated values of the slow states, $\boldsymbol{\eta}$, to the actual ones, $\mathbf{x}$, with transient behavior depending on the location of the spectrum of the matrix

$$\bar{\mathbf{C}}_L = \frac{\partial\tilde{\mathbf{F}}}{\partial\boldsymbol{\eta}_s}(\boldsymbol{\eta}_s)$$

and the controller of eq 19 (and thus, the controller of eq 17) can be simplified to

$$\dot{\boldsymbol{\eta}} = \mathbf{F}(\boldsymbol{\eta}) + \mathbf{G}(\boldsymbol{\eta})([\beta_r L_{\mathbf{G}} L_{\mathbf{F}}^{r-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{r} \beta_k L_{\mathbf{F}}^k h_1(\boldsymbol{\eta})\})$$

$$u = [\beta_r L_{\mathbf{G}} L_{\mathbf{F}}^{r-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{r} \beta_k L_{\mathbf{F}}^k h_1(\boldsymbol{\eta})\} \quad (20)$$

In this case, the result of theorem 2 establishes a robustness property of the controller of eq 20, with respect to uniformly exponentially stable and fast unmodeled dynamics.

Remark 9. The exponential stability of the closed-loop system guarantees that in the presence of small errors in process parameters, the states of the closed-loop system will be bounded. Furthermore, since the controller of eq 17 enforces an approximately linear input/output dynamics between y_1 and v, it is possible to implement a linear error feedback controller around the $(y_1 - v)$ loop to ensure asymptotic offsetless output tracking in the closed-loop system, in the presence of constant unknown process parameters and unmeasured disturbance inputs. In this case, one can use calculations similar to the ones in Kravaris et al. (1994) and Daoutidis and Christofides (1995) to derive the following two-time-scale mixed-error and output feedback controller, which possesses integral action, of the following form:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^{T}(\boldsymbol{\eta})]\boldsymbol{\omega} + \mathbf{l}(\boldsymbol{\eta})(\mathbf{y}_2 - \mathbf{p}(\boldsymbol{\eta})\boldsymbol{\omega}) + \mathbf{g}_2(\boldsymbol{\eta})([\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{e - \sum_{k=1}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^{k} h_1(\boldsymbol{\eta})\})$$

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})([\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{e - \sum_{k=1}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^{k} h_1(\boldsymbol{\eta})\}) + \mathbf{S}(y_1 - h_1(\boldsymbol{\eta}))$$

$$u = [\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{e - \sum_{k=1}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^{k} h_1(\boldsymbol{\eta})\} + \mathbf{k}^{T}(\boldsymbol{\eta})\boldsymbol{\omega} \tag{21}$$

that enforces exponential stability and asymptotic offsetless output tracking in the closed-loop system in the presence of constant modeling errors and disturbances. We finally note that for open-loop stable two-time-scale systems, the controller of eq 21 reduces to

$$\dot{\boldsymbol{\eta}} = \mathbf{F}(\boldsymbol{\eta}) + \mathbf{G}(\boldsymbol{\eta})([\beta_r L_{\mathbf{G}} L_{\mathbf{F}}^{r-1} h_1(\boldsymbol{\eta})]^{-1}\{e - \sum_{k=1}^{r} \beta_k L_{\mathbf{F}}^{k} h_1(\boldsymbol{\eta})\})$$

$$u = [\beta_r L_{\mathbf{G}} L_{\mathbf{F}}^{r-1} h_1(\boldsymbol{\eta})]^{-1}\{e - \sum_{k=1}^{r} \beta_k L^{k}{}_{\mathbf{F}} h_1(\boldsymbol{\eta})\} \tag{22}$$

which is a model-state feedback controller synthesized on the basis of the open-loop slow subsystem (Kravaris et al., 1994) (see also Coulibaly et al. (1995) for the synthesis of model-state feedback controllers for linear systems). For the controller of eq 22, similar arguments as in theorem 2 can be used to establish a robustness property with respect to uniformly exponentially stable and fast unmodeled dynamics.

Remark 10. The fundamental result of theorem 1 and the control algorithm of theorem 2 can be directly applied to two-time-scale processes with ϵ-dependent right-hand side, with the following state-space description:

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}, \epsilon\mathbf{z}, \epsilon) + \mathbf{Q}_1(\mathbf{x}, \epsilon\mathbf{z}, \epsilon)\mathbf{z} + \mathbf{g}_1(\mathbf{x}, \epsilon\mathbf{z}, \epsilon)u$$

$$\epsilon\dot{\mathbf{z}} = \mathbf{R}(\mathbf{x}, \mathbf{z})[\mathbf{f}_2(\mathbf{x}, \epsilon\mathbf{z}, \epsilon) + \mathbf{Q}_2(\mathbf{x}, \epsilon\mathbf{z}, \epsilon)\mathbf{z} + \mathbf{g}_2(\mathbf{x}, \epsilon\mathbf{z}, \epsilon)u] \tag{23}$$

$$y_1 = h_1(\mathbf{x}, \epsilon), \quad y_2 = \mathbf{p}(\mathbf{x}, \epsilon)\mathbf{z}$$

where $\mathbf{R}(\mathbf{x}, \mathbf{z})$ is a diagonal matrix of dimension $p \times p$, which is positive definite for all $\mathbf{x} \in \mathrm{IR}^n$ and $\mathbf{z} \in \mathrm{IR}^p$. This is possible because the stabilizability and observability requirements of assumption 1 suffice to ensure that the fast subsystem of the system of eq 23 can be made exponentially stable via fast dynamic output feedback of the form of eq 7, and the modified slow subsystem obtained after the stabilization of the fast subsystem is the same as the one of eq 11, which means that there is no need to modify assumption 2 in order to derive the result of theorem 1, and the control algorithm of theorem 2 (eq 17) remains the same.

Remark 11. The two-time-scale output feedback controller synthesis result of theorem 2 can be readily generalized to multi-input multi-output nonlinear two-time-scale systems with a state-space description of the following form:

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x})\mathbf{z} + \mathbf{G}_1(\mathbf{x})\mathbf{u}$$

$$\epsilon\dot{\mathbf{z}} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{G}_2(\mathbf{x})\mathbf{u} \tag{24}$$

$$y_{1i} = h_{1i}(\mathbf{x}), \quad i = 1, ..., m, \; \bar{\mathbf{y}}_2 = \mathbf{P}(\mathbf{x})\mathbf{z}$$

where $\mathbf{u} = [u_1 \cdots u_m]^T \in \mathrm{IR}^m$ denotes the manipulated input vector, y_{1i} is the ith controlled output, $\bar{\mathbf{y}}_2$ denotes the vector of fast measured outputs, $\mathbf{G}_1(\mathbf{x})$, $\mathbf{G}_2(\mathbf{x})$, and $\mathbf{P}(\mathbf{x})$ are matrices of appropriate dimensions, and $h_{1i}(\mathbf{x})$ are scalar functions. Even though the detailed theoretical development and solution of the output feedback controller synthesis problem for systems of eq 24 will not be presented here for brevity, in subsection 4.2, we design and implement a multivariable nonlinear output feedback controller on a fluidized catalytic cracking process which exhibits two-time-scale behavior.

3.4. Output Feedback Control of Linear Two-Time-Scale Processes. Corollary 1 that follows provides the counterpart of the result of theorem 2 for linear two-time-scale systems of the form of eq 5. The proof of the corollary is similar to the proof of theorem 2, and thus will be omitted for brevity.

Corollary 1. Consider the linear two-time-scale system of eq 5 and assume that the pairs ($\mathbf{Q}_2$, $\mathbf{g}_2$) and ($\mathbf{g}_2$, $\mathbf{p}$) are stabilizable and detectable, respectively, under a dynamic output feedback controller of the following form:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{A}_2\boldsymbol{\eta} + [\mathbf{Q}_2 + \mathbf{g}_2\mathbf{k}^{T}]\boldsymbol{\omega} + \mathbf{l}(\mathbf{y}_2 - \mathbf{p}\boldsymbol{\omega}) + \mathbf{g}_2([\beta_{\tilde{r}}\mathbf{c}\tilde{\mathbf{A}}^{\tilde{r}-1}\tilde{\mathbf{G}}]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k \mathbf{c}\tilde{\mathbf{A}}^{k-1}\boldsymbol{\eta}\})$$

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{A}}\boldsymbol{\eta} + \tilde{\mathbf{G}}([\beta_{\tilde{r}}\mathbf{c}\tilde{\mathbf{A}}^{\tilde{r}-1}\tilde{\mathbf{G}}]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k \mathbf{c}\tilde{\mathbf{A}}^{k-1}\boldsymbol{\eta}\}) + \mathbf{S}(y_1 - \mathbf{c}\boldsymbol{\eta}) \tag{25}$$

$$u = ([\beta_{\tilde{r}}\mathbf{c}\tilde{\mathbf{A}}^{\tilde{r}-1}\tilde{\mathbf{G}}]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k \mathbf{c}\tilde{\mathbf{A}}^{k-1}\boldsymbol{\eta}\}) + \mathbf{k}^{T}\boldsymbol{\omega}$$

where $\tilde{\mathbf{A}} = \mathbf{A}_1 - [\mathbf{Q}_1 + \mathbf{g}_1\mathbf{k}^T][\mathbf{Q}_2 + \mathbf{g}_2 k^T]^{-1}\mathbf{A}_2$, $\tilde{\mathbf{G}} = \mathbf{g}_1 - [\mathbf{Q}_1 + \mathbf{g}_1\mathbf{k}^T][\mathbf{Q}_2 + \mathbf{g}_2\mathbf{k}^T]^{-1}\mathbf{g}_2$, ($\mathbf{S}$, $\mathbf{k}^T$, $\mathbf{l}$) are vectors chosen so that the matrices $\tilde{\mathbf{A}} - \mathbf{Sc}$, $\mathbf{Q}_2 + \mathbf{g}_2\mathbf{k}^T$, and $\mathbf{Q}_2 - \mathbf{lp}$ are Hurwitz, respectively, $\tilde{r}$ is the relative order of the appropriate modified slow subsystem, and β_k are parameters chosen so that the roots of the polynomial $\beta_0 + \beta_1 s + ... + \beta_{\tilde{r}} s^{\tilde{r}} = 0$ have negative real part. Suppose also that the zero dynamics of the modified slow subsystem is exponentially stable. Then, there exists a positive real number $\bar{\epsilon}$, such that if $\epsilon \in (0, \bar{\epsilon})$, the closed-loop system is exponentially stable and the output satisfies

$$y_1(t) = y_1^{s}(t) + O(\epsilon), \qquad t \geq 0 \tag{26}$$

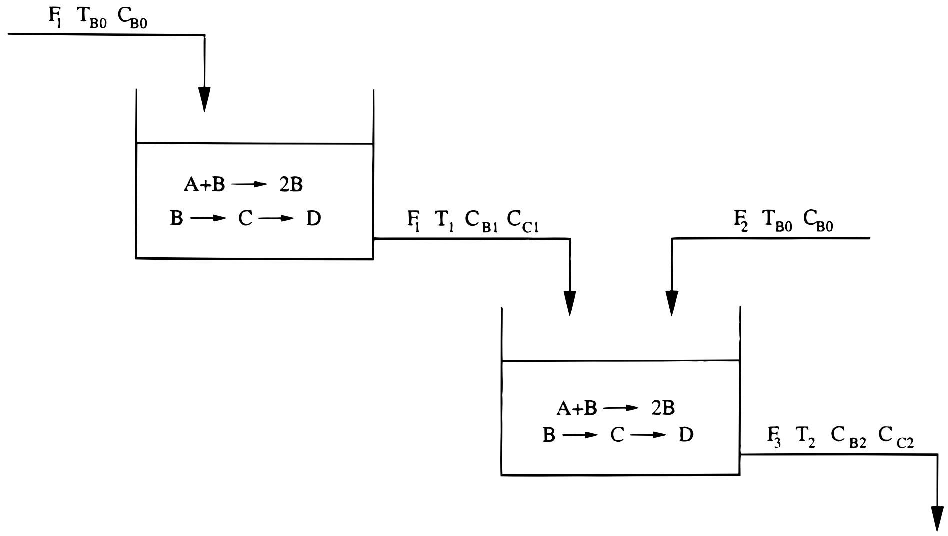

Figure 1. Two continuous-stirred tank reactors in series.

where y_1^s is the solution of

$$\sum_{k=0}^{\tilde{r}} \beta_k \frac{d^k y_1^s}{dt^k} = v$$

Remark 12. A direct comparison between the results of theorem 1 and corollary 1 reveals several fundamental differences in the nature of the solution of the output feedback control problem between linear and nonlinear two-time-scale processes. These differences are (a) the initialization requirement on the states of the slow observer, $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, required for the case of nonlinear systems, is not needed for linear systems, (b) global *asymptotic* stability, as expected, is shown for the closed-loop system in the case of linear systems, while semi-global (arbitrarily large initial conditions) *bounded* stability is shown for the closed-loop system in the case of nonlinear systems, and (c) for linear systems, the upper bound on the singular perturbation parameter is independent of the initial conditions of the two-time-scale system, while for nonlinear systems, this bound decreases as the bound on the initial conditions increases (cf. theorem 1).

4. Simulation Studies

4.1. Application to Two Chemical Reactors in Series. We consider two continuous stirred tank reactors in series (Figure 1), where a reaction scheme of the form: $A + B \xrightarrow{k_1} 2B$, $B \xrightarrow{k_2} C \xrightarrow{k_3} D$, takes place. A is a reactant, B is the desired product (autocatalytic species), and C and D are the undesired products. Assuming that the species A is in excess in the two reactors, the inlet streams consist of pure species B, the autocatalytic reaction is first-order, and the side reactions are zero order, a dynamic model of the process can be derived from material and energy balances and is of the following form:

$$\begin{aligned}
V_1 \frac{dC_{B1}}{dt} &= F_1(C_{B0} - C_{B1}) + k_{10} e^{-E_1/RT_1} C_{B1} V_1 - k_{20} e^{-E_2/RT_1} V_1 \\
V_1 \frac{dC_{C1}}{dt} &= -F_1 C_{C1} + k_{20} e^{-E_2/RT_1} V_1 - k_{30} e^{-E_3/RT_1} V_1 \\
\frac{dT_1}{dt} &= \frac{F_1}{V_1}(T_{B0} - T_1) + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{-E_1/RT_1} C_{B1} + \frac{\bar{Q}_1}{\rho_m c_{pm} V_1} + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} e^{-E_2/RT_1} + \frac{(-\Delta H_{r_3})}{\rho_m c_{pm}} k_{30} e^{-E_3/RT_1} \\
V_2 \frac{dC_{B2}}{dt} &= F_1 C_{B1} + F_2 C_{B0} - (F_1 + F_2) C_{B2} + k_{10} e^{-E_1/RT_2} C_{B2} V_2 - k_{20} e^{-E_2/RT_2} V_2 \\
V_2 \frac{dC_{C2}}{dt} &= F_1 C_{C1} - (F_1 + F_2) C_{C2} + k_{20} e^{-E_2/RT_2} V_1 - k_{30} e^{-E_3/RT_2} V_1 \\
\frac{dT_2}{dt} &= \frac{F_2}{V_2} T_{B0} + \frac{F_1}{V_2} T_1 - \frac{F_1 + F_2}{V_2} T_2 + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{-E_1/RT_2} C_{B2} + \frac{\bar{Q}_2}{\rho_m c_{pm} V_2} + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} e^{-E_2/RT_2} + \frac{(-\Delta H_{r_3})}{\rho_m c_{pm}} k_{30} e^{-E_3/RT_2}
\end{aligned} \tag{27}$$

where V_1 and V_2 are the liquid holdups of the first and second reactor, respectively, T_1, C_{B1}, and C_{C1} and T_2, C_{B2}, and C_{C2} denote temperature and concentrations of species B and C in the first and second reactor, respectively, C_{B0} and T_{B0} denote the inlet temperature and concentration of the species B, $\bar{Q}_1$ and $\bar{Q}_2$ denote the heat inputs to the reactors, k_{10}, k_{20}, k_{30}, E_1, E_2, E_3, ΔH_1, ΔH_2, and ΔH_3 denote the pre-exponential constants, the activation energies, and the enthalpies of the two reactions.

The control objective is the regulation of C_{B2} by manipulating the inlet concentration C_{B0}. The process exhibits two-time-scale behavior because the liquid holdup of the first reactor is smaller than the liquid holdup of the second reactor. Defining the parameter $\epsilon = V_1/V_2 = 0.2$ and setting $u = C_{B0} - C_{B0s}$, $x_1 = T_1$, $x_2 = C_{B2}$, $x_3 = T_2$, $x_4 = C_{C2}$, $z_1 = C_{B1}$, $z_2 = C_{C1}$, $y_1 = x_2$, and $y_2 = z_1$, the original set of equations can be put in the form of eq 1 with

$$\mathbf{f}_1(\mathbf{x}) = \begin{bmatrix} \frac{F_1}{V_1}(T_{B0} - x_1) + \frac{\bar{Q}_1}{\rho_m c_{pm} V_1} + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} e^{-E_2/Rx_1} + \frac{(-\Delta H_{r_3})}{\rho_m c_{pm}} k_{30} e^{-E_3/Rx_1} \\ \frac{F_2}{V_2} C_{B0s} - \frac{F_1 + F_2}{V_2} x_2 + k_{10} e^{-E_1/Rx_3} x_2 - k_{20} e^{-E_0/Rx_3} \\ \frac{F_2}{V_2} T_{B0} + \frac{F_1}{V_2} x_1 - \frac{F_1 + F_2}{V_2} x_3 + \frac{\bar{Q}_2}{\rho_m c_{pm} V_2} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{-E_1/Rx_3} x_2 + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} e^{-E_2/Rx_3} + \frac{(-\Delta H_{r_3})}{\rho_m c_{pm}} k_{30} e^{-E_3/Rx_3} \\ -\frac{F_1 + F_2}{V_2} x_4 + k_{20} e^{-E_2/Rx_3} - k_{30} e^{-E_3/Rx_3} \end{bmatrix}$$

$$\mathbf{Q}_1(\mathbf{x}) = \begin{bmatrix} \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{-E_1/Rx_1} & 0 \\ \frac{F_1}{V_2} & 0 \\ 0 & 0 \\ 0 & F_1 V_2 \end{bmatrix}, \quad \mathbf{g}_1(\mathbf{x}) = \begin{bmatrix} 0 \\ \frac{F_2}{V_2} \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_2(\mathbf{x}) = \begin{bmatrix} \frac{F_1}{V_2} C_{B0s} - \frac{k_{10} e^{-E_1/Rx_1} V_1}{V_2} \\ \frac{k_{20} e^{-E_2/Rx_1} V_1}{V_2} - \frac{k_{30} e^{-E_3/Rx_1} V_1}{V_2} \end{bmatrix},$$

$$\mathbf{Q}_2(\mathbf{x}) = \begin{bmatrix} -\frac{F_1}{V_2} + \frac{k_{10} e^{-E_1/Rx_1} V_1}{V_2} & 0 \\ 0 & -\frac{F_1}{V_2} \end{bmatrix}$$

$$\mathbf{g}_2(\mathbf{x}) = \begin{bmatrix} \frac{F_1}{V_2} \\ 0 \end{bmatrix}, \quad \mathbf{h}_1(\mathbf{x}) = [x_2], \quad \mathbf{p}(\mathbf{x}) = [1 \quad 0]$$

The values of the process parameters and the corresponding steady-state values of the process variables are given in Table 1. One can easily see that for these operating conditions the process exhibits slightly nonminimum phase behavior (i.e., possesses two-time-scale zero dynamics with unstable fast dynamics; see also Christofides and Daoutidis (1996) for a general treatment of this issue). Furthermore, the fast dynamics of the process are unstable and assumption 1 is satisfied. This suggested the use of a fast-dynamic output feedback controller of the following form:

$$\frac{d\omega}{d\tau} = \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^T(\boldsymbol{\eta})]\omega + \mathbf{l}(\boldsymbol{\eta})(y_2 - \mathbf{p}(\boldsymbol{\eta})\omega) + \mathbf{g}_2(\boldsymbol{\eta})\tilde{u}$$

$$u = \tilde{u} + \mathbf{k}^T(\boldsymbol{\eta})\omega \qquad (28)$$

with $\mathbf{l}(\boldsymbol{\eta}) = [7.5 \quad (-F_1 + k_{10} e^{-E_1/R\eta_1} V_1) \quad 0]^T$ and $\mathbf{k}^T(\boldsymbol{\eta}) = -3.0/F_1[-F_1 + k_{10} e^{-E_1/R\eta_1} V_1 \quad 0]$ to stabilize the fast dynamics. The nonlinear gains $\mathbf{l}(\boldsymbol{\eta})$ and $\mathbf{k}^T(\boldsymbol{\eta})$ were chosen so that assumption 1 is satisfied for the process in question.

Setting $\epsilon = 0$, the representation of the modified closed-loop slow subsystem was obtained (the explicit form of this system is omitted here for brevity). For this system, it was verified that it is exponentially unstable and minimum-phase (i.e., its corresponding zero dynamics are locally exponentially stable), while its relative order is $\tilde{r} = 1$. Therefore, the assumptions of theorem 2 hold and the controller of eq 21, that is,

$$\frac{d\omega}{d\tau} = \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^T(\boldsymbol{\eta})]\omega + \mathbf{l}(\boldsymbol{\eta})(y_2 - \mathbf{p}(\boldsymbol{\eta})\omega) + \mathbf{g}_2(\boldsymbol{\eta})([\beta_1 L_{\tilde{\mathbf{G}}} h_1(\boldsymbol{\eta})]-1\{v - \beta_0 h_1(\boldsymbol{\eta}) - \beta_1 L_{\tilde{\mathbf{F}}} h_1(\boldsymbol{\eta})\})$$

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})([\beta_1 L_{\tilde{\mathbf{G}}} h_1(\boldsymbol{\eta})]^{-1}\{v - \beta_0 h_1(\boldsymbol{\eta}) - \beta_1 L_{\tilde{\mathbf{F}}} h_1(\boldsymbol{\eta})\}) + \mathbf{S}(y_1 - h_1(\boldsymbol{\eta}))$$

$$u = [\beta_1 L_{\tilde{\mathbf{G}}} h(\boldsymbol{\eta})]^{-1}\{v - \beta_0 h_1(\boldsymbol{\eta}) - \beta_1 L_{\tilde{\mathbf{F}}} h_1(\boldsymbol{\eta})\} + \mathbf{k}^T(\boldsymbol{\eta})\omega \qquad (29)$$

with $\mathbf{l}(\boldsymbol{\eta}) = [7.5 \quad (-F_1 + k_{10} e^{-E_1 R\eta_1} V_1) \quad 0]^T$, $\mathbf{k}^T(\boldsymbol{\eta}) = -3.0/F_1[-F_1 + k_{10} e^{-E_1 R\eta_1} V_1 \quad 0]$, $\mathbf{S} = [10\ 10\ 10\ 10]^T$, $\beta_0 = 1.0$, and $\beta_1 = 15.0$, was employed in the simulations. The linear gain, $\mathbf{S}$, was chosen so that the nonlinear observer used to produce estimates of the slow states of the process (i.e., η-subsystem of the system of eq 29) is a stable system in the region of operation considered in the subsequent simulations, thereby ensuring stability of the closed-loop slow subsystem.

Several simulation runs were performed to test the controller and compare its performance with controller designs that do not account for the presence of time-scale multiplicity. The process was initially assumed to be at steady state. In the first simulation run, the output tracking capability of the controller was tested for a 0.3 mol/L decrease on the value of the setpoint (v

Table 1. Process Parameters and Steady-State Values (Two Chemical Reactors in Series)

$V_1 = 0.2$ m^3
$V_2 = 1.0$ m^3
$R = 1.987$ kcal kmol^{-1} K^{-1}
$C_{B0s} = 2.0$ kmol m^{-3}
$T_{B0} = 305.0$ K
$c_{pm} = 0.231$ kcal kg^{-1} K^{-1}
$\rho_m = 900.0$ kg m^{-3}
$\bar{Q}_1 = 1.1 \times 10^5$ kcal min^{-1}
$\bar{Q}_2 = 2.2 \times 10^4$ kcal min^{-1}
$\Delta H_{r_1} = 5.4 \times 10^3$ kcal kg mol^{-1}
$\Delta H_{r_2} = 10.67 \times 10^3$ kcal kg mol^{-1}
$\Delta H_{r_3} = 10.0 \times 10^3$ kcal kg mol^{-1}
$k_{10} = 2.012 \times 10^6$ kmol m^{-3} min^{-1}
$k_{20} = 2.25 \times 10^7$ min^{-1}
$k_{30} = 3.60 \times 10^6$ min^{-1}
$E_1 = 9.0 \times 10^3$ kcal kmol^{-1}
$E_2 = 8.0 \times 10^3$ kcal kmol^{-1}
$E_3 = 9.0 \times 10^3$ kcal kmol^{-1}
$F_1 = 0.2$ m^3 min^{-1}
$F_2 = 2.0$ m^3 min^{-1}
$C_{B1s} = 2.5$ kmol m^{-3}
$C_{C1s} = 6.0$ kmol m^{-3}
$T_{1s} = 300.0$ K
$C_{B2s} = 3.125$ kmol m^{-3}
$C_{C2s} = 3.273$ kmol m^{-3}
$T_{2s} = 300.0$ K

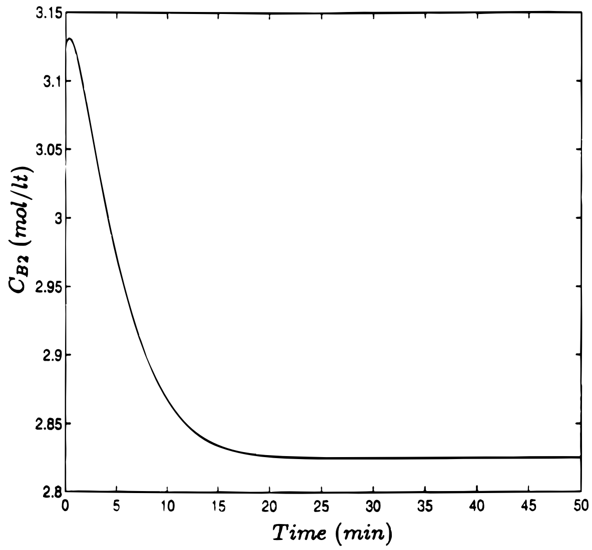

Figure 2. Closed-loop profile for set-point tracking in the case of using a two-time-scale output feedback controller (two chemical reactors in series).

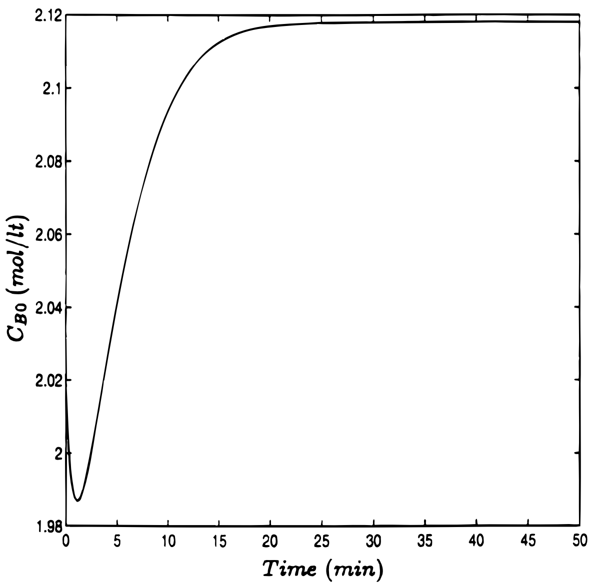

Figure 3. Manipulated input profile for set-point tracking in the case of using a two-time-scale output feedback controller (two chemical reactors in series).

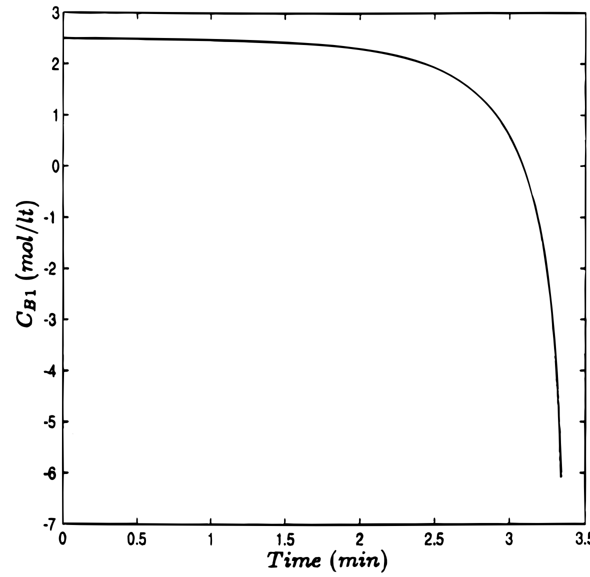

Figure 4. Closed-loop profile of fast state, z_1, in the case of using an output feedback controller synthesized on the basis of the slow subsystem (two chemical reactors in series).

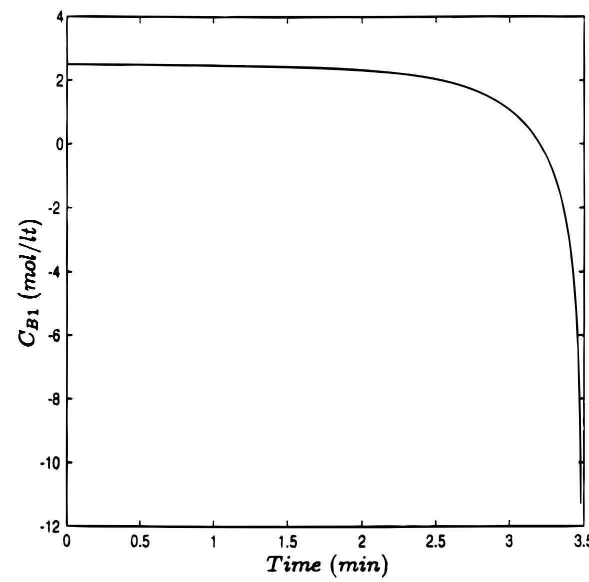

Figure 5. Closed-loop profile of fast state, z_1, in the case of using an output feedback controller synthesized on the basis of the two-time-scale system (two chemical reactors in series).

$= 2.825$). Figure 2 shows the profile of the controlled output of the process, while the manipulated input is shown in Figure 3. One can immediately observe that the controller drives the output of the system close to the new value of the setpoint (i.e., the requirement $\lim_{t\to\infty}|y - v| = O(0.2)$ is satisfied), while stabilizing the fast dynamics of the process.

For the sake of comparison, we implemented on the process the same controller without the *fast* observer and with $\mathbf{k}^T(\boldsymbol{\eta}) = 0$ (this is equivalent to synthesizing a dynamic output feedback controller on the basis of the slow subsystem of eq 2). A 0.3 mol/L decrease on the value of the setpoint ($v = 2.825$) was considered. The profile of the fast state, z_1, is given in Figure 4. It is clear that this controller leads to closed-loop instability, because it does not stabilize the unstable fast dynamics of the process. We also implemented on the process an input/output linearizing dynamic output feedback controller (Daoutidis and Christofides, 1995) synthesized on the basis of the two-time-scale process model, without accounting for the presence of time-scale multiplicity. This controller, as expected, leads to an unstable closed-loop system (see the profile of the fast state, z_1, in Figure 5), because the process with $y = C_{B1}$ and $u = C_{B0}$ exhibits slightly nonminimum phase behavior (i.e., possesses two-time-scale zero dynamics with unstable fast dynamics). Finally, we implemented a proportional integral (PI) controller with $K_c = 1.0$ and $\tau_1 = 2.0$. The closed-loop output profile is shown in Figure 6. The PI controller cannot stabilize the closed-loop system (the process output stabilizes at a different value than the new setpoint, $v = 2.825$). The reason is that the PI controller only uses feedback of the controlled output, C_{B2}, which is a slow variable, and thus, it cannot stabilize the unstable fast dynamics of the open-loop process, leading to closed-loop instability.

We also evaluated the output tracking capability of the controller in the presence of initialization errors in the states of the observer and noise in the measure-

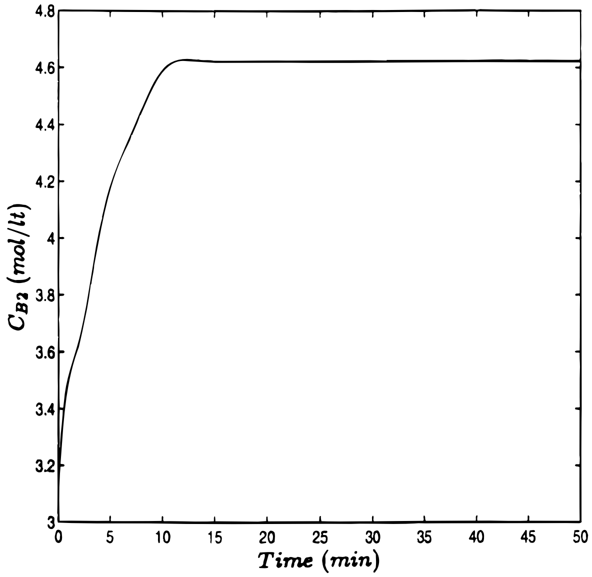

Figure 6. Closed-loop output profile for set-point tracking under proportional integral control (two chemical reactors in series).

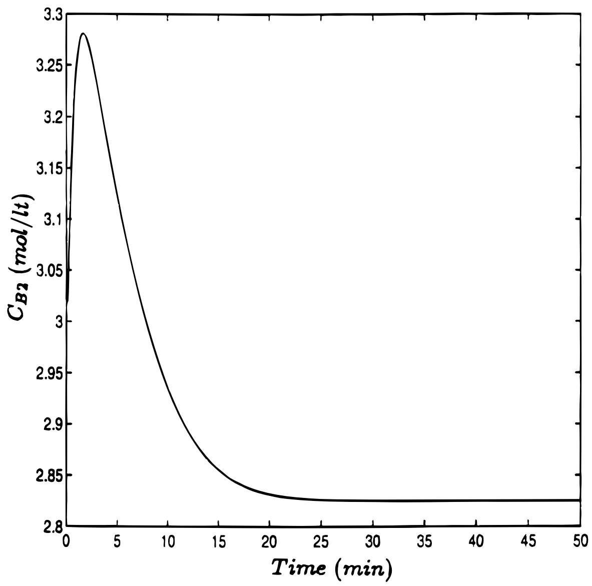

Figure 7. Closed-loop output profile for set-point tracking in the case of using a two-time-scale output feedback controller—effect of initialization errors (two chemical reactors in series).

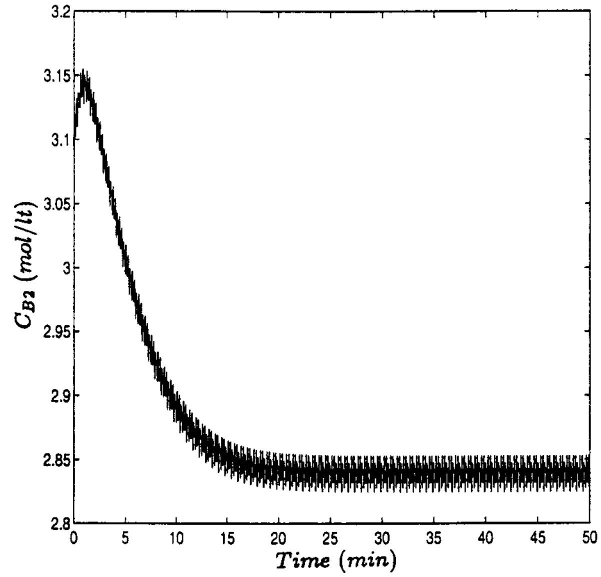

Figure 8. Closed-loop output profile for set-point tracking in the case of using a two-time-scale output feedback controller—effect of measurement noise (two chemical reactors in series).

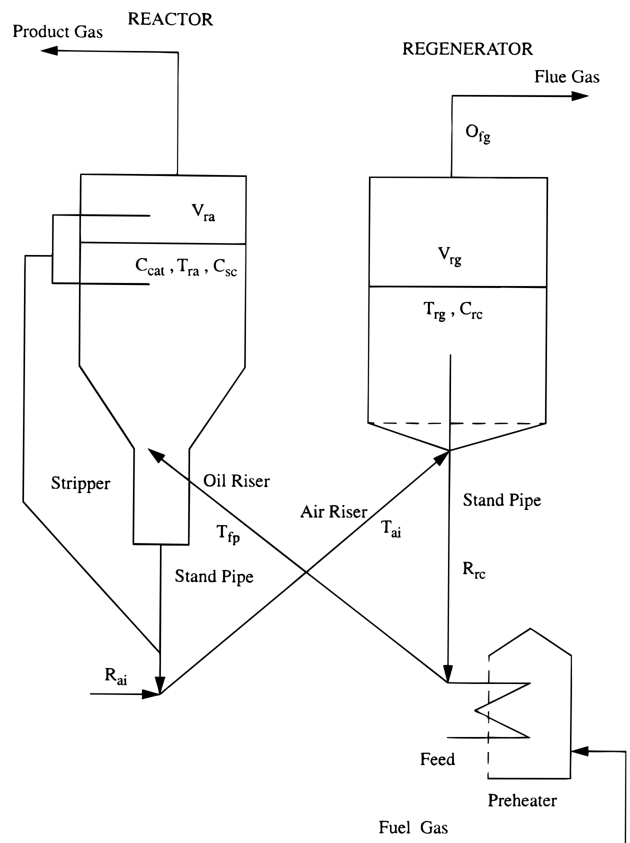

Figure 9. A fluidized catalytic cracker.

ments of C_{B1} and C_{B2}, for a 0.3 mol/L decrease on the value of the setpoint ($v = 2.825$). Initially, a 10% initialization error in the states ω_1 and η_1 was considered. Figure 7 shows the profile of the controlled output of the process. Clearly, the controller possesses a robustness property with respect to significant initialization errors, driving the output of the process close to the new setpoint value (i.e., the requirement $\lim_{t\to\infty}|y - v| = O(0.2)$ is satisfied), while stabilizing the unstable fast dynamics of the process. Finally, a 0.1 mol/L amplitude white noise was considered in the measurements of C_{B1} and C_{B2}. Figure 8 shows the profile of the controlled output of the process. It is clear that the controller is robust with respect to significant measurement noise, driving the output of the process close to the new setpoint value.

4.2. Application to a Fluidized Catalytic Cracker. In this section, we implement the developed control method on a fluidized catalytic cracker (FCC) unit. Previous works on control of the FCC process include the application of linear (e.g., Monge and Georgakis, 1987; Huq et al., 1995) and nonlinear (e.g., Christofides and Daoutidis, 1997) control methods. However, in these works, feedback control has been implemented under the assumption that measurements of the process-state variables are available. In the present study, this assumption is eliminated. The FCC unit considered here is shown in Figure 9 and consists of a cracking reactor where the cracking reactions take place and a regenerator where the carbon removal reactions take place (the reader may refer to Denn (1986) and McFar-

Table 2. Process Parameters and Steady-State Values (Fluidized Catalytic Cracker)

$E_{cc} = 18\,000.0$ Btu lb^{-1} mol^{-1}
$E_{cr} = 27\,000.0$ Btu lb^{-1} mol^{-1}
$E_{oc} = 63\,000.0$ But lb^{-1} mol^{-1}
$k_{cc} = 8.59$ Mlb h^{-1} psia^{-1} t^{-1} (wt %)$^{-1.06}$
$k_{cr} = 11\,600$ Mbbl day^{-1} psia^{-1} t^{-1} (wt %)$^{-1.15}$
$k_{or} = 3.5 \times 10^{10}$ Mlb h^{-1} psia^{-1} t^{-1}
$V_{rg} = 200.0$ t
$V_{ra} = 60.0$ t
$T_{fps} = 744.0$ °F
$T_{ai} = 175.0$ °F
$P_{rg} = 25.0$ psia
$P_{ra} = 40.0$ psia
$\Delta H_{fv} = 60.0$ Btu lb^{-1}
$\Delta H_{cr} = 77.3$ Btu lb^{-1}
$\Delta H_{rg} = 10\,561.0$ Btu lb^{-1}
$S_a = 0.3$ Btu lb^{-1} °F^{-1}
$S_c = 0.3$ Btu lb^{-1} °F^{-1}
$S_p = 0.7$ Btu lb^{-1} °F^{-1}
$F_{rc} = 40.0$ t min^{-1}
$R_{tf} = 100.0$ Mbbl/day
$D_{tf} = 7.0$ lb gal^{-1}
$R_{ai} = 400.0$ Mlb min^{-1}
$C_{cat} = 0.8723$ wt %
$C_{sc} = 1.5698$ wt %
$C_{rc} = 0.6975$ wt %
$T_{ra} = 930.6255$ °F
$T_{rg} = 1155.9605$ °F

lane et al. (1993) for details on process description). Under standard modeling assumptions (see Denn (1986) for details), a dynamic model for the process can be developed and is of the form (Denn, 1986; Christofides and Daoutidis, 1997):

$$V_{ra}\frac{dC_{cat}}{dt} = -60F_{rc}C_{cat} + 50R_{cf}$$

$$V_{ra}\frac{dC_{sc}}{dt} = +60F_{rc}(C_{rc} - C_{sc}) + 50R_{cf}$$

$$V_{ra}\frac{dT_{ra}}{dt} = 60F_{rc}(T_{rg} - T_{ra}) + 0.875\frac{S_f}{S_c}D_{tf}R_{tf}(T_{fp} - T_{ra}) + 0.875\frac{-\Delta H_{fv}}{S_c}D_{tf}R_{tf} + 0.5\frac{(-\Delta H_{cr})}{S_c}R_{oc} \quad (30)$$

$$V_{rg}\frac{dC_{rc}}{dt} = 60F_{rc}(C_{sc} - C_{rc}) - 50R_{cb}$$

$$V_{rg}\frac{dT_{rg}}{dt} = 60F_{rc}(T_{ra} - T_{rg}) + 0.5\frac{S_a}{S_c}R_{ai}(T_{ai} - T_{rg}) - 0.5\left(-\frac{\Delta H_{rg}}{S_c}\right)R_{cb}$$

where C_{cat}, C_{sc}, and C_{rc} denote the concentrations of catalytic carbon on spent catalyst, the total carbon on spent catalyst, and carbon on regenerated catalyst, T_{ra} and T_{rg} denote the temperatures in the reactor and the regenerator, D_{tf} is the density of the total feed, V_{ra} and V_{rg} denote the holdup of the reactor and the regenerator, ΔH_{rg} and ΔH_{cr} are heat of reactions, ΔH_{fv} is the heat of feed vaporization, F_{rc} denotes the flow rate of the catalyst from reactor to regenerator, S_a, S_c, and S_f denote specific heats, T_{fp} and T_{ai} denote the inlet temperatures of the feed in the reactor and the air in the regenerator, and R_{cf}, R_{oc}, and R_{cb} denote reaction rates. The analytic expressions for the reaction rates R_{cf}, R_{oc}, and R_{cb} can be found in Denn (1986) and Christofides and Daoutidis (1997) and are omitted here for brevity. The values of the process parameters and the corresponding steady-state values are given in Table 2. The FCC unit exhibits a two-time-scale behavior because the residence time in the reactor is smaller than the one in the regenerator. This implies that the regenerator is the process that essentially determines the dynamic response of the entire FCC unit, and therefore, the control problem must focus on the regenerator. To this end, the control objective is the regulation of the temperature in the regenerator, T_{rg}, and the concentration of the carbon on the regenerated catalyst, C_{rc}, by manipulating the inlet temperatures, T_{fp} and T_{ai}. Defining the singular perturbation parameter $\epsilon = V_{ra}/V_{rg}$ and setting $\mathbf{x} = [x_1 \;\; x_2]^T = [C_{rc} \;\; T_{rg}]^T$, $\mathbf{z} = [z_1 \;\; z_2 \;\; z_3]^T = [C_{cat} \;\; C_{sc} \;\; T_{ra}]^T$, $\mathbf{u} = [u_1 \;\; u_2]^T = [T_{fp} \;\; T_{ai}]^T$, $\mathbf{y} = [y_{11} \;\; y_{12}]^T = [C_{rc} \;\; T_{rg}]^T$, the system of eq 30 can be put into the standard singularly perturbed form:

$$\frac{dx_1}{dt} = f_{11}(x_1, x_2) + Q_{11}z_2$$

$$\frac{dx_2}{dt} = f_{12}(x_1, x_2) + Q_{12}z_3 + g_{12}u_2$$

$$\epsilon\frac{dz_1}{dt} = Q_{21}(z_1, z_2, z_3, x_1)$$

$$\epsilon\frac{dz_2}{dt} = f_{22}(x_1) + Q_{22}(z_1, z_2, z_3, x_1)$$

$$\epsilon\frac{dz_3}{dt} = f_{23}(x_2) + Q_{23}(z_1, z_2, z_3, x_1) + g_{23}u_1$$

$$y_{11} = x_1, y_{12} = x_2 \quad (31)$$

where f_{11}, Q_{11}, f_{12}, Q_{12}, g_{12}, Q_{21}, f_{22}, Q_{22}, f_{23}, Q_{23}, and g_{23} are functions whose specific form is omitted for brevity. It was verified that the system of eq 31 possesses an exponentially stable equilibrium manifold for the fast dynamics, which implies that it is not needed to utilize a preliminary feedback law to stabilize the fast dynamics. Setting $\epsilon = 0$, the equilibrium manifold of the fast dynamics can be calculated analytically and is of the form $\mathbf{z}_s = \mathbf{g}(x_1, x_2, u_1)$, where $\mathbf{g}$ is a smooth vector function (note that the input u_1 enters this algebraic equation in a nonlinear fashion, due to the nonlinear appearance of the fast state $\mathbf{z}$ in the system of eq 31). The reduced system can then be found to be of the following form:

$$\frac{dx_1}{dt} = F_1(x_1, x_2) + G_{11}(x_1, x_2, u_1)$$

$$\frac{dx_2}{dt} = F_2(x_1, x_2) + G_{21}(x_1, x_2, u_1) + G_{22}u_2 \quad (32)$$

with F_1, G_{11}, F_2, G_{21}, and G_{22} appropriately defined (their exact expressions are omitted for brevity). For the system of eq 32 the condition $r_1 + r_2 = 1 + 1 = 2$ (where r_i is the relative order of the ouput y_{1i} with respect to the manipulated input vector $\mathbf{u}$) holds, which implies that this system does not possess zero dynamics. It was verified that the system of eq 32 is a stable one, and thus, the necessary controller consists of a multivariable input–output linearizing controller designed on the basis of the system of eq 32 coupled with an open-

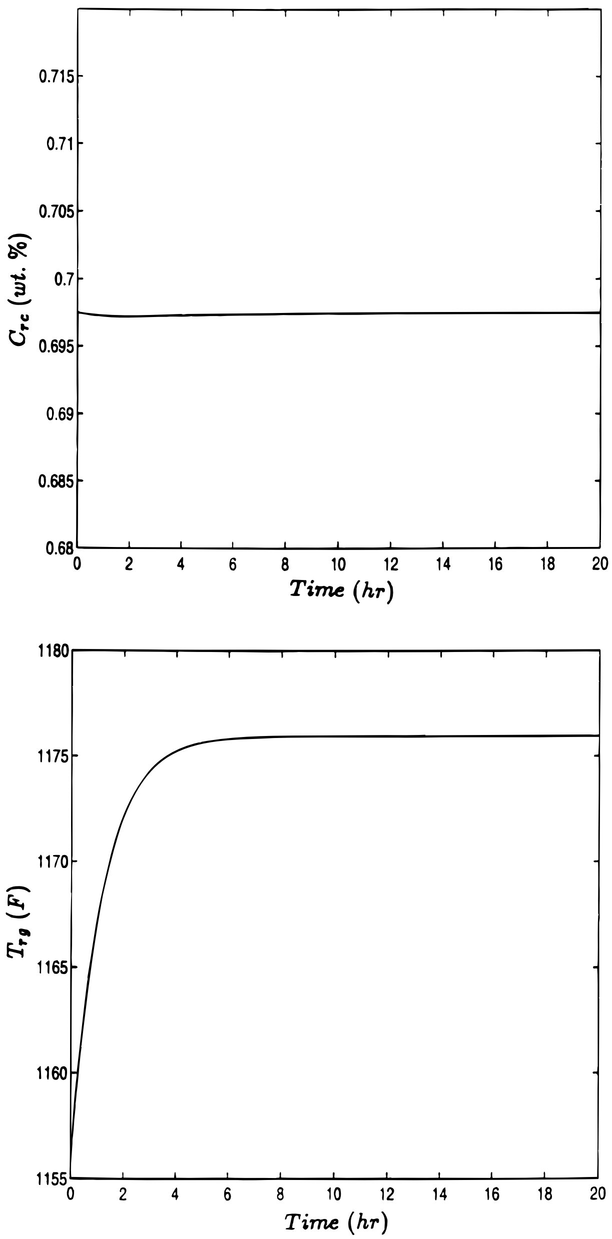

Figure 10. Closed-loop output profiles for set-point tracking (fluidized catalytic cracker).

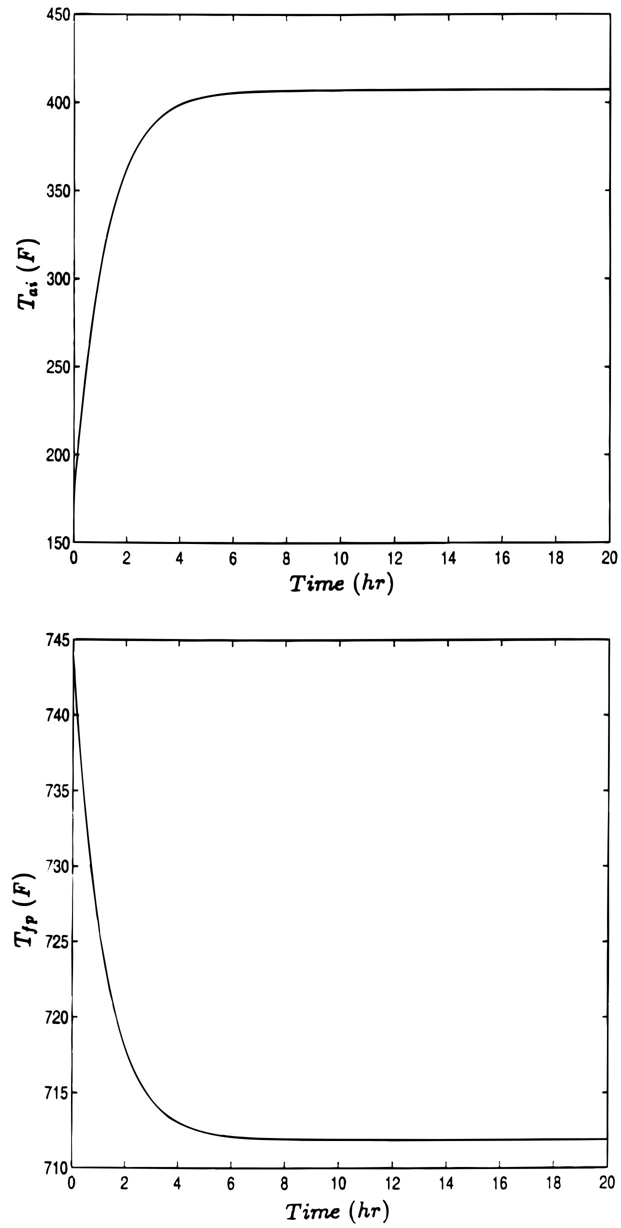

Figure 11. Manipulated input profiles for set-point tracking (fluidized catalytic cracker).

loop observer and is of the following form:

$$\frac{d\eta_1}{dt} = F_1(\eta_1, \eta_2) + G_{11}(\eta_1, \eta_2, u_1)$$

$$\frac{d\eta_2}{dt} = F_2(\eta_1, \eta_2) + G_{21}(\eta_1, \eta_2, u_1) + G_{22}u_2$$

$$u_1 = \bar{G}_{11}\left(\eta_1, \eta_2, \frac{1}{\beta_{11}}(v_1 - \eta_1) - F_1(\eta_1, \eta_2)\right)$$

$$u_2 = \frac{1}{G_{22}}\left\{\frac{1}{\beta_{21}}(v_2 - \eta_2) - \left[F_2(\eta_1, \eta_2) + G_{21}\left(\eta_1, \eta_2, \bar{G}_{11}\left(\eta_1, \eta_2, \frac{1}{\beta_{11}}(v_1 - \eta_1) - F_1(\eta_1, \eta_2)\right)\right)\right]\right\} \quad (33)$$

In the above controller, the formula for u_1 was computed by considering the algebraic equation $G_{11}(\eta_1, \eta_2, u_1) = \alpha$ and deriving its solution in terms of u_1 (i.e., $u_1 = \bar{G}_{11}(\eta_1, \eta_2, \alpha)$ (this was done due to the nonlinear appearance of the input u_1 in the system of eq 32)). Furthermore, referring to the controller of eq 33, we note that (a) it is ϵ-independent, (b) it does not use measurements of the states of the process since it employs an open-loop observer, and (c) its order is significantly lower than the one of the process model (which is expected, because it was synthesized on the basis of the two-dimensional slow system instead of the five-dimensional process model).

Two representative simulation runs of the closed-loop system under the controller of eq 33 with $\beta_{11} = 0.1$ and $\beta_{21} = 0.02$ are reported. In both runs, the process was initially ($t = 0.0$ h) assumed to be at steady state. In the first simulation run, we tested the set-point tracking capabilities of the controller. A 20.0 °F increase in the value of the output y_{12} was imposed at time $t = 0.0$ h. The output profiles are depicted in Figure 10 and the profiles of the corresponding manipulated inputs are given in Figure 11. It is clear that the controller drives

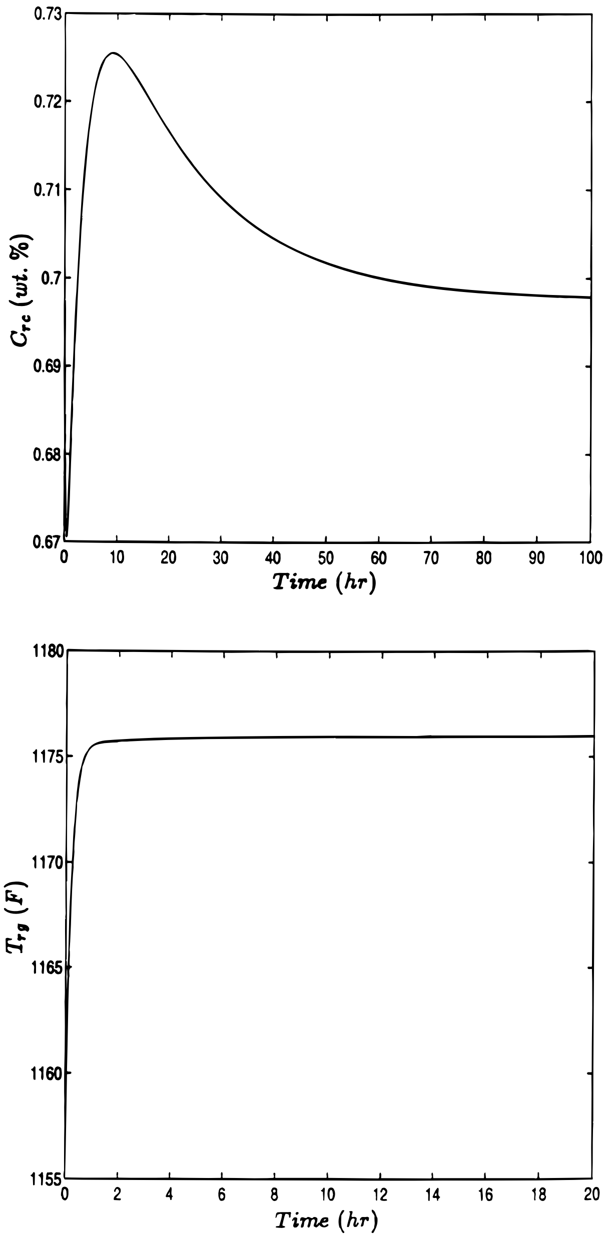

Figure 12. Closed-loop output profiles for set-point tracking—effect of initialization errors (fluidized catalytic cracker).

the output y_{12} to its new reference input value. It can also be observed that the output y_{11} stays at its set-point value. In the second simulation run, we tested the set-point tracking capabilities of the controller in the presence of initialization error in the observer states. A 5% error in the state η_1 and a 15 °F error in the state η_2 was considered. Figure 12 shows the closed-loop output profiles. Clearly, the controller regulates, y_{12}, at its new set-point value.

5. Conclusions

In this work, we studied the output feedback control problem for a broad class of nonlinear two-time-scale processes modeled within the singular perturbation framework. A sequential procedure was initially used to derive a nonlinear two-time-scale output feedback controller. The controller, which does not suffer from ill-conditioning and stiffness, globally asymptotically stabilizes the fast subsystem and enforces global asymptotic stability with asymptotic output tracking in the closed-loop slow subsystem. It was established that the controller enforces boundedness of the states and approximate asymptotic output tracking in the closed-loop two-time-scale system for arbitrarily large initial conditions, provided that the separation of the slow- and fast-dynamical phenomena of the process is sufficiently large. An explicit easy-to-implement realization of the controller that enforces local exponential stability and approximate output tracking, for all times, in the closed-loop two-time-scale system, was also derived. Differences in the nature of the solution of the output feedback control problem between linear and nonlinear two-time-scale systems were pointed out. Finally, the proposed controller was successfully applied to a series of two chemical reactors and a fluidized catalytic cracking unit.

Acknowledgment

Financial support from UCLA through the SEAS Dean's Fund is gratefully acknowledged.

Notation

Roman Letters

C_{B0} = inlet concentration of the autocatalytic species
C_{B1} = concentration of the autocatalytic species in the first reactor
C_{C1} = concentration of the side produce in the first reactor
C_{B2} = concentration of the autocatalytic species in the second reactor
C_{C2} = concentration of the side product in the second reactor
$C_{\rm cat}$ = concentration of catalytic carbon on spent catalyst
$C_{\rm sc}$ = concentration of total carbon on spent catalyst
$C_{\rm rc}$ = concentration of carbon on regenerated catalyst
c_{pm} = heat capacity of the reacting mixture
$D_{\rm tf}$ = density of total feed
E_1, E_2, E_3 = activation energies
$E_{\rm cc}$, $E_{\rm cr}$, $E_{\rm or}$ = activation energies
$\mathbf{F}$, $\tilde{\mathbf{F}}$, $\mathbf{f}_1$, $\mathbf{f}_2$ = vector fields
F_1, F_2 = flow rates
$\mathbf{G}$, $\tilde{\mathbf{G}}$, $\mathbf{g}_1$, $\mathbf{g}_2$ = vector fields associated with the input
$h_1(\mathbf{x})$ = output scalar field
$\mathbf{k}^{\rm T}(\mathbf{x})$ = nonlinear vector field
k_{10}, k_{20}, k_{30} = pre-exponential constants
$\mathbf{p}(\mathbf{x})$ = nonlinear vector field
$\mathbf{Q}_1(\mathbf{x})$ = matrix of dimension $n \times p$ associated with the slow-state vector $\mathbf{x}$
$\mathbf{Q}_2(\mathbf{x})$ = matrix of dimension $p \times p$ associated with the fast-state vector $\mathbf{z}$
Q_1, $\bar{Q}_2$ = heat inputs to the reactors
$R_{\rm ai}$ = air rate
$R_{\rm cb}$, $R_{\rm cf}$, $R_{\rm oc}$ = reaction rates
$R_{\rm tf}$ = total feed rate
r, $\tilde{r}$ = relative orders in the slow subsystems
$S_{\rm a}$, $S_{\rm c}$, $S_{\rm f}$ = specific heats
T_{B0} = inlet temperature of the autocatalytic species
T_{B1} = temperature in the first reactor
T_{B2} = temperature in the second reactor
$T_{\rm fp}$, $T_{\rm ai}$ = temperatures of the feed in the reactor and the air in the regenerator
$T_{\rm ra}$, $T_{\rm rg}$ = temperatures in reactor and regenerator
t = time
u = input
$\bar{u}$ = auxiliary input
V_1, V_2 = volumes of the liquid holdup in the reactors
v = external input

$\mathbf{x}$ = vector of the slow-state variables
y_1 = controlled output
y_2 = measured output
$\mathbf{z}$ = vector of the fast-state variables

Greek Letters

β_k = adjustable parameters
β_{ik} = adjustable parameters
ΔH_{r_1}, ΔH_{r_2}, ΔH_{r_3} = enthalpy of the reactions
ΔH_{rg}, ΔH_{cr} = heat of the reactions
ΔH_{fv} = heat of feed vaporization
ϵ = singular perturbation parameter
$\boldsymbol{\eta}$ = vector of the state observer
ρ_{m} = density of the reacting mixture
$\boldsymbol{\omega}$ = vector of the state observer

Math Symbols

$L_{\mathbf{f}}h$ = Lie derivative of a scalar field h with respect to the vector field $\mathbf{f}$
$L_{\mathbf{f}}^k h$ = kth order Lie derivative
$L_{\mathbf{g}}L_{\mathbf{f}}^{k-1}h$ = mixed Lie derivative
IR = real line
IR^i = i-dimensional Euclidean space
$\in$ = belongs to
T = transpose
$|\cdot|$ = standard Euclidean norm

Appendix

Notation—Definitions. (1) A function γ: $\mathrm{IR}_{\geq 0} \to \mathrm{IR}_{\geq 0}$ is said to be of class K if it is continuous and increasing and is zero at zero. It is of class K_∞, if in addition it is proper.

(2) A function β: $\mathrm{IR}_{\geq 0} \times \mathrm{IR}_{\geq 0} \to \mathrm{IR}_{\geq 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot,t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.

(3) A matrix $\mathbf{A}(\mathbf{x})$ of dimension $n \times n$ is said to be Hurwitz uniformly in $\mathbf{x} \in \mathrm{IR}^n$ if there exists a positive real number c such that $Re[\lambda_i(\mathbf{A}(\mathbf{x}))] \leq -c$, $i = 1, ..., n$ for all $\mathbf{x} \in \mathrm{IR}^n$, where λ_i denotes the ith eigenvalue of the matrix.

Definition 1. Referring to the nonlinear system of eq 2, the relative order of y_1^s with respect to u is defined as the smallest integer r for which

$$L_{\mathbf{G}}L_{\mathbf{F}}^{r-1}h_1(\mathbf{x}) \tag{34}$$

or $r = \infty$ if such an integer does not exist.

Definition 2 (Khalil, 1992). The equilibrium $x = 0$ of the system

$$\dot{\mathbf{x}} = \phi(\mathbf{x}) \tag{35}$$

where $\mathbf{x} \in \mathrm{IR}^n$ is asymptotically stable if for each $\mathbf{x}_0 \in \mathrm{IR}^n$, its solution with $\mathbf{x}(0) = \mathbf{x}_0$ exists for each $t \geq 0$ and satisfies

$$|\mathbf{x}(t)| \leq \beta(|\mathbf{x}(0)|,t) \tag{36}$$

The equilibrium $\mathbf{x} = 0$ is globally exponentially stable if the bound of eq 36 holds with $\beta(|\mathbf{x}(0)|,t) = K|\mathbf{x}(0)|e^{-at}$ where $K \geq 1$ and $a > 0$.

Proofs of Theorems 1 and 2. In what follows, we provide the proofs of theorems 1 and 2. To this end, we initially recall a theorem proved in Christofides and Teel (1996) which will be used in the proof of theorem 1.

Consider the singularly perturbed system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}) \\ \epsilon\dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}) \end{aligned} \tag{37}$$

where $\mathbf{x} \in \mathrm{IR}^n$ and $\mathbf{z} \in \mathrm{IR}^p$. The functions $\mathbf{f}$ and $\mathbf{g}$ are locally Lipschitz on $\mathrm{IR}^n \times \mathrm{IR}^p \times \mathrm{IR}^q$ and the algebraic equation $\mathbf{g}(\mathbf{x}, \mathbf{z}_s) = 0$ possesses a unique root, $\mathbf{z}_s = \mathbf{h}(\mathbf{x})$, with the properties that $\mathbf{h}(\mathbf{x})$ and $\partial\mathbf{h}/\partial\mathbf{x}$ are locally Lipschitz and $\mathbf{h}(0) = 0$.

Consider also the corresponding slow and fast subsystems:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})) \tag{38}$$

$$\frac{d\mathbf{y}}{d\tau} = g(x, \mathbf{h}(x) + \mathbf{y}) \tag{39}$$

where $\mathbf{y} = \mathbf{z} - \mathbf{h}(\mathbf{x})$. The assumptions that follow state our stability requirements on the slow and fast subsystems.

Assumption A1. The reduced system in eq 38 is asymptotically stable with the class KL function $\beta_{\mathbf{x}}$.

Assumption A2. The equilibrium $\mathbf{y} = 0$ of the boundary layer system in eq 39 is globally asymptotically stable, uniformly in $\mathbf{x} \in \mathrm{IR}^n$.

The main result is as follows.

Theorem A1. Consider the singularly perturbed system in eq 37 and let assumptions A1 and A2 hold. Then, there exists a function $\beta_{\mathbf{y}}$ of class KL such that, for each set of positive real numbers (δ, d), there is an $\epsilon^* > 0$ such that if $\epsilon \in (0, \epsilon^*)$ and $\max\{|\mathbf{x}(0)|, |\mathbf{y}(0)|\} \leq \delta$, then

$$|\mathbf{x}(t)| \leq \beta_{\mathbf{x}}(|\mathbf{x}(0)|, t) + d \tag{40}$$

$$|\mathbf{y}(\tau)| \leq \beta_{\mathbf{y}}(|\mathbf{y}(0)|, \tau) + d \tag{41}$$

where $\beta_{\mathbf{x}}$ is the function defined in assumption A1.

We note that theorem A1 has been generalized to two-time-scale systems with uncertain variables and ϵ-dependence in the right-hand side (see Christofides and Teel (1996) for details).

Proof of Theorem 1. Under the controller of eq 15, the closed-loop system takes the following form:

$$\begin{aligned} \epsilon\dot{\boldsymbol{\omega}} &= \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})]\omega + \mathbf{l}(\boldsymbol{\eta})(\mathbf{p}(\mathbf{x})\mathbf{z} - \mathbf{p}(\boldsymbol{\eta})\omega) + \mathbf{g}_2(\boldsymbol{\eta})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v) \\ \dot{\boldsymbol{\eta}} &= \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v) + \mathbf{s}(\boldsymbol{\eta})(h_1(\mathbf{x}) - h_1(\boldsymbol{\eta})) \\ \dot{\mathbf{x}} &= \mathbf{f}_1(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x})\mathbf{z} + \mathbf{g}_1(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})\omega + \mathbf{g}_1(\mathbf{x})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v) \\ \epsilon\dot{\mathbf{z}} &= \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})\omega + \mathbf{g}_2(\mathbf{x})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v) \end{aligned} \tag{42}$$

Obtaining the representation of the above system in the fast time scale, τ, and setting $\epsilon = 0$, the following fast subsystem is obtained:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})]\omega + \mathbf{l}(\boldsymbol{\eta})(\mathbf{p}(\mathbf{x})\mathbf{z} - \mathbf{p}(\boldsymbol{\eta})\omega) + \mathbf{g}_2(\boldsymbol{\eta})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v)$$

$$\frac{d\mathbf{z}}{dt} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})\omega + \mathbf{g}_2(\mathbf{x})(\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v) \quad (43)$$

From the assumption that $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, it follows that when $\epsilon = 0$, then $\boldsymbol{\eta}(\tau) = \mathbf{x}(\tau)$, $\tau \geq 0$, in the fast time scale, τ, and thus, the above system can be equivalently written as

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + [\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})]\mathbf{w} + \mathbf{l}(\mathbf{x})(\mathbf{p}(\mathbf{x})\mathbf{z} - \mathbf{p}(\mathbf{x})\omega) + \mathbf{g}_2(\mathbf{x})(\bar{p}(\mathbf{x}) + q(\mathbf{x})v)$$

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\omega + \mathbf{g}_2(\mathbf{x})(\bar{p}(\mathbf{x}) + q(\mathbf{x})v) \quad (44)$$

Defining the error vector $\mathbf{e}_{\mathbf{z}} = \boldsymbol{\omega} - \mathbf{z}$, the above system can be written as

$$\frac{d\mathbf{e}_{\mathbf{z}}}{d\tau} = [\mathbf{Q}_2(\mathbf{x}) - \mathbf{l}(\mathbf{x})\mathbf{p}(\mathbf{x})]\mathbf{e}_{\mathbf{z}}$$

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + [\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})]\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\mathbf{e}_{\mathbf{z}} + \mathbf{g}_2(\mathbf{x})(\bar{p}(\mathbf{x}) + q(\mathbf{x})v) \quad (45)$$

From the cascaded structure of the above system and assumption 1, we have that if the controller and observer gains, $\mathbf{k}^{\mathrm{T}}(\mathbf{x})$ and $\mathbf{l}(\mathbf{x})$, are chosen as in assumption 1, then the closed-loop fast subsystem of eq 44 is globally asymptotically stable uniformly in $\mathbf{x} \in \mathrm{IR}^n$.

Setting $\epsilon = 0$ in the system of eq 8, the following closed-loop slow subsystem is obtained:

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})[\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v] + \mathbf{s}(\boldsymbol{\eta})(y_1^{\mathrm{s}} - h_1(\boldsymbol{\eta}))$$

$$\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x}) + \tilde{\mathbf{G}}(\mathbf{x})[\bar{p}(\boldsymbol{\eta}) + q(\boldsymbol{\eta})v] \quad (46)$$

$$y_1^{\mathrm{s}} = h_1(\mathbf{x})$$

From assumption 2, it directly follows that the above system is globally asymptotically stable and $\lim_{t\to\infty}(y_1^{\mathrm{s}} - v) = 0$.

Since the fast and slow closed-loop subsystems are asymptotically stable, a direct application of the result of theorem A1 yields that for each pair of positive real numbers δ and d, there exists are $\epsilon^* > 0$ such that if $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, $\max\{|\mathbf{x}(0)|, |\mathbf{z}(0)|, |\omega(0)|, |\boldsymbol{\eta}(0)|, v\} \leq \delta$, and $\epsilon \in (0, \epsilon^*]$, the states of the two-time-scale closed-loop system are bounded and its output satisfies

$$\lim_{t\to\infty}|y_1(t) - v(t)| \leq d \quad (47)$$

The proof of theorem 1 is complete.

Proof of Theorem 2. Under the controller of eq 15, the closed-loop system takes the following form:

$$\epsilon\dot{\boldsymbol{\omega}} = \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})]\omega + \mathbf{l}(\boldsymbol{\eta})(\mathbf{p}(\mathbf{x})\mathbf{z} - \mathbf{p}(\boldsymbol{\eta})\omega) + \mathbf{g}_2(\boldsymbol{\eta})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\boldsymbol{\eta})\}$$

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\boldsymbol{\eta})\} + \mathbf{S}(h_1(\mathbf{x}) - h_1(\boldsymbol{\eta}))$$

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}) + \mathbf{Q}_1(\mathbf{x})\mathbf{z} + \mathbf{g}_1(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})\omega + \mathbf{g}_1(\mathbf{x})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta k L_{\tilde{\mathbf{F}}}^{k}h_1(\boldsymbol{\eta})\}$$

$$\epsilon\dot{\mathbf{z}} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})\omega + \mathbf{g}_2(\mathbf{x})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\boldsymbol{\eta})\} \quad (48)$$

Performing a two-time-scale decomposition in the above system, the fast subsystem takes the following form:

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\boldsymbol{\eta}) + [\mathbf{Q}_2(\boldsymbol{\eta}) + \mathbf{g}_2(\boldsymbol{\eta})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})]\omega + \mathbf{l}(\boldsymbol{\eta})(\mathbf{p}(\mathbf{x})\mathbf{z} - \mathbf{p}(\boldsymbol{\eta})\omega) + \mathbf{g}_2(\boldsymbol{\eta})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\boldsymbol{\eta})\}$$

$$\frac{dz}{d\tau} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\boldsymbol{\eta})\omega + \mathbf{g}_2(\mathbf{x})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\boldsymbol{\eta})\} \quad (49)$$

Since $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, it follows that $\boldsymbol{\eta}(\tau) = \mathbf{x}(\tau)$, $\tau \geq 0$, and thus, the above system can be equivalently written as

$$\frac{d\boldsymbol{\omega}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + [\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})]\omega + \mathbf{l}(\mathbf{x})(\mathbf{p}(\mathbf{x})\mathbf{z} - \mathbf{p}(\mathbf{x})\omega) + \mathbf{g}_2(\mathbf{x})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\mathbf{x})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\mathbf{x})\}$$

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x})\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\omega + \mathbf{g}_2(\mathbf{x})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\mathbf{x})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\mathbf{x})\} \quad (50)$$

In terms of the error vector $\mathbf{e}_{\mathbf{z}} = \boldsymbol{\omega} - \mathbf{z}$, the above system takes the following form:

$$\frac{d\mathbf{e}_{\mathbf{z}}}{d\tau} = [\mathbf{Q}_2(\mathbf{x}) - \mathbf{l}(\mathbf{x})\mathbf{p}(\mathbf{x})]\mathbf{e}_{\mathbf{z}}$$

$$\frac{d\mathbf{z}}{d\tau} = \mathbf{f}_2(\mathbf{x}) + [\mathbf{Q}_2(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})]\mathbf{z} + \mathbf{g}_2(\mathbf{x})\mathbf{k}^{\mathrm{T}}(\mathbf{x})\mathbf{e}_{\mathbf{z}} + \mathbf{g}_2(\mathbf{x})[\beta_{\tilde{r}}L_{\tilde{\mathbf{G}}}L_{\tilde{\mathbf{F}}}^{\tilde{r}-1}h_1(\mathbf{x})]^{-1}\{v - \sum_{k=0}^{\tilde{r}}\beta_k L_{\tilde{\mathbf{F}}}^{k}h_1(\mathbf{x})\} \quad (51)$$

which can be made globally asymptotically stable uniformly in $\mathbf{x} \in \mathrm{IR}^n$ through appropriate selection of the controller and observer gains $\mathbf{l}(\mathbf{x})$ and $\mathbf{k}^{\mathrm{T}}(\mathbf{x})$, respectively.

Setting $\epsilon = 0$ in the system of eq 48, the following closed-loop slow subsystem is obtained:

$$\dot{\boldsymbol{\eta}} = \tilde{\mathbf{F}}(\boldsymbol{\eta}) + \tilde{\mathbf{G}}(\boldsymbol{\eta})[\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^{k} h_1(\boldsymbol{\eta})\} + \mathbf{S}(y_1 - h_1(\boldsymbol{\eta}))$$

$$\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x}) + \tilde{\mathbf{G}}(\mathbf{x})[\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\boldsymbol{\eta})]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^{k} h_1(\boldsymbol{\eta})\} \tag{52}$$

$$y_1^{\mathrm{s}} = h_1(\mathbf{x})$$

Since for $\epsilon = 0$, $\boldsymbol{\eta}(0) = \mathbf{x}(0)$, the above system is equivalent to:

$$\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x}) + \tilde{\mathbf{G}}(\mathbf{x})[\beta_{\tilde{r}} L_{\tilde{\mathbf{G}}} L_{\tilde{\mathbf{F}}}^{\tilde{r}-1} h_1(\mathbf{x})]^{-1}\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{\mathbf{F}}}^{k} h_1(\mathbf{x})\}$$

$$y_1^{\mathrm{s}} = h_1(\mathbf{x}) \tag{53}$$

On the basis of the above system, one can easily show that y_1^{s} is the solution of the linear differential equation

$$\sum_{k=0}^{\tilde{r}} \beta_k \frac{\mathrm{d}^k y_1^{\mathrm{s}}}{\mathrm{d}t^k} = v$$

Furthermore, one can show, using the assumption of minimum-phase behavior of the modified closed-loop slow subsystem of eq 13 that the system of eq 53 is locally exponentially stable, which in turn implies that the system of eq 52 is locally exponentially stable.

Since the closed-loop fast and slow subsystems are exponentially stable, a direct application of the result of theorem 8.4 in Khalil (1992) yields that there exist positive real numbers $\bar{\delta}$ and $\bar{\epsilon}$, such that if $\boldsymbol{\eta}(0) = \mathbf{x}(0) + O(\epsilon)$, $\max\{|\mathbf{x}(0)|, |\mathbf{z}(0)|, |\boldsymbol{\omega}(0)|, |\boldsymbol{\eta}(0)|, v\} \le \bar{\delta}$ and $\epsilon \in (0, \bar{\epsilon}]$, the closed-loop two-time-scale system of eq 48 is exponentially stable and its output, y_1, satisfies the relation of eq 18. The proof of theorem 2 is complete.

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