

Robust output feedback control of quasi-linear parabolic PDE systems¹

Panagiotis D. Christofides*, James Baker

Department of Chemical Engineering, School of Engineering and Applied Sciences, University of California, Box 951592, Los Angeles, CA 90095-1592, USA

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Abstract

This paper proposes a methodology for the synthesis of nonlinear robust *output* feedback controllers for systems of quasi-linear parabolic partial differential equations with time-varying uncertain variables. The method is successfully applied to a typical diffusion–reaction process with uncertainty. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Galerkin's method; Lyapunov's direct method; Static output feedback; Diffusion–convection–reaction processes

1. Introduction

Quasi-linear parabolic partial differential equation (PDE) systems arise very frequently in the modeling of diffusion–convection–reaction processes (e.g., fluidized-bed and packed-bed reactors), and typically involve spatial differential operators whose eigenspectrum can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement. Therefore, the standard approach to the control of quasi-linear parabolic PDE systems involves the application of Galerkin's method to the PDE system to derive ODE systems that accurately describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for controller synthesis (e.g., [1, 12, 2]). The main disadvantage of this approach is that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation

may be very large, leading to complex controller design and high dimensionality of the resulting controllers. A natural approach for the derivation of accurate low-dimensional ODE approximations and the design of low-dimensional controllers for quasi-linear parabolic PDE systems is based on the concept of inertial manifold (see, for example, [17, 10]).

The majority of quasi-linear parabolic PDE systems which model diffusion–convection–reaction processes are uncertain due to the presence of unknown or partially known process parameters and disturbances. Uncertain variables, if they are not appropriately accounted for in the controller design, can significantly deteriorate the nominal closed-loop performance, and even lead to closed-loop instability. Motivated by this, the problem of synthesizing controllers that compensate for the effect of uncertainty (called robust controllers) in parabolic PDE systems has received considerable attention in the past. Initially, H^∞ control algorithms were developed for linear parabolic PDE systems in the frequency domain (e.g., [15, 18]). In [20], precise relations were derived between state-space and frequency-domain

* Corresponding author. Tel.: 001 310 794 1015; fax: 001 310 206 4107; e-mail: pdc@seas.ucla.edu.

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control theoretic concepts for a large class of linear infinite-dimensional systems, which allowed the development of the state-space counterparts of the frequency-domain H^∞ results (e.g., [25]). In a previous work [8], we proposed a Lyapunov-based robust controller design method for quasi-linear parabolic PDEs, under the assumption that measurements of the state variables are available. Other results on control of PDE systems with time-varying uncertainty include disturbance decoupling for linear parabolic PDEs [13, 14], Lyapunov-based control of hyperbolic PDE systems [11], Lyapunov-based [21] and H^∞ (e.g., [4, 22]) control of Navier–Stokes equations, as well as methods for the optimal selection of measurement sensors [3]. An alternative approach for the design of controllers for linear PDE systems with time-invariant (or slowly varying) uncertain variables involves the use of adaptive control methods (e.g., [5, 26, 16]).

In this paper, we extend the method presented in [8] to quasi-linear parabolic PDE systems with time-varying uncertain variables, for which only a finite number of measurements of the output variable is available for feedback. Initially, Galerkin’s method is used to derive an approximate ODE model, which is used to synthesize a robust state feedback controller via Lyapunov’s direct method. Then, under the assumption that the number of measurements is equal to the number of slow modes, we propose a procedure for obtaining estimates for the states of the approximate ODE model from the measurements. We show that the use of these estimates in the robust state feedback controller leads to a robust *output* feedback controller, which guarantees boundedness of the state and output tracking with arbitrary degree of asymptotic attenuation of the effect of the uncertain variables on the controlled output of the infinite-dimensional closed-loop system, provided that the separation between the slow and fast eigenvalues is sufficiently large. The theoretical results are successfully applied to a typical diffusion–reaction process with uncertainty.

2. Preliminaries

We consider quasi-linear parabolic PDE systems with uncertain variables of the form

$$\frac{\partial \bar{x}}{\partial t} = A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2} + wb(z)u + f(\bar{x}) + W(\bar{x}, r(z)\theta(t)),$$

$$y_c^i = \int_\alpha^\beta c^i(z)k\bar{x} dz, \quad i = 1, \dots, l,$$

$$y_m^\kappa = \int_\alpha^\beta s^\kappa(z)\omega\bar{x} dz, \quad \kappa = 1, \dots, p, \tag{1}$$

subject to the boundary conditions:

$$C_1\bar{x}(\alpha, t) + D_1 \frac{\partial \bar{x}}{\partial z}(\alpha, t) = R_1,$$

$$C_2\bar{x}(\beta, t) + D_2 \frac{\partial \bar{x}}{\partial z}(\beta, t) = R_2 \tag{2}$$

and the initial condition:

$$\bar{x}(z, 0) = \bar{x}_0(z), \tag{3}$$

where $\bar{x}(z, t) = [\bar{x}_1(z, t) \cdots \bar{x}_n(z, t)]^T$ denotes the vector of state variables, $[\alpha, \beta] \subset \mathbb{R}$ is the domain of definition of the process, $z \in [\alpha, \beta]$ is the spatial coordinate, $t \in [0, \infty)$ is the time, $u = [u^1 u^2 \cdots u^l]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, $\theta = [\theta_1 \cdots \theta_q] \in \mathbb{R}^q$ denotes the vector of uncertain variables, which may include uncertain process parameters or exogenous disturbances, $y_c^i \in \mathbb{R}$ denotes a controlled output, $y_m^\kappa \in \mathbb{R}$ denotes a measured output, $\partial \bar{x} / \partial z, \partial^2 \bar{x} / \partial z^2$ denote the first- and second-order spatial derivatives of \bar{x} , $f(\bar{x}), W(\bar{x}, r(z)\theta(t))$ are nonlinear vector functions, w, k, ω are vectors, A, B, C_1, D_1, C_2, D_2 are matrices, R_1, R_2 are column vectors, and $\bar{x}_0(z)$ is the initial condition.

The properties and role of the functions $b(z), r(z), s^i(z)$ are as follows: $b(z)$ is a known smooth vector function of z of the form $b(z) = [b^1(z) b^2(z) \cdots b^l(z)]$, where $b^i(z)$ describes how the control action $u^i(t)$ is distributed in the interval $[\alpha, \beta]$ (e.g., point/distributed actuation), $r(z) = [r_1(z) \cdots r_q(z)]$, where $r_k(z)$ is a known smooth function of z which specifies the position of action of the uncertain variable θ_k on $[\alpha, \beta]$, $c^i(z)$ is a known smooth function of z which is determined by the specification of the i th controlled output in the interval $[\alpha, \beta]$, and $s^i(z)$ is a known smooth function of z which is determined by the location and type of the measurement sensors (e.g., point/distributed sensing). A discussion on the practical implementation of point actuation (sensing) is given in Remark 8 below.

Throughout the paper, $O(\varepsilon)$ denotes the order of magnitude notation (i.e., $\delta(\varepsilon) = O(\varepsilon)$ if there exist positive real numbers k_1 and k_2 such that: $|\delta(\varepsilon)| \leq k_1|\varepsilon|, \forall |\varepsilon| < k_2$), $|\cdot|$ denotes the standard Euclidean norm, and $\|\theta\|$ denotes $\text{ess.sup.}|\theta(t)|, t \geq 0$ for any measurable function $\theta: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^q$. Moreover, we will use the Lie derivative notation: $L_f h$ denotes

the Lie derivative of a scalar field h with respect to the vector field f , $L_f^k h$ denotes the k th-order Lie derivative and $L_g L_f^{k-1} h$ denotes the mixed Lie derivative.

Defining the state x in the infinite-dimensional Hilbert space $\mathcal{H}([\alpha, \beta], \mathbb{R}^n)$ (with \mathcal{H} being the space of n -dimensional vector functions defined on $[\alpha, \beta]$ that satisfy the boundary condition of Eq. (2), with inner product and norm:

$$(\omega_1, \omega_2) = \int_{\alpha}^{\beta} (\omega_1(z), \omega_2(z))_{\mathbb{R}^n} dz,$$

$$\|\omega_1\|_2 = (\omega_1, \omega_1)^{1/2},$$

where ω_1, ω_2 are two elements of \mathcal{H} and the notation $(\cdot, \cdot)_{\mathbb{R}^n}$ denotes the standard inner product in \mathbb{R}^n , as

$$x(t) = \bar{x}(z, t), \quad t > 0, z \in [\alpha, \beta], \quad (4)$$

and the following operators

$$\mathcal{A}x = A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2},$$

$$x \in D(\mathcal{A}) = \left\{ x \in \mathcal{H}; C_1 \bar{x}(\alpha, t) + D_1 \frac{\partial \bar{x}}{\partial z}(\alpha, t) = R_1, \right.$$

$$\left. C_2 \bar{x}(\beta, t) + D_2 \frac{\partial \bar{x}}{\partial z}(\beta, t) = R_2 \right\}. \quad (5)$$

$$\mathcal{B}u = wbu, \quad \mathcal{C}x = (c, kx), \quad \mathcal{S}x = (s, \omega x),$$

where $c = [c^1 \ c^2 \ \dots \ c^l]$ and $s = [s^1 \ s^2 \ \dots \ s^p]$, the system of Eqs. (1)–(3) take the form

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + f(x) + \mathcal{W}(x, \theta), \quad x(0) = x_0, \quad (6)$$

$$y_c = \mathcal{C}x, \quad y_m = \mathcal{S}x,$$

where $f(x(t)) = f(\bar{x}(z, t))$, $\mathcal{W}(x(t), \theta) = W(\bar{x}, r\theta)$ and $x_0 = \bar{x}_0(z)$.

For \mathcal{A} , the eigenvalue problem is defined as

$$\mathcal{A}\phi_j = \lambda_j \phi_j, \quad j = 1, \dots, \infty, \quad (7)$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction; the eigenspectrum of \mathcal{A} , $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of \mathcal{A} , i.e., $\sigma(\mathcal{A}) = \{\lambda_1, \lambda_2, \dots\}$. We will assume that $\sigma(\mathcal{A})$ satisfies the following properties.

Assumption 1 (Christofides and Daoutidis [10]).

1. $\text{Re}\{\lambda_1\} \geq \text{Re}\{\lambda_2\} \geq \dots \geq \text{Re}\{\lambda_j\} \geq \dots$, where $\text{Re}\{\lambda_j\}$ denotes the real part of λ_j .

2. $\sigma(\mathcal{A})$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$, where $\sigma_1(\mathcal{A})$ consists of the first m (with m finite) eigenvalues, i.e., $\sigma_1(\mathcal{A}) = \{\lambda_1, \dots, \lambda_m\}$, and

$$|\text{Re}\{\lambda_1\}|/|\text{Re}\{\lambda_m\}| = O(1).$$

3. $\text{Re}\lambda_{m+1} < 0$ and $|\text{Re}\{\lambda_m\}|/|\text{Re}\{\lambda_{m+1}\}| = O(\varepsilon)$ where $\varepsilon := |\text{Re}\lambda_1|/|\text{Re}\lambda_{m+1}| < 1$ is a small positive number.

The assumptions of finite number of unstable eigenvalues and discrete eigenspectrum are always satisfied for parabolic PDE systems defined in finite spatial domains, while the assumption of existence of only a few dominant modes that describe the dynamics of the parabolic PDE system is usually satisfied by the majority of diffusion–convection–reaction processes (see the catalytic rod example of Section 5).

3. Galerkin’s method

We derive an m -dimensional approximation of the system of Eq. (6) using Galerkin’s method. Letting $\mathcal{H}_s, \mathcal{H}_f$ be two subspaces of \mathcal{H} , defined as $\mathcal{H}_s = \text{span}\{\phi_1, \phi_2, \dots, \phi_m\}$ and $\mathcal{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \dots\}$, and defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq. (6) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x. \quad (8)$$

Applying P_s and P_f to the system of Eq. (6) and using the above decomposition for x , the system of Eq. (6) can be equivalently written in the following form:

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f) + \mathcal{W}_s(x_s, x_f, \theta),$$

$$\frac{dx_f}{dt} = \mathcal{A}_f x_f + \mathcal{B}_f u + f_f(x_s, x_f) + \mathcal{W}_f(x_s, x_f, \theta), \quad (9)$$

$$y_c = \mathcal{C}x_s + \mathcal{C}x_f, \quad y_m = \mathcal{S}x_s + \mathcal{S}x_f,$$

$$x_s(0) = P_s x(0) = P_s x_0,$$

$$x_f(0) = P_f x(0) = P_f x_0,$$

where $\mathcal{A}_s = P_s \mathcal{A}$, $\mathcal{B}_s = P_s \mathcal{B}$, $f_s = P_s f$, $\mathcal{W}_s = P_s \mathcal{W}$, $\mathcal{A}_f = P_f \mathcal{A}$, $\mathcal{B}_f = P_f \mathcal{B}$, $f_f = P_f f$, $\mathcal{W}_f = P_f \mathcal{W}$. In the above system, \mathcal{A}_s is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = \text{diag}\{\lambda_j\}$, $f_s(x_s, x_f)$, $f_f(x_s, x_f)$ are Lipschitz vector functions, $\mathcal{W}_s(x_s, x_f, \theta)$, $\mathcal{W}_f(x_s, x_f, \theta)$ are Lipschitz matrix functions, and \mathcal{A}_f is a stable unbounded differential operator. Neglecting the infinite-dimensional x_f -subsystem in the system of Eq. (9), the following

m -dimensional slow system is obtained:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + f_s(x_s, 0) + \mathcal{B}_s u + \mathcal{W}_s(x_s, 0, \theta) \\ &=: F_0(x_s) + \sum_{i=1}^l \mathcal{B}_0^i u_0^i + \mathcal{W}_0(x_s, 0, \theta), \end{aligned} \quad (10)$$

$$y_{c_i} = \mathcal{C}^i x_s =: h_{i0}(x_s), \quad y_m = \mathcal{S} x_s,$$

where the subscript 0 in $(F_0, \mathcal{B}_0^i, u_0^i, \mathcal{W}_0, h_{i0})$ denotes that they are elements of the $O(\varepsilon)$ approximation of the x_s -subsystem of Eq. (9) (see proof of Theorem 1 below for a precise characterization of the accuracy of the system of Eq. (10)). We note that higher-order m -dimensional approximations of the system of Eq. (6) can be derived through combination of Galerkin’s method with approximate inertial manifolds (see, for example, [10, 8]).

Remark 1. We note that the above model reduction procedure which led to the approximate ODE system of Eq. (10) can also be used, when empirical eigenfunctions of the system of Eq. (6) computed through Karhunen–Loève expansion (known also as proper orthogonal decomposition) are used as basis functions in \mathcal{H}_s and \mathcal{H}_r , instead of the eigenfunctions of \mathcal{A} .

4. Robust output feedback controller synthesis

We consider the synthesis of robust output feedback control laws of the form

$$u_0 = p_0(y_m) + Q_0(y_m)\bar{v} + r_0(y_m, t), \quad (11)$$

where $p_0(y_m)$, $r_0(y_m, t)$ are vector functions, $Q_0(y_m)$ is a matrix, and \bar{v} is a vector of the form $\bar{v} = \mathcal{V}(v_i, v_i^{(1)}, \dots, v_i^{(r_i)})$ where $\mathcal{V}(v_i, v_i^{(1)}, \dots, v_i^{(r_i)})$ is a smooth vector function, $v_i^{(k)}$ is the k th time derivative of the external reference input v_i (which is assumed to be a smooth function of time) and r_i is a positive integer.

We now state three assumptions on the system of Eq. (10) which are needed to synthesize a robust controller of the form of Eq. (11) under the assumption that measurements of the states x_s are available (motivation for and explanations on the nature of these assumptions as well as details on controller design via Lyapunov’s direct method can be found in [8]).

Assumption 2. Referring to the system of Eq. (10), there exist a set of integers (r_1, r_2, \dots, r_l) and a coordinate transformation $(\zeta, \eta) = T(x_s, \theta)$ such that the representation of the system, in the coordinates

(ζ, η) , takes the form

$$\begin{aligned} \dot{\zeta}_1^{(1)} &= \zeta_2^{(1)}, \\ &\vdots \\ \dot{\zeta}_{r_1-1}^{(1)} &= \zeta_{r_1}^{(1)}, \\ \dot{\zeta}_{r_1}^{(1)} &= L_{F_0}^{r_1} h_{10}(x_s) + \sum_{i=1}^l L_{\mathcal{B}_0^i} L_{F_0}^{r_1-1} h_{10}(x_s) u_0^i \\ &\quad + L_{\mathcal{W}_0} L_{F_0}^{r_1-1} h_{10}(x_s), \\ &\vdots \\ \dot{\zeta}_1^{(l)} &= \zeta_2^{(l)}, \\ &\vdots \\ \dot{\zeta}_{r_l-1}^{(l)} &= \zeta_{r_l}^{(l)}, \\ \dot{\zeta}_{r_l}^{(l)} &= L_{F_0}^{r_l} h_{l0}(x_s) + \sum_{i=1}^l L_{\mathcal{B}_0^i} L_{F_0}^{r_l-1} h_{l0}(x_s) u_0^i \\ &\quad + L_{\mathcal{W}_0} L_{F_0}^{r_l-1} h_{l0}(x_s), \\ \dot{\eta}_1 &= \Psi_1(\zeta, \eta, \theta, \dot{\theta}), \\ &\vdots \\ \dot{\eta}_{m-\sum_i r_i} &= \Psi_{m-\sum_i r_i}(\zeta, \eta, \theta, \dot{\theta}), \\ y_{c_i} &= \zeta_1^{(i)}, \quad i = 1, \dots, l, \end{aligned} \quad (12)$$

where $x_s = T^{-1}(\zeta, \eta, \theta)$, $\zeta = [\zeta^{(1)} \dots \zeta^{(l)}]^T \in \mathbb{R}^{r_1+r_2+\dots+r_l}$, $\eta = [\eta_1 \dots \eta_{m-\sum_i r_i}]^T \in \mathbb{R}^{m-(r_1+r_2+\dots+r_l)}$.

The above assumption is always satisfied for systems for which $r_i = 1$, for all $i = 1, \dots, l$. In most practical applications, this requirement can be easily achieved by selecting the form of the actuator distribution functions $b_i(z)$ to be different than the eigenfunctions ϕ_j for $j = 2, \dots, \infty$ (i.e., pick $b_i(z)$ so that $b_i(z) \neq \phi_j$, for $i = 1, \dots, l$, $j = 2, \dots, \infty$). Referring to the system of Eq. (12), we will assume, in order to simplify the presentation of our results, that the matrix:

$$\begin{aligned} C_0(x_s) &= \begin{bmatrix} L_{\mathcal{B}_0^1} L_{F_0}^{r_1-1} h_{10}(x_s) & \cdots & L_{\mathcal{B}_0^l} L_{F_0}^{r_l-1} h_{10}(x_s) \\ \vdots & \cdots & \vdots \\ L_{\mathcal{B}_0^1} L_{F_0}^{r_1-1} h_{l0}(x_s) & \cdots & L_{\mathcal{B}_0^l} L_{F_0}^{r_l-1} h_{l0}(x_s) \end{bmatrix} \end{aligned} \quad (13)$$

is nonsingular uniformly in $x_s \in \mathcal{H}_s$ (see Remark 3).

Assumption 3. The dynamical system:

$$\begin{aligned} \dot{\eta}_1 &= \Psi_1(\zeta, 0, 0, 0), \\ &\vdots \\ \dot{\eta}_{m-\sum_i r_i} &= \Psi_{m-\sum_i r_i}(\zeta, 0, 0, 0) \end{aligned} \quad (14)$$

is locally exponentially stable.

Assumption 4. *There exists a known function $c_0(x_s, t)$ such that the following condition holds:*

$$[[L_{\mathcal{W}_0} L_{F_0}^{r_1-1} h_{10}(x_s) \cdots L_{\mathcal{W}_0} L_{F_0}^{r_l-1} h_{l0}(x_s)]^T] \leq c_0(x_s, t) \text{ for all } x_s \in \mathcal{H}_s, \theta \in \mathbb{R}^q, t \geq 0. \quad (15)$$

The following assumption is needed in order to obtain estimates of the states x_s of the system of Eq. (10) from the measurements $y_m^\kappa, \kappa = 1, \dots, p$.

Assumption 5. *$p = m$ (i.e., the number of measurements is equal to the number of slow modes), and the inverse of the operator S exists so that $\hat{x}_s = S^{-1} y_m$, where \hat{x}_s is an estimate of x_s .*

Theorem 1 that follows provides an explicit formula for the robust output feedback controller, conditions that ensure boundedness of the state, and a precise characterization of the ultimate uncertainty attenuation level. To simplify the statement of the theorem, we set $\bar{v}_i = [v_i \ v_i^{(1)} \ \cdots \ v_i^{(r_i)}]^T$ and $\bar{v} = [\bar{v}_1^T \ \bar{v}_2^T \ \cdots \ \bar{v}_m^T]^T$, while we will use the notation $|x_s|_2$ to denote that x_s belongs in a finite dimensional Hilbert space.

Theorem 1. *Consider the parabolic PDE system of Eq. (6) for which assumption 1 holds, and the finite-dimensional system of Eq. (10), for which assumptions 2, 3, 4 and 5 hold, under the robust output feedback controller:*

$$\begin{aligned} u_0 &= a_0(x_s, x_f, \bar{v}, t) \\ &:= [C_0(\hat{x}_s)]^{-1} \left\{ \sum_{i=1}^l \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (v_i^{(k)} - L_{F_0}^k h_{i0}(\hat{x}_s)) \right. \\ &\quad + \sum_{i=1}^l \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (v_i^{(k-1)}) \\ &\quad \left. - L_{F_0}^{k-1} h_{i0}(\hat{x}_s) - \chi [c_0(\hat{x}_s, t)] \right. \\ &\quad \left. \times \frac{\sum_{i=1}^l \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (L_{F_0}^{k-1} h_{i0}(\hat{x}_s) - v_i^{(k-1)})}{|\sum_{i=1}^l \sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}} (L_{F_0}^{k-1} h_{i0}(\hat{x}_s) - v_i^{(k-1)})| + \phi} \right\}, \end{aligned} \quad (16)$$

where

$$\hat{x}_s = S^{-1} y_m, \quad \frac{\beta_{ik}}{\beta_{ir_i}} = \begin{bmatrix} \beta_{ik}^1 & & \beta_{ik}^l \\ & \ddots & \\ \beta_{ir_i}^1 & & \beta_{ir_i}^l \end{bmatrix}^T$$

are column vectors of parameters chosen so that the roots of the equation $\det(B(s)) = 0$, where $B(s)$ is an $l \times l$ matrix, whose (i, j) th element is of the form $\sum_{k=1}^{r_i} (\beta_{jk}^i / \beta_{jr_i}^i) s^{k-1}$, lie in the open left-half of

the complex plane, and χ, ϕ are adjustable parameters with $\chi > 1$ and $\phi > 0$. Then, there exist positive real numbers (δ, ϕ^*) such that for each $\phi \leq \phi^*$, there exists $\varepsilon^*(\phi)$, such that if $\phi \leq \phi^*, \varepsilon \leq \varepsilon^*(\phi)$ and $\max\{|x_s(0)|_2, \|x_f(0)\|_2, \|\theta\|, \|\dot{\theta}\|, \|\bar{v}\|\} \leq \delta$,

(a) *the state of the infinite-dimensional closed-loop system is bounded, and*

(b) *the outputs of the infinite-dimensional closed-loop system satisfy:*

$$\limsup_{t \rightarrow \infty} |y_i - v_i| \leq d_0, \quad i = 1, \dots, l, \quad (17)$$

where $d_0 = O(\phi + \varepsilon)$ is a positive real number.

Proof of Theorem 1. Substituting the controller of Eq. (16) into the parabolic PDE system of Eq. (6), we obtain:

$$\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}a_0(x_s, x_f, \bar{v}, t) \\ &\quad + f(x) + \mathcal{W}(x, \theta), \quad x(0) = x_0, \quad (18) \\ y_c &= \mathcal{C}x, \quad y_m = \mathcal{S}x. \end{aligned}$$

One can easily verify that assumption 1 holds for the above system, and thus, a direct application of Galerkin's method yields the following infinite-dimensional system:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s a_0(x_s, x_f, \bar{v}, t) + f_s(x_s, x_f) \\ &\quad + \mathcal{W}_s(x_s, x_f, \theta), \\ \frac{dx_f}{dt} &= \mathcal{A}_f x_f + \mathcal{B}_f a_0(x_s, x_f, \bar{v}, t) + f_f(x_s, x_f) \\ &\quad + \mathcal{W}_f(x_s, x_f, \theta), \quad (19) \\ y_c &= \mathcal{C}_s x_s + \mathcal{C}_f x_f, \quad y_m = \mathcal{S}_s x_s + \mathcal{S}_f x_f, \\ x_s(0) &= P_s x(0) = P_s x_0, \quad x_f(0) = P_f x(0) = P_f x_0. \end{aligned}$$

Using that $\varepsilon = |\operatorname{Re} \lambda_i| / |\operatorname{Re} \lambda_{m+1}|$, the system of Eq. (19) can be written in the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s a_0(x_s, x_f, \bar{v}, t) + f_s(x_s, x_f) \\ &\quad + \mathcal{W}_s(x_s, x_f, \theta), \quad (20) \\ \varepsilon \frac{dx_f}{dt} &= \mathcal{A}_{f\varepsilon} x_f + \varepsilon \mathcal{B}_f a_0(x_s, x_f, \bar{v}, t) + \varepsilon f_f(x_s, x_f) \\ &\quad + \varepsilon \mathcal{W}_f(x_s, x_f, \theta), \end{aligned}$$

where $\mathcal{A}_{f\varepsilon}$ is an unbounded differential operator defined as $\mathcal{A}_{f\varepsilon} = \varepsilon \mathcal{A}_f$. Since $\varepsilon \ll 1$ (following from assumption 1, part 3) and the operators $\mathcal{A}_s, \mathcal{A}_{f\varepsilon}$ generate semigroups with growth rates which are of the same order of magnitude, the system of Eq. (20) is in the standard singularly perturbed form (see [23] for a precise definition of standard form), with x_s being the slow states and x_f being the fast states. Introducing

the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq. (20):

$$\frac{\partial x_f}{\partial \tau} = \mathcal{A}_{f\varepsilon} x_f. \quad (21)$$

From the fact that $\text{Re } \lambda_{m+1} < 0$ and the definition of ε , we have that the above system is globally exponentially stable. Setting $\varepsilon = 0$ in the system of Eq. (20), we have that $x_f = 0$ and thus, the finite-dimensional slow system takes the form

$$\frac{dx_s}{dt} = F_0(x_s) + \sum_{i=1}^l \mathcal{B}_0^i a_0^i(x_s, 0, \bar{v}, t) + \mathcal{W}_0(x_s, 0, \theta), \quad (22)$$

$$y_{c_i} = \mathcal{C}^i x_s = : h_{i0}(x_s).$$

For the above system we have shown in [8] that there exists a $\phi \in (0, \phi^*]$ such that if $\max\{|x_s(0)|_2, \|\theta\|, \|\dot{\theta}\|, \|\bar{v}\|\} \leq \delta$, then its state is bounded and its outputs satisfy $\limsup_{t \rightarrow \infty} |y_i - v_i| \leq O(\phi)$, $i = 1, \dots, l$. Finally, since the infinite-dimensional fast subsystem of Eq. (21) is exponentially stable, we can use standard singular perturbation arguments to obtain that there exists an $\varepsilon^*(\phi)$, such that if $\varepsilon \in (0, \varepsilon^*(\phi))$, $\max\{|x_s(0)|_2, \|x_f(0)\|_2, \|\theta\|, \|\dot{\theta}\|, \|\bar{v}\|\} \leq \delta$, then the state of the closed-loop parabolic PDE system of Eq. (18) is bounded and that its outputs satisfy the relation of Eq. (17). \square

Remark 2. We note that the controller of Eq. (16) uses static feedback of the measured outputs y_m^κ , $\kappa = 1, \dots, p$, and thus, it feeds back both x_s and x_f (this is in contrast to the robust state feedback controller designed in [8] which only uses feedback of the slow state x_s). However, even though the use of x_f feedback could lead to destabilization of the stable fast subsystem, the large separation of the slow and fast modes of the spatial differential operator (i.e., assumption that ε is sufficiently small) and the fact that the controller does not include terms of the form $O(1/\varepsilon)$ do not allow such a destabilization to occur.

Remark 3. The assumption that the characteristic matrix $C_0(x_s)$ is nonsingular is made to simplify the presentation of the controller synthesis results and can be relaxed if instead of static output feedback we use dynamic output feedback (the reader may refer to [19] for details). Moreover, the assumption that the zero dynamics of the finite-dimensional slow subsystem is exponentially stable is a standard one in geometric nonlinear control (see also [19]).

Remark 4. Even though static output feedback is more sensitive to measurement noise than dynamic output feedback, we prefer to use static feedback of y_m in the controller of Eq. (16) because the presence of the unknown variables does not allow the design of a robust state observer to obtain estimates of the slow state variables without imposing very restrictive conditions on the way $\theta(t)$ enters the finite-dimensional system of Eq. (10).

Remark 5. The approach followed here for the synthesis of robust output feedback controllers is not applicable to hyperbolic PDE systems (i.e., convection–reaction processes) where the eigenvalues cluster along vertical or nearly vertical asymptotes in the complex plane and thus, the controller synthesis problem has to be addressed directly on the basis of the hyperbolic PDE system (see [9, 11]).

Remark 6. We note that the result of Theorem 1 can be generalized to the case where the ODE systems used for controller design are obtained from combination of Galerkin’s method with approximate inertial manifolds (see [8] for details on the design of robust state feedback controllers on the basis of such ODE systems and [7, 6, 24] for other applications of inertial manifold theory to control of nonlinear parabolic PDEs).

Remark 7. The robust controller of Eq. (16) possesses a robustness property with respect to fast and asymptotically stable unmodeled dynamics (i.e., the controller enforces boundedness, output tracking and uncertainty attenuation in the closed-loop system, despite the presence of additional dynamics in the process model, as long as they are stable and sufficiently fast). This robustness property of the controller can be rigorously established by analyzing the closed-loop system with the unmodeled dynamics using singular perturbations, and is important for many practical applications where unmodeled dynamics occur due to actuator and sensor dynamics, fast process dynamics, etc.

5. Application to a diffusion–reaction process

We consider a long, thin rod in a furnace and assume that a zeroth-order exothermic catalytic reaction of the form $A \rightarrow B$ takes place on the rod. Because the reaction is exothermic a cooling medium which is in

contact with the rod is used for cooling. Under standard modeling assumptions, the spatiotemporal evolution of the dimensionless rod temperature is described by the following parabolic PDE:

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T e^{-\gamma/(1+\bar{x})} + \beta_U (b(z)u(t) - \bar{x}) - \beta_{T,n} e^{-\gamma} \quad (23)$$

subject to the Dirichlet boundary conditions:

$$\bar{x}(0, t) = 0, \quad \bar{x}(\pi, t) = 0 \quad (24)$$

and the initial condition:

$$\bar{x}(z, 0) = \bar{x}_0(z), \quad (25)$$

where \bar{x} denotes the dimensionless temperature of the rod, β_T denotes a dimensionless heat of reaction (which is assumed to be unknown and time-varying; uncertain variable), $\beta_{T,n}$ denotes a *nominal* dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, and u denotes the manipulated input (temperature of the cooling medium). The following typical values were given to the process parameters:

$$\beta_{T,n} = 50.0, \quad \beta_U = 2.0, \quad \gamma = 4.0. \quad (26)$$

Introducing the Hilbert space \mathcal{H} of square integrable functions that satisfy the boundary conditions of Eq. (24) and defining $x \in \mathcal{H}$ as

$$x(t) = \bar{x}(z, t), \quad \forall z \in [0, \pi], \quad (27)$$

the system of Eqs. (23)–(25) can be written in the form of Eq. (6), where the spatial differential operator takes the form:

$$\begin{aligned} \mathcal{A}x &= \frac{\partial^2 \bar{x}}{\partial z^2}, \\ x &\in D(\mathcal{A}) \\ &= \{x \in \mathcal{H}([0, \pi]; \mathbb{R}); \bar{x}(0, t) = 0, \bar{x}(\pi, t) = 0\}. \end{aligned} \quad (28)$$

The eigenvalue problem for \mathcal{A} can be solved analytically and its solution is of the form

$$\lambda_j = -j^2, \quad \phi_j(z) = \sqrt{\frac{2}{\pi}} \sin(jz), \quad j = 1, \dots, \infty. \quad (29)$$

Even though the eigenvalues of \mathcal{A} are all stable, the spatially uniform operating steady-state $\bar{x}(z, t) = 0$ of the system of Eq. (23) is unstable (i.e., the linearization of the system of Eq. (23) around $\bar{x}(z, t) = 0$ possesses one unstable eigenvalue owing to the

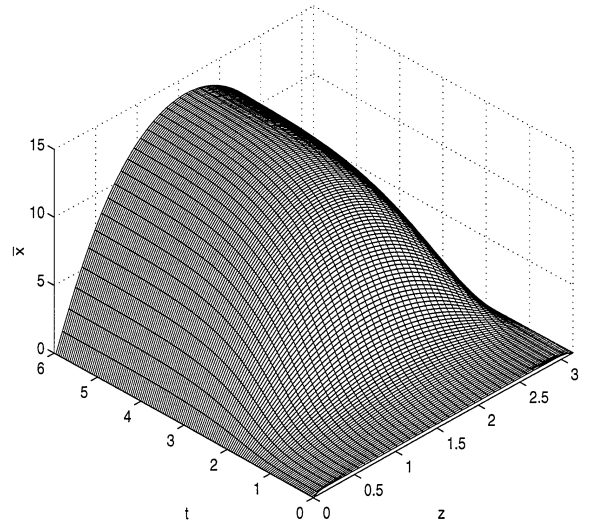


Fig. 1. Open-loop profile of \bar{x} with $\theta(t) = 0$.

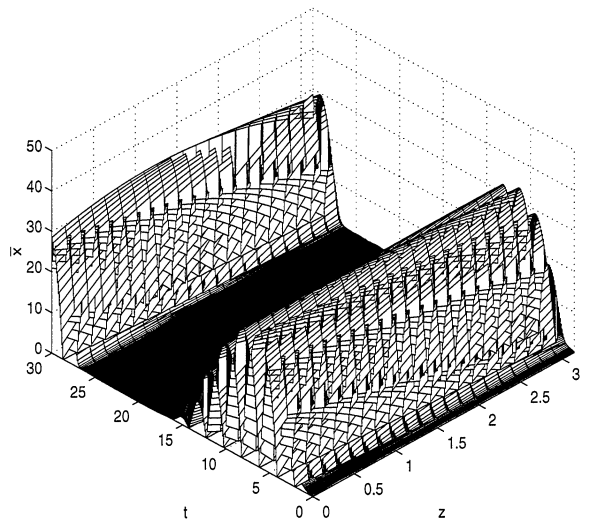


Fig. 2. Open-loop profile of \bar{x} with $\theta(t) = \beta_{T,n} \sin(0.524 t)$.

exothermicity of the reaction). Fig. 1 shows the spatiotemporal evolution of the state $\bar{x}(z, t)$ of the system of Eq. (23) when no uncertainty is present; the system moves to a spatially non-uniform steady state. We note that in all the simulation runs, a 20th order ODE approximation of the system of Eq. (23) derived from Galerkin’s method is used; higher-order approximations led to identical numerical results.

Due to the instability of the spatially uniform operating steady-state $\bar{x}(z, t) = 0$, we formulate the control problem as the one of stabilizing the rod temperature profile at $\bar{x}(z, t) = 0$ in the presence of time-varying uncertainty in the dimensionless heat of the reaction β_T ,

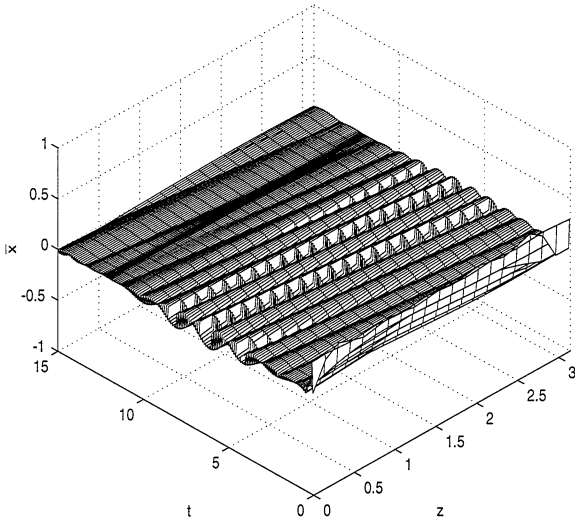


Fig. 3. Closed-loop profile of \bar{x} under nonlinear robust output feedback control-distributed control action.

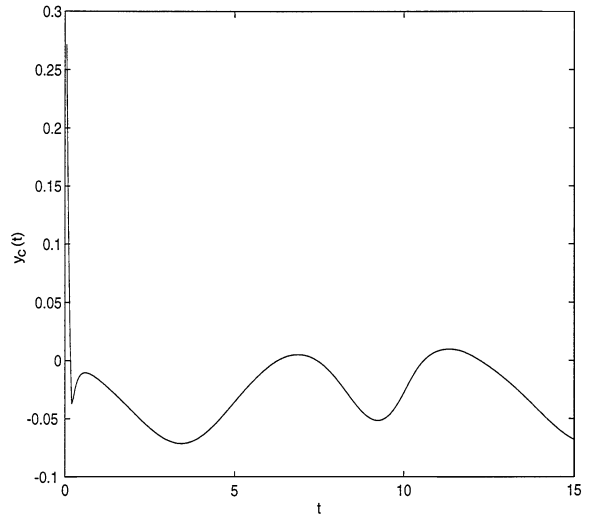


Fig. 5. Closed-loop output profile under nonlinear robust output feedback control-distributed control action.

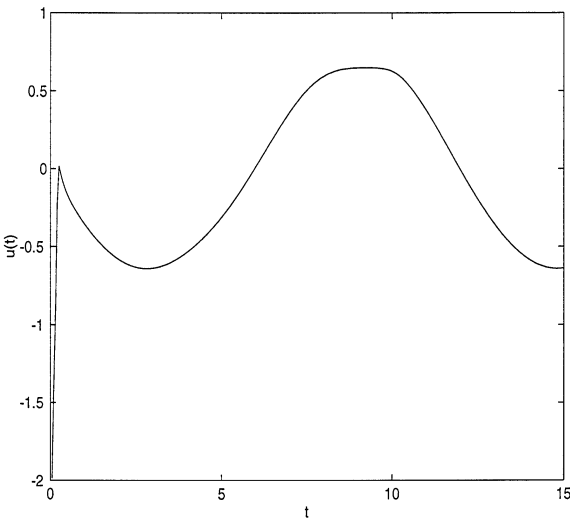


Fig. 4. Manipulated input profile for nonlinear robust output feedback controller-distributed control action.

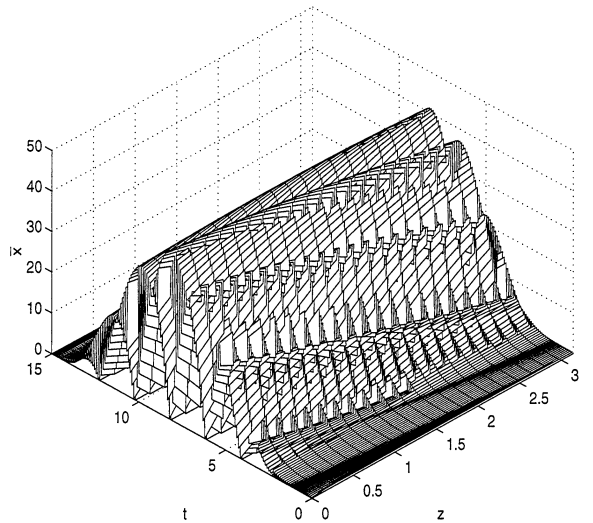


Fig. 6. Closed-loop profile of \bar{x} under nonlinear output feedback control (no uncertainty compensation)-distributed control action.

(i.e., $\beta_T = \beta_{T,n} + \theta(t)$ where $\theta(t) = \beta_{T,n} \sin(0.524 t)$). We note that this selection for $\theta(t)$ satisfies the requirements of Theorem 1 that $\theta(t), \dot{\theta}(t)$ should be sufficiently small (see closed-loop simulations below), while it leads to a very poor open-loop behavior for $\bar{x}(z, t)$ (see Fig. 2). Since the maximum open-loop temperature occurs in the middle of the rod, the controlled output was defined as

$$y_c(t) = \int_0^\pi \sqrt{\frac{2}{\pi}} \sin(z) \bar{x}(z, t) dz \quad (30)$$

and the actuator distribution function was taken to be $b(z) = \sqrt{(2/\pi)} \sin(z)$, in order to apply maximum cooling towards the middle of the rod. One measurement of x at $z = \pi/4$ was assumed to be available.

For the system of Eq. (23), we considered the first eigenvalue as the dominant one ($\varepsilon = 0.25$) and used Galerkin's method to derive a scalar ODE that was used for the synthesis of a nonlinear robust output feedback controller through an application of the formula of Eq. (16). This controller was employed in the

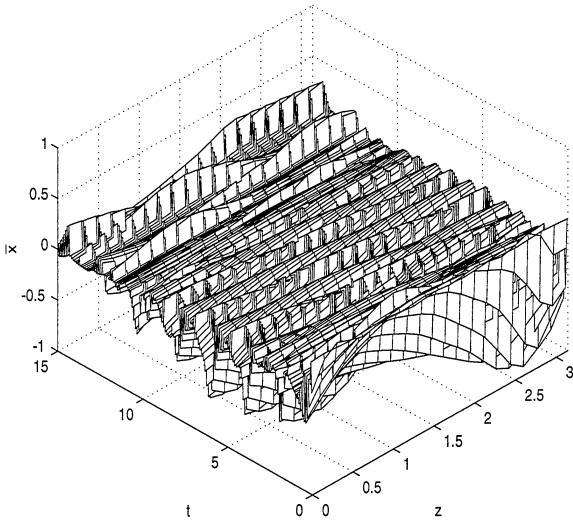


Fig. 7. Closed-loop profile of \bar{x} under nonlinear robust output feedback control-point control action.

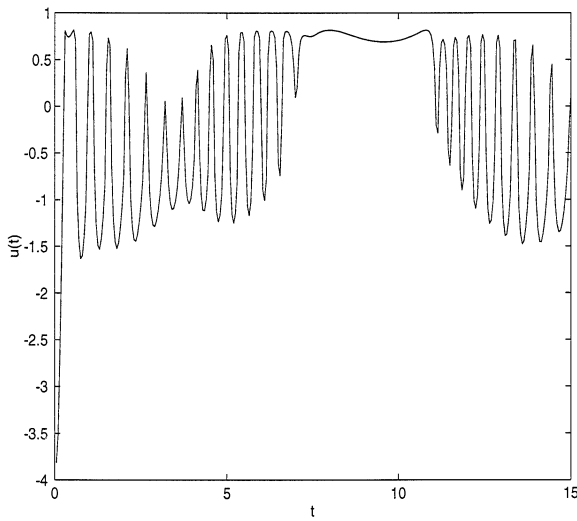


Fig. 8. Manipulated input profile for nonlinear robust output feedback controller-point control action.

simulations with $\chi = 1.2$,

$$c_0(\hat{x}_s, t) = \beta_{T,n} \int_0^\pi \phi_1(z) e^{-\gamma/(1+\hat{x}_s)} dz$$

and $\phi = 0.01$ to achieve an uncertainty attenuation level $d = 0.1$ (note that $0.1 = O(\varepsilon + \phi) = O(0.25 + 0.01)$).

Fig. 3 shows the evolution of the closed-loop rod temperature profile under the nonlinear robust output feedback controller, while Fig. 4 shows the corresponding manipulated input profile. Clearly, the proposed controller regulates the temperature profile at

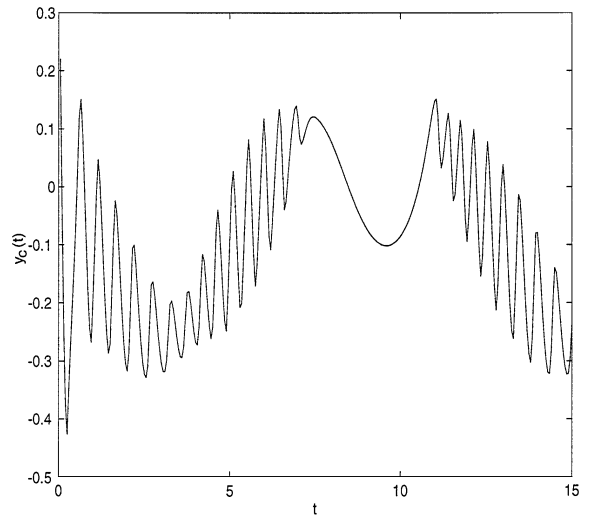


Fig. 9. Closed-loop output profile under nonlinear robust output feedback control-point control action.

$\bar{x}(z, t) = 0$, attenuating the effect of the uncertain variable (note that the requirement $\limsup_{t \rightarrow \infty} |y| \leq 0.1$ is enforced in the closed-loop system; Fig. 5). For the sake of comparison, we also implemented on the process the same controller as before without the term which compensates for the effect of the uncertainty (i.e., $c_0(\hat{x}_s, t) \equiv 0$). Fig. 6 shows the evolution of the closed-loop rod temperature profile. It is clear that this controller cannot regulate the temperature profile at the desired steady state, $\bar{x}(z, t) = 0$, because it does not compensate for the effect of the uncertainty.

Finally, in order to show that the proposed control method can be readily applied to the case of point control actuation, we consider the same control problem as above but with $b(z) = \delta(z - \pi/2)$ (i.e., a point control actuator influencing the rod at $z = \pi/2$ is used to stabilize the system at $\bar{x}(z, t) = 0$ in the presence of the uncertain variable). A nonlinear robust output feedback controller was synthesized on the basis of a scalar ODE model obtained from application of Galerkin's method to the system of Eq. (23) and implemented with $\chi = 1.2$,

$$c_0(\hat{x}_s, t) = \beta_{T,n} \int_0^\pi \phi_1(z) e^{-\gamma/(1+\hat{x}_s)} dz$$

and $\phi = 0.01$ to achieve an uncertainty attenuation level $d = 0.4$ (note that $0.4 = O(0.25 + 0.01)$).

Fig. 7 shows the evolution of the closed-loop rod temperature profile and Fig. 8 shows the corresponding manipulated input profile, for this case. The stabilization of the system at $\bar{x}(z, t) = 0$ with uncertainty attenuation has been

achieved (the requirement $\limsup_{t \rightarrow \infty} |y| \leq 0.4$ is satisfied; Fig. 9). Note that as expected, in the case of point control actuation $u(t)$ influences the states of the closed-loop system which are not used in the controller design model (spill-over effect), and thus, $\bar{x}(z, t)$ exhibits more oscillatory behavior (compare Fig. 7 with Fig. 3).

Remark 8. As a numerical note, we point out that in the case of point actuation (sensing) which influences (measures) the system at z_0 , the function $\delta(z - z_0)$ is assumed to have the finite value $1/2\mu$ in the interval $[z_0 - \mu, z_0 + \mu]$ (where μ is a small positive real number) and be zero elsewhere in $[0, \pi]$.

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