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Feedback control of two-time-scale nonlinear systems

PANAGIOTIS D. CHRISTOFIDES† and PRODROMOS DAOUTIDIS†

This article concerns a class of two-time-scale single-input single-output nonlinear systems. For such systems, combination of singular perturbation and geometric methods is employed to synthesize well-conditioned static state feedback laws that induce a well-characterized input–output behaviour and guarantee stability of the closed-loop system. The developed control laws are applied to two nonlinear chemical processes with time-scale multiplicity and their performance is evaluated through simulations.

Nomenclature

C_{B0}	inlet concentration of the autocatalytic species
C_{B1}	concentration of the autocatalytic species in the first reactor
C_{B2}	concentration of the autocatalytic species in the second reactor
C_C	concentration of the complex in the reactor
C_{E0}	inlet concentration of the enzyme
C_E	concentration of the enzyme in the reactor
C_P	concentration of the product in the reactor
C_{S0}	inlet concentration of the substrate
C_S	concentration of the substrate in the reactor
c_{pm}	heat capacity of the reacting mixture
E_0, E_1, E_2, E_3	activation energies
$F, \tilde{F}, f_1, f_2$	vector fields
F_1, F_2, F_3	flow rates
$G, \tilde{G}, g_1, g_2$	vector fields associated with the input
h	output scalar function
k^T	covector field
$k_0, k_{10}, k_{20}, k_{30}$	pre-exponential constants
Q_1	matrix of dimension $n \times p$ associated with the slow state vector x
Q_2	matrix of dimension $p \times p$ associated with the fast state vector z
$Q, \bar{Q}_1, \bar{Q}_2$	heat inputs to the reactors
$r, \hat{r}$	relative orders in the reduced systems
T_{B0}	inlet temperature of the autocatalytic species
T_1	temperature in the first reactor
T_2	temperature in the second reactor
T_{E0}	inlet temperature of the enzyme
T_r	temperature in the bioreactor

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† Department of Chemical Engineering and Materials Science, University of Minnesota, Minneapolis, MN 55455, U.S.A.

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T_{s0} inlet temperature of the substrate
t time
u input
$\tilde{u}, \hat{u}$ auxiliary inputs
V_r, V_1, V_2 volumes of the liquid hold-up in the reactors
v reference input
x vector of the slow state variables
y output
z vector of the fast state variables

Greek letters

β_k adjustable parameters
$\Delta H_{r_0}, \Delta H_{r_1}, \Delta H_{r_2}, \Delta H_{r_3}$ enthalpies of the reactions
ε singular perturbation parameter
ζ state vector in normal form coordinates
η state vector in normal form coordinates
$\bar{\eta}$ auxiliary variable
$\xi, \tilde{\xi}$ equilibrium manifolds for the fast dynamics in the closed-loop system
ρ_m density of the reacting mixture
φ_i auxiliary functions

Mathematical symbols

$L_f h$ Lie derivative of a scalar field h with respect to the vector field f
$L_f^k h$ kth-order Lie derivative
$L_g L_f^{k-1} h$ mixed Lie derivative
$\mathbb{R}$ real line
$\mathbb{R}^i$ i-dimensional euclidean space
$\in$ belongs to
$^{\mathrm{T}}$ transpose
$|\cdot|$ standard euclidean norm

1. Introduction

Most physicochemical systems are inherently nonlinear and are characterized by dynamical phenomena occurring in multiple time scales. Representative examples of nonlinear multiple-time-scale systems include catalytic continuous stirred-tank reactors (CSTRs) (Chang and Aluco 1984), catalytic cracking reactors (Georgakis 1977, Monge and Georgakis 1987), biochemical reactors (Heineken *et al.* 1967, Merril 1978, Baily and Ollis 1987, Bastin and Dochain 1990), high-purity distillation columns (Levine and Rouchon 1991), electromechanical networks (Chow 1982) and flexible mechanical systems (Da and Corless 1993).

Singular perturbation theory provides a natural framework for the analysis and control of multiple-time-scale systems (Kokotovic *et al.* 1986, Khalil 1992). Singular perturbation methods allow decomposing a multiple-time-scale system into separate reduced-order systems that evolve in different time scales, and inferring its asymptotic properties from knowledge of the behaviour of the reduced-order systems (Kokotovic *et al.* 1976). Within this framework, research in nonlinear two-time-scale systems has mainly focused on the study of stability issues and optimal control problems (see for example Kokotovic *et al.* (1986)), abstract geometric properties (Marino and Kokotovic 1988), and feedback control problems using the integral manifold approach (see for example Sharkey and O'Reilly (1987) and Barbot *et al.* (1992)) and sliding mode techniques (Heck 1991).

On the other hand, differential geometry has proven to be a natural framework for the analysis and control of nonlinear systems. Within this framework, several nonlinear control problems have been successfully addressed (see for example Isidori (1989), Nijmeijer and van der Schaft (1990) and Kravaris and Arkun (1991)). However, most of the developed nonlinear control methods do not take explicitly into account possible time-scale multiplicities and thus may lead to ill-conditioned controllers, significant performance deterioration or even destabilization of the closed-loop system in the presence of such time-scale multiplicity (Kokotovic *et al.* 1986). The combination of singular perturbation and differential geometric methods for the solution of the problem of exact state-space linearization of a class of nonlinear two-time-scale systems was proposed by Khorasani (1990).

This paper addresses the state feedback control problem for a class of two-time-scale nonlinear systems modelled within the mathematical framework of singular perturbations. The objective is to synthesize well-conditioned state feedback laws that induce a well-characterized input–output behaviour in the closed-loop system. The control laws are synthesized employing combination of singular perturbation and geometric methods. Conditions that guarantee the stability of the closed-loop system under the developed control laws are also derived. The developed control methodology is applied to two nonlinear chemical processes with time-scale multiplicity.

2. Two-time-scale systems: preliminaries

We shall consider two-time-scale nonlinear systems with the following state-space description:

$$\left.\begin{aligned} \dot{x} &= f_1(x) + Q_1(x)\,z + g_1(x)\,u \\ \varepsilon\dot{z} &= f_2(x) + Q_2(x)\,z + g_2(x)\,u \\ y &= h(x) \end{aligned}\right\} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$ and $z \in Z \subset \mathbb{R}^p$ denote vectors of state variables, with X and Z open and connected sets, $u \in \mathbb{R}$ denotes the input, $y \in \mathbb{R}$ denotes the controlled output, and ε is a small positive parameter. Furthermore, $f_1(x)$, $g_1(x)$, $f_2(x)$ and $g_2(x)$ are analytic vector fields, $Q_1(x)$ and $Q_2(x)$ are analytic matrices of dimensions $n \times p$ and $p \times p$ respectively, and $h(x)$ is an analytic scalar function.

Models of the form (1) are characterized by an explicit separation of the slow and fast dynamics, owing to the presence of the small parameter ε that multiplies the derivative of the state vector z. This fact allows decomposing such models into separate reduced-order systems evolving in separate time scales (Kokotovic *et al.* 1976). Specifically, assuming that the matrix $Q_2(x)$ is invertible uniformly in $x \in X$, and setting $\varepsilon = 0$, the *reduced system* or *slow subsystem* is obtained:

$$\left.\begin{aligned} \dot{x} &= F(x) + G(x)\,u \\ y^s &= h(x) \end{aligned}\right\} \tag{2}$$

where

$$\left.\begin{aligned} F(x) &= f_1(x) - Q_1(x)\,[Q_2(x)]^{-1} f_2(x) \\ G(x) &= g_1(x) - Q_1(x)\,[Q_2(x)]^{-1} g_2(x) \end{aligned}\right\} \tag{3}$$

and y^s denotes the output associated with the slow subsystem. Defining a fast time scale $\tau = t/\varepsilon$ and setting ε equal to zero, the *fast subsystem* is obtained:

$$\frac{dz}{d\tau} = f_2(x) + Q_2(x)\,z + g_2(x)\,u \tag{4}$$

where x can be considered approximately equal to its initial value $x(0)$. The following assumption states a stabilizability requirement on the system (4).

Assumption 1: *The pair* $[Q_2(x)\,g_2(x)]$ *is stabilizable, in the sense that there exists an analytic covector field* $k^{\mathrm{T}}(x)$ *such that the matrix* $Q_2(x) + g_2(x)\,k^{\mathrm{T}}(x)$ *is Hurwitz uniformly in* $x \in X$.

We shall now review the standard definition of relative order for nonlinear systems described by (2) and the definition of order of magnitude, which will be used in our development.

Definition 1: Referring to the nonlinear system (2), the relative order of y^s with respect to u is defined as the smallest integer r for which

$$L_G L_F^{r-1} h(x) \not\equiv 0 \tag{5}$$

□

or $r = \infty$ if such an integer does not exist.

Definition 2 (Khalil 1992): $\delta(\varepsilon) = O(\varepsilon)$ if there exist positive constants k and c such that

$$|\delta(\varepsilon)| \leqslant k|\varepsilon|, \quad \forall |\varepsilon| < c \tag{6}$$

□

A direct application of state feedback synthesis methods based on nonlinear inversion to systems of the form (1) may result in ill-conditioned control laws, that is control laws that become singular as $\varepsilon \to 0$, and/or closed-loop instability (for a detailed discussion, see Remarks 4 and 5). Motivated by the above considerations, in this paper, we shall address the problem of synthesizing *well-conditioned* static state feedback laws that guarantee exponential stability of the closed-loop system and ensure that the output of the closed-loop system satisfies a relation of the form

$$y(t) = y^s(t) + O(\varepsilon), \quad t \geqslant 0 \tag{7}$$

with $y^s(t)$ being the output of the closed-loop reduced system, where a well-characterized input–output response is enforced. These requirements will be obtained for sufficiently small ε.

The control problem will be initially addressed for systems in standard form, that is, systems for which the matrix $Q_2(x)$ is invertible uniformly in $x \in X$, and then for systems in non-standard form, that is, systems for which the matrix $Q_2(x)$ is singular for some $x \in X$. We shall establish that, for systems in standard form, the synthesis of the requisite state feedback law can be performed on the basis of the open-loop system (1) while, for systems in non-standard form, it has to be preceded by the feedback regularization of the fast dynamics.

3. Control of two-time-scale nonlinear systems in standard form

In this section, we shall address a state feedback synthesis problem for systems of the form (1) in standard form, with possibly unstable fast dynamics. Motivated by the possible instability of the fast dynamics and the affine appearance of the fast state z in the model of (1), we shall initially consider well-conditioned static state feedback laws of the form

$$u = \tilde{u} + k^{\mathrm{T}}(x)\, z \tag{8}$$

where $\tilde{u}$ is an auxiliary input and $k^{\mathrm{T}}(x)$ is an analytical vector field in $\mathbb{R}^p$ to stabilize the fast dynamics of the closed-loop system. Substitution of the control law (8) into the system (1) yields

$$\left.\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)\, k^{\mathrm{T}}(x)]\, z + g_1(x)\, \tilde{u} \\ \varepsilon\dot{z} &= f_2(x) + [Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]\, z + g_2(x)\, \tilde{u} \end{aligned}\right\} \tag{9}$$

One can immediately observe that the control law (8) preserves the two-time-scale nature of the system, and the linearity with respect to the state z and the auxiliary input $\tilde{u}$. Furthermore, performing a standard two-time-scale decomposition, the fast subsystem is given by

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = f_2(x) + [Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]\, z + g_2(x)\, \tilde{u} \tag{10}$$

while the slow subsystem takes the form

$$\left.\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\, \tilde{u} \\ y^s &= h(x) \end{aligned}\right\} \tag{11}$$

where

$$\left.\begin{aligned} \tilde{F}(x) &= f_1(x) - [Q_1(x) + g_1(x)\, k^{\mathrm{T}}(x)]\, [Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]^{-1} f_2(x) \\ \tilde{G}(x) &= g_1(x) - [Q_1(x) + g_1(x)\, k^{\mathrm{T}}(x)]\, [Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]^{-1} g_2(x) \end{aligned}\right\} \tag{12}$$

It is clear that the z-dependent state feedback law (8) allows modifying the stability characteristics of the fast dynamics of the original system by choosing the gain $k^{\mathrm{T}}(x)$ such that the matrix $Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)$ is Hurwitz uniformly in $x \in X$. Moreover, one can immediately observe that the reduced system (11) depends explicitly on the nonlinear feedback gain $k^{\mathrm{T}}(x)$. Therefore, before we proceed with the solution of the control problem in the slow time scale, we shall establish that the state feedback law (8) preserves the relative order r of the output y^s. The main result is given in Proposition 1 that follows. The proof of the proposition can be found in the Appendix.

Proposition 1: *Consider the two-time-scale system* (1), *assumed to be in standard form, for which Assumption* 1 *holds. Then, under a static state feedback law of the form* (8) *such that the matrix $Q_2(x)+g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$, the relative order of y^s with respect to $\tilde{u}$ in the reduced system* (11) *is equal to r.*

We shall now seek well-conditioned static state feedback laws of the form

$$\tilde{u} = p(x)+q(x)v \tag{13}$$

where $p(x)$ and $q(x)$ are scalar fields with $q(x) \neq 0$ for all $x \in X$, and v is a reference input, that enforce output tracking and guarantee stability of the closed-loop reduced system. Under a control law of the form (13) the closed-loop system is given by

$$\left.\begin{aligned} \dot{x} &= [f_1(x)+g_1(x)p(x)]+[Q_1(x)+g_1(x)k^T(x)]z+g_1(x)q(x)v \\ \varepsilon\dot{z} &= [f_2(x)+g_2(x)p(x)]+[Q_2(x)+g_2(x)k^T(x)]z+g_2(x)q(x)v \\ y &= h(x) \end{aligned}\right\} \tag{14}$$

Employing a two-time-scale decomposition for the system (14), the closed-loop fast subsystem is given by

$$\frac{dz}{d\tau} = [f_2(x)+g_2(x)p(x)]+[Q_2(x)+g_2(x)k^T(x)]z+g_2(x)q(x)v \tag{15}$$

and the closed-loop slow subsystem takes the form

$$\left.\begin{aligned} \dot{x} &= [\tilde{F}(x)+\tilde{G}(x)p(x)]+\tilde{G}(x)q(x)v \\ y^s &= h(x) \end{aligned}\right\} \tag{16}$$

Proposition 2 that follows allows specifying the order of the requested input–output response in the closed-loop reduced system.

Proposition 2: *Consider the two-time-scale system* (9), *where the matrix $Q_2(x)+g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$. Then, under a static state feedback control law of the form* (13) *the relative order of y^s with respect to v in the closed-loop reduced system* (16) *is equal to r.*

Proof. The proof of the proposition involves the application of the two-time-scale decomposition procedure to the resulting closed-loop system, and the standard argument for nonlinear systems of the form (2) under static state feedback of the form (13) (see for example Isidori (1989)). □

The result of Proposition 2 allows requesting an input–output behaviour of relative order r in the closed-loop reduced system. For simplicity, a linear minimal-order input–output behaviour will be postulated, of the form

$$\beta_r \frac{d^r y^s}{dt^r} + \ldots + \beta_1 \frac{dy^s}{dt} + \beta_0 y^s = v \tag{17}$$

where $\beta_0, \ldots, \beta_r$ are adjustable parameters, which can be chosen to guarantee input–output stability and to enforce desired performance specifications in the closed-loop reduced system.

We are now in a position to give an explicit synthesis formula for the state feedback controller (8)–(13) that enforces the requested control objectives in the closed-loop system. Theorem 1 that follows provides the main result of this section (the proof of the theorem can be found in the Appendix).

Theorem 1: *Consider the two-time-scale nonlinear system* (1), *assumed to be in standard form, for which Assumption* 1 *holds. Then the static state feedback law*

$$u = [1 + k^{\mathrm{T}}(x)\,[Q_2(x)]^{-1} g_2(x)]\,[\beta_r L_G L_F^{r-1} h(x)]^{-1}\left(v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right) + k^{\mathrm{T}}(x)\,[Q_2(x)]^{-1} f_2(x) + k^{\mathrm{T}}(x)\, z \tag{18}$$

where the feedback gain $k^{\mathrm{T}}(x)$ is such that the matrix $Q_2(x) + g_2(x)\,k^{\mathrm{T}}(x)$ is Hurwitz uniformly in $x \in X$,

(*a*) *guarantees local exponential stability of the fast dynamics of the closed-loop system and*

(*b*) *ensures that the output of the closed-loop system satisfies a relation of the form*

$$y(t) = y^{s}(t) + O(\varepsilon), \quad t \geqslant 0 \tag{19}$$

for ε sufficiently small, with $y^s(t)$ being the solution of (17).

Remark 1: The state feedback law (18) can be equivalently written as

$$u = [\beta_r L_G L_F^{r-1} h(x)]^{-1}\left(v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right) + k^{\mathrm{T}}(x)\,(z - \xi) \tag{20}$$

where

$$\xi = -[Q_2(x)]^{-1}\left[f_2(x) + g_2(x)\,[\beta_r L_G L_F^{r-1} h(x)]^{-1}\left(v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right)\right] \tag{21}$$

denotes a quasi-steady-state for the fast dynamics of the closed-loop system. It is clear that the above control law consists of two separate components: the component $k^{\mathrm{T}}(x)\,(z - \xi)$, which acts in the fast time scale and is responsible for the stabilization of the fast dynamics of the system, and the component

$$[\beta_r L_G L_F^{r-1} h(x)]^{-1}\left(v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right)$$

which acts in the slow time scale and induces the desired input–output behaviour in the closed-loop reduced system. Control laws that address control objectives in different time scales are called composite control laws (Kokotovic *et al.* 1986). The above analysis of the structure of the controller of Theorem 1 suggests that the two components of the controller can be synthesized independently, on the basis of the open-loop system (1). □

Remark 2: The fact that the component $k^{\mathrm{T}}(x)\,(z - \xi)$ acts in the fast time scale allows designing the feedback gain $k^{\mathrm{T}}(x)$ using standard control methods, such as pole placement and optimal control (for details see for example Kokotovic *et al.* 1986).

Remark 3: In the case of systems of the form (1) with stable fast dynamics, that is assuming that the eigenvalues of the matrix $Q_2(x)$ lie in the open left half of the complex plane uniformly in $x \in X$, the controller of Theorem 1 does not need to utilize feedback of the fast state vector z to stabilize the fast dynamics and thus simplifies to

$$u = [\beta_r L_G L_F^{r-1} h(x)]^{-1} \left(v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \right) \tag{22}$$

□

Remark 4: Referring to systems of the form (1) with stable fast dynamics, let ν and π denote the smallest integers for which $\partial y^{(\nu)}/\partial z \not\equiv 0$, $\partial y^{(\pi)}/\partial z \not\equiv 0$. Then, one can show following Christofides and Daoutidis (1994) that, if $\nu < \pi$, then $\nu = r$ and the control law (22) induces an input–output behaviour of the form (17) in the closed-loop *full-order* system. Furthermore, it can be established that, whenever $\nu = \pi$, a state feedback law that induces the input–output behaviour of (17) in the closed-loop full-order system requires feedback of the fast state vector z and thus may destabilize the fast dynamics of the system while, if $\nu > \pi$, such a control law will also be ill conditioned. □

Remark 5: In this remark, we shall address the implications of the instability of the fast dynamics on the stability of the zero dynamics of systems of the form (1), for which $\nu \leqslant \pi$. As noted in Remark 4, whenever $\nu > \pi$, an inversion-based state feedback law is ill-conditioned, and thus such an analysis is not particularly meaningful for this case. Referring to the system (1) with $\nu \leqslant \pi$, there exists (see for example Isidori 1989) a coordinate transformation

$$(\zeta, z) = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-\nu} \\ \zeta_1 \\ \vdots \\ \zeta_\nu \\ z \end{bmatrix} = T(x, z) = \begin{bmatrix} \varphi_1(x) \\ \vdots \\ \varphi_{n-\nu}(x) \\ h(x) \\ L_{f_1} h(x) \\ \vdots \\ L_{f_1}^{\nu-1} h(x) \\ z \end{bmatrix} \tag{23}$$

where $\varphi_i(x)$, $i = 1, \ldots, n-\nu$ are scalar fields, such that the system (1) takes the form

$$\left.\begin{aligned} \dot{\eta} &= L_{f_1} \varphi(\eta, \zeta) + \sum_{\kappa=1}^{p} L_{Q_{1\kappa}} \varphi(\eta, \zeta) z_k \\ \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{\nu-1} &= \zeta_v \\ \dot{\zeta}_\nu &= L_{f_1}^{\nu} h(\eta, \zeta) + L_{Q_1} L_{f_1}^{\nu-1} h(\eta, \zeta) z + L_{g_1} L_{f_1}^{\nu-1} h(\eta, \zeta) u \\ \varepsilon \dot{z} &= f_2(\eta, \zeta) + Q_2(\eta, \zeta) z + g_2(\eta, \zeta) u \\ y &= \zeta_1 \end{aligned}\right\} \tag{24}$$

where $\varphi(x) = [\varphi_1(x) \cdots \varphi_{n-\nu}(x)]^{\mathrm{T}}$, the (ζ, η) dependence in the right-hand side of (24) implies the evaluation of the functions at $(x,z)=T^{-1}(\eta,\zeta,z)$, and $L_{f_1}=[L_{f_1}\varphi_1\cdots L_{f_1}\varphi_{n-\nu}]^{\mathrm{T}}$, $L_{Q_{1\kappa}}\varphi=[L_{Q_{1\kappa}}\varphi_1\cdots L_{Q_{1\kappa}}\varphi_{n-\nu}]^{\mathrm{T}}$, $L_{Q_1}L_{f_1}^{\nu-1}h=[L_{Q_{11}}L_{f_1}^{\nu-1}h\cdots L_{Q_{1p}}L_{f_1}^{\nu-1}h]$, with Q_{1k} being the kth column vector of the matrix Q_1. Whenever $\nu<\pi$, the dynamical system

$$\left.\begin{aligned}\dot{\eta}&=L_{f_1}\varphi(\eta,0)+\sum_{\kappa=1}^{p}L_{Q_{1\kappa}}\varphi(\eta,0)\,z_k\\ \varepsilon\dot{z}&=f_2(\eta,0)+Q_2(\eta,0)\,z-g_2(\eta,0)\frac{L_{f_1}^{\nu}h(\eta,0)}{L_{g_1}L_{f_1}^{\nu-1}h(\eta,0)}\end{aligned}\right\}\tag{25}$$

is the zero dynamics of the system (1) (see for example Isidori (1989)). One can immediately observe that the modes of the zero dynamics contain the modes of the fast dynamics, and thus instability of the fast dynamics implies instability of the zero dynamics of the original system. Systems that possess a zero dynamics with a two-time-scale nature and unstable fast dynamics are called slightly non-minimum phase (see for example Sastry *et al.* (1989)). For such systems, one can immediately see that inversion-based state feedback laws lead to internal instability of the closed-loop system. On the other hand, under the state feedback law (18), the fast dynamics of the closed-loop system possess a locally exponentially stable equilibrium manifold, while the internal stability of the closed-loop system depends on the stability of the zero dynamics of the *open-loop reduced system* (see §5). Furthermore, whenever $\nu=\pi$, the zero dynamics of the system (1) in the coordinates of (23) take the form

$$\left.\begin{aligned}\dot{\eta}&=L_{f_1}\varphi(\eta,0)+\sum_{\kappa=1}^{p}L_{Q_{1\kappa}}\varphi(\eta,0)\,z_k\\ \varepsilon\dot{z}&=f_2(\eta,0)+\left(Q_2(\eta,0)-\frac{L_{Q_1}L_{f_1}^{\nu-1}h(\eta,0)}{L_{g_1}L_{f_1}^{\nu-1}h(\eta,0)}\right)z-g_2(\eta,0)\frac{L_{f_1}^{\nu}h(\eta,0)}{L_{g_1}L_{f_1}^{\nu-1}h(\eta,0)}\end{aligned}\right\}\tag{26}$$

and thus the system (1) may or may not be slightly non-minimum phase, depending on its specific structure. □

4. Control of two-time-scale nonlinear systems in non-standard form

In this section, we shall address a state feedback controller synthesis problem for two-time-scale nonlinear systems (1) in non-standard form, that is systems for which the matrix $Q_2(x)$ is singular. The immediate implication of the non-invertibility of the matrix $Q_2(x)$ is the lack of a well-defined quasi-steady-state for the fast state vector z (Khalil 1989). Therefore an application of the two-time-scale decomposition procedure to the system (1) will result in the derivation of a singular reduced system. Referring to this reduced system, it can be easily seen that the concept of relative order is not well defined, and the results of Propositions 1 and 2 that allowed the formulation and solution of the controller synthesis problems for two-time-scale systems in standard form do not hold.

The previous considerations imply that any attempt for the synthesis of state feedback laws for this class of systems requires the regularization of the fast dynamics (in the sense of inducing a well-defined quasi steady-state for the fast state vector z) through appropriate feedback of the fast state vector z. Motivated by this, in what follows, we shall initially employ appropriate feedback of the state vector z to induce an exponentially stable quasi-steady-state for the fast dynamics, and we shall subsequently formulate and solve the synthesis problem on the basis of the resulting two-time-scale system.

In particular, we shall initially consider a well-conditioned static state feedback law of the form

$$u = \hat{u} + k^{\mathrm{T}}(x)\, z \tag{27}$$

where $k^{\mathrm{T}}(x)$ is an analytic vector field in $\mathbb{R}^p$ and $\hat{u}$ is an auxiliary input, to regularize the fast dynamics. Under the control law (27) the system (1) takes the form

$$\left.\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)\, k^{\mathrm{T}}(x)]\, z + g_1(x)\, \hat{u} \\ \varepsilon \dot{z} &= f_2(x) + [Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]\, z + g_2(x)\, \hat{u} \end{aligned}\right\} \tag{28}$$

Performing a two-time-scale decomposition, the corresponding fast subsystem takes the form

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = f_2(x) + [Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]\, z + g_2(x)\, \hat{u} \tag{29}$$

while the corresponding slow subsystem is given by

$$\left.\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\, \hat{u} \\ y^{\mathrm{s}} &= h(x) \end{aligned}\right\} \tag{30}$$

where the vector fields $\tilde{F}(x)$ and $\tilde{G}(x)$ are given in (17). It is clear that the z-dependent state feedback law (27) allows us to regularize the fast dynamics of the system by choosing $k^{\mathrm{T}}(x)$ in such a manner that the matrix $Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)$ is Hurwitz uniformly in $x \in X$. A necessary and sufficient condition for the regularization of the fast dynamics is rank $[Q_2(x)\, g_2(x)] = m$, uniformly in $x \in X$ (Khalil 1989). Note that this condition is not an additional requirement, since it is directly implied by the stabilizability property of the pair $[Q_2(x)\, g_2(x)]$.

It is now possible to formulate and solve a controller synthesis problem on the basis of the system (28). In particular, we shall seek a well-conditioned static state feedback law of the form

$$\hat{u} = \tilde{p}(x) + \tilde{q}(x)\, v \tag{31}$$

where $\tilde{p}(x)$ and $\tilde{q}(x)$ are scalar fields, with $\tilde{q}(x) \neq 0$ for all $x \in X$, and v is the external reference input. Referring to the system (30), let $\tilde{r}$ denote the relative order of the output y^{s} with respect to the auxiliary input $\hat{u}$. It is then straightforward to show that the result of Proposition 2 holds for the two-time-scale system (28) under the state feedback law (31). This allows postulating the following input–output behaviour:

$$\beta_{\tilde{r}} \frac{\mathrm{d}^{\tilde{r}} \hat{y}^{\mathrm{s}}}{\mathrm{d}t^{\tilde{r}}} + \cdots + \beta_1 \frac{\mathrm{d}\hat{y}^{\mathrm{s}}}{\mathrm{d}t} + \beta_0\, \hat{y}^{\mathrm{s}} = v \tag{32}$$

in the closed-loop reduced system. Theorem 2 that follows summarizes the main result of this section. The proof of the theorem can be found in the Appendix.

Theorem 2: *Consider the two-time-scale nonlinear system* (1), *assumed to be in non-standard form, for which Assumption* 1 *holds. Then the static state feedback law*

$$u = [\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1} \left(v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) \right) + k^{\mathrm{T}}(x)\, z \tag{33}$$

where the feedback gain $k^{\mathrm{T}}(x)$ *is such that the matrix* $Q_2(x)+g_2(x)\,k^{\mathrm{T}}(x)$ *is Hurwitz uniformly in* $x \in X$,

(*a*) *guarantees local exponential stability of the fast dynamics of the closed-loop system and*

(*b*) *ensures that the output of the closed-loop system satisfies a relation of the form*

$$y(t) = \hat{y}^{\mathrm{s}}(t) + \mathrm{O}(\varepsilon), \quad t \geqslant 0 \tag{34}$$

for ε *sufficiently small, with* $\hat{y}^{\mathrm{s}}(t)$ *being the solution of* (32).

Remark 6: The result of Theorem 2 reveals a fundamental difference in the nature of the control problem between systems in standard and non-standard forms. In particular, in the case of systems in non-standard form, the formulation (relative order of the requested response) and solution of the control problem in the slow time scale are based on the reduced system (30) and thus explicitly depend on the gain $k^{\mathrm{T}}(x)$ used to regularize the fast dynamics, as opposed to systems in standard form where the control problem in the slow time scale is independent of the choice of $k^{\mathrm{T}}(x)$.

5. Conditions for closed-loop stability

In this section, we shall address the stability of the unforced ($v = 0$) closed-loop full-order system under the state feedback laws of Theorems 1 and 2. Initially, we shall consider two-time-scale systems of the form (1) in standard form and assume that

(1) the eigenvalues of the matrix $Q_2(x)+g_2(x)\,k^{\mathrm{T}}(x)$ lie in the open left-half of the complex plane uniformly in $x \in X$,

(2) the roots of the polynomial $\beta_0+\beta_1 s+\ldots+\beta_r s^r = 0$ lie in the open left half of the complex plane and

(3) the open-loop reduced system (2) is minimum phase, that is, its zero dynamics are locally exponentially stable.

Condition (1) guarantees that fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold. Moreover, it is straightforward to show that conditions (2) and (3) guarantee the local exponential stability of the unforced closed-loop reduced system, under the controller of theorem 1. The local exponential stability of the closed-loop reduced-order systems implies that the unforced closed-loop full-order system is locally exponentially stable, for a sufficiently small ε (see for example Isidori (1989) and Khalil (1992)).

In the case of systems in non-standard form, one can use similar arguments to show that the local exponential stability of the unforced closed-loop full-order system, for

Parameter	Value	Units
C_{S0}	12·92	kmol m^{-3}
C_{E0}	11·91	kmol m^{-3}
T_{E0}	310·0	K
T_{S0}	329·24	K
F_1	2·1	m^3 min^{-1}
F_2	0·1	m^3 min^{-1}
F_3	2·2	m^3 min^{-1}
V_r	1·0	m^3
R	1·987	kcal kmol^{-1} K^{-1}
c_{pm}	0·231	kcal kg^{-1} K^{-1}
ρ_m	900·0	kg m^{-3}
k_{10}	$3{\cdot}36\times10^6$	m^3 kmol^{-1} min^{-1}
k_{20}	$1{\cdot}80\times10^6$	min^{-1}
k_{30}	$5{\cdot}79\times10^7$	min^{-1}
E_1	$8{\cdot}0\times10^3$	kcal kmol^{-1}
E_2	$9{\cdot}0\times10^3$	kcal kmol^{-1}
E_3	$10{\cdot}0\times10^3$	kcal kmol^{-1}
ΔH_1	$1{\cdot}0\times10^3$	kcal kmol^{-1}
ΔH_2	$1{\cdot}0\times10^3$	kcal kmol^{-1}
ΔH_3	$5{\cdot}0\times10^3$	kcal kmol^{-1}
C_{C_s}	0·491	kmol m^{-3}
C_{E_s}	0·05	kmol m^{-3}
C_{S_s}	11·2	kmol m^{-3}
C_{P_s}	0·67	kmol m^{-3}
T_r	300·0	K
Q_s	$-3{\cdot}8\times10^3$	kcal min^{-1}

Table 1. Process parameters.

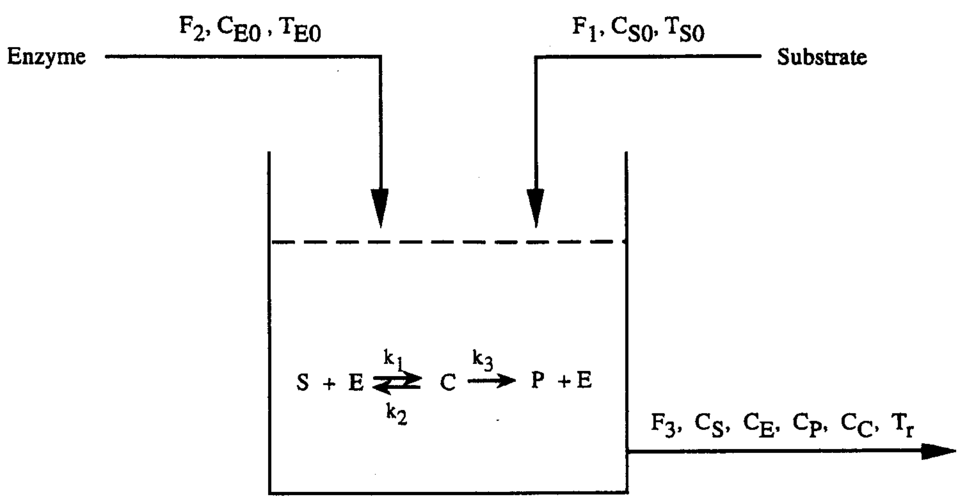

Figure 1. A biochemical CSTR.

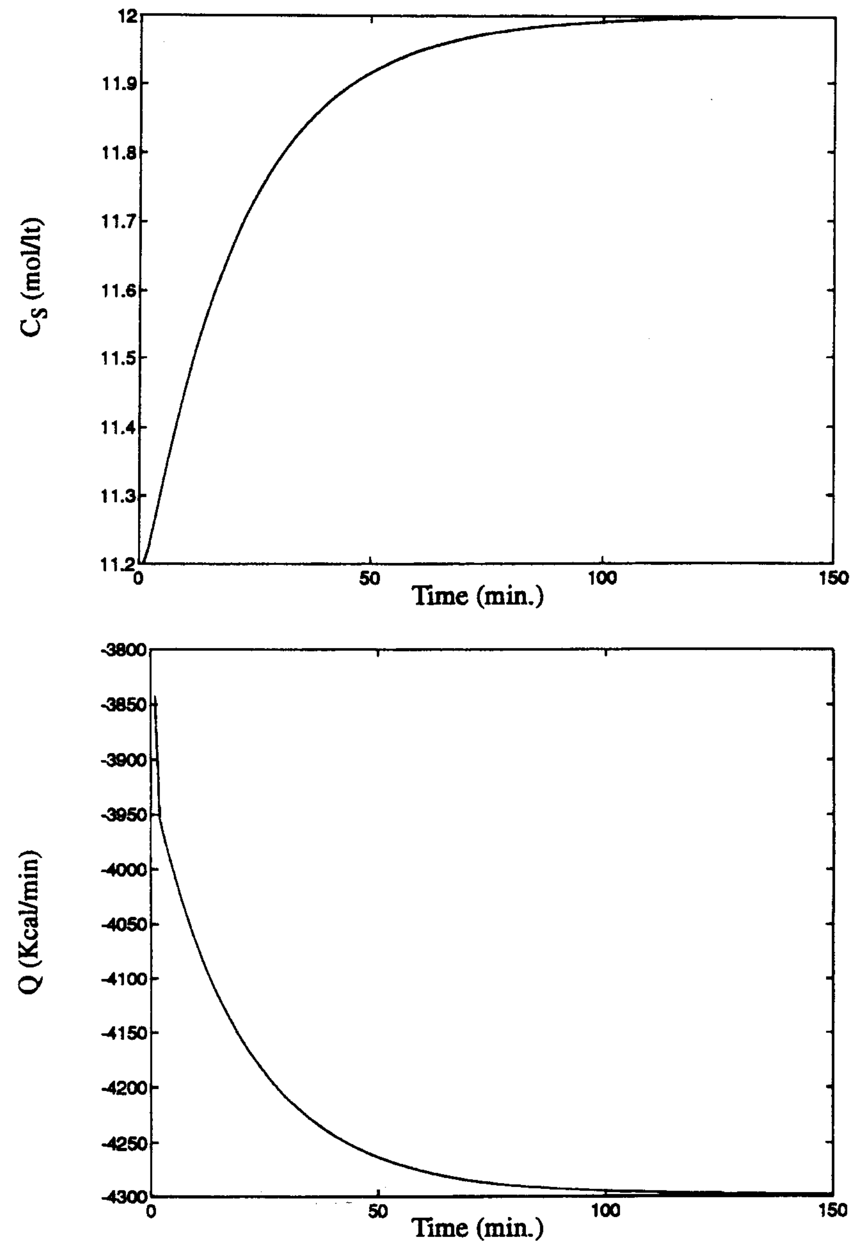

Figure 2. Output and input profiles for a positive change in the reference input.

ε sufficiently small, is guaranteed if condition (1) holds, the roots of the polynomial $\beta_0+\beta_1 s+\cdots+\beta_{\tilde{r}} s^{\tilde{r}}=0$ lie in the open left half of the complex plane, and the zero dynamics of the reduced system (30) are locally exponentially stable.

6. Simulation studies

6.1. *Application to a biochemical reactor*

Consider the biochemical CSTR shown in Fig. 1, where the following enzymatic reaction (Heineken *et al.* 1967) takes place:

$$\mathrm{S}+\mathrm{E} \rightleftharpoons \mathrm{C} \rightarrow \mathrm{P}+\mathrm{E} \tag{35}$$

where S, E, C and P denote the substrate, the enzyme, the complex and the product respectively. The inlet stream of flux rate F_1 consists of substrate of concentration C_{S0} and temperature T_{S0}, while the inlet stream of flux rate F_2 consists of enzyme of concentration C_{E0} and temperature T_{E0}. Under the assumptions

(*a*) perfect mixing in the reactor,

(*b*) uniform temperature in the reactor,

(*c*) constant volume of the liquid in the reactor, and

(*d*) constant density and heat capacity of the reacting liquid

the material and energy balances that describe the dynamical behaviour of the bioreactor take the following form:

$$
\left.\begin{aligned}
V_r\frac{dC_P}{dt} &= -F_3 C_P + k_{30}\exp\left(\frac{-E_3}{RT_r}\right)C_C V_r\\
V_r\frac{dC_S}{dt} &= F_1 C_{S0} - F_3 C_S - k_{10}\exp\left(\frac{-E_1}{RT_r}\right)C_S C_E V_r + k_{20}\exp\left(\frac{-E_2}{RT_r}\right)C_C V_r\\
V_r\frac{dC_E}{dt} &= F_2 C_{E0} - F_3 C_E - k_{10}\exp\left(\frac{-E_1}{RT_r}\right)C_S C_E V_r\\
&\quad + k_{20}\exp\left(\frac{-E_2}{RT_r}\right)C_C V_r + k_{30}\exp\left(\frac{-E_3}{RT_r}\right)C_C V_r\\
V_r\frac{dC_C}{dt} &= -F_3 C_C + k_{10}\exp\left(\frac{-E_1}{RT_r}\right)C_S C_E V_r - k_{20}\exp\left(\frac{-E_2}{RT_r}\right)C_C V_r\\
&\quad - k_{30}\exp\left(\frac{-E_3}{RT_r}\right)C_C V_r\\
V_r\frac{dT_r}{dt} &= F_1 T_{S0} + F_2 T_{E0} - F_3 T_r + \frac{Q}{\rho_m c_{pm}}\\
&\quad + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10}\exp\left(\frac{-E_1}{RT_r}\right)C_S C_E V_r\\
&\quad + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20}\exp\left(\frac{-E_2}{RT_r}\right)C_C V_r\\
&\quad + \frac{(-\Delta H_{r_3})}{\rho_m c_{pm}} k_{30}\exp\left(\frac{-E_3}{RT_r}\right)C_C V_r
\end{aligned}\right\} \qquad (36)
$$

where C_S, C_E, C_C and C_P denote the concentrations of the species S, E, C and P, and C_{S0} and C_{E0} denote the inlet concentrations of the species E and S, T_r and V_r denote the temperature and the volume of the reactor, F_3 denotes the outlet flow rate, and k_{10}, k_{20}, k_{30}, E_1, E_2, E_3, ΔH_1, ΔH_2 and ΔH_3 denote the pre-exponential constants, the activation energies and the enthalpies of the three reactions respectively. The control objective is the regulation of the concentration C_S of the substrate by manipulating the heat input

Q to the reactor. In this type of bioreaction, the inlet concentration of the enzyme is usually much smaller than that of the substrate (Heineken *et al.* 1967). Defining the parameter ε as

$$\varepsilon = \frac{C_{E0}}{C_{S0}} \tag{37}$$

and setting

$$u = Q - Q_s, \quad x_1 = \frac{C_P}{C_{S0}}, \quad x_2 = \frac{C_S}{C_{S0}}, \quad x_3 = \frac{C_C}{C_{S0}}, \quad x_4 = T_r, \quad z = \frac{C_E}{C_{E0}}, \quad y = x_2 \tag{38}$$

the original set of equations can be put in the form (1) with

$$f_1(x) = \begin{bmatrix} \dfrac{-F_3}{V_r} x_1 + k_{30} \exp\left(\dfrac{-E_3}{Rx_4}\right) x_3 \\[2ex] \dfrac{F_1}{V_r} - \dfrac{F_3}{V_r} x_2 + k_{20} \exp\left(\dfrac{-E_2}{Rx_4}\right) x_3 \\[2ex] -\dfrac{F_3}{V_r} x_3 - k_{20} \exp\left(\dfrac{-E_2}{Rx_4}\right) x_3 - k_{30} \exp\left(\dfrac{-E_3}{Rx_4}\right) x_3 \\[2ex] \dfrac{F_1}{V_r} T_{S0} + \dfrac{F_2}{V_r} T_{E0} - \dfrac{F_3}{V_r} x_4 + \dfrac{Q_s}{\rho_m c_{pm} V_r} + \dfrac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} \exp\left(\dfrac{-E_2}{Rx_4}\right) x_3 C_{S0} \\ + \dfrac{(-\Delta H_{r_3})}{\rho_m c_{pm}} k_{30} \exp\left(\dfrac{-E_3}{Rx_4}\right) x_3 C_{S0} \end{bmatrix}$$

$$Q_1(x) = \begin{bmatrix} 0 \\ -k_{10} \exp\left(\dfrac{-E_1}{Rx_4}\right) x_2 C_{E0} \\[2ex] k_{10} \exp\left(\dfrac{-E_1}{Rx_4}\right) x_2 C_{E0} \\[2ex] \dfrac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} \exp\left(\dfrac{-E_1}{Rx_4}\right) x_2 C_{E0} C_{S0} \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{\rho_m c_{pm} V_r} \end{bmatrix}$$

$$f_2(x) = \left[\frac{F_2}{V_r}\varepsilon + k_{20} \exp\left(\frac{-E_2}{Rx_4}\right) x_3 + k_{30} \exp\left(\frac{-E_3}{Rx_4}\right) x_3\right]$$

$$Q_2(x) = \left[\frac{-F_3}{V_r}\varepsilon - k_{10} \exp\left(\frac{-E_1}{Rx_4}\right) C_{E0} x_2\right]$$

$$g_2(x) = [0], \quad h(x) = [x_2]$$

It can be easily seen that $Q_2(x)$ is non-singular and thus the two-time-scale system is in standard form. Moreover, the matrix $Q_2(x)$ is Hurwitz since x_2 is always positive.

The values of the system parameters and the corresponding steady-state values of the system variables are given in Table 1. It was verified that these conditions correspond to a stable equilibrium point, while the zero dynamics of the reduced system are exponentially stable. Since the fast dynamics of the system are stable, the

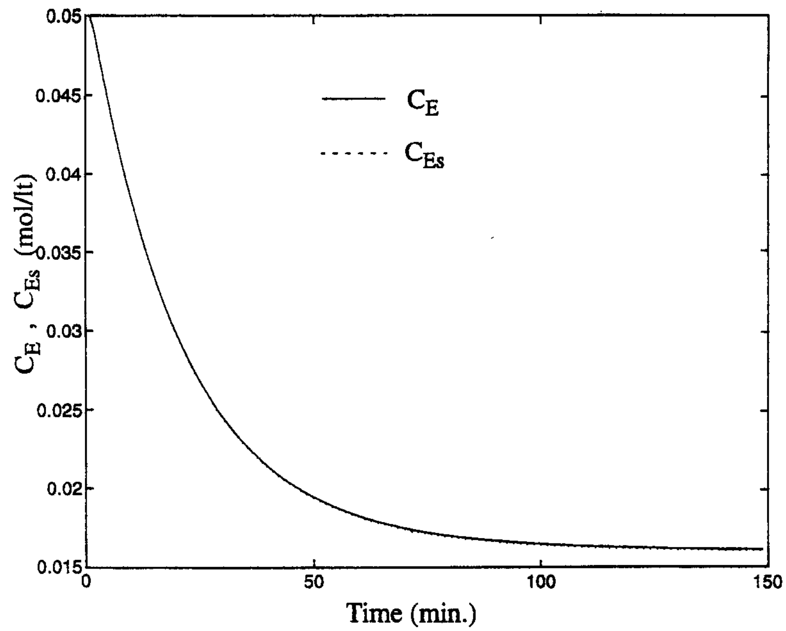

Figure 3. Profile of the concentration of the enzyme under the quasi steady-state assumption (dotted line). Comparison with the actual profile (solid line).

controller of (22) was employed in the simulations. Setting $\varepsilon = 0$, the open-loop reduced system (2) can be easily obtained with

$$F(x) = \begin{bmatrix} \dfrac{-F_3}{V_r} x_1 + k_{30} \exp\left(\dfrac{-E_3}{Rx_4}\right) x_3 \\[2ex] \dfrac{F_1}{V_r} C_{S0} - \dfrac{F_3}{V_r} x_2 - k_{30} \exp\left(\dfrac{-E_3}{Rx_4}\right) x_3 \\[2ex] -\dfrac{F_3}{V_r} x_3 \\[2ex] \dfrac{F_1}{V_r} T_{S0} + \dfrac{F_2}{V_r} T_{E0} - \dfrac{F_3}{V_r} x_4 + \dfrac{Q_s}{\rho_m c_{pm} V_r} + \dfrac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} \exp\left(\dfrac{-E_2}{Rx_4}\right) C_{S0}\, x_3 + \dfrac{(-\Delta H_{r_3})}{\rho_m c_{pm}} \\ \times k_{30} \exp\left(\dfrac{-E_3}{Rx_4}\right) C_{S0}\, x_3 + \dfrac{(-\Delta H_{r_1})\, C_{S0}}{\rho_m c_{pm}} \left[k_{20} \exp\left(\dfrac{-E_2}{Rx_4}\right) x_3 + k_{30} \exp\left(\dfrac{-E_3}{Rx_4}\right) x_3 \right] \end{bmatrix}$$

$$G(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{\rho_m c_{pm} V_r} \end{bmatrix}$$

It can be easily verified that the relative order is $r = 2$. The controller was tuned to give an overdamped response for changes in the reference input with time constant and damping factor

$$\tau = 3{\cdot}88 \text{ min}, \quad \zeta = 1{\cdot}93$$

through the following choice of the controller parameters:

$$\beta_0 = 1{\cdot}0, \quad \beta_1 = 15{\cdot}0 \text{ min}, \quad \beta_2 = 15{\cdot}0 \text{ min}^2$$

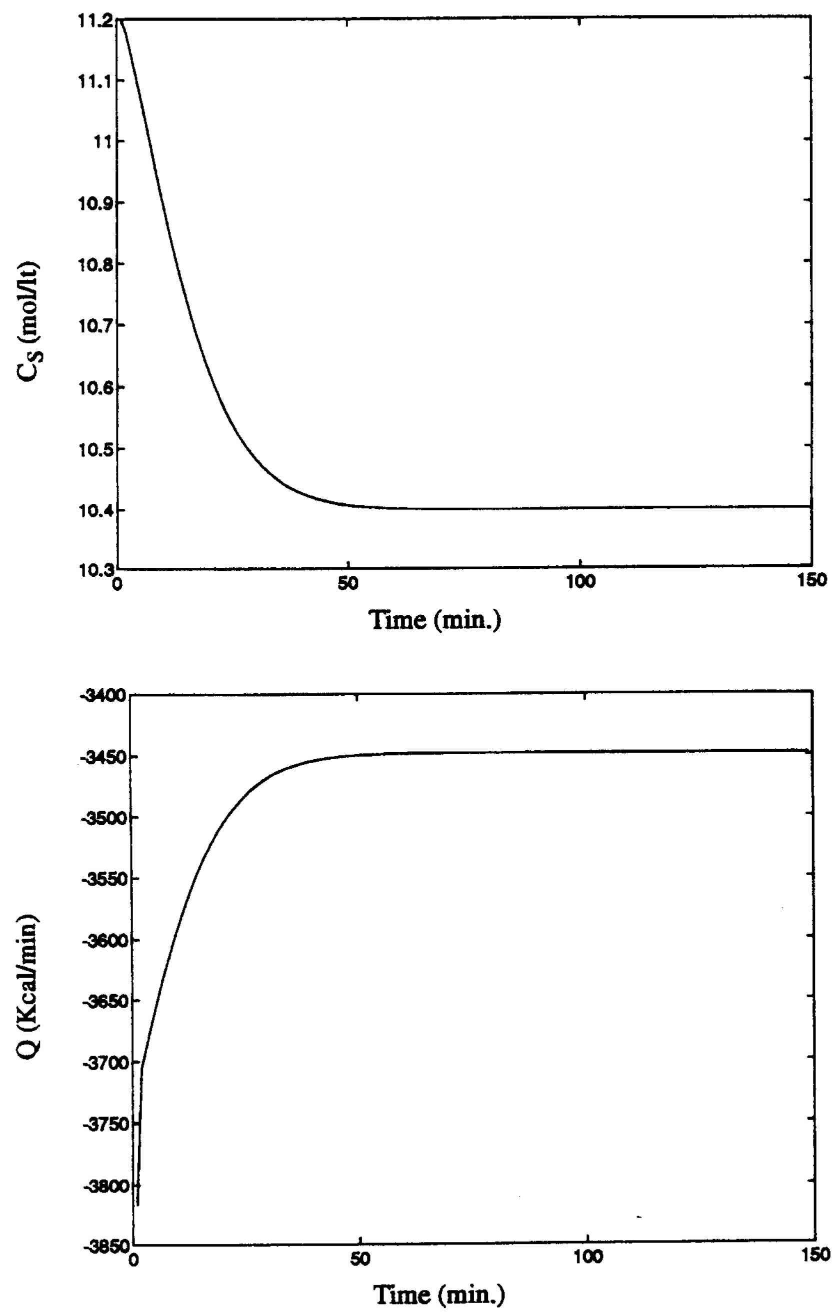

Figure 4. Output and input profiles for a negative change in the reference input.

Several simulations were performed to evaluate the performance of the controller. In the first simulation run, a 0·8 mol l^{-1} increase in the reference input value was imposed at time $t = 1{\cdot}0$ min. Figure 2 shows the corresponding output and input profiles. One can see that the controller regulates the output to the new value of the reference input. Figure 3 shows the profile of the concentration C_{Es}, of the enzyme under the quasi-steady-state assumption and the profile of the concentration C_{E} of the enzyme in the reactor. One can observe that the two profiles almost coincide which implies that the dynamics of the concentration of the enzyme are indeed negligible. In the second simulation run, a 0·8 mol l^{-1} decrease in the value of the reference input was imposed at time $t = 1{\cdot}0$ min. Figure 4 shows the corresponding output and input profiles. The controller regulates the output to the new value of the reference input.

Remark 7: A standard input–output linearizing controller synthesized on the basis of the full-order system yields

$$u = [\beta_2 L_{g_1} L_{f_1} h(x)]^{-1}\Big[y-\beta_0 h(x)-\beta_1 [L_{f_1} h(x)+L_{Q_1} h(x) z]-\beta_2\Big(L_{f_1}^2 h(x)+L_{Q_1} L_{f_1} h(x) z + [L_{f_1} L_{Q_1} h(x)+L_{Q_1}^2 h(x)+L_{G_1} L_{Q_1} h(x)] z+\frac{L_{Q_1} h(x) [f_2(x)+Q_2(x) z]}{\varepsilon}\Big)\Big] \tag{39}$$

which becomes singular as $\varepsilon \to 0$. Moreover, the implementation of this feedback law requires measurements of the concentration of the enzyme in the reactor, which are difficult to obtain in practice. □

6.2. *Application to a cascade of two continuous stirred-tank reactors*

Consider the cascade of the two CSTR shown in Fig. 5, where the following autocatalytic reaction (Gray and Scott 1990) takes place:

$$\mathrm{A}+\mathrm{B} \to 2\mathrm{B} \tag{40}$$

where A is a reactant, B is the autocatalytic species, followed by the zero-order side reaction

$$\mathrm{B} \to \mathrm{C} \tag{41}$$

where C is the undesired product. The species A is assumed to be in excess in the two reactors, while the inlet streams consist of autocatalytic species B of concentration C_{B0}. Under the assumptions

(*a*) perfect mixing in the reactor,
(*b*) uniform temperature in the reactor,
(*c*) constant volume of the liquid in the reactor and
(*d*) constant density and heat capacity of the reacting liquid

the material and energy balances that describe the dynamical behaviour of the system take the following form:

$$\left.\begin{aligned}
V_1 \frac{dC_{B1}}{dt} &= F_1 C_{B0} - F_1 C_{B1} + k_{10} \exp\left(\frac{-E_1}{RT_1}\right) C_{B1} V_1 - k_0 \exp\left(\frac{-E_0}{RT_1}\right) V_1 \\
\frac{dT_1}{dt} &= \frac{F_1}{V_1}(T_{B0}-T_1) + \frac{\bar{Q}_1}{\rho_m c_{pm} V_1} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} \exp\left(\frac{-E_1}{RT_1}\right) C_{B1} \\
&\quad + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}} k_0 \exp\left(\frac{-E_0}{RT_1}\right) \\
V_2 \frac{dC_{B2}}{dt} &= F_1 C_{B1} + F_2 C_{B0} - F_3 C_{B2} + k_{10} \exp\left(\frac{-E_1}{RT_2}\right) C_{B2} V_2 - k_0 \exp\left(\frac{-E_0}{RT_2}\right) V_2 \\
\frac{dT_2}{dt} &= \frac{F_2}{V_2} T_{B0} + \frac{F_1}{V_2} T_1 - \frac{F_3}{V_2} T_2 + \frac{\bar{Q}_2}{\rho_m c_{pm} V_2} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} \exp\left(\frac{-E_1}{RT_2}\right) C_{B2} \\
&\quad + \frac{(\Delta H_{r_0})}{\rho_m c_{pm}} k_0 \exp\left(\frac{-E_0}{RT_2}\right)
\end{aligned}\right\} \tag{42}$$

Parameter	Value	Units
V_1	0·2	m^3
V_2	1·0	m^3
R	1·987	kcal $kmol^{-1}$ K^{-1}
C_{B0_s}	2·0	kmol m^{-3}
T_{B0}	305·0	K
c_{pm}	0·231	kcal kg^{-1} K^{-1}
ρ_m	900·0	kg m^{-3}
Q_1	$1{\cdot}1 \times 10^5$	kcal min^{-1}
Q_2	$2{\cdot}2 \times 10^4$	kcal min^{-1}
ΔH_0	$5{\cdot}4 \times 10^3$	kcal $kmol^{-1}$
ΔH_1	$10{\cdot}67 \times 10^3$	kcal $kmol^{-1}$
k_0	$4{\cdot}71 \times 10^6$	kmol m^{-3} min^{-1}
k_{10}	$1{\cdot}08 \times 10^7$	min^{-1}
E_0	$8{\cdot}0 \times 10^3$	kcal $kmol^{-1}$
E_1	$9{\cdot}0 \times 10^3$	kcal $kmol^{-1}$
F_1	0·2	m^3 min^{-1}
F_2	2·0	m^3 min^{-1}
C_{B1_s}	2·5	kmol m^{-3}
T_{1_s}	300·0	K
C_{B2_s}	3·125	kmol m^{-3}
T_{2_s}	300·0	K

Table 2. Process parameters.

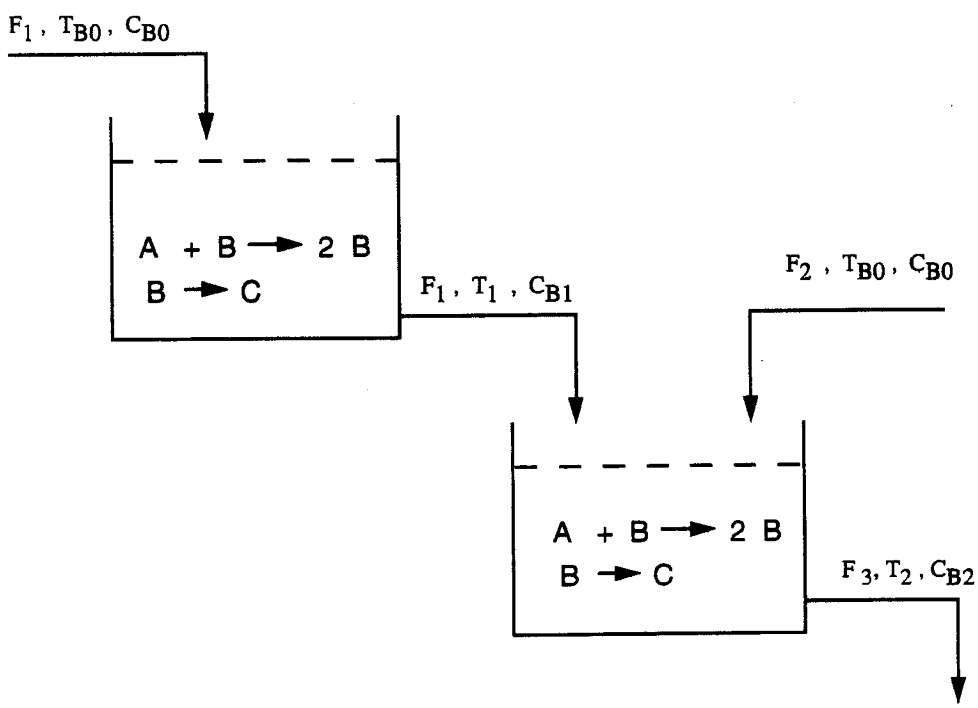

Figure 5. A cascade of two CSTRs.

where C_{B1}, T_1 and C_{B2}, T_2 denote the temperatures and the concentrations of the autocatalytic species in the first and second reactor, C_{B0} and T_{B0} denote the inlet temperature and concentration of the species B, F_3 is the outlet flowrate in the second reactor, $\bar{Q}_1$ and $\bar{Q}_2$ denote the heat inputs to the reactors, and k_{10}, k_0, E_1, E_0, ΔH_1 and

ΔH_0 denote the pre-exponential constants, the activation energies and the enthalpies of the two reactors respectively.

The control objective is the regulation of the concentration of the autocatalytic species B in the second reactor by manipulating the inlet concentration C_{B0}. In order to decrease the effect of the side reaction, that is to minimize the production of the species C, the liquid hold-up of the first reactor is smaller than the liquid hold-up of the second reactor (Levenspiel 1972). Defining the parameter ε as

$$\varepsilon = \frac{V_1}{V_2} \tag{43}$$

and setting

$$u = C_{B0} - C_{B0_s}, \quad x_1 = T_1, \quad x_2 = C_{B2}, \quad x_3 = T_2, \quad z_1 = C_{B1}, \quad y = x_2 \tag{44}$$

the original set of equations can be put in the form (1) with

$$F_1(x) = \begin{bmatrix} \frac{F_1}{V_1}(T_{B0} - x_1) + \frac{\bar{Q}_1}{\rho_m c_{pm} V_1} + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}} k_0 \exp\left(\frac{-E_0}{Rx_1}\right) \\ F_2 C_{B0_s} - F_3 x_2 + k_{10} \exp\left(\frac{-E_1}{Rx_3}\right) x_2 V_2 - k_0 \exp\left(\frac{-E_0}{Rx_3}\right) V_2 \\ \frac{F_2}{V_2} T_{B0} + \frac{F_1}{V_2} x_1 - \frac{F_3}{V_2} x_3 + \frac{\bar{Q}_2}{\rho_m c_{pm} V_2} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} \exp\left(\frac{-E_1}{Rx_3}\right) x_2 \\ + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}} k_0 \exp\left(\frac{-E_0}{Rx_3}\right) \end{bmatrix}.$$

$$Q_1(x) = \begin{bmatrix} \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} \exp\left(\frac{-E_1}{Rx_1}\right) \\ F_1 \\ 0 \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 \\ F_2 \\ 0 \end{bmatrix}$$

$$f_2(x) = \left[\frac{F_1}{V_2} C_{B0_s} - \frac{k_0 \exp\left(\frac{-E_0}{Rx_1}\right) V_1}{V_2} \right], \quad Q_2(x) = \left[-\frac{F_1}{V_2} + \frac{k_{10} \exp\left(\frac{-E_1}{Rx_1}\right) V_1}{V_2} \right]$$

$$g_2(x) = \left[\frac{F_1}{V_2}\right], \quad h(x) = [x_2]$$

The values of the system parameters and the corresponding steady-state values of the system variables are given in Table 2. One can easily see that for these operating conditions the matrix $Q_2(x)$ is invertible, and the fast dynamics of the system are unstable. Therefore the controller of Theorem 1 was employed in the simulations. Moreover, it was verified that the zero dynamics of the reduced system are exponentially stable.

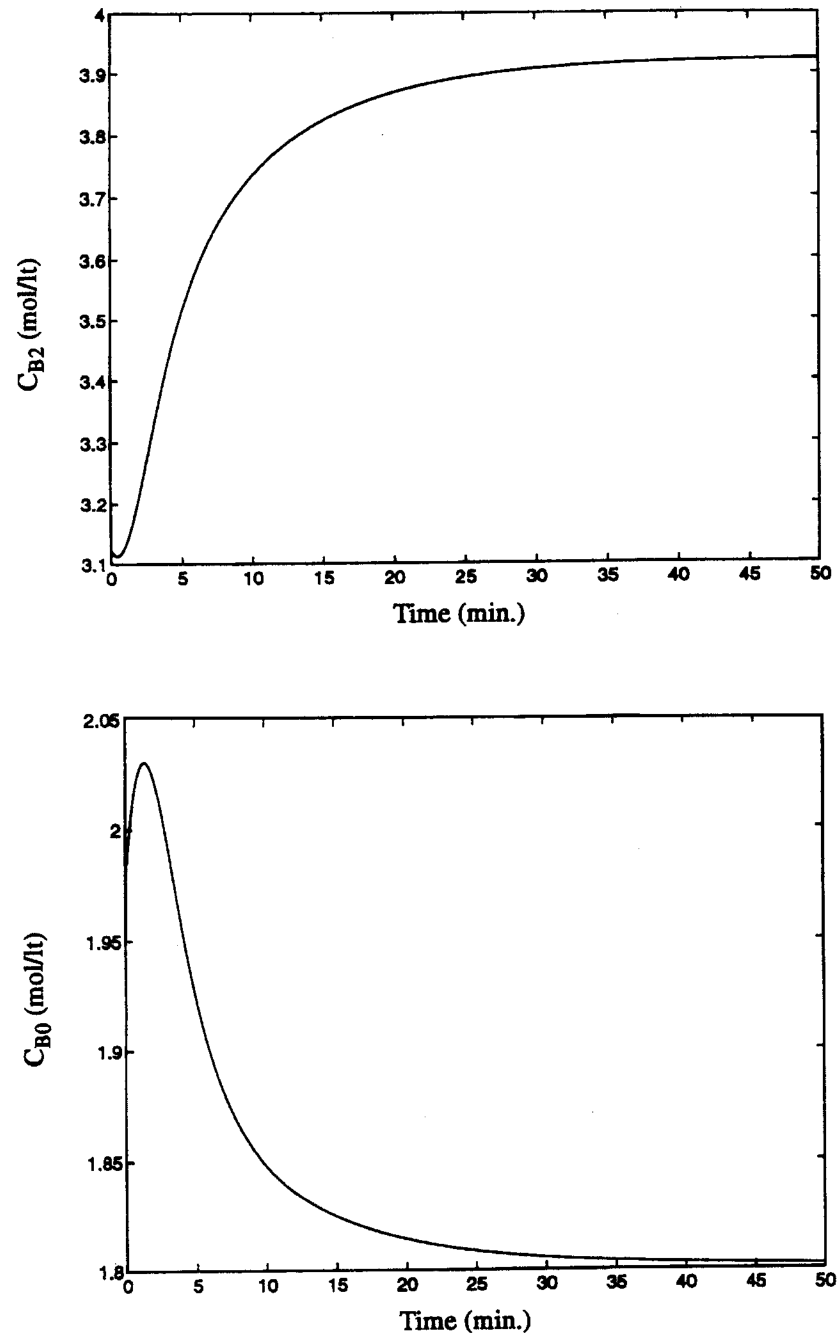

Figure 6. Output and input profiles for positive change in the reference input.

Setting $\varepsilon = 0$, the representation of the open-loop reduced system of the form (2) can be easily obtained with

$$F(x) = \begin{bmatrix} \frac{F_1}{V_1}(T_{B0}-x_1)+\frac{\bar{Q}_1}{\rho_m c_{pm} V_1}+\frac{(-\Delta H_{r_0})}{\rho_m c_{pm}}k_0 \exp\left(\frac{-E_0}{Rx_1}\right)+\frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}\exp\left(\frac{-E_1}{Rx_1}\right) \\ \times\left[F_1-k_{10}\exp\left(\frac{-E_1}{Rx_1}\right)V_1\right]^{-1}\left[F_1 C_{B0_s}-k_0\exp\left(\frac{-E_0}{Rx_1}\right)V_1\right] \\ F_2 C_{B0_s}-F_3 x_2+k_{10}\exp\left(\frac{-E_1}{Rx_3}\right)x_2 V_2-k_0\exp\left(\frac{-E_0}{Rx_3}\right)V_2+F_1 \\ \times\left[F_1-k_{10}\exp\left(\frac{-E_1}{Rx_1}\right)V_1\right]^{-1}\left[F_1 C_{B0_s}-k_0\exp\left(\frac{-E_0}{Rx_1}\right)V_1\right] \\ \frac{F_2}{V_2}T_{B0}+\frac{F_1}{V_2}x_1-\frac{F_3}{V_2}x_3+\frac{\bar{Q}_2}{\rho_m c_{pm} V_2}+\frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}\exp\left(\frac{-E_1}{Rx_3}\right)x_2 \\ +\frac{(-\Delta H_{r_0})}{\rho_m c_{pm}}k_0\exp\left(\frac{-E_0}{Rx_3}\right) \end{bmatrix}$$

$$G(x) = \begin{bmatrix} \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}\exp\left(\frac{-E_1}{Rx_1}\right)\left[F_1-k_{10}\exp\left(\frac{-E_1}{Rx_1}\right)V_1\right]^{-1}F_1 \\ F_2+F_1\left[F_1-k_{10}\exp\left(\frac{-E_1}{RX_1}\right)V_1\right]^{-1}F_1 \\ 0 \end{bmatrix}$$

It can be easily verified that the relative order is $r = 1$ and the controller of Theorem 1 takes the form

$$\begin{aligned} u = &\{1+k^{\mathrm{T}}(x)[Q_2(x)]^{-1}g_2(x)\}[\beta_1 L_G h(x)]^{-1}\left(v-\sum_{k=0}^{1}\beta_k L_F^k h(x)\right) \\ &+k^{\mathrm{T}}(x)[Q_2(x)]^{-1}f_2(x)+k^{\mathrm{T}}(x)z_1 \end{aligned} \tag{45}$$

The nonlinear gain $k^{\mathrm{T}}(x)$ was chosen as

$$k^{\mathrm{T}}(x) = -\frac{3{\cdot}0}{F_1}\left[-F_1+k_{10}\exp\left(\frac{-E_1}{Rx_1}\right)V_1\right] \tag{46}$$

to place the eigenvalue of the matrix $Q_2(x)+g_2(x)k^{\mathrm{T}}(x)$ in the open left half of the complex plane. The time constant for the input–output response was chosen to be $\tau = 15$ min, through the following choice of the controller parameters:

$$\beta_0 = 1{\cdot}0, \quad \beta_1 = 15{\cdot}0 \text{ min}$$

Several simulations were performed to evaluate the performance of the controller. In the first simulation run, a 0·8 mol l^{-1} increase in the value of the reference input was imposed at time $t = 0$. Figure 6 shows the output and the input profiles. Clearly, the

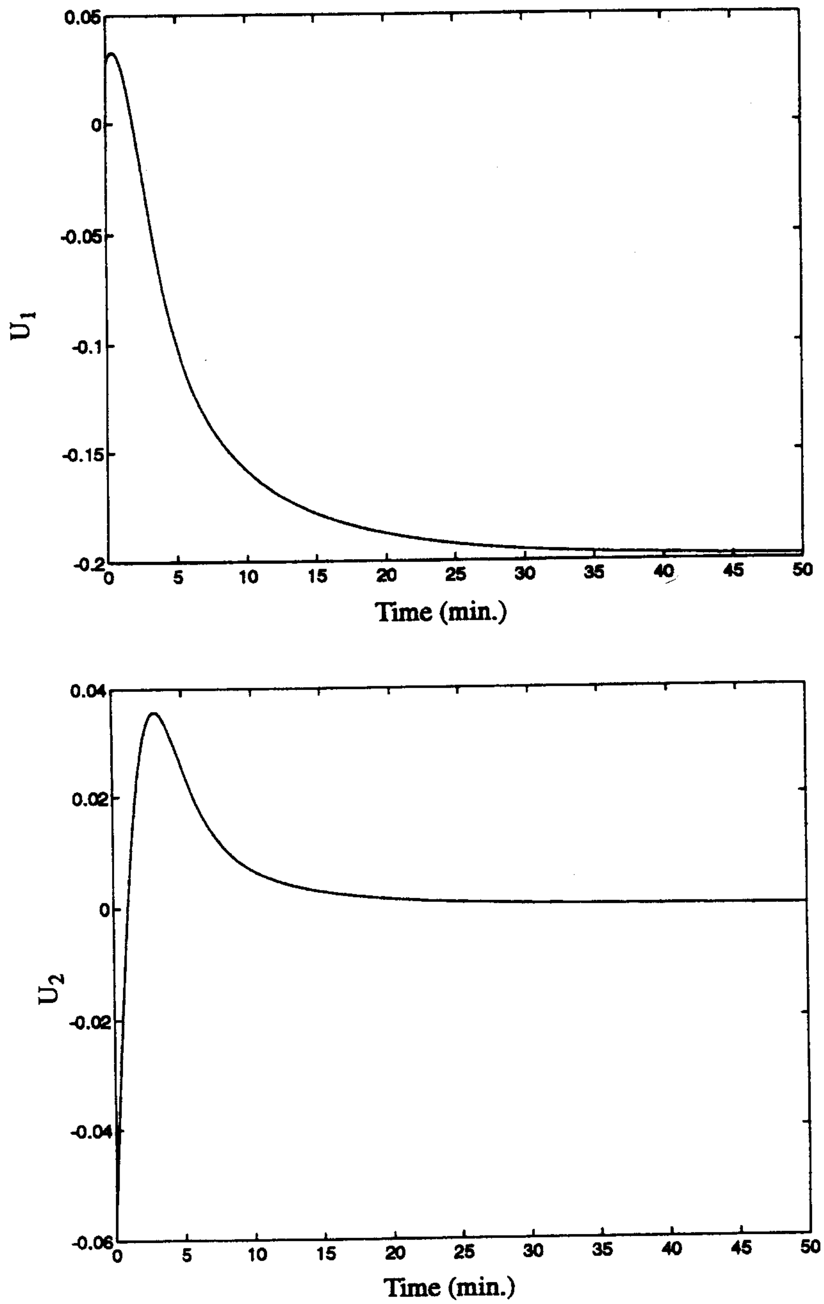

Figure 7. Fast (U_2) and slow (U_1) components of the input.

controller drives the output of the system to the new value of the reference input, while stabilizing the fast dynamics of the system. Figure 7 shows the profiles of the two components of the control law identified in Remark 1 for this particular simulation run. More specifically, U_1 denotes the component

$$U_1 = [\beta_1 L_G h(x)]^{-1} \left(v - \sum_{k=0}^{1} \beta_k L_F^k h(x) \right) \tag{47}$$

which acts in the slow time scale and induces the requested input–output behaviour in the closed-loop reduced system, while U_2 denotes the component

$$U_2 = k^{\mathrm{T}}(x) (z_1 - \xi_1) \tag{48}$$

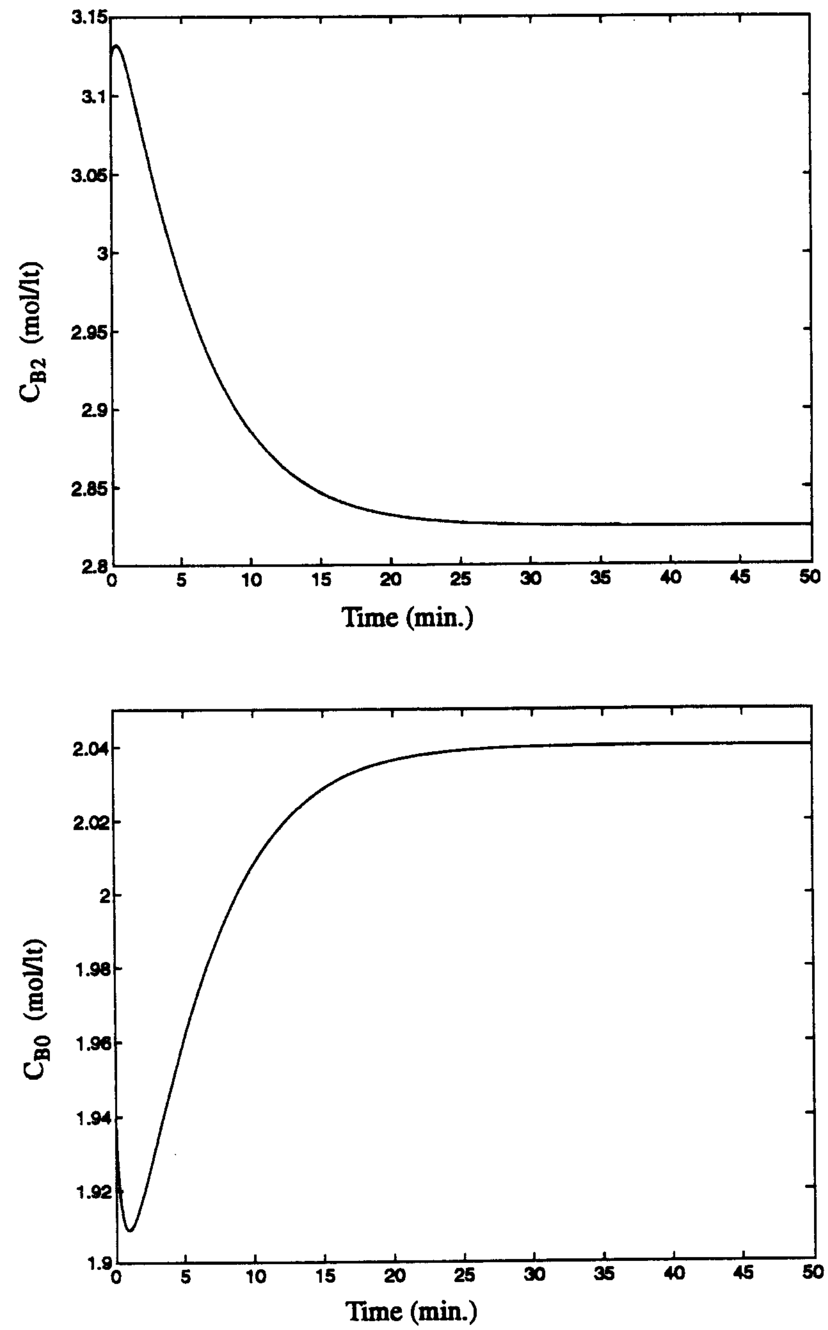

Figure 8. Output and input profiles for a negative change in the reference input.

where ξ_1 is defined in (21), which acts in the fast time scale and stabilizes the fast dynamics of the system. Clearly the behaviour of U_1 and U_2 conforms with the theoretical predictions. More specifically, the components U_1 acts for all times in order to drive the output of the system to the new value of the reference input, while the component U_2 approaches zero quickly.

In the second simulation run, a 0·3 mol l^{-1} decrease in the value of the reference input was imposed at time $t = 0$. The corresponding output and input profiles are shown in Fig. 8. The controller regulates the output to the new reference input value, while stabilizing the fast dynamics of the system.

Finally, an input–output linearizing control law, which was synthesized on the basis

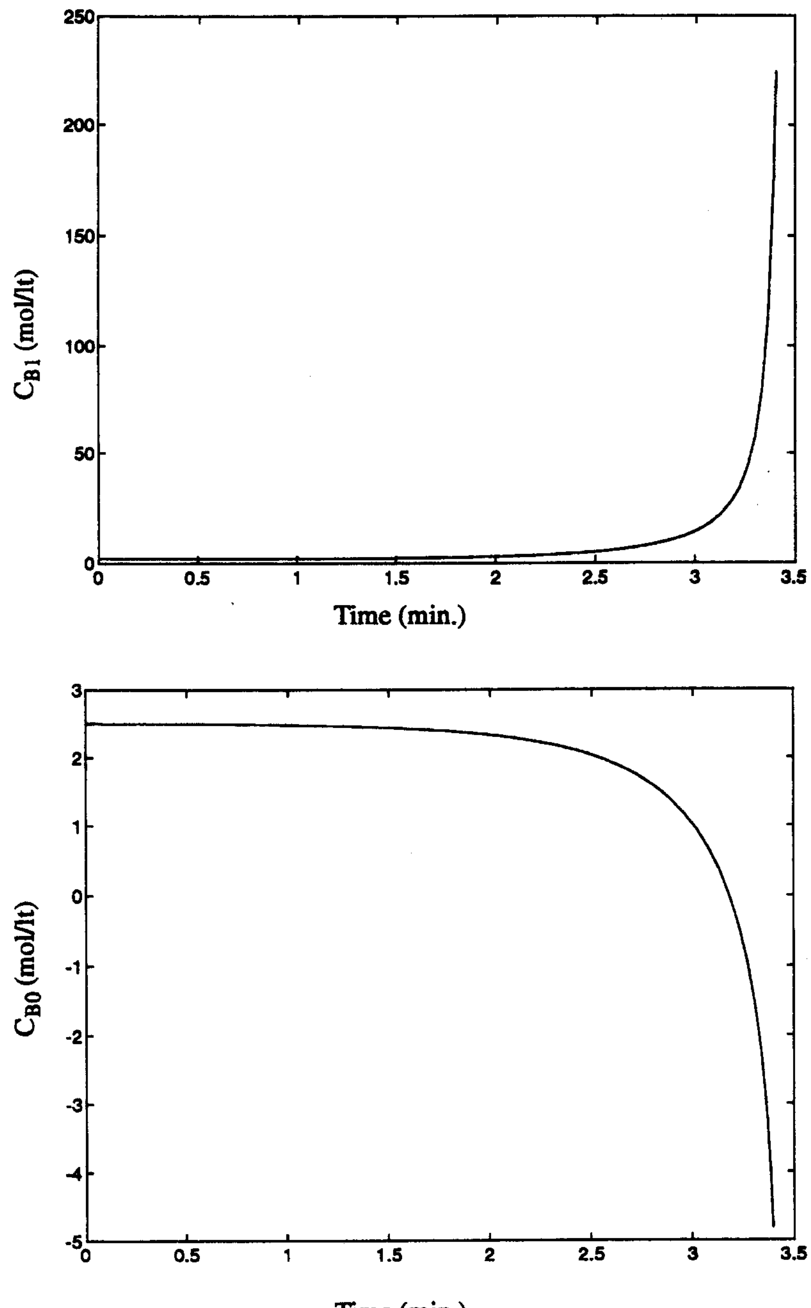

Figure 9. Fast state and input profiles for reference input tracking, under an input–output linearizing controller.

of the original system, was also employed in the simulations. A 0·3 mol l^{-1} decrease in the reference input value was imposed at time $t = 0$. Figure 9 shows the profile of the fast state z_1 which corresponds to the concentration of the species B in the first reactor, and the corresponding input. It is clear that the input–output linearizing controller leads to closed-loop instability, which is expected, since, as can be easily verified following the development of Remark 5, the process is slightly non-minimum phase.

7. Conclusions

In this article, we addressed and solved the problem of synthesizing well-conditioned static state feedback laws for a class of two-time-scale nonlinear systems, whose fast dynamics may be unstable or singular. The derived control laws guarantee exponential stability of the fast dynamics and induce a well-characterized input–output behaviour in the closed-loop reduced system. It was established that if the zero dynamics of the reduced system is exponentially stable, the closed-loop system is exponentially stable and the discrepancy between the output of the closed-loop full-order system and the output of the closed-loop reduced system is of order ε, for sufficiently small values of ε. The proposed control methodology was successfully applied to two representative nonlinear chemical processes with time-scale multiplicity and its superiority with respect to nonlinear control methods that neglect the presence of time-scale multiplicity was documented through simulations.

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Appendix

Proof of proposition 1: It is straightforward to show that the static state feedback law (8) on the quasi-steady-state $z_s = -[Q_2(x)]^{-1}[f_2(x)+g_2(x)]u$ takes the form

$$u = [1+k^{\mathrm{T}}(x)\,Q_2^{-1}(x)\,g_2(x)]^{-1}[\tilde{u}-k^{\mathrm{T}}(x)\,Q_2^{-1}(x)f_2(x)] \tag{A 1}$$

where the scalar function $1+k^{\mathrm{T}}(x)\,Q_2^{-1}(x)\,g_2(x)$ is non-zero uniformly in $x \in X$.

Under the state feedback law (A 1) the closed-loop reduced system takes the form

$$\left.\begin{aligned} \dot{x} &= F(x)+G(x)\,[1+k^{\mathrm{T}}(x)\,Q_2^{-1}(x)\,g_2(x)]^{-1}[\tilde{u}-k^{\mathrm{T}}(x)\,Q_2^{-1}(x)f_2(x)] \\ y^{s} &= h(x) \end{aligned}\right\} \tag{A 2}$$

where the vector fields $F(x)$ and $G(x)$ are given by (3). For the above system, one can directly show that the relative order of the output y with respect to the auxiliary input $\tilde{u}$ is exactly equal to r. It remains to be proved that the closed-loop reduced systems (11) and (A 2) are identical. Lemma 1 that follows states this result.

Lemma 1: *Consider the two-time-scale system of the form* (1) *in standard form, subject to the static state feedback law* (8), *and assume that the matrix* $Q_2(x)+g_2(x)\,k^{\mathrm{T}}(x)$ *is Hurwitz uniformly in* $x \in X$. *Then, the closed-loop reduced systems* (11) *and* (50), *derived by commuting the order of the operations closing the feedback loop and setting* $\varepsilon = 0$, *are identical.*

The proof of the lemma can be obtained along the lines of the proof which was given in the case of two-time-scale linear systems under linear static state feedback laws (see for example Corless *et al.* (1993)) and will be omitted for brevity. □

Proof of Theorem 1: Under the control law (18) the closed-loop system takes the form

$$\left.\begin{aligned}\dot{x} &= f_1(x)+Q_1(x)z+g_1(x)\Big[[1+k^{\mathrm{T}}(x)[Q_2(x)]^{-1}g_2(x)][\beta_r L_G L_F^{r-1}h(x)]^{-1} \\ &\quad\times\Big(v-\sum_{k=0}^{r}\beta_k L_F^k h(x)\Big)\Big]+g_1(x)\{k^{\mathrm{T}}(x)[Q_2(x)]^{-1}f_2(x)+k^{\mathrm{T}}(x)z\} \\ \varepsilon\dot{z} &= f_2(x)+Q_2(x)z+g_2(x)\Big[[1+k^{\mathrm{T}}(x)[Q_2(x)]^{-1}g_2(x)][\beta_r L_G L_F^{r-1}h(x)]^{-1} \\ &\quad\times\Big(v-\sum_{k=0}^{r}\beta_k L_F^k h(x)\Big)\Big]+g_2(x)\{k^{\mathrm{T}}(x)[Q_2(x)]^{-1}f_2(x)+k^{\mathrm{T}}(x)z\} \\ y &= h(x)\end{aligned}\right\} \quad (\text{A }3)$$

Using the set of variables ξ defined in (21), the closed-loop system (A 3) takes the form

$$\left.\begin{aligned}\dot{x} &= f_1(x)+Q_1(x)z+g_1(x)\Big[[\beta_r L_G L_F^{r-1}h(x)]^{-1}\Big(v-\sum_{k=0}^{r}\beta_k L_F^k h(x)\Big)+k^{\mathrm{T}}(x)(z-\xi)\Big] \\ \varepsilon\dot{z} &= f_2(x)+Q_2(x)z+g_2(x)\Big[[\beta_r L_G L_F^{r-1}h(x)]^{-1}\Big(v-\sum_{k=0}^{r}\beta_k L_F^k h(x)\Big)+k^{\mathrm{T}}(x)(z-\xi)\Big]\end{aligned}\right\} \quad (\text{A }4)$$

The properties of the above two-time-scale system can be analysed by performing a standard two-time-scale decomposition. More specifically, introducing a new set of variables $\bar{\eta}$, defined as $\bar{\eta}=z-\xi$, one can easily show that the closed-loop fast subsystem is given by

$$\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}\tau}=[Q_2(x)+g_2(x)k^{\mathrm{T}}(x)]\bar{\eta} \quad (\text{A }5)$$

Since the matrix $Q_2(x)+g_2(x)k^{\mathrm{T}}(x)$ is Hurwitz by the appropriate choice of the feedback gain $k^{\mathrm{T}}(x)$, the fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold of the form

$$\xi=-[Q_2(x)]^{-1}\Big\{f_2(x)+g_2(x)\Big[[\beta_r L_G L_F^{r-1}h(x)]^{-1}\Big(v-\sum_{k=0}^{r}\beta_k L_F^k h(x)\Big)\Big]\Big\} \quad (\text{A }6)$$

Furthermore, the closed-loop reduced system takes the form

$$\left.\begin{aligned}\dot{x} &= \{f_1(x)-Q_1(x)[Q_2(x)]^{-1}f_2(x)\}+\{g_1(x)-Q_1(x)[Q_2(x)]^{-1}g_2(x)\} \\ &\quad\times\Big[[\beta_r L_G L_F^{r-1}h(x)]^{-1}\Big(v-\sum_{k=1}^{r}\beta_k L_F^k h(x)\Big)\Big] \\ y^{\mathrm{s}} &= h(x)\end{aligned}\right\} \quad (\text{A }7)$$

Calculating the derivatives of the output on the basis of (A 7) yields the following expressions

$$\left.\begin{aligned}y^{\mathrm{s}} &= h(x) \\ y^{\mathrm{s}(1)} &= L_F h(x) \\ &\;\;\vdots \\ y^{\mathrm{s}(r-1)} &= L_F^{r-1}h(x) \\ y^{\mathrm{s}(r)} &= L_F^r h(x)+L_G L_F^{r-1}h(x)\Big[[\beta_r L_G L_F^{r-1}h(x)]^{-1}\Big(v-\sum_{k=0}^{r}\beta_k L_F^k h(x)\Big)\Big]\end{aligned}\right\} \quad (\text{A }8)$$

Substituting the above expressions into (17), it is straightforward to verify that the desired input–output behaviour is indeed enforced in the closed-loop reduced system.

We shall now establish that the relation (19) holds for the output of the closed-loop full-order system. To this end, let us denote by $x^{\mathrm{f}}(t)$ and $x^{\mathrm{s}}(t)$, the solutions of the x-subsystem (A 3) and of (A 7) respectively. Based on the stability results of §5, the closed-loop reduced system (A 7) is locally exponentially stable. Furthermore, as established earlier in this proof, the fast subsystem (A 5) is exponentially stable, uniformly in $x \in X$. Utilizing these two stability properties it can be shown (see Theorem 8.4 of Khalil (1992)) that under consistent initialization of the states x_i^{s}, x_i^{f}, that is $x_i^{\mathrm{s}}(0) = x_i^{\mathrm{f}}(0)$, $i = 1, \ldots, n$, the following estimate holds for the solutions of these systems

$$x^{\mathrm{f}}(t) = x^{\mathrm{s}}(t) + \mathrm{O}(\varepsilon), \quad t \geqslant 0 \tag{A 9}$$

for ε sufficiently small. Then, the analyticity of the scalar field $h(x)$ and the boundedness of the trajectories $x^{\mathrm{f}}(t)$ and $x^{\mathrm{s}}(t)$ for all times directly imply that the following estimate holds for the output y of the closed-loop full-order system

$$y(t) = y^{\mathrm{s}}(t) + \mathrm{O}(\varepsilon), \quad t \geqslant 0 \tag{A 10}$$

where $y(t)$ and $y^{\mathrm{s}}(t)$ denote the outputs of the systems (A 3) and (A 7) respectively. This completes the proof of Theorem 1. □

Proof of Theorem 2: Under the control law (33) the closed-loop system takes the form

$$\left.\begin{aligned} \dot{x} &= f_1(x) + Q_1(x)\, z + g_1(x)\left[[\beta_{\tilde{r}}\, L_{\tilde{G}}\, L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1}\left(v - \sum_{k=0}^{\tilde{r}} \beta_k\, L_{\tilde{F}}^{k} h(x)\right)\right] + g_1(x)\, k^{\mathrm{T}}(x)\, z \\ \varepsilon \dot{z} &= f_2(x) + Q_2(x)\, z + g_2(x)\left[[\beta_{\tilde{r}}\, L_{\tilde{G}}\, L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1}\left(v - \sum_{k=0}^{\tilde{r}} \beta_k\, L_{\tilde{F}}^{k} h(x)\right)\right] + g_2(x)\, k^{\mathrm{T}}(x)\, z \\ y &= h(x) \end{aligned}\right\} \tag{A 11}$$

Employing a standard two-time-scale decomposition, it is straightforward to show that the closed-loop fast subsystem takes the form

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = f_2(x) + [Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]\, z + g_2(x)\left[[\beta_{\tilde{r}}\, L_{\tilde{G}}\, L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1}\left(v - \sum_{k=0}^{\tilde{r}} \beta_k\, L_{\tilde{F}}^{k} h(x)\right)\right] \tag{A 12}$$

Clearly, the fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold of the form

$$\tilde{\xi} = -[Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)]^{-1}\left\{f_2(x) + g_2(x)\left[[\beta_{\tilde{r}}\, L_{\tilde{G}}\, L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1}\left(v - \sum_{k=0}^{\tilde{r}} \beta_k\, L_{\tilde{F}}^{k} h(x)\right)\right]\right\} \tag{A 13}$$

since the matrix $Q_2(x) + g_2(x)\, k^{\mathrm{T}}(x)$ is Hurwitz by the appropriate choice of the feedback gain $k^{\mathrm{T}}(x)$.

Furthermore, the closed-loop reduced system takes the form

$$\left.\begin{aligned}\dot{x} &= \tilde{F}(x)+\tilde{G}(x)\left[[\beta_{\bar r} L_{\tilde G} L_{\tilde F}^{\bar r-1} h(x)]^{-1}\left(v-\sum_{k=1}^{\bar r}\beta_k L_{\tilde F}^{k} h(x)\right)\right]\\ \hat{y}^{s} &= h(x)\end{aligned}\right\}\qquad \text{(A 14)}$$

It is then straightforward to show by a direct calculation of the derivatives of the output y on the basis of (A 14) that the requested input–output behaviour is indeed enforced in the closed-loop reduced system. Finally, using the same arguments as in theorem 1, it can be shown that the relation (34) holds for the output of the closed-loop system (A 11). □

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