# Feedback Control of Hyperbolic PDE Systems

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*This article deals with distributed parameter systems described by first-order hyperbolic partial differential equations (PDEs), for which the manipulated input, the controlled output, and the measured output are distributed in space. For these systems, a general output-feedback control methodology is developed employing a combination of theory of PDEs and concepts from geometric control. A concept of characteristic index is introduced and used for the synthesis of distributed state-feedback laws that guarantee output tracking in the closed-loop system. Analytical formulas of distributed output-feedback controllers are derived through combination of appropriate distributed state observers with the developed state-feedback controllers. Theoretical analogies between our approach and available results on stabilization of linear hyperbolic PDEs are also identified. The developed control methodology is implemented on a nonisothermal plug-flow reactor and its performance is evaluated through simulations.*

## Introduction

Chemical engineering processes are inherently nonlinear and very frequently involve state variables that change in both time and space. Representative examples of processes with significant spatial variations include plug-flow reactors (Ray, 1981), countercurrent absorbers–reactors (Rhee et al., 1986), fixed- and fluidized-bed reactors (Stangeland and Foss, 1970; Ray, 1981; Georgakis et al., 1977). The mathematical models of these processes are typically derived from the dynamic conservation equations and consist of nonlinear partial differential equations (PDEs).

The conventional approach for the control of PDE systems is based on the spatial discretization of the PDE model followed by the controller design on the basis of the resulting (linear or nonlinear) ordinary differential equation (ODE) model (Sorensen et al., 1980; Dochain et al., 1992; Patwardhan et al., 1992). However, there are certain well-known disadvantages associated with this approach. For example, fundamental control-theoretic properties, like controllability and observability, which should depend only on the location of sensors and actuators, may also depend on the discretization method and the number and location of discretization points (Ray, 1981). Moreover, neglecting the infinite-dimensional nature of the original system may lead to erroneous conclusions concerning the stability properties of the open-loop and/or the closed-loop system. Furthermore, in processes where the spatially distributed nature is very strong, due to the underlying convection and diffusion phenomena, such an approach limits the controller performance, and may lead to unacceptable control quality.

Motivated by the preceding considerations, significant research efforts have focused on the development of control methods for PDE systems that directly account for their spatially distributed nature. Excellent surveys of theoretical as well as application articles on this topic can be found in Balas (1982), Keulen (1993), and Ray (1978). Initially, systems of linear PDEs were considered, for which key system- and control-theoretic properties (e.g., existence and uniqueness of solutions, stability, controllability, and observability) were well-understood (Curtain and Pritchard, 1978). The well-known classification of PDE systems to hyperbolic, parabolic, and elliptic (Smoller, 1983), according to the properties of the spatial differential operator, essentially determined the approach followed for the solution of the control problem. Thus, for parabolic PDE systems (e.g., diffusion-reaction processes), the fact that the system dynamics is practically determined by a finite number of modes, motivated the use of modal decomposition techniques to derive ODE models that capture the dominant dynamics of the system (Curtain, 1982; Georgakis et al., 1977; Hanczyc and Palazoglu, 1992; Gay and Ray, 1995); the controller design problem was then addressed using methods for linear ODE systems. On the other hand, for hyperbolic PDE systems (e.g., convection-reaction processes), for which a modal decomposition is not possible, alternative approaches were followed, using mainly optimal control methods (Wang, 1966; Lo, 1973; Balas, 1986) on the basis of the original infinite-dimensional system, or ODE control

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methods on the basis of equivalent ODE realizations obtained by the method of characteristics (Ray, 1981).

Recently, considerable attention has focused on the understanding of system-theoretic properties and the dynamical behavior of nonlinear PDEs, by treating them as evolution equations in appropriate infinite dimensional spaces (Temam, 1988; Brown et al., 1991). Yet, available results on the control of systems of nonlinear PDEs are rather sparse, with the exception of optimal control approaches [(Balas, 1991; Burns and Kang, 1991; Kang and Ito, 1992) and the review paper (Lasiecka, 1995) for an overview of recent results in systems literature]. Due to their practical relevance and importance, quasi-linear PDE systems have attracted particular research interest. For quasi-linear parabolic PDE systems, an approach that utilizes a combination of eigenfunction expansion techniques and nonlinear control schemes was proposed in Chen and Chang (1992). For processes modeled by a single first-order quasi-linear hyperbolic PDE, an approach based on a combination of the method of characteristics and sliding mode techniques was proposed in Sira-Ramirez (1989). This method was further developed in Hanczyc and Palazoglu (1995) to account for possible discontinuous behavior of the control action and was applied to a heat exchanger. A Lyapunov-based approach, inspired from concepts of thermodynamics, to derive conditions that guarantee stability under linear control has also been proposed in Alonso and Ydstie (1995).

In this article, we address the feedback control problem for systems described by quasi-linear first-order hyperbolic PDEs, for which the manipulated input, the controlled output, and the measured output are distributed in space. Systems of this form arise naturally in chemical engineering as models of transport-reaction processes, whenever convective mechanisms dominate diffusive and dispersive ones [see the book by Rhee et al. (1986) for representative examples]. For such systems, our objective is to synthesize nonlinear distributed output-feedback controllers that enforce output tracking and guarantee stability in the closed-loop system.

The article is structured as follows: after reviewing the necessary preliminaries, a concept of characteristic index between the controlled output and the manipulated input (which can be thought of as the analogue of relative order) is introduced and used for the synthesis of distributed state-feedback controllers that induce output tracking in the closed-loop system. A notion of zero-output constraint dynamics for first-order hyperbolic PDEs is introduced and used to derive precise conditions that guarantee the stability of the closed-loop system. Then, output-feedback controllers are synthesized through a combination of appropriate distributed state observers with the developed state-feedback controllers. Theoretical analogies between the proposed approach and available feedback control methods for the stabilization of linear hyperbolic PDEs are pointed out. Controller implementation issues are also discussed. Finally, the application of the developed control method is illustrated through a nonisothermal plug-flow reactor example modeled by a system of three quasi-linear hyperbolic PDEs.

## First-Order Hyperbolic PDE Systems

### *Preliminaries*

We consider systems of quasi-linear first-order PDEs in one spatial dimension with the following state-space representation:

$$\frac{\partial x}{\partial t} = A(x)\frac{\partial x}{\partial z} + f(x) + g(x)u$$
$$y = h(x), \qquad q = p(x) \tag{1}$$

subject to the boundary condition:

$$C_1 x(a,t) + C_2 x(b,t) = R(t) \tag{2}$$

and the initial condition:

$$x(z,0) = x_0(z), \tag{3}$$

where $x(z,t) = [x_1(z,t) \ \cdots \ x_n(z,t)]^T$ denotes the vector of state variables, $x(z,t) \in \mathcal{H}^n[(a,b), \mathbb{R}^n]$, with $\mathcal{H}^n$ being the infinite-dimensional Hilbert space of $n$-dimensional vector functions defined on the interval $[a,b]$ whose spatial derivatives up to $n$th order are square integrable, $z \in [a,b] \subset \mathbb{R}$ and $t \in [0,\infty)$, denote position and time, respectively, $u(z,t)$ denotes the manipulated variable, $y(z,t)$ denotes the controlled variable, and $q(z,t)$ denotes the measured variable. $A(x)$ is a sufficiently smooth matrix, $f(x)$ and $g(x)$ are sufficiently smooth vector functions, $h(x)$, $p(x)$ are sufficiently smooth scalar functions, $R(t)$ is a column vector that is assumed to be a sufficiently smooth function of time, $x_0(z) \in \mathcal{H}[(a,b), \mathbb{R}^n]$, with $[(a,b), \mathbb{R}^n]$ being the Hilbert space of $n$-dimensional vector functions defined on the interval $[a,b]$ which are square integrable, and $C_1$, $C_2$ are constant matrices of dimension $n \times n$.

The model of Eq. 1 describes the majority of convection-reaction processes arising in chemical engineering (Rhee et al., 1986) and constitutes a natural generalization of *linear* PDE models (see Eq. 9 below), considered in Pell and Aris (1970) and Ray (1981) in the context of linear distributed state estimation and control. The distributed and affine appearance of the manipulated variable $u$ is typical in most practical applications (Pell and Aris, 1970; Gay and Ray, 1995; Ray, 1981), where the jacket temperature is usually selected as the manipulated variable (see below for a detailed discussion on how the jacket temperature is manipulated in practice). Furthermore, the possibility that the system of Eq. 1 may admit boundary conditions at two separate points (e.g., countercurrent processes) is captured by the boundary condition of Eq. 2.

Depending on the eigenvalues of the matrix $A(x)$, the system of Eq. 1 can be hyperbolic, parabolic, or elliptic (Smoller, 1983). Assumption 1, which follows, ensures that the system of Eq. 1 has a well-defined solution and specifies the class of systems considered in this manuscript.

*Assumption 1.* The matrix $A(x)$ is real symmetric and its eigenvalues satisfy:

$$\lambda_1(x) \le \cdots \le \lambda_k(x) < 0 < \lambda_{k+1}(x) \le \cdots \le \lambda_n(x) \tag{4}$$

for all $x \in \mathcal{H}^n[(a,b), \mathbb{R}^n]$.

Typical examples where Assumption 1 is satisfied include heat exchangers, plug-flow reactors, and countercurrent absorbers–reactors where the matrix $A(x)$ is constant and diagonal and its elements are the fluid velocities, as well as chromatography of two interacting solutes where $A(x)$ is a full matrix (Rhee et al., 1986). Systems of the form of Eq. 1 for which the eigenvalues of the matrix $A(x)$ are real and distinct are said to be *hyperbolic*, while systems for which some of the eigenvalues of the matrix $A(x)$ are identically equal are said to be *weakly hyperbolic* (Smoller, 1983).

### *Specification of the control problem*

Consider the system of quasi-linear PDEs of the form of Eq. 1, for which the manipulated variable $u(z,t)$, the measured variable $q(z,t)$, and the controlled variable $y(z,t)$ are distributed in space. Let's assume that for the control of the variable $y$, there exists a finite number of control actuators, $l$, and the same number of measurement sensors; clearly, it is not possible to control the variable $y(z,t)$ at all positions. Therefore, it is meaningful to formulate the control problem as the one of controlling $y(z,t)$ at a finite number of spatial intervals. In particular, referring to the single spatial interval $[z_i, z_{i+1}]$, we suppose that the manipulated input is $\bar{u}^i(t)$, with $\bar{u}^i \in \mathbb{R}$, and the measured output is $\bar{q}^i(t)$, with $\bar{q}^i \in \mathbb{R}$, while the controlled output is $\bar{y}^i(t)$, with $\bar{y}^i \in \mathbb{R}$, such that the following relations hold:

$$u(z,t) = b^i(z)\bar{u}^i(t), \qquad \bar{y}^i(t) = \mathcal{C}^i y(z,t),$$
$$\bar{q}^i(t) = \mathcal{Q}^i p(z,t), \qquad z_i \le z \le z_{i+1}, \tag{5}$$

where $b^i(z)$ is a known smooth function of $z$, and $\mathcal{C}^i$, $\mathcal{Q}^i$ are bounded linear operators, mapping $\mathcal{H}^n$ into $\mathbb{R}$. Figure 1 shows a pictorial representation of this formulation in the case of a prototype example. From a practical point of view, the function $b^i(z)$ describes how the control action $\bar{u}^i(t)$ is distributed in the spatial interval $[z_i, z_{i+1}]$, while the operator $\mathcal{Q}^i$ determines the structure of the sensor in the same spatial interval. Whenever the control action enters the system at a single point $z_0$ (e.g., lateral flow injections), with $z_0 \in [z_i, z_{i+1}]$ (i.e., point actuation), the function $b^i(z)$ is taken to be nonzero in a finite spacial interval of the form $[z_0 - \epsilon, z_0 + \epsilon]$, where $\epsilon$ is a small positive real number, and zero elsewhere in $[z_i, z_{i+1}]$. Similarly, in the case of a point sensor acting at $z_0$, the operator $\mathcal{Q}^i$ is assumed to act in $[z_0 - \epsilon, z_0 + \epsilon]$, and considered to be zero elsewhere. The operator $\mathcal{C}^i$

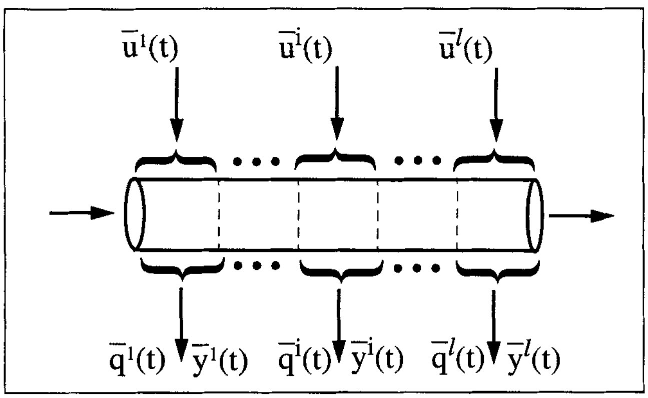


**Figure 1. Control problem specification in the case of a prototype example.**

depends on the desired performance specifications and in the majority of practical applications [see, e.g., Ray (1981), Hanczyc and Palazoglu (1992)], is of the following form:

$$\bar{y}^i(t) = \mathcal{C}^i h(x) = \int_{z_i}^{z_{i+1}} c^i(z) h(x(z,t))\, dz, \tag{6}$$

where $c^i(z)$ is a known smooth function of $z$. For simplicity, the functions $b^i(z)$, $c^i(z)$, $i = 1, \ldots, l$ will be assumed to be normalized in the interval $[a,b]$, that is,

$$\sum_{i=1}^{l} \int_{z_i}^{z_{i+1}} b^i(z)\, dz = \sum_{i=1}^{l} \int_{z_i}^{z_{i+1}} c^i(z)\, dz = 1.$$

Using the relations of Eqs. 5 and 6, the system of Eq. 1 takes the form

$$\begin{aligned} \frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\bar{u} \\ \bar{y} &= \mathcal{C}h(x), \qquad \bar{q} = \mathcal{Q}p(x) \\ C_1 x(a,t) &+ C_2 x(b,t) = R(t) \end{aligned} \tag{7}$$

where

$$\begin{aligned} \bar{u} &= [\bar{u}^1 \cdots \bar{u}^l]^T \\ \bar{y} &= [\bar{y}^1 \cdots \bar{y}^l]^T \\ b(z) &= \{[H(z-z_1) - H(z-z_2)]b^1(z) \\ &\quad \cdots [H(z-z_l) - H(z-z_{l+1})]b^l(z)\} \\ \mathcal{C} &= \{[H(z-z_1) - H(z-z_2)]\mathcal{C}^1 \\ &\quad \cdots [H(z-z_l) - H(z-z_{l+1})]\mathcal{C}^l\}^T \\ \mathcal{Q} &= \{[H(z-z_1) - H(z-z_2)]\mathcal{Q}^1 \\ &\quad \cdots [H(z-z_l) - H(z-z_{l+1})]\mathcal{Q}^l\}^T \end{aligned} \tag{8}$$

with $H(\cdot)$ being the standard Heaviside function.

Referring to the system of Eq. 7, we note that by setting $A(x) = A$, $f(x) = Bx$, $g(x) = w$, $h(x) = kx$, $p(x) = px$, where $A$ and $B$ are matrices, and $w$, $k$, $p$ are vectors of appropriate dimensions, it reduces to the following system of linear first-order hyperbolic PDEs:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx + wb(z)\bar{u} \\ \bar{y} &= \mathcal{C}kx, \qquad \bar{q} = \mathcal{Q}px, \end{aligned} \tag{9}$$

subject to the boundary condition of Eq. 2, and the initial condition of Eq. 3.

The following example will be used throughout the article to illustrate the various aspects of our methodology.

*Example.* Consider a steam-jacketed tubular heat exchanger (Ray, 1981). The dynamic model of the process is of the form:

$$\frac{\partial T}{\partial t} = -v_l \frac{\partial T}{\partial z} - aT + aT_j$$
$$T(0,t) = R(t), \tag{10}$$

where $T(z,t)$ denotes the temperature of the reactor, $z \in [0,1]$, $T_j(z,t)$ denotes the jacket temperature, $v_l$ denotes the fluid velocity in the exchanger, and $a$ is a positive constant. Considering $T_j$ as the manipulated variable, and $T$ as the controlled and measured variable, the preceding model can be put in the form of Eq. 1:

$$\frac{\partial x}{\partial t} = -v_l \frac{\partial x}{\partial z} - ax + au$$
$$y = x, \qquad q = x$$
$$x(0,t) = R(t). \tag{11}$$

Consider the case where these exists one actuator with distribution function $b(z)=1$, the controlled output is assumed to be $\bar{y}(t) = \int_0^1 x(z,t)\,dz$, and there is a point sensor located at $z = 0.5$. Utilizing these relations, the system of Eq. 11 takes the form

$$\frac{\partial x}{\partial t} = -v_l \frac{\partial x}{\partial z} - ax + a\bar{u}(t)$$
$$\bar{y}(t) = \int_0^1 x(z,t)\,dz, \qquad \bar{q} = \int_{0.5-\epsilon}^{0.5+\epsilon} x(z,t)\,dz. \tag{12}$$

△

### *Review of system-theoretic properties*

The objective of this subsection is to review basic system-theoretic properties of systems of first-order hyperbolic PDEs that will be used in the subsequent sections. For more details on these subjects, the reader may refer to Curtain and Pritchard (1978) and Russell (1978). We will start with the definitions of the inner product and the norm, with respect to which the notion of exponential stability for the systems under consideration will be defined.

• Let $\omega_1$, $\omega_2$ be two elements of $\mathcal{H}([a,b];\mathbb{R}^n)$. Then the inner product and the norm, in $\mathcal{H}([a,b];\mathbb{R}^n)$, are defined as follows:

$$(\omega_1, \omega_2) = \int_a^b (\omega_1(z), \omega_2(z))_{\mathbb{R}^n}\,dz$$
$$\|\omega_1\|_2 = (\omega_1, \omega_1)^{1/2}, \tag{13}$$

where the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in the Euclidean space $\mathbb{R}^n$.

Referring to the linear system of Eq. 9, for which Assumption 1 holds, it is well-established (Russell, 1978) that the operator

$$\mathcal{L}x = A\frac{\partial x}{\partial z} + Bx, \tag{14}$$

defined on the domain in $\mathcal{H}[(a,b);\mathbb{R}^n]$ consisting of functions $x \in \mathcal{H}^1[(a,b);\mathbb{R}^n]$ that satisfy the boundary condition of Eq. 2, generates a strongly continuous semigroup $U(t)$ of bounded linear operators on $\mathcal{H}[(a,b);\mathbb{R}^n]$. This fact implies the existence, uniqueness, and continuity of solutions for the system of Eq. 9. In particular, the generalized solution of this system is given by

$$x = U(t)x_0 + \int_0^t U(t-\tau)wb\bar{u}(\tau)\,d\tau + C(t)R, \tag{15}$$

where $C(t)$ is a bounded linear operator for each $t$ mapping $\mathcal{H}[(0,t);\mathbb{R}^n]$ into $\mathcal{H}[(a,b);\mathbb{R}^n]$. The notion of semigroup can be thought of as an analog of the notion of the state transition matrix used for linear finite-dimensional systems. As can be easily seen from Eq. 15, $U(t)$ evolves the initial condition $x_0$ forward in time. From general semigroup theory (Friedman, 1976), it is known that $U(t)$ satisfies the following growth property:

$$\|U(t)\|_2 \le Ke^{at}, \qquad t \ge 0, \tag{16}$$

where $K \ge 1$, $a$ is the largest real part of the eigenvalues of the operator $\mathcal{L}$, and an estimate of $K$, $a$ can be obtained utilizing the Hiller–Yoshida theorem (Friedman, 1976). Whenever the parameter $a$ is strictly negative, we will say that the operator of Eq. 14 generates an exponentially stable semigroup $U(t)$. We note that although there exist many stability concepts for PDEs [e.g., weak (asymptotic) stability (Friedman, 1976; Smoller, 1983)], we will focus throughout the article on exponential stability, because of its robustness to bounded perturbations, which is required in most practical applications, where there is always some uncertainty associated with the process model. The aforementioned concepts allow stating precisely a standard (Pell and Aris, 1970; Balas, 1986) detectability requirement for the system of Eq. 9, which will be exploited in the sixth section for the design of distributed state observers.

*Assumption 2.* The pair $[\mathcal{Q}k_p\ \mathcal{L}]$, is detectable, that is, there exists a bounded linear operator $\mathcal{P}$, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$, such that the linear operator $\mathcal{L}_o = \mathcal{L} - \mathcal{P}\mathcal{Q}k_p$ generates an exponentially stable semigroup.

This detectability assumption does not impose any restrictions on the form of the operator $\mathcal{Q}$ and thus, on the structure of the sensors (e.g., distributed, point sensors).

In closing this subsection, motivated by the lack of general stability results for systems of quasi-linear PDEs, we review a result that allows characterizing the local stability properties of the quasi-linear system of Eq. 7 on the basis of its corresponding linearized system. To this end, let's consider the linearization of the quasi-linear system of Eq. 7:

$$\frac{\partial x}{\partial t} = A(z)\frac{\partial x}{\partial z} + B(z)x + w(z)b(z)\bar{u}$$
$$\bar{y} = \mathcal{C}k(z)x, \qquad \bar{q} = \mathcal{Q}p(z)x, \tag{17}$$

where

$$A(x)=A(x_s(z)),\qquad B(z)=\left(\frac{\partial f(x)}{\partial x}\right)_{x=x_s(z)},$$

$$w(z)=g(x_s(z)),\qquad k(z)=\left(\frac{\partial h(x)}{\partial x}\right)_{x=x_s(z)},$$

$$p(z)=\left(\frac{\partial p(x)}{\partial x}\right)_{x=x_s(z)},$$

and $x_s(z)$ denote some steady-state profile.

*Proposition 1* (Smoller, 1983, p. 121). The system of Eq. 7 (with $\bar{u}=0$), for which Assumption 1 holds, subject to the boundary condition of Eq. 2, is locally exponentially stable if the operator of the linearized system of Eq. 17:

$$\bar{\mathfrak{L}}x = A\frac{\partial x}{\partial z}+B(z)x \tag{18}$$

generates an exponentially stable semigroup.

The following remark provides conditions, which can be easily verified in practice, that guarantee the open-loop stability of hyperbolic PDE systems of the form of Eq. 9 (Eq. 7).

*Remark 1.* Consider the system of linear (quasi-linear) first-order PDEs of the form of Eq. 9 (Eq. 7) with $\bar{u}=0$, and assume that the following conditions hold:

- $\lambda_1(x)\le\lambda_2(x)\le\cdots\le\lambda_n(x)<0$ for all $x\in\mathcal{K}^n[(a,b),\mathbb{R}^n]$.
- $C_2=0$.

In this case, it can be shown (Russell, 1978) that the eigenvalues of the operator of Eq. 18 (Eq. 14) are of the following form:

$$a_\mu=-\infty+\mu\pi i,\qquad \mu=-\infty,\cdots,\infty. \tag{19}$$

Thus, first-order PDE systems that satisfy the preceding conditions (physical examples include plug-flow reactors and cocurrent heat exchangers) possess eigenvalues that lie on a vertical line crossing the real axis at $s=-\infty$, which, according to Eq. 16, implies that they are exponentially stable.

### *Methodological framework*

Motivated by the fact that control methods for quasi-linear distributed parameter systems should explicitly account for their nonlinear and spatially varying nature, our methodology entails the following two steps:

1. Synthesize distributed nonlinear state-feedback controllers that enforce output tracking and derive conditions that guarantee the exponential stability of the closed-loop system.
2. Synthesize distributed nonlinear output-feedback controllers through a combination of the developed state-feedback controllers with appropriate distributed state observers.

Motivated by the mathematical properties of these systems, the state-feedback control problem is solved on the basis of the original PDE model, by following an approach conceptually similar to the one used for the synthesis of inversion-based controllers for ODE systems. In order to motivate the approach followed for the quasi-linear case and identify theoretical analogies between our approach and available results on feedback stabilization of linear hyperbolic PDEs, we will also present the development for the case of systems of linear PDEs of the form of Eq. 9. The development for the case of quasi-linear systems will be performed by essentially generalizing the results developed for the linear case in a nonlinear context.

## Characteristic Index

In this section, we will introduce the concept of characteristic index between the output $\bar{y}$ and the input $\bar{u}$ for systems of the form of Eq. 9, which will allow us to formulate and solve the state-feedback control problem. To reveal the origin and illustrate the role of this concept, we consider the operation of differentiation of the output $\bar{y}^i$ of the system of Eq. 9 with respect to time, which yields

$$\begin{aligned}\bar{y}^i&=\mathcal{C}^i kx\\ \frac{d\bar{y}^i}{dt}&=\frac{d}{dt}\mathcal{C}^i kx=\mathcal{C}^i k\frac{\partial x}{\partial t}\\ &=\mathcal{C}^i k\left(A\frac{\partial}{\partial z}+B\right)x+\mathcal{C}^i kwb^i(z)\bar{u}^i.\end{aligned} \tag{20}$$

Now, if the scalar $\mathcal{C}^i kwb^i(z)$ is nonzero, we will say that the characteristic index of $\bar{y}^i$ with respect to $\bar{u}^i$, denoted by $\sigma^i$, is equal to one. If $\mathcal{C}^i kwb^i(z)=0$, the characteristic index is greater than one, and from Eq. 20 we have

$$\frac{d\bar{y}^i}{dt}=\mathcal{C}^i k\left(A\frac{\partial}{\partial z}+B\right)x. \tag{21}$$

Performing one more time-differentiation, we obtain:

$$\frac{d^2\bar{y}^i}{dt^2}=\mathcal{C}^i k\left(A\frac{\partial}{\partial z}+B\right)^2 x+\mathcal{C}^i k\left(A\frac{\partial}{\partial z}+B\right)wb^i(z)\bar{u}^i. \tag{22}$$

In analogy with the preceding, if the scalar $\mathcal{C}^i k[A(\partial/\partial z)+B]wb^i(z)$ is nonzero, the characteristic index is equal to two, while if $\mathcal{C}^i k[A(\partial/\partial z)+B]wb^i(z)=0$, the characteristic index is greater than two.

Generalizing the preceding development, one can give the definition of characteristic index for systems of the form of Eq. 9.

*Definition 1.* Referring to the system of linear first-order PDEs of the form of Eq. 9, we define the characteristic index of the output $\bar{y}^i$ with respect to the input $\bar{u}^i$ as the smallest integer $\sigma^i$ for which

$$\mathcal{C}^i k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma^i-1}wb^i(z)\ne 0, \tag{23}$$

or $\sigma^i=\infty$ if such an integer does not exist.

*Remark 2.* According to Definition 1, the characteristic index is the smallest order time-derivative of the output $\bar{y}^i$ that explicitly depends on the manipulated input $\bar{u}^i$. In this sense, it can be thought of as a natural generalization of the concept of relative order for the systems under consideration.

For the case of linear ODE systems, the relative order can be interpreted as the difference in the degree of the denominator polynomial and numerator polynomial. Such an interpretation cannot be given for the concept of characteristic index because the frequency-domain representation of the system of Eq. 9 typically gives rise to transfer functions that involve complicated transcendental forms.

In analogy with the linear case, the following concept of characteristic index is introduced for the quasi-linear PDE system of Eq. 7.

*Definition 2.* Referring to the system of quasi-linear first-order PDEs of the form of Eq. 7, we define the characteristic index of the output $\bar{y}^i$ with respect to the input $\bar{u}^i$ as the smallest integer $\sigma^i$ for which

$$\mathcal{C}^i L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j}+L_f\right)^{\sigma^i-1}h(x)b^i(z)\not\equiv 0, \quad (24)$$

where $a_j$ denotes the $j$th column vector of the matrix $A(x)$, and $L_{a_j}$, $L_f$ denote the standard Lie derivative notation, or $\sigma^i=\infty$ if such an integer does not exist.

Throughout the article, it will be assumed that Eq. 24 holds for all $x\in\mathcal{H}^n$, $z\in[a,b]$.

From Definitions 1 and 2, one can immediately see that the characteristic index $\sigma^i$ depends on the structural properties of the process (the matrices $A$, $B$ and the vectors $w$, $k$ for the linear case, or the matrix $A(x)$ and the functions $f(x)$, $g(x)$, $h(x)$ for the quasi-linear case), as well as on the selection of the control system and objectives (the functions $b^i(z)$ and the output operators $\mathcal{C}^i$). Note that in the control problem specification of subsection titled "Specification of the Control Problem," we have implicitly assumed that $\mathcal{C}^i$ and $b^i(z)$ are chosen to act in the same spatial interval (collocated); in the case where $\mathcal{C}^i$ and $b^i(z)$ are chosen to act in different spatial intervals (noncollocated), it follows directly from Eqs. 23 and 24 that the characteristic index $\sigma^i=\infty$, which implies that this selection leads to loss of controllability of the output $\bar{y}^i$ from the input $\bar{u}^i$.

In most practical applications, the selection of $(b^i(z),\mathcal{C}^i)$ is typically consistent for all pairs $(\bar{y}^i,\bar{u}^i)$, in a sense that is made precise in the following assumption.

*Assumption 3.* Referring to the system of first-order PDEs of the form of Eq. 9 (Eq. 7), $\sigma^1=\sigma^2=\cdots=\sigma^l=\sigma$.

Given this assumption, $\sigma$ can be also thought of as the characteristic index between the output vector $\bar{y}$ and the input vector $\bar{u}$.

## State-Feedback Control

### *Linear systems*

In this subsection, we focus on systems of linear first-order PDEs of the form of Eq. 9 and address the problem of synthesizing a distributed state-feedback controller that forces the output of the closed-loop system to track a reference input in a prespecified manner. More specifically, we consider distributed state-feedback laws of the form:

$$\bar{u}=\mathcal{S}x+sv, \quad (25)$$

where $\mathcal{S}$ is a linear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, $s$ is an invertible diagonal matrix of functionals, and $v\in\mathbb{R}^l$ is the vector of reference inputs. The structure of the control law of Eq. 25 is motivated by available results on stabilization of linear PDEs systems via distributed state feedback (e.g., Wang, 1966; Balas, 1986) and the requirement of output tracking. Substituting the distributed state-feedback law of Eq. 25 into the system of Eq. 9, the following closed-loop system is obtained:

$$\frac{\partial x}{\partial t}=A\frac{\partial x}{\partial z}+Bx+wb(z)\mathcal{S}x+wb(z)sv$$
$$\bar{y}=\mathcal{C}kx. \quad (26)$$

It is clear that feedback laws of the form of Eq. 25 preserve the linearity with respect to the reference input vector $v$. We also note that the evolution of the linear PDE system of Eq. 26 is governed by a strongly continuous semigroup of bounded linear operators, because $\mathcal{L}$ generates a strongly continuous semigroup and $b(z)\mathcal{S}x$, $b(z)sv$ are bounded, finite-dimensional perturbations (Friedman, 1976), ensuring that the closed-loop system has a well-defined solution (see the subsection titled "Review of System-Theoretic Properties"). Proposition 2 that follows allows specifying the order of the input/output response in the closed-loop system (the proof is given in the Appendix).

*Proposition 2.* Consider the system of linear first-order PDEs of Eq. 9 subject to the boundary condition of Eq. 2, for which Assumptions 1 and 3 hold. Then, a distributed state-feedback control law of the form of Eq. 25 preserves the characteristic index $\sigma$, in the sense that the characteristic index of $\bar{y}$ with respect to $v$ in the closed-loop system of Eq. 26 is equal to $\sigma$.

The fact that the characteristic index between the output $\bar{y}$ and the reference input $v$ is equal to $\sigma$ suggests requesting the following input/output response for the closed-loop system:

$$\gamma_\sigma\frac{d^\sigma\bar{y}}{dt^\sigma}+\cdots+\gamma_1\frac{d\bar{y}}{dt}+\bar{y}=v, \quad (27)$$

where $\gamma_1,\gamma_2,\ldots,\gamma_\sigma$ are adjustable parameters. These parameters can be chosen to guarantee input/output stability and enforce desired performance specifications in the closed-loop system. Referring to Eq. 27, note that, motivated by physical arguments, we request, for each pair $(\bar{y}^i,v^i)$, $i=1,\ldots,l$, an input/output response of order $\sigma$ with the same transient characteristics [i.e., the parameters $\gamma_k$ are chosen to be the same for each pair $(\bar{y}^i,v^i)$]. This requirement can be readily relaxed if necessary to impose responses with different transient characteristics for the various pairs $(\bar{y}^i,v^i)$.

We are now in a position to state the main result of this subsection in the form of a theorem (the proof can be found in the Appendix).

*Theorem 1.* Consider the system of linear first-order PDEs of Eq. 9 subject to the boundary condition of Eq. 2, for which Assumptions 1 and 3 hold. Then, the distributed state-feedback law:

$$\bar{u} = \left[ \gamma_\sigma \mathcal{C} k \left( A \frac{\partial}{\partial z} + B \right)^{\sigma - 1} wb(z) \right]^{-1} \times \left\{ v - \mathcal{C} kx - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C} k \left( A \frac{\partial}{\partial z} + B \right)^{\nu} x \right\} \quad (28)$$

enforces the input/output response of Eq. 27 in the closed-loop system.

*Remark 3.* Referring to the controller of Eq. 28, it is clear that the calculation of the control action requires algebraic manipulations as well as differentiations and integrations in space, which is expected because of the distributed nature of the controller.

*Remark 4.* The distributed state-feedback controller of Eq. 28 was derived following an approach conceptually similar to the one employed for the synthesis of inversion-based controllers for ODE systems. We note that this is possible, because, for the system of Eq. 9, (a) the solution is well-defined (i.e., the evolution of the state is locally governed by a strongly continuous semigroup of bounded linear operators); (b) the input/output spaces are finite dimensional; and (c) the manipulated input and the controlled output are distributed in space. These three requirements are standard in most control theories for PDE systems (e.g., Balas, 1986, 1991) and only the third one poses some practical limitations excluding processes where the manipulated input appears in the boundary.

*Remark 5.* The class of distributed state-feedback laws of Eq. 25 is a generalization of control laws of the form

$$\bar{u} = \mathfrak{F} x, \quad (29)$$

where $\mathfrak{F}$ is a bounded linear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, which are used for the stabilization of linear PDEs. The usual approach followed for the design of the gain operator $\mathfrak{F}$ utilizes optimal control methods (e.g., Lo, 1973; Ray, 1981).

*Example (Continued).* In the case of the heat exchanger example introduced earlier, it can be easily verified that the characteristic index of the system of Eq. 12 is equal to one. Therefore, a first-order input/output response is requested in the closed-loop system:

$$\gamma_1 \frac{d\bar{y}}{dt} + \bar{y} = v. \quad (30)$$

Using the result of Theorem 1, the appropriate control law that enforces this response is

$$\bar{u} = \frac{1}{\gamma_1} \left\{ v - \int_0^1 x(z,t)\, dz - \gamma_1 \int_0^1 \left[ -v_l \frac{\partial x}{\partial z}(z,t) - ax(z,t) \right] dz \right\}. \quad (31)$$

### *Quasi-linear systems*

In this subsection, we consider systems of quasi-linear first-order PDEs of the form of Eq. 7 and control laws of the form:

$$\bar{u} = \bar{\mathbb{S}}(x) + \bar{s}(x) v, \quad (32)$$

where $\bar{\mathbb{S}}(x)$ is a nonlinear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, $\bar{s}(x)$ is an invertible diagonal matrix of functionals, and $v \in \mathbb{R}^l$ is the vector of reference inputs. The class of control laws of Eq. 32 is a natural generalization of the class of control laws considered for the case of linear systems (Eq. 25). Under the control law of Eq. 32, the closed-loop system takes the form:

$$\frac{\partial x}{\partial t} = A(x) \frac{\partial x}{\partial z} + f(x) + g(x) b(z) \bar{\mathbb{S}}(x) + g(x) b(z) \bar{s}(x) v$$
$$\bar{y} = \mathcal{C} h(x). \quad (33)$$

It is straightforward to show that the preceding system has locally a well-defined solution, and the counterpart of Proposition 2 also holds, that is, the characteristic index of the output $\bar{y}$ with respect to $v$ in the closed-loop system of Eq. 33 is equal to $\sigma$, which suggests seeking a linear input/output response of the form of Eq. 27 in the closed-loop system. Theorem 2 that follows states the controller synthesis result for this case.

*Theorem 2.* Consider the system of quasi-linear first-order PDEs of Eq. 7 subject to the boundary condition of Eq. 2, for which Assumptions 1 and 3 hold. Then, the distributed state-feedback law:

$$\bar{u} = \left[ \gamma_\sigma \mathcal{C} L_g \left( \sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma - 1} h(x) b(z) \right]^{-1} \times \left\{ v - \mathcal{C} h(x) - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C} \left( \sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\nu} h(x) \right\} \quad (34)$$

enforces the input/output response of Eq. 27 in the closed-loop system.

*Remark 6.* Theorem 2 provides an analytical formula of a distributed nonlinear state-feedback controller that enforces a linear input/output response in the closed-loop system. In this sense, the controller of Eq. 34 can be viewed as a counterpart of input/output linearizing control laws for nonlinear ODE systems (see Kravaris and Arkun, 1991, and the references therein), in the case of infinite-dimensional systems of the form of Eq. 7.

## Closed-Loop Stability

The goal of this section is to define a concept of zero dynamics and the associated notion of minimum phaseness for systems of first-order hyperbolic PDEs of the form of Eq. 9 (Eq. 7), subject to the boundary conditions of Eq. 2; this will allow us to state conditions that guarantee exponential stability of the closed-loop system. We will initially define the concept of zero dynamics for the case of linear systems (the definition for the case of quasi-linear systems is completely similar and will be omitted for brevity). Our definition is analogous with the one given in Byrnes et al. (1994) [see also Pohjolainen (1981)] for the case of linear parabolic PDE systems with boundary feedback control.

*Definition 3.* The zero dynamics associated with the system of linear first-order PDEs of Eq. 9 is the system obtained by constraining the output to zero, that is, the system:

$$\frac{\partial x}{\partial t} = A\frac{\partial x}{\partial z} + Bx - wb(z)\left[\mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \times \left\{\mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\sigma} x\right\} \quad (35)$$

$$\mathcal{C}kx \equiv 0$$

$$C_1x(a,t) + C_2x(b,t) = R(t).$$

From Definition 3, it is clear that the dynamical system that describes the zero dynamics is an infinite-dimensional one. The concept of zero dynamics allows us to define a notion of minimum-phaseness for systems of the form of Eq. 9. More specifically, if the zero dynamics is exponentially stable, the system of Eq. 9 is said to be minimum phase, while if the zero dynamics is unstable, the system of Eq. 9 is said to be nonminimum phase.

We have now introduced the necessary elements that will allow us to address the issue of closed-loop stability. Proposition 3 that follows provides conditions that guarantee the exponential stability of the closed-loop system (the proof can be found in the Appendix).

*Proposition 3.* Consider the system of linear first-order PDEs of Eq. 9 for which Assumptions 1 and 3 hold, under the controller of Eq. 28. Then, the closed-loop system is exponentially stable (i.e., the differential operator of the closed-loop system generates an exponentially stable semigroup) if the following conditions are satisfied:

1. The roots of the equation

$$1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0 \quad (36)$$

lie in the open left-half of the complex plane.

2. The system of Eq. 35 is exponentially stable.

*Remark 7.* Referring to the preceding proposition, we note that the first condition addresses the input/output stability of the closed-loop system and the second condition addresses its internal stability. Note also that the first condition is associated with the stability of a *finite* number of poles, while the second condition concerns the stability of an *infinite* number of poles. This is expected since the input/output spaces are finite dimensional, while the state of the system evolves in infinite dimensions.

*Remark 8.* From the result of Proposition 3, it follows that the controller of Theorem 1 places a finite number of poles of the open-loop infinite-dimensional system of Eq. 9 at prespecified (depending on the choice of parameters $\gamma_k$) locations, by essentially canceling an infinite number of poles, those included in the zero dynamics. Furthermore, the closed-loop system is exponentially stable if the zero dynamics of the original system is exponentially stable (Condition 2 of Proposition 3). This result is analogous to available results of stabilization of systems of linear PDEs of the form of Eq. 9 with feedback law of the form of Eq. 29. Specifically, it is well known (Russell, 1978; Balas, 1986) that control laws of the form of Eq. 29 allow placing a finite number of open-loop poles at prespecified locations, while also guaranteeing the exponentially stability of the closed-loop system, if the pair $[\mathcal{L}\ wb(z)]$ is stabilizable (i.e., the remaining infinite uncontrolled poles are in the open left-half of the complex plane).

*Remark 9.* From the result of Proposition 3 and the discussion of Remark 8, it is clear that the derivation of exponential-stability results for the closed-loop system under the control law of Eq. 25 (or the control law of Eq. 29) requires that the open-loop system be minimum phase (or stabilizable). The *a priori* verification of these properties can in principle be performed by utilizing spectral theory for operators in infinite-dimensions (Friedman, 1976; Pohjolainen, 1981). However, these calculations are difficult to perform in the majority of practical applications. In practice, the stabilizability and minimum-phase properties can be checked through simulations.

In closing this section, we address the issue of closed-loop stability for systems of quasi-linear PDEs. Proposition 4 that follows provides the counterpart of the result of Proposition 3 for the case of quasi-linear systems.

*Proposition 4.* Consider the system of quasi-linear first-order PDFs of Eq. 7 for which Assumptions 1 and 3 hold, under the controller of Eq. 34. Then, the closed-loop system is locally exponentially stable (i.e., the differential operator of the linearized closed-loop system generates an exponentially stable semigroup) if the following conditions are satisfied:

1. The roots of the equation

$$1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0 \quad (37)$$

lie in the open left-half of the complex plane.

2. The zero dynamics of the system of Eq. 7 is locally exponentially stable.

*Remark 10.* The exponential stability of the closed-loop system guarantees, in both the linear and the quasi-linear case, that in the presence of small modeling errors, the states of the closed-loop system will be bounded. Furthermore, since the input/output spaces of the closed-loop system are finite dimensional, and the controller of Eq. 34 enforces a linear input/output dynamics between $\bar{y}$ and $v$, it is possible to implement a linear error feedback controller with integral action around the $(\bar{y} - v)$ loop to ensure asymptotic offsetless output tracking in the closed-loop system, in the presence of constant unknown model parameters and disturbances.

## Output-Feedback Control

In this section, we will consider the synthesis of distributed output-feedback controllers for systems of the form of Eq. 9 (Eq. 7). The requisite controllers will be synthesized employing a combination of the developed distributed state-feedback controllers with distributed state observers. Analysis of the resulting closed-loop system allows us to derive precise conditions that guarantee that the requirements of exponential stability and output tracking are enforced in the closed-loop system.

The conventional approach followed for the design of state estimators for linear PDE systems is to discretize the system equations and then apply results from estimation theory for ODE systems [e.g., Sorensen et al. (1980)]. It has been shown, however, that methods for state estimation that treat the full distributed parameter system lead to state observers that yield significantly superior performance (Ray, 1981; Cooper et al., 1986). In this direction, available results on state estimation for systems of first-order hyperbolic PDEs concern mainly the use of Kalman filtering theory for the design of distributed state observers (Pell and Aris, 1970; Yu et al., 1974).

### *Linear systems*

We consider state observers with the following general state-space description (Pell and Aris, 1970):

$$\frac{\partial \eta}{\partial t} = A\frac{\partial \eta}{\partial z} + B\eta + wb(z)\bar{u} + \mathcal{P}(\bar{q} - \mathcal{Q}p\eta), \tag{38}$$

where $\mathcal{P}$ is a bounded linear operator, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$, that has to be designed so that the operator $\mathcal{L}_o = \mathcal{L} - \mathcal{P}\mathcal{Q}k$ generates an exponentially stable semigroup (note that this is possible by Assumption 2). The system of Eq. 38 consists of a replica of the process system and the term $\mathcal{P}(\bar{q} - \mathcal{Q}p\eta)$ used to enforce a fast decay of the discrepancy between the estimated and the actual values of the states of the system. In practice, the design of the operator $\mathcal{P}$ can be performed by (a) simple pole placement in the case where the output measurements are not corrupted by noise, or (b) Kalman filtering theory, in the case where the output measurements are noisy.

Theorem 3 that follows provides a state-space realization of the output-feedback controller resulting from the combination of the state observer of Eq. 38 with the state feedback controller of Eq. 28 (the proof can be found in the Appendix).

*Theorem 3.* Consider the system of linear first-order PDEs of Eq. 9 subject to the boundary condition of Eq. 2, for which Assumptions 1, 2, 3, and the conditions of Proposition 3 hold. Consider also the linear bounded operator $\mathcal{P}$ designed such that the operator $\mathcal{L}_o = \mathcal{L} - \mathcal{P}\mathcal{Q}p$ generates an exponentially stable semigroup. Then, the distributed output-feedback controller

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= A\frac{\partial \eta}{\partial z} + B\eta + wb(z)\left[\gamma_\sigma \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \\ &\quad \times \left\{v - \mathcal{C}k\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\nu}\eta\right\} + \mathcal{P}(\bar{q} - \mathcal{Q}p\eta) \\ \bar{\mu} &= \left[\gamma_\sigma \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \\ &\quad \times \left\{v - \mathcal{C}k\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\nu}\eta\right\}\end{aligned} \tag{39}$$

(a) Guarantees exponential stability of the closed-loop system

(b) Enforces the input/output response of Eq. 27 in the closed-loop system if $x(z,0) = \eta(z,0)$.

*Remark 11.* In the case of open-loop stable systems, a more convenient way to reconstruct the state of the system is to consider the observer of Eq. 38 with the operator $\mathcal{P}$ set identically equal to zero. This is motivated by the fact that the open-loop stability of the system guarantees the convergence of the estimated values to the actual ones with transient behavior depending on the location of the spectrum of the operator of Eq. 14.

*Remark 12.* Available results on stabilization of systems of linear hyperbolic PDEs via distributed output feedback [e.g., Ray (1981), Balas (1986)] concern the design of controllers with the following general state-space description:

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= A\frac{\partial \eta}{\partial z} + [B + wb(z)\mathcal{F}]\eta + \mathcal{R}(\bar{q} - \mathcal{Q}p\eta) \\ \bar{u} &= \mathcal{F}\eta,\end{aligned} \tag{40}$$

where $\mathcal{R}$ is a bounded linear operator, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$. We note that the main similarity between a controller of the form of Eq. 40 and the controller of Eq. 39 is that both are infinite dimensional (because of the state observers utilized), while their main difference lies in the fact that the controller of Eq. 39 guarantees exponential stability of the closed-loop system, if the open-loop system is minimum-phase and detectable, while a controller of the form of Eq. 40 will exponentially stabilize the closed-loop system, if the open-loop system is jointly stabilizable/detectable (Balas, 1986).

*Example (Continued).* Referring to the linear PDE system of Eq. 12, we note that $\lambda = -v_l < 0$, while $C_2 = 0$, and thus the system is open-loop stable according to the result of Remark 1. The open-loop stability of the system allows using a feedback controller, which consists of the distributed state-feedback controller coupled with an open-loop observer. The appropriate controller takes the form:

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= -v_l\frac{\partial \eta}{\partial z} - a\eta + a\frac{1}{\gamma_1} \\ &\quad \times \left\{v - \int_0^1 \eta(z,t)\,dz - \gamma_1\int_0^1\left(-v_l\frac{\partial \eta}{\partial z}(z,t) - a\eta(z,t)\right)dz\right\} \\ \bar{u} &= \frac{1}{\gamma_1}\left\{v - \int_0^1 \eta(z,t)\,dz - \gamma_1\right. \\ &\quad \left.\times \int_0^1\left(-v_l\frac{\partial \eta}{\partial z}(z,t) - a\eta(z,t)\right)dz\right\}.\end{aligned} \tag{41}$$

### *Quasi-linear systems*

In this subsection, we consider the synthesis of distributed output-feedback controllers for systems of the form of Eq. 7. Given the lack of available general results on state estimation of such systems, we will proceed with the design of a nonlinear state observer that guarantees local exponential convergence of the state estimates of the actual state values. In particular, the following state observer will be used to estimate the state vector of the system in space and time:

$$\frac{\partial \eta}{\partial t} = A(\eta)\frac{\partial \eta}{\partial z} + f(\eta) + g(\eta)b(z)\bar{u} + \bar{\mathcal{P}}(\bar{q} - \mathcal{Q}p(\eta)), \tag{42}$$

where $\eta$ denotes the observer state vector and $\bar{\mathcal{P}}$ is a linear operator, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$, designed on the basis of the linearization of the system of Eq. 42 so that the eigenvalues of the operator $\bar{\mathcal{L}}_o = \bar{\mathcal{L}} - \bar{\mathcal{P}}\mathcal{Q}p(z)$ lie in the left-half plane.

The state observer of Eq. 42 can be coupled with the state-feedback controller of Eq. 34 to derive an output-feedback controller that guarantees output tracking and closed-loop stability. The resulting controller is given in Theorem 4 that follows (the proof is given in the Appendix).

*Theorem 4.* Consider the system of quasi-linear first-order PDEs of Eq. 7 subject to the boundary condition of Eq. 2, for which Assumptions 1, 2, 3, and the conditions of Proposition 4 hold. Consider also the bounded operator $\bar{\mathcal{P}}$ designed such that the operator $\bar{\mathcal{L}}_o = \bar{\mathcal{L}} - \bar{\mathcal{P}}\mathcal{Q}p(z)$ generates an exponentially stable semigroup. Then, the distributed output feedback controller

$$\frac{\partial \eta}{\partial t} = A(\eta)\frac{\partial \eta}{\partial z} + f(\eta) + g(\eta)b(z)$$
$$\times \left[\gamma_\sigma \mathcal{C} L_g \left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\sigma-1} h(\eta) b(z)\right]^{-1}$$
$$\times \left\{ v - \mathcal{C}h(\eta) - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}\left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\nu} h(\eta)\right\}$$
$$+ \bar{\mathcal{P}}(\bar{q} - \mathcal{Q}p(\eta))$$
$$\bar{u} = \left[\gamma_\sigma \mathcal{C} L_g \left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\sigma-1} h(\eta) b(z)\right]^{-1}$$
$$\times \left\{ v - \mathcal{C}h(\eta) - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}\left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\nu} h(\eta)\right\} \quad (43)$$

(a) Guarantees local exponential stability of the closed-loop system
(b) Enforces the input/output response of Eq. 27 in the closed-loop system if $x(z,0) = \eta(z,0)$.

In analogy with the linear case, for open-loop stable systems, the operator $\bar{P}$ can be taken to be identically equal to zero, since the local exponential stability of the open-loop system guarantees the local convergence to the estimated values to the actual values.

*Remark 13.* Note that in the case of imperfect initialization of the observer states (i.e., $\eta(z,0) \neq x(z,0)$), although a slight deterioration of the performance may occur (i.e., the input/output response of Eq. 27 will not be exactly imposed in the closed-loop system), the output-feedback controllers of Theorems 3 and 4 guarantee exponential stability and asymptotic output tracking in the closed-loop system.

*Remark 14.* The nonlinear distributed output feedback controller of Eq. 43 is an infinite-dimensional one, due to the infinite-dimensional nature of the observer of Eq. 42. Therefore, a finite-dimensional approximation of the controller has to be derived for on-line implementation. This task can be performed utilizing standard discretization techniques such as finite differences and orthogonal collocation. It is expected that some performance deterioration will occur in this case, depending on the discretization method used and the number and location of discretization points (see the chemical reactor application presented in the next section). Finally, we note that it is well established (e.g., Balas, 1986) that as the number of discretization points increases, the closed-loop system resulting from the PDE model plus an approximate finite-dimensional controller converges to the closed-loop system resulting from the PDE model plus the infinite-dimensional controller, guaranteeing the well-posedness of the approximate finite-dimensional controller.

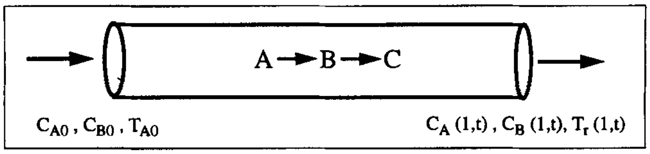


**Figure 2. Nonisothermal plug-flow reactor.**

## Application to a Nonisothermal Plug-Flow Reactor

### *Process description*

Consider the nonisothermal plug-flow reactor shown in Figure 2 where two first-order reactions in series take place:

$$A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C,$$

where $A$ is the reactant species, $B$ is the desired product, and $C$ is an undesired product. The inlet stream consists of pure $A$ of concentration $C_{A0}$ and temperature $T_{A0}$. The reactions are endothermic, and a jacket is used to heat the reactor. The reaction rate expressions are assumed to be of the following form:

$$r_1 = -k_{10}e^{-E_1/RT_r}C_A$$
$$r_2 = -k_{20}e^{-E_2/RT_r}C_B,$$

where $k_{10}$, $k_{20}$, $E_1$, $E_2$ denote the pre-exponential constants and the activation energies of the reactions. Under the following assumptions:

- Perfect radial mixing in the reactor
- Constant volume of the liquor in the reactor
- Constant density and heat capacity of the reacting liquid
- Negligible diffusive and dispersive phenomena

the material and energy balances that describe the dynamical behavior of the process take the following form:

- Mole balance for the species $A$

$$\frac{\partial C_A}{\partial t} = -v_l\frac{\partial C_A}{\partial z} - k_{10}e^{-E_1/RT_r}C_A.$$

- Mole balance for the species $B$

$$\frac{\partial C_B}{\partial t} = -v_l\frac{\partial C_B}{\partial z} + k_{10}e^{-E_1/RT_r}C_A - k_{20}e^{-E_2/RT_r}C_B.$$

- Reactor energy balance

$$\frac{\partial T_r}{\partial t} = -v_l\frac{\partial T_r}{\partial z} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{-E_1/RT_r}C_A$$
$$+ \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{-E_2/RT_r}C_B + \frac{U_w}{\rho_m c_{pm} V_r}(T_j - T_r)$$

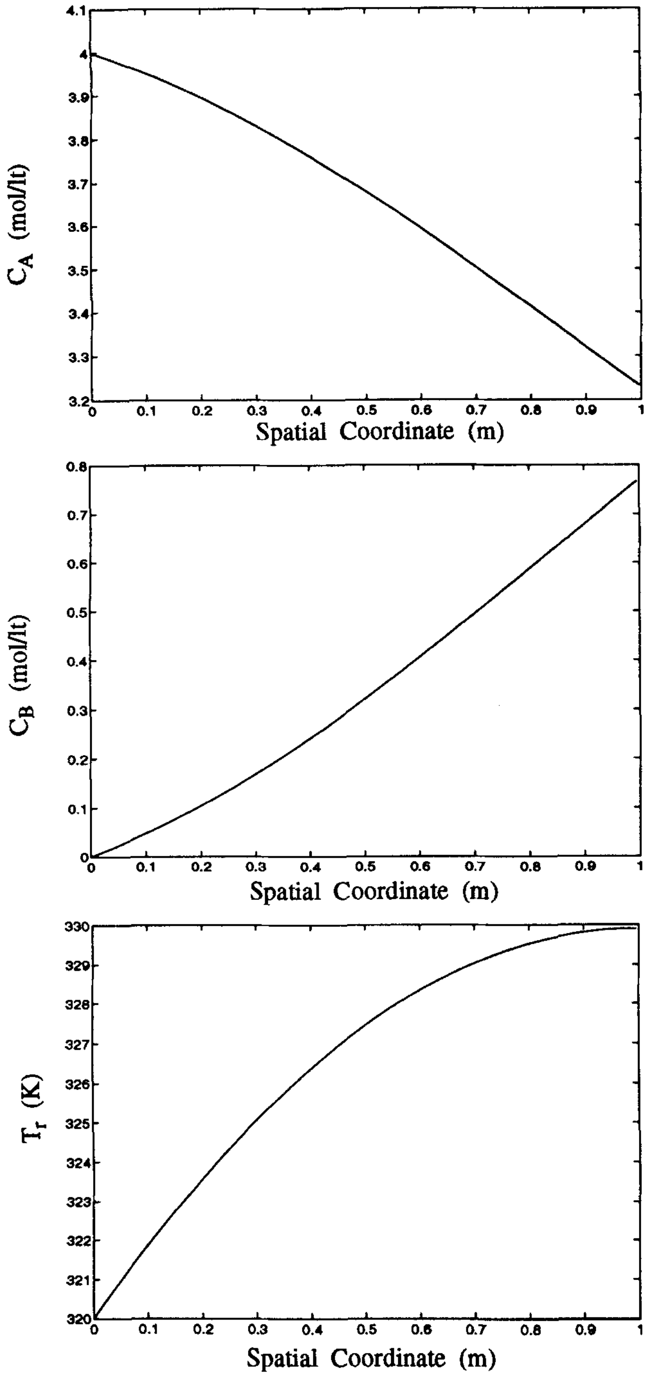


**Figure 3. Steady-state profiles of reactor state variables.**

subject to the following boundary conditions:

$$C_A(0,t)=C_{A0}, \qquad C_B(0,t)=0, \qquad T_r(0,t)=T_{A0},$$

where $C_A$ and $C_B$ denote the concentrations of the species $A$ and $B$ in the reactor; $T_r$ denotes the temperature of the reactor $C_{A_s}$, $C_{B_s}$, $T_{r_s}$ denote the steady-state profiles for the state variables; $\Delta H_{r_1}$, $\Delta H_{r_2}$ denote the enthalpies of the two reactions; $\rho_m$, $c_{pm}$ denote the density and heat capacity of the

**Table 1. Process Parameters**

| | |
|---|---|
| $v_l = 1.0$ | $\mathrm{m \cdot min^{-1}}$ |
| $L = 1.0$ | m |
| $V_r = 10.0$ | lt |
| $E_1 = 2.0\times10^4$ | $\mathrm{kcal \cdot kmol^{-1}}$ |
| $E_2 = 5.0\times10^4$ | $\mathrm{kcal \cdot kmol^{-1}}$ |
| $k_{10} = 5.0\times10^{12}$ | $\mathrm{min^{-1}}$ |
| $k_{20} = 5.0\times10^6$ | $\mathrm{min^{-1}}$ |
| $R = 1.987$ | $\mathrm{kcal \cdot kmol^{-1} \cdot K^{-1}}$ |
| $\Delta H_1 = 548{,}000.1$ | $\mathrm{kcal \cdot kmol^{-1}}$ |
| $\Delta H_2 = 986{,}000.1$ | $\mathrm{kcal \cdot kmol^{-1}}$ |
| $c_{pm} = 0.231$ | $\mathrm{kcal \cdot kg^{-1} \cdot K^{-1}}$ |
| $\rho_m = 0.09$ | $\mathrm{kg \cdot lt^{-1}}$ |
| $c_{pj} = 2.5$ | $\mathrm{kcal \cdot kg^{-1} \cdot K^{-1}}$ |
| $\rho_j = 1.44$ | $\mathrm{kg \cdot lt^{-1}}$ |
| $U_w = 2{,}000.0$ | $\mathrm{kcal \cdot min^{-1} \cdot K^{-1}}$ |
| $C_{A0} = 4.0$ | $\mathrm{mol \cdot lt^{-1}}$ |
| $C_{B0} = 0.0$ | $\mathrm{mol \cdot lt^{-1}}$ |
| $T_{A0} = 320.0$ | K |
| $T_{j0} = 350.0$ | K |
| $V_j^i = 1.0$ | lt |
| $\gamma_{jc} = 0.001$ | min |
| $F_{js}^1 = 28.87$ | $\mathrm{lt \cdot min^{-1}}$ |
| $F_{js}^2 = 43.66$ | $\mathrm{lt \cdot min^{-1}}$ |
| $F_{js}^3 = 59.63$ | $\mathrm{lt \cdot min^{-1}}$ |
| $F_{js}^4 = 72.34$ | $\mathrm{lt \cdot min^{-1}}$ |
| $F_{js}^5 = 77.23$ | $\mathrm{lt \cdot min^{-1}}$ |

fluid in the reactor; $V_r$ denotes the volume of the reactor; $U_w$ denotes the heat-transfer coefficient; $T_j$ denotes the spatially uniform temperature in the jacket; and $C_{A0}$ and $T_{A0}$ denote the concentration and temperature of the inlet stream in the reactor. The values used for process parameters are given in Table 1, while the corresponding steady-state profiles are shown in Figure 3. The control objective is the regulation of the concentration of the species $B$ throughout the reactor by manipulating the jacket temperature $T_j$. We note that, in practice, $T_j$ is usually manipulated indirectly through manipulation of the jacket inlet flow rate (this implementation issue is addressed in the subsection titled "Practical Implementation Issues").

Setting:

$$u=T_j-T_{j_s}, \qquad x_1=C_A, \qquad x_2=C_B, \qquad x_3=T_r, \qquad y=C_B \tag{44}$$

the original set of equations can be put in the form of Eq. 1 with:

$$f(x)=\begin{bmatrix} f_1(x_1,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} -k_{10}e^{-E_1/Rx_3}x_1 \\ k_{10}e^{-E_1/Rx_3}x_1 - k_{20}e^{-E_2/Rx_3}x_2 \\ \dfrac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{-E_1/Rx_3}x_1 \\ +\dfrac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{-E_2/Rx_3}x_2 - \dfrac{U_w}{\rho_m c_{pm} V_r}(x_3-T_{j_s}) \end{bmatrix}$$

$$A(x_1,x_2,x_3)=[a_1\ a_2\ a_3]=\begin{bmatrix}-v_l & 0 & 0\\ 0 & -v_l & 0\\ 0 & 0 & -v_l\end{bmatrix},$$

$$g(x)=\begin{bmatrix}g_1\\ g_2\\ g_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ U_w/\rho_m c_{pm} V_r\end{bmatrix},$$

$$h(x)=[x_2]. \quad (45)$$

It is clear that the matrix $A(x)$ is a real symmetric and its eigenvalues satisfy Eq. 4. Moreover, the three eigenvalues of $A(x)$ are identical, which implies that the preceding system of quasi-linear PDEs is weakly hyperbolic.

### *Control problem formulation — Controller synthesis*

In this subsection, we proceed with the formulation and solution of the control problem. More specifically, it will be assumed that there are five control actuators that are characterized by a unity distribution function, that is, $b^i(z)\bar{u}^i=\bar{u}^i$ for all $i=1,\ldots,5$. The control actuators are taken to act over equispaced intervals, that is:

$$\bar{u}(t)=\begin{cases}\bar{u}^1(t), & [0.0,0.2]\\ \bar{u}^2(t), & [0.2,0.4]\\ \bar{u}^3(t), & [0.4,0.6]\\ \bar{u}^4(t), & [0.6,0.8]\\ \bar{u}^5(t), & [0.8,1.0].\end{cases} \quad (46)$$

The desired performance requirement is to control the averaging outputs:

$$\bar{y}(t)=\begin{cases}\bar{y}^1(t)=\int_{0.0}^{0.2}5.0x_2(z,t)\,dz\\ \bar{y}^2(t)=\int_{0.2}^{0.4}5.0x_2(z,t)\,dz\\ \bar{y}^3(t)=\int_{0.4}^{0.6}5.0x_2(z,t)\,dz\\ \bar{y}^4(t)=\int_{0.6}^{0.8}5.0x_2(z,t)\,dz\\ \bar{y}^5(t)=\int_{0.8}^{1.0}5.0x_2(z,t)\,dz.\end{cases} \quad (47)$$

Using these relations, the model that will be used for the synthesis of the output-feedback controller is given by

$$\frac{\partial x}{\partial t}=A(x)\frac{\partial x}{\partial z}+f(x)+g(x)\sum_{\nu=1}^{5}(H(z-z_i)-H(z-z_{i+1}))\bar{u}^i$$

$$\bar{y}^i=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}x_2(z,t)\,dz, \qquad i=1,\ldots,5, \quad (48)$$

where the matrix $A(x)$ and the vector functions $f(x)$ and $g(x)$ are specified in Eq. 45. The reactor along with the control system is given in Figure 4. Referring to the system of Eq. 48, its characteristic index can be calculated using Definition 2. In particular, we have that:

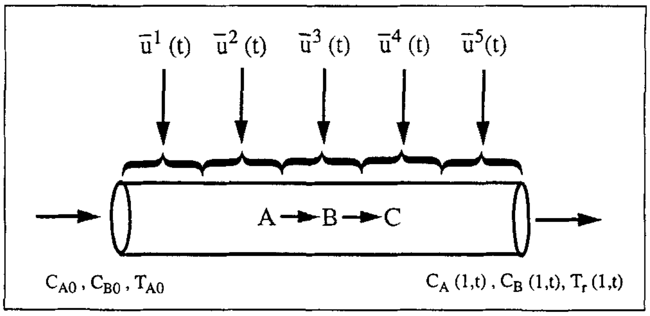


**Figure 4. Specification of the control problem for the plug-flow reactor.**

$$\mathcal{C}^iL_gh(x)=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}L_gh(x)=0, \quad \forall\, i=1,\ldots,5$$

$$\mathcal{C}^iL_g\left(\sum_{j=1}^{3}\frac{\partial x_j}{\partial z}L_{a_j}+L_f\right)h(x)$$

$$=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}\frac{\partial f_2}{\partial x_3}g_3\,dz\neq 0, \quad \forall\, i=1,\ldots,5. \quad (49)$$

Thus, the characteristic index of the system of quasi-linear PDEs of Eq. 48 is equal to 2. This allows us to request the following second-order response in the closed-loop system between $\bar{y}^i$ and $v^i$, for all $i=1,\ldots,5$:

$$\gamma_2\frac{d^2\bar{y}^i}{dt^2}+\gamma_1\frac{d\bar{y}^i}{dt}+\bar{y}^i=v^i. \quad (50)$$

Moreover, the eigenvalues of the matrix $A(x)$ are negative and the boundary conditions are specified in a single point. Thus, the result of Remark 1 applies directly, yielding that the system is open-loop stable. Furthermore, it was also verified through simulations that the process is minimum-phase. Therefore, the developed control method can be applied and the distributed output feedback controller of Theorem 4, with $\bar{\mathcal{P}}\equiv 0$, was employed in the simulations (note that due to open-loop stability of the process the controller does not use measured process outputs). The explicit form of the controller is as follows:

$$\frac{\partial\eta_1}{\partial t}=-v_e\frac{\partial\eta_1}{\partial z}-k_{10}e^{-E_1/R\eta_3}\eta_1$$

$$\frac{\partial\eta_2}{\partial t}=-v_e\frac{\partial\eta_2}{\partial z}+k_{10}e^{-E_1/R\eta_3}\eta_1-k_{20}e^{-E_2/R\eta_3}\eta_2$$

$$\frac{\partial\eta_3}{\partial t}=-v_e\frac{\partial\eta_3}{\partial z}+\frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{-E_1/R\eta_3}\eta_1$$

$$+\frac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{-E_2/R\eta_3}\eta_2+\frac{U_w}{\rho_m c_{pm}V_r}(T_{w_s}-w_3)$$

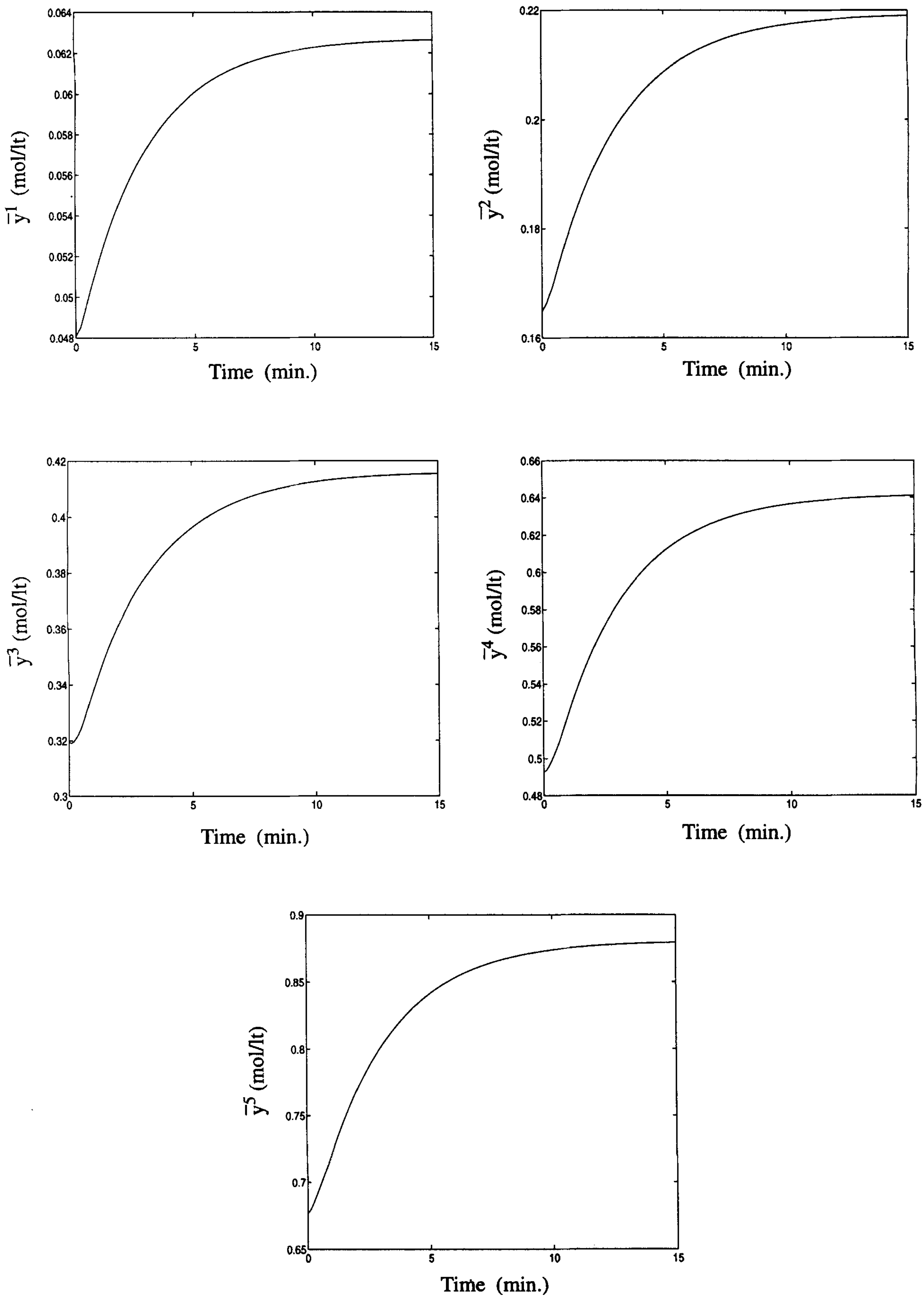


**Figure 5. Output profiles for a 30% change in reference inputs.**

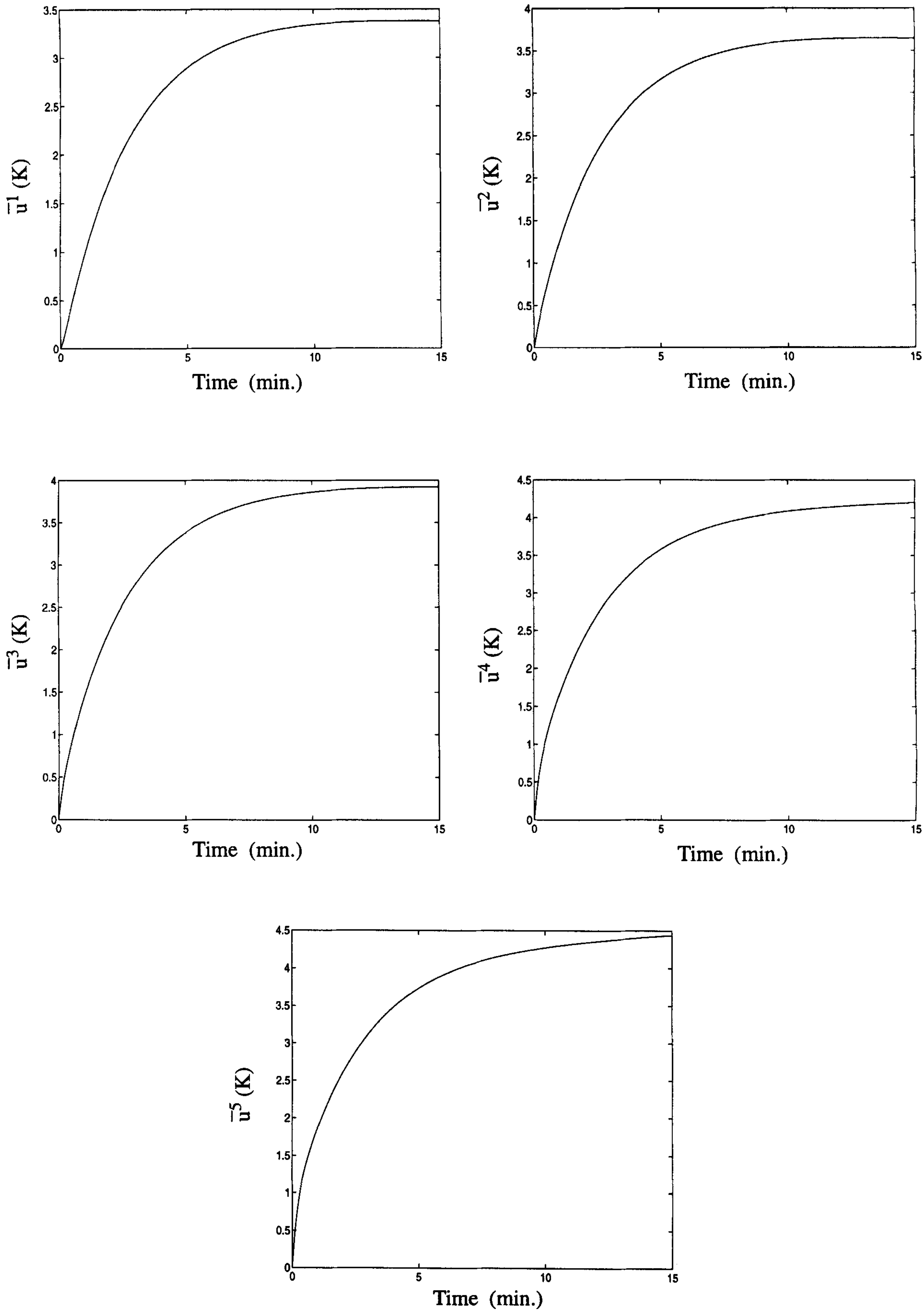


**Figure 6. Manipulated input profiles for a 30% increase in reference inputs.**

$$+\frac{U_w}{\rho_m c_{pm} V_r}\sum_{i=1}^{5}(H(z-z_i)-H(z-z_{i+1}))$$

$$\times\left[\gamma_2\mathcal{C}^i L_g\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)h(\eta)b^i(z)\right]^{-1}$$

$$\cdot\left\{v^i-\mathcal{C}^i h(\eta)-\gamma_1\mathcal{C}^i\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)h(\eta)\right.$$

$$\left.-\gamma_2\mathcal{C}^i\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)h(\eta)\right\}$$

$$\bar{u}^i=\left[\gamma_2\mathcal{C}^i L_g\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)h(\eta)b^i(z)\right]^{-1}$$

$$\times\left\{v^i-\mathcal{C}^i h(\eta)-\gamma_1\mathcal{C}^i\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)h(\eta)\right.$$

$$\left.-\gamma_2\mathcal{C}^i\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)^2 h(\eta)\right\},\quad i=1,\ldots,5,\qquad(51)$$

where the analytical expressions of the terms included in the controller are as follows:

$$\mathcal{C}^i L_g\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)h(\eta)b^i(z)=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}\frac{\partial f_2}{\partial\eta_3}g_3\,dz$$

$$\mathcal{C}^i\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}+L_f\right)h(\eta)$$

$$=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}\left(-v_l\frac{\partial\eta_2}{\partial z}+f_2(\eta_1,\eta_2,\eta_3)\right)dz$$

$$\mathcal{C}^i L_f\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}\right)h(\eta)$$

$$=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}(-v_l)\left(\frac{\partial f_2}{\partial\eta_1}\frac{\partial\eta_1}{\partial z}+\frac{\partial f_2}{\partial\eta_2}\frac{\partial\eta_2}{\partial z}+\frac{\partial f_2}{\partial\eta_3}\frac{\partial\eta_3}{\partial z}\right)dz$$

$$\mathcal{C}^i\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}\right)\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}\right)h(\eta)=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}v_l^2\frac{\partial^2\eta_2}{\partial z^2}$$

$$\mathcal{C}^i\left(\sum_{j=1}^{3}\frac{\partial\eta_j}{\partial z}L_{a_j}\right)L_f h(\eta)$$

$$=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}(-v_l)\left(\frac{\partial f_2}{\partial\eta_1}\frac{\partial\eta_1}{\partial z}+\frac{\partial f_2}{\partial\eta_2}\frac{\partial\eta_2}{\partial z}+\frac{\partial f_2}{\partial\eta_3}\frac{\partial\eta_3}{\partial z}\right)dz$$

$$\mathcal{C}^i L_f^2 h(\eta)=\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}$$

$$\times\left(\frac{\partial f_2}{\partial\eta_1}f_1(\eta_1,\eta_3)+\frac{\partial f_2}{\partial\eta_2}f_2(\eta_1,\eta_2,\eta_3)+\frac{\partial f_2}{\partial\eta_3}f_3(\eta_1,\eta_2,\eta_3)\right)dz.\qquad(52)$$

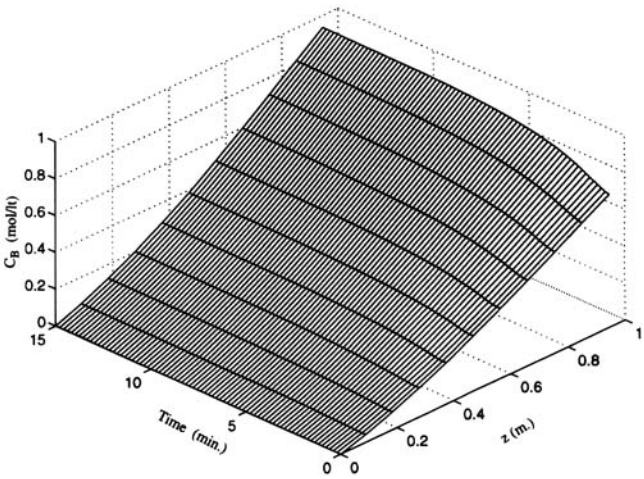


**Figure 7. Profile of concentration evolution of species *B* throughout the reactor, for a 30% increase in the reference inputs.**

The controller was tuned to give an overdamped response between the output $\bar{y}^i$ and the reference input $v^i$. In particular, the parameters $\gamma_1$ and $\gamma_2$ were chosen to be:

$$\gamma_1=3.0\,\text{min},\qquad\gamma_2=0.5\,\text{min}^2$$

to achieve the following time constant and damping factor:

$$\tau=0.707\,\text{min},\qquad\zeta=2.12$$

### *Evaluation of controller performance*

Several simulation runs were performed to evaluate the performance of the distributed output feedback controller of Eq. 51. The method of finite differences was employed to

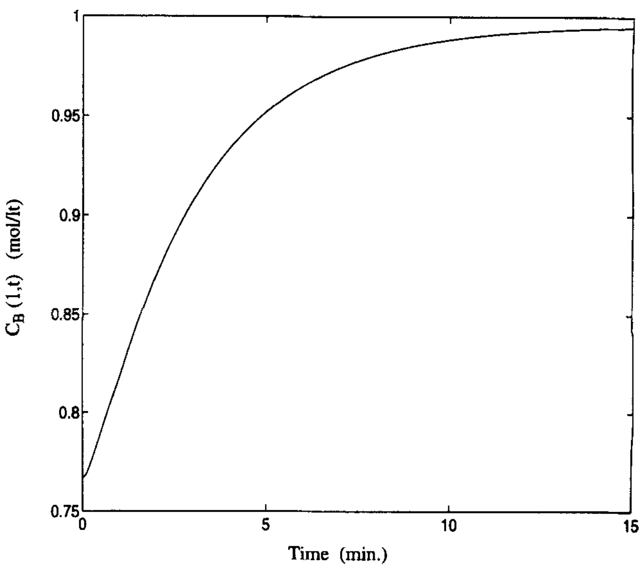


**Figure 8. Profile of concentration of species *B* in the outlet of the reactor, for a 30% increase in reference inputs.**

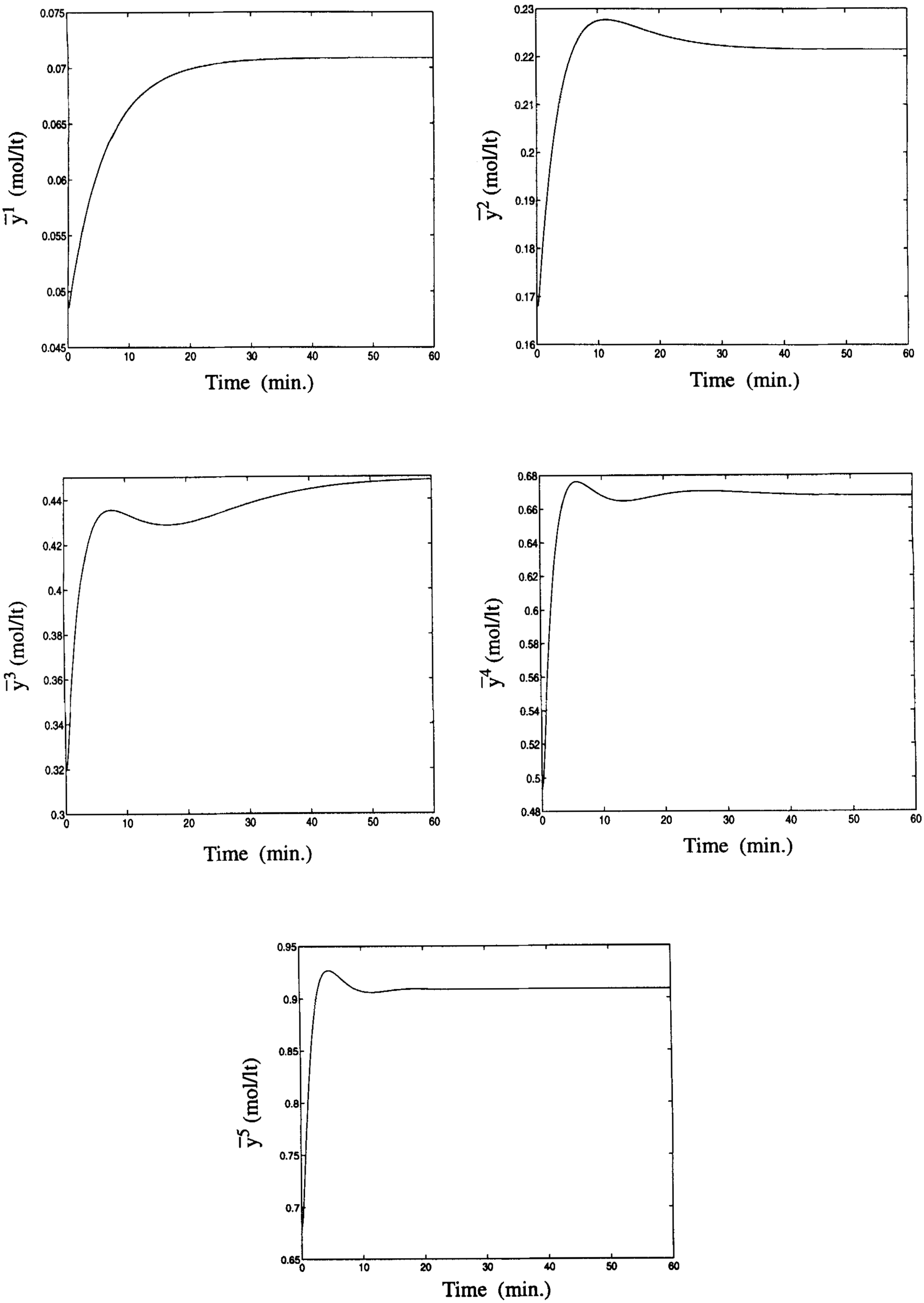


**Figure 9. Output profiles for a 30% increase in the reference inputs-discretization-based controller.**

derive a finite-dimensional approximation of the output-feedback controller of Eq. 51, with a choice of 200 discretization points. In all the simulation runs, the process was initially assumed to be at steady state.

In the first simulation run, we addressed the reference input tracking capabilities of the controller. Initially, a 30% increase in the reference inputs $v^i$, $i=1, \ldots, 5$ was imposed at time $t=0.0$ min. Figure 5 shows the corresponding output profiles. It is clear that the controller enforces the requested input/output response in the closed-loop system and regulates the output at the new reference input values. The corresponding input profiles for each control actuator are depicted in Figure 6. We observe that the control action, required by each actuator to drive the corresponding output to the new reference value, increases as we approach the outlet of the reactor. This is expected, because the amount of heat required to maintain the reaction rate that yields the necessary conversion increases along the length of the reactor. Figure 7 shows the evolution of the concentration of species $B$ throughout the reactor, while Figure 8 shows the profile of the concentration of species $B$ at the outlet of the reactor. We observe that by using a finite number of control actuators, we achieve satisfactory control of the output variable $C_B$ at all positions and times.

For the sake of comparison, we also consider the control of the reactor using a controller that was designed on the basis of a model resulting from discretization of the original PDE system in space. In particular, the method of finite differences was used to discretize the original PDE model of Eq. 48 into a set of five (equal to the number of control actuators) ordinary differential equations in time. Subsequently, an input/output linearizing controller (Kravaris and Arkun, 1991) was designed on the basis of the resulting ODE model. The corresponding output profiles are shown in Figure 9, while Figure 10 shows the profile of the concentration of species $B$ in the outlet of the reactor. It is clear that this controller leads to poor performance, because it does not explicitly take into account the spatially varying nature of the process.

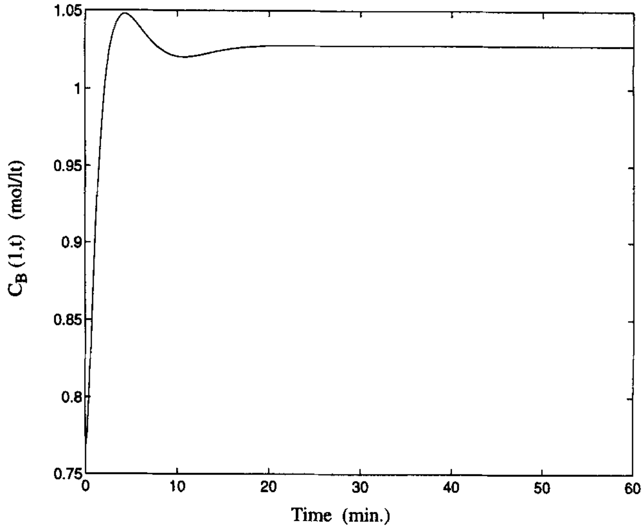


**Figure 10. Profile of concentration of species $B$ in the outlet of reactor, for a 30% increase in the reference inputs-discretization-based controller.**

### *Practical implementation issues*

The distributed output feedback controller of Eq. 51 assumes that the jacket temperature can be manipulated directly. In practice, the jacket temperature is usually manipulated indirectly through the jacket inlet flow rate. This can be achieved in a straightforward way by designing a controller to ensure that the jacket temperature obtains the values requested by the distributed controller of Eq. 51.

Specifically, under the assumption of perfect mixing and constant volume, the dynamic model of the jacket takes the form

$$\frac{dT_j^i}{dt}=\left(T_{j0}^i-T_j^i\right)\frac{F_{js}^i}{V_j^i}+\frac{U_w}{\rho_j c_{pj} V_j^i}\left(\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}T_r(z,t)\,dz-T_j^i\right)+\frac{\left(T_{j0}^i-T_j^i\right)}{V_j^i}u_{fl}^i, \qquad i=1,\ldots,5, \quad (53)$$

where $F_{js}^i$ is the steady-state jacket inlet flow rate; $V_j^i$ is the jacket volume; $\rho_j$, $c_{pj}$ are the density and heat capacity of the fluid in the jacket; $T_{j0}^i$ is the temperature of the inlet stream to a jacket; and $u_{fl}^i$ is the jacket inlet flow rate (chosen as the new manipulated input) in deviation variable form. Requesting a first-order response of the form:

$$\gamma_{jc}\frac{dT_j^i}{dt}+T_j^i=\bar{u}^i, \quad (54)$$

where $\gamma_{jc}$ is the time constant, the necessary controller takes the form:

$$u_{fl}^i=\frac{\left(\bar{u}^i-T_j^i\right)V_j^i-\gamma_{jc}\left[\left(T_{j0}^i-T_j^i\right)F_{js}^i+\frac{U_w}{\rho_j c_{pj}}\left(\int_{z_i}^{z_{i+1}}\frac{1}{z_{i+1}-z_i}T_r(z,t)dz-T_j^i\right)\right]}{\gamma_{jc}\left(T_{j0}^i-T_j^i\right)}. \quad (55)$$

The parameter $\gamma_{jc}$ should be chosen such that the response of Eq. 54 is sufficiently fast compared to the response of Eq. 50, while estimates of $T_r(z,t)$ can be obtained from the state observer of Eq. 51.

The performance of the control scheme resulting from the combination of this controller with the distributed controller of Eq. 51 was evaluated through simulations on the plug-flow reactor. The values used for the parameters and the steady-

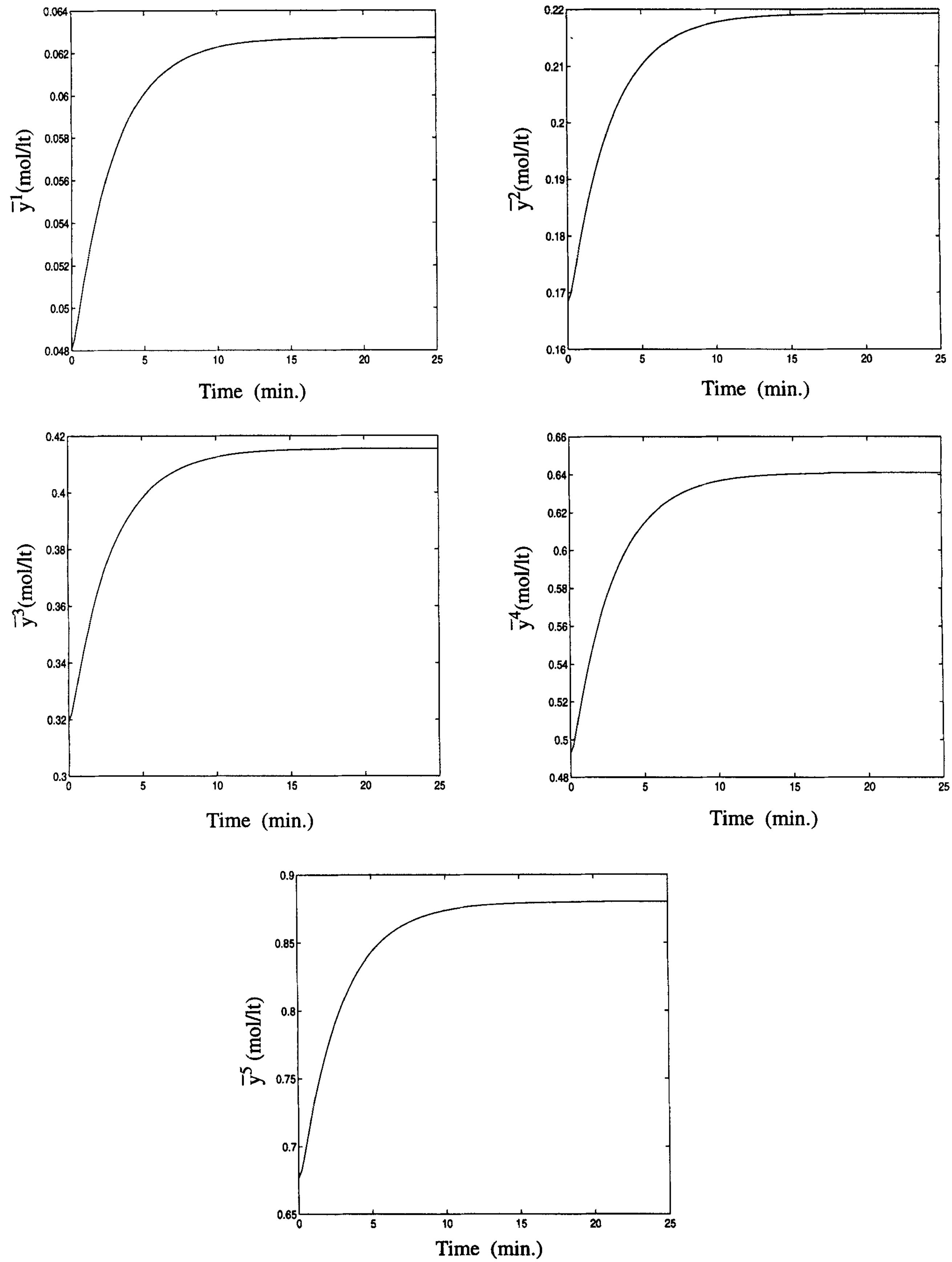


**Figure 11. Output profiles for a 30% change in reference inputs—jacket inlet flow rates are used as the manipulated inputs.**

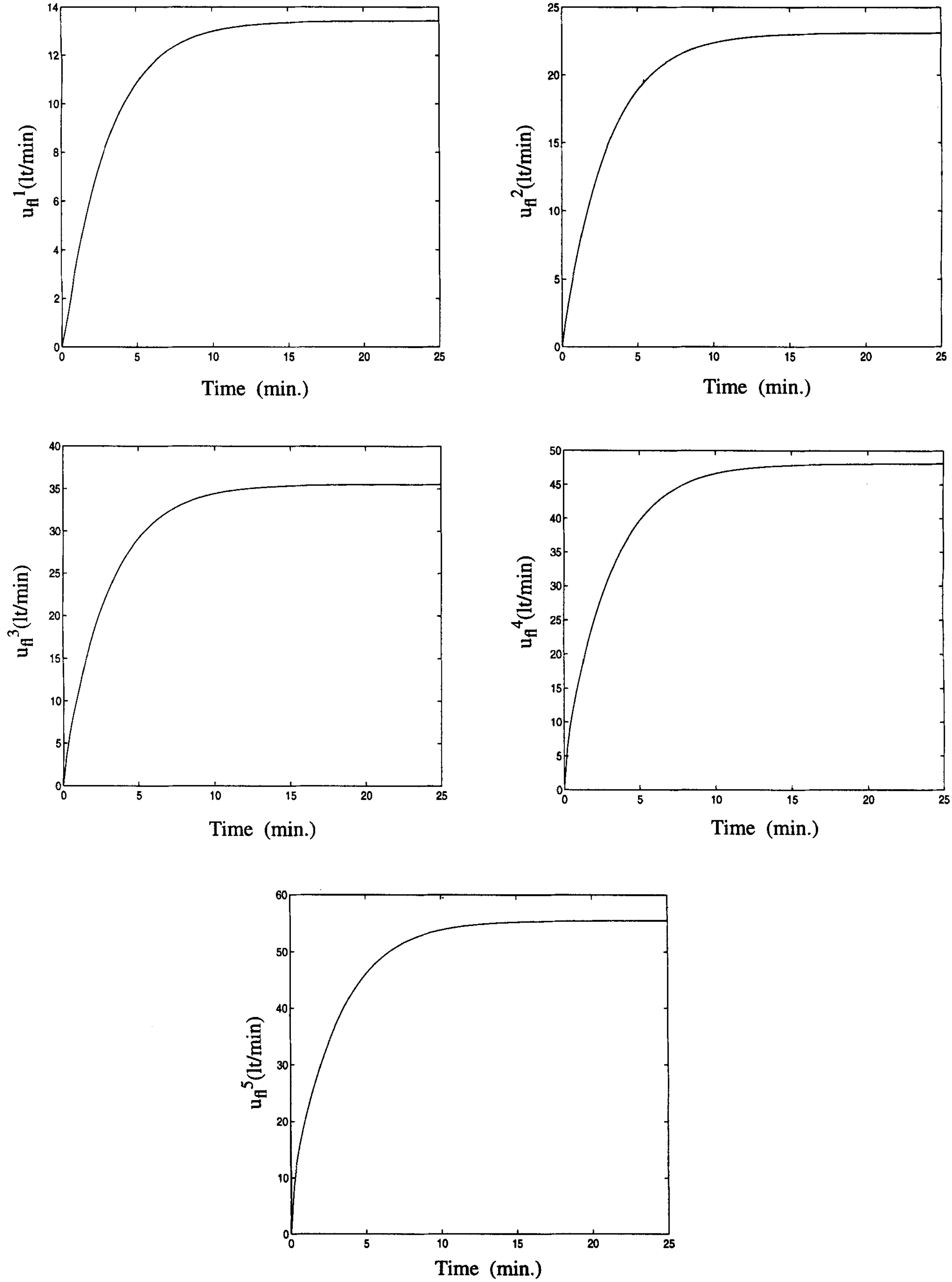


**Figure 12. Profiles of jacket inlet flow rates for a 30% increase in reference inputs.**

state values of the jacket inlet flow rate $F_{js}^i$, $i=1, \ldots, 5$, are given in Table 1. Figure 11 shows the output profiles for the same 30% increase in the value of the reference inputs as previously (the profiles for the jacket inlet flow rate are displayed in Figure 12). Clearly, the performance of the control scheme is excellent, enforcing closed-loop output responses that are very close to the ones obtained by neglecting the jacket dynamics (compare with Figure 5).

## Conclusions

In this work, we develop an output-feedback control methodology for systems described by quasi-linear first-order hyperbolic PDEs, for which the manipulated input, the controlled output, and the measurable output are distributed in space. The central idea of our approach is the combination of the theory of PDEs and concepts from geometric control. Initially, a concept of characteristic index was introduced and used for the synthesis of distributed state-feedback controllers that guarantee output tracking in the closed-loop system. Conditions that ensure exponential stability of the closed-loop system were derived. Analytical formulas of output-feedback controllers were also derived through a combination of suitable state observers with the developed distributed state-feedback controllers. The proposed control methodology was implemented, through simulations, on a nonisothermal plug-flow reactor, modeled by three quasi-linear hyperbolic PDEs. Comparisons with a control method that involves discretization in space of the original PDE model and application of standard nonlinear control methods for ODE systems established that the new control method yields superior performance.

## Acknowledgment

Financial support from the National Science Foundation (CTS-9624725) is gratefully acknowledged.

## Notation

$A(z)$ = matrices
$c^i(z)$ = performance specification function for the $i$th output
$F$ = outlet flow rate
$F_1$ = inlet flow rate
$T_w$ = temperature of the wall
$\bar{u}$ = manipulated input vector
$\bar{u}^i$ = $i$th manipulated input
$v^i$ = reference input for the $i$th actuator
$x$ = vector of state variables
$\gamma_k, \gamma_{jc}$ = adjustable parameters
$\sigma$ = characteristic index of $\bar{y}$ with respect to $\bar{u}$

### *Math symbols*

$L_f h$ = Lie derivative of a scalar field $h$ with respect to the vector field $f$
$L_f^k h$ = $k$th order Lie derivative
$L_g L_f^{k-1} h$ = mixed Lie derivative
$\mathbb{R}$ = real line
$\in$ = belongs to
$T$ = transpose
$\|\cdot\|_2$ = 2-norm in $\mathcal{H}$

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## Appendix

The proofs of Theorem 2 and Proposition 4, concerning the case of systems of quasi-linear PDEs, are conceptually similar to the ones given for the linear counterparts of these results, and will therefore be omitted for brevity.

### *Proof of Proposition 2*

Consider the closed-loop system of Eq. 26. Differentiating the output of this system with respect to time, we obtain the following set of equations:

$$
\begin{aligned}
\bar{y} &= \mathcal{C}kx \\
\frac{d\bar{y}}{dt} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B+wb(z)\mathcal{S}\right)x \\
\frac{d^2\bar{y}}{dt^2} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B+wb(z)\mathcal{S}\right)^2 x \\
&\vdots \\
\frac{d^{\bar{\sigma}}\bar{y}}{dt^{\bar{\sigma}}} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B+wb(z)\mathcal{S}\right)^{\bar{\sigma}} x \\
&\quad + \mathcal{C}k\left(A\frac{\partial}{\partial z}+B+wb(z)\mathcal{S}\right)^{\bar{\sigma}-1} wb(z)sv. \qquad \text{(A1)}
\end{aligned}
$$

It is sufficient to show that $\bar{\sigma}=\sigma$. Note that $\mathcal{C}k[A(\partial/\partial z)+B\ wb(z)\mathcal{S}] = \mathcal{C}k[A(\partial/\partial z)+B]+\mathcal{C}kwb(z)\mathcal{S} = \mathcal{C}k[A(\partial/\partial z)+B]$, because $\mathcal{C}kwb(z)=0$. Similarly, it can be shown, by induction, that $\mathcal{C}k[A(\partial/\partial z)+B+wb(z)\mathcal{S}]^i = \mathcal{C}k[A(\partial/\partial z)+B]^i$, for $i=1,\ldots,\sigma-1$. Using this simplification in the expressions for the time derivatives (Eq. A1) it follows directly that

$$
\begin{aligned}
\bar{y} &= \mathcal{C}kx \\
\frac{d\bar{y}}{dt} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)x \\
\frac{d^2\bar{y}}{dt^2} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^2 x \\
&\vdots \\
\frac{d^{\sigma}\bar{y}}{dt^{\sigma}} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma} x + \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\mathcal{S}x \\
&\quad + \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)sv. \qquad \text{(A2)}
\end{aligned}
$$

This completes the proof of the proposition.

### *Proof of Theorem 1*

Under the controller of Eq. 28, the closed-loop system takes the form:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z}+Bx+wb(z)\left[\gamma_{\sigma}\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \times\left\{v-\mathcal{C}kx-\sum_{\nu=1}^{\sigma}\gamma_{\nu}\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\nu}x\right\} \\
\bar{y} &= \mathcal{C}kx. \qquad \text{(A3)}
\end{aligned}
$$

From the result of Proposition 2, it follows that a differentiation of the output of the system of Eq. A3 yields the following equations:

$$
\begin{aligned}
\bar{y} &= \mathcal{C}kx \\
\frac{d\bar{y}}{dt} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)x \\
\frac{d^2\bar{y}}{dt^2} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^2 x \\
&\vdots \\
\frac{d^{\sigma}\bar{y}}{dt^{\sigma}} &= \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma} x + \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z) \\
&\quad \times\left[\gamma_{\sigma}\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \left\{v-\mathcal{C}kx-\sum_{\nu=1}^{\sigma}\gamma_{\nu}\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\nu}x\right\}. \qquad \text{(A4)}
\end{aligned}
$$

Substituting these relations to Eq. 27, one can easily show that the result of the theorem holds.

### *Proof of Proposition 3*

Utilizing the expressions for the output derivatives of Eq. A2 the closed-loop system of Eq. A3 takes the form:

$$\frac{\partial x}{\partial t} = A\frac{\partial x}{\partial z} + Bx + wb(z)\left[\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1}$$
$$\times\left\{\frac{1}{\gamma_\sigma}(v-\bar{y}) - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\frac{d^\nu \bar{y}}{dt^\nu} - \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma} x\right\}$$
$$\bar{y} = \mathcal{C}kx. \quad \text{(A5)}$$

Defining the state vectors $\zeta_i = (d^{i-1}\bar{y})/dt^{i-1}$, $i = 1, \ldots, \sigma$, the system of Eq. A5 can be equivalently written in the form of the following interconnection:

$$\dot{\zeta}_1 = \zeta_2$$
$$\vdots$$
$$\dot{\zeta}_{\sigma-1} = \zeta_\sigma$$
$$\dot{\zeta}_\sigma = -\frac{1}{\gamma_\sigma}\zeta_1 - \frac{\gamma_1}{\gamma_\sigma}\zeta_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_\sigma + \frac{1}{\gamma_\sigma}v$$
$$\frac{\partial x}{\partial t} = A\frac{\partial x}{\partial z} + Bx + wb(z)\left[\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1}$$
$$\times\left\{\bar{\zeta} - \mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma} x\right\}, \quad \text{(A6)}$$

where

$$\bar{\zeta} = \frac{1}{\gamma_\sigma}(v-\zeta_1) - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\zeta_{\nu+1}.$$

Condition 1 of the proposition guarantees that the $\zeta$-subsystem of the preceding interconnection is exponentially stable, and thus the following condition holds:

$$|\bar{\zeta}| \leq K_1|\bar{\zeta}_0|e^{-a_1 t}, \quad \text{(A7)}$$

where $|\cdot|$ denotes the standard Eucledian norm; $K_1$, $a_1$ are positive real numbers, with $K_1 \geq 1$; and $\bar{\zeta}_0$ is the value of the variable $\bar{\zeta}$ at time $t = 0$ {i.e., $\bar{\zeta}_0 = (1/\gamma_\sigma)[v(0) - \zeta_{(1)0}] - \Sigma_{\nu=1}^{\sigma-1}(\gamma_\nu/\gamma_\sigma)\zeta_{(\nu+1)0}$}. From Condition 2, we have that the differential operator of the system of Eq. 35 generates an exponentially stable semigroup $\bar{U}$, that is, $\|\bar{U}\|_2 \leq K_2 e^{-a_2 t}$, where $K_2$, $a_2$ are positive real numbers, with $K_2 \geq 1$. Utilizing Eq. 15, the following estimate can be written for the state $x$ of the system of Eq. A5:

$$\|x\|_2 \leq K_2\|x_0\|_2 e^{-a_2 t} + K_2\int_0^t e^{-a_2(t-\tau)}\|\bar{W}(z)\|_2|\bar{\zeta}(\tau)|\,d\tau, \quad \text{(A8)}$$

where

$$\bar{W}(z) = wb(z)\left[\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1}. \quad \text{(A9)}$$

Substituting Eq. A7 into Eq. A8, we have that:

$$\|x\|_2 \leq K_2\|x_0\|_2 e^{-a_2 t} + K_2K_1\int_0^t e^{-a_2(t-\tau)}\|\bar{W}(z)\|_2|\bar{\zeta}_0|e^{-a_1\tau}\,d\tau$$
$$= K_2\|x_0\|_2 e^{-a_2 t} + K_2K_1\|\bar{W}(z)\|_2|\bar{\zeta}_0|e^{-a_2 t}\int_0^t e^{(a_2-a_1)\tau}\,d\tau. \quad \text{(A10)}$$

Let $\delta = a_2 - a_1$. If $\delta = 0$, $\| x \|_2 \leq K_2 \| x_0 \|_2 e^{-a_2 t} + K_2K_1 \| \bar{W}(z) \|_2 | \bar{\zeta}_0 | e^{-a_2 t}t \leq K_2 \| x_0 \|_2 e^{-a_2 t} + K_2K_1 \| \bar{W}(z) \|_2 (| \bar{\zeta}_0 |/a_2 - a_3)e^{-a_3 t}$, where $0 < a_3 < a_2$. Clearly, in this case, the closed-loop system is exponentially stable. Furthermore, if $\delta > 0$, $\| x \|_2 \leq K_2 \| x_0 \|_2 e^{-a_2 t} + K_2K_1 \| \bar{W}(z)\|_2(|\bar{\zeta}_0|/\delta)e^{-a_1 t}(1-e^{-(a_2-a_1)t})$, while if $\delta < 0$, $\| x \|_2 \leq K_2 \| x_0 \|_2 e^{-a_2 t} + K_2K_1 \| \bar{W}(z) \|_2 (| \bar{\zeta}_0 |/| \delta |)e^{-a_2 t}(1-e^{(a_2-a_1)t})$. In either case ($\delta > 0$, $\delta < 0$), and we have that:

$$\|x\|_2 \leq K_2\|x_0\|_2 e^{-\tilde{a}t} + K_2K_1\|\bar{W}(z)\|_2\frac{|\bar{\zeta}_0|}{|\delta|}e^{-\tilde{a}t}, \quad \text{(A11)}$$

where $\tilde{a} = \min\{a_1, a_2\}$. From Eqs. A7–A11 the exponential stability of the closed-loop system of Eq. A5 follows directly.

### *Proof of Theorem 3*

*Part 1: Stability Analysis.* Substituting the controller of Eq. 39 in the system of Eq. 9, we have:

$$\frac{\partial x}{\partial t} = A\frac{\partial x}{\partial z} + Bx + wb(z)\left[\gamma_\sigma\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1}$$
$$\times\left\{v - \mathcal{C}k\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\nu}\eta\right\}$$
$$\frac{\partial \eta}{\partial t} = A\frac{\partial \eta}{\partial z} + B\eta + wb(z)\left[\gamma_\sigma\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1}$$
$$\times\left\{v - \mathcal{C}k\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\nu}\eta\right\} + \mathcal{P}(\bar{q} - \mathcal{Q}p\eta). \quad \text{(A12)}$$

Introducing the error coordinate $\bar{e} = x - \eta$, the preceding closed-loop system can be written as

$$\frac{\partial x}{\partial t} = A\frac{\partial x}{\partial z} + Bx + wb(z)\left[\gamma_\sigma\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1}$$
$$\times\left\{v - \mathcal{C}kx - \sum_{\nu=1}^{\sigma}\gamma_\nu\mathcal{C}k\left(A\frac{\partial}{\partial z}+B\right)^{\nu}x\right\} + \mathfrak{X}\bar{e}$$
$$\frac{\partial \bar{e}}{\partial t} = (\mathcal{L} - \mathcal{P}\mathcal{Q}k)\bar{e}, \quad \text{(A13)}$$

where

$$\mathfrak{X}\bar{e} = wb(z)\left[\gamma_\sigma \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1}$$

$$\times\left\{\mathcal{C}k\bar{e} + \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\nu} \bar{e}\right\}. \quad \text{(A14)}$$

Because the operator $\mathcal{P}$ is designed such that the operator $\mathcal{L} - \mathcal{P}\mathcal{Q}p$ generates an exponentially stable semigroup, the following estimate can be written for the evolution of the state $\bar{e}$ of the preceding system:

$$\|\bar{e}\|_2 \le K_3 \|\bar{e}_0\|_2 e^{-\bar{a}t}, \quad \text{(A15)}$$

where $\bar{e}_0$ denotes the vector of initial conditions and $K_3$, $\bar{a}$ are positive real numbers, with $K_3 \ge 1$. Furthermore, utilizing Eq. 15 and the fact that the conditions of Proposition 3 hold, the following inequality can be written for the state $x$ of the system of Eq. A13:

$$\|x\|_2 \le K_4 \|x_0\|_2 e^{-\hat{a}t} + K_4 \int_0^t e^{-\hat{a}(t-\tau)} \|\mathfrak{X}\bar{e}(\tau)\|_2 \, d\tau, \quad \text{(A16)}$$

where $K_4$, $\hat{a}$ are positive real numbers, with $K_4 \ge 1$. Substituting the inequality of Eq. A15 into Eq. A16, and performing similar calculations as in the proof of Proposition 3, it can be shown that the closed-loop system is exponentially stable.

*Part 2: Input/Output Response.* Under consistent initialization of the states $x$ and $\eta$, that is, $x(z,0) = \eta(z,0)$, it follows that $\bar{e}(z,0) = 0$. From the dynamical system for $\bar{e}$, it is clear that if $\bar{e}(z,0) = 0$, then $\bar{e}(z,t) = 0$ for all $t \ge 0$. Thus, the system of Eq. A13 reduces to

$$\frac{\partial x}{\partial t} = A\frac{\partial x}{\partial z} + Bx + wb(z)\left[\gamma_\sigma \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1}$$

$$\times\left\{v - \mathcal{C}kx - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}k\left(A\frac{\partial}{\partial z} + B\right)^{\nu} x\right\}. \quad \text{(A17)}$$

A direct application of Theorem 1 completes the proof of the theorem.

### *Proof of Theorem 4*

*Part 1: Stability Analysis.* Substituting the controller of Eq. 43 in the system of Eq. 7, we have:

$$\frac{\partial x}{\partial t} = A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)$$

$$\times\left[\gamma_\sigma \mathcal{C}L_g\left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\sigma-1} h(\eta)b(z)\right]^{-1}$$

$$\times\left\{v - \mathcal{C}h(\eta) - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}\left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\nu} h(\eta)\right\}$$

$$\frac{\partial \eta}{\partial t} = A(\eta)\frac{\partial \eta}{\partial z} + f(\eta) + g(\eta)b(z)$$

$$\times\left[\gamma_\sigma \mathcal{C}L_g\left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\sigma-1} h(\eta)b(z)\right]^{-1}$$

$$\times\left\{v - \mathcal{C}h(\eta) - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}\left(\sum_{j=1}^{n} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f\right)^{\nu} h(\eta)\right\}$$

$$+ \bar{\mathcal{P}}[\bar{q} - \mathcal{Q}p(\eta)]. \quad \text{(A18)}$$

In order to perform a local analysis of the stability properties of the preceding system, we consider its linearization:

$$\frac{\partial x}{\partial t} = A(z)\frac{\partial x}{\partial z} + B(z)x + w(z)b(z)$$

$$\times\left[\gamma_\sigma \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1}$$

$$\times\left\{v - \mathcal{C}k(z)\eta - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu} \eta\right\}$$

$$+ \bar{\mathcal{P}}(\bar{q} - \mathcal{C}p(z)\eta)$$

$$\frac{\partial \eta}{\partial t} = A(z)\frac{\partial \eta}{\partial z} + B(z)\eta + w(z)b(z)$$

$$\times\left[\gamma_\sigma \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1}$$

$$\times\left\{v - \mathcal{C}k(z)\eta - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu} \eta\right\}$$

$$+ \bar{\mathcal{P}}(\bar{q} - \mathcal{C}p(z)\eta). \quad \text{(A19)}$$

Introducing the error coordinate $\bar{e} = x - \eta$, the preceding closed-loop system can be written as

$$\frac{\partial x}{\partial t} = A(z)\frac{\partial x}{\partial z} + B(z)x + w(z)b(z)$$

$$\times\left[\gamma_\sigma \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1}$$

$$\times\left\{v - \mathcal{C}k(z)\eta - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu} x\right\} + \tilde{\mathfrak{X}}\bar{e}$$

$$\frac{\partial \bar{e}}{\partial t} = (\mathcal{L} - \bar{\mathcal{P}}\mathcal{Q}p(z))\bar{e}, \quad \text{(A20)}$$

where

$$\tilde{\mathfrak{X}}\bar{e} = w(z)b(z)\left[\gamma_\sigma \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1}$$

$$\times\left\{\mathcal{C}k(z)\bar{e} + \sum_{\nu=1}^{\sigma} \gamma_\nu \mathcal{C}k(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu} \bar{e}\right\}. \quad \text{(A21)}$$

From the conditions of the theorem, we have that the $x$-subsystem (with $\bar{e}=0$) of the preceding interconnection is exponentially stable and the $\bar{e}$-subsystem is also exponentially stable, because the operator $\bar{\mathcal{P}}$ is designed such that the operator $\bar{\mathcal{L}} - \bar{\mathcal{P}}\mathcal{Q}p(z)$ generates an exponentially stable semigroup. Following an approach analogous to the one used in the proof of Proposition 3, one can show that the system of Eq. A20 is exponentially stable. Utilizing the result of Proposition 1, we have that the closed-loop system of Eq. A18 is locally exponentially stable, because the linearized system of Eq. A20 is exponentially stable.

*Part 2: Input/Output Response.* Under consistent initialization of the states $x$ and $\eta$, that is, $x(z,0)=\eta(z,0)$, it follows that $\bar{e}(z,0)=0$. From the dynamical system for $\bar{e}$, it is clear that if $\bar{e}(z,0)=0$, then $\bar{e}(z,t)=0$ for all $t \geq 0$. Thus, the system of Eq. A20 reduces to

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z) \\
&\quad \times \left[\gamma_\sigma \mathcal{C} L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\quad \times \left\{v - \mathcal{C}h(x) - \sum_{\nu=1}^{\sigma}\gamma_\nu \mathcal{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(x)\right\} \\
\bar{y} &= \mathcal{C}h(x).
\end{aligned}
\tag{A22}
$$

Since the characteristic index between $\bar{y}$ and $v$ is $\sigma$, a differentiation of the output of the system of Eq. A22 yields the following expressions:

$$
\begin{aligned}
\bar{y} &= \mathcal{C}h(x) \\
\frac{d\bar{y}}{dt} &= \mathcal{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
\frac{d^2\bar{y}}{dt^2} &= \mathcal{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
&\;\;\vdots \\
\frac{d^\sigma\bar{y}}{dt^\sigma} &= \mathcal{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
&\quad + \mathcal{C}L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z) \\
&\quad \times \left[\gamma_\sigma \mathcal{C}L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\quad \times \left\{v - \mathcal{C}h(x) - \sum_{\nu=1}^{\sigma}\gamma_\nu \mathcal{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(x)\right\}.
\end{aligned}
\tag{A23}
$$

Substituting the preceding relation into Eq. 27, one can easily show that the result of the theorem holds.

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