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Compensation of Measurable Disturbances for Two-time-scale Nonlinear Systems*

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A feedforward/feedback control methodology is developed for nonlinear two-time-scale systems with disturbances, and is successfully applied to a catalytic continuous stirred tank reactor

Key Words—Singular perturbations; feedforward/feedback control; catalytic reactors.

Abstract—This paper deals with a class of two-time-scale nonlinear systems with time-varying disturbances, modeled within the framework of singular perturbations. Systems in both standard and nonstandard form are considered. For these systems, we synthesize well-conditioned control laws that utilize feedback of the full state vector and feedforward compensation of disturbances to exponentially stabilize the fast dynamics and enforce a pre-specified input/output behavior independently of the disturbances in the closed-loop slow subsystem. Singular perturbation methods are employed to establish that the discrepancy between the output of the closed-loop full-order system and the output of the closed-loop slow subsystem is proportional to the value of the singular perturbation parameter. The stability of the closed-loop system is analyzed using Lyapunov's direct method and precise conditions that guarantee boundedness of the trajectories, for sufficiently small values of the singular perturbation parameter, are derived. The application of the developed methodology is illustrated through a catalytic continuous stirred tank reactor example, modeled as a singularly perturbed system in nonstandard form. Copyright © 1996 Elsevier Science Ltd.

NOTATION

Roman letters

A_c	surface of the catalyst
A_r	surface of the reactor
C_{A0}	inlet concentration of the species A
C_{Ac}	concentration of the species A in the catalytic phase
c_{ph}	heat capacity of the homogeneous phase
c_{pc}	heat capacity of the catalytic phase
E_h, E_c	activation energies
d	disturbance input vector
$F, \tilde{F}, f_1, f_2$	vector fields
F_1	inlet flow rate
F_2	outlet flow rate
$G, \tilde{G}, g_1, g_2$	vector fields associated with the manipulated input
h	output scalar function
K_c	mass transfer coefficient
k^T	covector field
k_h, k_c	pre-exponential constants
L	Lyapunov function
Q_1, Q_2	matrices associated with the fast state vector z
r, $\hat{r}$	relative orders with respect to the input in the reduced system
S	matrix
T_{A0}	inlet temperature of the species A
T_h	temperature of the homogeneous phase
T_c	temperature of the catalytic phase
T_w	temperature of the wall
t	time
U_c, U_w	heat transfer coefficients
u	input
$\hat{u}$, $\tilde{u}$	auxiliary inputs
V	Lyapunov function
V_r	volume of the homogeneous phase
V_c	volume of the catalytic phase
$W, \tilde{W}, W_1, W_2$	matrices associated with the disturbance inputs
x	vector of the slow state variables
y	output
y^s	output associated with slow subsystem
z	vector of fast state variables

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Greek letters

β_k	adjustable parameters
ΔH_h, ΔH_c	enthalpy of the reactions
ϵ	singular perturbation parameter
η	state vector in normal-form coordinates
η_f	fast state vector
ξ, $\hat{\xi}$	equilibrium steady-states for the fast dynamics in the closed-loop system
ρ, $\hat{\rho}$, $\tilde{\rho}$	relative orders with respect to the disturbance input vector in the reduced system
ρ_h	density of the homogeneous phase
ρ_c	density of the catalytic phase
Φ_i, $\tilde{\Phi}_i$	auxiliary functions
Φ_ζ, Φ_η	matrices of smooth functions
ψ, Ψ, $\bar{\Psi}$	auxiliary functions
Ω	Lyapunov function
$\bar{\Omega}$	compact set

Mathematical symbols

$L_f h$	Lie derivative of a scalar function h with respect to the vector field f
$L_f^k h$	kth-order Lie derivative
$L_g L_f^{k-1} h$	mixed Lie derivative
$\mathbb{R}$	real line
$\mathbb{R}^i$	i-dimensional Euclidean space
$\in$	belongs to
T	transpose
$\lvert\cdot\rvert$	standard Euclidean norm
$\bar{0}$	zero covector of dimension q

1. INTRODUCTION

The majority of physicochemical processes are inherently nonlinear and are often characterized by the copresence of dynamical phenomena occurring in multiple time-scales. Typical examples of nonlinear multiple-time-scale systems include reaction networks (Breusegem and Bastin, 1991), catalytic reactors (Chang and Aluko, 1984; Denn, 1986), fluidized catalytic crackers (Monge and Georgakis, 1987), biochemical reactors (Bastin and Dochain, 1990; Christofides and Daoutidis, 1996), distillation columns (Levine and Rouchon, 1991), DC motor models (Kokotovic *et al.*, 1986), and electrical circuits (Khalil, 1992). Singular perturbation theory has proven to be the natural framework for the analysis and control of multiple-time-scale systems (Kokotovic *et al.*, 1986). Within this framework, stability issues (Saberi and Khalil, 1984; Christofides and Teel, 1995, 1996) and geometric properties (Fenichel, 1979) of nonlinear two-time-scale systems have been studied, and feedback control algorithms have been developed using optimal control (Kokotovic *et al.*, 1986), the integral manifold approach (Sharkey and O'Reilly, 1987), sliding mode techniques (Heck, 1991) and geometric control methods (Christofides and Daoutidis, 1996).

An additional problem encountered in the synthesis of control systems for actual processes is the presence of exogenous disturbances that may vary arbitrarily with time. It is well known that the presence of disturbances may cause significant degradation of the nominal performance and in some instances closed-loop instability. Motivated by this, the problem of disturbance decoupling for standard nonlinear systems has been studied extensively. Geometric conditions for the solvability of this problem via static state feedback (Isidori *et al.*, 1981) and static feedforward/static state feedback (Moog and Glumineau, 1983) have been derived, while the solution of this problem through dynamic feedforward/static state feedback (Daoutidis and Kravaris, 1993) has also been obtained. Other available results on the treatment of disturbance inputs deal with the use of high-gain feedback to achieve almost disturbance decoupling (see e.g. Marino *et al.*, 1988), and the use of feedforward compensation in the context of exact state-space linearization (Calvet and Arkun, 1988). However, a direct application of the aforementioned control methods to multiple-time-scale systems may lead to controller ill-conditioning and/or instability of the closed-loop system due to possible slightly non-minimum-phase behavior (Kokotovic *et al.*, 1986; Christofides and Daoutidis, 1996).

The problem of rejection of disturbances for linear two-time-scale systems has been addressed using H^∞ control methods (Khalil and Chen, 1992; Pan and Başar, 1993), stochastic control methods (see Kokotovic *et al.*, 1986, and references therein), and controller design via Lyapunov functions (Corless *et al.*, 1993). For nonlinear two-time-scale systems with stable fast dynamics, combination of singular perturbation theory and adaptive control schemes (Taylor *et al.*, 1989) has been proposed for this purpose. For nonlinear two-time-scale systems with unstable fast dynamics, an alternative approach that utilizes combination of singular perturbations and Lyapunov's direct method has been proposed in Khorasani (1989).

In this paper, a broad class of two-time-scale nonlinear systems modeled within the singular perturbation framework, with measurable time-varying disturbances, is considered; these include both systems in standard and nonstandard form. For these systems, we synthesize well-conditioned control laws that utilize feedback of the full state vector and feedforward compensation of disturbances to exponentially stabilize the fast dynamics and enforce a prespecified input/output behavior independently of the disturbances in the closed-loop slow subsystem. Singular perturbation methods are employed to establish that the discrepancy between the output of the closed-loop full-order system and the output of the closed-loop slow subsystem is proportional to the value of the singular perturbation parameter. Key differences in the nature of the control problem between systems in standard and nonstandard form are identified and discussed. The stability of the closed-loop system is analyzed using Lyapunov functions, and precise conditions that guarantee boundedness of the trajectories of the closed-loop system, for sufficiently small values of the singular perturbation parameter, are derived. Finally, the developed methodology is applied to a catalytic continuous stirred tank reactor modeled as a singularly perturbed system in nonstandard form.

2. PRELIMINARIES

We will consider two-time-scale nonlinear systems with the following state-space representation:

$$\begin{aligned} \dot{x} &= f_1(x) + Q_1(x)z + g_1(x)u + W_1(x)d(t), \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)u + W_2(x)d(t), \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$ and $z \in Z \subset \mathbb{R}^p$ denote vectors of state variables, with X and Z open and connected sets that contain the equilibrium point of interest, $u \in \mathbb{R}$ denotes the manipulated input, $d = [d_1(t) \ \dots \ d_q(t)] \in \mathbb{R}^q$ denotes the vector of disturbance inputs, which are assumed to be measurable and sufficiently smooth functions of time, and $y \in \mathbb{R}$ denotes the controlled output. ϵ is a small positive parameter, which can be interpreted as the speed ratio of the slow versus the fast dynamical phenomena of the system. Furthermore, $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ are analytic vector fields, $Q_1(x)$, $Q_2(x)$ and $W_1(x)$, $W_2(x)$ are analytic matrices of dimensions $n \times p$, $p \times p$ and $n \times q$, $p \times q$ respectively, and $h(x)$ is an analytic scalar function. In what follows, for simplicity, we shall suppress the time dependence in the notation of the disturbance input vector $d(t)$.

Modeling a two-time-scale process in a singularly perturbed form involves defining the singular perturbation parameter ϵ, taking into account the physicochemical characteristics of the process, so that the separation of the fast and slow variables becomes explicit, with ϵ multiplying the time derivatives of the fast variables z. ϵ usually represents small process parameters or the reciprocal of large process parameters (e.g. small/large masses, capacitances). The reader may also refer to Kokotovic *et al.* (1986) for further discussion of this issue. Referring to the specific singularly perturbed system (1), we note that the parameter ϵ appears only in the left-hand side (multiplying the time derivative $\dot{z}$), while the fast variable z enters in a linear fashion. The first assumption is made for notational simplicity, and can readily be relaxed (see Remark 8), while the second assumption is consistent with the fact that in many chemical processes the main nonlinearities are associated with the slow variables, and also allows explicitness and analytical insight in the theoretical development.

The intrinsic two-time-scale property of the system (1) can be analyzed by decomposing it into separate reduced-order systems evolving on different time scales (Kokotovic *et al.*, 1986). In particular, assuming that the system (1) is in standard form, i.e. the matrix $Q_2(x)$ is nonsingular uniformly in $x \in X$, and, setting $\epsilon = 0$, the system (1) takes the form

$$\dot{x} = f_1(x) + Q_1(x)z_s + g_1(x)u + W_1(x)d, \tag{2}$$

$$f_2(x) + Q_2(x)z_s + g_2(x)u + W_2(x)d = 0, \tag{3}$$

where z_s denotes a quasi-steady state for z. The invertibility of the matrix $Q_2(x)$ guarantees that the system of algebraic equations (3) admits a unique solution for z_s, of the form

$$z_s = -[Q_2(x)]^{-1}[f_2(x) + g_2(x)u + W_2(x)d]. \tag{4}$$

Substituting (4) into (2), the following *reduced system* or *slow subsystem* is obtained:

$$\begin{aligned} \dot{x} &= F(x) + G(x)u + W(x)d, \\ y^s &= h(x), \end{aligned} \tag{5}$$

where y^s denotes the output associated with the slow subsystem and

$$\begin{aligned} F(x) &= f_1(x) - Q_1(x)[Q_2(x)]^{-1}f_2(x), \\ G(x) &= g_1(x) - Q_1(x)[Q_2(x)]^{-1}g_2(x), \\ W(x) &= W_1(x) - Q_1(x)[Q_2(x)]^{-1}W_2(x). \end{aligned} \tag{6}$$

Note that the input u and the disturbance input vector d appear in an affine and separable fashion in the system (5) because of the linearity in z in the original system. To obtain a representation of the system that describes the fast dynamics of the system (1), we define a fast time scale

$$\tau = \frac{t}{\epsilon}. \quad (7)$$

In this new time scale, the original system takes the form

$$\frac{dx}{d\tau} = \epsilon[f_1(x) + Q_1(x)z + g_1(x)u + W_1(x)d], \quad (8)$$

$$\frac{dz}{d\tau} = f_2(x) + Q_2(x)z + g_2(x)u + W_2(x)d. \quad (9)$$

Setting ϵ equal to zero, the following *fast subsystem* is obtained:

$$\frac{dz}{d\tau} = f_2(x) + Q_2(x)z + g_2(x)u + W_2(x)d, \quad (10)$$

where x can be considered approximately equal to its initial value $x(0)$ and d is constant. The affine appearance of the fast state z in the system (10) implies that its bounded-input bounded-state stability depends exclusively on the eigenstructure of the matrix $Q_2(x)$. The following assumption states our stabilizability requirement on the open-loop fast subsystem (10).

Assumption 1. The pair $[Q_2(x)\ g_2(x)]$ is stabilizable, in the sense that there exists an analytic covector $k^T(x)$ such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$.

In what follows, we review various concepts of relative order that will allow us to formulate and solve specific controller synthesis problems.

Definition 1. Referring to the nonlinear system (5) the *relative order of the output* y^s *with respect to the input* u is defined as the smallest integer r for which

$$L_G L_F^{r-1} h(x) \not\equiv 0 \quad (11)$$

or $r = \infty$ if such an integer does not exist.

Throughout this paper, it will be assumed that (11) holds for all $x \in X$.

Definition 2. Referring to the nonlinear system (5), the *relative order of the output* y^s *with respect to the disturbance input vector* d is defined as the smallest integer ρ for which

$$[L_{W^1} L_F^{\rho-1} h(x)\ \ldots\ L_{W^q} L_F^{\rho-1} h(x)] \not\equiv [0\ \ldots\ 0], \quad (12)$$

where W^κ denotes the κth column vector of the matrix W, or $\rho = \infty$ if such an integer does not exist.

In closing this section, we recall the order of magnitude notation, which will be used in the subsequent sections.

Definition 3. (Khalil (1992).) $\delta(\epsilon) = O(\epsilon)$ if there exist positive constants k and c such that

$$|\delta(\epsilon)| \le k\,|\epsilon| \quad \forall\, |\epsilon| < c. \quad (13)$$

3. FORMULATION OF THE CONTROL PROBLEM

We shall address and solve the problem of synthesizing well-conditioned control laws that preserve the two-time-scale nature of the open-loop system, with the following objectives for the closed-loop system:

(i) boundedness of the trajectories;

(ii) the output of the closed-loop system satisfies a relation of the form

$$y(t) = y^s(t) + O(\epsilon), \quad t \ge 0, \quad (14)$$

with $y^s(t)$ being the output of the closed-loop reduced system, where a prespecified input/output response is enforced independently of the disturbances.

The above requirements will be enforced in the closed-loop system for sufficiently small values of the singular perturbation parameter ϵ. The control laws will utilize feedback of the state z to stabilize the fast dynamics and feedback of the state x combined with feedforward compensation of disturbances to enforce the desired behavior on the output.

The above-specified control problem will be initially addressed for systems in standard form and then for systems in nonstandard form. For systems in standard form, it will be established that the requisite control law can be synthesized utilizing exclusively information concerning the original system (1), while for systems in nonstandard form it has to be preceded by the feedback regularization of the fast dynamics.

4. DISTURBANCE COMPENSATION FOR SYSTEMS IN STANDARD FORM

In this section, we shall consider systems of the form (1) in standard form, with possibly unstable fast dynamics, i.e. systems for which some of the eigenvalues of the matrix $Q_2(x)$ may lie in the open right half of the complex plane for some $x \in X$. The instability of the fast dynamics dictates the need to employ feedback of the state z (Kokotovic *et al.*, 1986; Christofides and Daoutidis, 1996). In particular, motivated by the affine appearance of the fast state z in the model (1), let us initially consider control laws of the form

$$u = \tilde{u} + k^{\mathrm{T}}(x)z, \tag{15}$$

where $k^{\mathrm{T}}(x)$ is in $\mathbb{R}^p$, and $\tilde{u}$ is an auxiliary input, to achieve stabilization of the fast dynamics of the system. Under a control law of the form (15), the two-time-scale system (1) takes the form

$$\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)k^{\mathrm{T}}(x)]z \\ &\quad + g_1(x)\tilde{u} + W_1(x)d, \\ \epsilon\dot{z} &= f_2(x) + [Q_2(x) + g_2(x)k^{\mathrm{T}}(x)]z \\ &\quad + g_2(x)\tilde{u} + W_2(x)d. \end{aligned} \tag{16}$$

One can immediately observe that a control law of the form (15) preserves the two-time-scale nature of the process, and the linearity with respect to the state z, the auxiliary input $\tilde{u}$ and the disturbance vector d. Furthermore, performing a standard two-time-scale decomposition, one can easily show that the fast subsystem is given by

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}\tau} &= f_2(x) + [Q_2(x) + g_2(x)k^{\mathrm{T}}(x)]z \\ &\quad + g_2(x)\tilde{u} + W_2(x)d, \end{aligned} \tag{17}$$

while the reduced system takes the form

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\tilde{u} + \tilde{W}(x)d, \\ y^{\mathrm{s}} &= h(x), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \tilde{F}(x) &= f_1(x) - [Q_1(x) + g_1(x)]^{\mathrm{T}}k(x)[Q_2(x) \\ &\quad + g_2(x)k^{\mathrm{T}}(x)]^{-1}f_2(x), \\ \tilde{G}(x) &= g_1(x) - [Q_1(x) + g_1(x)k^{\mathrm{T}}(x)][Q_2(x) \\ &\quad + g_2(x)k^{\mathrm{T}}(x)]^{-1}g_2(x), \\ \tilde{W}(x) &= W_1(x) - [Q_1(x) + g_1(x)k^{\mathrm{T}}(x)][Q_2(x) \\ &\quad + g_2(x)k^{\mathrm{T}}(x)]^{-1}W_2(x). \end{aligned} \tag{19}$$

Clearly, the z-dependent control law (15) allows stabilization of the fast dynamics of the system by a choice of $k^{\mathrm{T}}(x)$ such that the matrix $Q_2(x) + g_2(x)k^{\mathrm{T}}(x)$ is Hurwitz uniformly in $x \in X$ (Assumption 1). Proposition 1 that follows establishes conditions for the invariance of relative orders r and ρ, under the control law (15). The proof of the proposition can be found in the Appendix.

Proposition 1. Consider the two-time-scale system of (16), assumed to be in standard form. Then, referring to the reduced system (18),

(a) the relative order of y^{s} with respect to the auxiliary input $\tilde{u}$ is equal to r, and
(b) the relative order of y^{s} with respect to the disturbance input vector d is equal to ρ, if $\rho \le r$.

In order to enforce the desired behavior on the output, let us now consider feedforward/state feedback laws of the form

$$\tilde{u} = p(x) + q(x)v + Q(x, d, d^{(1)}, \ldots), \tag{20}$$

where $p(x)$ and $q(x)$ are analytic scalar functions, with $q(x) \neq 0$ uniformly in $x \in X$, v is the reference input, and Q is a smooth algebraic function that is nonsingular under nominal conditions $(x_0, 0, \ldots)$ and well defined and finite for all possible smooth functions of time d. Given the asymptotic stability of the fast dynamics of the system (16), the control law (20) uses feedback of the slow state vector x only, to avoid destabilization of the fast dynamics, while dynamic feedforward terms are allowed in order to achieve complete elimination of the effect of d on y in the closed-loop reduced system.

Substitution of the control law (20) into the system (16) yields the following closed-loop system:

$$\begin{aligned} \dot{x} &= [f_1(x) + g_1(x)p(x)] + [Q_1(x) + g_1(x)k^{\mathrm{T}}(x)]z \\ &\quad + g_1(x)q(x)v + g_1(x)Q(x, d, d^{(1)}, \ldots) \\ &\quad + W_1(x)d, \\ \epsilon\dot{z} &= [f_2(x) + g_2(x)p(x)] \\ &\quad + [Q_2(x) + g_2(x)k^{\mathrm{T}}(x)]z + g_2(x)q(x)v \\ &\quad + g_2(x)Q(x, d, d^{(1)}, \ldots) + W_2(x)d, \\ y &= h(x). \end{aligned} \tag{21}$$

On the basis of (21), it is clear that the control law (20) does not modify the two-time-scale nature of the system, and the linearity with respect to the state vector z and the reference input v. Employing a two-time-scale decomposition for the system (21), it can be easily shown that the closed-loop fast subsystem is given by

$$\begin{aligned} \frac{dz}{\mathrm{d}\tau} &= [f_2(x) + g_2(x)p(x)] \\ &\quad + [Q_2(x) + g_2(x)k^{\mathrm{T}}(x)]z + g_2(x)[q(x)v \\ &\quad + Q(x, d, d^{(1)}, \ldots] + W_2(x)d, \end{aligned} \tag{22}$$

and the closed-loop reduced system takes the form

$$\begin{aligned} \dot{x} &= [\tilde{F}(x) + \tilde{G}(x)p(x)] + \tilde{G}(x)[q(x)v \\ &\quad + Q(x, d, d^{(1)}), \ldots] + \tilde{W}(x)d, \\ y^{\mathrm{s}} &= h(x), \end{aligned} \tag{23}$$

where $\tilde{F}(x)$, $\tilde{G}(x)$ and $\tilde{W}(x)$ are defined in (19).

Referring to (22), one can immediately see that the feedforward/state feedback law (20) preserves the stability characteristics of the fast dynamics of the original system. Proposition 2 that follows will allow the formulation of the controller synthesis problem in the closed-loop reduced system.

Proposition 2. Consider the two-time-scale system of (16), assumed to be in standard form. Then a feedforward/static state feedback control law of the form (20) preserves the relative order r, in the sense that the relative order of the output y^s with respect to the reference input v in the closed-loop reduced system (23) is equal to r.

Proof. The proof of the Proposition involves the application of the two-time-scale decomposition procedure to the resulting closed-loop system, and the standard argument for nonlinear systems of the form (23) under feedforward/state feedback of the form of (20) (see. e.g. Daoutidis and Kravaris, 1993). □

The result of proposition 2 suggests requesting an input/output response of relative order r in the closed-loop reduced system. For simplicity, the following linear input/output response will be postulated

$$\beta_r \frac{d^r y^s}{dt^r} + \ldots + \beta_1 \frac{dy^s}{dt} + \beta_0 y^s = v, \qquad (24)$$

where $\beta_0, \ldots, \beta_r$ are adjustable parameters, which can be chosen to guarantee input/output stability in the closed-loop reduced system and enforce desired performance characteristics.

Theorem 1 that follows summarizes the main result of this section. The proof of the theorem can be found in the Appendix.

Theorem 1. Consider the two-time-scale nonlinear system (1), assumed to be in standard form. Consider also the reduced system (5), and assume that $\rho < r$. Then the conditions

$$L_G \Phi_l(x, d) \equiv 0, \quad l = 0, 1, \ldots, r - \rho - 1, \qquad (25)$$

where

$$\Phi_l(x, d) = \sum_{\mu=0}^{l} L_F^{l-\mu}\left[\sum_{\kappa=1}^{q} d_\kappa(t) L_{W^\kappa} + \frac{\partial}{\partial t}\right] + \left[L_F + \sum_{\kappa=1}^{q} d_\kappa(t) L_{W^\kappa} + \frac{\partial}{\partial t}\right]^{\mu} L_F^{\rho-1} h(x) \qquad (26)$$

are necessary and sufficient in order for a control law of the form (20) to induce an input/output behavior of the form (24) in the closed-loop reduced system. If these conditions are satisfied and Assumption 1 also holds, the control law

$$\begin{aligned} u = \{1 + k^T(x)[Q_2(x)]^{-1} g_2(x)\}[\beta_r L_G L_F^{r-1} h(x)]^{-1} \\ \times \left[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\right] \\ + k^T(x)[Q_2(x)]^{-1}[f_2(x) + W_2(x)d] + k^T(x) z, \end{aligned} \qquad (27)$$

where the feedback gain $k^T(x)$ is such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$,

(a) guarantees exponential stability of the fast dynamics of the closed-loop system;

(b) ensures that the output of the closed-loop system satisfies a relation of the form

$$y(t) = y^s(t) + O(\epsilon), \quad t \geq 0, \qquad (28)$$

for ϵ sufficiently small, with $y^s(t)$ being the solution of (24).

Remark 1. The form of the functions defined in (26) and the pattern of the conditions of (25) can be found in Daoutidis and Kravaris (1993). From the result of the theorem, we note that the verification of the solvability conditions (25) and the evaluation of the functions Φ_l, $l = 0, 1, \ldots, r - \rho - 1$, are independent of the selection of the feedback gain $k^T(x)$ used for the stabilization of the fast dynamics, and thus can be performed on the basis of the open-loop reduced system (5). This fact is an immediate implication of the results of Propositions 1 and 2.

Remark 2. The feedforward/state feedback law (27) can be decomposed into two separate components: the components $k^T(x)(z - \xi)$, where ξ denotes a control-dependent quasi-steady state for the fast dynamics of the closed-loop system, defined as

$$\begin{aligned} \xi = -[Q_2(x)]^{-1}\Big\{f_2(x) + g_2(x)[\beta_r L_G L_F^{r-1} h(x)]^{-1} \\ \times \left[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\right] \\ + W_2(x)d\Big\}, \end{aligned} \qquad (29)$$

which acts in the fast time scale and stabilizes the fast dynamics of the system, and the component

$$[\beta_r L_G L_F^{r-1} h(x)]^{-1}\left[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\right], \quad (30)$$

which acts in the slow time scale and addresses the posed synthesis problem, i.e. induces the desired input/output behaviour in the closed-loop reduced system independently of the disturbances.

Remark 3. The fact that the component $k^T(x)(z - \xi)$ acts on the fast time scale, where the state vector x can be considered approximately equal to its initial value $x(0)$, allows design of the feedback gain $k^T(x)$ using standard control methods, such as pole placement, optimal control etc. (Kokotovic *et al.*, 1986).

Remark 4. In the case of two-time-scale systems of the form (1) with stable fast dynamics, i.e. when the matrix $Q_2(x)$ is Hurwitz uniformly in x, there is no need to employ feedback of the fast variable to stabilize the fast dynamics and the feedforward/feedback law (27) reduces to

$$u = [\beta_r L_G L_F^{r-1} h(x)]^{-1}\left[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\right] \quad (31)$$

In the remainder of this section, motivated by the available results on disturbance decoupling for standard nonlinear systems of the form (5) via static state feedback (Isidori *et al.*, 1981) of the form

$$u = p(x) + q(x)v, \quad (32)$$

we shall provide necessary and sufficient conditions for the solvability of the control problem formulated in Section 3, via well-conditioned static state feedback laws of the form

$$u = p(x) + q(x)v + k^T(x)z. \quad (33)$$

The main result is given in Proposition 3 that follows (the proof is included in the Appendix).

Proposition 3. Consider the two-time-scale system (1) in standard form, for which Assumption 1 holds. Consider also the reduced system (5). Then, the conditions

(i) $r < \rho$;

(ii) $k^T(x)[Q_2(x)]^{-1}W_2(x) \equiv \bar{0}$,

where $\bar{0}$ is the zero covector of dimension q, are necessary and sufficient in order for a well-conditioned static state feedback law of the form (33) to achieve approximate decoupling of the effect of d on y (in the sense made precise in requirement (ii) in Section 3) with stabilization of the fast dynamics in the closed-loop system.

Remark 5. Condition (i) of the proposition is expected to hold for the solvability of this problem, since it is necessary and sufficient for the solvability of the disturbance decoupling problem via static state feedback of the form (32) (see e.g. Isidori, 1989) and the approximate disturbance decoupling problem for two-time-scale systems (1) with exponentially stable fast dynamics, via the same class of control laws (see Levine and Rouchon, 1991). Condition (ii) of the proposition guarantees that the relative order of the output y^s with respect to the disturbance input vector d in the reduced system (18), say $\tilde{\rho}$, is $\tilde{\rho} = \rho > r$.

5. DISTURBANCE COMPENSATION FOR SYSTEMS IN NONSTANDARD FORM

In this section, we shall consider two-time-scale nonlinear systems (1) in nonstandard form, i.e. systems for which the matrix $Q_2(x)$ is singular for some $x \in X$. The direct consequence of the singularity of the matrix $Q_2(x)$ is the absence of a well-defined quasi-steady-state for the fast variables z (Khalil, 1989; Christofides and Daoutidis, 1996), and thus the lack of a well-defined open-loop reduced system. Therefore the results of Propositions 1 and 2 that allowed the verification of the solvability conditions (25) and the synthesis of the controller (27) on the basis of the open-loop reduced system (5) cannot be recovered. Motivated by this, we shall follow a two-step procedure for the synthesis of a feedforward/state feedback law that solves the posed problem. In the first step, appropriate feedback of the state vector z will be employed to regularize the fast dynamics, in the sense of inducing an exponentially stable quasi-steady-state for the fast dynamics. In the second step, we shall formulate and solve the synthesis problem on the basis of the resulting two-time-scale system in standard form.

More specifically, we shall initially consider a control law of the form

$$u = \hat{u} + k^{\mathrm{T}}(x)z, \tag{34}$$

where $k^{\mathrm{T}}(x)$ is a covector field in $\mathbb{R}^p$ and $\hat{u}$ is an auxiliary input, to achieve regularization of the fast dynamics. Under the control law (34), the system (1) takes the form

$$\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)k^{\mathrm{T}}(x)]z \\ &\quad + g_1(x)\hat{u} + W_1(x)d, \\ \epsilon\dot{z} &= f_2(x) + [Q_2(x) + g_2(x)k^{\mathrm{T}}(x)]z \\ &\quad + g_2(x)\hat{u} + W_2(x)d. \end{aligned} \tag{35}$$

Performing a two-time-scale decomposition, the corresponding fast subsystem takes the form

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}\tau} &= f_2(x) + [Q_2(x) + g_2(x)k^{\mathrm{T}}(x)]z \\ &\quad + g_2(x)\hat{u} + W_2(x)d, \end{aligned} \tag{36}$$

while the corresponding slow subsystem is given by

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\hat{u} + \tilde{W}(x)d, \\ \hat{y}^s &= h(x), \end{aligned} \tag{37}$$

where the vector fields $\tilde{F}(x)$, $\tilde{G}(x)$ and $\tilde{W}(x)$ are given in (19). It is clear that the z-dependent control law (34) allows us to regularize the fast dynamics of the system by choosing $k^{\mathrm{T}}(x)$ in such a manner that the matrix $Q_2(x) + g_2(x)k^{\mathrm{T}}(x)$ is Hurwitz uniformly in $x \in X$. It is now possible to formulate and solve a controller synthesis problem on the basis of the system (35). More specifically, we shall seek a feedforward/state feedback law of the form

$$\hat{u} = \tilde{p}(x) + \tilde{q}(x)v + \tilde{Q}(x, d, d^{(1)}, \ldots), \tag{38}$$

where $\tilde{p}(x)$ and $\tilde{q}(x)$ are scalar fields, with $\tilde{q}(x) \neq 0$ for all $x \in X$, $\tilde{Q}$ is a smooth algebraic nonsingular function and v is the reference input. Referring to the system (37), let $\hat{r}$ and $\hat{\rho}$ denote the relative orders of the output y with respect to the auxiliary input $\hat{u}$ and the disturbance input vector d, and let the input/output behavior

$$\beta_{\hat{r}} \frac{\mathrm{d}^{\hat{r}}\hat{y}^s}{\mathrm{d}t^{\hat{r}}} + \ldots + \beta_1 \frac{\mathrm{d}\hat{y}^s}{\mathrm{d}t} + \beta_0 \hat{y}^s = v \tag{39}$$

be postulated in the closed-loop reduced system.

Theorem 2 that follows summarizes the main result of this section. The proof of the theorem can be found in the Appendix.

Theorem 2. Consider the two-time-scale nonlinear system (1), assumed to be in nonstandard form. Consider also the nonlinear system (37), and assume that $\hat{\rho} < \hat{r}$. Then the conditions

$$L_{\tilde{G}}\tilde{\Phi}_l(x, d) \equiv 0, \quad l = 0, 1, \ldots, \hat{r} - \hat{\rho} - 1, \tag{40}$$

where

$$\begin{aligned} \tilde{\Phi}_l(x, d) = \sum_{\mu=0}^{l} L_{\tilde{F}}^{l-\mu}\left[\sum_{\kappa=1}^{q} d_\kappa(t) L_{\tilde{W}^\kappa} + \frac{\partial}{\partial t}\right] \\ \times \left[L_{\tilde{F}} + \sum_{\kappa=1}^{q} d_\kappa(t) L_{\tilde{W}^\kappa} + \frac{\partial}{\partial t}\right]^{\mu} L_{\tilde{F}}^{\hat{\rho}-1} h(x), \end{aligned} \tag{41}$$

are necessary and sufficient in order for a control law of the form (38) to induce the input/output behavior of the form (39) in the closed-loop reduced system. If these conditions are satisfied and Assumption 1 also holds, the control law

$$\begin{aligned} u = [\beta_{\hat{r}} L_{\tilde{G}} L_{\tilde{F}}^{\hat{r}-1} h(x)]^{-1} \Bigg[v - \sum_{k=0}^{\hat{r}} \beta_k L_{\tilde{F}}^k h(x) \\ - \sum_{k=\hat{\rho}}^{\hat{r}} \beta_k \tilde{\Phi}_{k-\hat{\rho}}(x, d, d^{(1)}, \ldots, d^{(k-\hat{\rho})}) \Bigg] \\ + k^{\mathrm{T}}(x)z, \end{aligned} \tag{42}$$

where the feedback gain $k^{\mathrm{T}}(x)$ is such that the matrix $Q_2(x) + g_2(x)k^{\mathrm{T}}(x)$ is Hurwitz uniformly in $x \in X$,

(a) guarantees exponential stability of the fast dynamics of the closed-loop system;

(b) ensures that the output of the closed-loop system satisfies a relation of the form

$$y(t) = \hat{y}^s(t) + O(\epsilon), \quad t \geq 0, \tag{43}$$

for ϵ sufficiently small, with $\hat{y}^s(t)$ being the solution of (39).

Remark 6. The result of Theorem 2 reveals a fundamental difference in the nature of the control problem between two-time-scale systems in standard and nonstandard form. In particular, in the case of systems in standard form, the solvability conditions of the problem can be checked in the open-loop reduced system (5), since they are independent of the feedback gain $k^{\mathrm{T}}(x)$ used to stabilize the fast dynamics. On the other hand, in the case of systems in nonstandard form, the solvability conditions should be checked in the reduced system (37), and thus depend explicitly on $k^{\mathrm{T}}(x)$ used to regularize the fast dynamics. These considerations imply that for systems in nonstandard form, it is possible, depending on the structure of the system under consideration, to select the feedback gain $k^{\mathrm{T}}(x)$ to guarantee stabilization of the fast dynamics as well as to ensure that the solvability conditions are satisfied.

Remark 7. In the case of two-time-scale systems of the form (1) in nonstandard form, one can show that the condition $\hat{r} < \hat{\rho}$ is necessary and sufficient for achieving approximate decoupling of the effect of the disturbances on the output of the closed-loop full-order system via well-conditioned static state feedback of the form (33). Notice that this condition depends explicitly on the feedback gain $k^T(x)$ utilized to regularize the fast dynamics, because of the lack of an analogue of Proposition 1 in the case of two-time-scale systems in nonstandard form.

Remark 8. The control algorithms of Theorems 1 and 2 can be directly applied to two-time-scale systems with ϵ-dependent right-hand side, with the following state-space description:

$$\begin{aligned} \dot{x} &= f_1(x, \epsilon z, \epsilon) + Q_1(x, \epsilon z, \epsilon)z \\ &\quad + g_1(x, \epsilon z, \epsilon)u + W_1(x, \epsilon z, \epsilon)d, \\ \epsilon\dot{z} &= R(x, z)[f_2(x, \epsilon z, \epsilon) \\ &\quad + Q_2(x, \epsilon z, \epsilon)z + g_2(x, \epsilon z, \epsilon)u \\ &\quad + W_2(x, \epsilon z, \epsilon)d], \\ y &= h(x), \end{aligned} \tag{44}$$

where $R(x, z)$ is a diagonal matrix of dimension $p \times p$, which is positive-definite for all $x \in X$, $z \in Z$. This is possible because (i) the stabilizability requirement of Assumption 1 suffices to ensure that the fast subsystem of the system (44) can be made exponentially stable, and (ii) in the case of nonsingular $Q_2(x, \epsilon z, \epsilon)$, the open-loop slow subsystem of the system (44) is the same as that of (5) (which ensures the applicability of Theorem 1), while in the case of singular $Q_2(x, \epsilon z, \epsilon)$, the slow subsystem obtained after the regularization of the fast dynamics is the same as that of (37) (which ensures the applicability of Theorem 2).

6. STABILITY ANALYSIS OF THE CLOSED-LOOP SYSTEM

In this section, the stability of the closed-loop full-order system under the derived control laws will be analyzed. Our objective is to specify conditions and provide an explicit formula for the calculation of the upper bound on ϵ, such that the states of the closed-loop system are bounded. The presence of time-varying disturbances in the model of the system does not allow utilizing standard stability results for two-time-scale systems (Saberi and Khalil, 1984; Khalil, 1992) (which are concerned with asymptotic stability), and requires further analysis of the closed-loop system to show boundedness of the states. Theorem 3 that follows states the main stability result for two-time-scale systems in standard form, under the control law (27) (the proof is given in the Appendix).

Theorem 3. Consider the two-time-scale nonlinear system with the state-space representation (1), assumed to be in standard form. Then, if $d(t)$ and its derivatives up to order $r - \rho + 1$ are sufficiently small and

(i) the roots of the polynomial

$$\beta_0 + \beta_1 s + \ldots + \beta_r s^r = 0 \tag{45}$$

lie in the open left half of the complex plane, and

(ii) the unforced zero dynamics of the open-loop reduced system (5) is exponentially stable,

there exists an ϵ^* such that the trajectories of the closed-loop system under the controller of Theorem 1 remain bounded for all $\epsilon \in (0, \epsilon^*]$.

Theorem 3 also holds for two-time-scale systems in nonstandard form, with the second condition imposed on the reduced system (37). The proof of this result is completely analogous to that of Theorem 3, and therefore is omitted for brevity. Finally, we note that the stability result of Theorem 3 holds even in the presence of sufficiently small constant parametric uncertainty and unmeasured disturbances (see the proof of the theorem for the detailed justification).

7. SIMULATION STUDY: A CATALYTIC CONTINUOUS STIRRED TANK REACTOR

In this section, the proposed control methodology will be applied to a representative chemical process with time-scale multiplicity. Consider the catalytic continuous stireed tank reactor shown in Fig. 1, where a homogeneous reaction A→B and a catalytic reaction A→C take place. The first reaction leads to the generation of the side-product B, while the second reaction leads to the production of the desired product C. The inlet stream F_1 consists of pure species A of concentration C_{A0}, and temperature T_{A0}. Under the assumptions

- perfect mixing in the reactor,
- uniform temperature in the homogeneous and catalytic phases,
- constant density and heat capacity of the reacting liquid and the catalyst,
- constant outlet flow rate of the reactor,

the process dynamic model consists of the following set of material and energy balances:

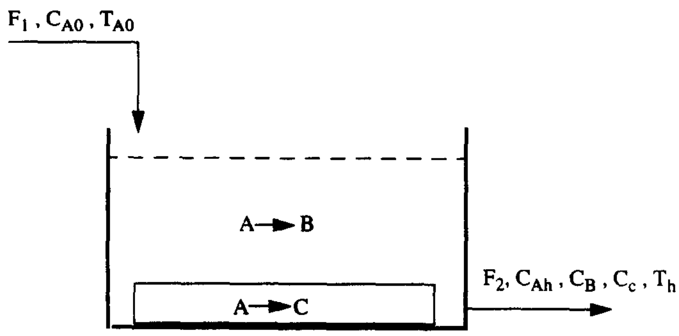

Fig. 1. A catalytic continuous stirred tank reactor.

- reactor mass balance:

$$\frac{\mathrm{d}V_r}{\mathrm{d}t} = F_1 - F_2; \tag{46}$$

- mole balance for species A (homogeneous phase):

$$\frac{\mathrm{d}C_{\mathrm{Ah}}}{\mathrm{d}t} = \frac{1}{V_r}\Big[F_1(C_{\mathrm{A0}} - C_{\mathrm{Ah}}) - k_h \exp\left(\frac{-E_h}{RT_h}\right)V_r - K_c A_c (C_{\mathrm{Ah}} - C_{\mathrm{Ac}})\Big] \tag{47}$$

- reactor energy balance (homogeneous phase):

$$\rho_h c_{ph}\frac{\mathrm{d}T_h}{\mathrm{d}t} = \frac{1}{V_r}\Big[\rho_h c_{ph} F_1 (T_{\mathrm{A0}} - T_h) + (-\Delta H_h) k_h \exp\left(\frac{-E_h}{RT_h}\right)V_r - U_w A_w (T_h - T_w) - U_c A_c (T_h - T_c)\Big]; \tag{48}$$

- mole balance for species A (catalytic phase):

$$\frac{\mathrm{d}C_{\mathrm{Ac}}}{\mathrm{d}t} = \frac{K_c A_c}{V_c}(C_{\mathrm{Ah}} - C_{\mathrm{Ac}}) - k_c \exp\left(\frac{-E_c}{RT_c}\right)C_{\mathrm{Ac}}; \tag{49}$$

- reactor energy balance (catalytic phase):

$$\rho_c c_{pc}\frac{\mathrm{d}T_c}{\mathrm{d}t} = \frac{U_c A_c}{V_c}(T_k - T_c) + (-\Delta H_c) k_c \exp\left(\frac{-E_c}{RT_c}\right)C_{\mathrm{Ac}}. \tag{50}$$

Here F_1 and F_2 denote the inlet and outlet flow rates, V_r and V_c denote the volumes of the homogeneous and catalytic phases, C_{Ah}, T_h and C_{Ac}, T_c denote the concentration and temperature of species A in the homogeneous and catalytic phases, k_h, k_c, E_h, E_c, ΔH_h and ΔH_c denote the pre-exponential factors, the activation energies and the enthalpies of the two reactions, K_c and U_c, U_w denote mass and heat transfer coefficients, and A_w and A_c denote the surfaces of the wall and the catalyst.

The control objective is the regulation of the temperature of the catalyst by manipulating the inlet flow rate F_1, in order to maintain the generation of the product species C at the desired level. The inlet concentration and temperature of the species A, C_{A0} and T_{A0} respectively, as well as the wall temperature T_w, are assumed to be the measurable disturbances. The values of the system parameters and the corresponding steady-state values of the system variables are given in Table 1. It was verified that these conditions correspond to a critically stable equilibrium point (the holdup of the homogeneous phase acts as an integrator). The process exhibits two-time-scale behavior owing to the large heat capacity of the catalytic phase (i.e. the term $\rho_c c_{pc}$ that multiplies the time derivative of the catalyst temperature is large). This implies that V_r, C_{Ah}, T_h and C_{Ac} are the fast process variables, while T_c is the slow process

Table 1. Process parameters

$F_2 = 500.0\ \mathrm{L\ min^{-1}}$
$c_{ph} = 0.231\ \mathrm{kcal\ kg^{-1}\ K^{-1}}$
$c_{pc} = 2.31\ \mathrm{kcal\ kg^{-1}\ K^{-1}}$
$\rho_h = 0.9\ \mathrm{kg\ L^{-1}}$
$\rho_c = 90.0\ \mathrm{kg\ L^{-1}}$
$K_c A_c = 1618.0\ \mathrm{L\ min^{-1}}$
$U_c A_c = 6667.0\ \mathrm{kcal\ min^{-1}\ K^{-1}}$
$U_w A_r = 3340.0\ \mathrm{kcal\ min^{-1}\ K^{-1}}$
$R = 1.987\ \mathrm{kcal\ kmol^{-1}\ K^{-1}}$
$k_h = 164.68\ \mathrm{L\ mol^{-1}\ min^{-1}}$
$E_h = 8.0 \times 10^3\ \mathrm{kcal\ kg^{-1}}$
$k_c = 2000.0\ \mathrm{min^{-1}}$
$E_c = 9.0 \times 10^3\ \mathrm{kcal\ kg^{-1}}$
$\Delta H_h = 69.2006\ \mathrm{kcal\ kmol^{-1}}$
$\Delta H_c = -99.0781\ \mathrm{kcal\ kmol^{-1}}$
$V_c = 145.1\ \mathrm{L}$
$V_s = 1000.0\ \mathrm{L}$
$C_{\mathrm{Ah}} = 5.0\ \mathrm{mol\ L^{-1}}$
$T_h = 690\ \mathrm{K}$
$C_{\mathrm{Ac}} = 3.75\ \mathrm{mol\ L^{-1}}$
$T_c = 720\ \mathrm{K}$
$F_{1s} = 500\ \mathrm{L\ min^{-1}}$
$C_{\mathrm{A0s}} = 10.0\ \mathrm{mol\ L^{-1}}$
$T_{\mathrm{A0s}} = 305.0\ \mathrm{K}$
$T_{ws} = 310.0\ \mathrm{K}$

variable (this fact was also validated through open-loop simulations). It was also verified that the process exhibits slightly non-minimum-phase behavior, i.e. its zero dynamics exhibits a two-time-scale behavior with critically stable fast dynamics (the fast process modes are part of the zero dynamics modes). In order to obtain a singularly perturbed representation of the process, where the partition to slow and fast variables is consistent with the dynamic behavior of the process, the parameter ϵ is defined as

$$\epsilon = \frac{1}{\rho_c c_{pc}} = 0.048\,\frac{\mathrm{K \cdot L}}{\mathrm{kcal}}. \tag{51}$$

Setting

$$x_1 = T_c,\quad z_1 = V_r,\quad z_2 = C_{Ah},\quad z_3 = T_h,$$
$$z_4 = C_{Ac},\quad u = F_1 - F_{1s},\quad d_1 = C_{A0} - C_{A0s},$$
$$d_2 = T_{A0} - T_{A0s},$$
$$d_3 = T_w - T_{ws},\quad y = x_1,\quad \bar{t} = \frac{t}{\rho_c c_{pc}},$$

the original set of equations can be put in the following singularly perturbed form (where the time derivatives are taken with respect to $\bar{t}$):

$$\begin{aligned}
\dot{x}_1 &= \frac{U_c A_c}{V_c}(z_3 - x_1) + (-\Delta H_c) k_c \exp\left(\frac{-E_c}{Rx_1}\right) z_4,\\
\epsilon \dot{z}_1 &= u,\\
\epsilon \dot{z}_2 &= \frac{1}{z_1}\Bigg[F_{1s} C_{A0s} + -k_h \exp\left(\frac{-E_h}{Rz_3}\right) z_1\\
&\quad + (-F_{1s} - K_c A_c) z_2 + K_c A_c z_3\\
&\quad + (C_{A0} - z_2) u + F_{1s} d_1\Bigg],\\
\epsilon \dot{z}_3 &= \frac{1}{z_1}\Bigg[F_{1s}(T_{A0} - z_3) - \frac{U_w A_w}{\rho_h c_{ph}}(z_3 - T_{ws})\\
&\quad - \frac{U_c A_c}{\rho_h c_{ph}}(z_3 - x_1) + \frac{(-\Delta H_h)}{\rho_h c_{ph}} k_h \exp\left(\frac{-E_h}{Rz_3}\right) z_1\\
&\quad + (T_{A0} - z_3) u + F_{1s} d_2 + \frac{U_w A_w}{\rho_h c_{ph}} d_3\Bigg],\\
\epsilon \dot{z}_4 &= \frac{K_c A_c}{V_c} z_2 + \left[-\frac{K_c A_c}{V_c} - k_c \exp\left(\frac{-E_c}{Rx_1}\right)\right] z_4.
\end{aligned} \tag{52}$$

From the structure of the differential equation for z_1 in the above system, it is clear that the fast dynamics of the process are singular, i.e. there exists no well-defined quasi-steady-state for the fast state vector z (this is an implication of the assumption that the outlet flow rate is constant). Since the process is in nonstandard form, the two-step procedure of Section 5 will be followed for the synthesis of the controller. In the first step, the regularizing feedback law of the form

$$u = \hat{u} - z_1 \tag{53}$$

was used to transform the original two-time-scale system into a new one in standard form with exponentially stable fast dynamics. Under this preliminary feedback, the original system takes the form (where the time derivatives are taken with respect to $\bar{t}$)

$$\begin{aligned}
\dot{x}_1 &= \frac{U_c A_c}{V_c}(z_3 - x_1) + (-\Delta H_c) k_c \exp\left(\frac{-E_c}{Rx_1}\right) z_4,\\
\epsilon \dot{z}_1 &= -z_1 + \hat{u},\\
\epsilon \dot{z}_2 &= \frac{1}{z_1}\Bigg\{F_{1s} C_{A0s} + \left[C_{A0} - z_2 - k_h \exp\left(\frac{-E_h}{Rz_3}\right)\right] z_1\\
&\quad + (-F_{1s} - K_c A_c) z_2\\
&\quad + K_c A_c z_3 + (C_{A0} - z_2)\hat{u} + F_{1s} d_1\Bigg\},\\
\epsilon \dot{z}_3 &= \frac{1}{z_1}\Bigg\{F_{1s}(T_{A0} - z_3) - \frac{U_w A_w}{\rho_h c_{ph}}(z_3 - T_{ws})\\
&\quad - \frac{U_c A_c}{\rho_h c_{ph}}(z_3 - x_1)\\
&\quad + \left[T_{A0} - z_3 + \frac{(-\Delta H_h) k_h}{\rho_h c_{ph}} \exp\left(\frac{-E_h}{Rz_3}\right)\right] z_1\\
&\quad + (T_{A0} - z_3)\hat{u} + F_{1s} d_2 + \frac{U_w A_w}{\rho_h c_{ph}} d_3\Bigg\},\\
\epsilon \dot{z}_4 &= \frac{K_c A_c}{V_c} z_2 + \left[-\frac{K_c A_c}{V_c} - k_c \exp\left(\frac{-E_c}{Rx_1}\right)\right] z_4.
\end{aligned} \tag{54}$$

Setting $\epsilon = 0$ in the above system and using the fact that for a physically meaningful problem $z_1 = V_r > 0$, the following set of differential and algebraic equations can be obtained:

$$\begin{aligned}
\dot{x}_1 &= \frac{U_c A_c}{V_c}(z_{3s} - x_1) + (-\Delta H_c) k_c \exp\left(\frac{-E_c}{Rx_1}\right) z_4,\\
0 &= F_{1s} C_{A0s} + \left[C_{A0} - z_{2s} - k_h \exp\left(\frac{-E_h}{Rz_{3s}}\right)\right]\hat{u}\\
&\quad + (-F_{1s} - K_c A_c) z_{2s} + K_c A_c z_{3s}\\
&\quad + (C_{A0} - z_{2s})\hat{u} + F_{1s} d_1\\
0 &= F_{1s}(T_{A0} - z_{3s}) - \frac{U_w A_w}{\rho_h c_{ph}}(z_{3s} - T_{ws})\\
&\quad - \frac{U_c A_c}{\rho_h c_{ph}}(z_{3s} - x_1) + \left[\frac{-(\Delta H_h) k_h}{\rho_h c_{ph}} \exp\left(\frac{-E_h}{Rz_{3s}}\right)\right]\hat{u}\\
&\quad + F_{1s} d_2 + \frac{U_w A_w}{\rho_h c_{ph}} d_3,\\
0 &= \frac{K_c A_c}{V_c} z_{2s} + \left[-\frac{K_c A_c}{V_c} - k_c \exp\left(\frac{-E_c}{Rx_1}\right)\right] z_{4s}.
\end{aligned} \tag{55}$$

Owing to the nonlinear appearance of the algebraic variable z_{3s} in the above model, the equilibrium manifold of the fast dynamics is of the form $z_s = \tilde{g}(x_1, \hat{u}, d_1, d_2, d_3)$, where $\tilde{g}$ is a smooth vector function. The slow subsystem takes then the form

$$\dot{x}_1 = \frac{U_c A_c}{V_c}[\tilde{g}_3(x_1, \hat{u}, d_1, d_2, d_3) - x_1] + (-\Delta H_c)k_c \exp\left(\frac{-E_c}{Rx_1}\right)\tilde{g}_4(x_1, \hat{u}, d_1, d_2, d_3) =: \mathscr{D}(x_1, \hat{u}, d_1, d_2, d_3) \quad (56)$$

where $\tilde{g}_3 = z_{3s}$ and $\tilde{g}_4 = z_{4s}$, and $\mathscr{D}(x_1, \hat{u}, d_1, d_2, d_3)$ is a nonlinear function. On the basis of this system, it is clear that the relative orders are $\hat{r} = 1$ and $\hat{\rho} = 1$. Owing to the nonlinear dependence of the auxiliary input $\hat{u}$ and the disturbances d_1, d_2 and d_3 in the system (56), the synthesis formula of Theorem 2 could not be readily used. Instead, the auxiliary input $\hat{u}$ was computed numerically by solving the nonlinear equation $\mathscr{D}(x_1, \hat{u}, d_1, d_2, d_3) = (v - \beta_0 x_1)/\beta_1$ for $\hat{u}$, so that the response

$$\beta_1 \dot{y}^s + \beta_0 y^s = v \quad (57)$$

is enforced in the closed-loop reduced system, where the parameters β_0 and β_1 were chosen to be

$$\beta_0 = 1.0, \quad \beta_1 = 1.87 \text{ min}.$$

Several simulations were performed to evaluate the performance of the controller.

In the first set of simulation runs, we addressed the capability of the controller to maintain the output of the system at the operating steady state in the presence of the following step disturbances: $d_1 = -0.8 \text{ mol L}^{-1}$,

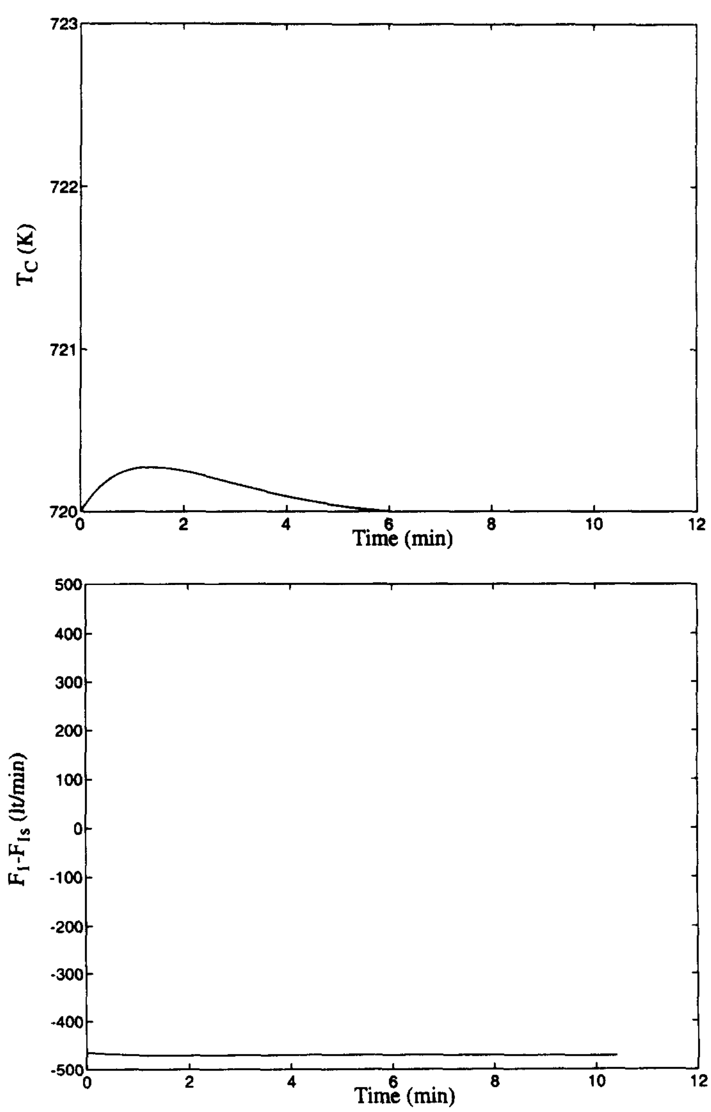

Fig. 2. Output and input profiles for regulation in the presence of constant disturbances.

$d_2 = -5.0\,\mathrm{K}$ and $d_3 = 5.0\,\mathrm{K}$, imposed at $t = 0\,\mathrm{min}$. The corresponding output and input profiles are shown in Fig. 2; clearly, the controller guarantees stability of the fast dynamics and regulates the output at the steady state, attenuating the effect of the disturbances. In the next set of simulation runs, we evaluated the reference input tracking capabilities of the controller in the presence of the above constant disturbances. A 10 K increase in the value of the reference input was imposed at time $t = 0\,\mathrm{min}$. Figure 3 displays the output and input profiles. The performance of the controller is excellent, guaranteeing the stability of the fast dynamics, regulating the output at the new reference input value and compensating for the effect of the disturbances. For the sake of comparison, we implemented the controller without measurements of the disturbance. Figure 4 depicts the output and input profiles. One can immediately see that the controller cannot attenuate the effect of the disturbances leading to offset. A feedforward/state feedback controller (Daoutidis and Kravaris, 1993) synthesized on the basis of the original two-time-scale system was also employed in the simulations. Figure 5 shows the output profile and the profile of the volume of the reactor. Clearly, this controller leads to instability of the closed-loop system, because the process exhibits a slightly non-minimum-phase behavior.

In the next set of simulation runs, we evaluated the capability of the controller to keep the output of the system at the operating steady state in the presence of time-varying disturbances. In particular, the following disturbances

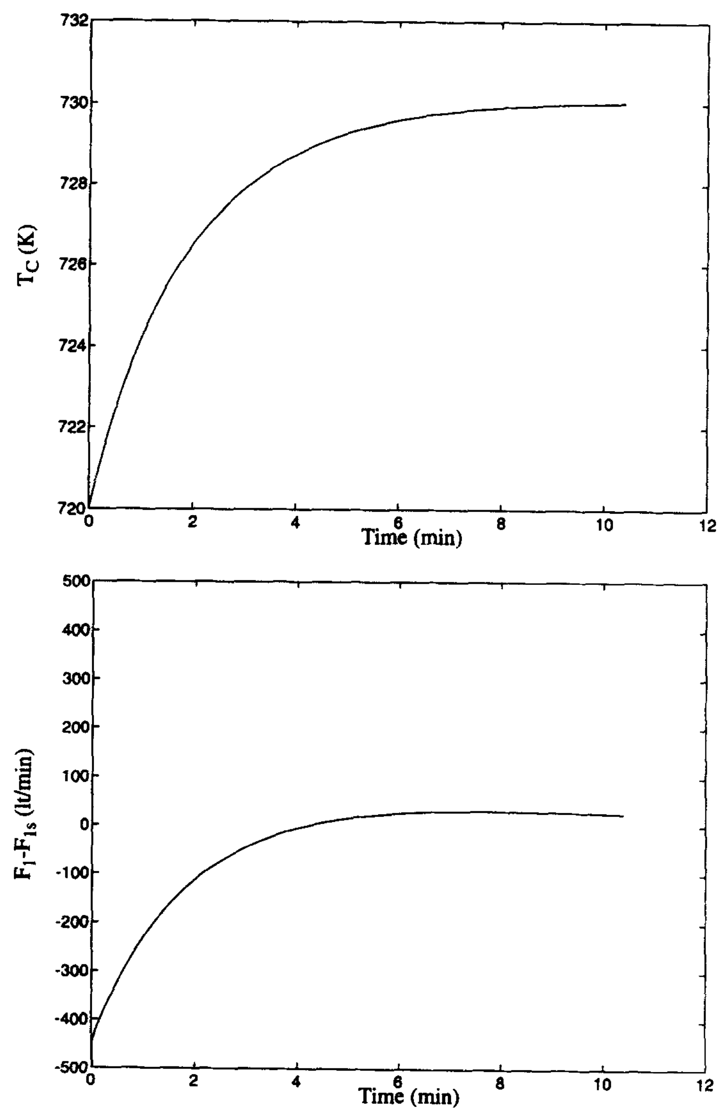

Fig. 3. Output and input profiles for reference input tracking in the presence of constant disturbances.

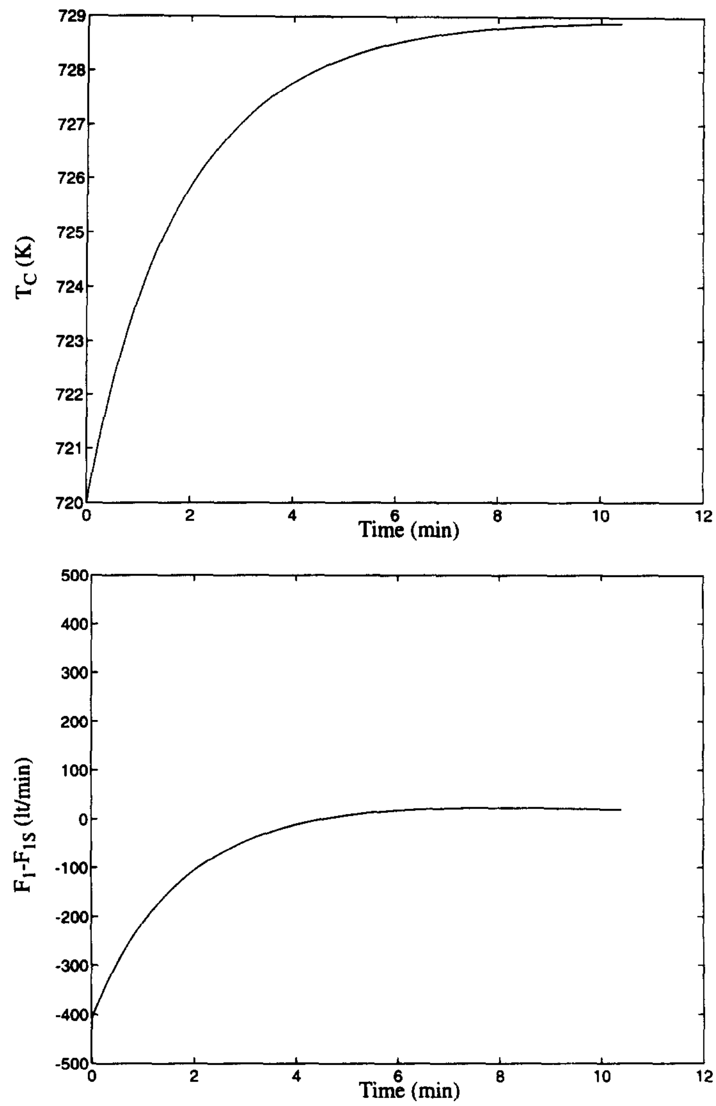

Fig. 4. Output and input profiles for reference input tracking in the presence of constant disturbances (pure feedback controller).

were imposed at $t = 0$ min:

$$d_1 = -0.8 + 0.5 \sin\left(\frac{2\pi}{T}t\right) \text{mol L}^{-1},$$

$$d_2 = -5.0 + \sin\left(\frac{2\pi}{T}t\right) \text{K},$$

$$d_3 = 5.0 + \sin\left(\frac{2\pi}{T}t\right) \text{K},$$

where $T = 0.2$ min. Figure 6 shows the output and input profiles. Clearly, the controller performance is very satisfactory, guaranteeing the stability of the fast dynamics, regulating the output at the steady state and attenuating the effect of the disturbances. Finally, the servo behavior of the controller was evaluated in the presence of the above time-varying disturbances. Figure 7 illustrtaes the resulting output and input profiles. As expected, the controller stabilizes the fast dynamics, while regulating the output at the new reference input value and attenuating the effect of the disturbances on the output.

Remark 9. Note that the implementation of the controller requires measurements of the volume of the reactor and the temperature of the catalytic phase for the feedback component, and measurements of the three disturbances for the feedforward component. It is important to compare these requirements with those of control methods that do not take explicitly into account the multiple-time-scale behavior exhibited by the process under consideration. For example, let us consider a feedforward/state feedback controller synthesized on the basis of the full-order system (Daoutidis and Kravaris, 1993). The explicit form of the controller takes

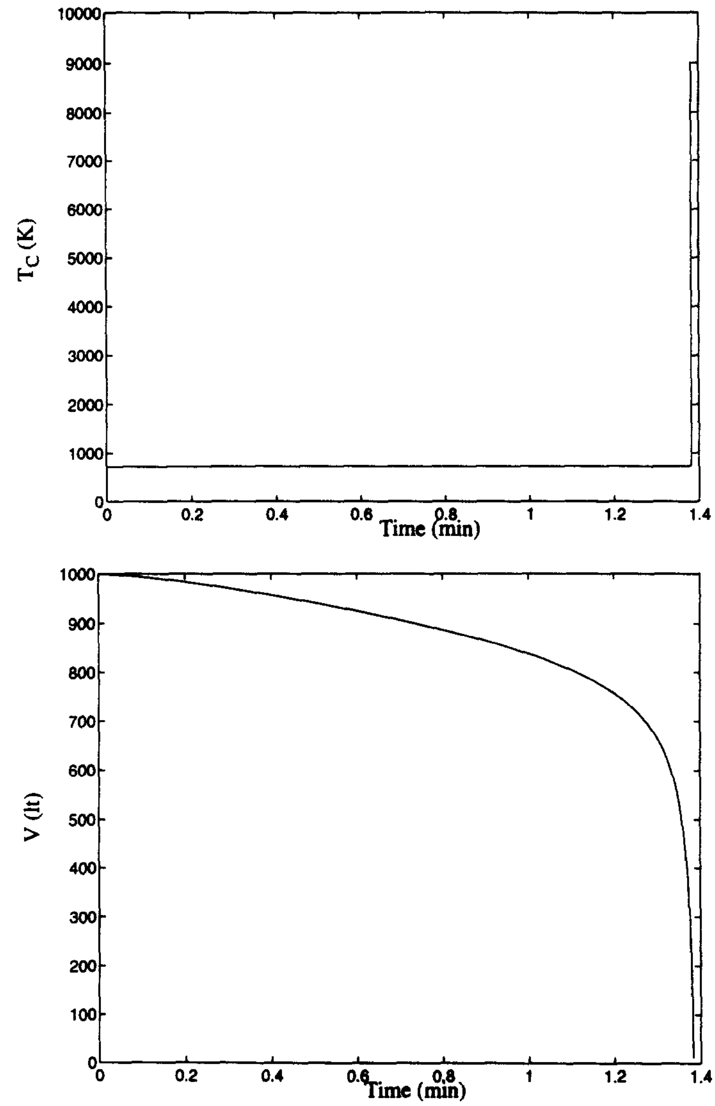

Fig. 5. Output and volume profiles for reference input tracking in the presence of constant disturbances (feedforward/feedback input/output linearizing controller).

the form

$$
\begin{aligned}
u = [\beta_2 L_{g_1} L_{f_1} h(x)]^{-1} \Big\{ & y - \beta_0 h(x) \\
& - \beta_1 \Big[L_{f_1} h(x) + L_{Q_1} h(x) z \\
& + \sum_{\kappa=1}^{3} L_{W_1} L_{f_1} h(x) d_\kappa(t) \Big] \\
& - \beta_2 \Big[L_{f_1}^2 h(x) + L_{Q_1} L_{f_1} h(x) z \\
& + [L_{f_1} L_{Q_1} h(x) + L_{Q_1}^2 h(x) + L_{g_1} L_{Q_1} h(x)] z \\
& + \frac{L_{Q_1} h(x)(f_2(x) + Q_2(x) z + W_2(x) d)}{\epsilon} \Big] \Big\},
\end{aligned}
\tag{58}
$$

which becomes singular as $\epsilon \to 0$. It can be also easily verified that the implementation of this control law requires measurements of the full state vector of the process (feedback component) as well as measurements of the three disturbance inputs (feedforward component). This includes measurements of concentrations of the species A in both the homogeneous and catalytic phases, which are difficult to obtain in practice.

From the results of the simulation study and the observations of Remark 9, it is obvious that the consideration of the two-time-scale nature of the process in question at the modeling and control level allows simplifying the synthesis and implementation of the control system as well as controlling effectively the process.

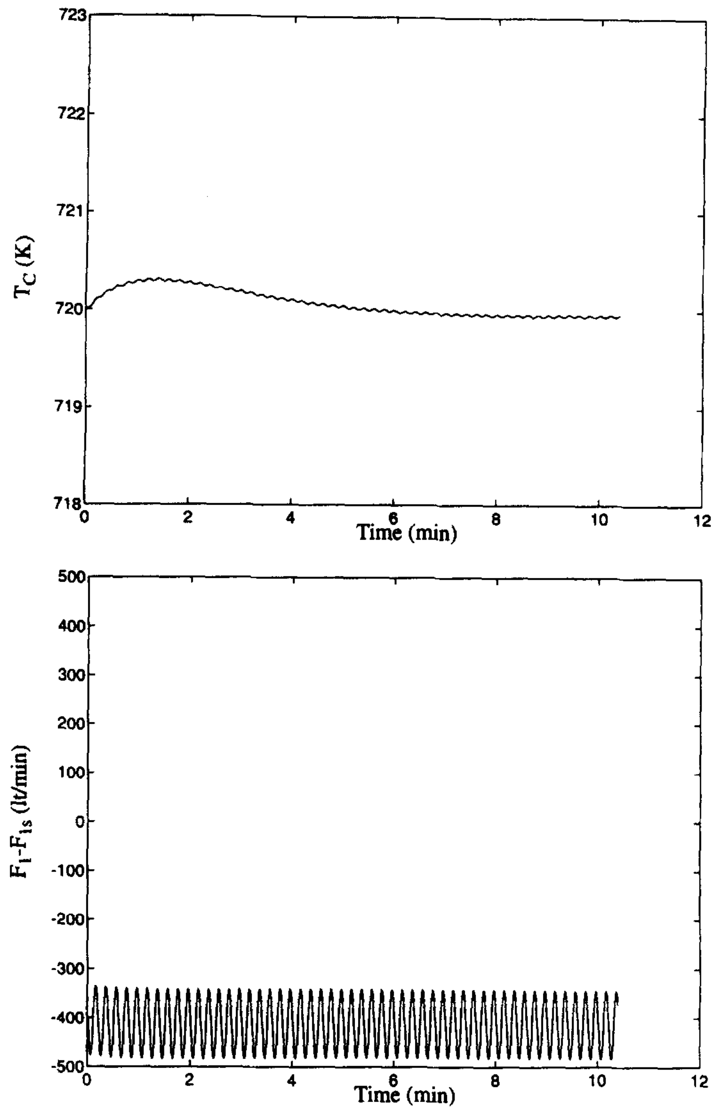

Fig. 6. Output and input profiles for regulation in the presence of time-varying disturbances.

8. CONCLUSIONS

In this article, a class of nonlinear two-time-scale control systems with external time-varying disturbances was considered. Systems in both standard and nonstandard form were studied. For such systems, we synthesized well-conditioned feedforward/state feedback laws that guarantee exponential stability of the fast dynamics and induce a prespecified input/output response in the closed-loop system independently of the disturbances in the limit as $\epsilon \rightarrow 0$. Utilizing singular perturbation methods, we established that the discrepancy between the output of the closed-loop full-order system and the output of the closed-loop reduced system is $O(\epsilon)$, for sufficiently small values of the singular perturbation parameter. We also identified and discussed fundamental differences in the nature of the control problem between systems in standard and nonstandard form. Lyapunov's direct method was used to study the stability properties of the closed-loop system and derive precise conditions that guarantee boundedness of the trajectories. Finally, the derived control methodology was successfully implemented on a typical nonlinear chemical process system with time-scale separation, modeled by a singularly perturbed system in nonstandard form. Comparison with an inversion-based control method, which does not take into account the time-scale multiplicity exhibited by the process, established that the developed control methodology yields significantly superior performance.

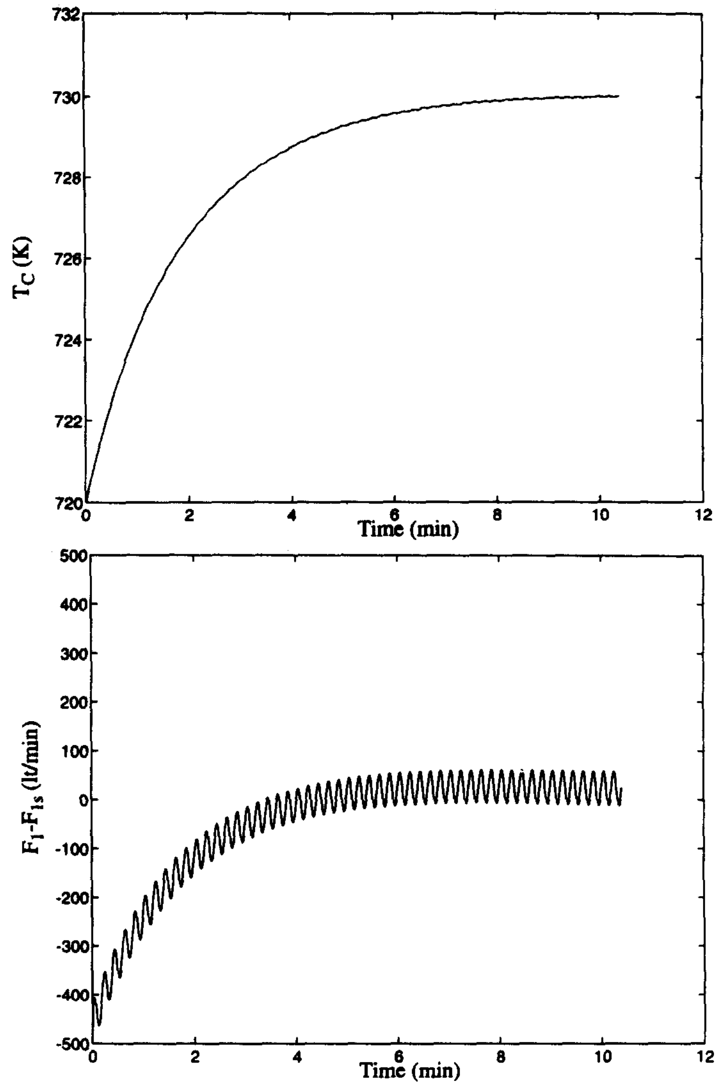

Fig. 7. Output and input profiles for reference input tracking in the presence of time-varying disturbances.

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APPENDIX

Proof of Proposition 1

It is straightforward to show that the control law (15) on the quasi-steady-state takes the forms

$$u = [1 + k^{\mathrm{T}}(x)Q_2^{-1}(x)g_2(x)]^{-1} \times \{\bar{u} - k^{\mathrm{T}}(x)Q_2^{-1}(x)[f_2(x) + W_2(x)d]\}, \quad \text{(A.1)}$$

where the scalar function $1 + k^{\mathrm{T}}(x)Q_2^{-1}(x)g_2(x)$ is nonzero uniformly in $x \in X$.

Under the state feedback law (A.1), the reduced system takes the form

$$\begin{aligned} \dot{x} &= F(x) + G(x)[1 + k^{\mathrm{T}}(x)Q_2^{-1}(x)g_2(x)]^{-1} \\ &\quad \times \{\bar{u} - k^{\mathrm{T}}(x)Q_2^{-1}(x)[f_2(x) + W_2(x)d]\} + W(x)d, \quad \text{(A.2)} \\ y^{\mathrm{s}} &= h(x), \end{aligned}$$

where the vector fields $F(x)$ and $G(x)$ are given by (6). For the system (A.2), $\bar{\rho}$ denotes the relative order of the output y^{s} with respect to the disturbance vector d. For this system, one can directly show that the relative order of the output y^{s} with respect to the auxiliary input $\bar{u}$ is exactly equal to r. On the other hand, if $\rho \le r$, direct differentiation of the output of the system of (A.2) yields the following expressions:

$$\begin{aligned} y^{\mathrm{s}} &= h(x), \\ y^{\mathrm{s}(1)} &= L_F h(x), \\ &\;\;\vdots \\ y^{\mathrm{s}(\bar{\rho}-1)} &= L_F^{\bar{\rho}-1}h(x), \\ y^{\mathrm{s}(\bar{\rho})} &= L_F^{\bar{\rho}}h(x) + \Phi_0(x, d), \\ y^{\mathrm{s}(\bar{\rho}+1)} &= L_F^{\bar{\rho}+1}h(x) + \Phi_1(x, d, d^{(1)}), \quad \text{(A.3)} \\ &\;\;\vdots \\ y^{\mathrm{s}(r-1)} &= L_F^{r-1}h(x) + \Phi_{r-\bar{\rho}-1}(x, d, d^{(1)}, \ldots, d^{(r-\bar{\rho}-1)}), \\ y^{\mathrm{s}(r)} &= L_F^{r}h(x) + L_G L_F^{r-1}h(x)[1 + k^{\mathrm{T}}(x)Q_2^{-1}(x)g_2(x)]^{-1} \\ &\quad \times \{\bar{u} - k^{T}(x)Q_2^{-1}(x)[f_2(x) + W_2(x)d]\} \\ &\quad + \Phi_{r-\bar{\rho}}(x, d, d^{(1)}, \ldots, d^{(r-\bar{\rho})}). \end{aligned}$$

On the basis of (A.3), it is clear that $\bar{\rho} = \rho$ whenever $\rho \le r$. It remains to be proved that the closed-loop reduced systems of (18) and (A.2) are identical. Lemma A.1 that follows establishes this result.

Lemma A.1. Consider the two-time-scale system of the form (1) in standard form, subject to the control law (15), and assume that the matrix $Q_2(x) + g_2(x)k^{\mathrm{T}}(x)$ is invertible uniformly in $x \in X$. Then the closed-loop reduced systems (18) and (A.2), derived by commuting the order of the operations (i) closing the feedback loop and (ii) setting $\epsilon = 0$, are identical.

Lemma A.1 generalizes a standard result for two-time-scale systems under linear static state feedback laws (see e.g. Corless *et al.*, 1993) to nonlinear two-time-scale systems under feedback laws of the form (15). The proof of the lemma can be readily obtained along the lines of the one given for the linear case (Corless *et al.*, 1993), and will be omitted for brevity.

Proof of Theorem 1

Part 1. Using the result of Lemma A.1, the reduced system (18) can be written equivalently in the form (A.2). Substitution of the control law (20) into the reduced system (A.2) then yields the following closed-loop reduced system:

$$\begin{aligned} \dot{x} &= F(x) + G(x)[1 + k^{\mathrm{T}}(x)Q_2^{-1}(x)g_2(x)]^{-1} \\ &\quad \times \{p(x) + q(x)v + Q(x, d, d^{(1)}, \ldots) \quad \text{(A.4)} \\ &\quad - k^{\mathrm{T}}(x)Q_2^{-1}(x)[f_2(x) + W_2(x)d]\} + W(x)d, \\ y^{\mathrm{s}} &= h(x). \end{aligned}$$

On the basis of (A.4) and following Daoutidis and Kravaris (1993), it can be shown that the input/output behavior of the form (24) can be enforced in the above closed-loop reduced system if and only if the conditions (25) are satisfied.

Part 2. Under the control law (27), the closed-loop system takes the form

$$\begin{aligned} \dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\Big\{\{1 + k^{\mathrm{T}}(x)[Q_2(x)]^{-1}g_2(x)\} \\ &\quad \times [\beta_r L_G L_F^{r-1}h(x)]^{-1} \times \Big[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\Big]\Big\} \\ &\quad + g_1(x)\{k^{\mathrm{T}}(x)[Q_2(x)]^{-1}[f_2(x) + W_2(x)d] \\ &\quad + k^{\mathrm{T}}(x)z\} + W_1(x)d, \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x) \quad \text{(A.5)} \\ &\quad \times \Big\{\{1 + k^{\mathrm{T}}(x)[Q_2(x)]^{-1}g_2(x)\}[\beta_r L_G L_F^{r-1}h(x)]^{-1} \\ &\quad \times \Big[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\Big]\Big\} \\ &\quad + g_2(x)\{k^{\mathrm{T}}(x)[Q_3(x)]^{-1}[f_2(x) \\ &\quad + W_2(x)d] + k^{\mathrm{T}}(x)z\} + W_2(x)d. \end{aligned}$$

Defining a set of auxiliary variables ξ by

$$\begin{aligned} \xi &= -[Q_2(x)]^{-1}\Big\{f_2(x) + g_2(x)[\beta_r L_G L_F^{r-1}h(x)]^{-1} \\ &\quad \times \Big[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\Big] + W_2(x)d\Big\} \quad \text{(A.6)} \end{aligned}$$

the closed-loop system (A.5) takes the form

$$\begin{aligned} \dot{x} &= f_1(x) + Q_1(x)z + g_1(x) \\ &\quad \times \Big\{[\beta_r L_G L_F^{r-1}h(x)]^{-1}\Big[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\Big] \\ &\quad + k^{\mathrm{T}}(x)(z - \xi)\Big\} + W_2(x)d, \quad \text{(A.7)} \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x) \\ &\quad \times \Big\{[\beta_r L_G L_F^{r-1}h(x)]^{-1}\Big[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\Big] \\ &\quad + k^{\mathrm{T}}(x)(z - \xi)\Big\} + W_2(x)d. \end{aligned}$$

The properties of the above two-time-scale system can be analyzed by performing a standard two-time-scale decomposition. More specifically, introducing a new set of variables η_f, defined as $\eta_f = z - \xi$, one can easily show that the closed-loop fast subsystem is given by

$$\frac{d\eta_f}{d\tau} = [Q_2(x) + g_2(x)k^T(x)]\eta_f \qquad (A.8)$$

Since the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$ by an appropriate choice of the feedback gain $k^T(x)$, the fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold of the form

$$\eta_f = 0 \qquad (A.9)$$

Furthermore, the closed-loop reduced system takes the form

$$\begin{aligned} \dot{x} &= F(x) + G(x) \\ &\quad \times \Big\{[\beta_r L_G L_F^{r-1} h(x)]^{-1}\Big[v - \sum_{k=0}^{r} \beta_k L_{\mathcal{F}}^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \dots, d^{(k-\rho)})\Big]\Big\} \\ &\quad + W(x)d =: \mathcal{F}(x, v, \bar{d}), \\ y^s &= h(x), \end{aligned} \qquad (A.10)$$

where $\bar{d} = [d \; d^{(1)} \; \dots \; d^{(r-\rho)}]^T$ denotes an extended disturbance vector. From part 1 of the proof, we have that the output of the above closed-loop reduced system is completely independent of d, and the input/output response of (24) is also enforced. Furthermore, from Proposition 2, the relative order of y^s with respect to v is equal to r, and thus there exists a coordinate transformation of the form

$$\begin{bmatrix} \zeta \\ \hline \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_r \\ \eta \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_{\mathcal{F}} h(x) \\ \vdots \\ L_{\mathcal{F}}^{r-1} h(x) \\ \psi(x) \end{bmatrix}, \qquad (A.11)$$

where $\eta \in \mathbb{R}^{n-r}$ and $\psi(x)$ is a smooth vector function, such that the closed-loop reduced system (A.10) takes the form

$$\begin{aligned} \dot{\zeta} &= A\zeta + bv, \\ \dot{\eta} &= \Psi(\zeta, \eta, \bar{d}), \\ y^s &= \zeta_1, \end{aligned} \qquad (A.12)$$

where A is a Hurwitz matrix of dimension $r \times r$, with β_k chosen to satisfy Condition (i) of Theorem 3, $b = [0 \; \dots \; 0 \; 1] \in \mathbb{R}^{r\times 1}$ is a column vector and $\Psi(\zeta, \eta, \bar{d})$ is a vector of smooth functions. Now, consider the following form of the closed-loop full-order system:

$$\begin{aligned} \dot{x} &= F(x) + G(x)\Big\{[\beta_r L_G L_F^{r-1} h(x)]^{-1}\Big[v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \dots, d^{(k-\rho)})\Big]\Big\} \\ &\quad + W(x)d + [Q_1(x) + g_1(x)k^T(x)](z - \xi), \\ \epsilon \dot{z} &= [Q_2(x) + g_2(x)k^T(x)](z - \xi), \end{aligned} \qquad (A.13)$$

or, in terms of the coordinates (ζ, η, z),

$$\begin{aligned} \dot{\zeta} &= A\zeta + bv + \Phi_\zeta(\zeta, \eta)\eta_f, \\ \dot{\eta} &= \Psi(\zeta, \eta, \bar{d}) + \Phi_\eta(\zeta, \eta)\eta_f, \\ \epsilon \dot{z} &= [Q_2(\zeta, \eta) + g_2(\zeta, \eta)k^T(\zeta, \eta)]\eta_f, \\ y &= \zeta_1, \end{aligned} \qquad (A.14)$$

where Φ_ζ and Φ_η, are matrices of smooth functions. From the stability result of Theorem 3, we have that if the hypotheses of the theorem are satisfied, there exists an $\epsilon^* > 0$ such that if $\epsilon \in (0, \epsilon^*]$ then the states (ζ, η, z) of the closed-loop full-order system of (A.14) are bounded. Let $\epsilon \in (0, \epsilon^*]$ and consider the singularly perturbed system, resulting from the states (ζ, z) of the system (A.14),

$$\begin{aligned} \dot{\zeta} &= A\zeta + bv + \Phi_\zeta(\zeta, \eta)\eta_f, \\ \epsilon \dot{z} &= [Q_2(\zeta, \eta) + g_2(\zeta, \eta)k^T(\zeta, \eta)]\eta_f, \\ y &= \zeta_1, \end{aligned} \qquad (A.15)$$

where η can be thought of as a bounded input. Performing a two-time-scale decomposition on the above system, it can be easily seen that the fast subsystem is exponentially stable uniformly in $x \in X$, and the reduced system takes the form

$$\begin{aligned} \dot{\zeta}^s &= A\zeta^s + bv, \\ y^s &= \zeta_1^s, \end{aligned} \qquad (A.16)$$

where the superscript s denotes state of the reduced system, which is also exponentially stable. Utilizing these two stability properties, it can be shown (see Theorem 8.4 in Khalil, 1992), that, under consistent initialization of the states ζ_i^s, ζ_i, i.e. $\zeta_i^s(0) = \zeta_i(0)$, $i = 1, \dots, r$, there exists an $\epsilon^{**} \in (0, \epsilon^*]$ such that if $\epsilon \in (0, \epsilon^{**}]$ then the following estimate holds for the solutions of systems (A.15) and (A.16):

$$\zeta(t) = \zeta^s(t) + O(\epsilon), \quad t \geq 0. \qquad (A.17)$$

From the above estimate, (28) follows directly. The proof of the theorem is complete. □

Proof of Proposition 3

Necessity. Consider the two-time-scale system (1), assumed to be in standard form. Consider also its corresponding reduced system (5), and assume that $\rho \leq r$. Under the control law (33), the two-time-scale system (1) yields

$$\begin{aligned} \dot{x} &= [f_1(x) + g_1(x)p(x)] + [Q_1(x) + g_1(x)k^T(x)]z \\ &\quad + g_1(x)q(x)v + W_1(x)d, \\ \epsilon \dot{z} &= [f_2(x) + g_2(x)p(x)] + [Q_2(x) + g_2(x)k^T(x)]z \\ &\quad + g_2(x)q(x)v + W_2(x)d. \end{aligned} \qquad (A.18)$$

One can now easily verify that the fast subsystem is exponentially stable, subject to an appropriate choice of $k^T(x)$. Moreover, the closed-loop reduced system takes the form

$$\begin{aligned} \dot{x} &= [\hat{F}(x) + \tilde{G}(x)p(x)] + \hat{G}(x)q(x)v + \tilde{W}(x)d, \\ y^s &= h(x). \end{aligned} \qquad (A.19)$$

For the above system, one can conclude, utilizing the results of Propositions 1 and 2, that the relative orders of the pairs (y^s, v) and (y^s, d) are equal to r and ρ respectively, for which, by assumption, $\rho \leq r$. But for nonlinear systems of the form (5), it is well established (see e.g. Isidori, 1989) that the necessary and sufficient condition for achieving exact disturbance decoupling via static state feedback laws of the form $u = p(x) + q(x)v$ is $r < \rho$, which establishes the necessity of this condition. Finally, if the condition $r < \rho$ holds, but the condition $k^T(x)[Q_2(x)]^{-1}W_2(x) \equiv \bar{0}$ does not, i.e., for some $x \in X$, $k^T(x)[Q_2(x)]^{-1}W_2(x) \neq \bar{0}$, we have from (A.3) that for the reduced system (18), $\tilde{\rho} = r < \rho$, and thus a static state feedback law of the form (33) does not suffice to completely eliminate the effect of d on y^s in the closed-loop reduced system of (A.19), which shows the necessity of this condition as well.

Sufficiency. Referring to the reduced system (5), assume that $r < \rho$ and $k^T(x)[Q_2(x)]^{-1}W_2(x) \equiv \bar{0}$. Then it is straightforward to show that the control law of Theorem 1 takes the form

$$\begin{aligned} u &= \{1 + k^T(x)[Q_2(x)]^{-1}g_2(x)\} \\ &\quad \times \{\beta_r L_G L_F^{r-1} h(x)]^{-1}\Big[v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\Big] \\ &\quad + k^T(x)[Q_2(x)]^{-1}f_2(x) + k^T(x)z, \end{aligned} \qquad (A.20)$$

which clearly belongs in the class of control laws described by (33), and does achieve stabilization of the fast dynamics with approximate decoupling of the effect of d on y.

Proof of Theorem 2

Part 1. Consider the closed-loop reduced system

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\tilde{p}(x) + \tilde{G}(x)\tilde{q}(x)v \\ &\quad + \tilde{G}(x)\tilde{Q}(x, d, d^{(1)}, \dots) + \tilde{W}(x)d, \\ y^s &= h(x). \end{aligned} \qquad (A.21)$$

Following Daoutidis and Kravaris (1993), it can be shown that the input/output behavior of the form (39) is enforced in the above closed-loop reduced system if and only if the conditions (40) are satisfied.

Part 2. Under the control law (42), the closed-loop system takes the form

$$\begin{aligned}\dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\Big\{[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1} \\ &\quad \times \Big[v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) \\ &\quad - \sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}(x, d, d^{(1)}, \ldots, d^{(k-\tilde{\rho})})\Big]\Big\} \\ &\quad + g_1(x)k^T(x)z + W_1(x)d, \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)\Big\{[\beta_{\tilde{r}} L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1} \\ &\quad \times \Big[v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) \\ &\quad - \sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}(x, d, d^{(1)}, \ldots, d^{(k-\tilde{\rho})})\Big]\Big\} \\ &\quad + g_2(x)k^T(x)z + W_2(x)d.\end{aligned} \tag{A.22}$$

Employing a standard two-time-scale decomposition, it is straightforward to show that the closed-loop fast subsystem takes the form

$$\begin{aligned}\frac{dz}{d\tau} &= f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x) \\ &\quad \times \Big\{\beta_{\tilde{r}} L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1}\Big[v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) \\ &\quad - \sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}(x, d, d^{(1)}, \ldots, d^{(k-\tilde{\rho})})\Big]\Big\} \\ &\quad + W_2(x)d.\end{aligned} \tag{A.23}$$

Clearly, the fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold of the form

$$\begin{aligned}\xi &= -[Q_2(x) + g_2(x)k^T(x)]^{-1}\Big(f_2(x) \\ &\quad + g_2(x)\Big\{[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1}\Big[v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) \\ &\quad - \sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}(x, d, d^{(1)}, \ldots, d^{(k-\tilde{\rho})})\Big]\Big\} + W_2(x)d\Big),\end{aligned} \tag{A.24}$$

since the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$, by an appropriate choice of the feedback gain $k^T(x)$.

Furthermore, the closed-loop reduced system takes the form

$$\begin{aligned}\dot{x} &= \tilde{F}(x) + \tilde{G}(x)\Big\{[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)]^{-1}\Big[v - \sum_{k=1}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) \\ &\quad - \sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}(x, d, d^{(1)}, \ldots, d^{(k-\tilde{\rho})})\Big]\Big\} + \tilde{W}(x)d, \\ \tilde{y}^s &= h(x).\end{aligned} \tag{A.25}$$

Finally, using the same arguments as in Theorem 1, it can be shown that the controller (42) enforces the input/output response of (39) in the closed-loop system in the limit as $\epsilon \to 0$.

□

Proof of Theorem 3

The proof of the theorem will be derived utilizing a Lyapunov function that is obtained as the composition of two Lyapunov functions for the fast and slow closed-loop subsystems (see also Saberi and Khalil (1984), Khalil (1992) and a Lyapunov argument used in Christofides and Teel (1995)).

The closed-loop system (A.13), with $v = 0$, in terms of the coordinates x, η_f takes the form

$$\begin{aligned}\dot{x} &= F(x) + G(x)\Big\{[\beta_r L_G L_F^{r-1} h(x)]^{-1}\Big[-\sum_{k=0}^{r} \beta_k L_F^k h(x) \\ &\quad - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\Big]\Big\} \\ &\quad + W(x)d + [Q_1(x) + g_1(x)k^T(x)]\eta_f, \\ \epsilon\dot{\eta}_f &= [Q_2(x) + g_2(x)k^T(x)]\eta_f + \epsilon\Big(\frac{\partial \xi}{\partial z}\dot{x} + \frac{\partial \xi}{\partial d}\dot{d}\Big)\end{aligned} \tag{A.26}$$

Under Conditions (i) and (ii), it can be shown (following e.g. Isidori, 1989) that the above closed-loop reduced system with $d(t) = 0$, that is

$$\begin{aligned}\dot{x} &= F(x) + G(x)\Big\{[\beta_r L_G L_F^{r-1} h(x)]^{-1} \\ &\quad \times \Big[-\sum_{k=0}^{r} \beta_k L_F^k h(x)\Big]\Big\} =: \mathcal{F}_1(x),\end{aligned} \tag{A.27}$$

is exponentially stable. Using Theorem 4.5 in Khalil (1992), we have that there exists a smooth Lyapunov function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(a_1, a_2, a_3, a_4, a_5)$ such that the conditions

$$\begin{aligned}a_1|x|^2 &\leq V(x) \leq a_2|x|^2, \\ \dot{V}(x) &= \frac{\partial V}{\partial x}\mathcal{F}_1(x) \leq -a_3|x|^2, \\ \Big|\frac{\partial V}{\partial x}\Big| &\leq a_4|x|\end{aligned} \tag{A.28}$$

hold for all $x \in X$ that satisfy $|x| \leq a_5$. Consider the closed-loop reduced system with $d(t) \neq 0$:

$$\begin{aligned}\dot{x} &= \mathcal{F}_1(x) + G(x)\Big\{[\beta_r L_G L_F^{r-1} h(x)]^{-1} \\ &\quad \times \Big[-\sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}(x, d, d^{(1)}, \ldots, d^{(k-\rho)})\Big]\Big\} \\ &\quad + W(x)d =: \mathcal{F}_1(x) + \mathcal{Q}(x, \bar{d}).\end{aligned} \tag{A.29}$$

Observing that the term $\mathcal{Q}(x, \bar{d})$ vanishes if $\bar{d}(t) \equiv 0$, and using the third part of (A.28) and $|x| \leq a_5$, direct computation of the time derivative of the function $V(x)$ along the trajectories of the above system yields

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x}[\mathcal{F}_1(x) + \mathcal{Q}(x, \bar{d})] \\ &= \frac{\partial V}{\partial x}[\mathcal{F}_1(x)] + \frac{\partial V}{\partial x}[\mathcal{Q}(x, \bar{d})] \\ &\leq -a_3|x|^2 + \frac{\partial V}{\partial x}[\mathcal{Q}(x, \bar{d})] \\ &\leq -a_3|x|^2 + a_4|x|a_6|\bar{d}|,\end{aligned} \tag{A.30}$$

where a_6 is positive real number. Since $d(t)$ and its derivatives are assumed to be sufficiently small, there exists a positive real number μ_1 that satisfies $\mu_1 \leq a_7 a_3/a_4 a_6$, where $a_7 < a_5$, such that $|\bar{d}| \leq \mu_1$. Whenever this condition holds, an application of Theorem 4.10 in Khalil (1992) gives that the state of the system (A.29) is bounded for all $x \in X$ that satisfy $|x| \leq a_5$. Let $|\bar{d}| \leq \mu_1$ and calculate the time derivative of $V(x)$ along the trajectories of the x subsystem of (A.26).

$$\dot{V}(x) \leq -a_3|x|^2 + a_4|x|a_6|\bar{d}| + a_8|x||\eta_f|, \tag{A.31}$$

where a_8 is a positive real number.

Since the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$, the fast subsystem

$$\epsilon\dot{\eta}_f = [Q_2(x) + g_2(x)k^T(x)]\eta_f \tag{A.32}$$

is exponentially stable. Using Lemma 5.7 in Khalil (1992), there exists a smooth Lyapunov function $\Omega: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ of the form

$$\Omega(x, \eta_f) = \eta_f^T S(x) \eta_f \tag{A.33}$$

and a set of positive real numbers $(b_1, b_2, b_3, b_4, b_5, b_6)$ such that the conditions

$$\begin{gathered} b_1 |\eta_f|^2 \leq \Omega(x, \eta_f) \leq b_2 |\eta_f|^2, \\ \frac{\partial \Omega}{\partial \eta_f}[Q_2(x) + g_2(x)k^T(x)]\eta_f \leq -b_3 |\eta_f|^2, \\ \left|\frac{\partial \Omega}{\partial \eta_f}\right| \leq b_4 |\eta_f|, \quad \left|\frac{\partial \Omega}{\partial x}\right| \leq b_5 |\eta_f|^2 \end{gathered} \tag{A.34}$$

hold for all η_f that satisfy $|\eta_f| \leq b_6$. Observe that the time derivative of the state vector x can be bounded as follows:

$$|\dot{x}| \leq w_1 + w_2 |x| + w_3 |x| |\eta_f|, \tag{A.35}$$

where w_1, w_2 and w_3 are positive constants, for all $x \in X$ that satisfy $|x| \leq a_5$. Furthermore, note that since d and its derivatives up to order $r - \rho + 1$ are sufficiently small, there exists a positive real number (possibly small) μ_2 such that $|\dot{\bar{d}}| \leq \mu_2$. Using this fact and the estimates $|\partial\xi/\partial x| \leq l_3$ and $|\partial\xi/\partial\bar{d}| \leq l_4 |x|$, where l_3 and l_4 are positive real numbers, we have

$$\begin{aligned} \left|\frac{\partial \xi}{\partial x}\dot{x} + \frac{\partial \xi}{\partial \bar{d}}\dot{\bar{d}}\right| &\leq l_3(w_1 + w_2|x| + w_3|x||\eta_f|) + \mu_2 l_4 |x| \\ &\leq l_3 w_1 + (l_3 w_2 + \mu_2 l_4)|x| + l_3 w_3 |x| |\eta_f|. \end{aligned} \tag{A.36}$$

Computing the time derivative of the function $\Omega(x, \eta_f)$ along the trajectories of the η_f subsystem of (A.26) and using the inequalities (A.34) and (A.35), we get

$$\begin{aligned} \dot{\Omega}(x, \eta_f) &= \frac{\partial \Omega}{\partial \eta_f}\dot{\eta}_f + \frac{\partial \Omega}{\partial x}\dot{x} \\ &\leq -\frac{b_3}{\epsilon}|\eta_f|^2 + b_4|\eta_f|\,[l_3 w_1 + (l_3 w_2 + \mu_2 l_4)|x| \\ &\quad + l_3 w_3 |x||\eta_f|] + b_5 |\eta_f|^2 (w_1 + w_2|x| + w_3|x||\eta_f|) \\ &\leq -\left(\frac{b_3}{\epsilon} - b_5 w_1\right)|\eta_f|^2 + (b_5 w_2 + b_4 l_3 w_3)|x||\eta_f|^2 \\ &\quad + b_5 w_3 |x||\eta_f|^3 + b_4 l_3 w_1 |\eta_f| \\ &\quad + b_4(l_3 w_2 + \mu_2 l_4)|x||\eta_f|. \end{aligned} \tag{A.37}$$

Consider now the smooth function (Saberi and Khalil, 1984) $L: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$,

$$L(x, \eta_f) = V(x) + \Omega(x, \eta_f), \tag{A.38}$$

as Lyapunov-function candidate for the system (A.26). From (A.28) and (A.34), we have that $L(x, \eta_f)$ is positive-definite and proper (tends to $+\infty$ as $|x| \to \infty$, or $|\eta_f| \to \infty$) with respect to its arguments. To establish boundedness of the trajectories, we shall follow an approach similar to that in Christofides and Teel (1995). To this end, we shall specify a region in state space where the analysis will be performed. First, let l_5 be some positive real number and define $\omega_4 := a_2(a_7)^2 + l_5$. Now, define

$$\omega_5 = \max_{\{(x,\eta_f) \in \mathbb{R}^n \times \mathbb{R}^p : |x| \leq a_5,\ |\eta_f| \leq b_6\}} L(x, \eta_f), \tag{A.39}$$

Let $\omega_6 > \max\{\omega_4, \omega_5\}$ and define the set $\bar{\Omega}$ as follows:

$$\begin{aligned} \bar{\Omega} = \{&(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q: \\ &|\bar{d}| \leq \mu_1, |\dot{\bar{d}}| \leq \mu_2, \omega_4 \leq L(x, \eta_f) \leq \omega_6\}. \end{aligned} \tag{A.40}$$

The set $\bar{\Omega}$ is compact. Computing the time derivative of L along the trajectories of the closed-loop system, the following expression can be easily obtained:

$$\dot{L}(x, \eta_f) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial \Omega}{\partial z}\dot{x} + \frac{\partial \Omega}{\partial \eta_f}\eta_f \tag{A.41}$$

Using the inequalities of (A.31) and (A.37), the following bound for $\dot{L}$ can be computed:

$$\begin{aligned} \dot{L}(x, \eta_f) \leq\ & -a_3|x|^2 + a_4|x|\,a_6|\bar{d}| \\ & + a_8|x||\eta_f| - \left(\frac{b_3}{\epsilon} - b_5 w_1\right)|\eta_f|^2 \\ & + (b_5 w_2 + b_4 l_3 w_2)|x||\eta_f|^2 \\ & + b_5 w_3 |x||\eta_f|^3 \\ & + b_4 l_3 w_1 |\eta_f| + b_4(l_3 w_2 + \mu_2 l_4)|x||\eta_f|. \end{aligned} \tag{A.42}$$

Using the fact that $|\eta_f| \leq b_6$, we get

$$\begin{aligned} \dot{L}(x, \eta_f) \leq\ & -a_3|x|^2 + a_4 a_6 |x||\bar{d}| \\ & - \left(\frac{b_3}{\epsilon} - b_5 w_1\right)|\eta_f|^2 + l_6|x||\eta_f| + b_4 l_3 w_1 |\eta_f|, \end{aligned} \tag{A.43}$$

where

$$l_6 = a_8 + b_6 b_5 w_2 + b_6 b_4 l_3 w_3 + b_6^2 b_5 w_3 + b_4 l_3 w_2 + b_4 \mu_2 l_4.$$

We shall now use a three-step argument to show that there exists a positive real number ϵ^* such that if $\epsilon \in (0, \epsilon^*]$ then $\dot{L}$ is negative for all $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega}$.

Step 1. $\eta_f = 0$. From the definition of the set $\bar{\Omega}$, $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega} \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}}): \eta_f = 0\}$ implies that $L(x, \eta_f) = V(x) \geq \omega_4$, which in turn implies that $|x| \geq \{[a_2(a_7)^2 + l_5]/a_2\}^{1/2} \geq a_7$. Using (A.43), we have, $\bar{\Omega} \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}})\}$,

$$\dot{L} = \dot{V} \leq -a_3|x|^2 + a_4 a_6 |x||\bar{d}| < 0 \tag{A.44}$$

Step 2. We shall now show that there exists $l_7 > 0$ such that for all $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega} \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}}): |\eta_f| \leq l_7\}$, $\dot{L}$ is negative. In this case, $\dot{L}$ takes the form

$$\dot{L} = -a_3|x|^2 + a_4 a_6 |x||\bar{d}| - \frac{b_3}{\epsilon}|\eta_f|^2 + \bar{\Psi}(x, \eta_f) \tag{A.45}$$

where $\bar{\Psi}(x, \eta_f)$ is a continuous scalar function that satisfies $\bar{\Psi}(x, 0) = 0$. Since $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \Delta \in \cap\{(x, \eta_f, d, \dot{\bar{d}}): \eta_f = 0\}$ implies that $|x| \geq \{[a_2(a_7)^2 + l_5]/a_2\}$ and, since $\dot{L}$ is continuous, there exists $l_8 > 0$ such that $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega} \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}}): |\eta_f| \leq l_8\}$ implies $|x| \geq a_7$. So, from the previous step, we already have that the sum of the first two terms in (A.45) is negative. Moreover, the sum of the third and the fourth terms is nonpositive. Then, using the properties of $\bar{\Psi}$, there exists $l_7 \leq l_8$ such that $\dot{L}$ is negative.

Step 3. $|\eta_f| \geq l_7$. Using (A.43) and the various bounds for $(x, \bar{d}, \eta_f)$, we have

$$\begin{aligned} \dot{L}(x, \eta_f) &\leq -a_3|x|^2 + a_4 a_6 |x||\bar{d}| - \left(\frac{b_3}{\epsilon} - b_5 w_1\right)|\eta_f|^2 \\ &\quad + l_6|x||\eta_f| + b_4 l_3 w_1 |\eta_f| \\ &\leq -a_3|x|^2 - \frac{b_3}{\epsilon}l_7^2 + l_9 \end{aligned} \tag{A.46}$$

where $l_9 = a_4 a_6 a_5 \mu_1 + b_5 w_1 b_6^2 + l_6 a_5 b_6 + b_4 l_3 w_1 b_6$. From (A.46), it is clear that $\dot{L}$ is negative for all $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega}$, provided that

$$\epsilon \leq \frac{b_3 l_7^2}{l_9} =: \epsilon^*. \tag{A.47}$$

Using a claim analogous to that in Christofides and Teel (1995), it can be shown that $L(x, \eta_f)$ is bounded for all $t \geq 0$. Since it is also proper with respect to (x, η_f), the boundedness of the trajectories follows directly for all $\epsilon \in (0, \epsilon^*]$. This completes the proof of the theorem. □