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Robust control of hyperbolic PDE systems

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Abstract—This work concerns systems of quasi-linear first-order hyperbolic partial differential equations (PDEs) with uncertain variables and unmodeled dynamics. For systems with uncertain variables, the problem of complete elimination of the effect of uncertainty on the output via distributed feedback (uncertainty decoupling) is initially considered; a necessary and sufficient condition for its solvability as well as explicit controller synthesis formulas are derived. Then, the problem of synthesizing a distributed robust controller that achieves asymptotic output tracking with arbitrary degree of attenuation of the effect of uncertain variables on the output of the closed-loop system is addressed and solved. Robustness with respect to unmodeled dynamics is studied within a singular perturbation framework. It is established that controllers which are synthesized on the basis of a reduced-order slow model, and achieve uncertainty decoupling or uncertainty attenuation, continue to enforce these objectives in the presence of unmodeled dynamics, provided that they are stable and sufficiently fast. The developed controller synthesis results are successfully implemented through simulations on a fixed-bed reactor, modeled by two quasi-linear first-order hyperbolic PDEs, where the reactant wave propagates through the bed with significantly larger speed than the heat wave, and the heat of reaction is unknown and time varying. © 1997 Elsevier Science Ltd

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1. INTRODUCTION

Systems of quasi-linear PDEs arise naturally in chemical engineering as models of transport-reaction processes.The conventional approach to the control of PDE systems involves the discretization of the original model to derive a set of ordinary differential equations (ODEs) followed by the application of linear/nonlinear control methods for ODEs. Although this approach may lead to satisfactory control quality in processes with mild spatial variations, it usually leads to poor performance in processes with strong spatial variations. Motivated by this, in the last five years, there is an emerging research activity on the development of control methods for quasi-linear PDE systems that explicitly account for the effect of spatial variations. The reader may refer to Lasiecka (1995) for a historical perspective on control of PDEs and to Christofides and Daoutidis (1996b) for a review of recent results in this area.

A class of quasi-linear PDE systems which can adequately describe the majority of convection-reaction processes is the one of first-order hyperbolic PDEs. Representative chemical processes modeled by such systems include plug-flow reactors (Ray 1981), fixed-bed reactors (Stangelland and Foss, 1970), pressure swing absorption processes (Ruthven and Sircar, 1994) etc. The distinct feature of hyperbolic PDEs is that all the eigenmodes of the spatial differential operator contain the same, or nearly the same, amount of energy, and thus an infinite number of modes is required to accurately describe their dynamic behavior. This feature distinguishes hyperbolic PDEs from parabolic PDEs (which arise as models of diffusion–convection-reaction processes), prohibits the application of modal decomposition techniques to derive reduced-order ODE models that approximately describe the dynamics of the PDE system and suggests addressing the control problem on the basis of the infinite-dimensional model itself. Following this approach, a methodology for the design of distributed state feedback controllers, based on combination of the method of characteristics and sliding mode techniques was proposed in Sira-Ramirez (1989) and Hanczyc and Palazoglu (1995). A geometric control methodology was developed in Christofides and Daoutidis (1996a) for the synthesis of nonlinear distributed output feedback controllers that enforce output tracking and guarantee stability of the closed-loop system. An alternative approach to the control of quasi-linear hyperbolic PDE systems is based on Lyapunov's direct method (Wang, 1964; Wang, 1966). The basic idea in this approach is to design

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a controller so that the time-derivative of an appropriate Lyapunov functional calculated along the trajectories of the closed-loop system is negative definite, which ensures that the closed-loop system is asymptotically stable. This approach was followed by Alonso and Ydstie (1995), where concepts from thermodynamics were employed to construct a Lyapunov functional candidate, which was used to derive conditions that guarantee asymptotic stability of the closed-loop system under boundary proportional-integral-derivative control.

One of the most important theoretical and practical problems in control is the one of designing a controller that compensates for the presence of mismatch between the model used for controller design and the actual process model. Typical sources of model uncertainty include unknown or partially known time-varying process parameters, exogenous disturbances and unmodeled dynamics. It is well known that the presence of uncertain variables and unmodeled dynamics, if not taken into account in the controller design, may lead to severe deterioration of the nominal closed-loop performance or even to closed-loop instability.

Research on robust control of PDEs with uncertain variables has been limited to linear systems. For linear parabolic PDEs, the problem of complete elimination of the effect of uncertain variables on the output via distributed state feedback (known as disturbance decoupling) was solved in Curtain (1984, 1986). Research on the problem of robust stabilization of linear PDE systems with uncertain variables led to the development of H^∞ control methods in the frequency-domain (e.g. Curtain and Glover, 1986; Palazoglu and Owens, 1987; Gauthier and Xu, 1989). The derivation of concrete relations between frequency-domain and state-space concepts for a wide class of PDE systems in Jacobson and Nett (1988) motivated research on the development of the state-space counterparts of the H^∞ results for linear PDE systems (see Keulen, 1993; Burns and King, 1994 for example). Within a state-space framework, an alternative approach for the design of controllers for linear PDE systems, that deals explicitly with time-invariant uncertain variables, involves the use of adaptive control methods (Wen and Balas, 1989; Hong and Bentsman, 1994; Demetriou, 1994; Balas, 1995).

On the other hand, the problem of robustness of control methods for PDE systems with respect to unmodeled dynamics is typically studied within the singular perturbation framework (e.g., Wen and Balas 1989). This formulation involves synthesizing a controller on the basis of a reduced-order PDE model (which captures the dominant (slow) dynamics of the process) and deriving conditions under which the same controller stabilizes the actual closed-loop system (which includes the unmodeled dynamics). This approach was employed in Wen and Balas (1989) to establish robustness of a class of finite-dimensional adaptive controllers, which asymptotically stabilize a PDE system with time-invariant uncertain variables, with respect to unmodeled dynamics, provided that they are stable and sufficiently fast. Furthermore, the singular perturbation framework allows studying the control and order reduction of PDE systems with fast and slow dynamics. Soliman and Ray (1979) utilized singular perturbation techniques to design well-conditioned state estimators for two-time-scale parabolic PDE systems, while Dochain and Bouaziz (1993) recently used singular perturbations to reduce the parabolic PDE model that describes a fixed-bed reactor with strong diffusive phenomena to an ODE one.

In this article, we consider systems of first-order hyperbolic PDEs with uncertainty, for which the manipulated input and the controlled output are distributed in space. The objective is to develop a framework for the synthesis of distributed robust controllers that handle explicitly time-varying uncertain variables and unmodeled dynamics. For systems with uncertain variables, the problem of complete elimination of the effect of uncertainty on the output via distributed feedback is initially considered; a necessary and sufficient condition for its solvability as well as explicit controller synthesis formulas are derived. Then, a distributed robust controller is derived that guarantees boundedness of the state and achieves asymptotic output tracking with arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the output of the closed-loop system. The controller is designed constructively using Lyapunov's direct method and requires that there exist known bounding functions that capture the magnitude of the uncertain terms and a matching condition is satisfied. The problem of robustness with respect to unmodeled dynamics is then addressed within the context of control of two-time-scale systems modeled in singularly perturbed form. Initially, a robustness result of the bounded stability property of a reduced-order PDE model with respect to stable and fast dynamics is proved. This result is then used to establish that the controllers which are synthesized on the basis of a reduced-order slow model, and achieve uncertainty decoupling or uncertainty attenuation, continue to enforce these control objectives in the full-order closed-loop system, provided that the unmodeled dynamics are stable and sufficiently fast. The developed control method is tested through simulations on a nonisothermal fixed-bed reactor, where the reactant wave propagates through the bed with significantly larger speed than the heat wave, and the heat of reaction is unknown and time varying.

2. FIRST-ORDER HYPERBOLIC PDEs

2.1. *Description of hyperbolic PDEs with uncertain variables*

We will focus on systems of quasi-linear first-order hyperbolic PDEs in one spatial variable, with the following state-space description:

$$\frac{\partial x}{\partial t} = A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\bar{u} + W(x)r(z)\theta(t)$$

$$\bar{y}_s = \mathscr{C}h(x) \qquad (1)$$

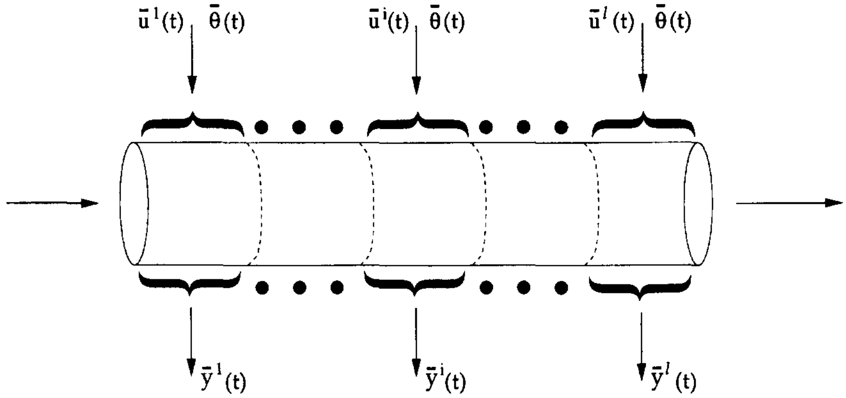

Fig. 1. Specification of the control problem in a prototype example.

subject to the boundary condition:

$$C_1 x(\alpha, t) + C_2 x(\beta, t) = R(t) \tag{2}$$

and the initial condition:

$$x(z, 0) = x_0(z). \tag{3}$$

In the above description, $x(z,t) = [x_1(z,t) \cdots x_n(z,t)]^T$ denotes the vector of state variables, $x(z,t) \in \mathscr{H}^n[(\alpha,\beta), \mathbb{R}^n]$,with $\mathscr{H}^n$ being the infinite-dimensional Hilbert space of n-dimensional vector functions defined on the interval $[\alpha, \beta]$ which satisfy the boundary condition of eq. (2) and whose spatial derivatives up to nth order are square integrable, and $z \in [\alpha, \beta] \subset \mathbb{R}$ and $t \in [0, \infty)$, denote position and time, respectively. $\bar{u} = [\bar{u}^1 \cdots \bar{u}^l]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, $\theta = [\theta_1 \cdots \theta_q] \in \mathbb{R}^q$ denotes the vector of uncertain variables, which may include uncertain process parameters or exogenous disturbances, $\bar{y}_s = [\bar{y}_s^1 \cdots \bar{y}_s^l]^T \in \mathbb{R}^l$ denotes the vector of controlled outputs. Figure 1 shows the location of the manipulated inputs and controlled outputs in the case of a prototype example. $A(x)$, $W(x)$ are sufficiently smooth matrices of appropriate dimensions, $f(x)$ and $g(x)$ are sufficiently smooth vector functions, $h(x)$ is a sufficiently smooth scalar function, C_1, C_2 are constant matrices of appropriate dimensions, $R(t)$ is a time-varying column vector, and $x_0(z) \in \mathscr{H}[(\alpha,\beta), \mathbb{R}^n]$, with $\mathscr{H}[(\alpha,\beta), \mathbb{R}^n]$ being the Hilbert space of n-dimensional vector functions which satisfy the boundary condition of eq. (2) and are square integrable on the interval $[\alpha, \beta]$. $b(z)$ is a known smooth vector function of the form:

$$b(z) = [(H(z - z_1) - H(z - z_2))b^1(z) \cdots (H(z - z_l) - H(z - z_{l+1}))b^l(z)] \tag{4}$$

where H denotes the standard Heaviside function and $b^i(z)$ describes how the control action $\bar{u}^i(t)$ is distributed in the spacial interval $[z_i, z_{i+1}]$. $r(z)$ is a known matrix whose (i,k)th element is of the form $r_k^i(z)$, where the function $r_k^i(z)$ specifies the position of action of the uncertain variable θ_k on $[z_i, z_{i+1}]$. $\mathscr{C}$ is a bounded linear operator, mapping $\mathscr{H}^n$ into $\mathbb{R}^l$, of the form

$$\mathscr{C} = [(H(z - z_1) - H(z - z_2))\mathscr{C}^1 \cdots (H(z - z_l) - H(z - z_{l+1}))\mathscr{C}^l]^T \tag{5}$$

where the operator $\mathscr{C}^i$ depends on the desired performance specifications and in most practical applications is assumed to be of the form

$$\bar{y}_s^i(t) = \mathscr{C}^i h(x) = \int_{z_i}^{z_{i+1}} c^i(z) h(x(z,t))\,dz \tag{6}$$

where $c^i(z)$ is a known smooth function. For simplicity, the functions $b^i(z), c^i(z), r_k^i(z)$, $i = 1, \ldots, l$, $k = 1, \ldots, 9$ are assumed to be normalized in the interval $[\alpha, \beta]$, i.e.,

$$\sum_{i=1}^{l} \int_{z_i}^{z_{i+1}} b^i(z)\,dz = \sum_{i=1}^{l} \int_{z_i}^{z_{i+1}} c^i(z)\,dz = \sum_{i=1}^{l} \int_{z_i}^{z_{i+1}} r_k^i(z)\,dz = 1.$$

The quasi-linear hyperbolic PDE system of eq. (1), with $\theta(t) \equiv 0$, describes the majority of convection-reaction processes arising in chemical engineering (Rhee *et al.*, 1986) and is a natural generalization of linear models considered in Ray (1981) and Christofides and Daoutidis (1996a) in the context of linear distributed state estimation and control. The assumption of affine and separable appearance of $\bar{u}$ and θ is a standard one in uncertainty decoupling and robust control studies for linear PDEs (e.g., Curtain, 1984, 1986), and is satisfied in most practical applications, where the jacket temperature is usually chosen to be the manipulated input (see, for example, Ray, 1981; Christofides and Daoutidis, 1996a), and the heat of reactions, the pre-exponential constants, the temperature and concentration of lateral inlet streams, etc. are typical uncertain variables.

2.2. *Mathematical preliminaries*

In this subsection, we specify precisely the class of systems of the form of eq. (1) considered in this work, and review some basic stability properties and results for these systems, including a converse Lyapunov theorem. First, we define the inner product and the norm in $\mathscr{H}([\alpha, \beta]; \mathbb{R}^n)$.

Let ω_1, ω_2 be two elements of $\mathscr{H}([\alpha, \beta]; \mathbb{R}^n)$. Then, the inner product and the norm in $\mathscr{H}([\alpha, \beta]; \mathbb{R}^n)$ are defined as follows:

$$(\omega_1, \omega_2) = \int_a^b (\omega_1(z), \omega_2(z))_{\mathbb{R}^n}\, dz$$

$$\| \omega_1 \|_2 = (\omega_1, \omega_1)^{1/2} \tag{7}$$

where the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$.

Our requirement that the PDE system of eq. (1) is hyperbolic is precisely formulated in the following assumption.

Assumption 1. *The matrix* A(x) *is real symmetric, and its eigenvalues satisfy*:

$$\lambda_1(x) \leqslant \cdots \leqslant \lambda_k(x) < 0 < \lambda_{k+1}(x) \leqslant \cdots \leqslant \lambda_n(x) \tag{8}$$

for all $x \in \mathscr{H}^n[(\alpha, \beta), \mathbb{R}^n]$.

Assumption 1 is satisfied by most convection-reaction processes, where the matrix $A(x)$ is usually diagonal and its eigenvalues are the fluid velocities (e.g. plug-flow reactors, heat exchangers, countercurrent absorbers-reactors). Assumption 1 also guarantees the local existence, uniqueness and continuity of solutions of the system of eq. (1) (Russell, 1978), in the sense that the operator of the corresponding linearized system

$$\mathscr{L}x = A(z)\frac{\partial x}{\partial z} + B(z)x \tag{9}$$

where $A(z) = A(x_s(z))$, $B(z) = (\partial f(x)/\partial x)_{x = x_s(z)}$, and $x_s(z)$ denotes some steady-state profile, defined on the domain in $\mathscr{H}[(\alpha, \beta); \mathbb{R}^n]$ consisting of functions $x \in \mathscr{H}^1[(\alpha, \beta); \mathbb{R}^n]$ which satisfy the boundary condition of eq. (2), generates a strongly continuous semigroup $U(t)$ of bounded linear operators on $\mathscr{H}[(\alpha, \beta); \mathbb{R}^n]$. In particular, the generalized solution of this system is given by

$$x = U(t) + \int_0^t U(t-\tau) f(x)\, d\tau + \int_0^t U(t-\tau) G(x) b\bar{u}(\tau)\, d\tau + \int_0^t U(t-\tau) W(x) r\theta(\tau)\, d\tau + C(t)R \tag{10}$$

where $C(t)$ is a bounded linear operator for each t mapping $\mathscr{H}[(0, t); \mathbb{R}^n]$ into $\mathscr{H}[(\alpha, \beta); \mathbb{R}^n]$. From eq. (10), it is clear that $U(t)$ evolves the initial condition x_0 forward in time, and thus $U(t)$ can be thought of as an analog of the notion of state transition matrix, used for linear finite-dimensional systems, in the case of infinite-dimensional systems. From general semigroup theory (Friedman, 1976), it is known that $U(t)$ satisfies the following growth property:

$$\| U(t) \|_2 \leqslant K e^{at}, \qquad t \geqslant 0 \tag{11}$$

where $K \geqslant 1$, a is the largest real part of the eigenvalues of the operator $\mathscr{L}$, and an estimate of K, a can be obtained utilizing the Hiller–Yoshida theorem (Friedman, 1976). Whenever the parameter a is strictly negative, we will say that the operator of eq. (9) generates an exponentially stable semigroup $U(t)$.

The fact that the operator of eq. (9) generates a strongly continuous semigroup allows applying Lyapunov stability theorems for infinite-dimensional systems to the system in eq. (1) (Wang, 1964, 1966). In particular, the following standard converse Lyapunov theorem for systems of the form of eq. (1) will be important for our purposes.

Theorem 1 (Wang, 1966). *Consider the system of eq.* (1), *with* ($|\bar{u}| = |\theta| \equiv 0$), *and assume that the operator of eq.* (9) *generates a locally exponentially stable semigroup. Then, there exists a smooth functional* V: $\mathscr{H}^n \times [\alpha, \beta] \to \mathbb{R}_{\geqslant 0}$ *of the form*

$$V(t) = \int_a^b x^T q(z) x\, dz \tag{12}$$

where q(z) *is a known positive definite function satisfying* $\sum_{i=1}^{l} \int_{z_i}^{z_{i+1}} q(z)\, dz = 1$, *and a set of positive real numbers* a_1, a_2, a_3, a_4, a_5 *such that if* $\| x \|_2 \leqslant a_5$ *the following conditions hold*:

$$a_1 \| x \|_2^2 \leqslant V(t) \leqslant a_2 \| x \|_2^2$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\left[A(x)\frac{\partial x}{\partial z} + f(x)\right] \leqslant -a_3 \| x \|_2^2$$

$$\left\| \frac{\partial V}{\partial x} \right\|_2 \leqslant a_4 \| x \|_2. \tag{13}$$

Remark 1. Note that, for infinite-dimensional systems, stability with respect to one norm does not necessarily imply stability with respect to another norm. This difficulty is not encountered in finite-dimensional systems since all norms defined in a finite dimensional vector space are equivalent. In our case, we choose to study stability with respect to the L_2-norm since for hyperbolic systems it represents a measure of the total energy of the system at any time, and thus exponential stability with respect to this norm implies that the system's total energy tends to zero as $t \to \infty$ (i.e., since $V(t)$ is a quadratic function: $V(t) \to 0$ as $t \to \infty \Rightarrow$ exponential stability).

2.3. *Concepts of characteristic index and zero dynamics*

We will initially review the concept of characteristic index between the output $\bar{y}_s^i$ and the input $\bar{u}^i$ introduced in Christofides and Daoutidis (1996a) that will be used in the formulation and solution of the uncertainty decoupling and robust uncertainty attenuation control problems.

Definition 1 (Christofides and Daoutidis, 1996a). *Referring to the system of eq.* (1), *we define the characteristic index of the output* $\bar{y}_s^i$ *with respect to the input* $\bar{u}^i$ *as the smallest integer* σ^i *for which*

$$\mathscr{C}^i L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma^i - 1} h(x) b^i(z) \not\equiv 0 \quad (14)$$

where a_j *denotes the* j*th column vector of the matrix* A(x), *and* L_{a_j}, L_f *denote the standard Lie derivative notation, or* $\sigma^i = \infty$ *if such an integer does not exist.*

From the above definition, it follows that σ^i depends on the structure of the process (matrix $A(x)$ and functions $f(x)$, $g(x)$, $h(x)$), as well as on the actuator and performance specification functions, $b^i(z)$ and $c^i(z)$, respectively. In most practical applications, the selection of $(b^i(z), c^i(z))$ is typically consistent for all pairs $(\bar{y}_s^i, \bar{u}^i)$, in a sense which is made precise in the following assumption.

Assumption 2. *Referring to the system of first-order partial differential equations of the form of eq.* (1), $\sigma^1 = \sigma^2 = \cdots = \sigma^l = \sigma$.

Given the above assumption, σ can be also thought of as the characteristic index between the output vector $\bar{y}_s$ and the input vector $\bar{u}$.

We will now define a concept of characteristic index between the output $\bar{y}_s^i$ and the uncertainty vector θ, for systems of the form of eq. (1) that will be used to express the solvability condition of the uncertainty decoupling problem via distributed state feedback and the matching condition in the robust uncertainty attenuation problem.

Definition 2. *Referring to the system of eq.* (1), *we define the characteristic index of the output* $\bar{y}_s^i$ *with respect to the vector of uncertainties* θ *as the smallest integer* δ^i *for which there exists* $k \in [1, q]$ *such that*

$$\mathscr{C}^i L_{W_k} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\delta^i - 1} h(x) r_k^i(z) \not\equiv 0 \quad (15)$$

where W_k *denotes the* k*th column vector of the matrix* W(x), *or* $\delta^i = \infty$ *if such an integer does not exist.*

From the above definition, it follows that δ^i depends on the structure of the process (matrices $A(x)$, $W(x)$ and functions $f(x)$, $h(x)$), as well as on the position of action of the uncertain variables θ (functions $r_k^i(z)$) and the function $c^i(z)$. In order to simplify the statement of our results, we define the characteristic index of the output vector $\bar{y}_s$ with respect to the vector of uncertainties θ as $\delta = \min\{\delta^1, \delta^2, \ldots, \delta^l\}$.

Finally, we recall a concept of zero dynamics for systems of the form of eq. (1), with $\theta(t) \equiv 0$, proposed in Christofides and Daoutidis (1996a); this will be used to state conditions that guarantee stability of the closed-loop system.

Definition 3 (Christofides and Daoutidis, 1996a). *The zero dynamics associated with the system of eq.* (1), *with* $\theta(t) \equiv 0$, *is the system obtained by constraining the output to zero, i.e., the system:*

$$\frac{\partial x}{\partial t} = A \frac{\partial x}{\partial z} + f(x) - g(x) b(z)$$
$$\times \left[\mathscr{C} L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma - 1} h(x) b(z) \right]^{-1}$$
$$\times \left\{ \mathscr{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma} h(x) \right\}$$
$$\mathscr{C} h(x) \equiv 0$$
$$C_1 x(\alpha, t) + C_2 x(\beta, t) = R(t). \quad (16)$$

3. UNCERTAINTY DECOUPLING

In this section, we consider systems of the form of eq. (1) and address the problem of synthesizing a distributed state feedback controller that stabilizes the closed-loop system and forces the output to track the external reference input in a prespecified manner for all times, independently of the uncertain variables. More specifically, we consider control laws of the form

$$\bar{u} = \mathscr{S}(x) + s(x) v \quad (17)$$

where $\mathscr{S}(x)$ is a smooth nonlinear operator mapping $\mathscr{H}^n$ into $\mathbb{R}^l$, $s(x)$ is an invertible matrix of smooth functionals, and $v \in \mathbb{R}^l$ is the vector of external reference inputs. Under the control law of eq. (17), the closed-loop system takes the form

$$\frac{\partial x}{\partial t} = A(x) \frac{\partial x}{\partial z} + f(x) + g(x) b(z) \mathscr{S}(x) + g(x) b(z) s(x) v$$
$$+ W(x) r(z) \theta(t)$$
$$\bar{y}_s = \mathscr{C} h(x) \quad (18)$$

It is clear that feedback laws of the form of eq. (17) preserve the linearity with respect to the external reference input v. We also note that the evolution of the closed-loop system of eq. (18) is locally governed by a strongly continuous semigroup of bounded linear operators, because $b(z)\mathscr{S}(x)$, $b(z)s(x)$, $r(z)\theta(t)$ are bounded, finite-dimensional perturbations, ensuring that the solution of the system of eq. (18) is well defined. Proposition 1 that follows allows specifying the order of the input/output response in the nominal closed-loop system (the proof can be found in the appendix).

Proposition 1. *Consider the system of eq.* (1), *for which assumption* 1 *holds, subject to the distributed state* *feedback law of eq.* (17). *Then, referring to the closed-loop system of eq.* (18):

(a) *the characteristic index of* $\bar{y}_s$ *with respect to the external reference input* v *is equal to* σ, *and*
(b) *the characteristic index of* $\bar{y}_s$ *with respect to* θ *is equal to* δ, *if* $\delta \leqslant \sigma$.

The fact that the characteristic index between the output $\bar{y}_s$ and the external reference input v is equal to σ suggests requesting the following input/output response for the closed-loop system:

$$\gamma_\sigma \frac{d^\sigma \bar{y}_s}{dt^\sigma} + \cdots + \gamma_1 \frac{d\bar{y}_s}{dt} + \bar{y}_s = v \tag{19}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_\sigma$ are adjustable parameters which can be chosen to guarantee input/output stability in the closed-loop system. Referring to eq. (19), note that, motivated by physical arguments, we request, for each pair $(\bar{y}_s^i, v^i)$, $i = 1, \ldots, l$, an input/output response of order σ with the same transient characteristics (i.e. the parameters γ_k are chosen to be the same for each pair $(\bar{y}_s^i, v^i)$). The following theorem provides the main result of this section.

Theorem 2. *Consider the system of eq.* (1) *for which assumption* 1 *holds, and assume that*: (i) *the roots of the equation* $1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0$ *lie in the open left-half of the complex plane,* (ii) *the zero dynamics* [*eq.* (16)] *is locally exponentially stable, and* (iii) *there exists a positive real number* $\hat{\delta}$ *such that* $\max\{\|\theta\|, \|\dot{\theta}\|\} \leqslant \hat{\delta}$. *Then, the condition* $\sigma < \delta$ *is necessary and sufficient in order for a distributed state feedback law of the form of eq.* (17) *to completely eliminate the effect of* θ *on* $\bar{y}_s$ *in the closed-loop system. Whenever this condition is satisfied, the control law*:

$$\bar{u} = \left[\gamma_\sigma \mathscr{C} L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\sigma-1} h(x) b(z)\right]^{-1} \times \left\{v - \mathscr{C}h(x) - \sum_{\nu=1}^{\sigma} \gamma_\nu \mathscr{C}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\nu} h(x)\right\} \tag{20}$$

(a) *guarantees boundedness of the state of the closed-loop system,*
(b) *enforces the input/output response of eq.* (19) *in the closed-loop system.*

Remark 2. Referring to the solvability condition of the problem, notice that it depends not only on the structural properties of the process but also on the shape of the actuator distribution functions $b(z)$, the position of action of the vector of uncertain variables [matrix function $r(z)$] and the performance specification functions $c(z)$, since σ and δ depend on $c(z)$, $b(z)$ and $c(z)$, $r(z)$, respectively. This implies that it is generally possible to select $(b(z), c(z))$ to allow achieving uncertainty decoupling via distributed state feedback (see illustrative example in Remark 5 below).

Remark 3. The solvability condition of the problem and the distributed state feedback controller of eq. (20) were derived following an approach conceptually similar to the one used to solve the counterpart of this problems in the case of ODE systems (e.g. Daoutidis and Christofides, 1996a). We note that this is possible, because, for the system of eq. (1): (a) the input/output spaces are finite dimensional; and (b) the input/output operators [functions $c^i(z)$, $b^i(z)$] are bounded.

Remark 4. We note that the distributed state feedback controller of eq. (20) that solves the uncertainty decoupling problem is identical to the one derived in Christofides and Daoutidis (1996a) which solves the output tracking problem via distributed state feedback, while the conditions (i) and (ii) of Theorem 2, as proved in Christofides and Daoutidis (1996a), ensure that the controller of eq. (20) exponentially stabilizes the nominal closed-loop system [i.e. $\theta(t) \equiv 0$].

Remark 5 (Illustrative example). Consider a plug-flow reactor with lateral feed, where a first-order reaction of the form $A \to B$ takes place. Assuming that the lateral flow rate is significantly smaller than the inlet flow rate (i.e. $v_l \gg F$, where v_l denotes the velocity of the inlet stream to the reactor and F denotes the lateral flow rate per unit volume), the dynamic model of the process is given by

$$\begin{aligned}
\frac{\partial C_A}{\partial t} &= -v_l \frac{\partial C_A}{\partial z} - kC_A + F(C_{Al} - C_A) \\
\frac{\partial T}{\partial t} &= -v_l \frac{\partial T}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}} kC_A + F(T_l - T) \\
&\quad + \frac{U_w}{\rho_f c_{pf} V_r}(T_j - T) \\
C_A(0, t) &= C_{A0}, \qquad T(0, t) = T_0
\end{aligned} \tag{21}$$

where $C_A(z, t)$ and $T(z, t)$ denote the concentration of the species A and the temperature in the reactor, $z \in [0, 1]$, $T_j(z, t)$ denotes the wall temperature, C_{Al}, T_l denote the concentration and the temperature of the lateral inlet stream, k denotes the reaction rate constant, ΔH denotes the enthalpy of the reaction, ρ_f, c_{pf} denote the density and heat capacity of the reacting liquid, U_w denotes the heat transfer coefficient, and V_r denotes the volume of the reactor. Set $x_1 = C_A$, $x_2 = T$, $u = T_j - T_{js}$, $\theta_1 = T_l - T_{ls}$, $\theta_2 = C_{Al} - C_{Als}$, $y = T$, $R_1(t) = C_{A0}(t)$ and $R_2(t) = T_0(t)$, where T_{js}, T_{ls}, C_{Als} denote steady-state values. Assume that for the control of the system there is available one control actuator with distribution function $b(z)$; let also $r_1(z)$ and $r_2(z)$ determine the position of action of θ_1 and θ_2, respectively, and

$\bar{y}(t) = \int_0^1 c(z)x_2(z,t)\,dz$; then the system of eq. (21) can be written as

$$\frac{\partial x_1}{\partial t} = -v_l\frac{\partial x_1}{\partial z} - kx_1 + F(C_{Als} - x_1) + F\theta_2(t)$$
$$\frac{\partial x_2}{\partial t} = -v_l\frac{\partial x_2}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}}kx_1 + \frac{U_w}{\rho_f c_{pf} V_r}(T_{js} - x_2) + F(T_{ls} - x_2) + \frac{U_w}{\rho_f c_{pf} V_r}b(z)\bar{u} + Fr_1(z)\theta_1(t)$$
$$\bar{y}(t) = \int_0^1 c(z)x_2(z,t)\,dz, \quad x_1(0,t) = R_1, x_1(0,t) = R_2. \tag{22}$$

Performing a time-differentiation of the output $\bar{y}(t)$, we have

$$\frac{d\bar{y}}{dt} = \int_0^1 c(z)\left(-v_l\frac{\partial x_2}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}}kx_1 + \frac{U_w}{\rho_f c_{pf} V_r}(T_{js} - x_2) + F(T_{ls} - x_2)\right)dz + \frac{U_w}{\rho_f c_{pf} V_r}\int_0^1 c(z)b(z)\bar{u}\,dz + F\int_0^1 c(z)r_1(z)\theta_1(t)\,dz. \tag{23}$$

From the above equation and the result of Theorem 2, it follows that if $(c(z), b(z)) = \int_0^1 c(z)b(z)\,dz \neq 0$ (in which case $\sigma = 1$), then the uncertainty decoupling problem via distributed state feedback is solvable as long as $\sigma < \delta$, that is when $(c(z), r_1(z)) = \int_0^1 c(z)r_1(z)\,dz = 0$. Whenever this condition holds, the necessary controller can be derived from the synthesis formula of eq. (20).

4. ROBUST CONTROL: UNCERTAIN VARIABLES

In this section, we consider systems of the form of eq. (1) for which the condition $\sigma < \delta$ is not necessarily satisfied and address the problem of synthesizing a distributed robust nonlinear controller of the form:

$$\bar{u} = \mathscr{P}(x) + \bar{s}(x)v + \mathscr{R}(x,t) \tag{24}$$

where $\mathscr{P}(x)$ is a smooth nonlinear operator mapping $\mathscr{H}^n$ into $\mathbb{R}^l$, $\bar{s}(x)$ is an invertible matrix of smooth functionals, where $\mathscr{R}(x,t)$ is a vector of nonlinear functionals, that guarantees closed-loop stability, enforces asymptotic output tracking, and achieves arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the output of the closed-loop system. The control law of eq. (24) consists of two components, the component $\mathscr{P}(x) + \bar{s}(x)v$ which is used to enforce output tracking and stability in the nominal closed-loop system, and the component $\mathscr{R}(x,t)$ which is used to asymptotically attenuate the effect of $\theta(t)$ on the output of the closed-loop system.The design of $\mathscr{P}(x) + \bar{s}(x)v$ will be performed employing the geometric approach presented in the previous section, while the design of $\mathscr{R}(x,t)$ will be performed constructively using Lyapunov's direct method. The central idea of Lyapunov-based controller design is to construct $\mathscr{R}(x,t)$, using the knowledge of bounding functions that capture the size of the uncertain terms and assuming a certain path under which the uncertainty may affect the output, so that the ultimate discrepancy between the output of the closed-loop system and the reference input can be made arbitrarily small by a suitable selection of the controller parameters.

We will assume that there exists a known smooth (not necessarily small) vector function, which captures the magnitude of the vector of uncertain variables θ for all times. Information of this kind is usually available in practice, as a result of experimental data, preliminary simulations, etc. Assumption 3 states precisely our requirement.

Assumption 3. *There exists a known vector* $\bar{\theta}(t) = [\bar{\theta}_1(t) \cdots \bar{\theta}_q(t)]$, *whose elements are continuous positive definite functions defined on* $t \in [0, \infty)$ *such that*

$$|\theta_k(t)| \leqslant \bar{\theta}_k(t), \qquad k = 1, \ldots, q. \tag{25}$$

Assumption 4 that follows characterizes precisely the class of systems of the form of eq. (1) for which our robust control methodology is applicable. It determines the path under which the vector of uncertain variables $\theta(t)$ may affect the output of the closed-loop system. More specifically, we assume that the vector of uncertain variables $\theta(t)$ enters the system in such a manner such that the expressions for the time-derivatives of the output $\bar{y}_s$ up to order $\sigma - 1$ are independent of θ. This assumption is significantly weaker than the solvability condition required for the uncertainty decoupling problem (cf. theorem 2) and is satisfied by many convection–reaction processes of practical interest (including the plug-flow and fixed-bed reactor examples studied in this paper).

Assumption 4. $\sigma \leqslant \delta$.

Theorem 3 provides the main result of this section (the proof is given in the Appendix).

Theorem 3. *Consider the system of eq.* (1), *for which assumptions* 1–4, *and the conditions* (i)–(iii) *of theorem* 2 *hold, under the distributed state feedback law of the form:*

$$\bar{u} = \left[\mathscr{C}L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1}h(x)b(z)\right]^{-1} \times\left\{\gamma_0\left[\frac{\gamma_1}{\gamma_\sigma}v - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\nu-1}h(x)\right] - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\nu}h(x) - 2K(t)\Delta(x,\phi)\sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\nu-1}h(x) - \frac{\gamma_1}{\gamma_\sigma}v\right\} \tag{26}$$

where ϕ, γ_k are adjustable parameters, and γ_k are chosen so that $\gamma_0 > 0$ and the polynomial $s^{\sigma-1} + [(\gamma_{\sigma-1})/\gamma_\sigma]s^{\sigma-2} + \cdots + (\gamma_2/\gamma_\sigma)s + \gamma_1/\gamma_\sigma = 0$ is Hurwitz, $\Delta(x, \phi)$ is an $l \times l$ diagonal matrix whose (i, i)th element is of the form

$$\Delta(x, \phi) = \left(\left| \sum_{v=1}^{\sigma-1} \frac{\gamma_v}{\gamma_\sigma} \mathscr{C}^i \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{v-1} h(x) - \frac{\gamma_1}{\gamma_\sigma} v^i \right| + \phi \right)^{-1}$$

and the time-varying gain $K(t)$ is of the form

$$K(t) = \left[\sum_{k=1}^{q} \mathscr{C} L_{W_k} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma-1} \times h(x) r_k(z) \bar{\theta}_k(t) \right]. \quad (27)$$

Then, for each positive real number d, there exist a pair of positive real numbers $(\tilde{\delta}, \phi^*)$ such that if $\max\{\|x_0\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \tilde{\delta}$ and $\phi \in (0, \phi^*]$, then: (a) *the state of the closed-loop system is bounded, and* (b) *the output of the closed-loop system satisfies:*

$$\lim_{t \to \infty} |\bar{y}_s^i - v^i| \leqslant d, \quad i = 1, \ldots, l. \quad (28)$$

Remark 6. The calculation of the control action from the controllers of eqs (20)–(26) requires algebraic manipulations as well as differentiations and integrations in space, which is expected because of their distributed nature.

Remark 7. The robust nonlinear controller of eq. (26) guarantees an arbitrary degree of asymptotic attenuation of the effect of a large class of uncertain *time-varying* variables on the output (those that satisfy assumption 4). In many practical applications, there exist unknown time-invariant process parameters which may not necessarily satisfy assumption 4. Since the manipulated input and controlled output spaces are finite-dimensional, the asymptotic rejection [in the sense that eq. (28) holds] of such constant uncertainties can be achieved by combining the controller of eq. (26) with an external linear error-feedback controller with integral action.

Remark 8. Comparing the uncertainty decoupling result of theorem 2 with the result of theorem 3 [eq. (28)], we note that the latter is clearly applicable to a larger class of systems (those satisfying the condition $\sigma \leqslant \delta$), achieving, however, a weaker performance requirement. More specifically, in the uncertainty decoupling problem a well-characterized input/output response is enforced for all times independently of $\theta(t)$, while in the case of robust uncertainty rejection the output error $(\bar{y}_s^i - v^i)$ is ensured to stay bounded and asymptotically approach arbitrarily close to zero (by appropriate choice of the parameter ϕ).

Remark 9 (Illustrative example (cont'd)). Whenever $(c(z), r_1(z)) \neq 0$ (which implies that $\delta = \sigma = 1$), the uncertainty decoupling problem via distributed state feedback is not solvable. However, since $\delta = \sigma$, we have that the matching condition of our robust control methodology is satisfied. Assuming that there exists a known upper bound $\bar{\theta}(t)$ on the size of the uncertain term $\theta(t)$, such that $|\theta| \leqslant \bar{\theta}(t)$, the control law of theorem 3 takes the form:

$$\bar{u} = \left[\frac{U_w}{\rho_f c_{pf} V_r} \int_0^1 c(z) b(z) \, dz \right]^{-1} \left\{ \gamma_0 \left(v - \int_0^1 x_2 \, dz \right) - \int_0^1 c(z) \left(-v_l \frac{\partial x_2}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}} k x_1 + \frac{U_w}{\rho_f c_{pf} V_r} (T_{js} - x_2) + F(T_{ls} - x_2) \right) dz - 2\bar{\theta}(t) \frac{\int_0^1 x_2(z, t) \, dz - v}{|\int_0^1 x_2(z, t) \, dz - v| + \phi} \right\}. \quad (29)$$

5. TWO-TIME-SCALE HYPERBOLIC PDE SYSTEMS

In the rest of the theoretical part of this paper, we will analyze the problem of robustness of the controllers of eqs (20)–(26) with respect to stable unmodeled dynamics. This problem will be addressed within the broader context of control of two-time-scale hyperbolic PDE systems, modeled within the framework of singular perturbations. Such a formulation provides a natural setting for addressing robustness with respect to unmodeled dynamics. To this end, in this section, we will derive two general stability results for two-time-scale hyperbolic PDE systems which will be used in the next section, to derive conditions that guarantee robustness of the controllers of eqs (20)–(26) to stable unmodeled dynamics.

5.1. *Two-time-scale decomposition*

We will focus on singularly perturbed hyperbolic PDE systems with the following state-space description:

$$\frac{\partial x}{\partial t} = A_{11}(x) \frac{\partial x}{\partial z} + A_{12}(x) \frac{\partial \eta}{\partial z} + f_1(x) + Q_1(x)\eta + g_1(x) b(z) \bar{u} + W_1(x) r(z) \theta(t)$$

$$\varepsilon \frac{\partial \eta}{\partial t} = A_{21}(x) \frac{\partial x}{\partial z} + A_{22}(x) \frac{\partial \eta}{\partial z} + f_2(x) + Q_2(x)\eta + g_2(x) b(z) \bar{u} + W_2(x) r(z) \theta(t)$$

$$\bar{y} = \mathscr{C} h(x) \quad (30)$$

subject to the boundary conditions:

$$C_{11} x(\alpha, t) + C_{12} x(\beta, t) = R_1(t)$$

$$C_{21} \eta(\alpha, t) + C_{22} \eta(\beta, t) = R_2(t) \quad (31)$$

and the initial conditions:

$$x(z,0) = x_0(z)$$
$$\eta(z,0) = \eta_0(z) \qquad (32)$$

where $x(z,t) = [x_1(z,t) \cdots x_n(z,t)]^T$, $\eta(z,t) = [\eta_1(z,t) \cdots \eta_p(z,t)]^T$, denote vectors of state variables, $x(z,t) \in \mathscr{H}^{(n)}[(\alpha,\beta),\mathbb{R}^n]$, $\eta(z,t) \in \mathscr{H}^{(n)}[(\alpha,\beta),\mathbb{R}^p]$, with $\mathscr{H}^{(i)}[(\alpha,\beta),\mathbb{R}^j]$ being the infinite-dimensional Hilbert space of j-dimensional vector functions defined on the interval $[\alpha,\beta]$ whose spatial derivatives up to ith order are square integrable, $A_{11}(x)$, $A_{12}(x)$, $A_{21}(x)$, $A_{22}(x)$ are sufficiently smooth matrices, $f_1(x), f_2(x), g_1(x), g_2(x)$ are sufficiently smooth vector functions, $Q_1(x)$, $Q_2(x)$, $W_1(x)$, $W_2(x)$ are sufficiently smooth matrices, ε is a small parameter which quantifies the degree of coupling between the fast and slow modes of the system, and $R_1(t)$, $R_2(t)$ are time-varying column vectors. The following assumption states that the two-time-scalesystem of eq. (30) is hyperbolic.

Assumption 5. *The* $(n+p)\times(n+p)$ *matrix*:

$$\mathscr{A}(x) = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix} \qquad (33)$$

is real symmetric, and its eigenvalues satisfy:

$$\bar{\lambda}_1(x) \leqslant \cdots \leqslant \bar{\lambda}_k(x) < 0 < \bar{\lambda}_{k+1}(x) \leqslant \cdots \leqslant \bar{\lambda}_{n+p}(x) \qquad (34)$$

for all $x \in \mathscr{H}^n[(\alpha,\beta),\mathbb{R}^n]$.

The time-scale multiplicity of the system of eq. (30) can be explicitly taken into account by decomposing it into separate reduced-order models associated with different time-scales. Defining $\mathscr{L}_{i1}x = A_{i1}(x)\partial x/\partial z + f_i(x)$, $\mathscr{L}_{j2} = A_{j2}\partial\eta/\partial z + Q_j(x)\eta$, $i,j = 1,2$, and setting $\varepsilon = 0$, the PDE system of eq. (30), reduces to a system of coupled partial and ordinary differential equations of the form

$$\frac{\partial x}{\partial t} = \mathscr{L}_{11}x + \mathscr{L}_{12}\eta_s + g_1(x)b(z)\bar{u} + W_1(x)r(z)\theta(t)$$
$$0 = \mathscr{L}_{21}x + \mathscr{L}_{22}\eta_s + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t)$$
$$\bar{y}_s = \mathscr{C}h(x) \qquad (35)$$

The solution of the ODE $\mathscr{L}_{21}x + \mathscr{L}_{22}\eta_s + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t) = 0$ subject to the boundary conditions of eq. (31) is of the form

$$\eta_s = \mathscr{L}_{22}^{-1}(\mathscr{L}_{21}x + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t)). \qquad (36)$$

The slow subsystem, which captures the slow dynamics of the system of eq. (30), takes the form

$$\frac{\partial x}{\partial t} = \mathscr{L}x + G(x)b(z)\bar{u} + W(x)r(z)\theta(t)$$
$$\bar{y}_s = \mathscr{C}h(x) \qquad (37)$$

where $\mathscr{L}x = \mathscr{L}_{11}x - \mathscr{L}_{12}\mathscr{L}_{22}^{-1}\mathscr{L}_{21}x$, $G(x) = g_1(x) - \mathscr{L}_{12}\mathscr{L}_{22}^{-1}g_2(x)$, $W(x) = W_1(x) - \mathscr{L}_{12}\mathscr{L}_{22}^{-1}W_2(x)$. Defining a fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, the fast subsystem, which describes the fast dynamics of the system of eq. (30), takes the form

$$\frac{\partial\eta}{\partial\tau} = \mathscr{L}_{21}x + \mathscr{L}_{22}\eta + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t) \qquad (38)$$

where x, θ are independent of time.

5.2. *Stability results*

In this subsection, we will give two basic stability results for systems of the form of eq. (30) which are important on their own right and will also be used in the subsequent sections to prove a robustness property of the controllers of eqs (20)–(26) with respect to exponentially stable unmodeled dynamics. We will begin with the statement of stability requirements on the fast and slow subsystems. In particular, we assume that the fast subsystem of eq. (38) and the unforced ($\bar{u} = \bar{\theta} = 0$) slow subsystem of eq. (37) are locally exponentially stable. Assumptions 6 and 7 that follow formalize these requirements.

Assumption 6. *The differential operator* $\mathscr{L}_{22}\eta$ *generates an exponentially stable strongly continuous semigroup* $U_f(t)$ *that satisfies*;

$$\|U_f(t)\|_2 \leqslant K_F e^{-a_f t/\varepsilon}, \quad t \geqslant 0 \qquad (39)$$

where $K_f \geqslant 1$, *and* a_f *is some positive real number.*

Assumption 7. *The differential operator* $\mathscr{L}x$ *generates a locally (i.e., for* $x \in \mathscr{H}$ *that satisfy* $\|x\|_2 \leqslant \delta_x$, *where* δ_x *is a positive real number) exponentially stable strongly continuous semigroup* $U_s(t)$ *that satisfies*:

$$\|U_s(t)\|_2 \leqslant K_s e^{-a_s t}, \quad t \geqslant 0 \qquad (40)$$

where $K_s \geqslant 1$, *and* a_s *is some positive real number.*

From the above assumption and using eq. (10), the smoothness of $G(x)$ and $W(x)$, and the fact that for $\|x\|_2 \leqslant \delta_x$, there exists a pair of positive real numbers $\bar{M}_1, \bar{M}_2$ such that $\|G(x)\|_2 \leqslant \bar{M}_1$ and $\|W(x)\|_2 \leqslant \bar{M}_2$, we have that the following estimate holds for the system of eq. (37):

$$\|x\|_2 \leqslant K_s\|x_0\|_2 e^{-a_s t} + K_s\int_0^t e^{-a_s(t-\tau)}\|G(x)b(z)\|_2|\bar{u}(\tau)|\,d\tau + K_s\int_0^t e^{-a_s(t-\tau)}\|W(x)r(z)\|_2|\theta(\tau)|\,d\tau$$
$$\leqslant K_s\|x_0\|_2 e^{-a_s t} + M_1\|\bar{u}\| + M_2\|\theta\| \qquad (41)$$

where

$$M_1 = \frac{K_s \bar{M}_1 \| b(z) \|_2}{a_s}, \quad M_2 = \frac{K_s \bar{M}_2 \| r(z) \|_2}{a_s}$$

provided that the initial condition ($\| x_0 \|_2$) and the inputs ($\bar{u}, \theta$) are sufficiently small to satisfy $K_s \| x_0 \|_2 + M_1 \| \bar{u} \| + M_2 \| \theta \| \leqslant \delta_x$. Theorem 4 provides the main stability result of this section (the proof is given in the appendix).

Theorem 4. *Consider the system of eq.* (30) *with* $\bar{u} \equiv 0$, *for which assumptions 5–7 hold, and define* $\eta_f := \eta - \eta_s$. *Then, there exist positive real numbers* ($\bar{K}_s, \bar{a}_s, \bar{K}_f, \bar{a}_f$), *such that for each positive real number* d, *there exist positive real numbers* ($\bar{\delta}, \varepsilon^*$) *such that if* $\max\{\| x_0 \|_2, \| \eta_{f_0} \|_2, \| \theta \|, \| \dot{\theta} \|\} \leqslant \bar{\delta}$ *and* $\varepsilon \in (0, \varepsilon^*]$, *then, for all* $t \geqslant 0$:

$$\| x \|_2 \leqslant \bar{K}_s \| x_0 \|_2 e^{-\bar{a}_s t} + M_2 \| \theta \| + d \qquad (42)$$

$$\| \eta_f \|_2 \leqslant K_f \| \eta_{f_0} \|_2 e^{-\bar{a}_f t/\varepsilon} + d. \qquad (43)$$

Theorem 4 establishes a robustness result of the boundedness property that the slow subsystem of eq. (37) possesses, with respect to exponentially stable singular perturbations, provided that they are sufficiently fast. Theorem 5 that follows establishes, another conceptually important result, namely that the exponential stability property of a reduced system [consider the system of eq. (37) with $|\bar{u}| = |\theta(t)| \equiv 0$] is preserved in the presence of exponentially stable singular perturbations. The proof of this theorem in analogous to the one of theorem 4 and will be omitted for brevity.

Theorem 5. *Consider the system of eq.* (30) *with* $|\bar{u}| = |\theta(t)| \equiv 0$, *for which assumptions 5–7 hold, and define* $\eta_f := \eta - \eta_s$. *Then, there exist positive real numbers* ($\bar{\delta}, \varepsilon^*$) *such that if* $\max\{\| x_0 \|_2, \| \eta_{f_0} \|_2\} \leqslant \bar{\delta}$ *and* $\varepsilon \in (0, \varepsilon^*]$, *the system of eq.* (30) *is exponentially stable* [*i.e. eqs* (42) *and* (43) *hold with* $d = 0$, $\theta(f) \equiv 0$].

6. ROBUSTNESS WITH RESPECT TO UNMODELED DYNAMICS

In this section, we utilize the general stability results of the previous section to establish robustness properties of the controllers of eqs (20)–(26) with respect to unmodeled dynamics. In particular, we use the hyperbolic singularly perturbed system of eq. (30) to represent the actual process model, and we assume that the model which is available for controller design is the slow system of eq. (37). Our objective is to establish that the controllers of eqs (20)–(26), synthesized on the basis of the slow system, are robust to unmodeled dynamics in the sense that they enforce approximate uncertainty decoupling and robust disturbance rejection in the actual model of eq. (30), provided that the separation of the fast and slow modes is sufficiently large.

6.1. *Robustness of uncertainty decoupling to unmodeled dynamics*

In this subsection, we consider the uncertainty decoupling problem for the two-time-scale hyperbolic system of eq. (30) whose fast dynamics is stable (assumption 6). Substitution of the distributed state feedback law of eq. (17) into the system of eq. (30) yields:

$$\begin{aligned}\frac{\partial x}{\partial t} &= \mathscr{L}_{11} x + \mathscr{L}_{12} \eta + g_1(x) b(z) \mathscr{S}(x) \\ &\quad + g_1(x) b(z) s(x) v + W_1(x) r(z) \theta(t) \\ \varepsilon \frac{\partial \eta}{\partial t} &= \mathscr{L}_{21} x + \mathscr{L}_{22} \eta + g_2(x) b(z) \mathscr{S}(x) \\ &\quad + g_2(x) b(z) s(x) v + W_2(x) r(z) \theta(t) \\ \bar{y} &= \mathscr{C} h(x). \end{aligned} \qquad (44)$$

From the above representation of the closed-loop system, it is clear that the control law of eq. (17) preserves the two-time-scale property of the open-loop system and guarantees that the system of eq. (44) has a well-defined solution. Performing a two-time-scale decomposition on the system of eq. (44), the closed-loop fast subsystem takes the form

$$\begin{aligned}\frac{\partial \eta}{\partial \tau} &= \mathscr{L}_{21} x + \mathscr{L}_{22} \eta + g_2(x) b(z) \mathscr{S}(x) \\ &\quad + g_2(x) b(z) s(x) v + W_2(x) r(z) \theta(t) \end{aligned} \qquad (45)$$

where x, θ are independent of time. From the representation of the closed-loop fast subsystem it is clear that the control law of eq. (44) preserves the exponential stability property of the fast dynamics of the closed-loop system. The closed-loop slow system takes the form

$$\begin{aligned}\frac{\partial x}{\partial t} &= \mathscr{L} x + G(x) b(z) \mathscr{S}(x) + G(x) b(z) s(x) v \\ &\quad + W(x) r(z) \theta(t) \\ \bar{y}_s &= \mathscr{C} h(x). \end{aligned} \qquad (46)$$

For the system of eq. (46) it is straightforward to show that the result of proposition 1 holds, which suggests requesting an input/output response of the form of eq. (19) in the closed-loop reduced system.

Theorem 6 that follows establishes a robustness property of the solvability condition of the uncertainty decoupling problem ($\sigma < \delta$) and the input/output response of eq. (19) with respect to stable, sufficiently fast, unmodeled dynamics. The proof of the theorem is given in the appendix.

Theorem 6. *Consider the system of eq.* (30) *for which assumptions 5 and 6 hold, under the controller of eq.* (17). *Consider also the slow subsystem of eq.* (37) *and assume that the condition* $\sigma < \delta$ *and the stability* *conditions of theorem 2 hold. Then, for each positive real number* d, *there exist positive real numbers* $(\tilde{\delta}, \varepsilon^*)$ *such that if* $\max\{\|x_0\|_2, \|\eta_{f_0}\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \tilde{\delta}$ *and* $\varepsilon \in (0, \varepsilon^*]$, *then*:

(a) *the state of the closed-loop system is bounded, and*
(b) *the output of the closed-loop system satisfies for all* $t \geqslant 0$

$$\bar{y}^i(t) = \bar{y}_s^i(t) + d, \quad i = 1, \ldots, l \tag{47}$$

where $\bar{y}_s^i(t)$ *are the solutions of eq.* (19).

Remark 10. Referring to the above theorem, note that we do not impose any assumptions on the way the uncertain variables θ enter the actual process model of eq. (30) [i.e. the condition $\sigma < \delta$ has to be satisfied only in the slow subsystem of eq. (37)]. This is achieved at the expense of the controller of eq. (17), which is synthesized on the basis of the slow subsystem of eq. (37), enforcing a type of approximate uncertainty decoupling in the actual process model, in the sense that the discrepancy between the output of the actual closed-loop system and the output of the closed-loop reduced system (which is independent of θ) can be made smaller than a given positive number d (possibly small) for all times, provided that ε is sufficiently small [eq. (47)].

6.2. *Robust control: uncertain variables and unmodeled dynamics*

In this subsection, we consider the robust uncertainty rejection problem for two-time-scale hyperbolic systems of the form of eq. (30) with stable fast dynamics. Under a control law of the form of eq. (24), the closed-loop system takes the form

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \mathscr{L}_{11}x + \mathscr{L}_{12}\eta \\
&\quad + g_1(x)b(z)[\mathscr{S}(x) + \bar{s}(x)v + \bar{\mathscr{R}}(x,t)] \\
&\quad + W_1(x)r(z)\theta(t) \\
\varepsilon\frac{\partial \eta}{\partial t} &= \mathscr{L}_{21}x + \mathscr{L}_{22}\eta \\
&\quad + g_2(x)b(z)[\mathscr{S}(x) + \bar{s}(x)v + \bar{\mathscr{R}}(x,t)] \\
&\quad + W_2(x)r(z)\theta(t) \\
\bar{y} &= \mathscr{C}h(x).
\end{aligned} \tag{48}$$

The fast dynamics of the above system is exponentially stable and the reduced system takes the form

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \mathscr{L}x + G(x)b(z)[\mathscr{S}(x) + \bar{s}(x)v + \bar{\mathscr{R}}(x,t)] \\
&\quad + W(x)r(z)\theta(t) \\
\bar{y}_s &= \mathscr{C}h(x).
\end{aligned} \tag{49}$$

Theorem 7 that follows establishes a robustness property of the controller of eq. (26) to sufficiently fast unmodeled dynamics (the proof is given in the appendix).

Theorem 7. *Consider the system of eq.* (30) *for which assumptions* 5 *and* 6 *hold, under the controller of eq.* (26). *Consider also the slow subsystem of eq.* (37) *and suppose that assumption* 4 *and the stability conditions of theorem* 2 *hold. Then, for each positive real number* d, *there exist positive real numbers* $(\tilde{\delta}, \phi^*)$ *such that if* $\max\{\|x_0\|_2, \|\eta_{f_0}\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \tilde{\delta}$, $\phi \in (0, \phi^*]$, *there exists a positive real number* $\varepsilon^*(\phi)$ *such that if* $\max\{\|x_0\|_2, \|\eta_{f_0}\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \tilde{\delta}$, $\phi \in (0, \phi^*]$, *and* $\varepsilon \in (0, \varepsilon^*(\phi)]$, *then*:

(a) *the state of the closed-loop system is bounded, and*
(b) *the output of the closed-loop system satisfies*:

$$\lim_{t\to\infty} |\bar{y}^i - v^i| \leqslant d, \quad i = 1, \ldots, l. \tag{50}$$

Remark 11. Regarding the result of the above theorem, a few observations are in order: (i) no matching ondition is imposed in the actual process model of eq. (30), (ii) the dependence of ε^* on ϕ is due to the presence of ϕ on the closed-loop fast subsystem, and (iii) a bound for the output error, for all times [and not only asymptotically as in eq. (50)] can be obtained from the proof of the theorem in terms of the initial conditions and d.

Remark 12. Whenever the open-loop fast subsystem of eq. (38) is unstable, i.e. the operator $\mathscr{L}_{22}\eta$ generates an exponentially unstable semigroup, a preliminary distributed state feedback law of the form

$$\bar{u} = \mathscr{F}\eta + \tilde{u} \tag{51}$$

can be used to stabilize the fast dynamics, under the assumption that the pair $[\mathscr{L}_{22}g_2(x), b(z)]$ is stabilizable, yielding thus a two-time-scale hyperbolic system for which assumption 7 holds. The design of the gain operator $\mathscr{F}$ can be performed using for example standard optimal control methods (Balas, 1983).

7. SIMULATION STUDY: A FIXED-BED REACTOR

Consider a fixed-bed reactor where an elementary reaction of the form $A \to B$ takes place, shown in Fig. 2. The reaction is endothermic and a jacket is used to heat the reactor. Under the assumptions of perfect radial mixing, constant density and heat capacity of the reacting liquid, and negligible diffusive phenomena, a dynamic model of the process can be derived from material and energy balances and has the form

$$\begin{aligned}
\rho_b c_{pb}\frac{\partial T}{\partial t} &= -\rho_f c_{pf} v_l \frac{\partial T}{\partial z} \\
&\quad + (-\Delta H)k_0 e^{-E/RT}C_A + \frac{U_w}{V_r}(T_j - T) \\
\varepsilon\frac{\partial C_A}{\partial t} &= -v_l\frac{\partial C_A}{\partial z} - k_0 e^{-E/RT}C_A
\end{aligned} \tag{52}$$

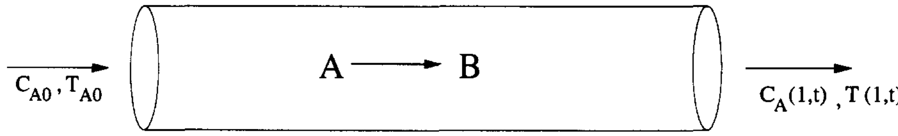

Fig. 2. A nonisothermal fixed-bed reactor.

subject to the initial and boundary conditions:

$$C_A(0,t) = C_{A0}, \qquad C_A(z,0) = C_{A_s}(z)$$

$$T(0,t) = T_{A0}, \qquad T(z,0) = T_s(z)$$

where C_A denotes the concentration of the species A, T denotes the temperature in the reactor, $\bar{\varepsilon}$ denotes the reactor porosity, ρ_b, c_{pb} denote the density and heat capacity of the bed, ρ_f, c_{pf} denote the density and heat capacity of the fluid phase, v_l denotes the velocity of the fluid phase, U_w denotes the heat transfer coefficient, T_j denotes the spatially uniform temperature in the jacket, V_r denotes the volume of the reactor, k_0, E, ΔH denote the pre-exponential factor, the activation energy, and the enthalpy of the reaction, C_{A0}, T_{A0} denote the concentration and temperature of the inlet stream, and $C_{A_s}(z)$, $T_s(z)$ denote the steady-state profiles for the concentration of the species A and temperature in the reactor.

The main feature of fixed-bed reactors is that the reactant wave propagates through the bed with a significantly larger speed than the heat wave, because the exchange of heat between the fluid and packing slows the thermal wave down (Stangeland and Foss, 1970). Therefore, the system of eq. (52) possesses an inherent two-time-scale property, i.e., the concentration dynamics are much faster than the temperature dynamics (this fact was also verified through open-loop simulations). This implies that C_A is the fast variable, while T is the slow variable. In order to obtain a singularly perturbed representation of the process, where the partition to fast and slow variables is consistent with the dynamic characteristics of the process, the singular perturbation parameter ε was defined as

$$\varepsilon = \frac{\bar{\varepsilon}}{\rho_b c_{pb}} \cdot \tag{53}$$

Setting $\bar{t} = t/\rho_b c_{pb}$, $x = T$, $\eta = C_A$, the system of eq. (52) can be written in the following singularly perturbed form:

$$\frac{\partial x}{\partial \bar{t}} = -\rho_f c_{pf} v \frac{\partial x}{\partial z} + (-\Delta H) k_0 \mathrm{e}^{-E/Rx}\eta + \frac{U_w}{V_r}(T_j - x)$$

$$\varepsilon \frac{\partial \eta}{\partial \bar{t}} = -v_l \frac{\partial \eta}{\partial z} - k_0 \mathrm{e}^{-E/Rx}\eta. \tag{54}$$

The values of the process parameters are given in Table 1. It was verified that they correspond to a stable steady-state for the open-loop system. The control problem considered is the one of controlling the temperature of the reactor (which is the variable that essentially determines the dynamics of the process) by manipulating the jacket temperature. Notice that T_j is chosen to be the manipulated input with the understanding that in practice its manipulation is achieved indirectly through manipulation of the jacket inlet flow rate (for more details see the discussion in remark 14). The enthalpy of the reaction is considered to be the main uncertain variable. Assuming that there is available one control actuator with distribution function $b(z) = 1$ and defining $\bar{u} = T_j - T_{js}$, $\bar{y} = \int_0^1 x\,\mathrm{d}z$, $\theta = \Delta H - \Delta H_0$, we have from eq. (54):

Table 1 Process parameters

$v_l = 30.0\ \mathrm{m\,hr^{-1}}$
$V_r = 1.0\ \mathrm{m^3}$
$\bar{\varepsilon} = 0.01$
$L = 1.0\ \mathrm{m}$
$E = 2.0 \times 10^4\ \mathrm{kcal\,kmol^{-1}}$
$k_0 = 5.0 \times 10^{12}\ \mathrm{hr^{-1}}$
$R = 1.987\ \mathrm{kcal\,kmol^{-1}\,K^{-1}}$
$\Delta H_0 = 35480.111\ \mathrm{kcal\,kmol^{-1}}$
$c_{pf} = 0.0231\ \mathrm{kcal\,kg^{-1}\,K^{-1}}$
$\rho_f = 90.0\ \mathrm{kg\,m^{-3}}$
$c_{pb} = 6.67 \times 10^{-4}\ \mathrm{kcal\,kg^{-1}\,K^{-1}}$
$\rho_b = 1500.0\ \mathrm{kg\,m^{-3}}$
$U_w = 500.0\ \mathrm{kcal\,hr^{-1}\,K^{-1}}$
$C_{A0} = 4.0\ \mathrm{kmol\,m^{-3}}$
$T_{A0} = 320.0\ \mathrm{K}$

$$\frac{\partial x}{\partial \bar{t}} = -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 \mathrm{e}^{-E/Rx}\eta + \frac{U_w}{V_r}(T_{js} - x) + \frac{U_w}{V_r}\bar{u} + k_0 \mathrm{e}^{-E/Rx}\eta\theta$$

$$\varepsilon \frac{\partial \eta}{\partial \bar{t}} = -v_l \frac{\partial \eta}{\partial z} - k_0 \mathrm{e}^{-E/Rx}\eta$$

$$\bar{y} = \int_0^1 x\,\mathrm{d}z. \tag{55}$$

Performing a two-time-scale decomposition of the above system, the fast subsystem takes the form

$$\frac{\partial \eta}{\partial \bar{\tau}} = -v_l \frac{\partial \eta}{\partial z} - k_0 \mathrm{e}^{-E/Rx}\eta \tag{56}$$

where $\bar{\tau} = \bar{t}/\varepsilon$ and the x in the above system depends only on the position z. From the system of eq. (56), it is clear that the fast dynamics of the system of eq. (55) are exponentially stable uniformly in x, and thus they can be neglected in the controller design. Setting $\varepsilon = 0$, the model of eq. (55) reduces to a set of a partial and an ordinary differential equation of the form:

$$\frac{\partial x}{\partial \bar{t}} = -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 e^{-E/Rx} \eta + \frac{U_w}{V_r}(T_{js} - x) + \frac{U_w}{V_r}\bar{u} + k_0 e^{-E/Rx} \eta \theta$$

$$0 = -v_l \frac{\partial \eta}{\partial z} - k_0 e^{-E/Rx} \eta. \qquad (57)$$

The structure of the ordinary differential equation allows an analytic derivation of its solution subject to the boundary condition $\eta(0, t) = C_{A0}(t)$, which is of the form

$$\eta(z) = C_{A0} e^{k_0 \int_0^z e^{-E/Rx} dz / v_l}. \qquad (58)$$

Substituting eq. (58) into eq. (57), the following reduced system can be obtained:

$$\frac{\partial x}{\partial \bar{t}} = -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 e^{-E/Rx} C_{A0} e^{k_0 \int_0^z e^{-E/Rx} dz / v_l} + \frac{U_w}{V_r}(T_{js} - x) + \frac{U_w}{V_r}\bar{u} + k_0 e^{-E/Rx} C_{A0} e^{k_0 \int_0^z e^{-E/Rx} dz / v_l} \theta$$

$$\bar{y}_s = \int_0^1 x \, dz. \qquad (59)$$

The above system is clearly in the form of eq. (1) and a straightforward calculation of the time-derivative of the output yields that the characteristic indices of the output $\bar{y}_s$ with respect to the manipulated input $\bar{u}$ and the uncertain variable θ are $\sigma = \delta = 1$. This implies that it is not possible to decouple the effect of θ on $\bar{y}_s$ via distributed state feedback, and a robust controller of the form of eq. (26) should be synthesized to attenuate the effect of $\bar{\theta}$ on $\bar{y}_s$ [this is possible because the matching condition, assumption 4, holds for the system of eq. (59)]. Moreover, it was verified through simulations that the zero dynamics of the system of eq. (59) are locally exponentially stable. The explicit form of the controller of eq. (26) is

$$\bar{u} = \frac{V_r}{U_w}\left\{\gamma_0\left(v - \int_0^1 x \, dz\right) - \int_0^1 \left(-\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 e^{-E/Rx} C_{A0} e^{k_0 \int_0^z e^{-E/Rx} dz / v_l} + \frac{U_w}{V_r}(T_{js} - x)\right) dz - K(t) \frac{\int_a^b x(z,t) \, dz - v}{\left|\int_a^b x(z,t) \, dz - v\right| + \phi}\right\} \qquad (60)$$

where

$$K(t) = 2\bar{\theta} \int_0^1 \frac{k_0 V_r}{U_w} e^{-E/Rx} C_{A0} e^{k_0 \int_0^z e^{-E/Rx} dz / v_l} dz.$$

A time-varying uncertainty was considered expressed by a sinusoidal function of the form

$$\theta = 0.5(-\Delta H_0)\sin(t) \qquad (61)$$

The upper bound on the uncertainty was taken to be $\bar{\theta} = 0.5|(-\Delta H_0)|$. In this application, the value of the singular perturbation parameter is fixed, i.e. $\varepsilon = 0.01$. From theorem 6, it is clear that for a given value of ε^*, there exists a lower bound on the level of asymptotic attenuation d that can be achieved. We performed a set of computer simulations (for the regulation problem) to calculate ϕ^* for certain values of d, and, in turn, the value of ε^* for $\phi \leqslant \phi^*$. The following set of parameters were found to give an $\varepsilon^* \leqslant 0.01$ and used in the simulations:

$$\gamma_0 = 0.2, \qquad \phi = 0.1 \qquad (62)$$

to achieve an ultimate degree of attenuation $d = 0.5$.

Two simulation runs were performed to test the regulatory, set-point tracking and uncertainty rejection capabilities of the controller. In both runs, the process was initially ($t = 0.0$ min) assumed to be at steady state. In the first simulation run, we tested the regulatory capabilities of the controller. Figure 3 shows the closed-loop output and manipulated input profiles, while Fig. 4 displays the evolution of the temperature profile through out the reactor. Clearly, the controller regulates the output at the operating steady-state compensating for the effect of uncertainty and satisfying the requirements $\lim_{t\to\infty}|\bar{y} - v| \leqslant 0.5$. For the sake of comparison, we also implemented the same controller without the term which is responsible for the compensation of uncertainty. The output and manipulated input profiles for this simulation run are given in Fig. 5. It is obvious that the controller cannot attenuate the effect of the uncertainty on the output of the process, leading to poor transient performance and offset. In the next simulation run, we tested the output tracking capabilities of the controller. A 6.6 K increase in the value of the reference input was imposed at time $t = 0.0$ h. The output and manipulated input profiles are shown in Fig. 6. It is clear that the controller drives the output $\bar{y}$ to its new reference input value, achieving the requirement $\lim_{t\to\infty}|\bar{y} - v| \leqslant 0.5$. Figure 7 shows the output and manipulated profiles in the case of using the same controller without the term which is responsible for the compensation of uncertainty. Clearly, the performance is very poor. From the results of the simulation study, we conclude that there is a need to compensate for the effect of the uncertainty as well as that the proposed robust control methodology is a very efficient tool for this purpose.

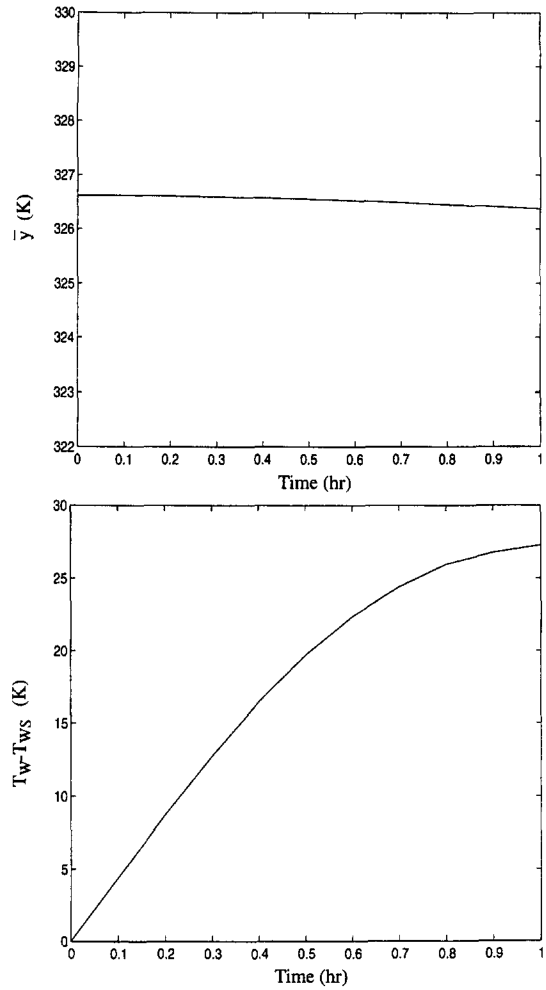

Fig. 3. Closed-loop output and manipulated input profiles for regulation.

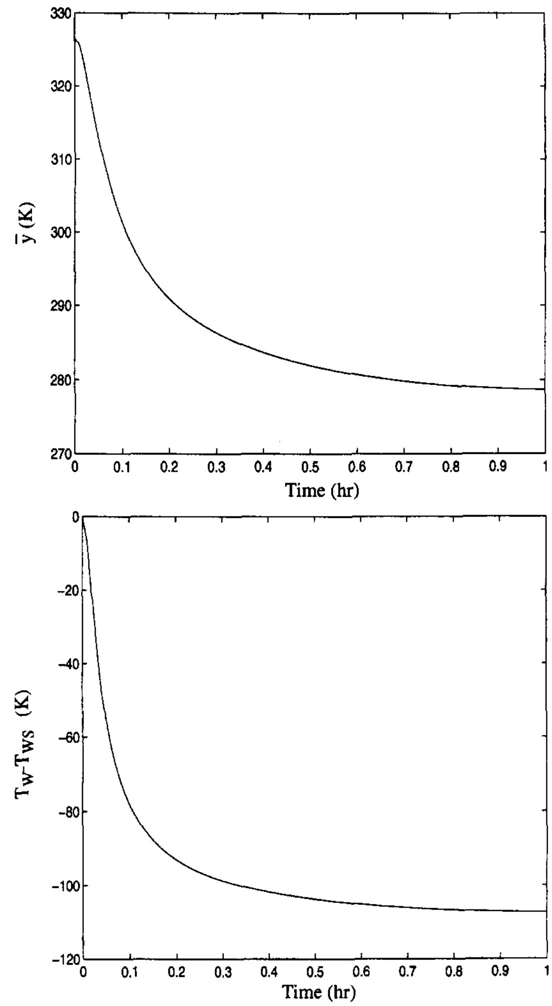

Fig. 5. Closed-loop output and manipulated input profiles for regulation (no uncertainty compensation).

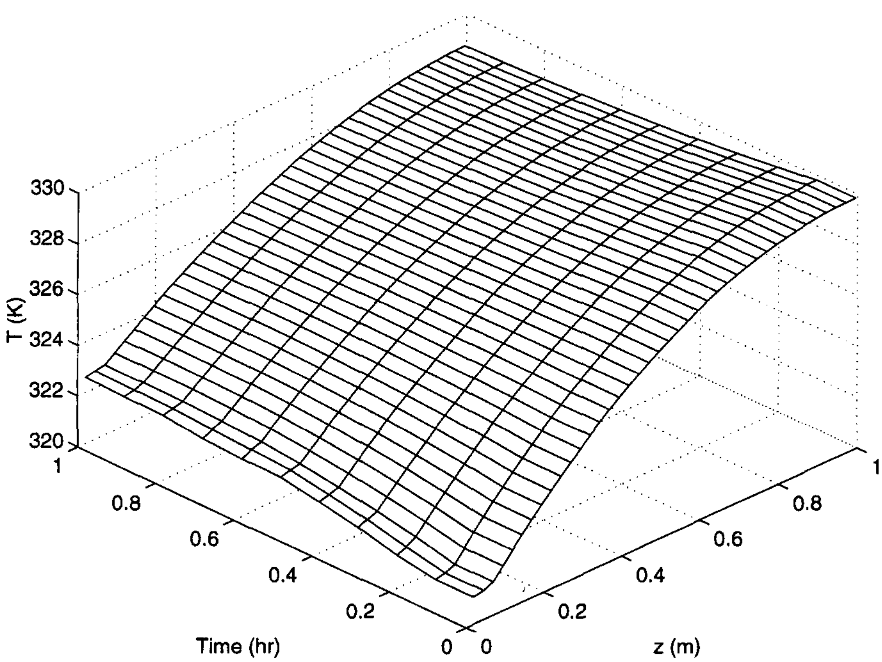

Fig. 4. Profile of evolution of reactor temperature for regulation.

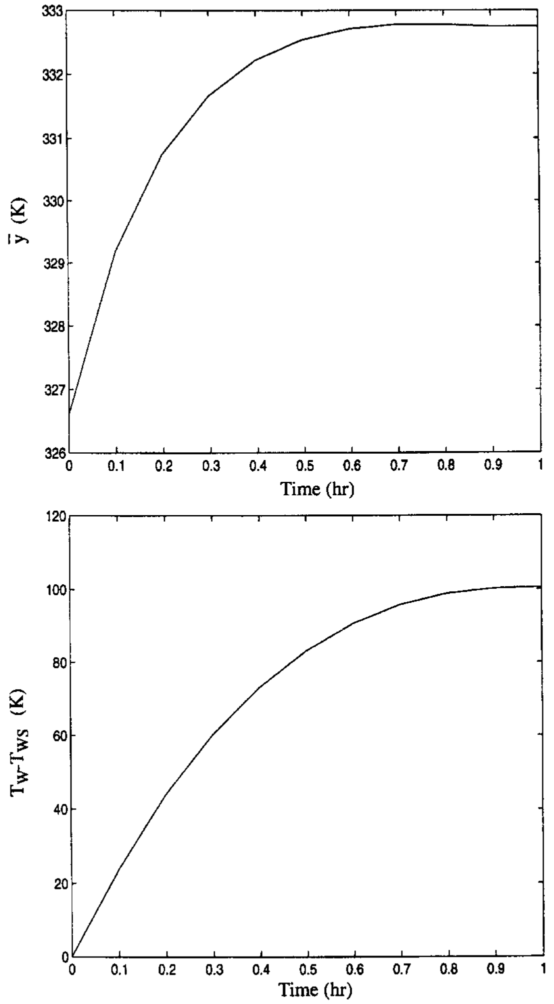

Fig. 6. Closed-loop output and manipulated input profiles for reference input tracking.

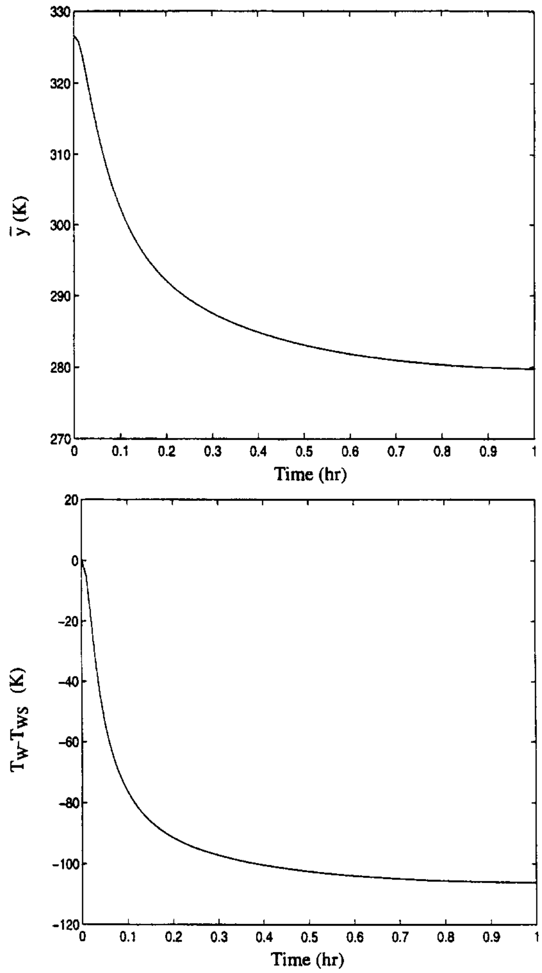

Fig. 7. Closed-loop output and manipulated input profiles for reference input tracking (no uncertainty compensation).

Remark 13. The reduction in the dimensionality of the original model of the reactor using the fact that the process exhibits a two-time-scale property eliminates the need for using measurements of the concentration of the species A in the controller of eq. (60), which greatly facilitates its practical implementation.

Remark 14. Regarding the practical implementation of the robust distributed controller of eq. (60), we note that the manipulated variable T_j cannot be manipulated directly, but indirectly through manipulation of the jacket inlet flow rate. To this end, a controller should be designed based on an ODE model that describes the jacket dynamics, that operates in an internal loop to manipulate the jacket inlet flow rate to ensure that the jacket temperature obtains the values computed by the distributed robust controller (Christofides and Daoutidis, 1996a). Of course, when such a controller is used, a slight deterioration of the closed-loop performance and robustness obtained under the assumption that T_j can be manipulated directly, will occur.

8. CONCLUSIONS

In this work, we considered systems of first-order hyperbolic PDEs with uncertainty, for which the manipulated input and the controlled output are distributed in space. Both uncertain variables and unmodeled dynamics were considered. In the case of uncertain variables, we initially derived a necessary and sufficient condition for the solvability of the problem of complete elimination of the effect of uncertainty on the output via distributed feedback as well as an explicit controller synthesis formula. Then, assuming that there exist known bounding functions that capture the magnitude of the uncertain terms and a matching condition is satisfied, we synthesized, using Lyapunov's direct method, a distributed robust-controller that guarantees boundedness of the state and asymptotic output tracking with arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the output of the closed-loop system. In the presence of uncertain variables and unmodeled dynamics, we established that the proposed distributed controllers enforce approximate uncertainty decoupling or uncertainty attenuation in the closed-loop system as long as the unmodeled dynamics are stable and sufficiently fast. A nonisothermal fixed-bed reactor with unknown heat of the reaction was used to illustrate the application of the proposed control method. The fact that the reactant wave propagates through the bed faster than the heat wave was explicitly taken into account, and the original model of the process, which consists of two quasi-linear hyperbolic PDEs that describe the spatio-temporal evolution of the reactant concentration and the reactor temperature, was reduced to one quasi-linear hyperbolic PDE that describes the evolution of the reactor temperature. A distributed robust controller was then designed on the basis of the reduced-order model to enforce output tracking and compensate for the effect of the uncertainty. Computer simulations were used to evaluate the performance and robustness properties of the controller.

Acknowledgement

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NOTATION

A, $A(x)$, $A(z)$	matrices
$b^i(z)$	smooth function of z
$c^i(z)$	smooth function of z
C_{A0}	inlet concentration of the species A
C_{Al}	concentration of the species A in lateral stream
C_1, C_2	constant matrices
C_A	concentration of the species A
c_{pb}	heat capacity of the bed
c_{pf}	heat capacity of the fluid phase
E	activation energy
f, f_1, f_2	vector fields
F	lateral inlet flow rate
g, g_1, g_2	vector fields associated with the manipulated variable
h	controlled output scalar field
k_0	pre-exponential constant
l	total number of control actuators
Q_1, Q_2	sufficiently smooth matrices
T_{A0}	inlet temperature of the fluid in the reactor
T_{Al}	temperature of lateral stream
T	temperature of the reactor
T_j	temperature of the jacket
t	time
U_w	heat transfer coefficients
u	manipulated variable
$\bar{u}$	manipulated input vector
$\bar{u}^i$	ith manipulated input
V_r	volume of the reactor
v	external reference input vector
v^i	external reference input for the ith actuator
v_l	fluid velocity
W, W_1, W_2	sufficiently smooth matrices
x	vector of state variables
$\bar{y}$, $\bar{y}_s$	controlled outputs
z	spatial coordinate

Greek letters

α	boundary of the spatial domain
β	boundary of the spatial domain
γ_k	adjustable parameters
ΔH	enthalpy of the reaction
η	observer state vector
δ^i	characteristic index of $\bar{y}^i$ with respect to $\bar{\theta}^i$
δ	characteristic index of $\bar{y}$ with respect to $\bar{\theta}$
θ	vector of uncertain variables
ρ_b	density of the bed
ρ_f	density of the fluid phase
σ^i	characteristic index of $\bar{y}^i$ with respect to $\bar{u}^i$
σ	characteristic index of $\bar{y}$ with respect to $\bar{u}$

Maths Symbols

$\mathscr{C}^i$	bounded linear operator
$\mathscr{H}$	infinite dimensional Hilbert space
$L_f h$	Lie derivative of a scalar field h with respect to the vector field f
$L_f^k h$	kth order Lie derivative
$L_g L_f^{k-1} h$	mixed Lie derivative
$\mathscr{S}$, $\bar{\mathscr{P}}$, $\bar{\mathscr{R}}$	nonlinear functionals
$\mathbb{R}$	real line
$\mathbb{R}_{\geq 0}$	positive real line
$\mathbb{R}^i$	i-dimensional Euclidean space
$\in$	belongs to
T	transpose
$\lvert\cdot\rvert$	standard Euclidean norm
$\lVert\cdot\rVert_2$	2-norm in $\mathscr{H}$
$(\cdot,\cdot)_{\mathbb{R}^n}$	inner product in $\mathbb{R}^n$
$\lVert\theta\rVert$	ess.sup. $\{\lvert\theta(t)\rvert, t \geq 0\}$, where $\theta: \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is any measurable function
$u(t)$	signal defined on $[0, T]$
u_τ	signal defined on $[0, \infty]$, given by $u_\tau(t) = u(t)$ *if* $t \in [0,\tau]$, and $u_\tau(t) = 0$ if $t \in (\tau, \infty)$

REFERENCES

Alonso, A. A. and Ydstie, E. B. (1995). Nonlinear control, passivity and the second law of thermodynamics. AIChE Annual Meeting, Paper 181i, Miami Beach, Florida.

Balas, M. J. (1983). The Galerkin method and feedback control of linear distributed parameter systems. *J. Math. Anal. Appl.* **91**, 527.

Balas, M. J. (1995). Finite-dimensional direct adaptive control for discrete-time infinite-dimensional linear systems. *J. Math. Anal. Appl.* **196**, 153.

Burns, J. A. and King, B. B. (1994). Optimal sensor location for robust control of distributed parameter systems. *Proceedings of 33rd IEEE Conference on Decision and Control*, 3965, Orlando, FL, U.S.A.

Christofides, P. D. and Daoutidis, P. (1996). Feedback control of hyperbolic PDE systems. *A.I.Ch.E. J.* **42**, 3063.

Christofides, P. D. and Daoutidis, P. (1997). Control of nonlinear distributed parameter processes: current results and future research directions. *Proceedings of 5th International Conference on Chemical Process Control* (in press).

Christofides, P. D. and Teel, A. R. (1996). Singular perturbations and input-to-state stability. *IEEE Trans. Automat. Control.* **41**, 1645.

Curtain, R. F. (1984). Disturbance decoupling for distributed systems by boundary control. *Proceedings of 2nd International Conference on Control Theory for Distributed Parameter Systems and Applications*, 109, Vorau, Austria.

Curtain, R. F., (1986). Invariance concepts in infinite dimensions, *SIAM J. Control Optim.* **24**, 1009.

Curtain, R. F. and Glover, K. (1986) Robust stabilization of infinite dimensional systems by finite dimensional controllers. *Systems Control Lett.* **7**, 41.

Daoutidis, P. and Christofides, P. D. (1995). Dynamic feedforward/output feedback control of nonlinear processes. *Chem. Engng Sci.* **50**, 1889.

Demetriou, M. A. (1994). Model reference adaptive control of slowly time-varying parabolic systems. *Proceedings of 33rd IEEE Conference on Decision and Control*, Orlando, Fl., U.S.A., p. 775.

Dochain, D. and Bouaziz, B. (1993). Approximation of the dynamical model of fixed-bed reactors via a singular perturbation approach. *Proceedings of IMACS International Symposium MIM-S2'93*, p. 34.

Friedman, A. (1976). Partial differential equations. Holt, Rinehart & Winston, New York.

Gauthier J. P. and Xu, C. Z. (1989). H^{∞}-control of a distributed parameter system with non-minimum phase. *Int. J. Control* **53**, 45.

Hanczyc, E. M. and Palazoglu, A. (1995). Sliding mode control of nonlinear distributed parameter chemical processes. *I & EC Res.* **34**, 455.

Hong K. and Bentsman, J. (1994). Application of averaging method for integro-differential equations to model reference adaptive control of parabolic systems. *Automatica* **36**, 1415.

Jacobson, C. A. and Nett, C. N. (1988). Linear state-space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability. *IEEE Trans. Automat. Control* **AC-33**, 541.

Keulen, B. (193). *H_{∞}-control for distributed parameter systems: a state-space approach.* Birkhauser, Boston.

Khalil, H. K. (1992). Nonlinear systems. Macmillan, New York, U.S.A.

Lasiecka, I. (1995). Control of systems governed by partial differential equations: a historical perspective. *Proceedings of 34th IEEE Conference Decision Control*, 2792, New Orleans, Louisiana.

Palazoglu A. and Owens, S. E. (1987). Robustness analysis of a fixed-bed tubular reactor: impact of modeling decisions, *Chem. Engng Comm.* **59**, 213.

Ray, W. H. (1981) Advanced process control, McGraw-Hill, New York. U.S.A.

Rhee, H. K., Aris, R. and Amundson, N. R. (1986). *First-order partial differential equations*: Vols I & II. Prentice-Hall, Englewood Cliffs, NJ.

Russell, D. L. (1978). Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Rev.* **20**, 639.

Ruthven, D. M. and Sircar, S. (1994) Design of membrane and PSA processes for bulk gas separation. *Proceedings of the 4th International Conference on the Foundations of Computer-Aided Process Design*, **29**, Snowmass, Co. U.S.A.

Sira-Ramirez, H., (1989). Distributed sliding mode control in systems described by quasilinear partial differential equations. *Systems Control Lett.* **13**, 177.

Soliman, M. A. and Ray, W. H. (1979). Non-linear filtering for distributed parameter systems having a small parameter. *Int. J. Control* **30**, 757.

Stangeland, B. E. and Foss, A. S. (1970). Control of a fixed-bed chemical reactor. *I & EC Res.* **38**, 38.

Wang, P. K. C., (1964). Control of distributed parameter systems. In *Advances in Control Systems*. Academic Press, New York, U.S.A.

Wang, P. K. C. (1966). Asymptotic stability of distributed parameter systems with feedback controls. *IEEE Trans. Automat. Control* **AC-11**, 46.

Wen, J. T. and Balas, M. J. (1989). Robust adaptive control in hilbert space. *J. Math. Anal. Appl.* **143**, 1.

APPENDIX

Proof of Proposition 1 (1) The proof of the first part of the proposition is given in (Christofides and Daoutidis, 1996; Proposition 2)

(2) Referring to the open-loop system of eq. (1), let $\delta \leqslant \sigma$. We will first show that the result holds in the case $\delta < \sigma$. Consider the closed-loop system of eq. (18). From part one of the proposition, we have that the characteristic index of the output $\bar{y}_s$ with respect to v in the closed-loop system is equal to σ. From the definition of characteristic index of $\bar{y}_s$ with respect to θ, we have that the following relations hold for the open-loop system:

$$\mathscr{C}^i L_{W_k}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\mu-1} h(x)\, r_k^i(z) \equiv 0,$$
$$\forall \mu = 1, \dots, \delta - 1, \quad i = 1, \dots, l. \tag{A1}$$

Differentiating the output of the closed-loop system with respect to time and using the relations of eq. (A1), we get

$$\bar{y}_s = \mathscr{C} h(x)$$
$$\frac{d\bar{y}_s}{dt} = \mathscr{C}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} \rightarrow L_f\right) h(x)$$
$$\frac{d^2\bar{y}_s}{dt^2} = \mathscr{C}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right) h(x)$$
$$\vdots$$
$$\frac{d^\delta\bar{y}_s}{dt^\delta} = \mathscr{C}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\delta-1}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right) h(x)$$
$$+ \mathscr{C} \sum_{k=1}^{q} L_{W_k}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\delta-1} h(x) r_k(z)\theta_k(t). \tag{A2}$$

From the above equations, it is clear that the result of the second part of the proposition holds if $\delta < \sigma$. The same argument can be used to show that the result is also true for the case $\delta = \sigma$. The details in this case are omitted for brevity. □

Proof of Theorem 2. (1) (*uncertainty decoupling*). *Necessity*. We will proceed by contradiction. Consider the system of eq. (1) and assume that $\delta \leqslant \sigma$. Referring to the closed-loop system of eq. (18), we have, from proposition 1, that the characteristic indices σ, δ are preserved, and the condition $\delta \leqslant \sigma$ holds. This fact implies that θ affects directly the δ-th time-derivative of $\bar{y}_s$ in the closed-loop system, and thus $\bar{y}_s$, which yields a contradiction.

Sufficiency: Referring to the system of eq. (1), suppose that $\sigma < \delta$. In this case, a time-differentiation of $\bar{y}$ up to σth order yields the following expressions:

$$\bar{y}_s = \mathscr{C}h(x)$$

$$\frac{d\bar{y}_s}{dt} = \mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x)$$

$$\frac{d^2\bar{y}_s}{dt^2} = \mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x)$$

$$\vdots$$

$$\frac{d^\sigma\bar{y}_s}{dt^\sigma} = \mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) + \mathscr{C}L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1}h(x)b(z)u. \quad \text{(A3)}$$

From the expression of the σth derivative of $\bar{y}_s$, it is clear that there exists a control law of the form of eq. (17) [e.g.], the controller of eq. (20)] which guarantees that $\bar{y}_s$ is independent of θ in the closed-loop system, for all times. Finally, one can easily show, utilizing the expressions of eq. (65), that the controller of eq. (20) enforces the input/output response of eq. (19) in the closed-loop system.

(2) (*boundedness*). First, we note that whenever $\theta(t) \equiv 0$, the conditions (i) and (ii) of the theorem guarantee that the nominal closed-loop system is locally exponentially stable [Christofides and Daoutidis, 1996; Proposition 4]. Since the nominal closed-loop system is locally exponentially stable, we have from theorem 1 that there exists a smooth Lyapunov functional $V: \mathscr{H}^n \times [\alpha, \beta] \to R$ of the form of eq. (12), and a set of positive real numbers a_1, a_2, a_3, a_4, a_5, such that the following properties hold:

$$a_1\|x\|_2^2 \leqslant V(t) \leqslant a_2\|x\|_2^2$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\mathscr{L}(x) \leqslant -a_3\|x\|_2^2$$

$$\left\|\frac{\partial V}{\partial x}\right\|_2 \leqslant a_4\|x\|_2 \quad \text{(A4)}$$

if $\|x\|_2 \leqslant a_5$. Whenever $(\theta(t) \neq 0)$ the closed-loop system, under the control law of eq. (20), takes the form

$$\frac{\partial x}{\partial t} = \mathscr{L}(x) + W(x)r(z)\theta(t) \quad \text{(A5)}$$

where $\mathscr{L}(x)$ is a nonlinear operator. Computing the time-derivative of the functional $V: \mathscr{H}^n \times [\alpha, \beta] \to R$ along the trajectories of the uncertain closed-loop system of eq. (A5), using eq. (A6), and using the fact that if $\|x\|_2 \leqslant a_5$ there exists a positive real number a_6 such that $\|W(x)\|_2 \leqslant a_6$, we get

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}(\mathscr{L}(x) + W(x)r(z)\theta(t)) \leqslant -a_3\|x\|_2^2 + a_4a_6\|x\|_2\|r(z)\|_2|\theta(t)|. \quad \text{(A6)}$$

From the last inequality of the above equation, we have that if

$$|\theta(t)| \leqslant \frac{a_3a_5}{4a_4a_6\|r(z)\|_2} \leqslant \hat{\delta}, \quad \forall t \geqslant 0,$$

then

$$\frac{dV}{dt} \leqslant -\frac{a_3}{2}\|x\|_2^2 \quad \text{(A7)}$$

if $a_5/2 \leqslant \|x\|_2 \leqslant a_5$. Using the result of theorem 4.10 reported in Khalil (1992), we have that $\|x\|_2$ is a bounded quantity, which implies that the state of the closed-loop system is bounded. □

Proof of Theorem 3. First, we define the state vectors $\zeta_v^i = (d^{v-1}\bar{y}_s^i)/(dt^{v-1})$, $i = 1, \ldots, l$, $v = 1, \ldots, \sigma$, $\zeta_v = [\zeta_v^1\ \zeta_v^2 \ldots \zeta_v^l]^T$, $\tilde{\zeta}_1 = \zeta_1 - v$, $\tilde{\zeta}_\sigma^i = \zeta_\sigma^i + (\gamma_1/\gamma_\sigma)\tilde{\zeta}_1^i + (\gamma_2/\gamma_\sigma)\zeta_2^i + \cdots + (\gamma_{\sigma-1}/\gamma_\sigma)\zeta_{\sigma-1}^i$, $\tilde{\zeta}_\sigma = [\tilde{\zeta}_\sigma^1\tilde{\zeta}_\sigma^2 \cdots \tilde{\zeta}_\sigma^l]^T$ and for ease of notation, we also set:

$$\bar{s}(x) = \left[\mathscr{C}L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1}h(x)b(z)\right]^{-1}. \quad \text{(A8)}$$

Using the above notation the controller of eq. (27) takes the form

$$\bar{u} = \bar{s}(x)\left\{-\gamma_0\tilde{\zeta}_\sigma - \sum_{v=1}^{\sigma-1}\frac{\gamma_v}{\gamma_\sigma}\zeta_{v+1} - 2K(t)\Delta(\tilde{\zeta}_\sigma, \phi)\tilde{\zeta}_\sigma\right\} \quad \text{(A9)}$$

where $\Delta(\tilde{\zeta}_\sigma, \phi)$ is an $l \times l$ diagonal matrix, whose (i, i)th element is of the form $(|\tilde{\zeta}_\sigma^i| + \phi)^{-1}$.

(1) *Asymptotic output tracking*. Using the definition for the state vectors ζ_v, $v = 1, \ldots, \sigma$, and $\tilde{\zeta}_1, \tilde{\zeta}_\sigma$, the closed-loop system

$$\frac{\partial x}{\partial t} = A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\bar{s}(x)\left\{-\gamma_0\tilde{\zeta}_\sigma - \sum_{v=1}^{\sigma-1}\frac{\gamma_v}{\gamma_\sigma}\zeta_{v+1} - 2K(t)\Delta(\tilde{\zeta}_\sigma, \phi)\tilde{\zeta}_\sigma\right\} + W(x)r(z)\theta(t)$$

$$\bar{y}_s = \mathscr{C}h(x) \quad \text{(A10)}$$

can be equivalently written in the following form

$$\dot{\tilde{\zeta}}_1 = \zeta_2$$

$$\vdots$$

$$\dot{\zeta}_{\sigma-1} = -\frac{\gamma_1}{\gamma_\sigma}\tilde{\zeta}_1 - \frac{\gamma_2}{\gamma_\sigma}\zeta_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_{\sigma-1} + \tilde{\zeta}_\sigma$$

$$\dot{\tilde{\zeta}}_\sigma = \sum_{v=1}^{\sigma-1}\frac{\gamma_v}{\gamma_\sigma}\zeta_{v+1} - \gamma_0\tilde{\zeta}_\sigma - \sum_{v=1}^{\sigma-1}\frac{\gamma_v}{\gamma_\sigma}\zeta_{v+1} - 2K(t)\Delta(\tilde{\zeta}_\sigma, \phi)\tilde{\zeta}_\sigma + K(t) \quad \text{(A11)}$$

$$\frac{\partial x}{\partial t} = \mathscr{L}x + g(x)b(z)\bar{s}(x)\left\{-\gamma_0\tilde{\zeta}_\sigma - \sum_{v=1}^{\sigma-1}\frac{\gamma_v}{\gamma_\sigma}\zeta_{v+1} - 2K(t)\Delta(\tilde{\zeta}_\sigma, \phi)\tilde{\zeta}_\sigma\right\} + W(x)r(z)\theta.$$

To establish the relation of eq. (28), we will first work with the $\tilde{\zeta}_\sigma$-subsystem of eq. (73) and establish a bound for the state of this system in terms of the initial condition and the parameter ϕ (in order to simplify the presentation we will focus on the ith input/output pair). To this end, we consider the following smooth functions $V^i: \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, $i = 1, \ldots, l$:

$$V^i = \tfrac{1}{2}(\tilde{\zeta}^i_\sigma)^2 \tag{A12}$$

to show that the time-derivative of V^i is negative definite outside of a region that includes the steady state in the space and this region can be made arbitrarily small by picking ϕ sufficiently small. Calculating the time-derivative of V^i along the trajectories of the $\tilde{\zeta}^i_\sigma$-subsystem of eq. (A10), we have

$$\begin{aligned}\frac{dV^i}{dt} &= \tilde{\zeta}^i_\sigma[-\gamma_0\tilde{\zeta}^i_\sigma - 2K(t)(|\tilde{\zeta}^i_\sigma| + \phi)^{-1}\tilde{\zeta}^i_\sigma + K(t)]\\ &\leqslant -\gamma_0(\tilde{\zeta}^i_\sigma)^2 - 2K(t)(|\tilde{\zeta}^i_\sigma| + \phi)^{-1}(\tilde{\zeta}^i_\sigma)^2 + K(t)|\tilde{\zeta}^i_\sigma|\\ &\leqslant -\gamma_0(\tilde{\zeta}^i_\sigma)^2 - 2K(t)(|\tilde{\zeta}^i_\sigma| + \phi)^{-1}(\tilde{\zeta}^i_\sigma)^2\\ &\quad + K(t)(|\tilde{\zeta}^i_\sigma| + \phi)^{-1}[(\tilde{\zeta}^i_\sigma)^2 + |\tilde{\zeta}^i_\sigma|\phi]\\ &\leqslant -\gamma_0(\tilde{\zeta}_\sigma)^2 - K(t)(|\tilde{\zeta}^i_\sigma| + \phi)^{-1}|\tilde{\zeta}_\sigma|(|\tilde{\zeta}^i_\sigma| - \phi).\end{aligned} \tag{A13}$$

Clearly, if $|\tilde{\zeta}^i_\sigma| \geqslant \phi$, $\dot{V}^i$ is negative definite, which implies [theorem 4.10] that there exist positive real numbers $\hat{K} \geqslant 1$, $\bar{a}$, γ, such that the following bound holds for the norm of the state of the $\tilde{\zeta}^i_\sigma$-subsystem:

$$|\tilde{\zeta}^i_\sigma| \leqslant \hat{K}|(\tilde{\zeta}^i_\sigma)_0|e^{-\bar{a}t} + \gamma\phi \tag{A14}$$

where $(\tilde{\zeta}^i_\sigma)_0$ denotes the value of $\tilde{\zeta}^i_\sigma$ at time $t = 0$. Consider now the following subsystem:

$$\begin{aligned}\dot{\tilde{\zeta}}^i_1 &= \zeta^i_2\\ &\vdots\\ \zeta^i_{\sigma-1} &= -\frac{\gamma_1}{\gamma_\sigma}\tilde{\zeta}^i_1 - \frac{\gamma_2}{\gamma_\sigma}\zeta^i_2 - \ldots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta^i_{\sigma-1} + \tilde{\zeta}^i_\sigma\end{aligned} \tag{A15}$$

where $\tilde{\zeta}^i_\sigma$ can be thought of as an external input. Since the parameters γ_k are chosen so that the polynomial $1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0$ is Hurwitz, we have that there exists positive real numbers K_ζ, a_ζ, $\gamma_{\tilde{\zeta}_\sigma}$ such that the following bound can be written for the state vector $\zeta^i = [\tilde{\zeta}^i_1 \zeta^i_2 \cdots \zeta^i_{\sigma-1}]$ of the system of eq. (A15):

$$|\zeta^i| \leqslant K_\zeta|\zeta^i_0|e^{-a_\zeta t} + \gamma_{\tilde{\zeta}_\sigma}\|\tilde{\zeta}^i_\sigma\|. \tag{A16}$$

Taking the limit as $t \to \infty$, using the property that $\lim_{t\to\infty}\sup_{t\geqslant 0}\{|\tilde{\zeta}^i_\sigma|\} = \sup_{t\geqslant 0}\{\lim_{t\to\infty}|\tilde{\zeta}^i_\sigma|\}$, and using eq. (A13) we have

$$\lim_{t\to\infty}|\zeta^i| \leqslant \gamma_{\tilde{\zeta}^i_\sigma}\gamma\phi. \tag{A17}$$

Picking $\phi^* = d/\gamma_{\tilde{\zeta}_\sigma}\gamma$, the relation of eq. (28) follows directly from the fact $\lim_{t\to\infty}|\tilde{\zeta}^i_1| \leqslant \lim_{t\to\infty}|\zeta^i|$.

(2) *Boundedness of the state.* The proof that the state is bounded, whenever the conditions of the theorem hold, can be obtained by using a standard contradiction argument. It starts by assuming that there exists a maximal time T such that for all $t \in [0, T)$, the states $(x, \zeta, \tilde{\zeta}_\sigma)$ of the system of eq. (A11) (note that T always exists because the system starts from bounded initial conditions). Then, the bounds that hold for the states $(x, \zeta, \tilde{\zeta}_\sigma)$ for $t \in [0, T)$ are derived and is shown that they continue to hold for $t \in [0, T + k_T]$, where k_T is some positive number. However, this contradicts the assumption that T is the maximal time in which the state is bounded, which implies that $T = \infty$, and thus the state of the closed-loop system is bounded for all times. □

Proof of Theorem 4. Using L_{ij} as defined in Section 7.1, the representation of the system of eq. (30), with $\bar{u} = 0$, in the coordinates (x, η_f) takes the form

$$\begin{aligned}\frac{\partial x}{\partial t} &= \mathscr{L}x + W(x)r(z)\theta(t) + \mathscr{L}_{12}\eta_f\\ \varepsilon\frac{\partial \eta_f}{\partial t} &= \mathscr{L}_{22}\eta_f + \varepsilon\left(\frac{\partial z_s}{\partial x}\dot{x} + \frac{\partial z_s}{\partial \theta}\dot{\theta}\right).\end{aligned} \tag{A18}$$

Since from assumption 8, the system:

$$\varepsilon\frac{\partial \eta_f}{\partial t} = \mathscr{L}_{22}\eta_f \tag{A19}$$

is exponentially stable, we have that if $(x, \theta, \dot{\theta})$ are bounded, then there exist a set of positive real numbers K_f, $\bar{a}_f$, γ_{η_f} such that the following bound can be written for the state of the η_f-subsystem of eq. (A20), for all $t \geqslant 0$:

$$\|\eta_f\|_2 \leqslant K_f\|\eta_{f0}\|_2 e^{-\bar{a}_f t/\varepsilon} + \gamma_{\eta_f}\varepsilon. \tag{A20}$$

Using assumption 7 and eq. (10), we have that the following bound holds for the state of the x-subsystem of eq. (A19), for all $t \geqslant 0$:

$$\begin{aligned}\|x\|_2 &\leqslant K_s\|x_0\|_2 e^{-a_s t}\\ &\quad + K_s\int_0^t e^{-a_s(t-\tau)}\|W(x)r(z)\|_2|\theta(\tau)|d\tau\\ &\quad + K_s\int_0^t e^{-a_s(t-\tau)}\|\mathscr{L}_{12}\|_2\|\eta_f\|_2 d\tau\\ &\leqslant K_s\|x_0\|_2 e^{-a_s t} + M_2\|\theta\|\\ &\quad + M_3\sup_{t\geqslant 0}\{\|\eta_f\|_2\}\end{aligned} \tag{A21}$$

where

$$M_2 = \frac{K_s\bar{M}_2\|r(z)\|_2}{a_s} \quad \text{and} \quad M_3 = \frac{K_s\|\mathscr{L}_{12}\|_2}{a_s}$$

provided that the initial condition $(\|x_0\|_2)$ and the inputs $(\|\theta\|, \|\dot{\theta}\|, \sup\{\|\eta_f\|_2\})$ are sufficiently small. Note that since we do not know *a priori* that the states (x, η_f) of the system of eq. (A19) are bounded, we have to work with truncations and exploit causality in order to prove boundedness. Let $\tilde{\delta}$ be as given in the statement of the theorem, so that $\max\{\|x_0\|_2, \|\eta_{f_0}\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \tilde{\delta}$, and let δ_x to be a positive real number that satisfies

$$\delta_x > K_s\tilde{\delta} + M_2\tilde{\delta} + d \tag{A22}$$

where d is a positive real number specified in the statement of the theorem. Note that since $K_s \geqslant 1$, $\delta_x > \tilde{\delta}$, and using continuity with respect to initial conditions, we define $[0, T)$ to be the maximal interval in which $\|x_t\|_2 \leqslant \delta_x$, for all $t \in [0, T)$ and suppose that T is finite. We will now show by contradiction that $T = \infty$, provided that ε is sufficiently small.

First, notice that from eq. (A20), we have that for all $t \in [0, T)$, $\|\eta_{f_t}\| \leqslant K_f\tilde{\delta} + \gamma_\eta\varepsilon$. Let ε_0 be a positive real number so that $\varepsilon \in (0, \varepsilon_0]$ and define $\delta_{\eta_f} := K_f \bar{\delta} + \gamma_{\eta_f} \varepsilon_0$. Note that $\delta_{\eta_f} > \bar{\delta}$.

Lemma (Christofides and Teel, 1996). *Referring to the* x*-subsystem of eq.* (A18), *let eq.* (A21) *hold. Then, for each pair of positive real numbers* $\bar{\delta}$, $\bar{d}$, *there exists a positive real number* ρ^* *such that for each* $\rho \in [0, \rho^*]$, *if* $\max\{\|x_0\|_2, \|\eta_f\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \bar{\delta}$, *then the solution of the* x*-subsystem of eq.* (A20) *with* $x(0) = x_0$ *exists for each* $t \geqslant 0$ *and satisfies*:

$$\|x\|_2 \leqslant K_s \|x_0\|_2 e^{-a_s t} + M_2 \|\theta\| + M_3 \sup_{t \geqslant \rho} \{\|\eta_f\|_2\} + \bar{d}. \tag{A23}$$

Applying the result of the above lemma, we have that there exists a positive real number ρ (assume, without loss of generality, that $\rho < T$), such that if $\max\{\|x_0\|_2, \|\eta_f\|, \|\theta\|, \|\dot{\theta}\|\} \leqslant \delta_{\eta_f}$, then the solution of the x-subsystem of eq. (A18) with $x(0) = x_0$ exists for each $t \in [0, T)$ and satisfies:

$$\|x\|_2 \leqslant K_s \|x_0\|_2 e^{-a_s t} + M_2 \|\theta\| + M_3 \sup_{t \geqslant \rho} \{\|\eta_f\|_2\} + \frac{d}{2}. \tag{A24}$$

Substituting eq. (A20) into eq. (A26), if $\varepsilon \in (0, \varepsilon_0]$, we have for all $t \in [0, T)$:

$$\|x\|_2 \leqslant K_s \|x_0\|_2 e^{-a_s t} + M_2 \|\theta\| + \frac{d}{2} + M_3 K_f \|\eta_{f0}\|_2 (e^{-\bar{a}_f \rho/\varepsilon} + \gamma_{\eta_f} \varepsilon). \tag{A25}$$

From the fact that the last term of the above equation vanishes as $\varepsilon \to 0$, we have that there exists an $\varepsilon_1 \in (0, \varepsilon_0]$ such that if $\varepsilon \in (0, \varepsilon_1]$, then for all $t \in [0, T)$:

$$\|x\|_2 \leqslant K_s \|x_0\|_2 e^{-a_s t} + M_2 \|\theta\| + d. \tag{A26}$$

From the definition of δ_x, the assumptionthat T is finite and continuity of x, there must exist some positive real number k such that $\sup_{t \in [0, T+k]} \{\|x\|_2\} < \delta_x$. This contradicts that T is maximal. Hence, $T = \infty$ and the inequality eq. (42) holds for all $t \geqslant 0$.

Finally, letting ε_3 be such that $\gamma_{\eta_f} \varepsilon \leqslant d$ for all $\varepsilon \in [0, \varepsilon_3]$ it follows that both inequalities of eqs (42) and (43) hold for $\varepsilon \in (0, \varepsilon^*]$ where $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. □

Proof of Theorem 6. Substituting the controller of eq. (20) into the system of eq. (30) and using the expression for the output derivatives of the closed-loop slow system of eq. (65), we have

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \mathscr{L} x + g(x) b(z) \\
&\times \left[\gamma_\sigma \mathscr{C} L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma-1} h(x) b(z) \right]^{-1} \\
&\times \left\{ \frac{1}{\gamma_\sigma} (v - \bar{y}) - \sum_{v=1}^{\sigma-1} \frac{\gamma_v}{\gamma_\sigma} \frac{d^v \bar{y}}{dt^v} - \mathscr{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma} h(x) \right\} \\
&+ W_1(x) r(z) \theta(t) + \mathscr{L}_{12} \eta_f \\
\varepsilon \frac{\partial \eta_f}{\partial t} &= \mathscr{L}_{22} \eta_f + g_2(x) b(z) \\
&\times \left[\gamma_\sigma \mathscr{C} L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma-1} h(x) b(z) \right]^{-1} \\
&\times \left\{ \frac{1}{\gamma_\sigma} (v - \bar{y}) - \sum_{v=1}^{\sigma-1} \frac{\gamma_v}{\gamma_\sigma} \frac{d^v \bar{y}}{dt^v} - \mathscr{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma} h(x) \right\} \\
&+ W_2(x) r(z) \theta(t) + \varepsilon \left(\frac{\partial z_s}{\partial x} \dot{x} + \frac{\partial z_s}{\partial \theta} \dot{\theta} \right) \\
\bar{y} &= \mathscr{C} h(x).
\end{aligned} \tag{A27}$$

Defining the state vectors $\zeta_v = (d^{v-1} \bar{y} / dt^{v-1})$, $v = 1, \ldots, \sigma$, the system of eq. (A27) can be equivalently written in the following form:

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 + \sum_{k=1}^{p} \mathscr{C} L_{\mathscr{L}_{12}^k} h(x) \eta_f \\
&\vdots \\
\dot{\zeta}_{\sigma-1} &= \zeta_\sigma + \sum_{k=1}^{p} \mathscr{C} L_{\mathscr{L}_{12}^k}^{\sigma-2} h(x) \eta_f \\
\dot{\zeta}_\sigma &= -\frac{1}{\gamma_\sigma} \zeta_1 - \frac{\gamma_1}{\gamma_\sigma} \zeta_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma} \zeta_\sigma + \frac{1}{\gamma_\sigma} v + \sum_{k=1}^{p} \mathscr{C} L_{\mathscr{L}_{12}^k}^{\sigma-1} h(x) \eta_f \\
\frac{\partial x}{\partial t} &= \mathscr{L} x + g(x) b(z) \\
&\times \left[\gamma_\sigma \mathscr{C} L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma-1} h(x) b(z) \right]^{-1} \\
&\times \left\{ \zeta - \mathscr{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma} h(x) \right\} \\
&+ \mathscr{L}_{12} \eta_f + W(x) r(z) \theta \\
\varepsilon \frac{\partial \eta_f}{\partial t} &= \mathscr{L}_{22} \eta_f + g_2(x) b(z) \\
&\times \left[\gamma_\sigma \mathscr{C} L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma-1} h(x) b(z) \right]^{-1} \\
&\times \left\{ \zeta - \mathscr{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma} h(x) \right\} \\
&+ W_2(x) r(z) \theta(t) + \varepsilon \left(\frac{\partial z_s}{\partial x} \dot{x} + \frac{\partial z_s}{\partial \theta} \dot{\theta} \right).
\end{aligned} \tag{A28}$$

Performing a two-time-scale decomposition, it can be shown that the fast dynamics of the above system is locally exponentially stable, and the reduced system takes the form:

$$\begin{aligned}
\dot{\zeta}_1^s &= \zeta_2^s \\
&\vdots \\
\dot{\zeta}_{\sigma-1}^s &= \zeta_\sigma^s \\
\dot{\zeta}_\sigma^s &= -\frac{1}{\gamma_\sigma} \zeta_1^s - \frac{\gamma_1}{\gamma_\sigma} \zeta_2^s - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma} \zeta_\sigma^s + \frac{1}{\gamma_\sigma} v
\end{aligned}$$

$$\frac{\partial x}{\partial t} = \mathscr{L}x + g(x)b(z)$$
$$\times\left[\gamma_\sigma \mathscr{C} L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1}$$
$$\times\left\{\zeta - \mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\}$$
$$+ W(x)r(z)\theta. \quad \text{(A29)}$$

From theorem 2, we have that the state of the above system is bounded. Using the result of theorem 4, we have that the state of the closed-loop system of eq. (A27) is bounded provided that the initial conditions, the uncertainty, the rate of change of uncertainty and the singular perturbation parameter are sufficiently small. Using the auxiliary variable $\tilde{\zeta}_v = \zeta_v - \zeta_v^s$, $v = 1, \ldots, \sigma$, the system of eq. (76) can be written as

$$\dot{\tilde{\zeta}}_1 = \tilde{\zeta}_2 + \sum_{k=1}^{p}\mathscr{C}L_{\mathscr{L}_{12}^k}h(x)\eta_f$$
$$\vdots$$
$$\dot{\tilde{\zeta}}_{\sigma-1} = \tilde{\zeta}_\sigma + \sum_{k=1}^{p}\mathscr{C}L_{\mathscr{L}_{12}^k}^{\sigma-2}h(x)\eta_f$$
$$\dot{\tilde{\zeta}}_\sigma = -\frac{1}{\gamma_\sigma}\tilde{\zeta}_1 - \frac{\gamma_1}{\gamma_\sigma}\tilde{\zeta}_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\tilde{\zeta}_\sigma + \frac{1}{\gamma_\sigma}v + \sum_{k=1}^{p}\mathscr{C}L_{\mathscr{L}_{12}^k}^{\sigma-1}h(x)\eta_f$$
$$\frac{\partial x}{\partial t} = \mathscr{L}x + g(x)b(z)$$
$$\times\left[\gamma_\sigma \mathscr{C} L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1}$$
$$\times\left\{\zeta - \mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\}$$
$$+ \mathscr{L}_{12}\eta_f + W(x)r(z)\theta$$
$$\varepsilon\frac{\partial \eta_f}{\partial t} = \mathscr{L}_{22}\eta_f + g_2(x)b(z)$$
$$\times\left[\gamma_\sigma \mathscr{C} L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1}$$
$$\times\left\{\zeta - \mathscr{C}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\}$$
$$+ W_2(x)r(z)\theta(t) + \varepsilon\left(\frac{\partial z_s}{\partial x}\dot{x} + \frac{\partial z_s}{\partial \theta}\dot{\theta}\right). \quad \text{(A30)}$$

Using the fact that the state of the above system is bounded and the equality $\zeta_v(0) = \zeta_v^s(0)$, holds $\forall v = 1, \ldots, \sigma$, the following bounds can be written for the evolution of the states $\tilde{\zeta} = [\tilde{\zeta}_1^{\mathrm{T}}\tilde{\zeta}_2^{\mathrm{T}} \ldots \tilde{\zeta}_\sigma^{\mathrm{T}}]^{\mathrm{T}}$, η_f of the above system:

$$|\tilde{\zeta}| \leqslant M_4\|\eta_f\|_2$$
$$\|\eta_f\|_2 \leqslant K_f\|\eta_{f0}\|_2 \mathrm{e}^{-\bar{a}_f t/\varepsilon} + \gamma_{\eta_f}\varepsilon \quad \text{(A31)}$$

where M_4 is a positive real number. Combining the above inequalities, we have

$$|\tilde{\zeta}| \leqslant M_4 K_f\|\eta_{f0}\|_2 \mathrm{e}^{-\bar{a}_f t/\varepsilon} + M_4\gamma_{\eta_f}\varepsilon. \quad \text{(A32)}$$

Since $M_4, K_f, \|\eta_{f0}\|_2, \gamma_{\eta_f}$ are some finite numbers of order one, and the right-hand side of the above inequality is a continuous function of ε which vanishes as $\varepsilon \to 0$, we have that there exists an ε^* such that if $\varepsilon \in (0, \varepsilon^*]$, then $M_4 K_f\|\eta_{f0}\|_2 \mathrm{e}^{-\bar{a}_f t/\varepsilon} + M_4\gamma_{\eta_f}\varepsilon \leqslant d$, $\forall t > 0$. Thus, $|\bar{y}^i(t) - \bar{y}_s^i(t)| = |\tilde{\zeta}_1^i(t)| \leqslant |\tilde{\zeta}_1(t)| \leqslant |\tilde{\zeta}(t)| \leqslant d$, $\forall t > 0$. □

Proof of Theorem 7. The detailed presentation of the proof of the theorem is too lengthy and will be omitted for brevity. Instead, we will provide a brief outline of the proof. Initially, it can be shown, following analogous steps as in the proof of theorem 3, that the state of the closed-loop reduced system of eq. (49) is bounded and there exist positive real numbers K_e, a_e, γ_e such that the following estimate holds for the ith output error for all $t \geqslant 0$:

$$|\bar{y}_s^i - v^i| \leqslant K_e|(\bar{y}_s^i - v^i)_0|\mathrm{e}^{-a_e t} + \gamma_e\phi \quad \text{(A33)}$$

where $(\bar{y}^i - v^i)_0$ is the output error at time $t = 0$. Picking $\phi^* = d/2\gamma_e$, we have that for $\phi \in (0, \phi^*]$:

$$|\bar{y}_s^i - v^i| \leqslant K_e|(\bar{y}_s^i - v^i)_0|\mathrm{e}^{-a_e t} + \frac{d}{2}. \quad \text{(A34)}$$

Now, referring to system of eq. (48) we have shown that satisfies the assumption of theorem 4 (the state of the uncertain closed-loop reduced system is bounded and the boundary layer is exponentially stable) and thus, the result of this theorem can be applied. This means that for each positive real number d, there exist positive real numbers $(\bar{K}_e, \bar{a}_e, \bar{\gamma}_e, \bar{\delta})$, such that if $\phi \in (0, \phi^*]$ there exists an $\varepsilon^*(\phi)$ such that, if $\max\{\|x_0\|_2, \|\eta_{f_0}\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \bar{\delta}$, $\phi \in (0, \phi^*]$ and $\varepsilon \in (0, \varepsilon^*(\phi)]$, the state of the closed-loop system of eq. (48) is bounded and the following estimate holds for the output error for all $t \geqslant 0$:

$$|\bar{y}^i - v^i| \leqslant \bar{K}_e|(\bar{y}^i - v^i)_0|\mathrm{e}^{-\bar{a}_e t} + \gamma_e\phi + \frac{d}{2} \quad \text{(A35)}$$

Taking the limit as $t \to 0$ of the above inequality, we have that if $\max\{\|x_0\|_2, \|\eta_{f_0}\|_2, \|\theta\|, \|\dot{\theta}\|\} \leqslant \bar{\delta}$, $\phi \in (0, \phi^*]$ and $\varepsilon \in (0, \varepsilon^*(\phi)]$, then:

$$\lim_{t\to\infty}|\bar{y}^i - v^i| \leqslant \gamma_e\phi + \frac{d}{2} \leqslant d \quad \text{(A36)}$$

□