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DISTRIBUTED OUTPUT FEEDBACK CONTROL OF TWO-TIME-SCALE HYPERBOLIC PDE SYSTEMS

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This article focuses on systems of two-time-scale hyperbolic partial differential equations (PDEs), modeled in singularly perturbed form, for which the manipulated inputs, the controlled and the measured outputs are distributed in space. The objective is to synthesize distributed output feedback controllers that guarantee closed-loop stability and enforce output tracking, provided that the speed ratio of the fast versus the slow dynamical phenomena of the two-time-scale system is sufficiently large. Initially, singular perturbation methods are used to derive two separate PDE models which describe the fast and slow dynamics of the original system. These models are then used as a basis for the synthesis of well-conditioned distributed state feedback controllers that guarantee stability and enforce output tracking in the closed-loop system. Then, two distributed state observers are independently designed on the basis of the fast and slow subsystems, to provide estimates of the fast and slow states of the system. These state observers are coupled with the distributed state feedback controller to yield a distributed output feedback controller that enforces the desired objectives in the closed-loop system. The proposed methodology is applied to a convection-reaction process with time-scale multiplicity.

1. Introduction

Convection-reaction processes can be adequately described by systems of quasi-linear first-order hyperbolic PDEs. The distinct feature of hyperbolic PDEs is that all the eigenmodes of the spatial differential operator contain the same or nearly the same amount of energy, and thus an infinite number of modes is required to accurately describe their dynamic behavior. This feature suggests addressing the control problem on the basis of the infinite-dimensional model itself. Following this approach, control methods were recently proposed for the synthesis of nonlinear distributed feedback controllers for quasi-linear hyperbolic PDEs utilizing sliding mode control (Sira-Ramirez, 1989; Hanczyc and Palazoglu, 1995), geometric control (Christofides and Daoutidis, 1996a), and Lyapunov-based control (Alonso and Ydstie, 1995; Christofides and Daoutidis, 1997).

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An important feature encountered in many convection-reaction processes is the presence of time-scale multiplicity. Typical examples of convection-reaction processes which exhibit multiple-time-scale behavior include fixed-bed reactors, pressure swing absorption processes, etc. For these processes, it is well-established that a direct application of distributed control methods (including the afore-mentioned specific ones) may lead to controller implementation and sensitivity problems. In particular, such an approach may lead to observer-based output feedback controllers which are ill-conditioned (i.e. the state feedback controller generates very large control actions in the presence of small measurement/modeling errors) and not easily implementable due to stiffness (i.e. the state observer is a stiff system due to ill-conditioning of the observer gain).

Singular perturbation methods have provided a natural framework for the analysis and well-conditioned controller synthesis for ordinary differential equations (ODE) systems (Kokotovic *et al.*, 1986). In this approach, the original two-time-scale process is decomposed into two separate lower-order well-conditioned models associated with different time-scales. These models are subsequently used for the synthesis of well-conditioned controllers. In the area of control and state estimation of two-time-scale distributed parameter systems, singular perturbation methods were initially used in (Soliman and Ray, 1979) for the design of well-conditioned distributed state estimators for a class of parabolic PDEs. More recently, Dochain and Bouaziz (1993) used singular perturbations to reduce the parabolic PDE model that describes a fixed-bed reactor with strong diffusive phenomena to an ODE one. The singular perturbation framework was also used in (Christofides and Daoutidis, 1997) to study robustness of a distributed robust state feedback controller, designed on the basis of a hyperbolic PDE model using Lyapunov-based control methods, with respect to fast and stable unmodeled dynamics.

In this paper, we consider systems of two-time-scale hyperbolic PDEs, modeled in singularly perturbed form, for which the manipulated inputs, the controlled and the measured outputs are distributed in space. The objective is to synthesize distributed output feedback controllers that guarantee closed-loop stability and enforce output tracking, provided that the speed ratio of the fast versus the slow dynamical phenomena of the two-time-scale system is sufficiently large. Initially, singular perturbation methods are used to derive two separate PDE models which describe the fast and slow dynamics of the original system. These models are then used as a basis for the synthesis of well-conditioned distributed state feedback controllers that guarantee stability and enforce output tracking. Then, two distributed state observers which incorporate well-conditioned observer gains are designed to provide estimates of the fast and slow states of the system. These state observers are coupled with the distributed state feedback controllers to yield distributed output feedback controllers that enforce the desired objectives in the closed-loop system. The proposed methodology is applied to a representative convection reaction process with time-scale multiplicity.

2. Two-Time-Scale Hyperbolic PDEs

2.1. Description of Class of Systems

We focus on two-time-scale first-order PDE systems with the following state-space description:

$$\begin{cases} \dfrac{\partial x}{\partial t} = A_{11}(x)\dfrac{\partial x}{\partial z} + A_{12}(x)\dfrac{\partial \eta}{\partial z} + f_1(x) + Q_1(x)\eta + g_1(x)b(z)\bar{u} \\ \epsilon\dfrac{\partial \eta}{\partial t} = A_{21}(x)\dfrac{\partial x}{\partial z} + A_{22}(x)\dfrac{\partial \eta}{\partial z} + f_2(x) + Q_2(x)\eta + g_2(x)b(z)\bar{u} \\ \bar{y} = \mathcal{C}h(x), \quad q_1 = \mathcal{Q}_1 p_1 x, \quad q_2 = \mathcal{Q}_2 p_2 \eta \end{cases} \tag{1}$$

subject to the boundary conditions

$$C_{11}x(\alpha,t) + C_{12}x(\beta,t) = R_1, \quad C_{21}\eta(\alpha,t) + C_{22}\eta(\beta,t) = R_2 \tag{2}$$

and the initial conditions

$$x(z,0) = x_0(z), \quad \eta(z,0) = \eta_0(z) \tag{3}$$

where $x(z,t) = [x_1(z,t) \ \cdots \ x_n(z,t)]^T$, $\eta(z,t) = [\eta_1(z,t) \ \cdots \ \eta_p(z,t)]^T$ denote vectors of state variables, $x(z,t) \in \mathcal{H}^{(n)}[(\alpha,\beta),\mathbb{R}^n]$, $\eta(z,t) \in \mathcal{H}^{(p)}[(\alpha,\beta),\mathbb{R}^p]$, with $\mathcal{H}^{(j)}[(\alpha,\beta),\mathbb{R}^j]$ being the infinite-dimensional Hilbert space of j-dimensional vector functions defined on the interval $[\alpha,\beta]$, $z \in [\alpha,\beta] \subset \mathbb{R}$ and $t \in [0,\infty)$ denote position and time, respectively. $\bar{u} = [\bar{u}^1 \ \cdots \ \bar{u}^l]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, $\bar{y} = [\bar{y}^1 \ \cdots \ \bar{y}^l]^T \in \mathbb{R}^l$ denotes the vector of controlled outputs, $\bar{q}_1 = [\bar{q}_1^1 \ \cdots \ \bar{q}_1^l]^T \in \mathbb{R}^l$, $\bar{q}_2 = [\bar{q}_2^1 \ \cdots \ \bar{q}_2^l]^T \in \mathbb{R}^l$ denote the vectors of measured outputs, and ϵ is a small parameter which quantifies the degree of coupling between the fast and slow modes of the system. $A_{11}(x), A_{12}(x), A_{21}(x), A_{22}(x), Q_1(x), Q_2(x)$ are sufficiently smooth matrices, $f_1(x), f_2(x), g_1(x), g_2(x)$ are sufficiently smooth vector functions, $h(x)$ is a sufficiently smooth scalar function, p_1, p_2, R_1, R_2 are constant vectors and $C_{11}, C_{12}, C_{21}, C_{22}$ are constant vectors. Furthermore, $b(z)$ is a known smooth vector function of the form

$$b(z) = \big[\big(H(z-z_1) - H(z-z_2)\big)b^1(z) \cdots \big(H(z-z_l) - H(z-z_{l+1})\big)b^l(z)\big] \tag{4}$$

where H denotes the standard Heaviside function and $b^i(z)$ describes how the control action $\bar{u}^i(t)$ is distributed in the space interval $[z_i, z_{i+1}]$. $\mathcal{C}$ is a bounded linear operator, mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, of the form:

$$\mathcal{C} = \big[\big(H(z-z_1) - H(z-z_2)\big)\mathcal{C}^1 \cdots \big(H(z-z_l) - H(z-z_{l+1})\big)\mathcal{C}^l\big]^T \tag{5}$$

where the operator $\mathcal{C}^i$ depends on the desired performance specifications and in most practical applications is taken to be of the form:

$$\bar{y}_s^i(t) = \mathcal{C}^i h(x) = \int_{z_i}^{z_{i+1}} c^i(z)h\big(x(z,t)\big)\,\mathrm{d}z \tag{6}$$

where $c^i(z)$ is a known smooth function. $\mathcal{Q}_1, \mathcal{Q}_2$ are bounded linear operators mapping $\mathcal{H}^n$ into $\mathbb{R}^l$ and $\mathcal{H}^p$ into $\mathbb{R}^l$, respectively, of the form

$$\mathcal{Q}_k = \left[(H(z-z_1) - H(z-z_2))\, \mathcal{Q}_k^1 \cdots (H(z-z_l) - H(z-z_{l+1}))\, \mathcal{Q}_k^l \right]^T, \; k = 1,2 \quad (7)$$

Systems of the form of eqn. (1) arise naturally in practice as models of convection-reaction processes which involve physicochemical phenomena occurring in different time-scales. The linear appearance of the manipulated input, which is also distributed in space, is typical in most practical applications where the wall-temperature is usually chosen to be the manipulated input. The linear appearance of the fast variables is also consistent with the fact that the main nonlinearities are usually associated with the slow dynamics. The separation of controlled and measured outputs ($\bar{y}$ and (q_1, q_2)) is done to allow defining the controlled outputs at different spatial positions from the measured outputs (e.g. consider control of temperature throughout a reactor using temperature measurements from several points in the reactor). Finally, the partition of the measured outputs to two different sets, q_1 and q_2, which exclusively include slow and fast variables, respectively, is also typical in practice where measurements of the slow variables are usually obtained independently (and very often at different spatial positions) of the ones of the fast variables.

The following assumption states that the two-time-scale system of eqn. (1) is hyperbolic.

Assumption 1. *The* $(n+p) \times (n+p)$ *matrix*

$$\mathcal{A}(x) = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix} \quad (8)$$

is real symmetric, and its eigenvalues satisfy

$$\bar{\lambda}_1(x) \leq \cdots \leq \bar{\lambda}_k(x) < 0 < \bar{\lambda}_{k+1}(x) \leq \cdots \leq \bar{\lambda}_{n+p}(x) \quad (9)$$

for all $x \in \mathcal{H}^n[(\alpha,\beta), \mathbb{R}^n]$.

Assumption 1 ensures (Russell, 1978) that the differential operator of the linearized open-loop system of eqn. (1) generates a strongly continuous semigroup $U(t)$ of bounded linear operators, which guarantees the local existence, uniqueness and continuity of solutions of the system (1). From general semigroup theory (Friedman, 1976), it is known that $U(t)$ satisfies the growth property $\|U(t)\|_2 \leq Ke^{at}, \;\; t \geq 0$, where K, a are real numbers with $K \geq 1$ and $\|\cdot\|_2$ denotes the inner product in $\mathcal{H}[(\alpha,\beta), \mathbb{R}^{n+p}]$ (see the Appendix for the definition of the inner product). An estimate of K, a can be obtained utilizing the Hille-Yoshida theorem (Friedman, 1976). Whenever the parameter a is strictly negative, we will say that $U(t)$ is an exponentially stable semigroup. Throughout the paper we will use the order of magnitude notation $O(\epsilon)$, i.e. $\delta(\epsilon) = O(\epsilon)$ if there exist positive constants k and c such that $|\delta(\epsilon)| \leq k|\epsilon|\,, \; \forall\, |\epsilon| < c$.

2.2. Two-Time-Scale Decomposition—Preliminaries

In this subsection, we initially use singular perturbation methods to decompose the two-time-scale system (1) into separate models which describe the fast and slow dynamics of this system. Then, we will state certain standard assumptions on these models that will be exploited in the synthesis of the distributed output feedback controller. To simplify the notation, we set $\mathcal{L}_{i1}x = A_{i1}(x)\frac{\partial x}{\partial z} + f_i(x)$, $\mathcal{L}_{j2}\eta = A_{j2}(x)\frac{\partial \eta}{\partial z} + Q_j(x)\eta$, $i,j = 1,2$. Defining a fast time-scale $\tau = t/\epsilon$ and setting $\epsilon = 0$, the fast subsystem, which describes the fast dynamics of the system (1), takes the form

$$\begin{cases} \dfrac{\partial \eta}{\partial \tau} = \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\bar{u} \\ q_2 = \mathcal{Q}_2 p_2 \eta \end{cases} \tag{10}$$

where x is independent of time. In order to ensure that it is possible to stabilize the above system through distributed output feedback, the following standard stabilizability (Assumption 2) and detectability (Assumption 3) requirements will be needed (see also (Balas, 1986; Pell and Aris, 1970)). The stabilizability requirement will be used in the next section to design a distributed state feedback controller to stabilize the fast subsystem, while the detectability requirement will be exploited in Section 4 for the design of a distributed state observer that provides estimates of the vector of fast state variables.

Assumption 2. *The pair* $[\mathcal{L}_{22}\ g_2(x)b(z)]$ *is stabilizable uniformly in* x, *i.e., there exist a bounded linear operator* $\mathcal{F}$, *mapping* $\mathcal{H}^p$ *into* $\mathbb{R}^l$, *such that the operator* $\mathcal{L}_s\eta = \mathcal{L}_{22}\eta + g_2(x)b(z)\mathcal{F}\eta$ *generates an exponentially stable semigroup.*

Assumption 3. *The pair* $[\mathcal{Q}_2 p_2\ \mathcal{L}_{22}]$, *is detectable, i.e., there exist a bounded linear operator* $\mathcal{P}_2$, *mapping* $\mathbb{R}^l$ *into* $\mathcal{H}^p$, *such that the operator* $\mathcal{L}_{2o}\eta = \mathcal{L}_{22}\eta - \mathcal{P}_2\mathcal{Q}_2 p_2\eta$ *generates an exponentially stable semigroup.*

Remark 1. In practice, the design of the feedback operator $\mathcal{F}$ can be performed utilizing standard optimal control methods (e.g. (Ray, 1981)), while the design of the estimator operator $\mathcal{P}_2$ can be performed via (a) simple pole placement in the case where the output measurements are not corrupted by noise, (b) Kalman filtering theory, in the case where the output measurements are noisy (Pell and Aris, 1970; Soliman and Ray, 1979). We also note that the detectability requirement of Assumption 3 does not impose any restrictions on the form of the operators $\mathcal{Q}_1, \mathcal{Q}_2$ and thus, on the structure of the measurement sensors (e.g. distributed, pointwise sensors).

Setting $\epsilon = 0$, the PDE system (1) reduces to a system of coupled partial and ordinary differential equations of the form

$$\begin{cases} \dfrac{\partial x}{\partial t} = \mathcal{L}_{11}x + \mathcal{L}_{12}\eta_s + g_1(x)b(z)\bar{u} \\ 0 = \mathcal{L}_{21}x + \mathcal{L}_{22}\eta_s + g_2(x)b(z)\bar{u} \end{cases} \tag{11}$$

The solution of the ODE $\mathcal{L}_{21}x + \mathcal{L}_{22}\eta_s + g_2(x)b(z)\bar{u} = 0$ subject to the boundary conditions (2) is of the form

$$\eta_s = \mathcal{L}_{22}^{-1}\left(\mathcal{L}_{21}x + g_2(x)b(z)\bar{u}\right) \tag{12}$$

The slow subsystem, which captures the slow dynamics of the system (1), takes then the form

$$\begin{cases} \dfrac{\partial x}{\partial t} = \mathcal{L}(x) + G(x)b(z)\bar{u} \\ \bar{y}_s = \mathcal{C}h(x), \qquad q_1 = \mathcal{Q}_1 p_1 x \end{cases} \tag{13}$$

where the subscript s in $\bar{y}_s$ denotes that this output is associated with slow subsystem, $\mathcal{L}(x) = \mathcal{L}_{11}x - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}x$, $G(x) = g_1(x) - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}g_2(x)$.

We will now recall a result developed in (Christofides and Daoutidis, 1997), which states that if the fast and slow hyperbolic PDE subsystems (10)–(13) are locally exponentially stable, then the two-time-scale hyperbolic PDE system (1) is locally exponentially stable and the discrepancy between the solutions of the system (1) and the subsystems (10)–(13) measured in appropriate norm is of $O(\epsilon)$ for almost all times, provided that ϵ is sufficiently small.

Theorem 1. (Christofides and Daoutidis, 1997) *Consider the system (1), for which Assumption 1 holds, suppose that the fast and slow hyperbolic PDE subsystems (10)–(13) are locally exponentially stable, and define $\eta_f := \eta - \eta_s$, where η_s is the solution of the system (10). Then, there exist positive real numbers $(\delta, \epsilon^*, a_f)$ such that if $\max\{\|x_0\|_2, \|\eta_{f0}\|_2\} \leq \delta$ and $\epsilon \in (0, \epsilon^*]$, the system (1) is locally exponentially stable, and*

$$\|x - x_s\|_2 = O(\epsilon) \tag{14}$$

$$\|\eta_f\|_2 = K_f\|\eta_{f0}\|_2 e^{-a_f t/\epsilon} + O(\epsilon) \tag{15}$$

Finally, we recall a concept of zero dynamics for systems of the form (13), proposed in (Christofides and Daoutidis, 1996a), which will be used to state conditions that guarantee the stability of the closed-loop system.

Definition 1. (Christofides and Daoutidis, 1996a) The zero dynamics associated with the system (13) is the system obtained by constraining the output to zero, i.e., the system

$$\begin{cases} \dfrac{\partial x}{\partial t} = \mathcal{L}(x) - G(x)b(z)\left[\gamma_\sigma \mathcal{C}L_G L_{\mathcal{L}}^{\sigma-1}h(x)b(z)\right]^{-1}\left\{\gamma_\sigma \mathcal{C}L_{\mathcal{L}}^{\sigma}h(x)\right\} \\ \mathcal{C}h(x) \equiv 0 \\ C_1 x(\alpha, t) + C_2 x(\beta, t) = R(t) \end{cases} \tag{16}$$

We note that the zero dynamics of the system (13) are infinite dimensional in nature. This is expected because the number of controlled outputs of the system (13) is finite (l), while the state of the system evolves in infinite dimensions.

2.3. Concept of Characteristic Index

Referring to the system (13), we now recall the concept of characteristic index between the output $\bar{y}^i$ and the input $\bar{u}^i$ introduced in (Christofides and Daoutidis, 1996a) that will be used in the formulation and solution to the distributed state feedback control problem for the slow subsystem (see the next section).

Definition 2. (Christofides and Daoutidis, 1996a) Referring to the system (13), we define the characteristic index of the output $\bar{y}_s^i$ with respect to the input $\bar{u}^i$ as the smallest integer σ^i for which

$$C^i L_G L_{\mathcal{L}}^{\sigma^i - 1} h(x) b^i(z) \not\equiv 0 \tag{17}$$

where $L_{\mathcal{L}}, L_f$ denote the standard Lie derivative notation, or $\sigma^i = \infty$ if such an integer does not exist.

In most practical applications, the choice of $(b^i(z), c^i(z))$ is typically consistent for all pairs $(\bar{y}_s^i, \bar{u}^i)$, in a sense which is made precise in the following assumption.

Assumption 4. *Referring to the system (13),* $\sigma^1 = \sigma^2 = \cdots = \sigma^l = \sigma$.

From the above assumption, it follows that σ can also be thought of as the characteristic index between the output vector $\bar{y}$ and the input vector $\bar{u}$.

3. Distributed State Feedback Control of Hyperbolic PDE Systems: Review

In this section, we briefly review a methodology developed in (Christofides and Daoutidis, 1996a) for the synthesis of distributed state feedback controllers for first-order hyperbolic PDE systems of the form of eqn. (13) that enforce closed-loop stability and output tracking. These results will be used in the present paper. We consider control laws of the form

$$\bar{u} = \mathcal{S}(x) + s(x)v \tag{18}$$

where $\mathcal{S}(x)$ is a smooth nonlinear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, $s(x)$ is an invertible matrix of smooth functionals, and $v \in \mathbb{R}^l$ is the vector of external reference inputs. Under the control law (18), the closed-loop system takes the form

$$\left\{ \begin{aligned} \frac{\partial x}{\partial t} &= \mathcal{L}(x) + G(x)b(z)\mathcal{S}(x) + G(x)b(z)s(x)v \\ \bar{y}_s &= \mathcal{C}h(x) \end{aligned} \right. \tag{19}$$

It is clear that feedback laws of the form (18) preserve the linearity with respect to the external reference input v. We also note that the evolution of the closed-loop system of eqn. (19) is locally governed by a strongly continuous semigroup of bounded linear operators, because $b(z)\mathcal{S}(x), b(z)s(x)$ are bounded, finite dimensional perturbations, ensuring that the solution of the system (19) is well-defined. Proposition 1 that follows allows specifying the order of the input/output response in the closed-loop system.

Proposition 1. (Christofides and Daoutidis, 1996a) *Consider the system (13), for which Assumption 1 holds, subject to the distributed state feedback law (18). Then, referring to the closed-loop system (19) the characteristic index of $\bar{y}_s$ with respect to the external reference input v is equal to σ.*

The fact that the characteristic index between the output $\bar{y}_s$ and the external reference input v is equal to σ suggests requesting the following input/output response for the closed-loop system:

$$\gamma_\sigma \frac{d^\sigma \bar{y}_s}{dt^\sigma} + \cdots + \gamma_1 \frac{d\bar{y}_s}{dt} + \bar{y}_s = v \tag{20}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_\sigma$ are adjustable parameters which can be chosen to guarantee input/output stability in the closed-loop system. Referring to eqn. (20), note that, motivated by physical arguments, we request, for each pair $(\bar{y}_s^i, v^i)$, $i = 1, \ldots, l$, an input/output response of order σ with the same transient characteristics (i.e. the parameters γ_k are chosen to be the same for each pair $(\bar{y}_s^i, v^i)$). The following theorem proved in (Christofides and Daoutidis, 1996a) provides the main result of this section.

Theorem 2. (Christofides and Daoutidis, 1996a) *Consider the system (13) for which Assumption 1 holds, under the distributed state feedback controller*

$$\bar{u} = \left[\gamma_\sigma C L_G L_{\mathcal{L}}^{\sigma-1} h(x) b(z)\right]^{-1} \left\{ v - Ch(x) - \sum_{\nu=1}^{\sigma} \gamma_\nu C L_{\mathcal{L}}^{\nu} h(x) \right\} \tag{21}$$

Suppose also that: (i) the roots of the equation $1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0$ lie in the open left-half of the complex plane, and (ii) the zero dynamics (16) is locally exponentially stable. Then, there exists a positive real number δ such that if $\|x_0\|_2 \leq \delta$, the controller of eqn. (21): (a) guarantees local exponential stability of the closed-loop system, and (b) enforces the input/output response of eqn. (20) in the closed-loop system.

Remark 2. We note that the approach followed for the synthesis of the distributed state feedback controller of eqn. (21), and the nature of the conditions for exponential stability of the closed-loop system, are conceptually similar to the ones employed for the synthesis of inversion-based controllers for nonlinear ODE systems which enforce a linear input/output response and guarantee exponential stability of the closed-loop system. This is possible, because for the system (13), the solution is well-defined, the number of manipulated inputs and controlled outputs is finite, and the manipulated inputs and the controlled outputs are distributed in space. The reader may refer to (Christofides and Daoutidis, 1996a) for more details on this issue and comparisons of this approach with existing approaches for the synthesis of stabilizing controllers for first-order hyperbolic PDE systems.

4. Distributed State Feedback Control of Two-Time-Scale Hyperbolic PDE Systems

The objective of this section is to synthesize distributed state feedback controllers for systems of two-time-scale hyperbolic PDEs of the form (1) that ensure stability and enforce reference input in the closed-loop system, as long as ϵ is sufficiently small. Motivated by the possible instability of the fast dynamics and the linear appearance of the fast variable η (1), we initially consider a preliminary distributed state feedback law of the form

$$\bar{u} = \mathcal{F}\eta + \tilde{u} \tag{22}$$

where $\tilde{u}$ is an auxiliary input, to stabilize the fast dynamics. Substituting the control law (22) into (1), we obtain:

$$\left\{\begin{aligned} \frac{\partial x}{\partial t} &= \mathcal{L}_{11}x + \mathcal{L}_{12}\eta + g_1(x)b(z)\mathcal{F}\eta + g_1(x)b(z)\tilde{u} \\ \epsilon\frac{\partial \eta}{\partial t} &= \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\mathcal{F}\eta + g_2(x)b(z)\tilde{u} \end{aligned}\right. \tag{23}$$

Performing a two-time-scale decomposition to the above system, the fast subsystem takes the form:

$$\frac{\partial \eta}{\partial \tau} = \mathcal{L}_{21}x + \left[\mathcal{L}_{22} + g_2(x)b(z)\mathcal{F}\right]\eta + g_2(x)b(z)\tilde{u} \tag{24}$$

which is exponentially stable by appropriate selection of $\mathcal{F}$ (Assumption 2). Furthermore, the slow subsystem takes the form

$$\left\{\begin{aligned} \frac{\partial x}{\partial t} &= \tilde{\mathcal{L}}(x) + \tilde{G}(x)b(z)\tilde{u} \\ \bar{y}_s &= \mathcal{C}h(x) \end{aligned}\right. \tag{25}$$

where $\tilde{\mathcal{L}}(x) = \mathcal{L}_{11}x - [\mathcal{L}_{12} + g_1(x)b(z)\mathcal{F}][\mathcal{L}_{22} + g_2(x)b(z)\mathcal{F}]^{-1}\mathcal{L}_{21}x$, $\tilde{G}(x) = g_1(x) - [\mathcal{L}_{12}+g_1(x)b(z)\mathcal{F}][\mathcal{L}_{22}+g_2(x)b(z)\mathcal{F}]^{-1}g_2(x)$. Referring to the above system, we define the characteristic index of the output $\bar{y}_s$ with respect to the input $\tilde{u}$ as $\tilde{\sigma}$.

Since the slow subsystem (25) is in the form (13), the methodology described in the previous section can be directly employed to synthesize a distributed state feedback law of the form

$$\tilde{u} = \bar{\mathcal{S}}(x) + \bar{s}(x)v \tag{26}$$

where $\bar{\mathcal{S}}(x)$ is a smooth nonlinear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, and $\bar{s}(x)$ is an invertible matrix of smooth functionals, to enforce exponential stability and output tracking for changes in the reference input in the closed-loop slow subsystem. Under the control law (26), the closed-loop slow subsystem takes the form

$$\left\{\begin{aligned} \frac{\partial x}{\partial t} &= \tilde{\mathcal{L}}(x) + \tilde{G}(x)b(z)\bar{\mathcal{S}}(x) + \tilde{G}(x)b(z)\bar{s}(x)v \\ \bar{y}_s &= \mathcal{C}h(x) \end{aligned}\right. \tag{27}$$

For the above system, it can be shown that the characteristic index of $\bar{y}_s$ with respect to v is equal to $\tilde{\sigma}$. This suggests requesting the following input/output response in the closed-loop slow subsystem:

$$\gamma_{\tilde{\sigma}} \frac{d^{\tilde{\sigma}} \bar{y}_s}{dt^{\tilde{\sigma}}} + \cdots + \gamma_1 \frac{d\bar{y}_s}{dt} + \bar{y}_s = v \tag{28}$$

where $\gamma_1, \gamma_2, \cdots, \gamma_{\tilde{\sigma}}$ are adjustable parameters.

Theorem 3 provides the formula of the distributed state feedback controller that guarantees closed-loop stability and enforces output tracking, provided that ϵ is sufficiently small (the proof is given in the Appendix).

Theorem 3. *Consider the system of quasi-linear partial differential equations (1), for which Assumptions 1, 2, and 4 hold. Suppose also that (a) the roots of the equation* $1+\gamma_1 s + \cdots + \gamma_{\tilde{\sigma}} s^{\tilde{\sigma}} = 0$ *lie in the open left-half of the complex plane, and (b) the zero dynamics of the system (25) is exponentially stable. Then, there exist positive real numbers* (δ, ϵ^*) *such that if* $\max\{\|x_0\|_2, \|\eta_{f0}\|_2\} \leq \delta$ *and* $\epsilon \in (0, \epsilon^*]$, *the distributed state feedback controller*

$$\bar{u} = \mathcal{F}\eta + \left[\gamma_{\tilde{\sigma}} C L_{\tilde{\mathcal{G}}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(x) b(z)\right]^{-1} \left\{ v - Ch(x) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(x) \right\} \tag{29}$$

(a) guarantees local exponential stability of the closed-loop system,

(b) ensures that the output of the closed-loop system satisfies

$$\bar{y}^i(t) = \bar{y}_s^i(t) + O(\epsilon), \quad i = 1, \ldots, l \tag{30}$$

for all $t \geq 0$, *where* $\bar{y}_s^i(t)$, $i = 1, \ldots, l$, *are the solutions of eqn. (28).*

Remark 3. Owing to the distributed nature of the controllers (21)–(29), the calculation of the control action requires algebraic manipulations as well as differentiations and integrations in space.

Remark 4. Theorem 3 provides an analytical formula of a distributed nonlinear state feedback controller that enforces an approximate linear input/output response in the closed-loop two-time-scale hyperbolic PDE system. In this sense, the controller (29) can be viewed as the counterpart of approximate input/output linearizing control laws for nonlinear two-time-scale ODE systems (Christofides and Daoutidis, 1996b), in the case of distributed parameter systems of the form of eqn. (1).

Remark 5. Whenever the fast dynamics of (1) are exponentially stable, there is no need to use the preliminary feedback (22) to stabilize them, and thus, the controller (29) becomes identical to the controller (21). In this case, Theorem 3 provides a fundamental robustness result of the controller (21) with respect to stable and sufficiently fast unmodeled dynamics (e.g. sensor and actuator dynamics, etc.).

5. Distributed Output Feedback Control of Two-Time-Scale Hyperbolic PDE Systems

In this section, we synthesize distributed output feedback controllers for systems of the form (1) that ensure stability and enforce reference input in the closed-loop system, as long as ϵ is sufficiently small. The requisite controllers will be synthesized employing combination of the developed distributed state feedback controllers with distributed state observers. Given the lack of general available results on state estimation of such systems, we will proceed with the design of well-conditioned distributed nonlinear state observers which guarantee local exponential convergence of the state estimates to the actual state values. In particular, the following well-conditioned distributed state observer will be used to estimate the fast variable η in the fast time scale, in space and time:

$$\frac{\partial \bar{\eta}}{\partial \tau} = \mathcal{L}_{21}\bar{x} + \mathcal{L}_{22}\bar{\eta} + g_2(\bar{x})b(z)\bar{u} + \mathcal{P}_2(\bar{q}_2 - \mathcal{Q}_2 p_2 \bar{\eta}) \tag{31}$$

where $\bar{\eta}$ denotes the estimate of η. $\mathcal{P}_2$ is an ϵ-independent linear operator designed on the basis of the linearization of the system (31) so that the eigenvalues of the operator $\mathcal{L}_{2o}\bar{\eta} = \mathcal{L}_{22}\bar{\eta} - \mathcal{P}_2\mathcal{Q}_2 p_2\bar{\eta}$ lie in the left-half plane (Assumption 3).

On the other hand, the following well-conditioned distributed state observer will be used to estimate the slow variable x in the slow time-scale, in space and time:

$$\frac{\partial \bar{x}}{\partial t} = \tilde{\mathcal{L}}(\bar{x}) + \tilde{G}(\bar{x})b(z)\tilde{u} + \mathcal{P}_1(\bar{q}_1 - \mathcal{Q}_1 p_1 \bar{x}) \tag{32}$$

where $\bar{x}$ denotes the estimate of x. $\mathcal{P}_1$ is an ϵ-independent linear operator, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$, designed on the basis of the linearization of the system (32) so that the eigenvalues of the operator $\bar{\mathcal{L}}_{1o}(\bar{x}) = \bar{\mathcal{L}}(\bar{x}) - \mathcal{P}_1\mathcal{Q}_1 p_1\bar{x}$ lie in the left-half plane. The state observers (31)–(32) consist of a replica of the fast and slow subsystems, respectively, and the term $\mathcal{P}_2(\bar{q}_2 - \mathcal{Q}_2 p_2\bar{\eta})$, $\mathcal{P}_1(\bar{q}_1 - \mathcal{Q}_1 p_1\bar{x})$ used to enforce a fast decay of the discrepancy between the estimated and the actual values of the states of the system.

Combining the distributed state feedback controller (29) with the state observers (31)–(32), we derive a distributed output feedback controller that enforces output tracking and guarantees closed-loop stability, provided that ϵ is sufficiently small. Theorem 4 below provides the main result of this paper (the proof of the theorem is given in the Appendix).

Theorem 4. *Consider the system of quasi-linear partial differential equations (1), for which Assumptions 1–4 hold. Suppose also that: (a) the roots of the equation* $1+\gamma_1 s + \cdots + \gamma_{\bar{\sigma}} s^{\bar{\sigma}} = 0$ *lie in the open left-half of the complex plane, (b) the zero dynamics of the system (25) is exponentially stable, and (c)* $\bar{x}(z,0) = x(z,0)$ *and* $\bar{\eta}(z,0) = \eta(z,0)$. *Then, there exist positive real numbers* (δ, ϵ^*) *such that if* $\max\{\|x_0\|_2, \|\eta_{f_0}\|_2\} \leq \delta$

and $\epsilon \in (0, \epsilon^*]$, the distributed output feedback controller:

$$\left\{ \begin{aligned} \frac{\partial \bar{x}}{\partial t} &= \tilde{\mathcal{L}}(\bar{x}) + \tilde{G}(\bar{x})b(z)\left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x})b(z)\right]^{-1} \\ &\quad \times \left\{ v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x}) \right\} + \mathcal{P}_1(\bar{q}_1 - \mathcal{Q}_1 p_1 \bar{x}) \\ \frac{\partial \bar{\eta}}{\partial \tau} &= \mathcal{L}_{21}\bar{x} + \mathcal{L}_{22}\bar{\eta} + g_2(\bar{x})b(z)\mathcal{F}\bar{\eta} + \mathcal{P}_2(\bar{q}_2 - \mathcal{Q}_2 p_2 \bar{\eta}) + g_2(\bar{x})b(z) \\ &\quad \times \left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x})b(z)\right]^{-1} \left\{ v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x}) \right\} \\ \bar{u} &= \mathcal{F}\bar{\eta} + \left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x})b(z)\right]^{-1} \left\{ v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x}) \right\} \end{aligned} \right. \tag{33}$$

(a) guarantees local exponential stability of the closed-loop system,

(b) ensures that the output of the closed-loop system satisfies for all $t \geq 0$:

$$\bar{y}^i(t) = \bar{y}_s^i(t) + O(\epsilon), \quad i = 1, \dots, l \tag{34}$$

where $\bar{y}_s^i(t)$, $i = 1, \dots, l$, are the solutions of (28).

Remark 6. The use of distributed output feedback to stabilize the fast dynamics can be avoided whenever they are exponentially stable, leading to the following simplification of the controller (33):

$$\left\{ \begin{aligned} \frac{\partial \bar{x}}{\partial t} &= \mathcal{L}(\bar{x}) + G(\bar{x})b(z)\left[\gamma_\sigma C L_G L_{\mathcal{L}}^{\sigma-1} h(x)b(z)\right]^{-1} \\ &\quad \times \left\{ v - Ch(x) - \sum_{\nu=1}^{\sigma} \gamma_\nu C L_{\mathcal{L}}^{\nu} h(x) \right\} + \mathcal{P}_1(\bar{q}_1 - \mathcal{Q}_1 p_1 \bar{x}) \\ \bar{u} &= \left[\gamma_\sigma C L_G L_{\mathcal{L}}^{\sigma-1} h(x)b(z)\right]^{-1} \left\{ v - Ch(x) - \sum_{\nu=1}^{\sigma} \gamma_\nu C L_{\mathcal{L}}^{\nu} h(x) \right\} \end{aligned} \right. \tag{35}$$

Notice that the above controller is identical to the distributed output feedback controller proposed in (Christofides and Daoutidis, 1996a, Th. 4) for hyperbolic PDE systems of the form (13). Therefore, in the case of two-time-scale systems with stable fast dynamics, Theorem 4 establishes a robustness property of the controller (35) (and thus, of the controller of Theorem 4 in (Christofides and Daoutidis, 1996a)) with respect to unmodeled dynamics, provided that they are stable and sufficiently fast.

Remark 7. We note that in the case of two-time-scale systems with exponentially stable fast and slow subsystems, the controller (35) can be further simplified by setting $\mathcal{P}_1$ identically equal to zero. The reason for this simplification is that the exponential stability of the open-loop slow subsystem guarantees the convergence of the estimated values of the slow states to the actual ones with transient behavior depending on the location of the spectrum of the differential operator of the open-loop slow subsystem.

Remark 8. The nonlinear distributed output feedback (33) is an infinite dimensional one, because the state observers (31), (32) are distributed parameter systems. Therefore, the on-line implementation of this controller requires to utilize discretization techniques such as finite differences, orthogonal collocation, etc., to derive an approximate finite-dimensional controller which can be implemented in practice. It is expected that some performance deterioration will occur in this case, depending on the discretization method used and the number and location of discretization points. We note that the deterioration of the performance decreases as the number of discretization points increases, guaranteeing the well-posedness of the approximate finite-dimensional controller (Balas, 1986).

6. Control of a Fixed-Bed Reactor

In this section, we present an application of the proposed method to a fixed-bed reactor where an elementary reaction of the form $A \to B$ takes place. This example was also considered in (Christofides and Daoutidis, 1997) in the context of distributed robust state feedback control of hyperbolic PDEs with uncertain variables and unmodeled fast dynamics (see also (Stangeland and Foss, 1970)). Under standard modeling assumptions, a dynamic model of the process can be derived from material and energy balances and has the form:

$$\begin{cases} \rho_b c_{pb} \dfrac{\partial T}{\partial t} = -\rho_f c_{pf} v_l \dfrac{\partial T}{\partial z} + (-\Delta H) k_0 e^{-E/RT} C_A + \dfrac{U_w}{V_r}(T_j - T) \\ \bar{\epsilon} \dfrac{\partial C_A}{\partial t} = -v_l \dfrac{\partial C_A}{\partial z} - k_0 e^{-E/RT} C_A \end{cases} \tag{36}$$

subject to the initial and boundary conditions:

$$C_A(z,0) = C_{A_s}(z), \qquad T(z,0) = T_s(z)$$
$$C_A(0,t) = C_{A0}, \qquad T(0,t) = T_{A0}$$

In the above model, C_A denotes the concentration of the species A in the fluid phase, T denotes the temperature in the bed, $\bar{\epsilon}$ denotes the reactor porosity, ρ_b, c_{pb} denote the density and heat capacity of the bed, ρ_f, c_{pf} denote the density and heat capacity of the fluid phase, v_l denotes the velocity of the fluid phase, U_w denotes the heat transfer coefficient, T_j denotes the spatially uniform temperature in the jacket, V_r denotes the volume of the reactor, $k_0, E, \Delta H$ denote the pre-exponential factor, the activation energy, and the enthalpy of the reaction, C_{A0}, T_{A0} denote the concentration and temperature of the inlet stream, and $C_{A_s}(z), T_s(z)$ denote the steady-state profiles for the concentration of the species A and temperature in the reactor. The values of the process parameters are given in Table 1 below and they correspond to a stable steady-state for the open-loop system.

Table 1. Process parameters.

$v_l = 30.0$	m hr^{-1}
$V_r = 1.0$	m^3
$\bar{\epsilon} = 0.01$	
$L = 1.0$	m
$E = 2.0 \times 10^4$	kcal kmol^{-1}
$k_0 = 5.0 \times 10^{12}$	hr^{-1}
$R = 1.987$	kcal kmol^{-1} K^{-1}
$\Delta H_0 = 35480.111$	kcal kmol^{-1}
$c_{pf} = 0.0231$	kcal kg^{-1} K^{-1}
$\rho_f = 90.0$	kg m^{-3}
$c_{pb} = 6.67 \times 10^{-4}$	kcal kg^{-1} K^{-1}
$\rho_b = 1500.0$	kg m^{-3}
$U_w = 500.0$	kcal hr^{-1} K^{-1}
$C_{A0} = 4.0$	kmol m^{-3}
$T_{A0} = 320.0$	K

An important feature of fixed-bed reactors is that the reactant wave propagates through the bed with a significantly larger speed than the heat wave, because the exchange of heat between the fluid and packing slows the thermal wave down. Therefore, the system (36) possesses an inherent two-time-scale property, i.e., the concentration dynamics are much faster than the temperature dynamics. Defining $\epsilon = \bar{\epsilon}/\rho_b c_{pb}$ and setting $\bar{t} = t/\rho_b c_{pb}$, $x = T$, $\eta = C_A$, the system (36) can be written in the following singularly perturbed form:

$$\left\{ \begin{aligned} \frac{\partial x}{\partial \bar{t}} &= -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H) k_0 e^{-E/Rx} \eta + \frac{U_w}{V_r}(T_j - x) \\ \epsilon \frac{\partial \eta}{\partial \bar{t}} &= -v_l \frac{\partial \eta}{\partial z} - k_0 e^{-E/Rx} \eta \end{aligned} \right. \tag{37}$$

The control problem considered is the one of controlling the temperature of the reactor by manipulating the jacket temperature. We assume that there is available one control actuator with distribution function $b(z) = 1$ and the manipulated input and the controlled output are defined as $\bar{u} = T_j - T_{js}$ and $\bar{y} = \int_0^1 x \, dz$.

Performing a two-time-scale decomposition of the system (37), the fast dynamics of this system are described by

$$\frac{\partial \eta}{\partial \bar{\tau}} = -v_l \frac{\partial \eta}{\partial z} - k_0 e^{-E/Rx} \eta \tag{38}$$

where $\bar{\tau} = \bar{t}/\epsilon$, and are clearly exponentially stable. The slow subsystem is of the form

$$\begin{aligned}\frac{\partial x}{\partial \bar{t}} &= -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + \frac{U_w}{V_r}(T_{js} - x) + \frac{U_w}{V_r}\bar{u} + (-\Delta H_0) \\ &\quad \times k_0 e^{-E/Rx} C_{A0} \exp\left(k_0 \int_0^z \exp(-E/Rx)\, \mathrm{d}z/v_l\right) \end{aligned} \tag{39}$$

For the above slow subsystem $\sigma = 1$ and the zero dynamics are locally exponentially stable. The controller (33), with $\mathcal{F}, \mathcal{P}_1$ identically equal to zero and without the $\bar{\eta}$-subsystem, was used in the simulations with γ_1=0.2, for a value of the singular perturbation $\epsilon = 0.01$. The explicit form of the controller is:

$$\left\{\begin{aligned}\frac{\partial \bar{x}}{\partial \bar{t}} &= -\rho_f c_{pf} v_l \frac{\partial \bar{x}}{\partial z} + \frac{U_w}{V_r}(T_{js} - \bar{x}) + \frac{U_w}{V_r}\bar{u} + (-\Delta H_0) \\ &\quad \times k_0 e^{-E/R\bar{x}} C_{A0} \exp\left(k_0 \int_0^z \exp(-E/R\bar{x})\, \mathrm{d}z/v_l\right) \\ \bar{u} &= \frac{V_r}{U_w}\left\{\gamma_0 \left(v - \int_0^1 \bar{x}\, \mathrm{d}z\right) - \int_0^1 \left(- \rho_f c_{pf} v_l \frac{\partial \bar{x}}{\partial z} + (-\Delta H_0) \right.\right. \\ &\quad \times k_0 \exp^{-E/R\bar{x}} C_{A0} \exp\left(k_0 \int_0^z \exp(-E/R\bar{x})\, \mathrm{d}z/v_l\right) \\ &\quad \left.\left. + \frac{U_w}{V_r}(T_{js} - \bar{x})\right) \mathrm{d}z \right\}\end{aligned}\right. \tag{40}$$

The method of finite differences, with a choice of 200 equispaced discretization points, was employed to derive a finite dimensional approximation of the above output feedback controller. A simulation run was performed to test the output tracking capability of the controller. The process was initially assumed to be at steady-state, and at $t = 0\,\mathrm{hr}$ a 2% increase on the value of the reference input was considered. Figure 1 shows the profile of the controlled output of the process and the profile of the manipulated input. One can immediately observe that the controller drives the output at the new steady-state satisfying the requirement $\lim_{t\to\infty} |y - v| = O(\epsilon)$.

7. Conclusions

In this work, we considered systems of quasi-linear two-time-scale hyperbolic PDEs, for which the manipulated inputs, the controlled and the measured outputs are distributed in space. For these systems, we synthesized well-conditioned distributed output feedback controllers that enforce closed-loop stability and output tracking, provided that the speed ratio of the fast versus the slow dynamical phenomena of the two-time-scale system is sufficiently large. These controllers were synthesized through combination of well-conditioned distributed state feedback controllers with distributed state observers which include ϵ-independent observer gains. The developed control method was successfully applied to a fixed-bed reactor.

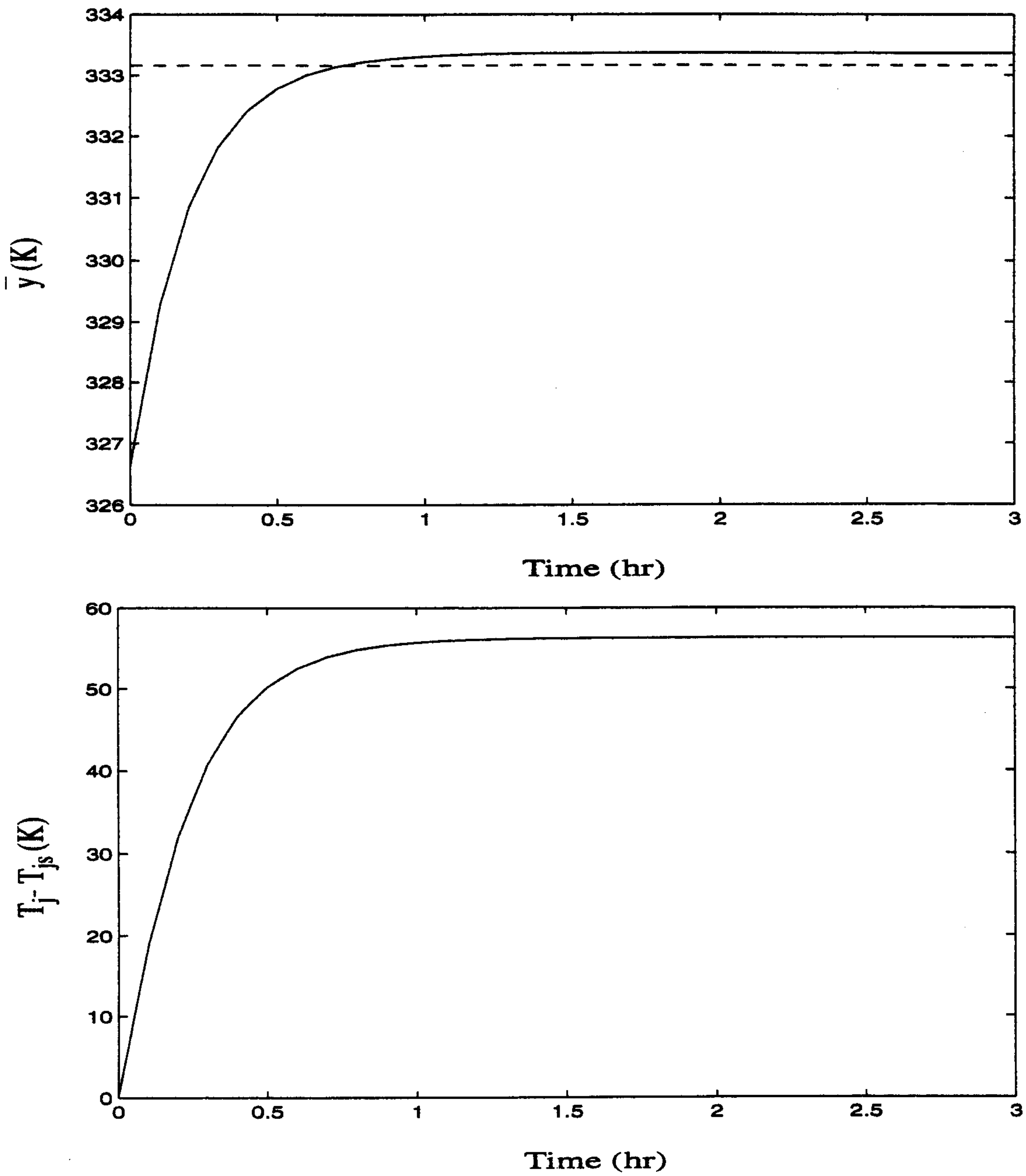

Fig. 1. Closed-loop output profile and input profile for reference input tracking.

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Appendices

A. Definitions

Let ω_1, ω_2 be two elements of $\mathcal{H}([\alpha, \beta]; \mathbb{R}^\kappa)$, where κ is a positive integer. Then, the inner product and the norm in $\mathcal{H}([\alpha, \beta]; \mathbb{R}^\kappa)$ are defined as follows:

$$(\omega_1, \omega_2) = \int_a^b (\omega_1(z), \omega_2(z))_{\mathbb{R}^\kappa} \, dz, \quad \|\omega_1\|_2 = (\omega_1, \omega_1)^{1/2} \tag{A1}$$

where the notation $(\cdot, \cdot)_{\mathbb{R}^\kappa}$ denotes the standard inner product in $\mathbb{R}^\kappa$.

B. Proof of Theorem 3

Under the controller (29), the closed-loop system takes the form:

$$\begin{cases} \dfrac{\partial x}{\partial t} = \mathcal{L}_{11}x + \mathcal{L}_{12}\eta + g_1(x)b(z)\mathcal{F}\eta + g_1(x)b(z)\left[\gamma_{\bar{\sigma}}CL_{\bar{G}}L_{\bar{\mathcal{L}}}^{\bar{\sigma}-1}h(x)b(z)\right]^{-1} \\ \qquad \times \left\{v - Ch(x) - \displaystyle\sum_{\nu=1}^{\bar{\sigma}} \gamma_\nu CL_{\bar{\mathcal{L}}}^{\nu}h(x)\right\} \\ \epsilon\dfrac{\partial \eta}{\partial t} = \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\mathcal{F}\eta + g_2(x)b(z)\left[\gamma_{\bar{\sigma}}CL_{\bar{G}}L_{\bar{\mathcal{L}}}^{\bar{\sigma}-1}h(x)b(z)\right]^{-1} \\ \qquad \times \left\{v - Ch(x) - \displaystyle\sum_{\nu=1}^{\bar{\sigma}} \gamma_\nu CL_{\bar{\mathcal{L}}}^{\nu}h(x)\right\} \end{cases} \tag{B1}$$

Performing a two-time-scale decomposition the fast subsystem takes the form:

$$\begin{aligned} \frac{\partial \eta}{\partial \tau} &= \mathcal{L}_{21}x + [\mathcal{L}_{22} + g_2(x)b(z)\mathcal{F}]\eta + g_2(x)b(z)\left[\gamma_{\bar{\sigma}}CL_{\bar{G}}L_{\bar{\mathcal{L}}}^{\bar{\sigma}-1}h(x)b(z)\right]^{-1} \\ &\quad \times \left\{v - Ch(x) - \sum_{\nu=1}^{\bar{\sigma}} \gamma_\nu CL_{\bar{\mathcal{L}}}^{\nu}h(x)\right\} \end{aligned} \tag{B2}$$

where x is independent of time and thus, it is clearly locally exponentially stable by an appropriate selection of the operator $\mathcal{F}$ (Assumption 3). Moreover, the slow subsystem takes the form

$$\begin{cases} \dfrac{\partial x}{\partial t} = \tilde{\mathcal{L}}(x) + \tilde{G}(x)b(z)\left[\gamma_{\bar{\sigma}}CL_{\bar{G}}L_{\bar{\mathcal{L}}}^{\bar{\sigma}-1}h(x)b(z)\right]^{-1} \\ \qquad \times \left\{v - Ch(x) - \displaystyle\sum_{\nu=1}^{\bar{\sigma}} \gamma_\nu CL_{\bar{\mathcal{L}}}^{\nu}h(x)\right\} \\ \bar{y}_s = Ch(x) \end{cases} \tag{B3}$$

Using the result of Proposition 4 in (Christofides and Daoutidis, 1996a), it can be shown that the above system is locally exponentially stable. Moreover, differentiating the output of the above system with respect to time up to order $\tilde{\sigma}$ and substituting the resulting expressions into (20), it is straightforward to show that the input/output response (20) is enforced in the closed-loop slow subsystem.

Finally, since the closed-loop fast and slow subsystems are locally exponentially stable, there exists (Theorem 1) a sufficiently small ϵ^* such that if $\epsilon \in (0, \epsilon^*]$, the closed-loop system (B1) is locally exponentially stable and the closeness of solutions result (30) holds.

C. Proof of Theorem 4

Under the controller (33), the closed-loop system takes the form

$$\left\{\begin{aligned}
\frac{\partial \bar{x}}{\partial t} &= \tilde{\mathcal{L}}(\bar{x}) + \tilde{G}(\bar{x})b(z)\left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\
&\quad \times \left\{v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x})\right\} + \mathcal{P}_1(\bar{q}_1 - \mathcal{Q}_1 p_1 \bar{x}) \\
\frac{\partial \bar{\eta}}{\partial \tau} &= \mathcal{L}_{21}\bar{x} + \mathcal{L}_{22}\bar{\eta} + g_2(\bar{x})b(z)\mathcal{F}\bar{\eta} + g_2(\bar{x})b(z)\left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\
&\quad \times \left\{v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x})\right\} + \mathcal{P}_2(\bar{q}_2 - \mathcal{Q}_2 p_2 \bar{\eta}) \\
\frac{\partial x}{\partial t} &= \mathcal{L}_{11}x + \mathcal{L}_{12}\eta + g_1(x)b(z)\mathcal{F}\bar{\eta} + +g_1(x)b(z)\left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\
&\quad \times \left\{v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x})\right\} \\
\epsilon\frac{\partial \eta}{\partial t} &= \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\mathcal{F}\bar{\eta} + g_2(x)b(z)\left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\
&\quad \times \left\{v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x})\right\}
\end{aligned}\right. \tag{C1}$$

Performing a two-time-scale decomposition the fast subsystem takes the form

$$\left\{\begin{aligned}
\frac{\partial \bar{\eta}}{\partial \tau} &= \mathcal{L}_{21}\bar{x} + \mathcal{L}_{22}\bar{\eta} + g_2(\bar{x})b(z)\mathcal{F}\bar{\eta} + g_2(\bar{x})b(z)\left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\
&\quad \times \left\{v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x})\right\} + \mathcal{P}_2(\bar{q}_2 - \mathcal{Q}_2 p_2 \bar{\eta}) \\
\frac{\partial \eta}{\partial \tau} &= \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\mathcal{F}\bar{\eta} + g_2(x)b(z)\left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\
&\quad \times \left\{v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x})\right\}
\end{aligned}\right. \tag{C2}$$

where $x, \bar{x}$ are independent of time. Defining the deviation vector $e_\eta(z,t) = \bar{\eta}(z,t) - \eta(z,t)$ and using that the assumption $x(z,0) = \bar{x}(z,0)$ implies that $x(z,t) = \bar{x}(z,t)$ in the fast time scale τ, the above system can be written in the following form:

$$\left\{\begin{aligned} \frac{\partial e_\eta}{\partial \tau} &= \left[\mathcal{L}_{22} + \mathcal{P}_2 \mathcal{Q}_2 p_2\right] e_\eta \\ \frac{\partial \eta}{\partial \tau} &= \mathcal{L}_{21} x + [\mathcal{L}_{22} + g_2(x) b(z) \mathcal{F}]\eta + g_2(x) b(z) \left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\ &\quad \times \left\{ v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x}) \right\} + g_2(x) b(z) \mathcal{F} e_\eta \end{aligned}\right. \tag{C3}$$

From the detectability Assumption 3, we have that the operator $[\mathcal{L}_{22} - \mathcal{P}_2 \mathcal{Q}_2 p_2] e_\eta$ generates a locally exponentially stable semi-group, which implies that the e_η-subsystem of the system (C3) is locally exponentially stable. On the other hand, the stabilizability Assumption 2 states that the operator $[\mathcal{L}_{22} + g_2(x) b(z) \mathcal{F}]\eta$ generates an exponentially stable semi-group, which implies that the η-subsystem of the system (C3) is also locally exponentially stable. Thus, the fast subsystem (C3) consists an interconnection of two locally exponentially stable subsystems, which implies that this is also locally exponentially stable.

The closed-loop slow subsystem takes the form

$$\left\{\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= \tilde{\mathcal{L}}(\bar{x}) + \tilde{G}(\bar{x}) b(z) \left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\ &\quad \times \left\{ v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x}) \right\} + \mathcal{P}_1(\bar{q}_1 - \mathcal{Q}_1 p_1 \bar{x}) \\ \frac{\partial x}{\partial t} &= \tilde{\mathcal{L}}(x) + \tilde{G}(x) b(z) \left[\gamma_{\tilde{\sigma}} C L_{\tilde{G}} L_{\tilde{\mathcal{L}}}^{\tilde{\sigma}-1} h(\bar{x}) b(z)\right]^{-1} \\ &\quad \times \left\{ v - Ch(\bar{x}) - \sum_{\nu=1}^{\tilde{\sigma}} \gamma_\nu C L_{\tilde{\mathcal{L}}}^{\nu} h(\bar{x}) \right\} \end{aligned}\right. \tag{C4}$$

Using the result of Theorem 4 in (Christofides and Daoutidis, 1996a), it is straightforward to show that the above closed-loop slow subsystem is locally exponentially stable.

Finally, since the closed-loop fast and slow subsystems are locally exponentially stable, the result of Theorem 1 of this paper can be applied to get the existence of a sufficiently small ϵ^* such that if $\epsilon \in (0, \epsilon^*]$, the closed-loop system (C1) is locally exponentially stable and the closeness of solutions result (34) holds.

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