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Finite-Dimensional Control of Parabolic PDE Systems Using Approximate Inertial Manifolds

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This paper introduces a methodology for the synthesis of nonlinear finite-dimensional output feedback controllers for systems of quasi-linear parabolic partial differential equations (PDEs), for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement. Combination of Galerkin's method with a novel procedure for the construction of approximate inertial manifolds for the PDE system is employed for the derivation of ordinary differential equation (ODE) systems (whose dimension is equal to the number of slow modes) that yield solutions which are close, up to a desired accuracy, to the ones of the PDE system, for almost all times. These ODE systems are used as the basis for the synthesis of nonlinear output feedback controllers that guarantee stability and enforce the output of the closed-loop system to follow up to a desired accuracy, a prespecified response for almost all times.

1. INTRODUCTION

Parabolic PDE systems arise naturally as models of diffusion-convection-reaction processes [16] and typically involve spatial differential operators whose eigenspectrum can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement [11, 1]. This implies that the dynamic behavior of such systems can be approximately described by finite-dimensional systems. Motivated by this, the standard approach to the control of parabolic PDEs involves the application of Galerkin's method to the PDE system to derive ODE systems that describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for the synthesis of finite-dimen-

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sional controllers [1, 16]. However, there are two key controller implementation and closed-loop performance problems associated with this approach. First, the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to high dimensionality of the resulting controllers [12]. Second, there is a lack of a systematic way to characterize the discrepancy between the solutions of the PDE system and the approximate ODE system in finite time, which is essential for characterizing the transient performance of the closed-loop PDE system.

A natural framework to address the problem of deriving low-dimensional ODE systems that accurately reproduce the solutions of dissipative PDEs is based on the concept of inertial manifold (IM) (see [18] and the references therein). An IM is a positively invariant, finite-dimensional Lipschitz manifold, which attracts every trajectory exponentially. If an IM exists, the dynamics of the parabolic PDE system restricted on the inertial manifold is described by a set of ODEs called the inertial form. Hence, stability and bifurcation studies of the infinite-dimensional PDE system can be readily performed on the basis of the finite-dimensional inertial form [18]. However, the explicit derivation of the inertial form requires the computation of the analytic form of the IM. Unfortunately, IMs have been proven to exist only for certain classes of PDEs (for example, the Kuramoto−Sivashinsky equation and some diffusion-reaction equations [18]), and even then it is almost impossible to derive their analytic form. In order to overcome the problems associated with the existence and construction of IMs, the concept of approximate inertial manifold (AIM) has been introduced [9, 10] and used for the derivation of ODE systems whose dynamic behavior approximates the one of the inertial form.

In the area of control of nonlinear parabolic PDE systems, few papers have appeared in the literature dealing with the application of IM for the synthesis of finite-dimensional controllers. In particular, in [17] the problem of stabilization of a parabolic PDE with boundary finite-dimensional feedback was studied; a standard observer-based controller augmented with a residual mode filter [3] was used to induce an inertial manifold in the closed-loop system, and thus reduce the stabilization problem for the PDE system to a stabilization problem for the finite-dimensional inertial form. In [5], the theory of inertial manifolds was utilized to determine the extent to which linear boundary proportional control influences the dynamic and steady-state response of the closed-loop system.

In this paper, we introduce a methodology for the synthesis of nonlinear finite-dimensional output feedback controllers for systems of quasi-linear parabolic PDEs. Singular perturbation methods are initially employed to establish that the discrepancy between the solutions of an ODE system of dimension equal to the number of slow modes, obtained through Galerkin's method, and the PDE system is proportional to the degree of separation of the fast and slow modes of the spatial operator. Then, a procedure, motivated by the theory of singular perturbations, is proposed for the construction of AIMs for the PDE system. The AIMs are used for the derivation of ODE systems of dimension equal to the number of slow modes, that yield solutions which are close, up to a desired accuracy, to the ones of the PDE system, for almost all times. These ODE systems are used as the basis for the synthesis of nonlinear output feedback controllers that guarantee stability and enforce the output of the closed-loop system to follow up to a desired accuracy, a prespecified response for almost all times.

2. PRELIMINARIES

We consider quasi-linear parabolic PDE systems of the form

$$\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2} + wb(z)u + f(\bar{x}) \\ y^i &= \int_{z_i}^{z_{i+1}} c^i(z)k\bar{x}\,dz, \qquad i = 1,\ldots,l \end{aligned} \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} C_1\bar{x}(\alpha,t) + D_1\frac{\partial \bar{x}}{\partial z}(\alpha,t) &= R_1 \\ C_2\bar{x}(\beta,t) + D_2\frac{\partial \bar{x}}{\partial z}(\beta,t) &= R_2 \end{aligned} \tag{2}$$

and the initial condition

$$\bar{x}(z,0) = \bar{x}_0(z), \tag{3}$$

where $\bar{x}(z,t) = [\bar{x}_1(z,t)\cdots \bar{x}_n(z,t)]^T$ denotes the vector of state variables, $[\alpha,\beta] \subset \mathbb{R}$ is the domain of definition of the process, $z \in [\alpha,\beta]$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $u = [u^1 \ u^2 \ \cdots \ u^l]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, and $y^i \in \mathbb{R}$ denotes a controlled output. $\partial\bar{x}/\partial z$, $\partial^2\bar{x}/\partial z^2$ denote the first- and second-order spatial derivatives of $\bar{x}$, $f(\bar{x})$ is a vector function, w, k are constant vectors, A, B, C_1, D_1, C_2, D_2 are constant matrices, R_1, R_2 are column vectors, and $\bar{x}_0(z)$ is the initial condition. $b(z)$ is a known smooth vector function of z of the form $b(z) = [b^1(z)b^2(z)\cdots b^l(z)]$, where $b^i(z)$ describes how the control action $u^i(t)$ is distributed in the spatial interval $[z_i, z_{i+1}] \subset [\alpha, \beta]$, and $c^i(z)$ is a known smooth function of z which is determined by the desired performance specifications in the interval $[z_i, z_{i+1}]$. Whenever the control action enters the system at a single point z_0, with $z_0 \in [z_i, z_{i+1}]$ (i.e., point actuation), the function $b^i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_0 - \epsilon, z_0 + \epsilon]$, where ϵ is a small positive real number, and zero elsewhere in $[z_i, z_{i+1}]$. Throughout the paper, we will use the order of magnitude notation $O(\epsilon)$. In particular, $\delta(\epsilon) = O(\epsilon)$ if there exist positive real numbers k_1 and k_2 such that $|\delta(\epsilon)| \leq k_1|\epsilon|$, $\forall |\epsilon| < k_2$.

We formulate the system of Eqs. (1)–(3) in a Hilbert space $\mathcal{H}([\alpha, \beta], \mathbb{R}^n)$, with $\mathcal{H}$ being the space of n-dimensional vector functions defined on $[\alpha, \beta]$ that satisfy the boundary condition of Eq. (2), with inner product and norm

$$\begin{aligned} (\omega_1, \omega_2) &= \int_\alpha^\beta (\omega_1(z), \omega_2(z))_{\mathbb{R}^n}\, dz \\ \|\omega_1\|_2 &= (\omega_1, \omega_1)^{1/2}, \end{aligned} \tag{4}$$

where ω_1, ω_2 are two elements of $\mathcal{H}([\alpha, \beta]; \mathbb{R}^n)$ and the notation $(\cdot, \cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function x on $\mathcal{H}([\alpha, \beta], \mathbb{R}^n)$ as

$$x(t) = \bar{x}(z, t), \qquad t > 0,\ z \in [\alpha, \beta], \tag{5}$$

the operator $\mathcal{A}$ in $\mathcal{H}([\alpha, \beta], \mathbb{R}^n)$ as

$$\begin{aligned} \mathcal{A}x &= A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2}, \\ x \in D(\mathcal{A}) &= \Big\{ x \in \mathcal{H}([\alpha, \beta]; \mathbb{R}^n);\ C_1 x(\alpha) + D_1 \frac{\partial x}{\partial z}(\alpha) = R_1, \\ &\qquad C_2 x(\beta) + D_2 \frac{\partial x}{\partial z}(\beta) = R_2 \Big\} \end{aligned} \tag{6}$$

and the input and output operators as

$$\mathcal{B}u = wbu, \qquad \mathcal{C}x = (c, kx), \tag{7}$$

where $c = [c^1 c^2 \dots c^l]$, the system of Eqs. (1)–(3) takes the form

$$\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}u + f(x) \\ y &= \mathcal{C}x \\ x(0) &= x_0, \end{aligned} \tag{8}$$

where $f(x(t)) = f(\bar{x}(z,t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear term $f(x)$ satisfies $f(0) = 0$ and is also locally Lipschitz continuous, i.e., there exist positive real numbers a_0, K_0 such that for any $x_1, x_2 \in \mathcal{H}$ that satisfy $\max\{\|x_1\|_2, \|x_2\|_2\} \leq a_0$, we have that

$$\| f(x_1) - f(x_2) \|_2 \leq K_0 \|x_1 - x_2\|_2. \tag{9}$$

For $\mathcal{A}$, the standard eigenvalue problem is of the form

$$\mathcal{A}\phi_j = \lambda_j \phi_j, \qquad j = 1, \ldots, \infty, \tag{10}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction;. the eigenspectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of $\mathcal{A}$, i.e., $\sigma(\mathcal{A}) = \{\lambda_1, \lambda_2, \ldots, \}$. Assumption 1 that follows states our hypotheses for the properties of $\sigma(\mathcal{A})$.

Assumption 1. (1) $\text{Re}\,\lambda_1 \geq \text{Re}\,\lambda_2 \geq \cdots \geq \text{Re}\,\lambda_j \geq \cdots$, where $\text{Re}\,\lambda_j$ denotes the real part of λ_j.

(2) $\sigma(\mathcal{A})$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) + \sigma_2(\mathcal{A})$, where $\sigma_1(\mathcal{A})$ consists of the first m (with m finite) eigenvalues, i.e., $\sigma_1(\mathcal{A}) = \{\lambda_1, \ldots, \lambda_m\}$, and $|\text{Re}\,\lambda_1|/|\text{Re}\,\lambda_m| = O(1)$.

(3) $\text{Re}\,\lambda_{m+1} < 0$ and $|\text{Re}\,\lambda_m|/|\text{Re}\,\lambda_{m+1}| = 0(\epsilon)$ where $\epsilon := |\text{Re}\,\lambda_1|/|\text{Re}\,\lambda_{m+1}| < 1$ is a small positive parameter.

Assumption 1 states that the eigenspectrum of $\mathcal{A}$ can be partitioned into a finite-dimensional part consisting of m slow, possibly unstable, eigenvalues, $\sigma_1(\mathcal{A}) = \{\lambda_1, \ldots, \lambda_m\}$ and a stable infinite-dimensional part containing the remaining fast eigenvalues, $\sigma_2(\mathcal{A}) = \{\lambda_{m+1}, \ldots, \}$, and that the separation between slow and fast eigenvalues of $\mathcal{A}$ is large. The assumption of the finite number of unstable eigenvalues is always satisfied for parabolic PDEs [11], while the existence of only a few dominant modes that capture the dominant dynamics of a parabolic PDE system is well-established for the majority of diffusion-convection-reaction processes (see, for example, the applications in the book [16], and the packed-bed reactor example studied in [8]).

Assumption 1 guarantees [11] that $\mathcal{A}$ generates a strongly continuous semigroup of bounded linear operators $U(t)$ which implies that the generalized solution of the system of Eq. (8) is given by

$$x = U(t)x_0 + \int_0^t U(t-s)(\mathcal{B}u(s) + f(x(s)))\,ds. \tag{11}$$

$U(t)$ satisfies the growth property

$$\|U(t)\|_2 \leq K_1 e^{a_1 t}, \qquad \forall t \geq 0, \tag{12}$$

where K_1, a_1 are real numbers, with $K_1 \geq 1$ and $a_1 \geq \text{Re}\,\lambda_1$. If a_1 is strictly negative, we will say that $\mathcal{A}$ generates an exponentially stable semigroup $U(t)$. Throughout the manuscript, we will focus on local exponential stability, and not on weak (asymptotic) stability [11], because of its robustness to bounded perturbations (e.g., uncertain variables, disturbances), which are always present in most practical applications [3].

We will now review the application of Galerkin's method to the system of Eq. (8) to derive an approximate finite-dimensional system. Let $\mathcal{H}_s$, $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as $\mathcal{H}_s = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \ldots,\}$ (the existence of $\mathcal{H}_s$, $\mathcal{H}_f$ follows from Assumption 1). Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq. (8) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x. \tag{13}$$

Applying P_s and P_f to the system of Eq. (8) and using the above decomposition for x, the system of Eq. (8) can be equivalently written in the form

$$\begin{aligned}
\frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f) \\
\frac{\partial x_f}{\partial t} &= \mathcal{A}_f x_f + \mathcal{B}_f u + f_f(x_s, x_f) \\
y &= \mathcal{C} x_s + \mathcal{C} x_f \\
x_s(0) &= P_s x(0) = P_s x_0, \qquad x_f(0) = P_f x(0) = P_f x_0,
\end{aligned} \tag{14}$$

where $\mathcal{A}_s = P_s \mathcal{A} P_s$, $\mathcal{B}_s = P_s \mathcal{B}$, $f_s = P_s f$, $\mathcal{A}_f = P_f \mathcal{A} P_f$, $\mathcal{B}_f = P_f \mathcal{B}$, and $f_f = P_f f$ and the notation $\partial x_f / \partial t$ is used to denote that the state x_f belongs in an infinite-dimensional space. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = \text{diag}\{\lambda_j\}$, $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are Lipschitz vector functions, and $\mathcal{A}_f$ is an unbounded differential operator which generates a strongly continuous exponentially stable semigroup (following from part (3) of Assumption 1 and the selection of $\mathcal{H}_s, \mathcal{H}_f$). Neglecting the fast modes, the following finite-dimensional system is derived,

$$\begin{aligned}
\frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, 0) \\
y_s &= \mathcal{C} x_s,
\end{aligned} \tag{15}$$

where the subscript s in y_s denotes that the output is associated with the slow system. The above system can be directly used for controller design employing standard control methods for ODEs [1, 16, 7].

Remark 1. We note that the large separation of slow and fast modes of the spatial operator in parabolic PDEs ensures that a controller which exponentially stabilizes the closed-loop ODE system, also stabilizes the closed-loop infinite-dimensional system [1]. This is in contrast to the application of this approach to hyperbolic PDEs where the eigenmodes cluster along nearly vertical asymptotes in the complex plane and thus, the controller has to be modified to compensate for the destabilizing effect of the residual modes [3].

3. ACCURACY OF ODE SYSTEM OBTAINED FROM GALERKIN'S METHOD

In this section, we use singular perturbation methods to establish that if the finite-dimensional system of Eq. (15) is exponentially stable, then the system of Eq. (14) is also exponentially stable and the discrepancy between the solution of the x_s-subsystem of the system of Eq. (14) and the solution of the system of Eq. (15) is proportional to the spectral separation of the slow and fast eigenvalues.

Using that $\epsilon = |\text{Re } \lambda_1| / |\text{Re } \lambda_{m+1}|$, the system of Eq. (14) can be written in the form

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f) \\ \epsilon \frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon} x_f + \epsilon \mathcal{B}_f u + \epsilon f_f(x_s, x_f), \end{aligned} \tag{16}$$

where $\mathcal{A}_{f\epsilon}$ is an unbounded differential operator defined as $\mathcal{A}_{f\epsilon} = \epsilon \mathcal{A}_f$. Since ϵ is a small positive number less than unity (following from Assumption 1, part (3)) and the operators $\mathcal{A}_s$, $\mathcal{A}_{f\epsilon}$ generate semigroups with growth rates which are of the same order of magnitude, the system of Eq. (16) is in the standard singularly perturbed form, with x_s being the slow states and x_f being the fast states.

Introducing the fast time-scale $\tau = t/\epsilon$ and setting $\epsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq. (16):

$$\frac{\partial x_f}{\partial \tau} = \mathcal{A}_{f\epsilon} x_f. \tag{17}$$

From the fact that Re $\lambda_{m+1} < 0$ and the definition of ϵ, we have that the above system is globally exponentially stable. Setting $\epsilon = 0$ in the system of Eq. 16 and using the fact that the inverse operator $\mathcal{A}_{f\epsilon}^{-1}$ exists and is also bounded (it follows from the fact that zero is in the resolvent of $\mathcal{A}_{f\epsilon}$), we have that

$$x_f = 0 \tag{18}$$

and thus the finite-dimensional slow system takes the form

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, 0). \tag{19}$$

We note that the above system is identical to the one obtained by applying the standard Galerkin's method to the system of Eq. (8), keeping the first m ODEs and completely neglecting the x_f-subsystem. Assumption 2 that follows states a stability requirement on the system of Eq. (19).

Assumption 2. The finite-dimensional system of Eq. (19) with $u(t) \equiv 0$ is exponentially stable, i.e., there exist positive real numbers (K_2, a_2, a_4), with $a_4 > K_2 \geq 1$ such that for all $x_s \in \mathcal{H}_s$ that satisfy $|x_s| \leq a_4$, the following bound holds:

$$|x_s(t)| \leq K_2 e^{-a_2 t}|x_s(0)|, \qquad \forall t \geq 0. \tag{20}$$

Proposition 1 that follows establishes that the solutions of the open-loop systems of Eqs. (19)–(17), after a short finite time interval required for the trajectories of the system of Eq. (16) to approach the quasi steady-state of Eq. (18), consist of an $O(\epsilon)$ approximation of the solutions of the open-loop system of Eq. (16). The proof is given in the Appendix.

PROPOSITION 1. *Consider the system of Eq.* (16) *with* $u(t) \equiv 0$ *and suppose that Assumptions* 1 *and* 2 *hold. Then, there exist positive real numbers* μ_1, μ_2, ϵ^* *such that if* $|x_s(0)| \leq \mu_1$, $\|x_f(0)\|_2 \leq \mu_2$, *and* $\epsilon \in (0, \epsilon^*]$, *then the solution* $x_s(t), x_f(t)$ *of the system of Eq.* (16) *satisfies for all* $t \in [0, \infty)$

$$\begin{aligned} x_s(t) &= \bar{x}_s(t) + O(\epsilon) \\ x_f(t) &= \bar{x}_f\left(\frac{t}{\epsilon}\right) + O(\epsilon), \end{aligned} \tag{21}$$

where $\bar{x}_s(t), \bar{x}_f(t)$ *are the solutions of the slow and fast subsystems of Eqs.* (19)–(17) *with* $u(t) \equiv 0$, *respectively.*

Remark 2. The counterpart of the result of Proposition 1 in finite-dimensional spaces is well known (Tikhonov's theorem [19]), while a similar result has also been established for linear infinite-dimensional systems [2]. The main technical difference in establishing this result between linear and quasi-linear infinite-dimensional systems is that, for quasi-linear systems the proof is based on Lyapunov arguments, while for linear systems the proof is obtained using combination of estimates of the states, obtained from the application of variations of constants formula [2]. This is a consequence of the fact that for quasi-linear systems it is not possible to derive a coordinate change that transforms the system of Eq. (16) into a cascaded interconnection where the fast modes are decoupled from the slow modes, which allows us to derive an exponentially decaying estimate, for sufficiently small ϵ, for the fast state, which is independent of the one of the slow state, and thus to prove the result through a direct combination of these estimates.

Remark 3. We note that it is possible, using standard results from center manifold theory for infinite-dimensional systems of the form of Eq. (8) [6], to show that if the system of Eq. (19) is asymptotically stable, then the system of Eq. (8) is also asymptotically stable and the discrepancy between the solution of the system of Eq. (19) and the x_s-subsystem of the system of Eq. 16 is asymptotically (as $t \to \infty$) proportional to ϵ. Although this result is important because it allows establishing asymptotic stability of the closed-loop infinite-dimensional system by performing a stability analysis on a low-order finite-dimensional system, it does not provide any information about the discrepancy between the solutions of these two systems for finite t.

4. CONSTRUCTION OF ODE SYSTEMS OF DESIRED ACCURACY VIA AIMs

In this section, we propose an approach originating from the theory of inertial manifolds for the construction of ODE systems of dimension m which yield solutions that are arbitrarily close (closer than $O(\epsilon)$) to the ones of the infinite-dimensional system of Eq. (8), for almost all times. An inertial manifold $\mathcal{M}$ for the system of Eq. (8) is a subset of $\mathcal{H}$, which satisfies the following properties [18]: (i) $\mathcal{M}$ is a finite-dimensional Lipschitz manifold; (ii) $\mathcal{M}$ is a graph of a Lipschitz function $\Sigma(x_s, u, \epsilon)$ mapping $\mathcal{H}_s \times \mathbb{R}^l \times (0, \epsilon^*]$ into $\mathcal{H}_f$ and for every solution $x_s(t), x_f(t)$ of Eq. (16) with $x_f(0) = \Sigma(x_s(0), u, \epsilon)$, then

$$x_f(t) = \Sigma(x_s(t), u, \epsilon), \qquad \forall t \geq 0; \tag{22}$$

and (iii) $\mathcal{M}$ attracts every trajectory exponentially. The evolution of the state x_f on $\mathcal{M}$ is given by Eq. (22), while the evolution of the state x_s is governed by the finite-dimensional inertial form

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, \Sigma(x_s, u, \epsilon)). \tag{23}$$

Assuming that $u(t)$ is smooth, differentiating Eq. (22), and utilizing Eq. (16), $\Sigma(x_s, u, \epsilon)$ can be computed as the solution of the partial differential equation

$$\epsilon \frac{\partial \Sigma}{\partial x_s}\left[\mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f)\right] + \epsilon \frac{\partial \Sigma}{\partial u}\dot{u} = \mathcal{A}_{f\epsilon} x_f + \epsilon\, \mathcal{B}_f u + \epsilon f_f(x_s, x_f) \tag{24}$$

which Σ has to satisfy for all $x_s \in \mathcal{H}_s, u \in \mathbb{R}^l, \epsilon \in (0, \epsilon^*]$. However, even for parabolic PDEs for which it is known that $\mathcal{M}$ exists, the derivation of an explicit analytic form of $\Sigma(x_s, u, \epsilon)$ is an extremely difficult (if not impossible) task.

Motivated by this, we will now propose a procedure, motivated by singular perturbations [15], to compute approximations of $\Sigma(x_s, u, \epsilon)$ (approximate inertial manifolds) and approximations of the inertial form, of desired accuracy. To this end, consider an expansion of $\Sigma(x_s, u, \epsilon)$ and u in a power series in ϵ,

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots + \epsilon^k u_k + O(\epsilon^{k+1}) \\ \Sigma(x_s, u, \epsilon) &= \Sigma^0(x_s, u) + \epsilon \Sigma^1(x_s, u) + \epsilon^2 \Sigma^2(x_s, u) + \cdots \\ &\quad + \epsilon^k \Sigma^k(x_s, u) + O(\epsilon^{k+1}), \end{aligned} \tag{25}$$

where u_k, Σ^k are smooth functions. Substituting the expressions of Eq. (25) into Eq. (24), and equating terms of the same power in ϵ, one can obtain approximations of $\Sigma(x_s, u, \epsilon)$ up to a desired order. Substituting the expansion for $\Sigma(x_s, u, \epsilon)$ and u up to order k into Eq. (23), the following approximation of the inertial form is obtained:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s\left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots + \epsilon^k u_k\right) \\ &\quad + f_s\left(x_s, \Sigma^0(x_s, u) + \epsilon \Sigma^1(x_s, u) + \epsilon^2 \Sigma^2(x_s, u) + \cdots \right. \\ &\quad \left. + \epsilon^k \Sigma^k(x_s, u)\right) \end{aligned} \tag{26}$$

In order to characterize the discrepancy between the solution of the open-loop finite-dimensional system of Eq. (26) and the solution of the x_s-subsystem of the open-loop infinite-dimensional system of Eq. (16), we will impose a stability requirement on the system of Eq. (26).

Assumption 3. The finite-dimensional system of Eq. (26) with $u(t) \equiv 0$ is exponentially stable, i.e., there exist positive real numbers $(\bar{K}_2, \bar{a}_2, \bar{a}_4)$, with $\bar{a}_4 > \bar{K}_2 \geq 1$ such that for all $x_s \in \mathcal{H}_s$ that satisfy $|x_s| \leq \bar{a}_4$, the following bound holds:

$$\left| x_s(t) \right| \leq \bar{K}_2 e^{-\bar{a}_2 t} \left| x_s(0) \right|, \qquad \forall t \geq 0. \tag{27}$$

Proposition 2 that follows establishes that the discrepancy between the solutions obtained from the open-loop system of Eq. (26) and the expansion for $\Sigma(x_s, u, \epsilon)$ of Eq. (29), and the solutions of the infinite-dimensional open-loop system of Eq. (16) is of $O(\epsilon^{k+1})$, for almost all times. The proof is given in the Appendix.

PROPOSITION 2. *Consider the system of Eq.* (16) *with* $u(t) \equiv 0$ *and suppose that Assumptions* 1 *and* 3 *hold. Then, there exist positive real numbers* $\bar{\mu}_1, \bar{\mu}_2, \bar{\epsilon}^*$ *such that if* $|x_s(0)| \leq \bar{\mu}_1$, $\|x_f(0)\|_2 \leq \bar{\mu}_2$, *and* $\epsilon \in (0, \bar{\epsilon}^*]$, *then the solution* $x_s(t), x_f(t)$ *of the system of Eq.* (16) *satisfies for all* $t \in [t_b, \infty)$

$$\begin{aligned} x_s(t) &= \tilde{x}_s(t) + O(\epsilon^{k+1}) \\ x_f(t) &= \tilde{x}_f(t) + O(\epsilon^{k+1}), \end{aligned} \tag{28}$$

where t_b *is the time required for* $x_f(t)$ *to approach* $\tilde{x}_f(t)$, $\tilde{x}_s(t)$ *is the solution of Eq.* (26) *with* $u(t) \equiv 0$, *and* $\tilde{x}_f(t) = \epsilon\Sigma^1(\tilde{x}_s, 0) + \epsilon^2\Sigma^2(\tilde{x}_s, 0) + \cdots + \epsilon^k\Sigma^k(\tilde{x}_s, 0)$.

Remark 4. The result of Proposition 2 provides the means for characterizing the discrepancy between the solution of the open-loop infinite-dimensional system of Eq. (8), $x(t)$ (and thus the solution of the parabolic PDE system of Eq. (1) with $u(t) \equiv 0$, and the solution $\tilde{x}(t) = \tilde{x}_s(t) + \tilde{x}_f(t) = \tilde{x}_s(t) + \epsilon\Sigma^1(\tilde{x}_s(t), 0) + \cdots + \epsilon^k\Sigma^k(\tilde{x}_s(t), 0)$, which is obtained from the $O(\epsilon^k)$ approximation of the open-loop inertial form (i.e., Eq. (26) with $u(t) \equiv 0$). In particular, substituting Eq. (28) into the equation $x(t) = x_s(t) + x_f(t)$, we have that $x(t) = \tilde{x}_s(t) + \tilde{x}_f(t) + O(\epsilon^{k+1})$ for $t \geq t_b$. Utilizing the definition of order of magnitude, we finally obtain the following characterization for the discrepancy between $x(t)$ and $\tilde{x}(t)$: $\|x(t) - \tilde{x}(t)\|_2 \leq k_1\epsilon^{k+1}$ for $t \geq t_b$, where k_1 is a positive real number.

Remark 5. Following the proposed approximation procedure, it can be shown that the $O(\epsilon)$ approximation of $\Sigma(x_s, 0, \epsilon)$ is $\Sigma^0(x_s, 0) = 0$ and the corresponding approximate inertial form is identical to the system of Eq. (19) (obtained via Galerkin's method) with $u(t) \equiv 0$. This system does not utilize any information about the structure of the fast subsystem, thus yielding solutions which are only $O(\epsilon)$ close to the solutions of the open-loop system of Eq. (8) (Proposition 1). On the other hand, the $O(\epsilon^2)$ approximation of $\Sigma(x_s, 0, \epsilon)$ can be shown to be of the form

$$\Sigma(x_s, 0, \epsilon) = \Sigma^0(x_s, 0) + \epsilon\Sigma^1(x_s, 0) = \epsilon\left(\mathcal{A}_{f\epsilon}\right)^{-1}\left[-f_f(x_s, 0)\right]. \tag{29}$$

The corresponding open-loop approximate inertial form does utilize information about the structure of the fast subsystem, and thus allows us to obtain solutions which are $O(\epsilon^2)$ close to the solutions of the open-loop system of Eq. (8) (Proposition 2).

Remark 6. The standard approach followed in the literature for the construction of AIMs for systems of the form of Eq. (14) with $u(t) \equiv 0$ (see, for example, [4]) is to directly set $\partial x_f / \partial t \equiv 0$, solve the resulting algebraic equations for x_f, and substitute the solution for x_f to the x_s-subsystem of Eq. (14), to derive the ODE system

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + f_s\Big(x_s, (\mathcal{A}_f)^{-1}[-f_f(x_s)]\Big). \tag{30}$$

It is straightforward to show that the slow system of Eq. (30) is identical to the one obtained by using the $O(\epsilon^2)$ approximation for $\Sigma(x, 0, \epsilon)$ for the construction of the approximate inertial form.

Remark 7. The expansion of u in a power series in ϵ is motivated by our intention to modify the synthesis of the feedback controller appropriately such that the discrepancy between the output of the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form and the output of the closed-loop PDE system will be of $O(\epsilon^{k+1})$ for almost all times (see also Remark 8).

5. FINITE-DIMENSIONAL CONTROL

In this section, we use the result of Proposition 2 to establish that a nonlinear finite-dimensional output feedback controller, that guarantees stability and enforces output tracking in the ODE system of Eq. (26), exponentially stabilizes the closed-loop PDE system and ensures that the discrepancy between the output of the closed-loop ODE system and the output of the closed-loop PDE system is of $O(\epsilon^{k+1})$, provided that ϵ is sufficiently small.

The finite-dimensional output feedback controller which achieves the desired objectives for the system of Eq. (26) is constructed through a standard combination of a state feedback controller with a state observer. In particular, we consider a state feedback control law of the general form

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \cdots + \epsilon^k u_k \\ &= p_0(x_s) + Q_0(x_s)v + \epsilon[p_1(x_s) + Q_1(x_s)v] + \cdots \\ &\quad + \epsilon^k[p_k(x_s) + Q_k(x_s)v], \end{aligned} \tag{31}$$

where $p_0(x_s), \ldots, p_k(x_s)$ are smooth vector functions, $Q_0(x_s), \ldots, Q_k(x_s)$ are smooth matrices, and $v \in \mathbb{R}^l$ is the constant reference input vector (see Remark 8 for a procedure for the synthesis of the control law, i.e., the explicit computation of $[p_0(x_s), \ldots, p_k(x_s), Q_0(x_s), \ldots, Q_k(x_s)]$). The following m-dimensional state observer is also considered for the implementation of the state feedback law of Eq. (31),

$$\begin{aligned}\frac{d\eta}{dt} = {} & \mathcal{A}_s\eta + \mathcal{B}_s\big(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v] + \cdots \\ & \qquad + \epsilon^k[p_k(\eta) + Q_k(\eta)v]\big) \\ & + f_s\big(\eta, \epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) + \cdots + \epsilon^k\Sigma^k(\eta,u)\big) \\ & + L\big(y - \big[\mathcal{C}\eta + \mathcal{C}\{\epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) + \cdots \\ & \qquad + \epsilon^k\Sigma^k(\eta,u)\}\big]\big),\end{aligned} \tag{32}$$

where $\eta \in \mathcal{H}_s$ denotes the observer state vector and L is a matrix chosen so that the eigenvalues of the matrix $C_L = \mathcal{A}_s + (\partial f_s/\partial\eta_{(\eta=\eta_s)}) - L[\mathcal{C}\eta_s + \mathcal{C}\{(\partial/\partial\eta)(\epsilon\Sigma^1(\eta, u(\eta)) + \epsilon^2\Sigma^2(\eta, u(\eta)) + \cdots + \epsilon^k\Sigma^k(\eta, u(\eta)))_{(\eta=\eta_s)}\}]$ lie in the open left-half of the complex plane, where η_s denotes the steady state for the system of Eq. (32). The finite-dimensional output feedback controller resulting from the combination of the state feedback controller of Eq. (31) with the state observer of Eq. (32) takes the form

$$\begin{aligned}\frac{d\eta}{dt} = {} & \mathcal{A}_s\eta + \mathcal{B}_s\big(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v] + \cdots \\ & \qquad + \epsilon^k[p_k(\eta) + Q_k(\eta)v]\big) \\ & + f_s\big(\eta, \epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) + \cdots + \epsilon^k\Sigma^k(\eta,u)\big) \\ & + L\big(y - \big[\mathcal{C}\eta + \mathcal{C}\{\Sigma^0(\eta,u) + \epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) \\ & \qquad + \cdots + \epsilon^k\Sigma^k(\eta,u)\}\big]\big) \\ u = {} & p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v] + \cdots \\ & + \epsilon^k[p_k(\eta) + Q_k(\eta)v].\end{aligned} \tag{33}$$

We note that the static component of the above controller does not use feedback of the fast state vector x_f in order to avoid destabilization of the fast modes of the closed-loop system. Assumption 4 states the desired control objectives under the controller of Eq. (33).

Assumption 4. The finite-dimensional output feedback controller of the form of Eq. (33) exponentially stabilizes the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form and ensures that its outputs $y_s^i(t)$, $i = 1, \ldots, l$, are the solutions of a known l-dimensional ODE system of the form $\phi(y_s^{i(r_i)}, y_s^{i(r_i-1)}, \ldots, y_s^i, v) = 0$, where ϕ is a vector function and r_i is an integer.

Theorem 1 provides a precise characterization of the stability and closed-loop transient performance enforced by the controller of Eq. (33) in the closed-loop PDE system (the proof is given in the Appendix).

THEOREM 1. *Consider the PDE system of Eq.* (8), *for which Assumptions* 1 *and* 4 *hold. Then, there exist positive real numbers* $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\tilde{\epsilon}^*$ *such that if* $|x_s(0)| \leq \tilde{\mu}_1, \|x_f(0)\|_2 \leq \tilde{\mu}_2$, *and* $\epsilon \in (0, \tilde{\epsilon}^*]$, *then the controller of Eq.* (33):

(a) *guarantees exponential stability of the closed-loop system, and*

(b) *ensures that the outputs of the closed-loop system satisfy for all* $t \in [t_b, \infty)$

$$y^i(t) = y_s^i(t) + O(\epsilon^{k+1}), \qquad i = 1, \dots, l, \tag{34}$$

where $y_s^i(t)$ *is the ith output of the* $O(\epsilon^{k+1})$ *approximation of the closed-loop inertial form.*

Remark 8. The construction of the state feedback law of Eq. (31), to ensure that the control objectives stated in Assumption 4 are enforced in the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form, can be performed following a sequential procedure. Specifically, the component $u_0 = p_0(x_s) + Q_0(x_s)v$ can be initially synthesized on the basis of the $O(\epsilon)$ approximation of the inertial form (Eq. (19)); then the component $u_1 = p_1(x_s) + Q_1(x_s)v$ can be synthesized on the basis of the $O(\epsilon^2)$ approximation of the inertial form. In general, at the kth step, the component $u_k = p_k(x_s) + Q_k(x_s)v$ can be synthesized on the basis of the $O(\epsilon^k)$ approximation of the inertial form (Eq. 26). The synthesis of $[p_\nu(x_s), Q_\nu(x_s)]$, $\nu = 0, \dots, k$, can be performed, at each step, utilizing standard geometric control methods for nonlinear ODEs (see [13], for example).

Remark 9. The implementation of the controller of Eq. (33) requires to explicitly compute the vector function $\Sigma^k(\eta, u)$. However, $\Sigma^k(\eta, u)$ has an infinite-dimensional range and therefore cannot be implemented in practice. Instead a finite-dimensional approximation of $\Sigma^k(\eta, u)$, say $\Sigma_t^k(\eta, u)$, can be derived by keeping the first $\overline{m}$ elements of $\Sigma^k(\eta, u)$ and neglecting the remaining infinite ones. Clearly, as $\overline{m} \to \infty$, $\Sigma_t^k(\eta, u)$ approaches $\Sigma^k(\eta,u)$. This implies that by picking $\overline{m}$ to be sufficiently large, the controller of Eq. (33) with $\Sigma_t^k(\eta, u)$ instead of $\Sigma^k(\eta, u)$ guarantees stability and enforces the requirement of Eq. (34) in the closed-loop infinite-dimensional system.

Remark 10. The proposed control methodology was implemented through simulations on a packed-bed reactor modeled by two quasi-linear parabolic PDEs [8]. In particular, it was shown that a finite-dimensional output feedback controller of dimension 4, of the form of Eq. (33) with an $O(\epsilon^2)$ approximation for $\Sigma(\eta, \epsilon, u)$ and $\overline{m} = 3$ provides a superior performance compared to a finite-dimensional output feedback controller of the same dimension, with an $O(\epsilon)$ approximation for $\Sigma(\eta, \epsilon, u)$.

6. CONCLUSIONS

In this work, we developed a methodology for the synthesis of nonlinear finite-dimensional output feedback controllers for systems of quasi-linear parabolic PDEs, for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement. Combination of Galerkin's method with a novel procedure for the construction of AIMs was used, for the derivation of ODE systems of dimension equal to the number of slow modes, that yield solutions which are close, up to a desired accuracy, to the ones of the PDE system, for almost all times. These ODE systems were used as the basis for the synthesis of nonlinear output feedback controllers that guarantee stability and enforce the output of the closed-loop system to follow up to a desired accuracy, a prespecified response for almost all times.

APPENDIX

Proof of Proposition 1. The proof of the proposition will be obtained in two steps. In the first step, we will show that the system of Eq. (16) is exponentially stable, provided that the initial conditions and ϵ are sufficiently small. In the second step, we will use the exponential stability property to prove closeness of solutions (Eq. 21).

Exponential Stability. First, the system of Eq. (16) with $u(t) \equiv 0$ can be equivalently written as

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + f_s(x_s, 0) + \left[f_s(x_s, x_f) - f_s(x_s, 0) \right] \\ \epsilon \frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon} x_f + \epsilon f_f(x_s, x_f). \end{aligned} \tag{35}$$

Let μ_1^*, μ_2^* with $\mu_1^* \geq a_4$ be two positive real numbers such that if $|x_s| \leq \mu_1^*$ and $\|x_f\|_2 \leq \mu_2^*$, then there exist positive real numbers (k_1, k_2, k_3) such that

$$\begin{aligned} \left| f_s(x_s, x_f) - f_s(x_s, 0) \right| &\leq k_1 \|x_f\|_2 \\ \left\| f_f(x_s, x_f) \right\|_2 &\leq k_2 |x_s| + k_3 \|x_f\|_2. \end{aligned} \tag{36}$$

Pick $\mu_1 < a_4 < \mu_1^*$ and $\mu_2 < \mu_2^*$. From Assumption 2 and the converse Lyapunov theorem for finite-dimensional systems [14, Theorem 4.10], we have that there exists a smooth Lyapunov function $V: \mathcal{H}_s \to \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(a_1, a_2, a_3, a_4, a_5)$, such that for all $x_s \in \mathcal{H}_s$ that satisfy $|x_s| \leq a_4$, the following conditions hold:

$$\begin{aligned} &a_1 |x_s|^2 \leq V(x_s) \leq a_2 |x_s|^2 \\ &\dot{V}(x_s) = \frac{\partial V}{\partial x_s} \left[\mathcal{A}_s x_s + f_s(x_s, 0) \right] \leq -a_3 |x_s|^2 \\ &\left| \frac{\partial V}{\partial x_s} \right| \leq a_5 |x_s|. \end{aligned} \tag{37}$$

From the global exponential stability property of the fast subsystem of Eq. (17) and the converse Lyapunov theorem for infinite-dimensional systems [21, 22], we have that there exists a Lyapunov functional $W: \mathcal{H}_f \to \mathbb{R}_{\geq 0}$ and a set of positive real numbers (b_1, b_2, b_3, b_4), such that for all $x_f \in \mathcal{H}_f$ the following conditions hold:

$$\begin{aligned} &b_1 \|x_f\|_2^2 \leq W(x_f) \leq b_2 \|x_f\|_2^2 \\ &\dot{W}(x_f) = \frac{1}{\epsilon} \frac{\partial W}{\partial x_f} \mathcal{A}_f^{\epsilon} x_f \leq -\frac{b_3}{\epsilon} \|x_f\|_2^2 \\ &\left\| \frac{\partial W}{\partial x_f} \right\|_2 \leq b_4 \|x_f\|_2. \end{aligned} \tag{38}$$

Consider now the smooth function $L: \mathcal{H}_s \times \mathcal{H}_f \to \mathbb{R}_{\geq 0}$,

$$L(x_s, x_f) = V(x_s) + W(x_f) \tag{39}$$

as a Lyapunov function candidate for the system of Eq. (35). From Eq. (37) and Eq. (38), we have that $L(x_s, x_f)$ is positive definite and proper (tends to $+\infty$ as $|x_s| \to \infty$, or $\|x_f\|_2 \to \infty$), with respect to its arguments. Computing the time-derivative of L along the trajectories of this system, and using the bounds of Eq. (37) and Eq. (38) and the estimates of Eq. (36), the following expressions can be easily obtained:

$$
\begin{aligned}
\dot{L}(x_s, x_f) &= \frac{\partial V}{\partial x_s}\dot{x} + \frac{\partial W}{\partial x_f}\dot{x}_f \\
&\leq \frac{\partial V}{\partial x_s}\left[\mathcal{A}_s x_s + f_s(x_s, 0)\right] + \frac{\partial V}{\partial x_s}\left[f_s(x_s, x_f) - f_s(x_s, 0)\right] \\
&\quad + \frac{\partial W}{\partial x_f}\mathcal{A}_f^{\epsilon} x_f + \frac{\partial W}{\partial x_f} f_f(x_s, x_f) \\
&\leq -a_3|x_s|^2 + a_5 k_1 |x_s| \, \|x_f\|_2 - \frac{b_3}{\epsilon}\|x_f\|_2^2 \\
&\quad + b_4\|x_f\|_2\left(k_2|x_s| + k_3\|x_f\|_2\right) \\
&\leq -a_3|x_s|^2 + (a_5 k_1 + b_4 k_2)|x_s| \, \|x_f\|_2 - \left(\frac{b_3}{\epsilon} - b_4 k_3\right)\|x_f\|_2 \\
&\leq -\left[|x_s| \quad \|x_f\|_2\right]\begin{bmatrix} a_3 & -\dfrac{a_5 k_1 + b_4 k_2}{2} \\ -\dfrac{a_5 k_1 + b_4 k_2}{2} & \dfrac{b_3}{\epsilon} - b_4 k_3 \end{bmatrix}\begin{bmatrix} |x_s| \\ \|x_f\|_2 \end{bmatrix}.
\end{aligned}
\tag{40}
$$

Defining $\epsilon_1 = a_3 b_3/(a_3 b_4 k_3 + ((a_5 k_1 + b_4 k_2)/2)^2)$, we have that if $\epsilon \in (0, \epsilon_1)$ then $\dot{L}(x_s, x_f) < 0$, which from the properties of L directly implies that the state of the system of Eq. (35) is exponentially stable, i.e., there exists a positive real number σ such that

$$
\begin{bmatrix} |x_s| \\ \|x_f\|_2 \end{bmatrix} \leq e^{-\sigma t}\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}. \tag{41}
$$

Closeness of Solutions. First, we define the error coordinate $e_f(\tau) = x_f(\tau) - \bar{x}_f(\tau)$. Differentiating $e_f(\tau)$ with respect to τ, the following dynamical system can be obtained:

$$
\frac{\partial e_f}{\partial \tau} = \mathcal{A}_{f\epsilon} e_f + \epsilon f_f\left(x_s, e_f + \bar{x}_f\right). \tag{42}
$$

Referring to the above system with $\epsilon = 0$, we have from the properties of the unbounded operator $\mathcal{A}_{f\epsilon}$ and the converse theorem of [21, 22], that there exists a Lyapunov functional $\overline{W} : \mathcal{H}_f \to \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$, such that for all $e_f \in \mathcal{H}_f$ the following conditions hold:

$$\begin{gathered} \bar{b}_1 \| e_f(\tau) \|_2^2 \leq \overline{W}\big(e_f(\tau)\big) \leq \bar{b}_2 \| e_f(\tau) \|_2^2 \\ \frac{\partial \overline{W}}{\partial \tau} = \frac{\partial \overline{W}}{\partial e_f} \mathcal{A}_f^{\epsilon} e_f \leq -\bar{b}_3 \| e_f(\tau) \|_2^2 \\ \left\| \frac{\partial \overline{W}}{\partial e_f} \right\|_2 \leq \bar{b}_4 \| e_f(\tau) \|_2 . \end{gathered} \tag{43}$$

Computing the time-derivative of $\overline{W}(e_f)$ along the trajectories of the system of Eq. (42) and using that $\|f_f(x_s, e_f + \bar{x}_f)\|_2 \leq k_4 \|e_f\|_2 + k_5$, where k_4, k_5 are positive real numbers (which follows from the fact that the states $(x_s, \bar{x}_f)$ are bounded), we have

$$\begin{aligned} \frac{\partial \overline{W}}{\partial \tau} &\leq -\bar{b}_3 \|e_f\|_2^2 + \epsilon \bar{b}_4 \|e_f\|_2 \big(k_4 \|e_f\|_2 + k_5\big) \\ &\leq -\big(\bar{b}_3 - \epsilon \bar{b}_4 k_4\big) \|e_f\|_2^2 + \epsilon \bar{b}_4 k_5 \|e_f\|_2 . \end{aligned} \tag{44}$$

Set $\epsilon_2 = \bar{b}_3 / \bar{b}_4 k_4$ and $\epsilon^* = \min\{\epsilon_1, \epsilon_2\}$. From the above inequality, using Theorem 4.10 in [14], we have that if $\epsilon \in (0, \epsilon^*)$, the following bound holds for $\|e_f(\tau)\|_2$ for all $t \in [0, \infty)$,

$$\| e_f(\tau) \|_2 \leq K_3 \| e_f(0) \|_2 e^{-a_3(t/\epsilon)} + \epsilon \delta_{ef}, \tag{45}$$

where δ_{ef} is a positive real number. From the above inequality and the fact that $\|e_f(0)\|_2 = 0$, the estimate $x_f(t) = \bar{x}_f(t/\epsilon) + O(\epsilon)$ follows directly.

Defining the error coordinate $e_s(t) = x_s(t) - \bar{x}_s(t)$ and differentiating $e_s(t)$ with respect to time, the following system can be obtained:

$$\frac{de_s}{dt} = \mathcal{A}_s e_s + f_s\big(\bar{x}_s + e_s, x_f\big) - f_s(\bar{x}_s). \tag{46}$$

The representation of the system of Eq. (46) in the fast time-scale τ takes the form

$$\frac{de_s}{d\tau} = \epsilon \big[\mathcal{A}_s e_s + f_s\big(\bar{x}_s + e_s, x_f\big) - f_s(\bar{x}_s) \big], \tag{47}$$

where $(e_s, \bar{x}_s)$ can be considered approximately constant, and thus using that $|e_s(0)| = 0$ and continuity of solutions for $e_s(\tau)$, the following bound can be written for $e_s(\tau)$ for all $\tau \in [0, \tau_b]$,

$$|e_s(\tau)| \leq \epsilon k_6 \tau_b, \qquad \forall \tau \in [0, \tau_b), \tag{48}$$

where k_6 is a positive real number and $\tau_b = t_b/\epsilon = O(1)$, with $t_b = O(\epsilon) > 0$ is the time required for $x_f(t)$ to approach $\bar{x}_p(t)$, i.e., $\|x_f(t)\|_2 \leq k_7 \epsilon$ for $t \in [t_b, \infty)$, where k_7 is a positive real number. The system of Eq. (46) with $\bar{x}_s(t) = x_f(t) \equiv 0$ is exponentially stable (Assumption 2). Moreover, since $\bar{x}_s(t)$ decays exponentially, the system of Eq. (46) is also exponentially stable if $x_f(t) \equiv 0$. This implies that for the system

$$\frac{de_s}{dt} = \mathcal{A}_s e_s + f_s(\bar{x}_s + e_s, 0) - f_s(\bar{x}_s) \tag{49}$$

there exists a smooth Lyapunov function $\bar{V}: \mathcal{H}_s \to \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5)$, such that for all $e_s \in \mathcal{H}_s$ that satisfy $|e_s| \leq \bar{a}_4$ the following conditions hold:

$$\begin{aligned}
&\bar{a}_1 |e_s|^2 \leq \bar{V}(e_s) \leq \bar{a}_2 |e_s|^2 \\
&\dot{\bar{V}}(e_s) = \frac{\partial \bar{V}}{\partial e_s}\left[\mathcal{A}_s e_s + f_s(\bar{x}_s + e_s, 0) - f_s(\tilde{x}_s)\right] \leq -\bar{a}_3 |e_s|^2 \\
&\left|\frac{\partial \bar{V}}{\partial e_s}\right| \leq \bar{a}_5 |e_s|.
\end{aligned} \tag{50}$$

Computing the time-derivative of $\bar{V}(e_s)$ along the trajectories of the system of Eq. (46) and using that for $t \in [t_b, \infty)$ $|f_s(\bar{x}_s + e_s, x_f) - f_s(\bar{x}_s + e_s, 0)| \leq k_8 \|x_f\|_2 \leq k_7 k_8 \epsilon$, where k_8 is a positive real number (which follows from the fact that the states $(\bar{x}_s, x_f)$ are bounded), we have for all $t \in [t_b, \infty)$

$$\dot{\bar{V}}(e_s) \leq -\bar{a}_3 |e_s|^2 + k_7 k_8 \bar{a}_5 |e_s| \epsilon. \tag{51}$$

From the above inequality, using Theorem 4.10 in [14], and that $|e_s(t_b)| = O(\epsilon)$, we have that the following bound holds for $|e_s(t)|$ for all $t \in [t_b, \infty)$,

$$|e_s(t)| \leq \epsilon \delta_{e_s}, \tag{52}$$

where δ_{e_s} is a positive real number. From inequalities of Eqs. (48)–(52), the estimate $x_s(t) = \bar{x}_s(t) + O(\epsilon)$ for all $t \geq 0$ follows directly. ∎

Proof of Proposition 2. The proof of the proposition will be obtained following a two-step approach similar to the one used in the proof of Proposition 1.

Exponential Stability. This part of the proof of the proposition is completely analogous to the proof of exponential stability in the case of Proposition 1, and thus, it will be omitted for brevity.

Closeness of Solutions. From the first part of the proof, we have that exponential stability is guaranteed provided that $\epsilon \in (0, \bar{\epsilon}^*)$, where $\bar{\epsilon}^*$ is a positive real number. Defining the error coordinate $\tilde{e}_s(t) = x_s(t) - \tilde{x}_s(t)$ and differentiating $\tilde{e}_s(t)$ with respect to time, the following system can be obtained,

$$\frac{d\tilde{e}_s}{dt} = \mathcal{A}_s \tilde{e}_s + f_s\left(\tilde{x}_s + \tilde{e}_s, x_f\right) - f_s\left(\tilde{x}_s, \tilde{x}_f\right), \tag{53}$$

where $\tilde{x}_f = \Sigma^0(\tilde{x}_s) + \epsilon\Sigma^1(\tilde{x}_s) + \epsilon^2\Sigma^2(\tilde{x}_s) + \cdots + \epsilon^k\Sigma^k(\tilde{x}_s)$. From Assumption 3 and the fact that $\tilde{x}_s(t)$ decays exponentially to zero, we have that the system

$$\frac{d\tilde{e}_s}{dt} = \mathcal{A}_s \tilde{e}_s + f_s\left(\tilde{x}_s + \tilde{e}_s, \tilde{x}_f\right) - f_s\left(\tilde{x}_s, \tilde{x}_f\right) \tag{54}$$

is exponentially stable, which implies that there exists a smooth Lyapunov function $\tilde{V}: \mathcal{H}_s \to \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5)$, such that for all $\tilde{e}_s \in \mathcal{H}_s$ that satisfy $|\bar{e}_s| \leq \tilde{a}_4$, the following conditions hold:

$$\begin{aligned}
&\tilde{a}_1|\tilde{e}_s|^2 \leq \tilde{V}(\tilde{e}_s) \leq \tilde{a}_2|\tilde{e}_s|^2 \\
&\dot{\tilde{V}}(\tilde{e}_s) = \frac{\partial\tilde{V}}{\partial\tilde{e}_s}\left[\mathcal{A}_s\tilde{e}_s + f_s\left(\tilde{x}_s + \tilde{e}_s, \tilde{x}_f\right) - f_s\left(\tilde{x}_s, \tilde{x}_f\right)\right] \leq -\tilde{a}_3|\tilde{e}_s|^2 \\
&\left|\frac{\partial\tilde{V}}{\partial\tilde{e}_s}\right| \leq \hat{a}_5|\tilde{e}_s|.
\end{aligned} \tag{55}$$

Computing the time-derivative of $\tilde{V}(\tilde{e}_s)$ along the trajectories of the system of Eq. (53) and using that for $t \in [0, \infty)$ $|f_s(\tilde{x}_s + \tilde{e}_s, x_f) - f_s(\tilde{x}_s + \tilde{e}_s, \tilde{x}_f)| \leq (\tilde{k}_1\epsilon^{k+1} + \tilde{k}_2 e^{\lambda_1 t/\epsilon})$, where $\tilde{k}_1, \tilde{k}_2$ are positive real numbers and λ_i is a negative real number, we have for all $t \in [0, \infty)$

$$\dot{\tilde{V}}(\tilde{e}_s) \leq -\tilde{a}_3|\tilde{e}_s|^2 + \left(\tilde{k}_1\epsilon^{k+1} + \tilde{k}_2 e^{\lambda_1 t/\epsilon}\right)\tilde{a}_5|\tilde{e}_s|. \tag{56}$$

From the above inequality, using Theorem 4.10 in [14] and the fact that $|\tilde{e}_s(0)| = 0$, we have that the following bound holds for $|\tilde{e}_s(t)|$ for all $t \in [0, \infty)$,

$$|\tilde{e}_s(t)| \leq \bar{K}e^{\lambda_1 t/\epsilon} + \tilde{K}\epsilon^{k+1}, \tag{57}$$

where $\bar{K}, \tilde{K}$ are positive real numbers. Since the term $\bar{K}e^{\lambda_1 t/\epsilon}$ vanishes outside the interval $[0, t_b]$ (where t_b is the time required for x_f to approach $\tilde{x}_f$), it follows from Eq. (57) that for all $t \in [t_b, \infty)$

$$|\tilde{e}_s(t)| \leq \tilde{K}\epsilon^{k+1}. \tag{58}$$

From the above inequality, the estimate $x_s(t) = \tilde{x}_s(t) + O(\epsilon^{k+1})$, for $t \geq t_b$, follows directly. ∎

Proof of Theorem 1. Substituting the output feedback controller of Eq. (33) into the system of Eq. (16), we get

$$\begin{aligned}
\frac{d\eta}{dt} &= \mathcal{A}_s\eta + \mathcal{B}_s\big(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v] \\
&\qquad + \cdots \\
&\qquad + \epsilon^k[p_k(\eta) + Q_k(\eta)v]\big) \\
&\quad + f_s(\eta, \epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)) \\
&\quad + L\big(y - [\mathcal{C}\eta + \mathcal{C}\epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)]\big) \\
\frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s\big(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v] + \cdots \\
&\qquad + \epsilon^k[p_k(\eta) + Q_k(\eta)v]\big) + f_s(x_s, x_f) \\
\epsilon\frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon}x_f + \epsilon\,\mathcal{B}_f\big(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v + \cdots \\
&\qquad + \epsilon^k[p_k(\eta) + Q_k(\eta)v]\big) + \epsilon f_f(x_s, x_f) \\
y^i &= \mathcal{C}x_s + \mathcal{C}x_f, \qquad i = 1, \ldots, l.
\end{aligned} \tag{59}$$

Performing a two-time-scale decomposition in the above system, the fast subsystem takes the form

$$\frac{\partial x_f}{\partial \tau} = \mathcal{A}_{f\epsilon}x_f \tag{60}$$

which is exponentially stable. Furthermore, the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form is given by

$$\begin{aligned}\frac{d\eta}{dt} &= \mathcal{A}_s\eta + \mathcal{B}_s\big(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v] + \cdots \\ &\qquad + \epsilon^k[p_k(\eta) + Q_k(\eta)v]\big) \\ &\quad + f_s\big(\eta, \epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) + \cdots + \epsilon^k\Sigma^k(\eta,u)\big) \\ &\quad + L\big(y - \big[\mathcal{C}\eta + \mathcal{C}\Sigma^0(\eta,u) + \epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) + \cdots \\ &\qquad + \epsilon^k\Sigma^k(\eta,u)\big]\big) \\ \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s\big(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta) + Q_1(\eta)v] + \cdots \\ &\qquad + \epsilon^k[p_k(\eta) + Q_k(\eta)v]\big) \\ &\quad + f_s\big(x_s, \epsilon\Sigma^1(x_s,u) + \epsilon^2\Sigma^2(x_s,u) + \cdots + \epsilon^k\Sigma^k(x_s,u)\big) \\ y_s^i &= \mathcal{C}x_s + \mathcal{C}\big[\epsilon\Sigma^1(x_s,u) + \epsilon^2\Sigma^2(x_s,u) + \cdots + \epsilon^k\Sigma^k(x_s,u)\big], \\ &\qquad i = 1,\ldots,l. \end{aligned} \tag{61}$$

Referring to the above closed-loop ODE system, Assumption 4 yields that it is exponentially stable and the output y_s^i, $i = 1,\ldots,l$, changes in a prespecified manner. A direct application of the result of Proposition 2 yields that there exist positive real numbers $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\epsilon}^*$ such that if $|x_s(0)| \leq \tilde{\mu}_1$, $\|x_f(0)\|_2 \leq \tilde{\mu}_2$ and $\epsilon \in (0, \tilde{\epsilon}^*]$, such that the closed-loop infinite-dimensional system is exponentially stable and the relation of Eq. (34) holds. ∎

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