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Robust control of multivariable two-time-scale nonlinear systems

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This paper focuses on the synthesis of well-conditioned multivariable robust controllers for a broad class of multi-input multi-output two-time-scale nonlinear systems with time-varying uncertain variables, modeled within the framework of singular perturbations. The proposed controller stabilizes the fast dynamics, guarantees boundedness of trajectories, and ensures that the ultimate discrepancy between the outputs and the external reference inputs in the closed-loop system can be made arbitrarily small by an appropriate choice of controller parameters, provided that the singular perturbation parameter is sufficiently small. The developed control method is tested, through simulations, on a fluidized catalytic cracking reactor. © 1997 Elsevier Science Ltd

All practical control systems must be robust with respect to uncertainty. Uncertain time-varying variables arise naturally in chemical engineering applications from unknown or partially known process parameters (e.g. heat transfer coefficients, kinetic constants, etc.) and unmeasured disturbance inputs (e.g. concentration of inlet streams). It is well-established that controllers that guarantee offsetless output tracking and stability in the nominal closed-loop system may lead to poor transient performance, offset and even closed-loop instability in the presence of uncertain variables. The traditional approach followed to ensure asymptotic offsetless rejection of uncertain variables is to incorporate integral action in the controller. This approach is adequate in the case of *constant* uncertain variables but it may lead to severe failure in the presence of *time-varying* uncertainty.

Motivated by this, the design of robust controllers that are capable of coping with uncertain variables has received considerable attention in the past. Research initially focused on the development of robust control methods for linear systems in the frequency-domain[1], including H_∞, μ-synthesis, etc. More recently, the state-space counterparts of the H_∞ frequency-domain results have been developed[2]. This motivated research on the extension of H_∞ control methods to certain classes of uncertain nonlinear systems. In this direction, theoretical results have been derived[3,4], but their practical applicability is still in question because the explicit construction of the controllers requires the analytical solution of nonlinear partial differential equations. The interested reader may refer to Allgöwer *et al.*[5] for a critical analysis on the applicability of nonlinear H_∞ control methods to engineering applications.

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An alternative approach for the synthesis of robust control systems for linear/nonlinear uncertain systems is based on Lyapunov's direct method. The basic idea is to design a controller, utilizing the knowledge of certain bounding functions on the size of the uncertainty, so that the time-derivative of an appropriate Lyapunov function calculated along the trajectories of the uncertain closed-loop system is negative definite as long as the state of the system is larger than a constant which can be made arbitrarily small by a suitable choice of controller parameters. This guarantees that the ultimate discrepancy between the output of the closed-loop system and the reference input is arbitrarily small. This idea was originally proposed in Corless and Leitman[6] and was further explored for robust controller design[7,8]. For single-input single-output input-output linearizable nonlinear systems, combination of this approach with geometric control schemes was studied[9–11].

Another important feature of many engineering applications is that they involve physicochemical phenomena occurring in different time-scales (e.g. catalytic reactors, biochemical reactors, flexible mechanical systems, etc.). It is well-established that a direct application of standard nonlinear control algorithms to multiple-time-scale systems, neglecting the presence of time-scale multiplicity, may lead to controller ill-conditioning and/or closed-loop instability[12]. Singular perturbation methods have proven to be a successful analytic tool for stability analysis and the synthesis of well-conditioned controllers for multiple-time-scale processes[13]. We employed a combination of singular perturbations and geometric control methods to synthesize well-conditioned feedback controllers for two-time-scale nonlinear systems that enforce a prespecified input/output response in the closed-loop system, provided that the time-scale separation is sufficiently large[12]. This methodology was extended[14] within a feedforward/feedback control framework to synthesize well-conditioned controllers that achieve an arbitrary degree of elimination of the effect of measurable disturbances on the output of the process for all times.

The combination of singular perturbations and control methods for uncertain systems, for the synthesis of well-conditioned controllers for stabilization of uncertain single-input single-output singularly perturbed systems, has been studied[15–17]. In Christofides and Daoutides[18], robust control laws were synthesized that ensure restricted (i.e. the initial conditions, uncertainty, and rate of change of uncertainty should be sufficiently small) asymptotic output tracking for nonlinear single-input single-output singularly perturbed systems, as long as the time-scale separation is sufficiently large. In Christofides and Teel[19], a robustness result for the input-to-state stability property with respect to singular perturbations was established and used in Christofides *et al.*[20] to prove that, under appropriate assumptions on the inverse dynamics of the reduced system, the controller derived in Christofides and Daoutides[18] ensures semi-global (i.e. the initial conditions, the uncertainty, and the rate of change of uncertainty can be arbitrarily large) asymptotic output tracking for the same class of singularly perturbed systems.

In this work, we extend the robust output tracking control methodology previously proposed[20] to multi-input multi-output two-time-scale nonlinear systems, for which the fast dynamics may be unstable or singular. The uncertainty is allowed to be time-varying. The objective is to provide an explicit formula of a multivariable state feedback controller that guarantees boundedness of the trajectories and an arbitrarily small ultimate discrepancy between the outputs and the reference inputs in the closed-loop system by a suitable selection of controller parameters. The resulting controller is a continuous function of the state of the system, and its construction requires the knowledge of appropriate bounding functions on the size of the uncertain terms. In the case of two-time-scale systems with stable fast dynamics, our main result establishes a fundamental robustness property of a controller synthesized on the basis of the low-dimensional slow model, with respect to both the parametric uncertainty and the fast (unmodeled) dynamics. The performance and robustness properties of the developed robust controller design method are evaluated through simulations in an fluidized catalytic cracking reactor, modeled in the standard singularly perturbed form, with unknown heat of the combustion reaction.

Mathematical preliminaries

We consider multi-input multi-output two-time-scale nonlinear systems with the following state-space representation:

$$\begin{aligned} \dot{x} &= f_1(x,\theta(t)) + Q_1(x,\theta(t))z + G_1(x,\theta(t))u \\ \epsilon\dot{z} &= f_2(x,\theta(t)) + Q_2(x,\theta(t))z + G_2(x,\theta(t))u \\ y_i &= h_i(x), \quad i = 1,\dots,m \end{aligned} \tag{1}$$

where $x \in \Re^n$ and $z \in \Re^p$ denote vectors of state variables, $u = [u_1 \cdots u_m]^T \in \Re^m$ denotes the vector of manipulated inputs, $\theta(t) = [\theta_1(t) \cdots \theta g(t)] \in \Re^q$ denotes the vector of uncertain variables, $y_i \in \Re$, denotes the ith controlled output, and ϵ is a small positive parameter, which quantifies the degree of coupling between the fast and slow dynamical phenomena of the system. $f_1(x,\theta(t)), f_2(x,\theta(t))$ are sufficiently smooth vector functions, $Q_1(x,\theta(t))$, $Q_2(x,\theta(t))$, $G_1(x,\theta(t))$, $G_2(x,\theta(t))$ are sufficiently smooth matrices of appropriate dimensions, and $h_i(x)$, $i = 1,\dots,m$ are sufficiently smooth scalar fields. The notation and some definitions used in the paper are given in the Appendix.

The fact that ϵ multiplies the time-derivative of the state z allows decomposing the system of Equation (1) into separate lower-order systems evolving in different time-scales[13]. Defining a fast time-scale, $\tau = \frac{t}{\epsilon}$, and setting equal to zero, the following *fast subsystem* is obtained:

$$\frac{dz}{d\tau} = f_2(x,\theta) + Q_2(x,\theta)z + G_2(x,\theta)u \tag{2}$$

where x can be considered equal to its initial value $x(0)$ and θ can be thought of as constant. Assumption 1 states a stabilizability requirement on the fast subsystem.

Assumption 1. The pair $[Q_2(x,\theta)G_2(x,\theta)]$ is stabilizable uniformly in $x \in \Re^n, \theta \in \Re^q$, in the sense that there exists a sufficiently smooth matrix $K(x)$ such that the matrix $Q_2(x,\theta) + G_2(x,\theta)K(x)$ is Hurwitz uniformly in $x \in \Re^n, \theta \in \Re^q$.

The fact that (x,θ) can be considered as constant in the fast subsystem of Equation (2) implies that the matrix $K(x)$ can be designed utilizing standard pole placement or optimal control methods[13]. Assuming that the matrix $Q_2(x,\theta)$ is invertible uniformly in $x \in \Re^n$, $\theta \in \Re^q$ and setting $\epsilon = 0$, the following representation for the equilibrium manifold of the fast dynamics of the system of Equation (1) is obtained:

$$z_s = -[Q_2(x,\theta)]^{-1}[f_2(x,\theta) + G_2(x,\theta)u] \tag{3}$$

where z_s denotes a quasi-steady-state for the fast state vector z. Utilizing the above equation, the *reduced system* or *slow subsystem* takes the form:

$$\begin{aligned} \dot{x} &= F(x,\theta) + G(x,\theta)u \\ y_i^s &= h_i(x), i = 1,\dots,m \end{aligned} \tag{4}$$

where the superscript s in y_i^s denotes that this output is associated with the slow subsystem and

$$F(x,\theta) = f_1(x,\theta) - Q_1(x,\theta)[Q_2(x,\theta)]^{-1}f_2(x,\theta)$$
$$G(x,\theta) = G_1(x,\theta) - Q_1(x,\theta)[Q_2(x,\theta)]^{-1}G_2(x,\theta) \quad (5)$$

Robust controller design

Control objectives

In this work, we will address the problem of synthesizing well-conditioned (i.e. independent of the singular perturbation parameter ϵ) robust static state feedback laws of the form:

$$u = \mathcal{R}(x, v_i^{(k)}) + K(x)z \quad (6)$$

where $\mathcal{R}(x, v_i^{(k)})$ is a vector function, $v_i^{(k)}$ denotes the kth time derivative of the external reference input v_i, which is assumed to be a smooth function of time and $K(x)$ is a sufficiently smooth matrix, which preserve the two-time-scale property of the open-loop system of Equation (1) and enforce the following objectives in the closed-loop system:

1. Boundedness of the trajectories.
2. Output tracking of external reference inputs with arbitrary degree of asymptotic attenuation of the effect of uncertainty on the outputs.

The above requirements will be enforced in the closed-loop system for sufficiently small values of the singular perturbation parameter ϵ.

Assumptions

In the general case where $Q_2(x,\theta)$ is singular or unstable, a preliminary control law of the form[11,12]:

$$u = \tilde{u} + K(x)z \quad (7)$$

where $\tilde{u}$ denotes an auxiliary input and $K(x)$ is a sufficiently smooth matrix of dimension $m \times p$ can be used to stabilize the fast dynamics, and thus induce a well-defined stable quasi-steady-state for the states z. More specifically, substitution of the control law of Equation (7) into the singularly perturbed system of Equation (1) yields:

$$\dot{x} = f_1(x,\theta) + [Q_1(x,\theta) + G_1(x,\theta)K(x)]z + G_1(x,\theta)\tilde{u}$$
$$\epsilon\dot{z} = f_2(x,\theta) + [Q_2(x,\theta) + G_2(x,\theta)K(x)]z + G_2(x,\theta)\tilde{u}$$
$$y_i = h_i(x),\ i = 1, ..., m \quad (8)$$

From Assumption 1, $K(x)$ can be chosen so that the matrix $Q_2(x,\theta) + G_2(x,\theta)K(x)$ is Hurwitz, uniformly in $x \in \Re^n$, $\theta \in \Re^q$, and thus the fast dynamics of the above system is exponentially stable. The reduced system then takes the form:

$$\dot{x} = \tilde{F}(x,\theta) + \tilde{G}(x,\theta)\tilde{u}$$
$$\tilde{y}_i^s = h_i(x), \quad i = 1, ..., m \quad (9)$$

where

$$\tilde{F}(x,\theta) = f_1(x,\theta) - A(x,\theta)f_2(x,\theta)$$
$$\tilde{G}(x,\theta) = G_1(x,\theta) - A(x,\theta)G_2(x,\theta) \quad (10)$$

and $A(x,\theta) = [Q_1(x,\theta) + G_1(x,\theta)K(x)][Q_2(x,\theta) + G_2(x,\theta)K(x)]^{-1}$. Referring to the system of Equation (9) we will state three assumptions that will allow us to proceed with the robust controller design.

Assumption 2. Referring to the system of Equation (9), there exist a set of integers $(\bar{r}_1, \bar{r}_2, ..., \bar{r}_m)$ and a co-ordinate transformation $(\zeta, \eta) = T(x,\theta)$ such that the representation of the system, in the co-ordinates (ζ, η), takes the form:

$$\begin{aligned}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\
&\vdots \\
\dot{\zeta}_{\bar{r}_1-1}^{(1)} &= \zeta_{\bar{r}_1}^{(1)} \\
\dot{\zeta}_{\bar{r}_1}^{(1)} &= L_{\tilde{F}}^{\bar{r}_1} h_1(T^{-1}(\zeta,\eta,\theta)) + \sum_{j=1}^{m} L_{\tilde{G}^j} L_{\tilde{F}}^{\bar{r}_1-1} h_1(T^{-1}(\zeta,\eta,\theta))\tilde{u}_j \\
&\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} \\
&\vdots \\
\dot{\zeta}_{\bar{r}_m-1}^{(m)} &= \zeta_{\bar{r}_m}^{(m)} \\
\dot{\zeta}_{\bar{r}_m}^{(m)} &= L_{\tilde{F}}^{\bar{r}_m} h_m(T^{-1}(\zeta,\eta,\theta)) + \sum_{j=1}^{m} L_{\tilde{G}^j} L_{\tilde{F}}^{\bar{r}_m-1} h_m(T^{-1}(\zeta,\eta,\theta))\tilde{u}_j \\
\dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\
\dot{\eta}_{n-\sum_i \bar{r}_i} &= \Psi_{n-\sum_i \bar{r}_i}(\zeta,\eta,\theta,\dot{\theta}) \\
\tilde{y}_i^s &= \zeta_1^{(i)},\ i = 1, ..., m
\end{aligned} \quad (11)$$

where $x = T^{-1}(\zeta,\eta,\theta)$, $\zeta = [\zeta^{(1)} ... \zeta^{(m)}]^T$, $\eta = [\eta_1 ... \eta_{n-\sum_i \bar{r}_i}]^T$, and $\tilde{G}^j$ denotes the jth column vector of the matrix $\tilde{G}(x,\theta)$. Moreover, for each $\theta \in \Re^q$, the states ζ, η are bounded if and only if the state x is bounded.

Assumption 2 states that the nominal system of Equation (9) is input/output linearizable and also incorporates the matching condition of our methodology, namely that a direct time-differentiation of the output y_i up to order $\bar{r}_i - 1$ yields a set of equations which are independent of the vector of uncertain variables θ. We note that, from an application point of view, this matching condition is less restrictive from the standard one imposed[6], which restricts $\theta(t)$ to enter the system in the same differential equations as u. The consequence of this generalization is that the η-subsystem depends on $\dot{\theta}(t)$, which implies that we need to assume that $\dot{\theta}(t)$ exists and is bounded (cf. Theorem 1).

Referring to the system of Equation (11), we will assume, for simplicity, that the matrix:

$$\tilde{C}(x,\theta)=\begin{bmatrix} L_{\tilde{G}^1}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(x,\theta) & \dots & L_{\tilde{G}^m}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(x,\theta) \\ \vdots & \dots & \vdots \\ L_{\tilde{G}^1}L_{\tilde{F}}^{\tilde{r}_m-1}h_m(x,\theta) & \dots & L_{\tilde{G}^m}L_{\tilde{F}}^{\tilde{r}_m-1}h_m(x,\theta) \end{bmatrix} \quad (12)$$

is non-singular uniformly in $x \in \Re^n, \theta \in \Re^q$ (see Remark 6 for a discussion on how to handle the case of singular $\tilde{C}(x,\theta)$).

Assumption 3. The dynamical system:

$$\begin{aligned} \dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\sum_i \tilde{r}_i} &= \Psi_{n-\sum_i \tilde{r}_i}(\zeta,\eta,\theta,\dot{\theta}) \end{aligned} \quad (13)$$

is input-to-state stable with respect to $\zeta, \theta, \dot{\theta}$.

The above assumption is more restrictive compared to the standard one of asymptotic stability of the zero dynamics without inputs[6]. However, it is necessary in order to prove that the states of the closed-loop system are bounded for initial conditions, uncertainty and rate of change of uncertainty arbitrarily large (see also Teel and Praly[21]). Furthermore, this assumption could be relaxed to local asymptotic stability of the η-subsystem with $(\zeta, \theta, \dot{\theta})$ identically equal to zero, leading to local stability results for the closed-loop system (see Remark 2).

We will now quantify possible knowledge about the uncertainty by assuming the existence of known bounding functions that capture the size of the uncertain terms in the system of Equation (11). To this end, we define the variable $\tilde{\delta}(x,\theta)$ as:

$$\tilde{\delta}(x,\theta) = [\tilde{F}(x,\theta) - \tilde{F}_{nom}(x)] \quad (14)$$

where $\tilde{F}_{nom}(x)$ denotes the vector field resulting from $\tilde{F}(x,\theta)$ by setting θ equal to its nominal value θ_0.

Using the fact that $L_{\tilde{F}}^{k-1}h_i(x) = L_{\tilde{F}_{nom}}^{k}h_i(x)$, $i=1,\dots,m$, $k=1,\dots,\tilde{r}_i$, Equation (11) can be written as follows:

$$\begin{aligned} \dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\ &\vdots \\ \dot{\zeta}_{\tilde{r}_1-1}^{(1)} &= \zeta_{\tilde{r}_1}^{(1)} \\ \dot{\zeta}_{\tilde{r}}^{(1)} &= L_{\tilde{F}_{nom}}^{\tilde{r}_1}h_1(T^{-1}(\zeta,\eta,\theta)) \\ &\quad + \sum_{j=1}^{m} L_{\tilde{G}^j}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(T^{-1}(\zeta,\eta,\theta))\tilde{u}_j \\ &\quad + L_{\tilde{\delta}}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(T^{-1}(\zeta,\eta,\theta)) \\ &\vdots \\ \dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} \\ &\vdots \\ \dot{\zeta}_{\tilde{r}_m}^{(m)} &= \zeta_{\tilde{r}_m}^{(m)} \\ \dot{\zeta}_{\tilde{r}_m}^{(m)} &= L_{\tilde{F}_{nom}}^{\tilde{r}_m}h_m(T^{-1}(\zeta,\eta,\theta)) \\ &\quad + \sum_{j=1}^{m} L_{\tilde{G}^j}L_{\tilde{F}}^{\tilde{r}_m-1}h_m(T^{-1}(\zeta,\eta,\theta))\tilde{u}_j \\ &\quad + L_{\tilde{\delta}}L_{\tilde{F}}^{\tilde{r}-1}h_\mu(T^{-1}(\zeta,\eta,\theta)) \\ \dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\sum_i \tilde{r}_i} &= \Psi_{n-\sum_i \tilde{r}_i}(\zeta,\eta,\theta,\dot{\theta}) \\ \tilde{y}_i^s &= \zeta_1^{(i)}, \quad i=1,\dots,m \end{aligned} \quad (15)$$

where $\zeta_k^{(i)} = L_{\tilde{F}_{nom}}^{k-1}h_1(T^{-1}(\zeta_1)\eta_s$, $i=1,\dots,m$, $k=1,\dots,\tilde{r}\theta_L)$.

Assumption 4. The terms $L_{\tilde{G}^j}L_{\tilde{F}}^{\tilde{r}_i-1}h_i(x)$, $i \in [1,m]$, have the same known constant sign, sgn_j, for all $x \in \Re^n$, $\theta \in \Re^q$, for each $j \in [1,m]$, (some of these terms may be zero, but the requirement that $\tilde{C}(x,\theta)$ is non-singular has to be satisfied), and there exist known functions $\tilde{c}_1(x,t)$, $\tilde{c}_{2j}(x,t)$, such that the following conditions hold:

$$\begin{aligned} &|[L_{\tilde{\delta}}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(x)\dots L_{\tilde{\delta}}L_{\tilde{F}}^{\tilde{r}_m-1}h_m(x)]^T| \le \tilde{c}_1(x,t) \\ &0 < \tilde{c}_{2j}(x,t) \le \min_{i\in[1,m]}|L_{\tilde{G}^j}L_{\tilde{F}}^{\tilde{r}_i-1}h_i(x)| \end{aligned} \quad (16)$$

and there exists a known positive real number b such that the $m \times m$ matrix $M(x,\theta,t) = \tilde{C}(x,\theta)\Delta(x,t)$, where $\Delta(x,t)$ is an $m \times m$ diagonal matrix whose (j,j)th element is of the form $\frac{\mathrm{sgn}_j}{\tilde{c}_{2j}}$, satisfies the property $1 \le \sigma_{min}\{M(x,\theta,t)\} \le \sigma_{max}\{M(x,\theta,t)\} \le b$, for all $x \in \Re^n$, $\theta \in \Re^q$, $t \ge 0$.

Assumption 4 requires the existence of a nonlinear function that captures the size of the additive uncertain terms in the system of Equation (11). This requirement is standard in all Lyapunov-based robust control methods[6,9,10], but in contrast we do not impose any growth conditions on the nonlinear function $\tilde{c}_1(x,t)$, such as quadratic growth, etc. Assumption 4 also requires the existence of a nonlinear function $\tilde{c}_{2j}(x,t)$ for each manipulated input u_j that lower bounds the m-coefficients $(L_{\tilde{G}^j}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(x),\dots,L_{\tilde{G}^j}L_{\tilde{F}}^{\tilde{r}_m-1}h_m(x))$ (note that this requirement does not have to be satisfied for the coefficients that are zero). This requirement is less restrictive than the ones imposed in Refs 6, 8, 10, where an upper bound (typically small) on the size of the uncertain terms that enter the system coupled with the manipulated inputs is assumed to exist.

Main result

We are now in position to proceed with the design of the auxiliary input $\tilde{u}$ to achieve the aforementioned control objectives in the closed-loop reduced system. Specifically, motivated by the requirement of output tracking, and the presence of additive and multiplicative uncertainties in the systems of Equation (11), we consider static state feedback laws of the form:

$$\tilde{u} = R_1(x,t)[p(x) + Q(x, v_i^{(k)}) + R_2(x,t)] \tag{17}$$

where $p(x)$, $R_2(x,t)$ are vector functions, $R_1(x,t)$ is a matrix, $Q(x, v_i^{(k)})$ is a column vector, and $v_i^{(k)}$ denotes the kth time derivative of the external reference input v^i, which is assumed to be a smooth function of time. The motivation for considering control laws of the form of Equation (17) is provided by the requirement of stabilization with output tracking in the nominal reduced system (term $p(x) + Q(x, v_i^{(k)})$), and the fact that the system in Equation (9) includes uncertainties entering coupled with the input (term $R_1(x,t)$) as well as additive ones (term $R_2(x,t)$).

Theorem 1 that follows provides an explicit formula of the controller that solves the control problem formulated in the previous section (the proof is given in the Appendix). To simplify the statement of the theorem, we set $\bar{v}_i = [v_i v_i^{(1)} \dots v_i^{(\bar{r}_i)}]^T$ and $\bar{v} = [\bar{v}_1^T \bar{v}_2^T \dots \bar{v}_m^T]^T$.

Theorem 1. Consider the uncertain singularly perturbed nonlinear system of Equation (1), for which assumptions 1, 2, 3 and 4 hold, and the static feedback law:

$$\begin{aligned} u = K(x)z + \Delta(x,t)\Bigg\{ & \sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i} \frac{\beta_{ik}}{\beta_{i\bar{r}_i}} (v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x)) \\ & + \sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i} \frac{\beta_{ik}}{\beta_{i\bar{r}_i}} (v_i^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1} h_i(x)) - (2+b)[\tilde{c}(x,t) \\ & + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i} \frac{\beta_{ik}}{\beta_{i\bar{r}_i}} (v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x))|]w(x,\phi) \Bigg\} \end{aligned} \tag{18}$$

where the feedback matrix $K(x)$ is such that the matrix $Q_2(x,\theta) + G_2(x,\theta)K(x)$ is Hurwitz uniformly in $x \in \Re^n$, $\theta \in \Re^q$, $\frac{\beta_{ik}}{\beta_{i\bar{r}_i}} = [\frac{\beta_{ik}^1}{\beta_{i\bar{r}_i}^1} \dots \frac{\beta_{ik}^m}{\beta_{i\bar{r}_i}^m}]^T$ are column vectors of parameters chosen so that the roots of the equation $\det(B(s)) = 0$, where $B(s)$ is an $m \times m$ matrix, whose (i,j)th element is of the form $\sum_{k=1}^{\bar{r}_i} \frac{\beta_{jk}^i}{\beta_{j\bar{r}_i}^i} s^{k-1}$, lie in the open left-half of the complex plane, and the vector function $w(x,\phi)$ is given by:

$$w(x,\phi) = \frac{\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i} \frac{\beta_{ik}}{\beta_{i\bar{r}_i}} (L_{\tilde{F}_{nom}}^{k-1} h_i(x) - v_i^{(k-1)})}{|\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i} \frac{\beta_{ik}}{\beta_{i\bar{r}_i}} (L_{\tilde{F}_{nom}}^{k-1} h_i(x) - v_i^{(k-1)})| + \phi} \tag{19}$$

where ϕ is an adjustable parameter. Then, for each set of positive real numbers $\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d$, there exists ϕ^* and for each $\phi \le \phi^*$, there exists $\epsilon^*(\phi)$, such that if $\phi \le \phi^*, \epsilon \le \epsilon^*(\phi)$ and $|x(0)| \le \delta_x$, $|z(0)| \le \delta_z$, $||\theta|| \le \delta_\theta$, $||\dot{\theta}|| \le \delta_{\dot{\theta}}$, $||\bar{v}|| \le \delta_{\bar{v}}$,

a) the state of the closed-loop system is bounded, and
b) the outputs of the closed-loop system satisfy:

$$\limsup_{t\to\infty} |y_i - v_i| \le d, \quad i = 1, \dots, m \tag{20}$$

Remark 1. Referring to the result of Theorem 1, we note that the dependence of the upper bound on the singular perturbation parameter, ϵ^*, on the adjustable parameter, ϕ, is due to the presence of ϕ on the dynamical system that describes the fast dynamics of the closed-loop system.

Remark 2. Theorem 1 establishes a semi-global type result (the initial conditions, uncertainty and rate of change of uncertainty can be in arbitrarily large compact sets), while it does not impose any kind of interconnection or growth conditions on the nonlinearities of the system. We note that if the co-ordinate transformation of Assumption 2 holds locally and the unforced zero dynamics

$$\begin{aligned} \dot{\eta}_1 &= \Psi_1(\zeta, 0, 0, 0) \\ &\vdots \\ \dot{\eta}_{n-\sum_i \bar{r}} &= \Psi_{n-\sum_i \bar{r}_i}(\zeta, 0, 0, 0) \end{aligned} \tag{21}$$

are locally exponentially stable, then the result of Theorem 1 holds locally (i.e. for sufficiently small initial conditions, uncertainty and rate of change of uncertainty).

Remark 3. Regarding the practical application of Theorem 1, one has to initially verify Assumptions 1-4 on the basis of the process model used for controller design, and utilize the synthesis formula of Equation (18) to derive the explicit form of the controller. Then, given the desired ultimate bound d, the proof of the theorem can be used to calculate ϕ^* from the desired $(\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d)$ and, in turn, the value ϵ^* for $\phi \le \phi^*$. If this ϵ^* is less than ϵ_p (the value of the singular perturbation parameter for the process), then d may need to be increased so that $\epsilon^* \ge \epsilon_p$. Of course, if ϵ_p is too large, there may be no value of d that works. On the other hand, the value of ϵ^*, calculated from the proof of the theorem, is typically conservative, and so it may be useful to check the appropriateness of ϕ^* (and d) through computer simulations (see the next section for an application of this procedure to the fluidized catalytic cracking process).

Remark 4. Referring to the control law of Equation (18), one can show that if the matrix $\tilde{C}(x,\theta)$ in the system of Equation (9) is independent of θ, then the following simplifications can be made: $\Delta(x,t) = \{\tilde{C}(x)\}^{-1}, b = 0$, and $[\tilde{c}_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i} \frac{\beta_{ik}}{\beta_{i\bar{r}_i}} (v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x))|] = \tilde{c}_1(x,t)$.

Remark 5. Referring to the control law of Equation (18), we note that the explicit expression of the term which is responsible for the attenuation of the effect of additive uncertainties is nonlinear not only in the x-co-ordinates but also in the (ζ, η) co-ordinates (see Equations (46) and (47)). This is in agreement with the robust control methods[6,9] and in contrast to the robust control methodology proposed in Arkun and Calvet[10], where the expression of the term $R_2(x, t)$ is linear in (ζ, η) co-ordinates, and is a consequence of the fact that we allow $\bar{c}_1(x, t)$ to be a general nonlinear function.

Remark 6. The singular perturbation formulation provides a natural setting for addressing robustness with respect to unmodeled dynamics[13]. In particular, if the open-loop fast subsystem of Equation (2) is exponentially stable uniformly in $x \in \Re^n, \theta \in \Re^q$, the controller of Equation (18) can be simplified to

$$u = \Delta(x,t)\left\{ \sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^k h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1} h_i(x)) - (2+b)[c_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{r_i} \frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^k h_i(x)) + ||w(x,\phi) \right\} \tag{22}$$

which can be explicitly synthesized utilizing information about the open-loop reduced system of Equation (4). In this case, the result of Theorem 1 establishes a fundamental robustness property of the controller of Equation (22), with respect to uniformly exponentially stable unmodeled dynamics, provided that they are sufficiently fast. This robustness property is very important in many practical applications, where fast dynamics, such as sensor and actuator dynamics, are usually neglected in the controller design.

Remark 7. Whenever the characteristic matrix $\tilde{C}(x, \theta)$ of the reduced system of Equation (9) is singular, one can utilize available dynamic extension algorithms[22] to design a dynamic state feedback law of the general form:

$$\begin{aligned} \dot{\xi} &= a(x,\xi) + b(x,\xi)\bar{v} \\ \tilde{u} &= c(x,\xi) + d(x,\xi)\bar{v} \end{aligned} \tag{23}$$

where $a(x, \xi), b(x, \xi), c(x, \xi), d(x, \xi)$ are vector functions of appropriate dimensions, $\bar{v} \in \Re^m$ denotes a vector of auxiliary inputs, such that the augmented system

$$\begin{aligned} \dot{\xi} &= a(x,\xi) + b(x,\xi)\bar{v} \\ \dot{x} &= \tilde{F}(x,\theta) + \tilde{G}(x,\theta)c(x,\xi) + \tilde{G}(x,\theta)d(x,\theta)\bar{v} \\ \tilde{y}_i^s &= h_i(x), \quad i = 1,\dots,m \end{aligned} \tag{24}$$

with $\tilde{y}^s \in \Re^m$ as output vector and $\bar{v} \in \Re^m$ as input vector possesses a non-singular characteristic matrix. Then, the robust control methodology of this paper can be directly applied to the system of Equation (24).

Application to a fluidized catalytic cracker

In this section, we illustrate the implementation of the developed control methodology on the industrially significant process of fluidized catalytic cracking (FCC) shown in *Figure 1*. The FCC unit consists of a cracking reactor, where the desired reactions include cracking of high boiling gas oil fractions into lighter hydrocarbons (e.g. gasoline) and the undesired include carbon formation reactions, and a regenerator, where the carbon removal reactions take place. Detailed discussions on the features of the FCC unit exist[23,24], and applications of linear control methods to the process can be found[25,26]. Under the following standard modeling assumptions:

- well-mixed reactive catalyst in the reactor
- small-size catalyst particles
- constant solid holdup in reactor and regenerator
- uniform and constant pressure in reactor and regenerator,

the process dynamic model takes the form[23]:

$$\begin{aligned} V_{ra}\frac{dC_{cat}}{dt} &= -60F_{rc}C_{cat} + 50R_{cf} \\ V_{ra}\frac{dC_{sc}}{dt} &= -60F_{rc}(C_{rc} - C_{sc}) + 50R_{cf} \\ V_{ra}\frac{dT_{ra}}{dt} &= 60F_{rc}(T_{rg} - T_{ra}) \\ &\quad + 0.875\frac{S_f}{S_c}D_{tf}R_{tf}(T_{fp} - T_{ra}) \\ &\quad + 0.875\frac{(-\Delta H_{fv})}{S_c}D_{tf}R_{tf} \\ &\quad + 0.5\frac{(-\Delta H_{cr})}{S_c}R_{oc} \\ V_{rg}\frac{dC_{rc}}{dt} &= 60F_{rc}(C_{sc} - C_{rc}) - 50R_{cb} \\ V_{rg}\frac{dT_{rg}}{dt} &= 60F_{rc}(T_{ra} - T_{rg}) \\ &\quad + 0.5\frac{S_a}{S_c}R_{ai}(T_{ai} - T_{rg}) \\ &\quad + 0.5(-\frac{\Delta H_{rg}}{S_c})R_{cb} \end{aligned} \tag{25}$$

where C_{cat}, C_{sc}, C_{rc} denote the concentrations of catalytic carbon on spent catalyst, the total carbon on spent catalyst, and carbon on regenerated catalyst, T_{ra}, T_{rg} denote the temperatures in the reactor and the regenerator, D_{tf} is the density of total feed, V_{ra}, V_{rg} denote the holdup of the reactor and the regenerator, $\Delta H_{rg}, \Delta H_{cr}$ are the heat of reactions, ΔH_{fv} is the heat of feed vaporization, F_{rc} denotes the flow rate of catalyst from reactor to regenerator, S_a, S_c, S_f denote specific heats, T_{fp}, T_{ai} denote the inlet temperatures of the feed in the reactor and the air in the regenerator, and R_{cf}, R_{oc}, R_{cb} denote reaction rates. The analytic expressions for the reaction rates R_{cf}, R_{oc}, R_{cb} are[23]:

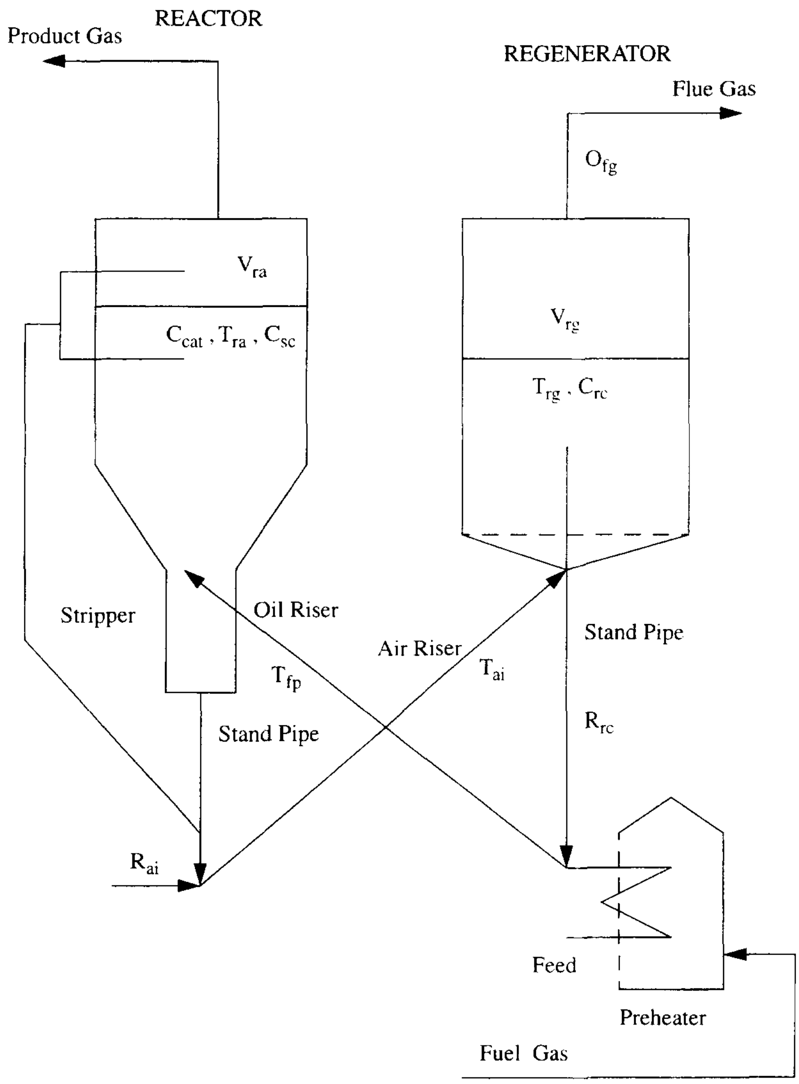

Figure 1 A fluidized catalytic cracker

$$
\begin{aligned}
R_{cf} &= \frac{k_{cc} V_{ra} P_{ra}}{C_{cat} C_{rc}^{0.06}} \exp\left\{ \frac{-E_{cc}}{R(T_{ra} + 460.0)} \right\} \\
R_{oc} &= \frac{k_{cr} V_{ra} P_{ra} R_{tf} D_{tf} K_{cr}}{R_{tf} + V_{ra} P_{ra} K_{cr}} \\
K_{cr} &= \frac{k_{cr}}{C_{cat} C_{rc}^{0.15}} \exp\left\{ \frac{-E_{cr}}{R(T_{ra} + 460.0)} \right\} \\
R_{cb} &= \frac{R_{ai}(21 - O_{fg})}{200} \\
O_{fg} &= 21 \exp\left\{ \frac{-\frac{V_{rg} P_{rg}}{R_{ai}}}{\frac{10^6}{4.76 R_{ai}^2} + \frac{100}{k_{or}} \exp\left\{ \frac{-E_{or}}{R(T_{rg}+460.0)} C_{rc} \right\}} \right\}
\end{aligned}
\qquad (26)
$$

where k_{cc}, k_{cr}, k_{or} are pre-exponential kinetic rate constants, E_{cc}, E_{cr}, E_{or} are activation energies, O_{fg} is the oxygen in flue gas, R_{tf} is the total feed rate, and R_{ai} is the air rate. The values of the process parameters and the corresponding steady-state values are given in *Table 1*. The process exhibits a two-time-scale behaviour because the residence time in the reactor is significantly smaller than the one in the regenerator [23]. This

Table 1 Process parameters and steady-state values

E_{cc}	=	18 000.0	Btu lb^{-1} mole^{-1}
E_{cr}	=	27 000.0	Btu lb^{-1} mole^{-1}
E_{oc}	=	63 000.0	Btu lb^{-1} mole^{-1}
k_{cc}	=	8.59	Mlb h^{-1} psia^{-1} ton^{-1} (wt%)$^{-1.06}$
k_{cr}	=	11 600	Mbbl day^{-1} psia^{-1} ton^{-1}(wt%)$^{-1.15}$
k_{or}	=	$3,15 \times 10^{10}$	Mlb h^{-1} psia^{-1} ton^{-1}
V_{rg}	=	60.0	ton
V_{ra}	=	200.0	ton
T_{fps}	=	744.0	F
T_{ai}	=	175.0	F
P_{rg}	=	25.0	psia
P_{ra}	=	40.0	psia
ΔH_{fv}	=	60.0	Btu lb^{-1}
ΔH_{cr}	=	77.2	Btu lb^{-1}
ΔH_{rg}	=	10 561.0	Btu lb^{-1}
S_a	=	0.3	Btu lb^{-1}F^{-1}
S_c	=	0.7	Btu lb^{-1}F^{-1}
F_{rc}	=	40.0	ton min^{-1}
R_{tf}	=	100.0	mbbl day^{-1}
D_{tf}	=	7.0	lb gal^{-1}
R_{ai}	=	400.0	mlb min^{-1}
C_{cat}	=	0.8723	wt%
C_{sc}	=	1.5698	wt%
C_{rc}	=	0.6975	wt%
T_{ra}	=	930.6255	F
T_{rg}	=	1155.9605	F

implies that although the operator focus is on the reactor, the control problem must focus on the regenerator which is the process that essentially determines the dynamic response of the entire FCC unit. To this end, the control objective is the regulation of the temperature in the regenerator, T_{rg}, and the concentration of the carbon on the regenerated catalyst, C_{rc}, by manipulating the inlet temperatures, T_{fp} and T_{ai}. The uncertain variable for the system is taken to be the heat of combustion in the regeneration, which is also assumed to be time-varying. Defining the singular perturbation parameter ϵ as

$$\epsilon = \frac{V_{ra}}{V_{rg}} \tag{27}$$

and setting $x = [x_1\ x_2]^T = [C_{rc}\ T_{rg}]^T$, $z = [z_1\ z_2\ z_3]^T = [C_{cat}\ C_{sc}\ T_{ra}]^T$, $u = [u_1\ u_2]^T = [T_{fp}\ T_{ai}]^T$, $y = [y_1\ y_2]^T = [C_{rc}\ T_{rg}]^T$, $\theta = [\Delta h_{rg}]$, the system of Equation (25) can be put into the standard singularly perturbed from:

$$\begin{aligned}
\frac{dx_1}{dt} &= f_{11}(x_1, x_2) + Q_{11}z_2 \\
\frac{dx_2}{dt} &= f_{12}(x_1, x_2, \theta) + Q_{12}z_3 + g_{12}u_2 \\
\epsilon\frac{dz_1}{dt} &= Q_{21}(z_1, z_2, z_3, x_1) \\
\epsilon\frac{dz_2}{dt} &= f_{22}(x_1) + Q_{22}(z_1, z_2, z_3, x_1) \\
\epsilon\frac{dz_3}{dt} &= f_{23}(x_2) + Q_{23}(z_1, z_2, z_3, x_1) + g_{23}u_1
\end{aligned} \tag{28}$$

where $f_{11}, Q_{11}, f_{12}, Q_{12}, g_{12}, Q_{21}, f_{22}, Q_{22}, f_{23}, Q_{23}, g_{23}$ are functions whose specific form is omitted for brevity. It was verified that the system of Equation (28) possesses an exponentially stable equilibrium manifold for the fast dynamics, which implies that it is not needed to utilize a preliminary feedback law of the form of Equation (7) to stabilize the fast dynamics. Setting $\epsilon = 0$, the equilibrium manifold of the fast dynamics can be calculated analytically and is of the form: $z_s = g(x_1, x_2, u_1)$, where g is a smooth vector function (note that the input u_1 enters this algebraic equation in a nonlinear fashion, due to the nonlinear appearance of the fast state z in the system of Equation (28). The reduced system can then be found to be of the form:

$$\begin{aligned}
\frac{dx_1}{dt} &= F_1(x_1, x_2) + G_{11}(x_1, x_2, u_1) \\
\frac{dx_2}{dt} &= F_2(x_1, x_2, \theta) + G_{21}(x_1, x_2, u_1) + G_{22}u_2
\end{aligned} \tag{29}$$

with $F_1, G_{11}, F_2, G_{21}, G_{22}$ appropriately defined (their exact expressions are omitted for brevity) . The system of Equation (29) is already in the form of Equation (11), with u appearing in a nonlinear fashion, and the condition $r_1 + r_2 = 1 + 1 = 2$ holds, which implies that this system does not possess zero dynamics. Furthermore, Assumption 4 is also satisfied and the function $\tilde{c}_1(x)$ takes the form:

$$\tilde{c}_1(x) = F_2(x_1, x_2, |\theta|) \tag{30}$$

where $|\theta|$ denotes the upper bound on the size of the uncertain variable, which is assumed to be constant. Clearly, the assumptions required for the application of the result of the theorem are satisfied. Referring to the system of Equation (29), we also note that the uncertain variable θ does not appear coupled with the input variables u_1, u_2 and that the differential equation that describes the evolution of x_1 is independent of θ and u_2. The former fact allows using the simplifications discussed in Remark 3 in the controller formula, while the latter fact suggests that there is no need for an uncertainty compensation term to be included in the formula that calculates the control action for the manipulated input u_1. Note that the synthesis formula of Equation (18) cannot be readily used due to the nonlinear appearance of the input u_1 in the system of Equation (29). To resolve this problem, we first considered the algebraic equation $G_{11}(x_1, x_2, u_1) = \alpha$ and derived its solution in terms of u_1, i.e. $u_1 = \bar{G}_{11}(x_1, x_2, \alpha)$. Then, the necessary controller was found to be:

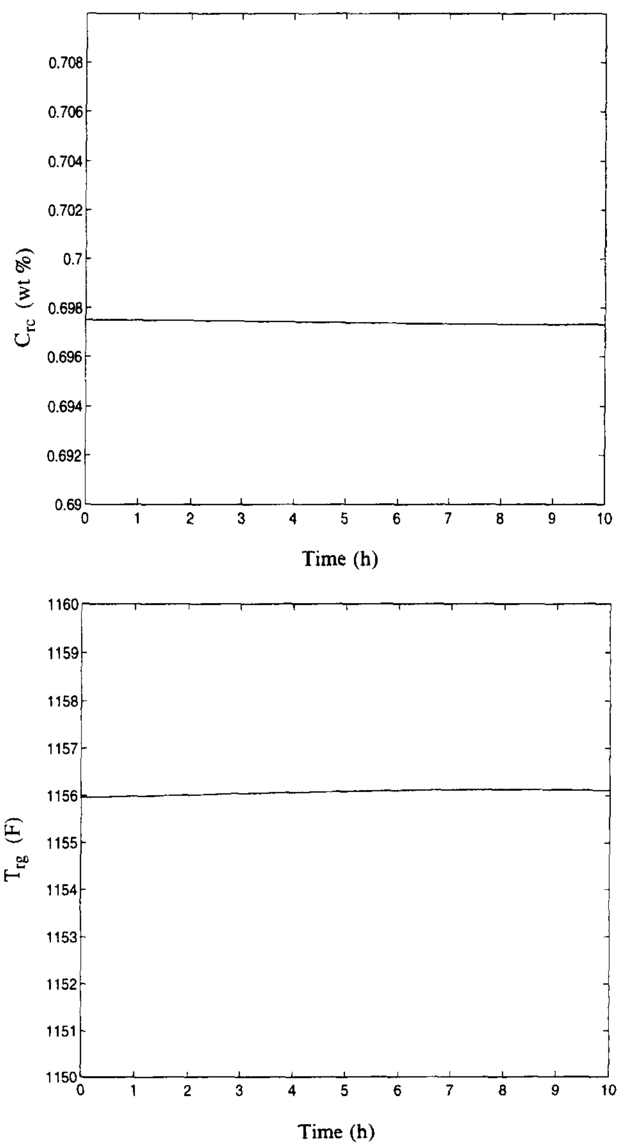

Figure 2 Closed-loop output profiles for regulation

$$
\begin{aligned}
u_1 &= \bar{G}_{11}(x_1, x_2, \frac{1}{\beta_{11}}(v_1 - x_1) - F_1(x_1, x_2)) \\
u_2 &= \frac{1}{G_{22}}\left\{\frac{1}{\beta_{21}}(v_2 - x_2) - [F_2(x_1, x_2, \theta_0)\right. \\
&\quad + G_{21}(x_1, x_2, \bar{G}_{11}(x_1, x_2, \frac{1}{\beta_{11}}(v_1 - x_1) \\
&\quad \left. - F_1(x_1, x_2))] - 2\tilde{c}_1(x)\frac{x_2 - v_2}{|x_2 - v_2| + \phi}\right\}
\end{aligned} \tag{31}
$$

For the above robust controller, we note that its practical implementation requires measurements of only two of the states of the process (concentration of carbon on regenerated catalyst and temperature of the regenerator). A slowly-varying uncertainty was considered expressed by a sinusoidal function of the form:

$$\theta = 180.0 \sin(0.2t) \tag{32}$$

The upper bound on the uncertainty was taken to be $|\theta| = 180.0$. From Theorem 1, it is clear that there exists a trade-off between the upper bound on the value of the singular perturbation parameter ϵ^* and the level of asymptotic attenuation d that can be achieved. In the application in question, the value of the singular perturbation parameter is fixed by the design of the process, i.e. $\epsilon_p = 0.3$, and thus there exists a lower bound on the selection of the level d. We performed a set of computer simulations (for the regulation problem) to calculate ϕ^* for certain values of d and, in turn, the value of ϵ^* for $\phi \leq \phi^*$. The following set of parameters were found to give an $\epsilon \leq \epsilon_p$ and used in the simulations

$$\beta_{11} = 0.1, \beta_{21} = 0.02, \phi = 0.5 \tag{33}$$

to achieve an ultimate degree of attenuation $d = 0.1$, for a value of the singular perturbation parameter $\epsilon = 0.3$.

Two representative simulation runs are reported. In both runs, the process was initially ($t = 0.0$ h) assumed to be at steady-state. In the first simulation run, we tested the regulatory capabilities of the controller. *Figure 2* shows the closed-loop output profiles, while *Figure 3* displays the corresponding manipulated input

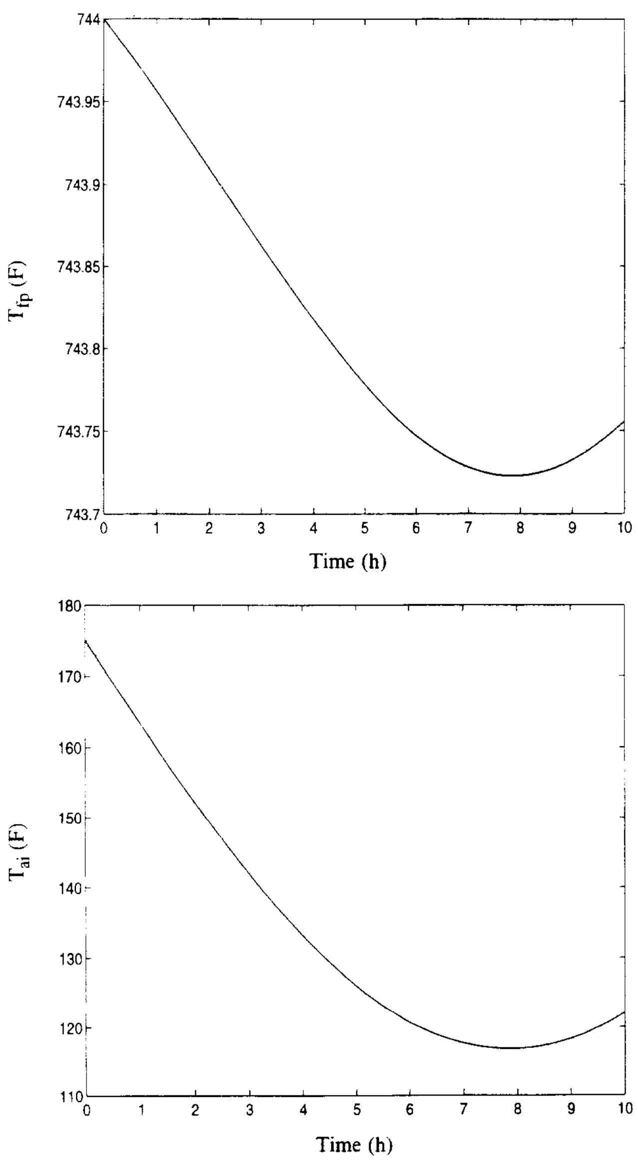

Figure 3 Closed-loop input profiles for regulation

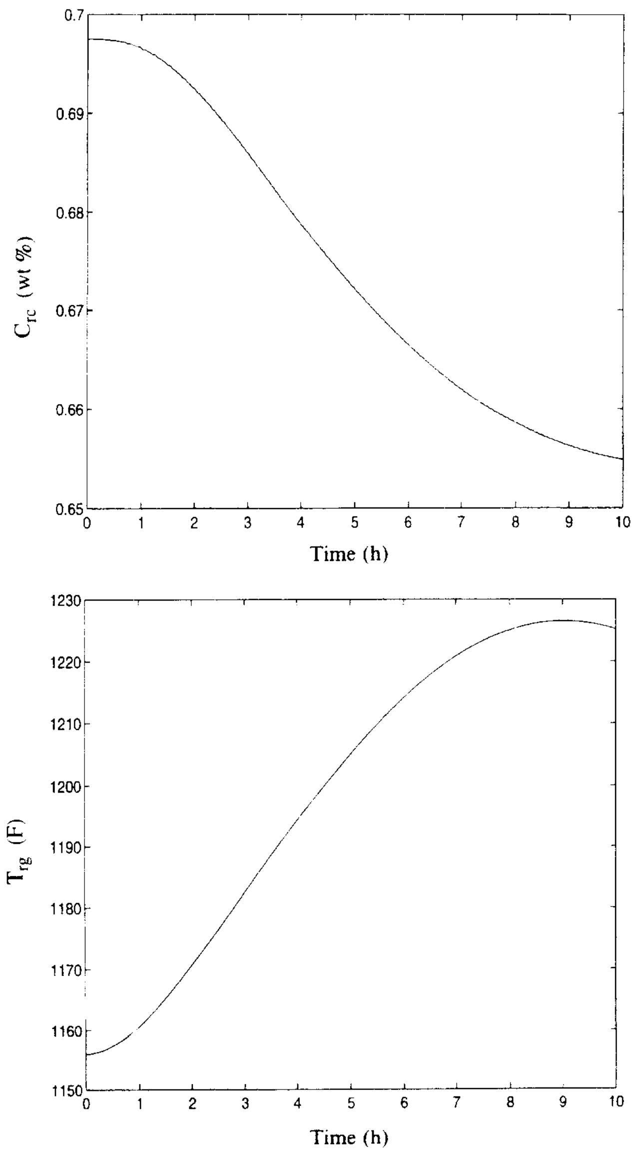

Figure 4 Closed-loop output profiles for regulation (no uncertainty compensation)

profiles. Clearly the controller regulates the output at the operating steady-state compensating for the effect of uncertainty and satisfying the requirements $\limsup_{t\to\infty}|y_1 - v_1| \leq 0.1$, $\limsup_{t\to\infty}|y_2 - v_2| \leq 0.1$.

For the sake of comparison, we also implemented the same controller without the term which is responsible for the compensation of uncertainty, i.e. a decoupling input/output linearizing controller for the nominal open-loop reduced system. The output profiles for this simulation run are shown in *Figure 4*. One can immediately see how strong is the effect of the uncertainty on the outputs of the process, leading to poor transient performance and offset. In the next simulation run, we tested the output tracking capabilities of the controller. A 25.0 *F* increase in the value of the output y_2 was imposed at time $t = 0.0$ h. The output profiles are depicted in *Figure 5* and the profiles of the corresponding manipulated inputs are given in *Figure 6*. It is clear that the controller drives the output y_2 to its new reference input value; achieving the requirement $\limsup_{t\to\infty}|y_2 - v_2| \leq 0.01$. It can be also observed that the output y_1 stays very close to its reference input value (i.e. the requirement $\limsup_{t\to\infty}|y_1 - v_1| \leq 0.01$ is satisfied). Finally, *Figure 7* shows the closed-loop output profiles for the same simulation run in the case of implementing the decoupling input/output linearizing controller to the process. It is clear that this controller cannot attenuate the effect of the uncertainty yielding unacceptable performance. From the results of the simulation study, we conclude that the proposed methodology is a powerful tool for the synthesis of nonlinear controllers that compensate for the effect of uncertainty.

Conclusions

This paper proposes a robust multivariable controller-design methodology for a broad class of multi-input multi-output two-time-scale nonlinear processes with uncertain variables, modeled in singularly perturbed form. The proposed controller guarantees boundedness of the state and asymptotic output tracking with an

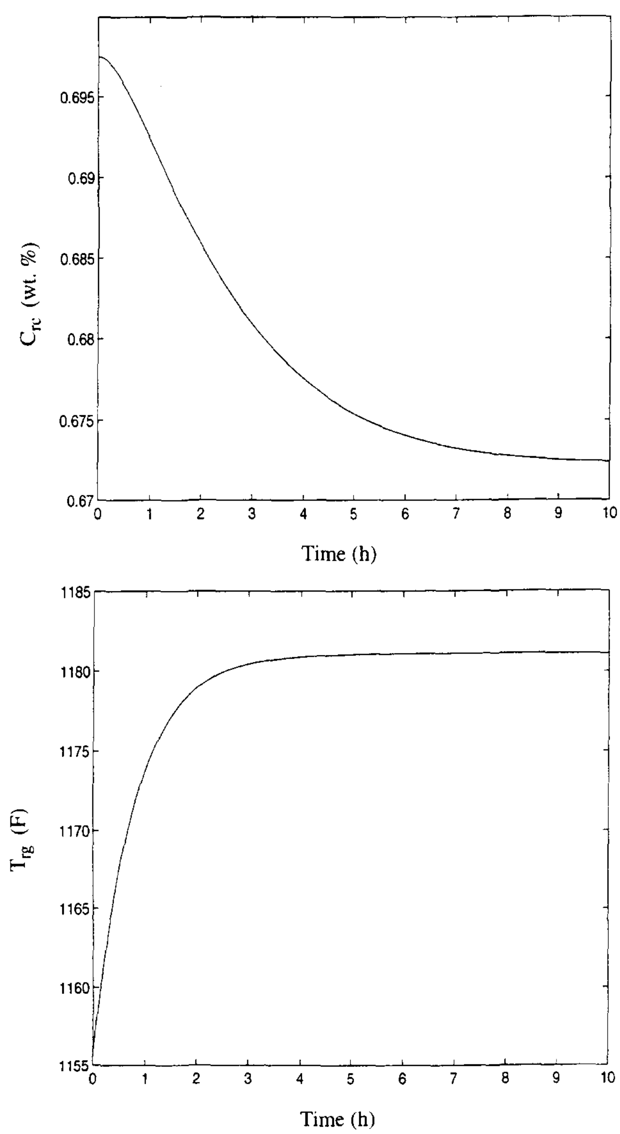

Figure 5 Closed-loop output profiles for reference input tracking

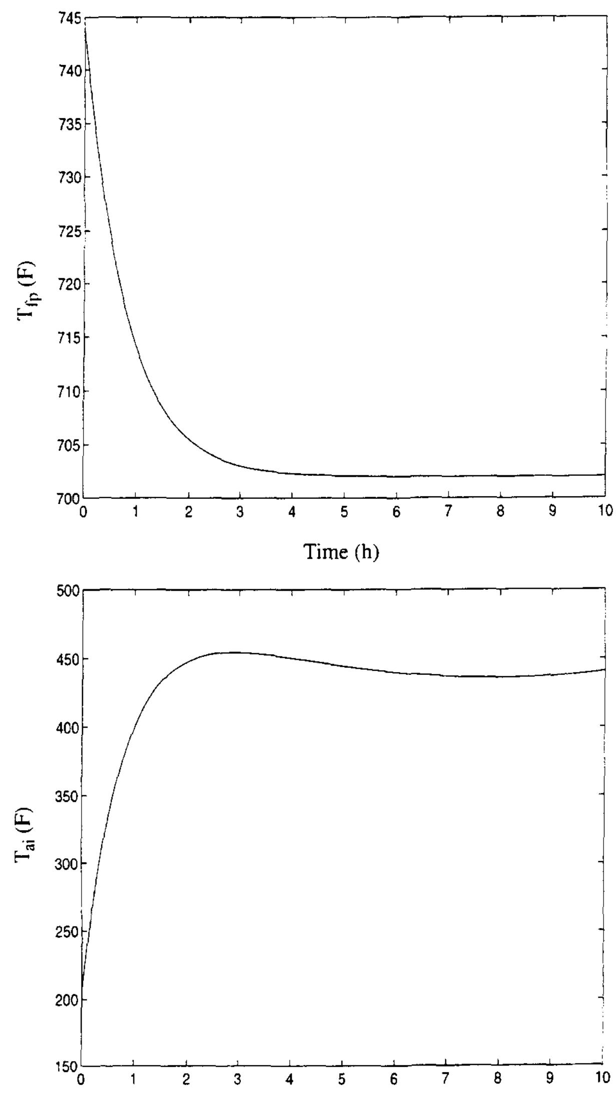

Figure 6 Closed-loop input profiles for reference input tracking

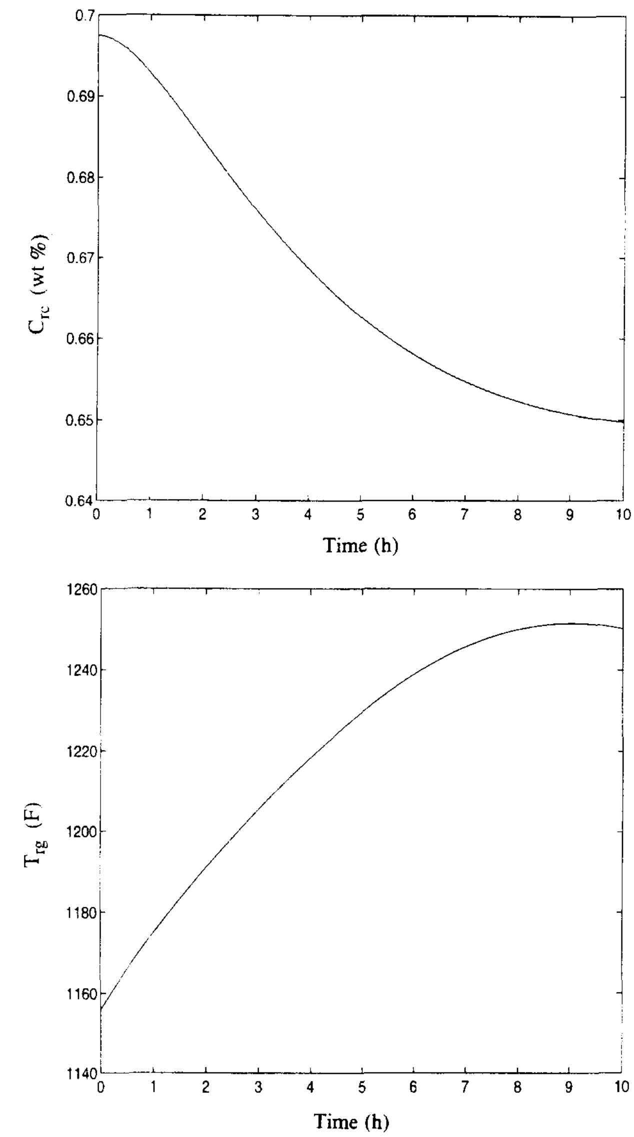

Figure 7 Closed-loop output profiles for reference input tracking (no uncertainty compensation)

arbitrary degree of attenuation of the effect of the uncertainty on the output by a suitable choice of controller parameters, as long as the singular perturbation parameter is sufficiently small. The method was applied to a fluidized catalytic cracking reactor with unknown heat of the combustion reaction and its performance was successfully tested through computer simulations.

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APPENDIX

Definitions

- For any measurable (with respect to the Lebesgue measure) function $\theta : \Re_{\geq 0} \rightarrow \Re^m$, $||\theta||$ denotes ess.sup.$|\theta(t)|$, $t \geq 0$.
- A function $\gamma : \Re_{\geq 0} \rightarrow \Re_{\geq 0}$ is said to be of class K if it is continuous, increasing and is zero at zero.
- A function $\beta : \Re_{\geq 0} \times \Re_{\geq 0} \rightarrow \Re_{\geq 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.
- A matrix $A(x, \theta)$ of dimension $n \times n$ is said to be Hurwitz uniformly in $x \in \Re^n, \theta \in \Re^q$, if there exists a positive real number c such that $\mathrm{Re}[\lambda_i(A(x, \theta))] \leq -c$, $i = 1, ..., n$ for all $x \in \Re^n$, $\theta \in \Re^q$, where λ_i denotes the ith eigenvalue of the matrix.
- Let $x \in \Re^n, y \in \Re^n$ be two vectors and $A \in \Re^{n \times n}$ be a matrix, then if $\sigma_{max}\{A\}, \sigma_{min}\{A\}$ denote the maximum and minimum singular values of A, the following relations hold:

$$x^T A y \leq \sigma_{max}\{A\}|x||y|, \quad -x^T A y \leq -\sigma_{min}\{A\}|x||y| \tag{34}$$

Definition[26]*:* The system in Equation (4) (with $u \equiv 0$) is said to be input-to-state stable (ISS) with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_0 \in \Re^n$ and for each measurable, essentially bounded input $\theta(\cdot)$ on $[0,\infty)$ the solution of Equation (4) with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(||\theta||), \forall t \geq 0 \tag{35}$$

Proof of Theorem 1: The proof of the theorem is conceptually analogous (although notationally more involved) to the one given in Christofides *et al.*[20] for single-input single-output systems. In order to stress the issues associated with the multivariable nature of the problem and avoid repetitions, we will refer without proof to some results established in Christofides *et al.*[20]. The proof consists of three parts: initially, the global exponential stability of the fast dynamics of the closed-loop system is established; then, the closed-loop reduced system is analyzed using Lyapunov techniques to derive bounds that capture the evolution of the states in terms of the initial conditions and the inputs, and a direct application of a result developed in Christofides and Teel[19] is made to establish that these bounds continue to hold up to an arbitrarily small offset for the singularly perturbed system. Finally, the resulting bounds are utilized, using techniques (small gain theorem-type calculations) similar to those used in Teel[27] and Jiang *et al.*[28], to show boundedness of the trajectories and establish the inequality of Equation (20). All the above results will be obtained for sufficiently small values of ϕ and ϵ.

Part 1: In this part of the proof, we will establish that the closed-loop fast subsystem is globally exponentially stable. Under the control law of Equation (18) the closed-loop system takes the form:

$$\begin{aligned}\dot{x} = {} & \tilde{F}(x,\theta) + \tilde{G}(x,\theta)\Bigg\{\Delta(x,t)\Bigg\{\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} \\ & - L^k_{\tilde{F}_{nom}}h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k-1)} - L^{k-1}_{\tilde{F}_{nom}}h_i(x)) \\ & - (2+b)[\tilde{c}_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L^k_{\tilde{F}_{nom}}h_i(x))|] \\ & w(x,\phi)\Bigg\}\Bigg\} + [Q_1(x,\theta) + G_1(x,\theta)K(x)] \\ & [z - C(x,\theta,\phi,v_i^{(k)})]\end{aligned} \tag{36}$$

$$\epsilon\dot{z} = [Q_2(x,\theta) + G_2(x,\theta)K(x)][z - C(x,\theta,\phi,v_i^{(k)})] \tag{37}$$

where

$$\begin{aligned}C(x,\theta,\phi,v_i^{(k)}) = {} & -[Q_2(x,\theta) + G_2(x,\theta)K(x)]^{-1}\Bigg[f_2(x,\theta) \\ & + G_2(x,\theta)\Bigg\{\Delta(x,t)\Bigg\{\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} \\ & - L^k_{\tilde{F}}h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L^k_{\tilde{F}_{nom}}h_i(x)) \\ & - (2+b)[\tilde{c}(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} \\ & - L^k_{\tilde{F}_{nom}}h_i(x))|]w(x,\phi)\Bigg\}\Bigg\}\Bigg]\end{aligned} \tag{38}$$

It is clear that if the matrix $K(x)$ is chosen so that the matrix $Q_2(x,\theta) + G_2(x,\theta)K(x)$ is Hurwitz uniformly in $x \in \Re^n, \theta \in \Re^q$ the closed-loop fast subsystem

$$\frac{dz}{d\tau} = [Q_2(x,\theta) + G_2(x,\theta)K(x)][z - C(x,\theta,\phi,v_i^{(k)})] \tag{39}$$

possesses a globally exponentially stable equilibrium manifold of the form of Equation (38).

Part 2: To simplify the notation, set $\tilde{C}_i(x,\theta) = [L_{\tilde{G}1}L_{\tilde{F}}^{\tilde{r}_i-1}h_i(T^{-1}(\zeta,\eta,\theta)) \ldots L_{\tilde{G}m}L_{\tilde{F}}^{\tilde{r}_i-1}h_i(T^{-1}(\zeta,\eta,\theta))]$, $i = 1, \ldots, m$, $e_z = (z - C(x,\theta,\phi,v_i^{(k)}))$. Observing the similarity in the structure of the system of Equation (36) and the x-subsystem of Equation (9) and using Assumption 2, the representation of the closed-loop system of Equations (36) and (37) in (ζ,η,z) co-ordinates takes the form:

$$
\begin{aligned}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} + e_z\bar{\Psi}_1^{(1)}(\zeta,\eta,\theta,v_i^{(k)}) \\
&\vdots \\
\dot{\zeta}_{\tilde{r}_1-1}^{(1)} &= \zeta_{\tilde{r}_1}^{(1)} + e_z\bar{\Psi}_{\tilde{r}_1}^{(1)}(\zeta,\eta,\theta,v_i^{(k)}) \\
\dot{\zeta}_{\tilde{r}_1}^{(1)} &= L_{\tilde{F}}^{\tilde{r}_1}h_1(T^{-1}(\zeta,\eta,\theta)) + e_z\bar{\Psi}_{\tilde{r}_1}^{(1)}(\zeta,\eta,\theta,v_i^{(k)}) \\
&\quad + \tilde{C}_1(x,\theta)\Bigg\{\Delta(x,t)\Bigg\{\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} \\
&\quad - L_{\tilde{F}_{nom}}^{k}h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k-1)} \\
&\quad - L_{\tilde{F}_{nom}}^{k-1}h_i(x)) - (2+b)[\bar{c}(x,t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^{k}h_i(x))|]w(x,\phi)\Bigg\}\Bigg\} \\
&\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} + e_z\bar{\Psi}_1^{(m)}(\zeta,\eta,\theta,v_i^{(k)}) \\
&\vdots \\
\dot{\zeta}_{\tilde{r}_m-1}^{(m)} &= \zeta_{\tilde{r}}^{(m)} + e_z\bar{\Psi}_{\tilde{r}_m-1}^{(m)}(\zeta,\eta,\theta,v_i^{(k)}) \\
\dot{\zeta}_{\tilde{r}_m}^{(m)} &= L_{\tilde{F}}^{\tilde{r}_m}h_m(T^{-1}(\zeta,\eta,\theta)) + e_z\bar{\Psi}_{\tilde{r}_m}^{(m)}(\zeta,\eta,\theta,v_i^{(k)}) \\
&\quad + \tilde{C}_m(x,\theta)\Bigg\{\Delta(x,t)\Bigg\{\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} \\
&\quad - L_{\tilde{F}_{nom}}^{k}h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k-1)} \\
&\quad - L_{\tilde{F}_{nom}}^{k-1}h_i(x)) - (2+b)[\bar{c}(x,t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^{k}h_i(x))|]w(x,\phi)\Bigg\}\Bigg\} \\
\dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_{\sum_i\tilde{r}_i+1}(\zeta,\eta,\theta,v_i^{(k)}) \\
&\vdots \\
\dot{\eta}_{n-\sum_i\tilde{r}_i} &= \Psi_{n-\sum_i\tilde{r}_i}(\zeta,\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_n(\zeta,\eta,\theta,v_i^{(k)}) \\
\epsilon\dot{z} &= [Q_2(T^{-1}(\zeta,\eta,\theta),\theta) \\
&\quad + G_2(T^{-1}(\zeta,\eta,\theta),\theta)K(T^{-1}(\zeta,\eta,\theta))]e_z \\
y_i &= \zeta_1^{(i)}, \quad i=1,\ldots m
\end{aligned}
\tag{40}
$$

where $\bar{\Psi}_k^{(i)}$, $i=1,\ldots,m$, $k=1,\ldots,\tilde{r}_i$, and $\bar{\Psi}_{\sum_i\tilde{r}_i+1},\ldots,\bar{\Psi}_n$, are Lipschitz functions of their arguments.

Introducing the variables $e_k^{(i)} = \zeta_k^{(i)} - v_i^{(k-1)}$, $i=1,\ldots,m$, $k=1,\ldots,\tilde{r}_i$, $\tilde{e}_{\tilde{r}_i}^{(i)} = \sum_{i=1}^{m} e_{\tilde{r}_i}^{(i)} + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i-1}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_k^{(i)}$, and the notation $\bar{e}^{(i)} = [e_1^{(i)}e_2^{(i)}\ldots e_{\tilde{r}_i-1}^{(i)}]^T$, $\bar{e} = [e^{(1)^T}e^{(2)^T}\ldots e^{(m)^T}]^T$, $\bar{\eta} = [\bar{e}^T\eta^T]$, $\tilde{e}_{\tilde{r}} = [\tilde{e}_{\tilde{r}}^{(1)}\tilde{e}_{\tilde{r}}^{(2)}\ldots\tilde{e}_{\tilde{r}}^{(m)}]^T$, $v_i = [v_i^{(0)}\ v_i^{(1)}\ldots v_i^{(\tilde{r}_i-1)}]^T$, $\tilde{v} = [v_1^{(T)}v_2^{(T)}\ldots v_m^{(T)}]^T$, $M_i(x,\theta,t) = \tilde{C}_i(x,\theta)\ \Delta(x,t)$, the system of Equation (40) takes the form:

$$
\begin{aligned}
\dot{e}_1^{(1)} &= e_2^{(1)} + e_z\bar{\Psi}_1^{(1)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
&\vdots \\
\dot{e}_{\tilde{r}_1-1}^{(1)} &= -\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i-1}\frac{\beta_{ik}^1}{\beta_{i\tilde{r}_i}^1}e_k^{(i)} + \sum_{i=1}^{m}\tilde{e}_{\tilde{r}_1}^{(1)} \\
&\quad + e_z\bar{\Psi}_{\tilde{r}_1}^{(1)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
\dot{\tilde{e}}_{\tilde{r}_1}^{(1)} &= e_z\bar{\Psi}_{\tilde{r}_1}^{(1)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
&\quad + L_{\tilde{\delta}}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)) \\
&\quad + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}^{(1)}}{\beta_{i\tilde{r}_i}^{(1)}}e_{k+1}^{(i)} \\
&\quad + M_1(x,\theta,t)\Bigg\{-\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} - \tilde{e}_{\tilde{r}} \\
&\quad - (2+b)[\bar{c}_1((\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]w(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),\phi)\Bigg\} \\
&\vdots \\
\dot{e}_1^{(m)} &= e_2^{(m)} + e_z\bar{\Psi}_1^{(m)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
&\vdots \\
\dot{e}_{\tilde{r}_m-1}^{(m)} &= \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i-1}\frac{\beta_{ik}^{(m)}}{\beta_{i\tilde{r}_i}^{(m)}}e_k^{(i)} + \sum_{i=1}^{m}\tilde{e}_{\tilde{r}_m}^{(m)}e_z\bar{\Psi}_{\tilde{r}_m-1}^{(m)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
\dot{\tilde{e}}_{\tilde{r}_m}^{(m)} &= e_z\bar{\Psi}_{\tilde{r}_m}^{(m)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
&\quad + L_{\tilde{\delta}}L_{\tilde{F}}^{\tilde{r}_m-1}h_m(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)) \\
&\quad + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}^{(m)}}{\beta_{i\tilde{r}_i}^{(m)}}e_{k+1}^{(i)} \\
&\quad + M_m(x,\theta,t)\Bigg\{-\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} - \tilde{e}_{\tilde{r}} \\
&\quad - (2+b)[\bar{c}_1((\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]w(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),\phi)\Bigg\} \\
\dot{\eta} &= \Psi_1(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_{\sum_i\tilde{r}_i+1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
&\vdots \\
\dot{\eta}_{n-\sum_i\tilde{r}_i} &= \Psi_{n-\sum_i\tilde{r}_i}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_n(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) \\
\epsilon\dot{z} &= [Q_2(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)\theta) \\
&\quad + G_2K(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta))]e_z \\
y_i &= \zeta_1^{(i)}, \quad i=1,\ldots,m
\end{aligned}
\tag{41}
$$

We will proceed with a two-step procedure to establish the ISS bounds that capture the evolution of the states $\bar{e}$, $\tilde{e}_{\tilde{r}}$, $\bar{\eta}$ of the above system. In particular, we will initially obtain these bounds in the absence of singular perturbations (i.e. when the singular perturbation parameter ϵ is equal to zero). Once these bounds are derived, we will apply the main result reported in Christofides and Teel[19] to establish that these bounds continue to hold up to an arbitrarily small offset, for initial conditions and uncertainties in an arbitrarily large compact set, provided that ϵ is sufficiently small.

Step 1: First, note that the linear structure of $\bar{e}$ subsystem of the reduced system of Equation (41) and the fact that it is exponentially stable when $\tilde{e}_{\bar{r}} \equiv 0$ allows using a direct Lyapunov function argument to show that there exist positive real numbers, $k_1, a, \gamma_{\tilde{e}_{\bar{r}}}$ such that the following ISS bound holds for the reduced $\bar{e}$ subsystem:

$$|\bar{e}(t)| \leq k_1 e^{-at}|\bar{e}(0)| + \gamma_{\tilde{e}_{\bar{r}}}||\tilde{e}_{\bar{r}}|| \tag{42}$$

In the rest of this step, we will show that the controller of Equation (18) ensures that the subsystem consisting of the state vector $\tilde{e}_{\bar{r}} = [\tilde{e}^{(1)}_{\bar{r}_1}\tilde{e}^{(2)}_{\bar{r}_2}\ldots\tilde{e}^{(m)}_{\bar{r}_m}]^T$ of the reduced system of Equation (41) possesses an ISS property with respect to $\tilde{e}_{\bar{r}}, \bar{e}, \bar{v}, \dot{\bar{v}}, \eta, \theta$ and, moreover, the gain function saturates at ϕ. To this end consider the following singularly perturbed system:

$$\begin{aligned}
\dot{\bar{e}} &= e_z\bar{\Psi}_{\bar{r}}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta) + L_{\bar{g}}L_{\bar{F}}^{\bar{r}-1}h(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta)) \\
&\quad + \sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1} + M(x,\theta,t)\Bigg\{-\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1} \\
&\quad - \tilde{e}_{\bar{r}} - (2+b)[\bar{c}_1((\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}|]w(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta)), \phi)\Bigg\} \\
\epsilon\dot{z} &= [Q_2(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), \theta) \\
&\quad + G_2(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), \theta)K(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta))]e_z
\end{aligned} \tag{43}$$

where $\bar{\Psi}_{\bar{r}}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta) = [\bar{\Psi}^{(1)}_{\bar{r}_1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta)\ldots\bar{\Psi}^{(m)}_{\bar{r}_m}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta)]^T$, $v^{(\bar{r})} = [v_1^{(\bar{r}_1)}\ldots v_m^{(\bar{r}_m)}]^T$, $L_{\bar{g}}L_{\bar{F}}^{\bar{r}-1}h(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta)) = L_{\bar{g}}L_{\bar{F}}^{\bar{r}_1-1}h_1(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta))\ldots L_{\bar{g}}L_{\bar{F}}^{\bar{r}_m-1}h_m(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta))]^T$. From Part 1, we have that the fast dynamics of the above system is globally exponentially stable. Consider the reduced system corresponding to the singularly perturbed system of Equation (43):

$$\begin{aligned}
\dot{\tilde{e}}_{\bar{r}} &= L_{\bar{g}}L_{\bar{F}}^{\bar{r}-1}h(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta)) + \sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1} \\
&\quad + M\Bigg\{-\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1} - \tilde{e}_{\bar{r}} - (2+b) \\
&\quad [\bar{c}((\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t) + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}|] \\
&\quad w(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), \phi)\Bigg\}
\end{aligned} \tag{44}$$

To establish that the above system is ISS with respect to $\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta$, we use the following smooth function $V : \Re^m \to \Re_{\geq 0}$:

$$V = \frac{1}{2}\tilde{e}_{\bar{r}}^2 \tag{45}$$

Calculating the time-derivative of V along the trajectory of the system of Equation (44), we have:

$$\begin{aligned}
\dot{V} &= \tilde{e}_{\bar{r}}^T\Bigg[L_{\bar{g}}L_{\bar{F}}^{\bar{r}-1}h(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta)) \\
&\quad + \sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1} + M\Bigg\{-\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1} \\
&\quad - \tilde{e}_{\bar{r}} - (2+b)[\bar{c}_1((\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}|]w(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), \phi)\Bigg\}\Bigg]
\end{aligned} \tag{46}$$

Furthermore, it is straightforward to show that the representation of the vector function $w(x, \phi)$ in terms of the vector $\tilde{e}_{\bar{r}}$ is given by

$$\omega(\tilde{e}_{\bar{r}}, \phi) = \frac{\tilde{e}_{\bar{r}}}{|\tilde{e}_{\bar{r}}| + \phi} \tag{47}$$

Substituting Equation (47) into Equation (46) and using the fact that $1 \leq \sigma_{min}\{M(x,\theta,t)\} \leq \sigma_{max}\{M(x,\theta,t)\} \leq b$ and the inequalities of Equation (34), we have:

$$\begin{aligned}
\dot{V} &\leq \tilde{e}_{\bar{r}}^T\Bigg\{-M(x,\theta,t)\tilde{e}_{\bar{r}} - (2+b)M(x,\theta,t) \\
&\quad [\bar{c}(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t) + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}|]\frac{\tilde{e}_{\bar{r}}}{|\tilde{e}_{\bar{r}}| + \phi} \\
&\quad + (M(x,\theta,t) - I_{m\times m})\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1} \\
&\quad + L_{\bar{g}}L_{\bar{F}}^{\bar{r}-1}h(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta))\Bigg\} \\
&\leq \Bigg\{-\tilde{e}_{\bar{r}}^2 - (2+b)[\bar{c}(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}|]\frac{\tilde{e}_{\bar{r}}^2}{|\tilde{e}_{\bar{r}}| + \phi} + (1+b) \\
&\quad (\tilde{e}_{\bar{r}}|(|\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}| + \bar{c}(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t))\Bigg\} \\
&\leq \Bigg\{-\tilde{e}_{\bar{r}}^2 - [\bar{c}_1(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t) \\
&\quad + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}|]\frac{\tilde{e}_{\bar{r}}^2}{|\tilde{e}_{\bar{r}}| + \phi} + \phi(1+b) \\
&\quad [\bar{c}(T^{-1}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta), t) + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}e^{(i)}_{k+1}|]\frac{\tilde{e}_{\bar{r}}}{|\tilde{e}_{\bar{r}}| + \phi}\Bigg\}
\end{aligned} \tag{48}$$

From the last inequality, it follows directly that if $|\tilde{e}_{\tilde{r}}| \geq \phi(1+b)$, the time-derivative of the Lyapunov function satisfies $\dot{V} \leq -\tilde{e}_{\tilde{r}}^2$. This fact implies that the ultimate bound on the state $\tilde{e}_{\tilde{r}}$ of the system of Equation (44) depends only on the parameter ϕ and is independent of the states $\bar{e}, \eta$.

We will now analyze the time-derivative of V for $|\tilde{e}_{\tilde{r}}| \geq \phi(1+b)$. For ease of notation, we set $U = [\tilde{e}_{\tilde{r}}^T \bar{\eta}^T \theta^T \bar{v}^T]^T$. Then Equation (48) can be written as

$$\begin{aligned} \dot{V} \leq & \Big\{ -\tilde{e}_{\tilde{r}}^2 + (1+b)|\tilde{e}_{\tilde{r}}|[\tilde{c}_1(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta), t) \\ & + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} e_{k+1}^{(i)}|] \Big\} \\ \leq & -e_{\tilde{r}}^2 + (1+b)|\tilde{e}_{\tilde{r}}|\rho(|U|) \end{aligned} \tag{49}$$

where ρ is a class K_∞ function. Summarizing, we have that $\dot{V}$ satisfies the following properties:

$$\begin{aligned} &\dot{V} \leq -\frac{|\tilde{e}_{\tilde{r}}|^2}{2}, \\ &|\tilde{e}_{\tilde{r}}| \geq \min\{\phi(1+b), (2(1+b)\rho(|U|))\} =: \tilde{\gamma}_u(|U|) \end{aligned} \tag{50}$$

Using the result of theorem 4.10 in Khalil[29], we get that the following ISS bound holds for the state $\tilde{e}_{\tilde{r}}$ of the system of Equation (44):

$$|\tilde{e}_{\tilde{r}}(t)| \leq e^{-0.5t}|\tilde{e}_{\tilde{r}}(0)| + \tilde{\gamma}u(||U||) \tag{51}$$

Before we proceed with the rest of this step, we will assume that $\phi \in (0, \phi^*]$, where

$$\phi^* = \frac{d}{(1+b)(1+2_{\gamma\tilde{e}_{\tilde{r}}})} \tag{52}$$

Consider the singularly perturbed system consisting of the states $(\bar{e}, \eta, z)$ of the system of Equation (41). For this system, it can be shown that its fast dynamics are globally exponentially stable and the η-subsystem of the reduced system is ISS with respect to $\bar{e}, \tilde{e}_{\tilde{r}}, \theta, \dot{\theta}$. Furthermore, the state $\bar{\eta} = [\bar{e}^T\eta^T]^T$ of this system possesses an ISS property with respect to $\tilde{e}_{\tilde{r}}, \theta, \dot{\theta}$[28,30]. Utilizing the converse theorem developed in Sontag and Wang[31], we have that there exists a converse function for the system comprised of the states $(\bar{e}, \eta)$ and the existence of this function implies that there exist a function $\beta_{\bar{\eta}}$ of class KL and a function $\bar{\gamma}_s$ of class K such that the following ISS inequality holds for the state $\bar{\eta}$:

$$\begin{aligned} |\bar{\eta}(t)| &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_s(||[\tilde{e}_{\tilde{r}}^T \theta^T \dot{\theta}^T]^T||) \\ &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_{\tilde{r}}}(||\tilde{e}_{\tilde{r}}||) + \bar{\gamma}_\theta(||\theta||) + \bar{\gamma}_{\dot{\theta}}(||\dot{\theta}||) \end{aligned} \tag{53}$$

where $\bar{\gamma}_{\tilde{e}_{\tilde{r}}}, \bar{\gamma}_\theta, \bar{\gamma}_{\dot{\theta}}$ are class K functions, respectively, defined as $\bar{\gamma}_{\tilde{e}_{\tilde{r}}}(s) = \bar{\gamma}_\theta(s) = \bar{\gamma}_{\dot{\theta}}(s) = \bar{\gamma}_s(3s)$.

Step 2: We will now utilize the main result reported in Christofides and Teel[19] to establish that the ISS inequalities of Equations (51)–(53) continue to hold up to an arbitrarily small offset, for the states $\tilde{e}_{\tilde{r}}, \bar{\eta}$ of the singularly perturbed system of Equation (41). Following Christofides *et al.*[20], it can be shown that, given the set of positive real numbers $(\delta_{\tilde{e}_{\tilde{r}}}, \delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, \tilde{\delta}_{\bar{\eta}}, \tilde{\delta}_{\tilde{e}_{\tilde{r}}}, d_{\tilde{e}_{\tilde{r}}}, d_{\bar{\eta}})$ (which can be specified from the data of the theorem; for details see Christofides *et al.*[20]), that there is an $\epsilon^{\tilde{e}_{\tilde{r}}}(\phi)$ such that if $\epsilon \in (0, \epsilon^{\tilde{e}_{\tilde{r}}}(\phi)]$ and $|\tilde{e}_{\tilde{r}}(0)| \leq \delta_{\tilde{e}_{\tilde{r}}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\bar{v}|| \leq \delta_{\bar{v}}$, $||\bar{\eta}|| \leq \tilde{\delta}_{\bar{\eta}}$, $||\tilde{e}_{\tilde{r}}|| \leq \tilde{\delta}_{\tilde{e}_{\tilde{r}}}$, then

$$|\tilde{e}_{\tilde{r}}(t)| \leq e^{-0.5t}|\tilde{e}_{\tilde{r}}(0)| + \tilde{\gamma}_u(||U||) + \phi \tag{54}$$

Furthermore, it can be also shown using the result from Christofides and Teel[19] that the singularly perturbed system comprised of $\bar{e}, \eta, z$, with the same converse function which exists for the reduced system comprised of the states $\bar{e}, \eta$ and its resulting $(\beta_{\bar{\eta}}, \bar{\gamma}_s)$. Thus we have that if $\epsilon \in (0, \epsilon^{\bar{\eta}}(\phi)]$ and $|\bar{\eta}(0)| \leq \delta_\eta$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$, $||\tilde{e}_{\tilde{r}}|| \leq \tilde{\delta}_{\tilde{e}_{\tilde{r}}}$, then

$$\begin{aligned} |\bar{\eta}(t)| &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_s(||[\tilde{e}_{\tilde{r}}^T \theta^T \dot{\theta}^T]^T||) + d_{\bar{\eta}} \\ &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0), t) + \bar{\gamma}_{\tilde{e}_{\tilde{r}}}(||\tilde{e}_{\tilde{r}}||) + \bar{\gamma}_\theta(||\theta||) \\ &\quad + \bar{\gamma}_{\dot{\theta}}(||\dot{\theta}||) + d_{\bar{\eta}} \end{aligned} \tag{55}$$

Part 3: The proof of the theorem can be completed by showing that for any given set of positive real numbers $\delta_{\tilde{e}_{\tilde{r}}}$, $\delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d$ (already specified) and with ϕ^* defined as in Equation (52), there exists $\epsilon^*(\phi) \in (0, \epsilon^{\bar{\eta}}(\phi)]$, such that if $\epsilon \in (0, \epsilon^*(\phi)]$ and $|\tilde{e}_{\tilde{r}}(0)| \leq \delta_{\tilde{e}_{\tilde{r}}}$, $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$ the output of the closed-loop system of Equation (37) satisfies the relation of Equation (20), for each $\phi \in (0, \phi^*]$.

This result can be established by analyzing the behavior of the dynamical system comprised of the states $\tilde{e}_{\tilde{r}}, \bar{\eta}$ of the system of Equation (41), for which the inequalities of Equations (54) and (55) hold, using calculations similar to those used in Teel[27] and Jiang[28]. First, a contradiction argument can be used to show that if $\phi \in (0, \phi^*]$, the evolution of the states $\tilde{e}_{\tilde{r}}, \bar{\eta}$ starting from initial conditions that satisfy $|\tilde{e}_{\tilde{r}}(0)| \leq \delta_{\tilde{e}_{\tilde{r}}}$, $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$ and for $\theta, \dot{\theta}, \bar{v}$ such that $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$ satisfies the following inequalities:

$$|\tilde{e}_{\tilde{r}}(t)| \leq \tilde{\delta}_{\tilde{e}_{\tilde{r}}}, \ |\bar{\eta}(t)| \leq \tilde{\delta}_{\bar{\eta}} \tag{56}$$

for all times. Second, the asymptotic behavior of the system comprised of the states $\bar{e}, \tilde{e}_r$ of Equation (41) is analyzed to establish that the inequality of Equation (20) holds. These calculations are similar to the ones in Christofides *et al.*[20] and are omitted for reasons of brevity.

Nomenclature

Roman letters

C_{cat}	concentration of catalytic carbon on spent catalyst

C_{sc}	concentration of total carbon on spent catalyst
C_{rc}	concentration of carbon on regenerated catalyst
D_{tf}	density of total feed
$d, d_{\bar{e}_{\bar{r}}}, d_{\bar{e}}, d_{\bar{\eta}}$	positive real numbers
E_{cc}, E_{cr}, E_{or}	activation energies
$F, \tilde{F}, \tilde{F}_{nom}, f_1, f_2$	vector fields
F_{rc}	flow rate of catalyst from reactor to regenerator
$G, \tilde{G}, G_1, G_2$	vector fields associated with the inputs
h_i	ith output scalar field
$K(x)$	sufficiently smooth matrix
k_{cc}, k_{cr}, k_{or}	pre-exponential kinetic constants
O_{fg}	oxygen in flue gas
Q_1	matrix of dimension $n \times m$ associated with the slow state vector x
Q_2	matrix of dimension $m \times m$ associated with the fast state vector z
R_{ai}	air rate
R_{cb}, R_{cf}, R_{oc}	reaction rates
R_{tf}	total feed rate
$\tilde{r}_i$	integer associated with the input vector $\tilde{u}$ in the reduced system
S_a, S_c, S_f	specific heats
T_{fp}, T_{ai}	temperatures of the feed in the reactor and the air in the regenerator
T_{ra}, T_{rg}	temperatures in reactor and regenerator
t	time
$u, \bar{v}$	manipulated input vectors
$\tilde{u}$	auxiliary input vector
V_{ra}, V_r	reactor and regenerator catalyst holdups
v	external reference input vector
x	vector of the slow state variables
y_i	ith output
z	vector of the fast state variables

Greek letters

β_{ik}	adjustable parameters
$\Delta H_{rg} \Delta H_{cr}$	heat of the reactions
ΔH_{fv}	heat of feed vaporization
$\delta, \delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{e}_{\bar{r}}}, \delta_{\bar{\eta}}, \delta_{\bar{e}}, \delta_\eta, \delta_{\tilde{\bar{e}}_{\bar{r}}}, \delta_{\tilde{\bar{\eta}}}$	positive real numbers
$\epsilon, \epsilon^{\bar{e}_{\bar{r}}}, \epsilon^{\bar{\eta}}, \epsilon^{\bar{e}}$	singular perturbation parameters
ζ	state vector
η	state vector
ϕ	adjustable parameter

Math symbols

$\lvert \cdot \rvert$	standard Euclidean norm
$sgn(\cdot)$	sign function
Id	identity function
$L_f h$	Lie derivative of a scalar field h with respect to the vector field f
$L_f^k h$	kth order Lie derivative
$L_g L_f^{k-1} h$	mixed Lie derivative
$\Re$	real line
$\Re^i$	i-dimensional Euclidean space
$\in$	belongs to
T	transpose