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Automatica 41 (2005) 209–217
www.elsevier.com/locate/automatica

Brief paper

Robust hybrid predictive control of nonlinear systems☆

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Received 13 January 2004; received in revised form 12 August 2004; accepted 16 August 2004
Available online 8 December 2004

Abstract

In this work, we consider nonlinear systems with input constraints and uncertain variables, and develop a robust hybrid predictive control structure that provides a safety net for the implementation of any model predictive control (MPC) formulation, designed with or without taking uncertainty into account. The key idea is to use a Lyapunov-based bounded robust controller, for which an explicit characterization of the region of robust closed-loop stability can be obtained, to provide a stability region within which any available MPC formulation can be implemented. This is achieved by devising a set of switching laws that orchestrate switching between MPC and the bounded robust controller in a way that exploits the performance of MPC whenever possible, while using the bounded controller as a fall-back controller that can be switched in at any time to maintain robust closed-loop stability in the event that the predictive controller fails to yield a control move (due, e.g., to computational difficulties in the optimization or infeasibility) or leads to instability (due, e.g., to inappropriate penalties and/or horizon length in the objective function). The implementation and efficacy of the robust hybrid predictive control structure are demonstrated through simulations using a chemical process example.

Keywords: Input constraints; Lyapunov-based bounded control; Model predictive control; Controller switching; Hybrid systems and control; Stability region

1. Introduction

Stabilization of nonlinear systems subject to uncertainty and manipulated input constraints is a fundamental control problem that has been the subject of significant research work. One of the control methods suited for handling constraints within an optimal control setting is model predictive control (MPC). Numerous research studies have investigated the stability properties of model predictive controllers for systems without uncertainty (e.g., see Keerthi & Gilbert, 1988; De oliveira & Biegler, 1994; Adetola & Guay, 2003 and the review paper Mayne, Rawlings, Rao, & Scokaert, 2000).

The problem of analysis and design of predictive controllers for uncertain linear systems has been extensively investigated (e.g., see Genceli & Nikolaou, 1993; Lee & Yu, 1997; Scokaert & Mayne, 1998; Lee & Kouvaritakis, 2002; Bemporad, Borrelli, & Morari, 2003 and Bemporad & Morari, 1999; Mayne et al., 2000 for surveys of results in this area). For uncertain nonlinear systems, the problem of robust MPC design continues to be an area of ongoing research (see, e.g., Michalska & Mayne, 1993; Fernando, Fontes, & Magni, 2003; Magni, Nicolao, Scattolini, & Allgower, 2003; Alamo, Muoz de la Pea, Limon, & Camacho, 2003; Sakizlis, Kakalis, Dua, Perkins, & Pistikopoulos, 2004). While min–max formulations provide a natural setting within which to address this problem, computational problems with these approaches are well known, and stem in part from the nonlinearity of the model which typically makes the optimization problem nonconvex and in part from performing the min–max optimization over the nonconvex problem. Furthermore, the robust stability guarantee in various MPC formulations (with or without stability conditions and

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor M. Guay under the direction of Editor F. Allgower.

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0005-1098/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2004.08.020

with or without robustness considerations) is contingent upon the assumption of initial feasibility, and the set of initial conditions starting from where feasibility and stability is guaranteed is not explicitly characterized.

Stabilizing control laws that provide explicitly-defined regions of attraction for the closed-loop system have been developed using Lyapunov techniques; the reader may refer to Kokotovic and Arcak (2001) for a survey of results in this area. In El-Farra and Christofides(2003a,b), a class of Lyapunov-based bounded robust nonlinear controllers, inspired by the results on bounded control originally presented in Lin and Sontag (1991), was developed. While these Lyapunov-based controllers have well-characterized stability and constraint-handling properties, they cannot, in general, be designed to be optimal with respect to a prespecified, arbitrary cost function.

From the above discussion, it is clear that both MPC and Lyapunov-based analytic control approaches possess, by design, their own distinct stability and optimality properties. Recently, we proposed a hybrid predictive control structure that employs switching between bounded control and MPC for the stabilization of linear systems under state (El-Farra, Mhaskar, & Christofides, 2004a) and output feedback (Mhaskar, El-Farra, & Christofides, 2004), and in El-Farra, Mhaskar, and Christofides (2004b) for nonlinear systems, subject to input constraints while providing a priori (off-line) the set of initial conditions, for which closed-loop stability is guaranteed (through bounded control).

The presence of uncertainty, however, may alter the stability region of the nominal controllers (designed without taking the uncertainty into account) or even render the closed-loop system unstable. Furthermore, simply replacing the fall-back controller by an appropriate robust controller and implementing the same switching logics proposed in El-Farra et al.(2004a,b) may lead to switching that is too conservative, resulting in the implementation of the fall-back controller for almost all times. Motivated by these considerations, we consider in this work nonlinear systems with input constraints and uncertain variables, and develop a robust hybrid predictive control structure. The proposed method provides a safety net for the implementation of any available MPC formulation, designed with or without taking uncertainty into account, and allows for an explicit characterization of the set of initial conditions starting from where the closed-loop system is guaranteed to be stable. The key idea is to use a Lyapunov-based robust controller, for which an explicit characterization of the closed-loop stability region can be obtained, to provide a stability region within which MPC can be implemented. A set of switching laws are designed that exploit the performance of MPC whenever possible, while using the bounded controller to provide the stability guarantees.

The rest of the paper is organized as follows. In Section 2, we present the class of systems under consideration and review how the robust constrained control problem is addressed within the bounded control and MPC frameworks. In Section 3, two controller switching schemes, that address (with varying degrees of flexibility) robust stability and performance objectives, are described. Finally in Section 4, the implementation and efficacy of the robust hybrid predictive control structure are demonstrated through a chemical process example.

2. Preliminaries

We consider nonlinear systems with uncertain variables and input constraints, described by

$$\dot{x} = f(x) + G(x)u + W(x)\theta(t), \quad u \in \mathbf{U}, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}^m$ denotes the vector of constrained manipulated inputs, taking values in a nonempty convex subset $\mathbf{U}$ of $\mathbb{R}^m$, where $\mathbf{U} = \{u \in \mathbb{R}^m : \|u\| \leqslant u_{\max}\}$, $\|\cdot\|$ is the Euclidean norm of a vector, $u_{\max} > 0$ is the magnitude of input constraints, and $\theta(t) = [\theta^1(t) \ldots \theta^q(t)]^{\mathrm{T}} \in \Theta \subset \mathbb{R}^q$ denotes the vector of uncertain (possibly time-varying) but bounded variables taking values in a nonempty compact convex subset of $\mathbb{R}^q$ and $f(0) = 0$. The vector function $f(x)$, the matrices $G(x)=[g^1(x) \ldots g^m(x)]$ and $W(x)=[w^1(x) \ldots w^q(x)]$, where $g^i(x) \in \mathbb{R}^n$, $i = 1 \cdots m$, and $w^i(x) \in \mathbb{R}^n$, $i = 1 \cdots q$, are assumed to be sufficiently smooth on their domains of definition. The notation $L_f h$ denotes the standard Lie derivative of a scalar function $h(\cdot)$ with respect to the vector function $f(\cdot)$, the notation $x(T^-)$ denotes the limit of the trajectory $x(t)$ as T is approached from the left, i.e., $x(T^-)=\lim_{t \to T^-} x(t)$ and the notation $\partial\Omega$ is used to donate the boundary of a closed set, Ω. Throughout the manuscript, we assume that for any $u \in \mathbf{U}$ the solution of the system of Eq. 1 exists and is continuous for all t, and we focus on the state feedback problem where measurements of the entire state, $x(t)$, are assumed to be available for all t.

2.1. Bounded robust Lyapunov-based control

Referring to the system of Eq. (1), we assume that the uncertain variable term, $W(x)\theta$, is non-vanishing (in the sense that the origin is no longer the equilibrium point of the uncertain system) and that a robust control Lyapunov function (RCLF) (Freeman & Kokotovic, 1996), V exists. Consider also, the bounded state feedback control law (see El-Farra & Christofides, 2001, 2003a, b, for details on controller design):

$$u = -\left(\frac{\alpha(x) + \sqrt{(\alpha_1(x))^2 + (u_{\max}\beta(x))^4}}{(\beta(x))^2[1 + \sqrt{1 + (u_{\max}\beta(x))^2}]}\right)(L_G V)^{\mathrm{T}} \tag{2}$$

when $L_G V \neq 0$ and $u = 0$ when $L_G V = 0$, where $\alpha(x) = L_f V + (\rho\|x\| + \chi\theta_b\|(L_W V)^{\mathrm{T}}\|)\left(\frac{\|x\|}{\|x\|+\phi}\right)$, $\alpha_1(x) = L_f V + \rho\|x\| + \chi\theta_b\|(L_W V)^{\mathrm{T}}\|$, $\beta(x) = \|(L_G V)^{\mathrm{T}}\|$,

$L_G V = [L_{g^1} V \ \cdots \ L_{g^m} V]$ and $L_W V = [L_{w^1} V \ \cdots \ L_{w^q} V]$ are row vectors, θ_b is a positive real number such that $\|\theta(t)\| \leqslant \theta_b$, for all $t \geqslant 0$, and ρ, χ and ϕ are adjustable parameters that satisfy $\rho > 0$, $\chi > 1$ and $\phi > 0$. Let Π be the set defined by $\Pi(\theta_b, u_{\max}) = \{x \in \mathbb{R}^n : \alpha_1(x) \leqslant u_{\max}\beta(x)\}$ and assume that $\Omega := \{x \in \mathbb{R}^n : V(x) \leqslant c^{\max}\} \subseteq \Pi(\theta_b, u_{\max})$ for some $c^{\max} > 0$. Then, given any positive real number, d, such that:

$$\mathbb{D} := \{x \in \mathbb{R}^n : \|x\| \leqslant d\} \subset \Omega \tag{3}$$

and for any initial condition $x_0 \in \Omega$, it can be shown that there exists a positive real number ε^* such that if $\phi/(\chi - 1) < \varepsilon^*$, the states of the closed-loop system of Eqs. (1) and (2) satisfy $x(t) \in \Omega \ \forall \ t \geqslant 0$ and $\limsup_{t\to\infty} \|x(t)\| \leqslant d$.

Remark 1. Referring to the above controller design, it is important to make the following remarks. First, a general procedure for the construction of RCLFs for nonlinear systems of the form of Eq. (1) is currently not available. Yet, for several classes of nonlinear systems that arise commonly in the modeling of engineering applications, it is possible to exploit system structure to construct RCLFs. For example, for feedback linearizable systems, quadratic Lyapunov functions can be chosen as candidate RCLFs and can be made RCLFs with appropriate choice of the function parameters based on the process parameters (see, for example, Freeman & Kokotovic, 1996). Also, for nonlinear systems in strict feedback form, backstepping techniques can be employed for the construction of RCLFs (Krstic, Kanellakopoulos, & Kokotovic, 1995). Second, given that an RCLF, V, has been obtained for the system of Eq. (1), it is important to clarify the essence and scope of the additional assumption that there exists a level set, Ω, of V that is contained in Π. Specifically, the assumption that the set, Π, contains an invariant subset around the origin, is necessary to guarantee the existence of a set of initial conditions for which closed-loop stability is guaranteed (note that even though $\dot{V} < 0 \ \forall \ x \in \Pi \backslash \mathbb{D}$, there is no guarantee that trajectories starting within Π remain within Π for all times). Moreover, the assumption that Ω is a level set of V is made only to simplify the construction of Ω. This assumption restricts the applicability of the proposed control method because a direct method for the construction of an RCLF with level sets contained in Π is not available. However, the proposed control method remains applicable if the invariant set Ω is not a level set of V but can be constructed in some other way (which, in general, is a difficult task).

Remark 2. Regarding the choice of the above controller design, we note that the problem of designing control laws that guarantee stability in the presence of input constraints has been extensively studied (see, for example, Lin & Sontag, 1991; Teel, 1992; Liberzon, Sontag & Wang, 2002; El-Farra & Christofides, 2003a,b). The bounded robust controller design of Eq. (2), proposed in El-Farra and Christofides(2003a,b) (inspired by the results on bounded control in Lin & Sontag, 1991 for systems without uncertainty) is an example of a controller design that (1) guarantees robust stability in the presence of constraints, and (2) allows for an explicit characterization of the closed-loop stability region. The results of this paper are not limited to this particular choice of controllers and any other robust controller that satisfies (1) and (2) above, can be used.

2.2. Model predictive control

The model predictive control approach provides a framework with the ability to handle, among other issues, multivariable interactions, constraints on controls, and optimization requirements, all in a consistent, systematic manner. For the purpose of illustrating our results, we describe here a symbolic MPC formulation that incorporates most existing MPC formulations as special cases. This is not a new formulation of MPC; the general description is only intended for the purpose of highlighting the fact that the robust hybrid predictive control structure (to be proposed in the next section) can incorporate any available MPC formulation. In MPC, the control action at time t is conventionally obtained by solving, on-line, a finite horizon optimal control problem. The generic form of the optimization problem can be described as

$$\begin{aligned} &u(\cdot) = \arg\min\{\max\{J_s(x, t, u(\cdot))|\theta(\cdot) \in \Theta\}|u(\cdot) \in S\} \\ &s.t. \ \dot{x}(t) = f(x(t)) + G(x)u + W(x)\theta(t), \\ &x(0) = x_0, \quad x(t+T) \in \Omega_{\text{MPC}}(x, t, \theta), \end{aligned} \tag{4}$$

where

$$\begin{aligned} &J_s(x, t, u(\cdot)) \\ &= \int_t^{t+T} (x'(s)Qx(s) + u'(s)Ru(s))\,\mathrm{d}s \\ &\quad + F(x(t+T)) \end{aligned} \tag{5}$$

and $S = S(t, T)$ is the family of piecewise continuous functions, with period Δ, mapping $[t, t+T]$ into the set of admissible controls, T is the horizon length and θ is the bounded uncertainty assumed to belong to a set Θ. A control $u(\cdot)$ in S is characterized by the sequence $\{u[k]\}$, where $u[k] := u(k\Delta)$ and satisfies $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. J_s is the performance index, R and Q are strictly positive definite, symmetric matrices and the function $F(x(t+T))$ represents a penalty on the states at the end of the horizon. The maximization over θ may not be carried out if the MPC version used is not a min–max type of formulation. The set $\Omega_{\text{MPC}}(x, t, \theta)$ could be a fixed, terminal set, or may represent inequality constraints (as in the case of MPC formulations that require some norm of the state, or a Lyapunov function for the system, to decrease at the end of the horizon). This stability constraint may or may not account for uncertainty. The stability guarantees in MPC formulations (with or without explicit stability conditions, and with or without robustness considerations, and whether or not it is a min–max type of formulation) are dependent on the assumption of initial feasibility. Obtaining an explicit characterization of the closed-loop stability region of the predictive controller under uncertainty and constraints remains a difficult task.

3. Robust hybrid predictive control structure

We first cast the system of Eq. (1) as a switched system of the form

$$\dot{x} = f(x) + G(x)u_{i(t)} + W(x)\theta(t), \ \ \|u_{i(t)}\| \leqslant u_{\max}, \tag{6}$$

where $i : [0, \infty) \to \{1, 2\}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, implying that only a finite number of switches between the two controllers is allowed on any finite-time interval. The index, $i(t)$, represents a discrete state that indexes the control input, u, with the understanding that $i(t) = 1$ if and only if $u_i(x(t)) = M_s(x(t))$ (i.e., MPC is used) and $i(t) = 2$ if and only if $u_i(x(t)) = b(x(t))$ (i.e., bounded control is used). Our goal is to construct a switching law that provides the supervisor with the switching times that ensure stabilizing transitions between the two controllers. We first present a scheme in Theorem 1 that focuses on the issue of robust closed-loop stability, and then in Theorem 2, provide more flexible switching rules that, in addition, enhance the overall closed-loop performance. The proofs of both theorems are given in the appendix.

3.1. Stability-based controller switching

Theorem 1. *Consider the switched nonlinear system of Eq.* (6), *the model predictive controller of Eqs.* (4) *and* (5) *and the bounded controller of Eq.* (2). *Let* $x(0) = x_0 \in \Omega$, *and initially set* $T_s = T_D = T_{\inf} = \infty$. *At the earliest time* $t \geqslant 0$ *for which the closed-loop state under MPC satisfies* $V(x(t^-)) = c^{\max}$ *set* $T_s = t$. *At the earliest time for which the closed-loop state under MPC satisfies* $\|x(t)\| \leqslant d$, *where* d *was defined in Eq.* (3), *set* $T_D = t$. *Finally, at the earliest time* t *that MPC is infeasible, set* $T_{\inf} = t$. *Define* $T_{\text{switch}} = \min\{T_s, T_D, T_{\text{design}}, T_{\inf}\}$, *where* $0 \leqslant T_{\text{design}} < \infty$ *is arbitrary. Then, the switching rule*

$$i(t) = \begin{cases} 1, & 0 \leqslant t < T_{\text{switch}} \\ 2, & t \geqslant T_{\text{switch}} \end{cases} \tag{7}$$

guarantees that $x(t) \in \Omega \ \forall \ t \geqslant 0$ *and* $\limsup_{t\to\infty} \|x(t)\| \leqslant d$.

Remark 3. The robust hybrid predictive controller of Theorem 1 is designed and implemented as follows:

- Given the nonlinear system of Eq. (1), θ_b and $u_{\max}$, design the bounded robust controller of Eq. (2), and calculate an estimate of its stability region Ω.
- Design/pick an MPC formulation (the MPC formulation could be min–max optimization based, linear or nonlinear, and with or without stability constraints). For convenience, we refer to the general MPC formulation of Eqs. (4) and (5).
- Given any $x_0 \in \Omega$, check the feasibility of the optimization in Eqs. (4) and (5) at $t = 0$, and if feasible, start implementing MPC (i.e., set $u(0) = M_s(x_0)$).
- If at any time, MPC becomes infeasible ($t = T_{\inf}$), or the states of the closed-loop system approach the boundary of Ω ($t = T_s$) or the closed-loop states enter the set $\mathbb{D}$ ($t = T_D$) then switch to the bounded robust controller, else keep MPC active in the closed-loop system until a time T_{design}.
- Switch to the bounded robust controller at T_s, T_D, T_{design}, or $T_{\inf}$, whichever comes earliest, to achieve practical closed-loop stability.

Remark 4. The purpose of switching to the bounded robust controller after the time T_{design} is to ensure convergence to $\mathbb{D}$ and avoid possible cases where the closed-loop states, under MPC, could wander inside Ω without actually converging to, and staying within, $\mathbb{D}$. Convergence to $\mathbb{D}$ could also be achieved (see, for example, El-Farra et al., 2004a,b), by switching to the bounded controller when $\dot{V} \geqslant 0$ under MPC. However, in the presence of uncertainty, such a condition might be very restrictive in the sense that it may terminate MPC implementation too early. Note that if an MPC design, that guarantees robust stability for the uncertain nonlinear system if initially feasible, is used it could be implemented for all time (T_{design} can be chosen to be practically infinity) to stabilize the closed-loop system. The stability safeguards, provided by the bounded controller, are still required because the stability of any MPC formulation, robust or otherwise, is based on the assumption of initial feasibility, which cannot be verified short of testing, via simulation, an initial condition for feasibility.

Remark 5. We note that while the MPC framework provides a transparent way of specifying a performance objectives, the various MPC formulations, in general, may not be optimal, and only approximate the infinite horizon optimal cost to varying degrees of success. The choice of a particular MPC design can be made entirely on the basis of the desired tradeoff between performance and computational complexity because the stability guarantees of the robust hybrid predictive controller are independent of the specific MPC formulation being used.

3.2. Enhancing closed-loop performance

In this section we relax the switching rules of Theorem 1, in order to take better advantage of the MPC performance. The relaxed switching scheme uses a family of l fall back controllers together with the MPC and allows for multiple switchings between MPC and the bounded controllers. We recast the system of Eq. (1) as a switched system of the form:

$$\dot{x} = f(x) + G(x)u_{i(t)} + W(x)\theta(t)$$
$$\|u_{i(t)}\| \leqslant u_{\max}, \;\; i(t) \in \{1, 2, \ldots, l+1\}, \tag{8}$$

where $i(t) = k$ if and only if $u_i(x(t)) = b_k(x(t))$ (i.e., the kth bounded controller is used) and $i(t) = l + 1$ if and only if $u_i(x(t)) = M_s(x(t))$ (i.e., MPC is used). We also define, for the l bounded controllers, and the robust control Lyapunov functions, $V_k, \;\; k=1, \ldots, l$, $0 < \alpha < 1$, $c_{k,j}^{\max} = \alpha^j c_k^{\max}$, $\Omega_k^j = \{x \in \mathbb{R}^n : V_k(x) \leqslant c_{k,j}^{\max}\}$ and $\Omega_L^j = \bigcup_{k=1}^l \Omega_k^j$, for $j = 0, \ldots, N-1$, where $N < \infty$ is an integer, and $d^{\max} = \max_{k=1,\ldots,l}\{d_k\}$, where d_k are arbitrarily small positive numbers such that $\mathbb{D}_k = \{x \in \mathbb{R}^n : \|x\| \leqslant d_k\} \subset \Omega_k^{N-1}$.

Theorem 2. *Consider the constrained switched nonlinear system of Eq.* (8), *the model predictive controller of Eqs.* (4) *and* (5) *and the l bounded controllers of the form of Eq.* (2), *designed using the robust control Lyapunov functions $V_k, \;\; k=1, \ldots, l$. Consider any initial condition $x(0) \equiv x_0 \in \Omega_L^0$ and initially set $T_0 = 0$, $T_1 = T_2 = \cdots = T_N = T_D = \infty$. At the earliest time, $t \geqslant 0$, for which the closed-loop state under MPC satisfies $\|x(t)\| \leqslant d^{\max}$, set $T_D = t$. At the earliest time, $t \geqslant 0$, under MPC such that*

$$x(t) \in \partial\Omega_L^{j-1}, \quad x(t^-) \in \Omega_L^{j-1} \quad \text{and} \quad T_{j-1} < \infty \tag{9}$$

for some $j \in \{1, \ldots, N\}$, or MPC is infeasible, set $T_j = t$. Define $T_{\text{switch}} = \min\{T_D, T_N, T_{\text{design}}\}$, where $0 \leqslant T_{\text{design}} < \infty$ is arbitrary. Then the switching rule

$$i(t) = \begin{Bmatrix} l+1, & 0 \leqslant T_j \leqslant t \leqslant T_{j+1} \leqslant T_{\text{switch}}, \; x \in \Omega_L^j \\ a, & 0 \leqslant T_j \leqslant t \leqslant T_{j+1} \leqslant T_{\text{switch}}, \; x \notin \Omega_L^j \\ f, & t > T_{\text{switch}} \end{Bmatrix} \tag{10}$$

for some $a \in \{1, \ldots, l\}$ for which $x(T_j) \in \Omega_a^{j-1}$, and for some $f \in \{1, \ldots, l\}$ for which $x(T_{\text{switch}}) \in \Omega_f^0$, guarantees that $x(t) \in \Omega_L^0 \forall \; t \geqslant 0$ and $\lim\sup_{t\to\infty} \|x(t)\| \leqslant d^{\max}$.

Remark 6. The robust hybrid predictive controller of Theorem 2 is designed and implemented as follows (see also the schematic representation in Fig. 1 for an example when $l = N = 2$):

- For each Lyapunov function V_k, $k = 1, \ldots, l$: construct a bounded controller using Eq. (2), a set of N concentric level sets, Ω_k^j, $j = 0, \ldots, N-1$, with $\Omega_k^{N-1} \subset \Omega_k^{N-2} \subset \cdots \subset \Omega_k^1 \subset \Omega_k^0$, and a terminal set, $\mathbb{D}_k \subset \Omega_k^{N-1}$.
- Initialize the closed-loop system using MPC at any initial condition, x_0, within Ω_L^0 (the union of the level sets Ω_k^0) and monitor the Lyapunov function value.
- At the earliest time that either the closed-loop state approaches the boundary of Ω_L^0 or MPC is infeasible, switch to bounded control (this corresponds to encountering T_1; see Fig. 1).

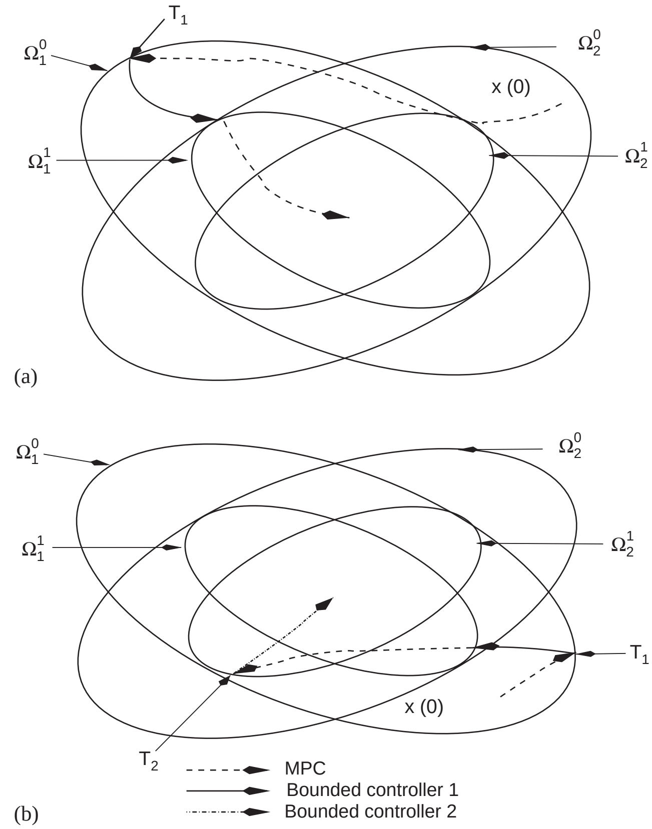

Fig. 1. Examples of the evolution of closed-loop state trajectory under the switching scheme of Theorem 2.

- Implement any of the bounded controllers whose stability region contains the state at T_1 until the state enters Ω_L^1 (the union of the level sets Ω_k^1) after which switch back to MPC (in Fig. 1 this happens when the state enters Ω_1^1).
- If the closed-loop state under MPC does not escape the boundary of Ω_L^1 and MPC continues to be feasible, keep MPC in the closed-loop system (this happens in Fig. 1(a), where the state continues to stay within Ω_L^1). If the closed-loop state, however, tries to escape the boundary of Ω_L^1 (this happens in Fig. 1(b) when the state hits the boundary of Ω_2^1 at $t = T_2$) or MPC is infeasible again, switch to bounded control and implement it until the states are driven inside the next safety zone where MPC is re-activated and confined (note that each time MPC is switched back in, its implementation is confined within a smaller safety zone since $\Omega_L^1 \supset \Omega_L^2 \supset \cdots \supset \Omega_L^{N-1}$). Repeat this process until either the state under MPC is about to escape the Nth safety zone, Ω_L^{N-1}, or the state enters the largest terminal set, $\mathbb{D}_{\max}$, or a time greater than the design parameter, T_{design}, has elapsed. At the earliest time that any of these events takes place, switch permanently to any of the bounded controllers whose stability region contains the state at the time to ensure practical stability of the closed-loop system.

Table 1
Process parameters and steady-state values.

$V_{\mathrm{R}} = 0.1\,\mathrm{m}^3$	$R = 8.314\,\mathrm{J/mol}\cdot\mathrm{K}$
$C_{A0s} = 1.0\,\mathrm{Kmol/m}^3$	$T_{A0s} = 310.0\,\mathrm{K}$
$\Delta H_{\mathrm{n}} = -4.78\times 10^4\,\mathrm{J/mol}$	$k_0 = 72\times 10^9\,\mathrm{min}^{-1}$
$E = 8.314\times 10^4\,\mathrm{J/mol}$	$c_p = 0.239\,\mathrm{J/g}\cdot\mathrm{K}$
$\rho = 1000.0\,\mathrm{Kg/m}^3$	$F = 0.1\,\mathrm{m}^3/\,\mathrm{min}$
$T_{\mathrm{R}s} = 395.3\,\mathrm{K}$	$C_{As} = 0.57\,\mathrm{Kmol/m}^3$

4. Application to a chemical reactor

Consider the following model of an irreversible elementary exothermic reaction of the form $A \xrightarrow{k} B$ in a well-mixed continuous stirred tank reactor:

$$V_{\mathrm{R}}\frac{\mathrm{d}C_{\mathrm{A}}}{\mathrm{d}t} = F(C_{A0} - C_{\mathrm{A}}) - k_0\,\exp\left(\frac{-E}{RT}\right)C_{\mathrm{A}}V,$$

$$V_{\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}t} = F(T_{A0} - T) - \frac{\Delta H}{\rho c_p}k_0\,\exp\left(\frac{-E}{RT}\right)C_A V + \frac{Q}{\rho c_p}, \tag{11}$$

where C_{A} denotes the concentration of species A, T, V_{R} denote the temperature and volume of the reactor, respectively, Q denotes the rate of heat input to the reactor, k_0, E, ΔH denote the pre-exponential constant, the activation energy, and the enthalpy of the reaction, respectively, and c_p and ρ denote the heat capacity and density of the fluid in the reactor, respectively. Table 1 lists the steady-state values and process parameters. The control objective is to regulate both the reactor temperature and reactant concentration at the (open-loop) unstable equilibrium point by manipulating both the rate of heat input/removal and the inlet reactant concentration. Defining

$$x_1 = C_{\mathrm{A}} - C_{As},\quad x_2 = T - T_s,\, u_1 = C_{A0} - C_{A0s},$$

$$u_2 = Q,\, \theta_1(t) = T_{A0} - T_{A0s},\, \theta_2(t) = \Delta H - \Delta H_{\mathrm{n}},$$

where the subscript s denotes the steady-state value and ΔH_{n} denotes the nominal value of the heat of reaction, the process model of Eq. (11) can be cast in the form of Eq. (1). In all simulation runs, $\theta_1(t) = \theta_0\,\sin(3t)$, where $\theta_0 = 0.08T_{A0s}$ and $\theta_2(t) = 0.5(-\Delta H_{\mathrm{nom}})$, and the manipulated input constraints were $|u_1| \leqslant 1.0\,\mathrm{Kmol/m}^3$ and $|u_2| \leqslant 92\,\mathrm{KJ/s}$.

A quadratic Lyapunov function was used to design the bounded robust controller of Eq. (2), using $\chi = 1.01$, $\phi = 0.0001$, and $\rho = 0.01$. The term β^2 in the denominator of the control law of Eq. (2) was replaced by the number $v = 0.001$ close to the origin to alleviate chattering of the control action (note that for this example, the denominator term $\beta^2 = 0$ if and only if $x = 0$). The set of nonlinear ODEs was integrated using the MATLAB solver, ODE45, and the optimization problem in MPC was solved using the MATLAB nonlinear constrained optimization solver, *fmincon*.

The first set of simulation runs demonstrate the use of the robust hybrid predictive control strategy of Theorem 1, with

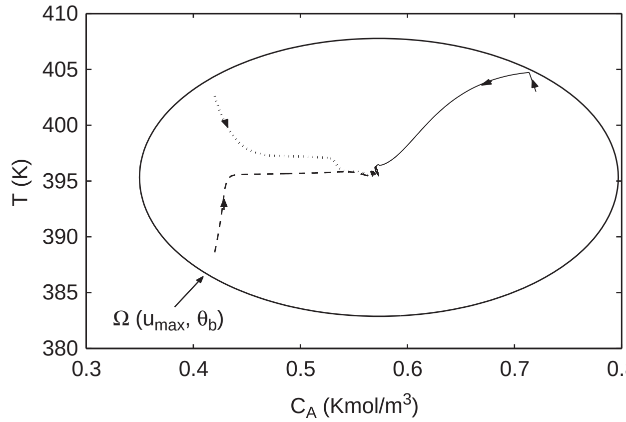

Fig. 2. Closed-loop state trajectory: implementation of the robust hybrid predictive controller of Theorem 1 using a nominal MPC formulation without stability constraints (solid line) and an MPC formulation with stability constraints, from two different initial conditions (dashed and dotted lines).

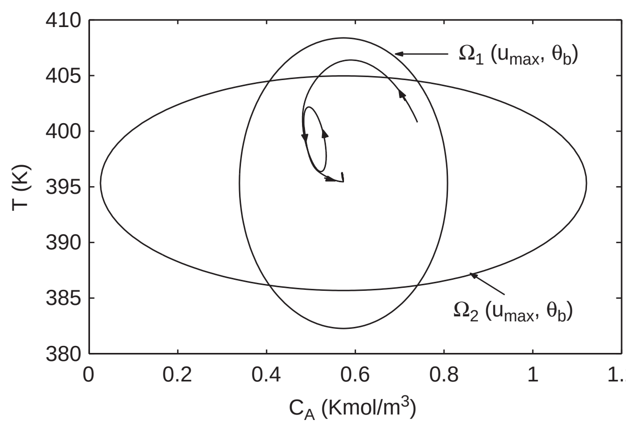

Fig. 3. Closed-loop state trajectory: implementation of the robust hybrid predictive controller of Theorem 2, with $l = 2$, $N = 1$, using a nominal MPC formulation without stability constraints (solid line).

the closed-loop system initialized from a point within the stability region $\Omega(u_{\max}, \theta_b)$. In the first scenario, a nominal nonlinear MPC formulation, without stability constraints, is used as part of the robust predictive control structure (setting $\Omega_{\mathrm{MPC}} = \mathbb{R}^n$, $F(x(t+T)) = 0$, $T = 0.02$ min in Eqs. (4) and (5)) and with the design parameter $T_{\mathrm{design}} = 10$ min. Shortly after the initial implementation of MPC, the supervisor detects, at $t = 0.6$ s, that the closed-loop states are close to the boundary of $\Omega(u_{\max}, \theta_b)$ and therefore switches to the bounded robust controller to stabilize the closed-loop system (solid lines in Figs. 2 and 4). Note that the stability region information is completely contained in the value of the level set obtained at the time of the computation of the stability region ($c^{\max}$) and the supervisor reaches this inference by simply evaluating the Lyapunov function, and comparing it to $c^{\max}$. In the next scenario, a stabilizing formulation of MPC is used (requiring the states to go to a small invariant set at the end

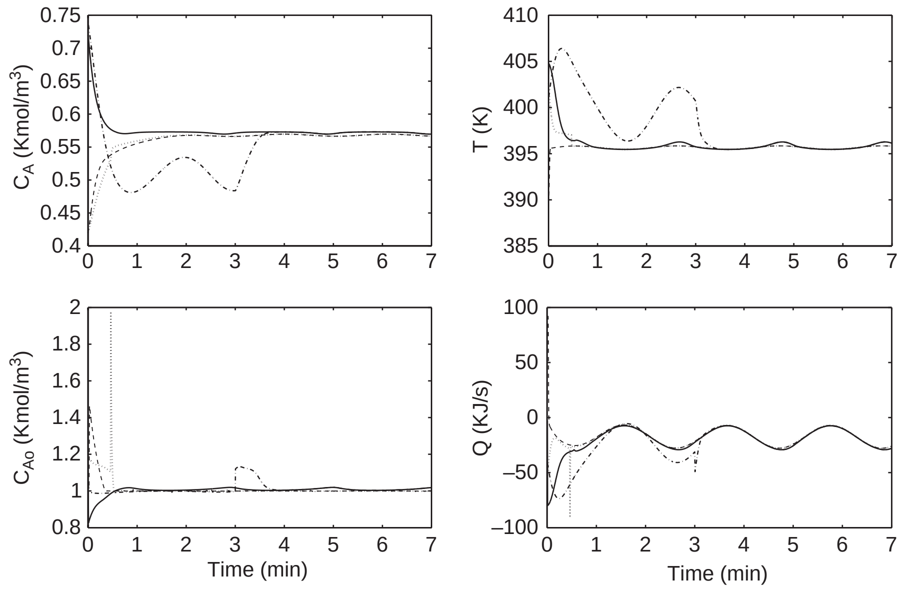

Fig. 4. Closed-loop state (top) and input (bottom) profiles: implementation of the robust hybrid predictive controller of Theorem 1 using a nominal MPC formulation without stability constraints (solid lines), an MPC formulation with stability constraints, from two different initial conditions (dashed and dotted lines) and the robust hybrid predictive controller of Theorem 2, with $l = 2$, $N = 1$, using a nominal MPC formulation without stability constraints (dash–dotted lines).

of the horizon), with a horizon length of $T = 0.02$ min and a $T_{\text{design}} = 20$ min. For the initial condition of the trajectory shown by the dashed lines in Figs. 2 and 4, the MPC yields a feasible solution and drives the states close to the origin. For the initial condition depicted by the dotted lines in Figs. 2 and 4, however, the MPC does not yield a feasible solution, and therefore the supervisor initially implements the bounded robust controller, switching to the MPC at $t = 0.465$ min, when the MPC becomes feasible, and leads to closed-loop stability.

Finally, we demonstrate the implementation of the relaxed switching scheme of Theorem 2 using two Lyapunov functions. We use a nominal nonlinear MPC formulation, without stability constraints (setting $\Omega_{\text{MPC}} = \mathbb{R}^n$, $F(x(t+T)) = 0$, $T = 0.02$ min in Eqs. (4) and (5)) and with the design parameter $T_{\text{design}} = 3$ min. Starting from an initial condition within Ω_2 under MPC, the switching logic allows MPC to be implemented in the closed-loop even though the states escape out of Ω_2, since they are still within Ω_1 (see solid line in Fig. 3 and dashed lines in Fig. 4). Note also that in this case the nominal MPC does not enforce the desired degree of uncertainty attenuation (note the oscillations). After the time T_{design} has elapsed, the supervisor implements the bounded controller (associated with Ω_2) in the closed-loop system to achieve practical stabilization.

Acknowledgements

Financial support from the National Science Foundation, CTS–0129571, is gratefully acknowledged.

Appendix A.

Proof of Theorem 1. The proof uses the fact that if the state of the closed-loop system resides in Ω when the bounded controller is switched in, i.e., $x(T_{\text{switch}}) \in \Omega$, then $x(t) \in \Omega \; \forall \; t \geqslant T_{\text{switch}}$, and convergence to the set $\mathbb{D}$ under the bounded robust controller are guaranteed (for a proof, see El-Farra & Christofides, 2003a,b). Therefore, we need only show that, under the switching scheme of Eq. (7), the closed-loop state is always within Ω until the time the bounded controller is switched in. For this purpose, we enumerate the four possible values that T_{switch} could take, and show that $x(T_{\text{switch}}) \in \Omega$ in all cases.

- *Case* 1: If $T_{\text{switch}} = T_s$, then from the definition of T_s, $x(T_s^-) \in \Omega$. By continuity of the solution of the system of Eq. (1) (which follows from the fact that the right-hand side of Eq. (6) is continuous in x and piecewise continuous in time, since only a finite number of switches is allowed), we have $x(T_s) \in \Omega$, i.e., $x(T_{\text{switch}}) \in \Omega$.
- *Case* 2: If $T_{\text{switch}} = T_D$, then from the definition of T_D, $x(T_D) \in \mathbb{D} \subset \Omega$; hence $x(T_{\text{switch}}) \in \mathbb{D} \subset \Omega$.
- *Case* 3: If $T_{\text{switch}} = T_{\text{design}}$, then from the definition of T_{switch}, $T_{\text{design}} \leqslant T_s$. To prove that $x(T_{\text{design}}) \in \Omega$, we proceed by contradiction. Assume $x(T_{\text{design}}) \notin \Omega$. Then $V(x(T_{\text{design}})) > c^{\max}$ (from the definition of Ω). By continuity of the solution, continuity of $V(\cdot)$ and the fact that $V(x_0) \leqslant c^{\max}$, there exists a time $0 \leqslant T_s' \leqslant T_{\text{design}}$ for which $V(x(T_s'^-)) = c^{\max}$. Since T_s is the

earliest time for which $V(x(t^{'-})) = c^{\max}$, then it must be that $T_s \leqslant T_s' \leqslant T_{\text{design}}$, which leads to a contradiction ($T_{\text{design}} \leqslant T_s$). Therefore, once again, $x(T_{\text{switch}}) \in \Omega$.
- *Case* 4: If $T_{\text{switch}} = T_{\text{inf}}$, the same argument as in Case 3 can be used (replacing T_{design} by T_{inf}) to show that $x(T_{\text{switch}}) \in \Omega$. □

Proof of Theorem 2. The proof of this theorem uses the fact that if the state of the closed-loop system resides in Ω_f^0 at the time that the fth bounded controller is switched in (and kept in the closed-loop for all future times), then under the bounded robust controller f, the region Ω_f^0 is invariant (for a proof, see El-Farra & Christofides, 2003a,b), and convergence to $\mathbb{D}$ is guaranteed. We now show that for $t \leqslant T_{\text{switch}}$ (i.e., up until the time that the switching between the controllers is governed by Eqs. (9) and (10)), the closed-loop state, initialized within Ω_L^0, stays within Ω_L^0. We only need to show that the switching logic ensures that the bounded controller is switched in before the states have escaped Ω_L^0 under MPC. To this end, we first note that $c_{k,j}^{\max} < c_{k,0}^{\max}$, for all $k = 1, \ldots, l$. Therefore, $\Omega_L^j \subset \Omega_L^0$. This implies that if $x \in \Omega_L^j$ then $x \in \Omega_L^0$.

Consider any time, t, when $T_j \leqslant t < T_{j+1} < T_{\text{switch}}$, for some $j \geqslant 1$. Between T_j and T_{j+1}, we have from Eq. (10) that MPC is switched in (and the bounded controller is switched out) only when $x(t) \in \Omega_L^j$. Now consider the time T_{j+1} at which time the bounded controller is switched back in, either due to Eq. (9) or because MPC is infeasible. In the first case, $x(T_{j+1}) \in \Omega_L^j \subset \Omega_L^0$. In the second case, since the right-hand side of the Eq. (8) is a piecewise continuous function of time and a continuous function of its states, and the closed-loop system is initialized within Ω_L^j under MPC, any time the MPC optimization problem is infeasible, $x(t) \in \Omega_L^j \subset \Omega_L^0$. At the time $t = T_{\text{switch}}$, the state of the closed-loop system can be shown to be within Ω_L^0 following the same line of reasoning used in the proof of Theorem 1, with the additional possibility of $T_{\text{switch}} = T_N$. As shown above, $x(t) \in \Omega_L^0$ for all $T_{j-1} \leqslant t \leqslant T_j$ and therefore $x \in \Omega_L^0$ for $T_{N-1} \leqslant t \leqslant T_N$ and therefore $x(T_N) \in \Omega_L^0$ implying once again that$x(T_{\text{switch}}) \in \Omega_L^0$. For all $t \geqslant T_{\text{switch}}$, the closed-loop state evolves under some bounded controller $u(x) = b_k(x)$, for which $x(T_{\text{switch}}) \in \Omega_k^0$. Therefore, $\limsup_{t\to\infty} \|x(t)\| \leqslant d_k$ (see El-Farra & Christofides, 2003a,b for the proof). Note that since $d^{\max}$ is chosen to be the largest of d_k, $k = 1, \cdots, l$, $\|x(t)\| \leqslant d_k$ implies $\|x(t)\| \leqslant d^{\max}$; hence $\limsup_{t\to\infty} \|x(t)\| \leqslant d^{\max}$. This completes the proof of Theorem 2. □

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