Predictive Control of Switched Nonlinear Systems With Scheduled Mode Transitions

Prashant Mhaskar, Nael H. El-Farra, and Panagiotis D. Christofides, *Member, IEEE*

***Abstract*—In this work, a predictive control framework is proposed for the constrained stabilization of switched nonlinear systems that transit between their constituent modes at prescribed switching times. The main idea is to design a Lyapunov-based predictive controller for each constituent mode in which the switched system operates and incorporate constraints in the predictive controller design which upon satisfaction ensure that the prescribed transitions between the modes occur in a way that guarantees stability of the switched closed–loop system. This is achieved as follows: For each constituent mode, a Lyapunov-based model predictive controller (MPC) is designed, and an analytic bounded controller, using the same Lyapunov function, is used to explicitly characterize a set of initial conditions for which the MPC, irrespective of the controller parameters, is guaranteed to be feasible, and hence stabilizing. Then, constraints are incorporated in the MPC design which, upon satisfaction, ensure that: 1) the state of the closed–loop system, at the time of the transition, resides in the stability region of the mode that the system is switched into, and 2) the Lyapunov function for each mode is nonincreasing wherever the mode is reactivated, thereby guaranteeing stability. The proposed control method is demonstrated through application to a chemical process example.**

***Index Terms*—Bounded Lyapunov-based control, input constraints, model predictive control (MPC), multiple Lyapunov functions, stability regions, switched systems.**

I. Introduction

THE operation of processes often involves controlled, discrete transitions between multiple, continuous modes of operation in order to handle, for example, changes in raw materials, energy sources, product specifications and market demands. The superposition of discrete events on the continuous process dynamics gives rise to a hybrid system behavior, i.e., intervals of piecewise continuous behavior interspersed by discrete transitions. Compared with purely continuous systems, the hybrid nature of these systems makes them more difficult to describe, analyze, and control. Motivated by these challenges as well as the abundance of situations where hybrid systems arise in practice, significant research efforts have focused on the study of hybrid systems over the last decade, covering a wide range of problems including, for example, modeling [1], optimization [2], stability analysis [3]–[6], and control [7]–[10].

A class of hybrid systems that has attracted significant attention recently, because it can model several practical control problems that involve integration of supervisory logic-based control schemes and feedback control algorithms, is the class of switched (or multimodal) systems. For this class, results have been developed for stability analysis using the tools of multiple Lyapunov functions (MLFs), for linear [11] and nonlinear [3] systems, and the concept of dwell time [5]; the reader may refer to [4] and [6] for a survey of results in this area. These results have motivated the development of methods for control of various classes of switched systems (see, e.g., [12], [13], and [9]).

In [14], a framework for coordinating feedback and switching for control of hybrid nonlinear systems with input constraints was developed. The key feature of the control methodology in [14] is the integrated synthesis of: 1) a family of lower-level bounded nonlinear controllers that stabilize the continuous dynamical modes, and provide an explicit characterization of the stability region associated with each mode, and 2) upper-level switching laws that determine, on the basis of the stability regions, whether or not a transition will preserve stability of the overall switched closed-loop system. While the approach allows one to *determine* whether or not a switch can be made at any given time without loss of stability, it does not address the problem of *ensuring* that such a switch be made safely at some predetermined time.

Guiding the system through a prescribed switching sequence requires a control algorithm that can incorporate both state and input constraints in the control design. One such control method is model predictive control (MPC) and has been studied extensively (see, for example, [15]–[17] and [18]). One of the important issues that arise in the possible use of predictive control policies for the purpose of stabilization of the individual modes, however, is the difficulty they typically encounter in identifying, a priori (i.e., before controller implementation), the set of initial conditions starting from where feasibility and closed-loop stability are guaranteed. This typically results in the need for extensive closed-loop simulations to search over the whole set of possible initial conditions, thus adding to the overall computational load. This difficulty is more pronounced when considering MPC of hybrid systems that involve switching between multiple modes.

For linear systems, the switched system can be transformed into a mixed logical dynamical system, and a mixed-integer linear program can be solved to come up with an optimal

Manuscript received March 16, 2004; revised April 25, 2005 and June 14, 2005. Recommended by Associate Editor J. P. Hespanha. This work was supported by the National Science Foundation under Grant CTS–0129571.

P. Mhaskar was with the Department of Chemical Engineering, the University of California, Los Angeles, CA 90095 USA. He is now with the Department of Chemical Engineering, McMaster University, Hamilton, ON L8S 4L7, Canada.

N. H. El-Farra was with the Department of Chemical Engineering, the University of California, Los Angeles, CA 90095 USA. He is now with the Department of Chemical Engineering and Material Science, the University of California, Davis, CA 95616 USA.

P. D. Christofides are with the Department of Chemical Engineering, the University of California, Los Angeles, CA 90024 USA (e-mail: pdc@seas.ucla.edu).

Digital Object Identifier 10.1109/TAC.2005.858692

switching sequence and switching times [19], [20]. For nonlinear systems, one can, in principle, set up the mixed integer nonlinear programming problem, where the decision variables include the control action together with the switching schedule. The presence of nonlinear dynamics, that typically make the optimization problem nonconvex, together with the discrete decision variables results in high computational complexity, and the computation time requirements render it unsuitable for the purpose of real-time control.

In many systems of practical interest, the switched system is required to follow a prescribed switching schedule, where the switching times are no longer decision variables, but are prescribed via an operating schedule. Motivated by this practical problem, we propose a predictive control framework for the constrained stabilization of switched nonlinear systems that transit between their constituent modes at prescribed switching times. The main idea is to design a Lyapunov–based predictive controller for each constituent mode in which the switched system operates and incorporate constraints in the predictive controller design which upon satisfaction ensure that the prescribed transitions between the modes occur in a way that guarantees stability of the switched closed–loop system. This is achieved as follows: for each constituent mode, a Lyapunov-based model predictive controller (MPC) is designed, and an analytic bounded controller, using the same Lyapunov function, is used to explicitly characterize a set of initial conditions for which the MPC, irrespective of the controller parameters, is guaranteed to be feasible and, hence, stabilizing. Then, constraints are incorporated in the MPC design which, upon satisfaction, ensure that: 1) the state of the closed–loop system, at the time of the transition, resides in the stability region of the mode that the system is switched into, and 2) the Lyapunov function for each mode is nonincreasing wherever the mode is reactivated, thereby guaranteeing stability. The proposed control method is demonstrated through application to a chemical process example.

II. PRELIMINARIES

We consider the class of switched nonlinear systems represented by the following state–space description:

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))u_{\sigma(t)}(t) \\ u_{\sigma(t)} &\in \mathcal{U}_\sigma \\ \sigma(t) &\in \mathcal{K} := \{1,\ldots,p\} \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of continuous-time state variables, $u_\sigma(t) = [u_\sigma^1(t)\cdots u_\sigma^m(t)]^T \in \mathcal{U}_\sigma \subset \mathbb{R}^m$ denotes the vector of constrained manipulated inputs taking values in a nonempty compact convex set $\mathcal{U}_\sigma := \{u_\sigma \in \mathbb{R}^m : \|u_\sigma\| \le u_\sigma^{\max}\}$, where $\|\cdot\|$ is the Euclidian norm, $u_\sigma^{\max} > 0$ is the magnitude of the constraints, $\sigma : [0,\infty) \to \mathcal{K}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, i.e., $\sigma(t_k) = \lim_{t\to t_k^+}\sigma(t)$ for all k, implying that only a finite number of switches is allowed on any finite interval of time. p is the number of modes of the switched system, $\sigma(t)$, which takes different values in the finite index set $\mathcal{K}$, represents a discrete state that indexes the vector field $f(\cdot)$, the matrix $G(\cdot)$, and the control input $u(\cdot)$, which altogether determine $\dot{x}$. Throughout this paper, we use the notations $t_{k_r^{\text{in}}}$ and $t_{k_r^{\text{out}}}$ to denote the time at which, for the rth time, the kth subsystem is switched in and out, respectively, i.e., $\sigma(t^+_{k_r^{\text{in}}}) = \sigma(t^-_{k_r^{\text{out}}}) = k$. With this notation, it is understood that the continuous state evolves according to $\dot{x} = f_k(x)+G_k(x)u_k$ for $t_{k_r^{\text{in}}} \le t < t_{k_r^{\text{out}}}$. $\mathcal{T}_{k,in} = \{t_{k_1^{\text{in}}}, t_{k_2^{\text{in}}},\ldots\}$ and $\mathcal{T}_{k,out} = \{t_{k_1^{\text{out}}}, t_{k_2^{\text{out}}},\ldots\}$ denote the set of switching times at which the kth subsystem is switched in and out, respectively. It is assumed that all entries of the vector functions $f_k(x)$, and the $n\times m$ matrices $G_k(x)$, are sufficiently smooth and that $f_k(0) = 0$ for all $k \in \mathcal{K}$. Throughout the paper, the notation $L_f\bar{h}$ denotes the standard Lie derivative of a scalar function $\bar{h}(x)$ with respect to the vector function $f(x)$, $L_f\bar{h}(x) = (\partial\bar{h}/\partial x)f(x)$, and $\limsup_{t\to\infty} f(x(t)) = \lim_{t\to\infty}\{\sup_{\tau\ge t} f(x(\tau))\}$.

In order to provide the necessary background for our main results in Section III, we will briefly review in the remainder of this section a bounded controller design the stability properties of which are exploited in the design of a Lyapunov–based model predictive controller and in explicitly characterizing its stability region. For simplicity, we focus on the state feedback control problem where measurements of $x(t)$ are assumed to be available for all t.

A. Bounded Lyapunov-Based Control

Consider the system of (1), for a fixed $\sigma(t) = k$ for some $k \in \mathcal{K}$, for which a control Lyapunov function, V_k, exists. Using the results in [21] (see also [22]), the following continuous bounded control law can be constructed:

$$u_k(x) = -k_k(x)(L_{G_k}V_k)^T(x) := b_k(x) \tag{2}$$

with

$$k_k(x) = \frac{L^*_{f_k}V_k(x) + \sqrt{\left(L^*_{f_k}V_k(x)\right)^2 + (u_k^{\max}\|(L_{G_k}V_k)^T(x)\|)^4}}{\|(L_{G_k}V_k)^T(x)\|^2\left[1+\sqrt{1+(u_k^{\max}\|(L_{G_k}V_k)^T(x)\|)^2}\right]} \tag{3}$$

$L_{G_k}V_k(x) = [L_{g_k^1}V_k \cdots L_{g_k^m}V_k]$ is a row vector, where g_k^i is the ith column of G_k, $L^*_{f_k}V_k = L_{f_k}V_k + \rho_k V_k$ and $\rho_k > 0$. For the previous controller, one can compute an estimate of the stability region that is described by

$$\Omega_k(u_k^{\max}) = \{x \in \mathbb{R}^n : V_k(x) \le c_k^{\max}\} \tag{4}$$

where $c_k^{\max} > 0$ is the largest number for which $\Phi_k(u_k^{\max}) \supseteq \Omega_k(u_k^{\max})\backslash\{0\}$, and

$$\Phi_k(u_k^{\max}) = \{x\in\mathbb{R}^n : L^*_{f_k}V_k(x) < u_k^{\max}\|(L_{G_k}V_k)^T(x)\|\}. \tag{5}$$

The bounded controller of (2)–(3) possesses a robustness property with respect to measurement errors, that preserves closed–loop stability when the control action is implemented in a discrete (sample and hold) fashion with a sufficiently small hold time (Δ). Specifically, the control law ensures that, for all initial conditions in Ω_k, the closed–loop state remains in Ω_k and eventually converges to some neighborhood of the origin whose size depends on Δ. This robustness property, formalized below in Proposition 1, will be exploited in the Lyapunov-based predictive controller design of Section II-B. For further results on the analysis and control of sampled-data nonlinear systems, the reader may refer to [23]–[26].

Proposition 1: Consider the constrained system of (1) for a fixed value of $\sigma(t) = k$*, under the bounded control law of (2)–(3) designed using the Lyapunov function* V_k *and* $\rho_k > 0$*, and the stability region estimate* Ω_k *under continuous implementation. Let* $u_k(t) = u_k(j\Delta)$ *for all* $j\Delta \le t < (j+1)\Delta$ *and* $u_k(j\Delta) = b_k(x(j\Delta))$, $j = 0, \ldots, \infty$*. Then, given any positive real number* d_k*, there exists a positive real number* Δ_k^* *such that if* $x(0) := x_0 \in \Omega_k$ *and* $\Delta \in (0, \Delta_k^*]$*, then* $x(t) \in \Omega_k \, \forall t \ge 0$ and $\limsup_{t\to\infty} \|x(t)\| \le d_k$.

Proof of Proposition 1: The proof consists of two parts. In the first part, we establish that the bounded state feedback control law of (2)–(3) enforces asymptotic stability for all initial conditions in Ω_k with a certain robustness margin. In the second part, given the size, d_k, of a ball around the origin that the system is required to converge to, we show the existence of a positive real number Δ_k^*, such that if the discretization time Δ is chosen to be in $(0, \Delta_k^*]$, then Ω_k remains invariant under discrete implementation of the bounded control law, and also that the state of the closed–loop system converges to the ball $\|x\| \le d_k$.

Part 1: Substituting the control law of (2)–(3) into the system of (1) for a fixed $\sigma(t) = k$ it can be shown that

$$\dot{V}_k(x) = L_{f_k}V_k(x) + L_{G_k}V_k(x)u(x) \le \frac{-\rho_k V_k}{\left[1 + \sqrt{1 + (u_k^{\max}\|(L_{G_k}V_k)^T(x)\|)^2}\right]} \quad (6)$$

for all $x \in \Omega_k$, where Ω_k was defined in (4). Since the denominator term in (6) is bounded in Ω_k, there exists a positive real number, ρ_k^*, such that $\dot{V}_k \le -\rho_k^* V_k$ for all $x \in \Omega_k$, which implies that the origin of the closed–loop system, under the control law of (2)–(3), is asymptotically stable, with Ω_k as an estimate of the domain of attraction.

Part 2: Note that since $V_k(\cdot)$ is a continuous function of the state, one can find a finite, positive real number, δ_k', such that $V_k(x) \le \delta_k'$ implies $\|x\| \le d_k$. In the rest of the proof, we show the existence of a positive real number Δ_k^* such that all state trajectories originating in Ω_k converge to the level set of V_k $(V_k(x) \le \delta_k')$ for any value of $\Delta \in (0, \Delta_k^*]$ and hence we have that $\limsup_{t\to\infty} \|x(t)\| \le d_k$.

To this end, consider a "ring" close to the boundary of the stability region, described by $\mathcal{M}_k := \{x \in \mathbb{R}^n : (c_k^{\max} - \delta_k) \le V_k(x) \le c_k^{\max}\}$ (see also Fig. 1), for a $0 \le \delta_k < c_k^{\max}$. Let the control action be computed for some $x(0) := x_0 \in \mathcal{M}_k$ and held constant until a time Δ_k^{**}, where Δ_k^{**} is a positive real number $(u_k(t) = u_k(x_0) := u_0 \, \forall t \in [0, \Delta_k^{**}])$. Then, $\forall t \in [0, \Delta_k^{**}]$

$$\begin{aligned}\dot{V}_k(x(t)) &= L_{f_k}V_k(x(t)) + L_{G_k}V_k(x(t))u_0 \\ &= L_{f_k}V_k(x_0) + L_{G_k}V_k(x_0)u_0 \\ &\quad + (L_{f_k}V_k(x(t)) - L_{f_k}V_k(x_0)) \\ &\quad + (L_{G_k}V_k(x(t))u_0 - L_{G_k}V_k(x_0)u_0). \end{aligned} \quad (7)$$

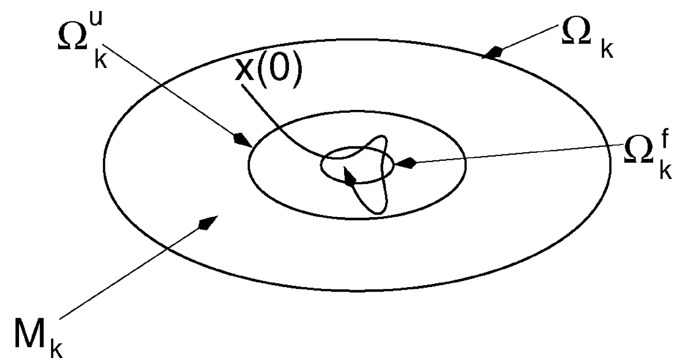

Fig. 1. Schematic representation of the stability region and the evolution of the closed–loop state trajectory under discrete implementation of the bounded controller.

Since the control action is computed based on the states in $\mathcal{M}_k \subseteq \Omega_k$, $L_{f_k}V_k(x_0) + L_{G_k}V_k(x_0)u_0 \le -\rho_k^* V_k(x_0)$. By definition, for all $x_0 \in \mathcal{M}_k$, $V_k(x_0) \ge c_k^{\max} - \delta_k$, therefore $L_{f_k}V(x_0) + L_{G_k}V_k(x_0)u_0 \le -\rho_k^*(c_k^{\max} - \delta_k)$.

Since the function $f_k(\cdot)$ and the elements of the matrix $G_k(\cdot)$ are continuous, $\|u_k\| \le u_k^{\max}$, and $\mathcal{M}_k$ is bounded, then one can find, for all $x_0 \in \mathcal{M}_k$ and a fixed Δ_k^{**}, a positive real number K_k^1, such that $\|x(t) - x_0\| \le K_k^1 \Delta_k^{**}$ for all $t \le \Delta_k^{**}$.

Since the functions $L_{f_k}V_k(\cdot)$, $L_{G_k}V_k(\cdot)$ are continuous, then given that $\|x(t) - x_0\| \le K_k^1 \Delta_k^{**}$, one can find positive real numbers K_k^2 and K_k^3 such that $\|L_{f_k}V_k(x(t)) - L_{f_k}V_k(x_0)\| \le K_k^3 K_k^1 \Delta_k^{**}$ and $\|L_{G_k}V_k(x(t))u_0 - L_{G_k}V_k(x_0)u_0\| \le K_k^2 K_k^1 \Delta_k^{**}$. Using these inequalities in (7), we get

$$\dot{V}_k(x(t)) \le -\rho_k^*(c_k^{\max} - \delta_k) + (K_k^1 K_k^2 + K_k^1 K_k^3)\Delta_k^{**}. \quad (8)$$

For a choice of $\Delta_k^{**} < (\rho_k^*(c_k^{\max} - \delta_k) - \epsilon_k)/(K_k^1 K_k^2 + K_k^1 K_k^3)$ where ϵ_k is a positive real number such that

$$\epsilon_k < \rho_k^*(c_k^{\max} - \delta_k) \quad (9)$$

we get that $\dot{V}_k(x(t)) \le -\epsilon_k < 0$ for all $t \le \Delta_k^{**}$. This implies that, given δ_k', if we pick δ_k such that $c_k^{\max} - \delta_k < \delta_k'$ and find a corresponding value of Δ_k^{**} then if the control action is computed for any $x \in \mathcal{M}_k$, and the "hold" time is less than Δ_k^{**}, we get that $\dot{V}_k$ remains negative during this time and, therefore, the state of the closed–loop system cannot escape Ω_k (since Ω_k is a level set of V_k). We now show the existence of Δ_k' such that for all $x_0 \in \Omega_k^f := \{x \in \mathbb{R}^n : V_k(x_0) \le c_k^{\max} - \delta_k\}$, we have that $x(\Delta) \in \Omega_k^u := \{x_0 \in \mathbb{R}^n : V_k(x_0) \le \delta_k'\}$, where $\delta_k' < c_k^{\max}$, for any $\Delta \in (0, \Delta_k']$.

Consider Δ_k' such that

$$\delta_k' = \max_{V_k(x_0) \le c_k^{\max} - \delta_k, u_k \in \mathcal{U}_k, t \in [0, \Delta_k']} V_k(x(t)). \quad (10)$$

Since V_k is a continuous function of x, and x evolves continuously in time, then for any value of $\delta_k < c_k^{\max}$, one can choose a sufficiently small Δ_k' such that (10) holds. Let $\Delta_k^* = \min\{\Delta_k^{**}, \Delta_k'\}$. We now show that for all $x_0 \in \Omega_k^u$ and $\Delta_k \in (0, \Delta_k^*]$, $x(t) \in \Omega_k^u$ for all $t \ge 0$.

For all $x_0 \in \Omega_k^u \bigcap \Omega_k^f$, by definition $x(t) \in \Omega_k^u$ for $0 \leq t \leq \Delta_k$ (since $\Delta_k \leq \Delta_k'$). For all $x_0 \in \Omega_k^u \backslash \Omega_k^f$ (and therefore $x_0 \in \mathcal{M}_k$), $\dot{V}_k < 0$ for $0 \leq t \leq \Delta$ (since $\Delta_k \leq \Delta_k^{**}$). Since Ω_k^u is a level set of V_k, then $x(t) \in \Omega_k^u$ for $0 \leq t \leq \Delta_k$. Either way, for all initial conditions in Ω_k^u, $x(t) \in \Omega_k^u$ for all future times.

We note that for x such that $x \in \Omega_k \backslash \Omega_k^u$, negative definiteness of $\dot{V}_k$ is guaranteed for $\Delta \leq \Delta_k^* \leq \Delta_k^{**}$. Hence, all trajectories originating in Ω_k converge to Ω_k^u, which has been shown to be invariant under the bounded control law with a hold time Δ less than Δ_k^* and, therefore, for all $x(0) \in \Omega_k$, $\limsup_{t\to\infty} V_k(x(t)) \leq \delta_k'$. Finally, since $V_k(x) \leq \delta_k'$ implies $\|x\| \leq d_k$, therefore we have that $\limsup_{t\to\infty} \|x(t)\| \leq d_k$. This completes the proof of Proposition 1.

Remark 1: Control Lyapunov function (CLF)-based stabilization of nonlinear systems has been studied extensively in the nonlinear control literature (e.g., see [27], [21], and [28]). The construction of constrained CLFs (i.e., CLFs that take the constraints into account) remains a difficult problem (especially for nonlinear systems) that is the subject of ongoing research. For several classes of nonlinear systems that arise commonly in the modeling of engineering systems, systematic, and computationally feasible methods are available for constructing unconstrained CLFs (CLFs for the unconstrained system) by exploiting the system structure. Examples include the use of quadratic functions for feedback linearizable systems and the use of back-stepping techniques to construct CLFs for systems in strict feedback form. In this work, the bounded controllers in (2)–(3) are designed using unconstrained CLFs, which are also used to explicitly characterize the associated regions of stability via (4)–(5). While the resulting estimates do not necessarily capture the entire domain of attraction (this remains an open problem even for linear systems), we will use them throughout the paper for a concrete illustration of the results. It is possible to obtain substantially improved estimates by using a combination of several CLFs (see, for example, [29] and [30]).

B. Lyapunov-Based Predictive Control

In this section, we consider predictive control of the system of (1), for a fixed $\sigma(t) = k$ for some $k \in \mathcal{K}$. We present here a Lyapunov–based design of MPC that guarantees feasibility of the optimization problem and hence constrained stabilization of the closed–loop system from an explicitly characterized set of initial conditions. For this predictive control design, the control action at state x and time t is obtained by solving, online, a finite horizon optimal control problem of the form

$$P(x,t) : \min\{J(x,t,u_k(\cdot)) \mid u_k(\cdot) \in S_k\} \quad (11)$$

$$\text{s.t. } \dot{x} = f_k(x) + G_k(x)u_k \quad (12)$$

$$\dot{V}_k(x(\tau)) \leq -\epsilon_k \text{ if } V_k(x(t)) > \delta_k',\ \tau \in [t, t+\Delta) \quad (13)$$

$$V_k(x(\tau)) \leq \delta_k' \text{ if } V_k(x(t)) \leq \delta_k',\ \tau \in [t, t+\Delta) \quad (14)$$

where ϵ_k, δ_k' are defined in (9) and Proposition 1, respectively, $S_k = S_k(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into $\mathcal{U}_k$, T is the specified horizon and V_k is the Lyapunov function used in the bounded controller design. A control $u_k(\cdot)$ in S_k is characterized by the sequence $\{u_k[j]\}$ where $u_k[j] := u_k(j\Delta)$ and satisfies $u_k(t) = u_k[j]$ for all $t \in [j\Delta, (j+1)\Delta)$. The performance index is given by

$$J(x,t,u_k(\cdot)) = \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u_k(s)\|_R^2\right] ds \quad (15)$$

where Q is positive–semidefinite symmetric matrix, and R is strictly positive–definite symmetric matrix. $x^u(s;x,t)$ denotes the solution of (1), due to control u, with initial state x at time t. The minimizing control $u_k^0(\cdot) \in S_k$ is then applied to the plant over the interval $[t, t+\Delta)$ and the procedure is repeated indefinitely. Stability properties of the closed–loop system under the Lyapunov–based predictive controller are inherited from the bounded controller under discrete implementation and are formalized in Proposition 2.

Proposition 2: Consider the constrained system of (1) for a fixed value of $\sigma(t) = k$ under the MPC control law of (11)–(15), designed using a control Lyapunov function V_k that yields a stability region Ω_k under continuous implementation of the bounded controller of (2)–(3) with a fixed $\rho_k > 0$. Then, given any positive real number d_k, there exist positive real numbers Δ_k^* and δ_k', such that if $x(0) \in \Omega_k$ and $\Delta \in (0, \Delta_k^*]$, then $x(t) \in \Omega_k \forall t \geq 0$ and $\limsup_{t\to\infty} \|x(t)\| \leq d_k$.

Proof of Proposition 2: From the proof of Proposition 1, we infer that given a positive real number, d_k, there exists an admissible manipulated input trajectory (provided by the bounded controller), and values of Δ_k^* and δ_k', such that for any $\Delta \in (0, \Delta_k^*]$ and $x(0) \in \Omega_k$, $\limsup_{t\to\infty} V_k(x(t)) \leq \delta_k'$ and $\limsup_{t\to\infty} \|x(t)\| \leq d_k$. The rest of the proof is divided in three parts. In the first part we show that for all $x_0 \in \Omega_k$, the predictive controller of (11)–(15) is feasible. We then show that Ω_k is invariant under the predictive control algorithm of (11)–(15). Finally, we prove practical stability for the closed–loop system.

Part 1: Consider some $x_0 \in \Omega_k$ under the predictive controller of (11)–(15), with a prediction horizon $T = N\Delta$, where Δ is the hold time and $1 \leq N < \infty$ is the number of prediction steps. The initial condition can either be such that $V_k(x_0) \leq \delta'$ or $\delta_k' < V_k(x_0) \leq c_k^{\max}$.

Case 1: If $\delta_k' < V_k(x_0) \leq c_k^{\max}$, the control input trajectory under the bounded controller of (2)–(3) provides a feasible solution to the constraint of (13) (see Proposition 1), given by $u(j\Delta) = b_k(x(j\Delta))$, $j = 1, \ldots, N$. Note that if $u = b_k$ for $t \in [0, \Delta]$, and $\Delta \in (0, \Delta_k^*]$, then $\dot{V}_k(x(t)) \leq -\epsilon_k < 0 \,\forall\, t \in [0, \Delta]$ and $b_k(x(t)) \in U_k$ [since b_k is computed using the bounded controller of (2)–(3)].

Case 2: If $V_k(x_0) \leq \delta_k'$, once again we infer from Proposition 1 that the control input trajectory provided by the bounded controller of (2)–(3) provides a feasible initial guess, given by $u(j\Delta) = b_k(x(j\Delta))$, $j = 1, \ldots, N$ (recall from Proposition 1, that under the bounded controller of (2)–(3), if $V_k(x_0) \leq \delta_k'$ then $V_k(x(t)) \leq \delta_k'\ \forall t \geq 0$). This shows that for all $x_0 \in \Omega_k$, the Lyapunov-based predictive controller of (11)–(15) is feasible.

Part 2: As shown in Part 1, for any $x_0 \in \Omega_k \backslash \Omega_k^u$, the constraint of (13) in the optimization problem is feasible. Upon implementation, therefore, the value of the Lyapunov function decreases. Since Ω_k is a level set of V_k, the state trajectories cannot escape Ω_k. On the other hand, if $x_0 \in \Omega_k^u$, feasibility of the constraint of (14) guarantees that the closed–loop state trajectory stays in $\Omega_k^u \subset \Omega_k$. In both cases, Ω_k continues to be an invariant region under the Lyapunov-based predictive controller of (11)–(13).

Part 3: Finally, consider an initial condition $x_0 \in \Omega_k \backslash \Omega_k^u$. Since the optimization problem continues to be feasible, we have that $\dot{V}_k < 0$ for all $x(t) \notin \Omega_k^u$ i.e., $V_k(x(t)) > \delta_k'$. All trajectories originating in Ω_k, therefore converge to Ω_k^u. For $x_0 \in \Omega_k^u$, the feasibility of the optimization problem implies $x(t) \in \Omega_k^u$, i.e., $V_k(x(t)) \leq \delta_k'$. Therefore, for all $x_0 \in \Omega_k$, $\limsup_{t\to\infty} V_k(x(t)) \leq \delta_k'$. Also, since $V_k(x) \leq \delta_k'$ implies $\|x\| \leq d_k$, we have that $\limsup_{t\to\infty} \|x(t)\| \leq d_k$. This completes the proof of Proposition 2.

Remark 2: Note that Lyapunov-based predictive control approaches (see, for example, [16] and [17]) typically incorporate a similar Lyapunov function decay constraint, albeit requiring the constraint of (13) to hold at the *end* of the prediction horizon as opposed to during the first time step, and assume the initial feasibility of this constraint. In contrast, the predictive controller formulation of (11)–(15) requires that the value of the Lyapunov function decrease during the first step only, allowing for the use of the bounded controller as an auxiliary controller to explicitly characterize the set of initial conditions stating from where the predictive controller is guaranteed to be feasible.

Remark 3: The fact that only practical stability is achieved is not a limitation of the MPC formulation, but is due to discrete-time implementation of the control action into a continuous-time dynamical system. Even if the bounded controller is used instead, under the same implement-and-hold time of Δ, the bounded controller can also only guarantee that the state of the closed–loop system converges to the set Ω_k^u, the size of which is limited by the value of the hold time, Δ (in the limit as Δ goes to zero – continuous implementation– the bounded controller and the predictive controller enforce asymptotic stability). Note also that any other Lyapunov-based analytic control design that provides an explicit characterization of the stability region and is robust with respect to discrete implementation can be used as an auxiliary controller.

Remark 4: One of the key challenges that impact on the practical implementation of nonlinear model predictive control (NMPC) is the inherent difficulty of characterizing, *a priori*, the set of initial conditions starting from where a given NMPC controller is guaranteed to stabilize the closed–loop system, or for a given set of initial conditions, to identify the value of the prediction horizon for which the optimization problem will be feasible. Use of conservatively large horizon lengths to address stability only increases the size and complexity of the nonlinear optimization problem and could make it intractable. Owing to the fact that closed–loop stability is guaranteed by the Lyapunov-based predictive controller from an explicitly characterized set of initial conditions, irrespective of the prediction horizon, the time required for the computation of the control action, if so desired, can be made smaller by reducing the size of the optimization problem by decreasing the prediction horizon.

Remark 5: In the event that measurements are not continuously available, but are available only at sampling times $\Delta_s > \Delta_k^*$, i.e., greater than what a given bounded control design can tolerate (and, therefore, greater than the maximum allowable discretization for the Lyapunov-based predictive controller), it is necessary to redesign the bounded controller to increase the robustness margin, and generate a revised estimate of the feasibility (and stability) region under the predictive controller. A larger value of Δ_k^* may be achieved by increasing the value of the parameter ρ_k in the design of the bounded controller (see proof of Proposition 1). If the value of the sampling time is reasonable, an increase in the value of the parameter ρ_k, while leading to a shrinkage in the stability region estimate, can increase Δ_k^* to a value greater than Δ_s and preserve the desired feasibility and stability guarantees of the Lyapunov-based predictive controller.

III. Predictive Control of Switched Nonlinear Systems

Consider now the nonlinear switched system of (1), with a prescribed switching sequence (including the switching times) defined by $\mathcal{T}_{k,in} = \{t_{k_1^{\text{in}}}, t_{k_2^{\text{in}}}, \ldots\}$ and $\mathcal{T}_{k,out} = \{t_{k_1^{\text{out}}}, t_{k_2^{\text{out}}}, \ldots\}$. Also, assume that for each mode of the switched system, a Lyapunov–based predictive controller of the form of (11)–(15) has been designed and an estimate of the stability region generated. The control problem is formulated as the one of designing a Lyapunov-based predictive controller that guides the closed–loop system trajectory in a way that the schedule described by the switching times is followed and stability of the closed–loop system is achieved. A predictive control algorithm that addresses this problem is presented in Theorem 1.

Theorem 1: Consider the constrained nonlinear system of (1), the control Lyapunov functions V_k, $k = 1, \ldots, p$, and the stability region estimates Ω_k, $k = 1, \ldots, p$ under continuous implementation of the bounded controller of (2)–(3) with fixed $\rho_k > 0$, $k = 1, \ldots, p$. Let $0 < T_{\text{design}} < \infty$ be a design parameter. Let t be such that $t_{k_r^{\text{in}}} \leq t < t_{k_r^{\text{out}}}$ and $t_{m_j^{\text{in}}} = t_{k_r^{\text{out}}}$ for some m, k. Consider the following optimization problem:

$$P(x,t) : \min\{J(x,t,u_k(\cdot)) \mid u_k(\cdot) \in S_k\} \tag{16}$$

$$J(x,t,u_k(\cdot)) = \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u_k(s)\|_R^2\right] ds \tag{17}$$

where T is the prediction horizon given by $T = t_{k_r^{\text{out}}} - t$, if $t_{k_r^{\text{out}}} < \infty$ and $T = T_{\text{design}}$ if $t_{k_r^{\text{out}}} = \infty$, subject to the following constraints:

$$\dot{x} = f_k(x) + G_k(x)u_k \tag{18}$$

$$\dot{V}_k(x(\tau)) \leq -\epsilon_k \text{ if } V_k(x(t)) > \delta_k',\ \tau \in [t, t+\Delta_{k_r}) \tag{19}$$

$$V_k(x(\tau)) \leq \delta_k' \text{ if } V_k(x(t)) \leq \delta_k',\ \tau \in [t, t+\Delta_{k_r}) \tag{20}$$

and if $t_{k_r^{\text{out}}} = t_{m_j^{\text{in}}} < \infty$ then

$$V_m(x(t_{m_j^{\text{in}}})) \leq \begin{cases} V_m(x(t_{m_{j-1}^{\text{in}}})) - \epsilon^*, & j > 1,\ V_m(x(t_{m_{j-1}^{\text{in}}})) > \delta_m' \\ \delta_m', & j > 1,\ V_m(x(t_{m_{j-1}^{\text{in}}})) \leq \delta_m' \\ c_m^{\max}, & j = 1 \end{cases} \tag{21}$$

where ϵ^* is a positive real number. Then, given a positive real number $d^{\max}$, there exist positive real numbers Δ^* and δ_k', $k =$

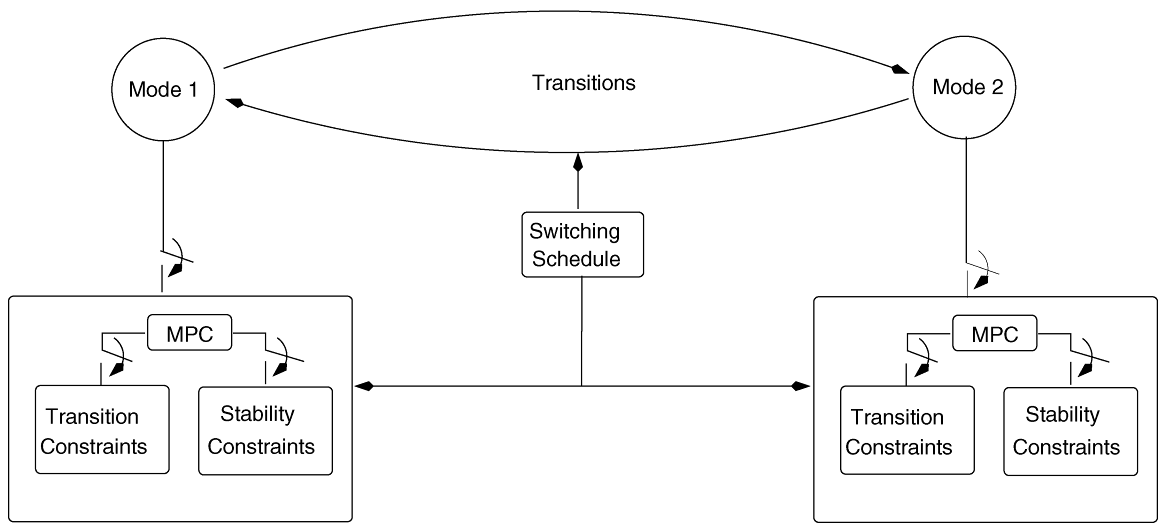

Fig. 2. Schematic representation of the predictive control structure comprised of the predictive and bounded controllers for the constituent modes, together with transition constraints.

$1, \ldots, m$ *such that if the optimization problem of (16)–(21) is feasible at all times, the minimizing control is applied to the system over the interval* $[t, t+\Delta_{k_r}]$, *where* $\Delta_{k_r} \in (0, \Delta^*]$ *and* $t_{k_r^{\rm out}} - t_{k_r^{\rm in}} = l_{k_r}\Delta_{k_r}$ *for some integer* $l_{k_r} > 0$ *and the procedure is repeated, then,* $\limsup_{t\to\infty} \|x(t)\| \le d^{\max}$.

Proof of Theorem 1: The proof of this theorem follows from the assumption of feasibility of the constraints of (19)–(21) at all times. Given the radius of the ball around the origin, $d^{\max}$, the value of δ'_k and Δ^*_k for all $k \in \mathcal{K}$ is computed the same way as in the proof of Proposition 1. Then, for the purpose of MPC implementation, a value of $\Delta_{k_r} \in (0, \Delta^*]$ is chosen where $\Delta^* = \min_{k=1,\ldots,p} \Delta^*_k$ and $t_{k_r^{\rm out}} - t_{k_r^{\rm in}} = l_{k_r}\Delta_{k_r}$ for some integer $l_{k_r} > 0$ (note that given any two positive real numbers $t_{k_r^{\rm out}} - t_{k_r^{\rm in}}$ and Δ^*, one can always find a positive real number $\Delta_{k_r} \le \Delta^*$ such that $t_{k_r^{\rm out}} - t_{k_r^{\rm in}} = l_{k_r}\Delta_{k_r}$ for some integer $l_{k_r} > 0$).

Part 1: First, consider the case when the switching is infinite. Let t be such that $t_{k_r^{\rm in}} \le t < t_{k_r^{\rm out}}$ and $t_{m_j^{\rm in}} = t_{k_r^{\rm out}} < \infty$. Consider the active mode k. If $V_k(x) > \delta'_k$, the continued feasibility of the constraint of (19) implies that $V_k(x(t_{k_r^{\rm out}})) < V_k(x(t_{k_r^{\rm in}}))$. The transition constraint of (21) ensures that if this mode is switched out and then switched back in, then $V_k(x(t_{k_{r+1}^{\rm in}})) < V_k(x(t_{k_r^{\rm in}}))$. In general $V_k(x(t_{k_l^{\rm in}})) < V_k(x(t_{k_{l-1}^{\rm in}})) < \cdots < c_k^{\max}$. Under the assumption of feasibility of the constraints of (19)–(21) for all future times, therefore, the value of $V_k(x)$ continues to decrease. If the mode of this Lyapunov function is not active, there exists at least some $j \in 1, \ldots, p$ such that mode j is active and Lyapunov function V_j continues to decrease until the time that $V_j \le \delta'_j$ (this happens because there are a finite number of modes, even if the number of switches may be infinite). From this point onwards, the constraint of (20) ensures that V_j continues to be less than δ'_j. Hence, $\limsup_{t\to\infty} \|x(t)\| \le d^{\max}$.

Part 2: For the case of a finite switching sequence, consider a t such that $t_{k_r^{\rm in}} \le t < t_{k_r^{\rm out}} = \infty$. Under the assumption of continued feasibility of (19)–(21), $V_k(x(t_{k_r^{\rm in}})) < V_k(x(t_{k_{r-1}^{\rm in}})) < \cdots < c_k^{\max}$. At the time of the switch to mode k, therefore, $x(t_{k_r^{\rm in}}) \in \Omega_k$. From this point onwards, the Lyapunov based controller is implemented using the Lyapunov function V_k, and the constraint of (21) is removed, in which case the predictive controller of Theorem 1 reduces to the predictive controller of (11)–(15). Since the value of Δ_{k_r} is chosen to be in $(0, \Delta^*]$, where $\Delta^* = \min_{k=1,\ldots,p} \Delta^*_k$, therefore $\Delta_{k_r} < \Delta^*_k$, which guarantees feasibility and convergence to the ball $\|x\| \le d^{\max}$ for any value of the prediction horizon (hence, for a choice of $T = T_{\rm design}$), and leads to $\limsup_{t\to\infty} \|x(t)\| \le d^{\max}$. This completes the proof of Theorem 1.

Remark 6: Note that the constraint of (19) is guaranteed to be feasible between mode transitions, provided that the system is initialized within the stability region, and does not require the assumption of feasibility. This stability constraint ensures that the value of the Lyapunov function of the currently active mode keeps decreasing (recall that one of the criteria in the multiple Lyapunov-function stability analysis is that the individual modes of the switched system be stable). The constraint of (19) expresses two transition requirements simultaneously: 1) the MLF constraints that requires that the value of the Lyapunov function be less than what it was the last time the system switched into that mode (required when the switching sequence is infinite, see [3] for details), and 2) the stability region constraint that requires that the state of the process reside within the stability region of the target mode at the time of the switch; since the stability regions of the modes are expressed as level sets of the Lyapunov functions, the MLF-constraint also expresses the stability region constraint. The understanding that it is a reasonably chosen switching schedule (that is, one that does not result in closed–loop instability), motivates assuming the feasibility of the transition constraints for all times. Note that the feasibility of the transition constraints can also be used to validate the switching schedule, and can be used to abort the switching schedule (i.e., to decide that the remaining switches should not be carried out) in the interest of preserving closed–loop stability.

Remark 7: Fig. 2 is a schematic that depicts the main features of the predictive controller for mode transitions, for representative switching between two modes, where the switching schedule dictates the transitions, and is accounted for in the

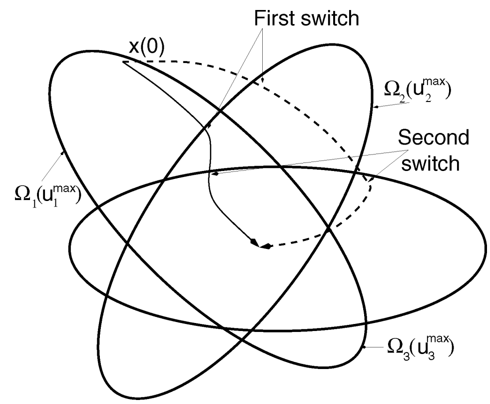

Fig. 3. Schematic representation of the closed–loop trajectory of a switched system under the implementation of the predictive control algorithm of Theorem 1 (solid line) and using the relaxed constraints of Remark 9 (dashed line).

predictive control design. A representative implementation of the predictive control algorithm of Theorem 1 is also depicted schematically by the solid lines in Fig. 3. The algorithm that follows explains the implementation of the predictive controller of Theorem 1, including the course of action that needs to be taken if the predictive controller does not yield a feasible solution.

1) Given the system model of **(1)** and the constraints on the input, design the bounded controller of **(2)-(3)** for each mode and compute the stability region estimate, $\Omega_k(u_k^{\max})$, for the bounded controller using **(4)-(5)**.
2) Given the size of the ball, $d^{\max}$, that the state is required to converge to, compute Δ_k^*, $k = 1,\ldots,p$ for the predictive controller of **(11)-(15)**, such that under single mode operation for each mode, $\limsup_{t\to\infty}\|x(t)\| \leq d^{\max}$. Compute $\Delta^* = \min_{k=1}^{p}\{\Delta_k^*\}$, and use a $\Delta_{k_r} \in (0,\Delta^*]$ for which $t_{k_r^{\rm out}} - t_{k_r^{\rm in}} = l_{k_r}\Delta_{k_r}$ for some integer $l_{k_r} > 0$, for the purpose of MPC implementation.
3) Consider the time $t_{k_r^{\rm in}}$, which designates the time that the closed-loop system switches to the kth mode for the rth time, and the state belongs to the stability region of the kth mode (for the purpose of initialization, i.e., at $t = 0$, this would correspond to $t_{1_1^{\rm in}}$, and the state belonging to the stability region $\Omega_1(u_1^{\max})$).
4) From the set of prescribed switching times, pick $t_{m_j^{\rm in}} = t_{k_r^{\rm out}}$ ($t_{m_j^{\rm in}}$, therefore, is the time that the next switch takes place, and that the system, upon exiting from the current mode enters mode m for the jth time).
5) Consider the predictive controller of Theorem 1. The constraint $V_m(t_{m_j^{\rm in}}) \leq V_m(t_{m_{j-1}^{\rm in}}) - \epsilon^*$ requires that when the closed-loop system enters the mode m, the value of V_m is less than what it was at the time that the system last entered mode m (this is a version of the multiple Lyapunov function stability condition, see **[3]**). If the system has never entered mode m before, i.e., for $j = 1$, set $V_m(t_{m_{j-1}^{\rm in}}) = c_m^{\max}$ (this requires that the state belongs to the stability region corresponding to mode m). If the closed-loop state has already entered the desired ball around the origin, implement $V_m(t_{m_j^{\rm in}}) \leq \delta'_m$, that ensures that the state stays within the ball, $\|x\| \leq d^{\max}$.
6) If at any time, the predictive controller of Theorem 1 does not yield a feasible solution, or the switching schedule does not prescribe another switch, go to step 7); else implement the predictive controller up-to time $t_{k_r^{\rm out}}$, i.e., until the time that the system switches into mode m and go back to step 4) to proceed with the rest of the switching sequence.
7) Implement the Lyapunov-based predictive controller of **(11)-(15)** for the current mode to stabilize the closed-loop system.

Remark 8: Note that since the switching times are fixed, the prediction of the states in the controller needs to be carried out from the current time up-to the time of the next switch only. The predictive controller is therefore implemented with a shrinking horizon between successive switching times. Note, however, that the value of the horizon is *not* a decision variable (and, therefore, does not incur any computational burden); its value is obtained simply by evaluating the difference between the next switching time and the current time. However, in the case that the switching schedule terminates, then after the last switch has been made and the system is evolving in the terminal mode, the horizon is fixed and set equal to a preset design parameter $T_{\rm design}$. Note that from this point onward, the controller design of (16)–(20) reduces to the Lyapunov based predictive controller of (11)–(15) for the terminal mode, and guarantees practical stability of the closed–loop system for any value of prediction horizon, including for $T = T_{\rm design}$. Note also that picking the hold time (Δ_{k_r}) during the operation between $t_{k_r^{\rm in}}$ and $t_{k_r^{\rm out}}$ such that the time interval between switches be an integer multiple of the sampling time is made to ensure that the system does not go through a transition while the final control move in a given mode is being implemented. To see the reasoning behind this, note first that, due to the discrete nature of controller implementation, the final control in the pre-switching mode is implemented for Δ_{k_r} time. Therefore, if the system undergoes a transition during that time (i.e., if $t+\Delta_{k_r}$ is greater than $t_{k_r^{\rm out}}$), then the control could cause the closed–loop state to leave the stability region of the target mode after switching has been executed. Picking Δ_{k_r} as before precludes such a possibility.

Remark 9: The constraints of (19)–(20) require that the closed–loop state evolve so that the value of the Lyapunov function for the current mode continues to decay, and therefore the closed–loop state trajectory evolves in a way that it enters the stability region of the target mode [due to (21)], while, up-to the time of the switch, continuing to evolve in the stability region of the current mode. Owing to this, if at any time the transition constraints are not feasible, and the switching sequence is aborted, the Lyapunov-based predictive controller for the then-current mode is guaranteed to stabilize the closed-loop system. While this condition safeguards against instability, it may be restrictive, and hamper the feasibility of the optimization problem, especially if the boundaries of the stability regions of the constituent modes do not have a significant overlap. When dealing with a finite switching sequence, the constraints of (19)–(20) can be relaxed for all times before the final switch takes place and also a relaxed version of the constraint of (21) may be used up to the time of the terminal switch, requiring only that $V_m(t_{m_j^{\text{in}}}) \leq c_m^{\max}$ (i.e., the closed–loop state resides in the stability region of the target mode, see dashed lines in Fig. 3).

Remark 10: For purely continuous systems, the problem of implementing various predictive control algorithms (not necessarily ones using Lyapunov-based stability constraints) with guaranteed stability regions was recently addressed for linear systems under state [31] and output [32] feedback control, nonlinear systems [29] and for nonlinear systems in the presence of uncertainty [30], by means of a hybrid control structure that uses bounded control as a fall-back controller in the event of infeasibility or instability of the predictive controller. In this paper, the bounded controller design is not used as a fall back controller, but rather for the purpose of providing an estimate of the stability region for the Lyapunov-based predictive controller, and feasible initial guesses for the control moves (the decision variables in the optimization problem). Note that the Lyapunov-based predictive controller of (11)–(15) is guaranteed to be feasible and stabilizing from an explicitly characterized set of initial conditions.

Remark 11: Note that the use of the predictive controller is both beneficial and essential to the problem of implementing a prescribed switching schedule using the proposed approach. In particular, while the bounded controller can achieve stabilization under single mode operation, the bounded controller framework simply does not address performance considerations, and more importantly does not allow for incorporating transition constraints, and there is no guarantee that the bounded controller (or even the Lyapunov-based predictive controller, if it does not incorporate the transition constraints) can stabilize the closed–loop system when following a prescribed switching schedule (see the simulation example in the next section for a demonstration of this point). In contrast, the predictive control approach provides a natural framework for specifying appropriate transitions constraints, which upon being feasible, ensure closed–loop stability.

Remark 12: For a nonlinear switched system of the form of (1), while one can in principle set up a mixed integer nonlinear programming problem, where the decision variables (and, hence, the solution to the optimization problem) include the control action together with the switching schedule, it is not possible to use it for the purpose of online implementation.

TABLE I
PROCESS PARAMETERS AND STEADY-STATE VALUES

V	=	0.1	m^3
R	=	8.314	$kJ/kmol \cdot K$
C_{A1_s}	=	0.79	$kmol/m^3$
C_{A2_s}	=	1.0	$kmol/m^3$
T_{A1}	=	352.6	K
T_{A2}	=	310.0	K
Q_{1_s}	=	0.0	KJ/hr
Q_{2_s}	=	0.0	KJ/hr
ΔH	=	-4.78×10^4	$kJ/kmol$
k_0	=	1.2×10^9	s^{-1}
E	=	8.314×10^4	$kJ/kmol$
c_p	=	0.239	$kJ/kg \cdot K$
ρ	=	1000.0	kg/m^3
F_1	=	3.34×10^{-3}	m^3/s
F_2	=	1.67×10^{-3}	m^3/s
T_{Rs}	=	395.33	K
C_{As}	=	0.57	$kmol/m^3$

In many systems of practical interest, however, the switched system may be only required to follow a prescribed switching schedule. The predictive controller algorithm provides an implementable controller for switched nonlinear systems with a prescribed switching sequence by incorporating appropriate constraints on the implemented control action in each of the individual modes. Note also that if it is possible to solve the mixed integer optimization problem offline, it can be used to provide a set of optimal switching times, and an associated switching sequence, which can be then implemented online using the proposed approach.

IV. APPLICATION TO A CHEMICAL PROCESS EXAMPLE

We consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The operation schedule requires switching between two available inlet streams consisting of pure A at flow rates F_1, F_2, concentrations C_{A1}, C_{A2}, and temperatures T_{A1}, T_{A2}, respectively. For each mode of operation, the mathematical model for the process takes the form

$$\begin{aligned}\dot{C}_A &= \frac{F_\sigma}{V}(C_{A\sigma} - C_A) - k_0 e^{-E/RT_R} C_A \\ \dot{T}_R &= \frac{F_\sigma}{V}(T_{A\sigma} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{-E/RT_R} C_A + \frac{Q_\sigma}{\rho c_p V}\end{aligned} \quad (22)$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, Q_σ is the heat removed from the reactor, V is the volume of the reactor, k_0, E, ΔH are the pre-exponential constant, the activation energy, and the enthalpy of the reaction, c_p and ρ, are the heat capacity and fluid density in the reactor and $\sigma(t) \in \{1, 2\}$ is the discrete variable. The values of all process parameters can be found in Table I. The control objective is to stabilize the reactor at the unstable equilibrium point $(C_A^s, T_R^s) = (0.57, 395.3)$ using the rate of heat input, Q_σ, and change in inlet concentration of species A, $\Delta C_{A\sigma} = C_{A\sigma} - C_{A\sigma_s}$ as manipulated inputs with constraints: $|Q_\sigma| \leq 1$ KJ/hr and $|\Delta C_{A\sigma}| \leq 1$ mol/l, $\sigma = 1, 2$. For both the modes, we considered quadratic Lyapunov functions of the form $V(x) = x' P_k x$ where $x = (C_A - C_A^s, T_R - T_R^s)$, with

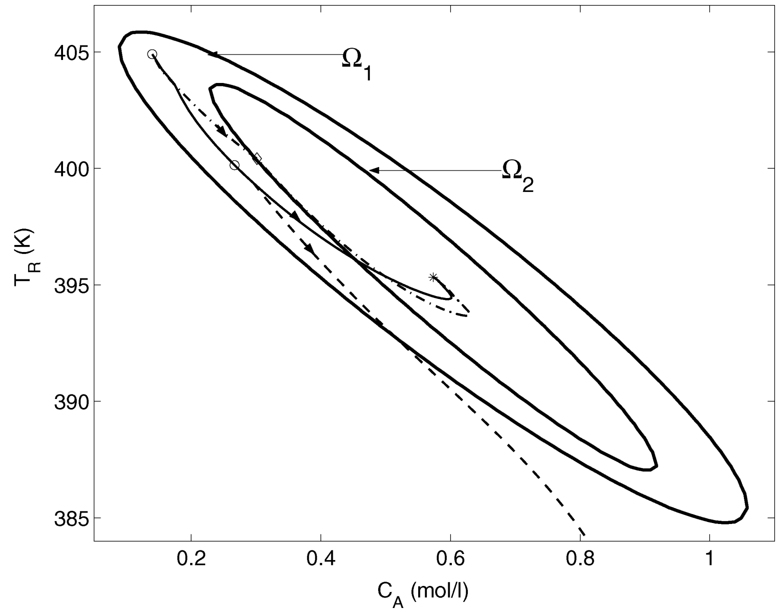

Fig. 4. Closed–loop state trajectory when the reactor is operated in mode 1 for all times under the stabilizing MPC formulation of (11)–(15) (solid line), when the reactor operation involves switching from mode 1 to mode 2 at $t = 0.1$ hr, under the predictive controller design of (11)–(15) (dashed line), and when the reactor operation involves switching from mode 1 to mode 2 at $t = 0.1$ hr, under the predictive controller of Theorem 1 (dashed–dotted line).

$P_1 = \begin{bmatrix} 33.24 & 1.32 \\ 1.32 & 0.06 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 7.39 & 0.32 \\ 0.32 & 0.016 \end{bmatrix}$ and used these in the Lyapunov-based controller design to compute the stability regions for the two modes, Ω_1 and Ω_2, shown in Fig. 4. The matrices P_k were computed by solving a Riccatti inequality using the linearized system matrices. The computation of the stability region [using (4) and (5)], however, was done using the nonlinear system dynamics. The parameters in the objective function of (15) are chosen as $Q = qI$, with $q = 1$, and $R = rI$, with $r = 1.0$. The constrained nonlinear optimization problem is solved using the MATLAB subroutine fmincon, and the set of ODEs is integrated using the MATLAB solver ODE45.

We first demonstrate the implementation of the Lyapunov-based predictive controller to a single mode operation of the chemical reactor, i.e., one in which the process is operated for all times in mode 1. To this end, we consider an initial condition that belongs to the stability region of the predictive controller for mode 1. As shown by the solid line in Fig. 4 (see Fig. 5 for the state and input profiles), starting from the initial condition $(C_A, T_R) = (0.14, 404.9)$, which belongs to the stability region of the predictive controller for mode 1, successful stabilization of the closed–loop system is achieved.

To demonstrate the need to account for the switched nature of the system, we choose a schedule involving a switch from inlet stream 1 (mode 1) to inlet stream 2 (mode 2) at time $t = 0.1$ hr. Once again the system is initialized within the stability region of mode 1, and the predictive controller for mode 1 is implemented. Up until $t = 0.1$ hr, the state of the closed–loop system moves toward the desired steady state (as seen from the dashed–lines in Fig. 4); however, when the system switches to mode 2, the MPC controller, designed for the stabilization of the process in mode 2, does not yield a feasible solution. If the bounded controller for mode 2 is implemented, the resulting control action is not able to stabilize the closed–loop system (dashed lines in Figs. 4–5). This happens because at the time of the transition, the state of the closed–loop system (marked by the o in Fig. 4) does not belong to the stability region of mode 2. Note also that while the predictive formulation of (11)–(15) guarantees stabilization for all initial conditions belonging to the stability region of mode 1, it does not incorporate constraints which enable or ensure a safe transition to mode 2.

Finally, the predictive control algorithm of Theorem 1 (which incorporates constraints that account for switching) is implemented (dashed–dotted lines in Figs. 4 and 5). The MPC controller of mode 1 is designed to drive the state of the closed–loop system such that the state belongs to the stability region of mode 2 at the switching time. Consequently, when the system switches to mode 2 at $t = 0.1$ hr, the closed–loop system state at the switching time (marked by the $\diamondsuit$ in Fig. 4) belongs to the stability region of the MPC designed for mode 2. At this time, when the process switches to mode 2 and the corresponding predictive controller is implemented, closed–loop stability is achieved.

V. CONCLUSION

In this work, a predictive control framework was proposed for the constrained stabilization of switched nonlinear systems that transit between their constituent modes at prescribed switching times. The main idea is to design a Lyapunov–based predictive controller for each of the constituent modes in which the switched system operates, and incorporate constraints in the predictive controller design which upon satisfaction ensure stability of the switched closed–loop system. This was achieved as follows: For each constituent mode, a Lyapunov-based MPC is designed, and an analytic bounded controller, using the same Lyapunov function is used to explicitly characterize the set of initial conditions for which the MPC, irrespective of the

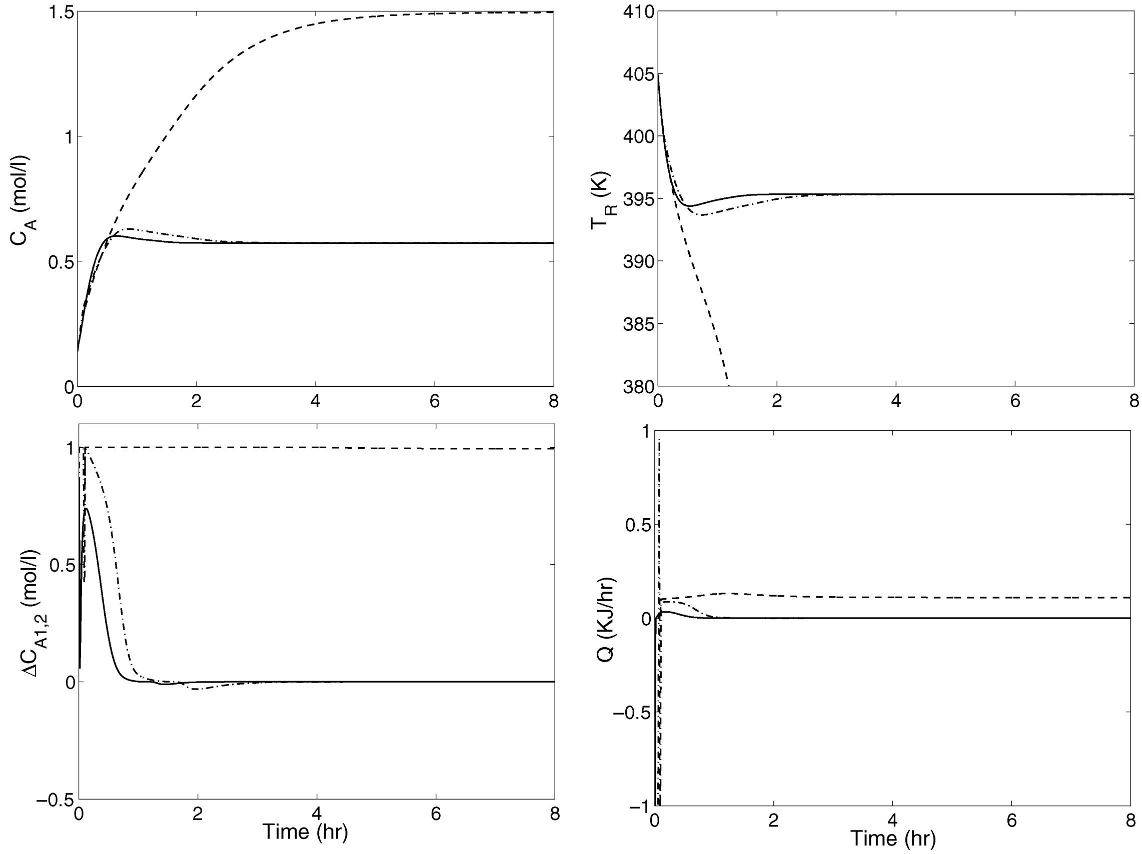

Fig. 5. Closed-loop state (top plots) and manipulated input (bottom plots) profiles: when the reactor is operated in mode 1 for all times under the stabilizing MPC formulation of (11)–(15) (solid line), when the reactor operation involves switching from mode 1 to mode 2 at $t = 0.1$ hr, under the predictive controller design of (11)–(15) (dashed line), and when the reactor operation involves switching from mode 1 to mode 2 at $t = 0.1$ hr, under the predictive controller of Theorem 1 (dashed–dotted line).

controller parameters, is guaranteed to be feasible, and hence stabilizing. Then, constraints were incorporated in the MPC design which, upon satisfaction, ensure that: 1) the state of the closed–loop system, at the time of the transition, resides in the stability region of the mode that the system is switched into, and 2) the value of the Lyapunov function for each of the modes eventually decays to zero, thereby guaranteeing stability. The proposed control method was demonstrated through application to a chemical process example.

REFERENCES

[1] E. C. Yamalidou and J. Kantor, "Modeling and optimal control of discrete-event chemical processes using petri nets," *Comput. Chem. Eng.*, vol. 15, pp. 503–519, 1990.

[2] I. E. Grossmann, S. A. van den Heever, and I. Harjukoski, "Discrete optimization methods and their role in the integration of planning and scheduling," in *Proc. 6th Int. Conf. Chemical Process Control*, Tucson, AZ, 2001, pp. 124–152.

[3] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Autom. Control*, vol. 43, no. 4, pp. 475–482, Apr. 1998.

[4] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Syst. Mag.*, vol. 19, no. 5, pp. 59–70, Oct. 1999.

[5] J. Hespanha and A. S. Morse, "Stability of switched systems with average dwell time," in *Proc. 38th IEEE Conf. Decision and Control*, Phoenix, AZ, 1999, pp. 2655–2660.

[6] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Leenartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proc. IEEE*, vol. 88, no. 7, pp. 1069–1082, Jul. 2000.

[7] J. Lygeros, D. N. Godbole, and S. S. Sastry, *A Game Theoretic Approach to Hybrid System Design*. ser. Lecture Notes in Computer Science, R. Alur and T. Henzinger, Eds. Berlin, Germany: Springer-Verlag, 1996, vol. 1066, pp. 1–12.

[8] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A unified framework for hybrid control: Model and optimal control theory," *IEEE Trans. Autom. Control*, vol. 43, no. 1, pp. 31–45, Jan. 1998.

[9] J. Hespanha and A. S. Morse, "Switching between stabilizing controllers," *Automatica*, vol. 38, pp. 1905–1917, 2004.

[10] X. P. Xu and P. J. Antsaklis, *Results and Perspectives on Computational Methods for Optimal Control of Switched Systems*, ser. Lecture Notes in Computer Science. New York: Springer-Verlag, 2003, vol. 2623, pp. 540–555.

[11] P. Peleties and R. DeCarlo, "Asymptotic stability of m-switched systems using Lyapunov-like functions," in *Proc. Amer. Control Conf.*, Boston, MA, 1991, pp. 1679–1684.

[12] M. A. Wicks, P. Peleties, and R. A. DeCarlo, "Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems," *Eur. J. Control*, vol. 4, pp. 140–147, 1998.

[13] B. Hu, X. Xu, P. J. Antsaklis, and A. N. Michel, "Robust stabilizing control law for a class of second-order switched systems," *Syst. Control Lett.*, vol. 38, pp. 197–207, 1999.

[14] N. H. El-Farra and P. D. Christofides, "Coordinated feedback and switching for control of hybrid nonlinear processes," *AIChE J.*, vol. 49, pp. 2079–2098, 2003.

[15] H. Michalska and D. Q. Mayne, "Robust receding horizon control of constrained nonlinear systems," *IEEE Trans. Automa. Control*, vol. 38, no. 11, pp. 1623–1633, Nov. 1993.

[16] S. L. D. Kothare and M. Morari, "Contractive model predictive control for constrained nonlinear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 6, pp. 1053–1071, Jun. 2000.

[17] J. A. Primbs, V. Nevistic, and J. C. Doyle, "A receding horizon generalization of pointwise min-norm controllers," *IEEE Trans. Autom. Control*, vol. 45, no. 5, pp. 898–909, May 2000.

[18] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[19] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics and constraints," *Automatica*, vol. 35, pp. 407–427, 1999.

[20] V. Dua, N. A. Bozinis, and E. N. Pistikopoulos, "A multiparametric programming approach for mixed-integer quadratic engineering problems," *Comput. Chem. Eng.*, vol. 26, pp. 715–733, 2002.

[21] Y. Lin and E. D. Sontag, "A universal formula for stabilization with bounded controls," *Syst. Control Lett.*, vol. 16, pp. 393–397, 1991.

[22] N. H. El-Farra and P. D. Christofides, "Integrating robustness, optimality and constraints in control of nonlinear processes," *Chem. Eng. Sci.*, vol. 56, pp. 1841–1868, 2001.

[23] J. W. Grizzle and P. V. Kokotovic, "Feedback linearization of sampled-data systems," *IEEE. Trans. Autom. Control*, vol. 33, no. 9, pp. 857–859, Sep. 1988.

[24] D. Nesic, A. R. Teel, and P. V. Kokotovic, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations," *Syst. Control Lett.*, vol. 38, pp. 259–270, 1999.

[25] N. Kazantzis, "A functional equations approach to nonlinear discrete-time feedback stabilization through pole-placement," *Syst. Control Lett.*, vol. 43, pp. 361–369, 2001.

[26] L. Zaccarian, A. R. Teel, and D. Nesic, "On finite gain L_P stability of nonlinear sampled-data systems," *Syst. Control Lett.*, vol. 49, pp. 201–212, 2003.

[27] Z. Artstein, "Stabilization with relaxed controls," *Nonlinear Anal.*, vol. 7, pp. 1163–1173, 1983.

[28] R. Sepulchre, M. Jankovic, and P. Kokotovic, *Constructive Nonlinear Control*. Berlin, Germany: Springer-Verlag, 1997.

[29] N. H. El-Farra, P. Mhaskar, and P. D. Christofides, "Hybrid predictive control of nonlinear systems: Method and applications to chemical processes," *Int. J. Robust Nonlinear Control*, vol. 4, pp. 199–225, 2004.

[30] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Robust hybrid predictive control of nonlinear systems," *Automatica*, vol. 41, pp. 209–217, 2005.

[31] N. H. El-Farra, P. Mhaskar, and P. D. Christofides, "Uniting bounded control and MPC for stabilization of constrained linear systems," *Automatica*, vol. 40, pp. 101–110, 2004.

[32] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Hybrid predictive control of process systems," *AIChE J.*, vol. 50, pp. 1242–1259, 2004.

Prashant Mhaskar was born in Varanasi, India, in 1977. He received the B.Tech. degree in chemical engineering from the Indian Institute of Technology, Bombay, in 1999, the M.S. degree in chemical engineering from Louisiana State University, Baton Rouge, in 2001, and the Ph. D. degree in chemical engineering from the University of California, Los Angeles (UCLA), in 2005.

Since September 2005, he has been an Assistant Professor in the Department of Chemical Engineering, McMaster University, Hamilton, ON, Canada. His research interests include model predictive control and hybrid control.

Dr. Mhaskar is a recipient of the UCLA Dissertation Year Fellowship for the academic year 2004–2005, Best Presentation in Session Awards at the 2004 and 2005 American Control Conferences, and the 2003 Teaching Assistant of the Year Award from the AIChE Student Chapter of UCLA.

Nael H. El-Farra was born in Kuwait in 1972. He received the B.Sc. and Ph.D. degrees in chemical engineering, with highest honors, from the University of California, Los Angeles (UCLA), in 1998 and 2004, respectively.

Since July 2004, he has been an Assistant Professor in the Department of Chemical Engineering and Materials Science at the University of California, Davis. His research interests include constrained nonlinear control and estimation, fault-tolerant control, robust control of hybrid systems, analysis and design of networked process control systems, and computational modeling, simulation and systems-level analysis of biological networks. He has published over 20 refereed articles and is coauthor of the book *Control of Nonlinear and Hybrid Process Systems: Designs for Uncertainty, Constraints and Time-Delays* (New York: Springer-Verlag, 2005).

Dr. El-Farra is a recipient of the 2004 Edward K. Rice Outstanding Doctoral Student Award from the UCLA Henry Sameuli School of Engineering and Applied Science, Best Presentation in Session awards at the 2000 and 2003 American Control Conference, the 2000 Teaching Assistant of the Year Award from the AIChE student chapter at UCLA, the UCLA Chancellor's Fellowship, for 1998–2002, and the UCLA School of Engineering and Applied Science Outstanding Bachelor of Science Award in 1998.

Panagiotis D. Christofides (M'02) was born in Athens, Greece, in 1970. He received the Diploma in chemical engineering from the University of Patras, Patras, Greece, in 1992, the M.S. degrees in electrical engineering and mathematics in 1995 and 1996, respectively, and the Ph.D. degree in chemical engineering in 1996, all from the University of Minnesota, Minneapolis.

Since July 1996, he has been with the Department of Chemical Engineering, the University of California, Los Angeles (UCLA), where he is currently a Professor. His theoretical research interests include nonlinear control, singular perturbations, and analysis and control of distributed parameter systems, multiscale systems, and hybrid systems, with applications to advanced materials processing, particulate processes, biological systems, and fluid flows. His research work has resulted in a large number of articles in leading scientific journals and conferences and two books entitled *Nonlinear and Robust Control of PDE Systems: Methods and Applications to Transport-Reaction Processes* (Norwell, MA: Birkhäuser, 2001) and *Model-Based Control of Particulate Processes* (Boston, MA: Kluwer, 2002). He has also coauthored (with N. H. El-Farra) the recent book *Control of Nonlinear and Hybrid Process Systems: Designs for Uncertainty, Constraints and Time-Delays* (New York: Springer-Verlag, 2005). A description of his research interests and a list of his publications can be found at http://www.chemeng.ucla.edu/pchristo/index.html.

Dr. Christofides has been a Member of the Control Systems Society Conference Editorial Board, the 2004 Program Coordinator of the Applied Mathematics and Numerical Analysis Area of AIChE and the Program Vice-Chair for Invited Sessions for the 2004 American Control Conference. He is currently an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He has received the Teaching Award from the AIChE Student Chapter of UCLA in 1997, a Research Initiation Award from the ACS-Petroleum Research Fund in 1998, a CAREER award from the National Science Foundation in 1998, the Ted Peterson Student Paper Award from the Computing and Systems Technology Division of AIChE in 1999 and a Young Investigator Award from the Office of Naval Research in 2001. He has also received twice the O. Hugo Schuck Best Paper Award in 2000 (with A. Armaou) and 2004 (with D. Ni, Y. Lou, L. Sha, S. Lao and J. P. Chang), and the Donald P. Eckman Award in 2004, all from the American Automatic Control Council. He was a Plenary Speaker in the 2005 American Control Conference.