

## Robust predictive control of switched systems: Satisfying uncertain schedules subject to state and control constraints

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### SUMMARY

In this work, we consider robust predictive control of switched uncertain nonlinear systems required to satisfy a prescribed switching sequence with uncertainty in the switching times subject to state and input constraints. To illustrate our approach, we consider first the problem of satisfying a prescribed schedule subject to uncertainty only in the switching times. Predictive controllers that guarantee the satisfaction of state and input constraints from an explicitly characterized set of initial conditions are first designed. The performance and constraint-handling capabilities of the predictive controllers are subsequently utilized in ensuring the satisfaction of the switching schedule while preserving stability. The results are then generalized to address the problem in the presence of parametric uncertainty and exogenous time-varying disturbances in the dynamics of the constituent modes. The proposed control method is demonstrated through application to a scheduled chemical process example. Copyright © 2007 John Wiley & Sons, Ltd.

Received 6 November 2006; Revised 28 March 2007; Accepted 25 April 2007

**KEY WORDS:** switched systems; uncertainty; input constraints; robust model predictive control; bounded Lyapunov-based control; multiple Lyapunov functions; stability regions

### 1. INTRODUCTION

Chemical processes often have to undergo discrete switches between different modes of operation to satisfy, for instance, the requirements of different product grades and to process different streams

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Contract/grant sponsor: National Science Foundation; contract/grant number: CTS-0529295

through the same unit. The operation schedule has to be executed while preserving process stability and respecting constraints on variables that arise due to performance and safety considerations. Furthermore, uncertainty in switching times due, for example, to the uncertainty in stream availabilities from the upstream processing units must be accounted for in the process operation. Even in the absence of such uncertainties, the hybrid nature of the process arising due to switching between different modes of operation makes the operation and control of the process a challenging problem. Motivated by these challenges, as well as the abundance of situations where hybrid systems arise in practice, significant research efforts have focused on the study of hybrid systems over the past decade, covering a wide range of problems including, for example, modelling [1], optimization [2], stability analysis [3–6], and control [7–10]. When considering switched systems, which constitute an important class of hybrid systems, the stability of the constituent modes of operation as well as that of the switching sequence is important. The stability analysis of switched systems has been addressed within the multiple Lyapunov function (MLF) framework for linear [11] as well as nonlinear [3] systems, and issues such as dwell time [5] have been studied (see [4, 6] for a survey of results in this area). These results have motivated the development of methods for control of various classes of switched systems (see, for example, [9, 12–14]).

In [15], a framework for coordinating feedback and switching for control of hybrid nonlinear systems with input constraints was developed, and switching laws were derived to determine (though not enforce), whether a scheduled transition will preserve stability of the overall switched closed-loop system. Enforcing a prescribed switching schedule, however, requires guiding the system trajectory *via* a control algorithm that incorporates both state and input constraints in the control design. One such control method is model predictive control (MPC), which has been studied extensively in the context of purely continuous and purely discrete time linear and nonlinear systems (see, for example, [16–20] and the survey paper, [21]).

An important issue that arises when implementing predictive control policies for the purpose of stabilization of the constituent modes, however, is the difficulty in identifying, *a priori* (i.e. before controller implementation), the set of initial conditions starting from where feasibility and closed-loop stability under the predictive controller are guaranteed. In [22, 23], predictive controllers that guarantee stabilization of the constituent modes from an explicitly characterized set of initial conditions subject to input and state constraints and uncertainty were designed, but did not consider the problem of satisfying a prescribed switching schedule. The constraint-handling capabilities of the predictive controller were utilized in [24] to ensure that the prescribed switching schedule is satisfied while preserving closed-loop stability. The work in [24], however, did not account for uncertainty in the switching times (the switching times were assumed to be explicitly known), and the controllers of the constituent modes of operation did not account for the presence of state constraints and uncertainty in the dynamics of the constituent modes. The presence of state constraints and uncertainty negatively impact the feasibility and stability properties of the controllers of the constituent modes in [24] and may destabilize the overall switched closed-loop system.

Motivated by these considerations, we consider in this work predictive control of switched nonlinear systems required to satisfy a prescribed switching sequence with uncertainty in the switching times, parametric uncertainty and time-varying exogenous disturbances in the dynamics of the constituent modes, and state and control constraints. The rest of the paper is organized as follows. Following some preliminaries in Section 2, we address first the problem of uncertainty in the switching times in Section 3. In Section 3.1, we review a predictive controller design that guarantees the satisfaction of state and input constraints from an explicitly characterized set of

initial conditions. The performance and constraint-handling capabilities of the predictive controllers are subsequently utilized in Section 3.2 to ensure the satisfaction of the switching schedule subject to uncertainty while preserving stability. Specifically, the system trajectory is required to reside in the stability region of the target mode by the earliest possible time that a switch out from the current mode can occur and to stay there up until the switch out occurs. The presence of uncertainty in the constituent modes of operation is then accounted for in Section 4, and its impact on the control policy demonstrated. To this end, robust predictive controllers are presented in Section 4.1 and the predictive controller formulated to satisfy the switching sequence is presented in Section 4.2. The proposed control method is demonstrated through application to a scheduled chemical process example in Section 5.

## 2. PRELIMINARIES

We consider the class of switched nonlinear systems represented by the following state-space description:

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))u_{\sigma(t)}(t) + W_{\sigma(t)}(x)\theta_{\sigma(t)}(t) \\ u_{\sigma(t)} &\in \mathcal{U}_{\sigma}, \quad \theta_{\sigma(t)} \in \Theta_{\sigma}, \quad \sigma(t) \in \mathcal{K} := \{1, \dots, p\} \end{aligned} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  denotes the vector of continuous-time state variables,  $u_{\sigma}(t) = [u_{\sigma}^1(t) \dots u_{\sigma}^m(t)]^T \in \mathcal{U}_{\sigma} \subset \mathbb{R}^m$  denotes the vector of constrained manipulated inputs taking values in a non-empty compact convex set  $\mathcal{U}_{\sigma} := \{u_{\sigma} \in \mathbb{R}^m : u_{\sigma, \min} \leq u_{\sigma} \leq u_{\sigma, \max}\}$ ,  $u_{\sigma, \min}, u_{\sigma, \max} \in \mathbb{R}^m$  denote the minimum and maximum allowable values for the manipulated inputs,  $u_{\sigma}^{\text{norm}} > 0$  is a positive real number such that  $\|u_{\sigma}\| \leq u_{\sigma}^{\text{norm}}$  implies  $u_{\sigma} \in \mathcal{U}_{\sigma}$  where  $\|\cdot\|$  is the Euclidian norm,  $\theta_{\sigma(t)}(t) = [\theta_{\sigma(t)}^1(t) \dots \theta_{\sigma(t)}^q(t)]^T \in \Theta_{\sigma} \subset \mathbb{R}^q$  denotes the vector of uncertain (possibly time varying) but bounded variables taking values in a non-empty compact convex subset of  $\mathbb{R}^q$ ,  $\sigma : [0, \infty) \rightarrow \mathcal{K}$  is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, i.e.  $\sigma(t_k) = \lim_{t \rightarrow t_k^+} \sigma(t)$  for all  $k$ , implying that only a finite number of switches is allowed on any finite interval of time.  $p$  is the number of modes of the switched system,  $\sigma(t)$ , which takes different values in the finite index set  $\mathcal{K}$ , represents a discrete state that indexes the vector field  $f(\cdot)$ , the matrices  $G(\cdot)$  and  $W(\cdot)$ , and the control input  $u(\cdot)$ , which altogether determine  $\dot{x}$ . Throughout the paper, we use the notations  $t_{k_r^{\text{in}}}$  and  $t_{k_r^{\text{out}}}$  to denote the (*a priori*) unknown time at which the  $k$ th subsystem is switched in and out, respectively, for the  $r$ th time, i.e.  $\sigma(t_{k_r^{\text{in}}}^+) = \sigma(t_{k_r^{\text{out}}}^-) = k$ . The times  $t_{k_r^{\text{in}}^{\text{min}}}, t_{k_r^{\text{in}}^{\text{max}}}$  denote the known range within which a transition into mode  $k$  can take place for the  $r$ th time, and  $t_{k_r^{\text{out}}^{\text{min}}}$  and  $t_{k_r^{\text{out}}^{\text{max}}}$  denote the known range within which the transition out of mode  $k$  can take place. With this notation, it is understood that the continuous state evolves according to  $\dot{x} = f_k(x) + G_k(x)u_k + W_k(x)\theta_k(t)$  for  $t \in [t_{k_r^{\text{in}}}, t_{k_r^{\text{out}}})$  where  $t_{k_r^{\text{in}}^{\text{min}}} \leq t_{k_r^{\text{in}}} \leq t_{k_r^{\text{in}}^{\text{max}}}$  and  $t_{k_r^{\text{out}}^{\text{min}}} \leq t_{k_r^{\text{out}}} \leq t_{k_r^{\text{out}}^{\text{max}}}$ . It is assumed that all entries of the vector functions  $f_k(x)$ , and the  $n \times m$  and  $n \times q$  matrices  $G_k(x)$  and  $W_k(x)$ , are sufficiently smooth and that  $f_k(0) = 0$  for all  $k \in \mathcal{K}$ . Throughout the paper, the notation  $L_f \bar{h}$  denotes the standard Lie derivative of a scalar function  $\bar{h}(x)$  with respect to the vector function  $f(x)$ ,  $L_f \bar{h}(x) = (\partial \bar{h} / \partial x) f(x)$ , and  $\limsup_{t \rightarrow \infty} f(x(t)) = \lim_{t \rightarrow \infty} \{\sup_{\tau \geq t} f(x(\tau))\}$ . To simplify the development and presentation of our results, we focus on the state feedback control problem (extensions to the output feedback control problem are possible and are the subject of other research work) where measurements of  $x(t)$  are assumed to be available for all  $t$ .

### 3. IMPLEMENTING A PRESCRIBED SWITCHING SEQUENCE AT UNCERTAIN TIMES

In this section, we first consider the problem where the uncertainty appears only in the switching times, and not in the dynamics of the constituent modes of operation. We first present a predictive controller design that guarantees the satisfaction of state and input constraints from an explicitly characterized set of initial conditions and then employ it to ensure stabilization while following the prescribed switching sequence.

#### 3.1. Predictive controller guaranteeing state and control constraints satisfaction

In this section, we consider predictive control of the system of Equation (1) without the uncertainty term (i.e. with  $\theta_{\sigma(t)} \equiv 0$ ) for a fixed  $\sigma(t) = k$  for some  $k \in \mathcal{K}$ . We briefly review a Lyapunov-based design of MPC that guarantees feasibility of the optimization problem and hence constrained stabilization of the closed-loop system from an explicitly characterized set of initial conditions subject to state and control constraints. For this predictive control design, the control action at state  $x$  and time  $t$  is obtained by solving, online, a finite horizon optimal control problem of the form

$$P(x, t) : \min\{J(x, t, u_k(\cdot)) \mid u_k(\cdot) \in S_k, x(\cdot) \in X_k\} \quad (2)$$

$$\text{s.t. } \dot{x} = f_k(x) + G_k(x)u_k \quad (3)$$

$$\dot{V}_k(x(\tau)) \leq -\varepsilon_k \quad \text{if } V_k(x(t)) > \delta'_k, \quad \tau \in [t, t + \Delta] \quad (4)$$

$$V_k(x(\tau)) \leq \delta'_k \quad \text{if } V_k(x(t)) \leq \delta'_k, \quad \tau \in [t, t + \Delta] \quad (5)$$

where  $X_k$  denotes the constraints that the state variables are required to respect in each mode; note that this is part of the control objective and not inherent to the dynamics of the system (it is therefore not included in the system description of Equation (1)). The parameters  $\varepsilon_k, \delta'_k$  are to be determined later,  $S_k = S_k(t, T)$  is the family of piecewise continuous functions (functions continuous from the right), with period  $\Delta$ , mapping  $[t, t + T]$  into  $\mathcal{U}_k$ ,  $T$  is the specified horizon and  $V_k$  is a control Lyapunov function. A control  $u_k(\cdot)$  in  $S_k$  is characterized by the sequence  $\{u_k[j]\}$  where  $u_k[j] := u_k(j\Delta)$  and satisfies  $u_k(\tau) = u_k[j]$  for all  $\tau \in [t + j\Delta, t + (j + 1)\Delta]$ . The performance index is given by

$$J(x, t, u_k(\cdot)) = \int_t^{t+T} [\|x^u(s; x, t)\|_Q^2 + \|u_k(s)\|_R^2] ds \quad (6)$$

where  $Q$  is a positive semidefinite symmetric matrix, and  $R$  is a strictly positive definite symmetric matrix.  $x^u(s; x, t)$  denotes the solution of Equation (1), due to control  $u$ , with initial state  $x$  at time  $t$ . The minimizing control  $u_k^0(\cdot) \in S_k$  is then applied to the plant over the interval  $[t, t + \Delta]$  and the procedure is repeated indefinitely.

To characterize the stability and constraint-handling properties of the predictive controller of Equations (2)–(6), an auxiliary-bounded controller is employed. Specifically, for the system of Equation (1) under a mode  $k$  (the subscript  $k$  will henceforth be dropped with the understanding that the controller for mode  $k$  is being described), using the results in [25], the following bounded

control law [26] (see also [27]) can be constructed:

$$u_{bc}(x) = \begin{cases} -k(x)(L_G V)'(x), & \|(L_G V)^T(x)\| \neq 0 \\ 0, & \|(L_G V)^T(x)\| = 0 \end{cases} \quad (7)$$

where

$$k(x) = \frac{L_f^* V(x) + \sqrt{(L_f^* V(x))^2 + (u^{\text{norm}} \|(L_G V)^T(x)\|)^4}}{\|(L_G V)^T(x)\|^2 [1 + \sqrt{1 + (u^{\text{norm}} \|(L_G V)^T(x)\|)^2}]} \quad (8)$$

$L_G V(x) = [L_{g^1} V \dots L_{g^m} V]$  is a row vector, where  $g^i$  is the  $i$ th column of  $G$ ,  $L_f^* V = L_f V + \rho V$  and  $\rho > 0$ . One can show, using a standard Lyapunov argument, that the set

$$\Omega_{x,u} = \{x \in \mathbb{R}^n : V(x) \leq c_{x,u}^{\text{max}}\} \quad (9)$$

such that  $c_{x,u}^{\text{max}} > 0$  is the largest number for which  $\Omega_{x,u} \subseteq \Phi_{x,u}$  with

$$\Phi_{x,u} = \{x \in X : L_f^* V(x) \leq u^{\text{norm}} \|(L_G V)^T(x)\|\} \quad (10)$$

provides an estimate of the stability region, starting from where the origin of the constrained closed-loop system, under the control law of Equations (7)–(8), is guaranteed to be asymptotically stable and state and input constraints are satisfied for all time. Utilizing the robustness of the control law of Equation (7) with respect to discrete implementation, the following result can be derived for the predictive controller of Equations (2)–(6) (see [22] for the proof).

*Theorem 1 (Mhaskar et al. [22])*

Consider the constrained system of Equation (1) for a fixed value of  $\sigma(t) = k$  under the MPC control law of Equations (2)–(6), designed using a control Lyapunov function  $V_k$  that yields a stability region  $\Omega_k$  under continuous implementation of the bounded controller of Equations (7)–(8) with a fixed  $\rho_k > 0$ . Then, given any positive real number  $d_k$ , there exist positive real numbers  $\varepsilon_k$ ,  $\Delta_k^*$  and  $\delta'_k$ , such that if  $x(0) \in \Omega_k$  and  $\Delta \in (0, \Delta_k^*]$ , then  $x(t) \in \Omega_k \forall t \geq 0$  and  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq d_k$ .

*Remark 1*

Some of the distinguishing features of the predictive controller formulation of Equations (2)–(6) are that the stability guarantees do not assume initial feasibility of the problem and the feasibility of the problem does not rely on the horizon being sufficiently large. Furthermore, the stability is ensured simply from a feasible solution to the optimization problem, and does not require the solution to be the global optimum (see [22] for further discussion and explanation of these points). Finally, the most important characteristic, and the one that makes it most suitable for use in the present work, is the explicit characterization of the stability region that is possible for each constituent mode of the switched system. It is this characterization that is utilized in formulating the constraints to ensure that the prescribed switching sequence can be executed while preserving closed-loop stability.

### 3.2. Handling uncertain switching times

Consider now the nonlinear switched system of Equation (1) with  $\theta(t) \equiv 0$ , with a prescribed switching sequence defined by  $\mathcal{T}_{k,\text{in}} = \{t_{k_1^{\text{in}}}, t_{k_2^{\text{in}}}, \dots\}$  and  $\mathcal{T}_{k,\text{out}} = \{t_{k_1^{\text{out}}}, t_{k_2^{\text{out}}}, \dots\}$ , with  $t_{k_r,\text{min}}^{\text{in}} \leq t_{k_r^{\text{in}}} \leq t_{k_r,\text{max}}^{\text{in}}$  and  $t_{k_r,\text{min}}^{\text{out}} \leq t_{k_r^{\text{out}}} \leq t_{k_r,\text{max}}^{\text{out}}$ . Assume that for each mode of the switched system, a Lyapunov-based predictive controller of the form of Equations (2)–(6) has been designed and an estimate of the

stability region generated. The control problem is formulated as the one of designing a Lyapunov-based predictive controller that guides the closed-loop system trajectory in a way that the schedule described by the switching times is followed in the presence of uncertainty in the switching times and stability of the closed-loop system is preserved. A predictive control algorithm that addresses this problem is presented in Theorem 2.

*Theorem 2*

Consider the constrained nonlinear system of Equation (1) with  $\theta(t) \equiv 0$ , the control Lyapunov functions  $V_k, k = 1, \dots, p$ , and the stability region estimates  $\Omega_k, k = 1, \dots, p$ , under continuous implementation of the bounded controller of Equations (7)–(8) with fixed  $\rho_k > 0, k = 1, \dots, p$ . Let  $0 < T_{\text{design}} < \infty$  be a design parameter. Let the system be evolving in mode  $k$ , scheduled to switch to mode  $m$  between  $t_{k,r,\min}^{\text{out}}$  and  $t_{k,r,\max}^{\text{out}}$ . Consider the following optimization problem:

$$P(x, t) : \min\{J(x, t, u_k(\cdot)) | u_k(\cdot) \in S_k\} \tag{11}$$

$$J(x, t, u_k(\cdot)) = \int_t^{t+T} [\|x^u(s; x, t)\|_Q^2 + \|u_k(s)\|_R^2] ds \tag{12}$$

where  $T$  is the prediction horizon given by  $T = t_{k,r,\max}^{\text{out}} - t$ , if  $t_{k,r,\max}^{\text{out}} < \infty$  and  $T = T_{\text{design}}$  if  $t_{k,r,\max}^{\text{out}} = \infty$ , subject to the following constraints:

$$\dot{x} = f_k(x) + G_k(x)u_k \tag{13}$$

$$\dot{V}_k(x(\tau)) \leq -\varepsilon_k \quad \text{if } V_k(x(t)) > \delta'_k, \quad \tau \in [t, t + \Delta_{k_r}] \tag{14}$$

$$V_k(x(\tau)) \leq \delta'_k \quad \text{if } V_k(x(t)) \leq \delta'_k, \quad \tau \in [t, t + \Delta_{k_r}] \tag{15}$$

And if  $t_{k,r,\max}^{\text{out}} < \infty$  then

$$V_m(x(\tau)) \leq \left\{ \begin{array}{ll} V_m(x(t_{m,j-1}^{\text{in}})) - \varepsilon^*, & j > 1, \quad V_m(x(t_{m,j-1}^{\text{in}})) > \delta'_m \\ \delta'_m, & j > 1, \quad V_m(x(t_{m,j-1}^{\text{in}})) \leq \delta'_m \\ c_m^{\text{max}}, & j = 1 \end{array} \right\} \tag{16}$$

$\forall \tau \in [t_{k,r,\min}^{\text{out}}, t_{k,r,\max}^{\text{out}}]$ , where  $\varepsilon^*$  is a positive real number. Then, given a positive real number  $d^{\text{max}}$ , there exist positive real numbers  $\delta'_k, k = 1, \dots, m$  such that if the optimization problem of Equations (11)–(16) is feasible at all times, the minimizing control is applied to the system over the interval  $[t, t + \Delta_{k_r}]$ , where  $\Delta_{k_r} \in (0, \Delta^*]$  with  $t_{k,r,\min}^{\text{out}} = t_{k_r}^{\text{in}} + l_{k_r,\min} \Delta_{k_r}, t_{k_r,\text{out}} = t_{k_r}^{\text{in}} + l_{k_r} \Delta_{k_r}$  and  $t_{k_r,\max}^{\text{out}} = t_{k_r}^{\text{in}} + l_{k_r,\max} \Delta_{k_r}$  for some integers  $l_{k_r,\min}, l_{k_r}, l_{k_r,\max} > 0$  and the procedure is repeated, then,  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq d^{\text{max}}$ .

*Proof*

The proof of this theorem follows from the assumption of feasibility of the constraints of Equations (14)–(16) at all times, and is divided into two parts. The first part considers the case where the switching sequence is infinite, while the second part considers the case of a finite switching sequence. In both cases, given the radius of the ball around the origin,  $d^{\text{max}}$ , the values of  $\delta'_k$  for all  $k \in \mathcal{K}$  are first computed such that  $V_k(x) \leq \delta'_k$  implies  $\|x\| \leq d^{\text{max}}$ .

*Part 1.* First consider the case when the switching sequence is infinite. Let  $t$  be such that  $t_{k_r^{\text{in}}} \leq t < t_{k_r^{\text{out}}}$  and  $t_{m_j^{\text{in}}} = t_{k_r^{\text{out}}} < \infty$ . Consider the active mode  $k$ . If  $V_k(x) > \delta'_k$ , the continued feasibility of the constraint of Equation (14) implies that  $V_k(x(t_{k_r^{\text{out}}})) < V_k(x(t_{k_r^{\text{in}}}))$ . The transition constraint of Equation (16) ensures that if this mode is switched out and then switched back in, then  $V_k(x(t_{k_{r+1}^{\text{in}}})) < V_k(x(t_{k_r^{\text{in}}}))$ . In general,  $V_k(x(t_{k_l^{\text{in}}})) < V_k(x(t_{k_{l-1}^{\text{in}}})) < \dots < c_k^{\text{max}}$ . Under the assumption of feasibility of the constraints of Equations (14)–(16) for all future times, therefore, the value of  $V_k(x)$  continues to decrease. If the mode of this Lyapunov function is not active, there exists at least some  $j \in 1, \dots, p$  such that mode  $j$  is active and Lyapunov function  $V_j$  continues to decrease until the time that  $V_j \leq \delta'_j$  (this happens because there are finite number of modes, even if the number of switches may be infinite). From this point onwards, the constraint of Equation (15) ensures that  $V_j$  continues to be less than  $\delta'_j$ . Hence,  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq d^{\text{max}}$ .

*Part 2.* For the case of a finite switching sequence, consider a  $t$  such that  $t_{k_r^{\text{in}}} \leq t < t_{k_r^{\text{out}}} = \infty$ . Under the assumption of continued feasibility of Equations (14)–(16),  $V_k(x(t_{k_r^{\text{in}}})) < V_k(x(t_{k_{r-1}^{\text{in}}})) < \dots < c_k^{\text{max}}$ . At the time of the switch to mode  $k$ , therefore,  $x(t_{k_r^{\text{in}}}) \in \Omega_k$ . From this point onwards, the Lyapunov-based controller is implemented using the Lyapunov function  $V_k$ , and the constraint of Equation (16) is removed, in which case the predictive controller of Theorem 2 reduces to the predictive controller of Equations (2)–(6). Since the value of  $\Delta_{k_r}$  is chosen to be in  $(0, \Delta^*]$ , where  $\Delta^* = \min_{k=1, \dots, p} \Delta_k^*$ , therefore  $\Delta_{k_r} < \Delta_k^*$ , which guarantees feasibility and convergence to the ball  $\|x\| \leq d^{\text{max}}$  for any value of the prediction horizon (hence, for a choice of  $T = T_{\text{design}}$ ), and leads to  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq d^{\text{max}}$ . This completes the proof of Theorem 2.  $\square$

### Remark 2

Note that the stability properties of the predictive controller of Equations (2)–(6) ensure that the constraints of Equations (14)–(15) are feasible between mode transitions, provided that the system is initialized within the stability region. These constraints formulate one of the criteria in the MLF stability analysis, which is to enforce the continued decay of the Lyapunov function of a given mode whenever the mode is active. The constraint of Equation (16) then requires that during the time that the switch out may take place: (1) the state of the process reside within the stability region of the target mode if it is the first switch to the target mode; (2) if the system did switch to the target mode before but did not reach the desired neighbourhood of the origin then for the value of the Lyapunov function to be less than what it was the last time the system switched into that mode (required when the switching sequence is infinite, see [3] for details); and finally, (3) to reside in the desired neighbourhood of the origin under the target mode if the system did reach the desired neighbourhood of the origin the last time the system switched into the target mode. The exact time when the switch is going to take place, however, is not known, and only a bound on the minimum and maximum value of the switching time is known. The constraints therefore are required to hold during the entire interval where the switch can take place. Once the switch takes place, however, this constraint is ‘updated’ for the next switch.

### Remark 3

The algorithm below explains the implementation of the predictive controller of Theorem 2:

1. Given the system model of Equation (1) and the constraints on the input, design the predictive controller of Equations (2)–(6) and estimate its stability region  $\Omega_k$  as well as the discretization time  $\Delta_k$ , such that if  $x(0) \in \Omega_k$  then  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq d^{\text{max}}$ .

2. For a time  $t_{k^{\text{in}}}$ , which designates the time that the closed-loop system switches to the  $k$ th mode for the  $r$ th time, and the state belongs to the stability region of the  $k$ th mode, consider the predictive controller of Theorem 2. The first part of the constraint of Equation (16) requires that during the time when the closed-loop system may enter the mode  $m$ , the value of  $V_m$  is less than what it was at the time that the system last entered mode  $m$  (this is a version of the MLF stability condition, see [3]). If the system has never entered mode  $m$  before, i.e. for  $j = 1$ , require the value of the Lyapunov function during the possible time of transition to be less than  $c_m^{\text{max}}$  (i.e. for the state to evolve in the stability region of the target mode during the possible time of switch). Finally, if the closed-loop state has already entered the desired ball around the origin, require the state to evolve within the neighbourhood during the possible time of switch.
3. If subsequent switches are scheduled, go back to step 2 else implement the Lyapunov-based predictive controller of Equations (2)–(6) for the current mode to stabilize the closed-loop system.

*Remark 4*

Note that the use of the predictive controller is imperative to the problem of implementing a prescribed switching schedule subject to uncertainty in the switching schedule. In particular, while bounded controllers can achieve stabilization under single-mode operation, the bounded controller framework simply does not allow for incorporating transition constraints, and there is no guarantee that the bounded controller (or even the Lyapunov-based predictive controller, if it does not incorporate the transition constraints) can stabilize the closed-loop system when following a prescribed switching schedule subject to uncertainty (see the simulation example for a demonstration). In contrast, the predictive controller approach provides a natural framework for specifying appropriate transition constraints, which upon being feasible, ensure closed-loop stability.

*Remark 5*

Note that for the purpose of illustration, it is assumed that the (uncertain) switching time is an integer multiple of the discretization time and this assumption can be easily removed. This can be achieved by requiring the constraints to hold starting from the largest time which is an integer multiple of the discretization time and is less than the uncertain switching time. The stability guarantees would not be compromised because the stability and transition constraints are required to be satisfied during the possible times that the switch out may take place, and even if the switch out takes place at a time which is not an integer multiple of the implementation time, the satisfaction of the transition constraints preserves stability.

#### 4. IMPLEMENTING A PRESCRIBED SWITCHING SEQUENCE AT UNCERTAIN TIMES SUBJECT TO UNCERTAINTY IN THE MODES

In the previous section, the uncertainty was present only in the switching times. We now consider the problem where the uncertainty appears not only in the switching times, but also in the dynamics of the constituent modes of operation. To handle parametric uncertainty and time-varying exogenous disturbances in the dynamics of the constituent modes, we first present a robust predictive controller design that guarantees the satisfaction of state and input constraints from an explicitly characterized



set of initial conditions. Subsequently, the controller implementing the prescribed schedule is developed in a way that it accounts for the uncertainty in the switching times as well as in the modes of operation.

4.1. Robust predictive controller

Consider the system of Equation (1) and assume that the uncertain variable term is vanishing in the sense that  $W_k(0)\theta_k(t) = 0$  for any  $\theta_k \in \Theta_k$ , i.e. disturbances that do not perturb the system from its nominal equilibrium point, and that a robust control Lyapunov function [28],  $V_k$  exists. Consider the receding horizon implementation of the control action computed by solving an optimization problem of the form:

$$P(x, t) : \min\{J(x, t, u(\cdot)) | u(\cdot) \in S\} \tag{17}$$

$$\text{s.t. } \dot{x} = f_k(x) + G_k(x)u \tag{18}$$

$$L_G V_k(x)u(t) \leq L_G V_k u_{bc}(x(t)) \tag{19}$$

where

$$u_{bc} = - \left( \frac{\alpha_k(x) + \sqrt{(\alpha_{1,k}(x))^2 + (u_k^{\text{norm}} \beta_k(x))^4}}{(\beta_k(x))^2 [1 + \sqrt{1 + (u_k^{\text{norm}} \beta_k(x))^2}]} \right) (L_G V_k(x))^T \tag{20}$$

when  $L_G V_k(x) \neq 0$  and  $u_{bc} = 0$  when  $L_G V_k(x) = 0$ , where  $\alpha_k(x) = L_f V_k(x) + (\rho_k \|x\| + \chi_k \theta_{b,k} \|L_W V_k^T\|)(\|x\| / (\|x\| + \phi_k))$ ,  $\alpha_{1,k}(x) = L_f V_k + \rho_k \|x\| + \chi_k \theta_{b,k} \|L_W V_k^T\|$ ,  $\beta_k(x) = \|(L_G V_k)^T\|$ ,  $L_G V_k = [L_{g^1} V_k \dots L_{g^m} V_k]$  and  $L_W V_k = [L_{w^1} V_k \dots L_{w^q} V_k]$  are row vectors,  $\theta_{b,k}$  is a positive real number such that  $\|\theta_k(t)\| \leq \theta_b$ , for all  $t \geq 0$ , and  $\rho_k$ ,  $\chi_k$ , and  $\phi_k$  are adjustable parameters that satisfy  $\rho_k > 0$ ,  $\chi_k > 1$  (note that larger values of  $\rho_k$  and  $\chi_k$  enforce faster decay of the Lyapunov function, albeit at the cost of lower stability region estimates) and  $\phi_k > 0$ ,  $S = S(t, T)$  is the family of piecewise continuous functions (functions continuous from the right), with period  $\Delta$ , mapping  $[t, t + T]$  into  $U$ . Equation (18) is the ‘nominal’ nonlinear model (without the uncertainty term) describing the time evolution of the state  $x$ . A control  $u(\cdot)$  in  $S$  is characterized by the sequence  $\{u[j]\}$  where  $u_k(\tau) = u_k[j]$  for all  $\tau \in [t + j\Delta, t + (j + 1)\Delta)$ . The performance index is given by

$$J(x, t, u(\cdot)) = \int_t^{t+T} [\|x^u(s; x, t)\|_Q^2 + \|u(s)\|_R^2] ds \tag{21}$$

where  $Q$  and  $R$  are positive semidefinite, and strictly positive definite, symmetric matrices, respectively, and  $x^u(s; x, t)$  denotes the solution of Equation (18), due to control  $u$ , with initial state  $x$  at time  $t$  and  $T$  is the specified horizon. The minimizing control  $u^0(\cdot) \in S$  is then applied to the plant over the interval  $[t, t + \Delta)$  and the procedure is repeated indefinitely.

Preparatory to the formalization of the stability and feasibility properties of the robust MPC, we recall the following result for the control law described by Equation (20) [26]. Specifically, let  $\Pi_k$  be the set defined by  $\Pi_k = \{x \in \mathbb{R}^n : \alpha_{1,k}(x) \leq u_k^{\text{norm}} \beta_k(x)\}$  and assume that  $\Omega_k := \{x \in \mathbb{R}^n : V(x) \leq c_k^{\text{max}}\} \subseteq \Pi_k(\theta_{b,k}, u_k^{\text{norm}})$  for some  $c_k^{\text{max}} > 0$ . Then, for any initial condition  $x_0 \in \Omega_k$ , it can

be shown that there exists a positive real number  $\gamma_k^*$  such that if  $\phi_k/(\chi_k - 1) < \gamma_k^*$ , the states of the closed-loop system of Equations (1) and (20) satisfy  $x(t) \in \Omega \forall t \geq 0$  and the origin of the closed-loop system is asymptotically stable. The stability properties of the predictive controller design follow from ensuring that  $\dot{V}$  under the predictive controller is at least as negative as that under the implementation of the controller of Equation (20) (expressed in Equation (19)); feasibility of the optimization problem and stability properties of the closed-loop system under the predictive controller are formalized in Theorem 3, for proof, see [23].

*Theorem 3 (Mhaskar [23])*

Consider the constrained system of Equation (1) under the mode  $k$  under the MPC law of Equations (17)–(21). Then, given any positive real number  $d$ , there exists a positive real number  $\Delta_k^{**}$  such that if  $\Delta \in (0, \Delta_k^{**}]$  and  $x(0) := x_0 \in \Omega$ , then the optimization problem of Equations (17)–(21) is guaranteed to be initially and successively feasible,  $x(t) \in \Omega \forall t \geq 0$  and  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq d$ .

*Remark 6*

The two key characteristics of the stability constraint of Equation (19) include: (1) the stability constraint can be guaranteed to be feasible, while respecting the other constraints, from an explicitly characterized set of initial conditions; and (2) the feasibility of the stability constraint guarantees closed-loop stability. In contrast, if the min–max approach was to be followed to handle uncertainty, one possible form that the stability constraint would take is  $\dot{V}(x(\tau)) \leq 0, \forall \tau \in [t, t + \Delta), \forall \theta(t) \in \Theta$ , i.e. to require the control action to enforce negative definiteness of the Lyapunov function derivative during the next step for all possible realizations of the uncertainty. The continued satisfaction of such a constraint would then imply stabilization. The presence of such a constraint, however, turns the optimization problem into a computationally expensive problem. Furthermore, the presence of such a constraint by itself would not allow for explicitly characterizing the set of initial conditions starting from where the optimization problem is guaranteed to be feasible. The stability constraint of Equation (19), in contrast, allows the characterization of the set of initial conditions starting from where the optimization problem is guaranteed to be feasible, without turning it into a min–max problem. Note that the constraint of Equation (19) essentially requires the control action to be computed such that it renders the time derivative of the Lyapunov function, evaluated at the value of the current state, to be more negative than the time derivative of the Lyapunov function that one would achieve if one were to implement the Lyapunov-based bounded robust controller of Equation (20). Specifically, the bounded robust design uses the bound on the uncertainty to prescribe a control action  $u_{bc}$  that yields  $\dot{V}_{bc} = L_f V + L_G V u_{bc} + L_W \theta < 0$ . The Lyapunov-based predictive controller, by requesting a  $u$  such that  $L_G V u \leq L_G V u_{bc}$ , essentially renders  $\dot{V} \leq \dot{V}_{bc} < 0$ . Furthermore, negative definiteness of  $\dot{V}$  during the first time step, not just for the initial value of the state, is ensured by exploiting the robustness margin of the control law of Equation (20). The robustness margin ensures that for a sufficiently small hold time (i.e. there exists a  $\Delta^*$  such that if  $\Delta < \Delta^*$ ),  $\dot{V}$  continues to remain negative during the first time step. For further details on the robust predictive control design and its stability and optimality properties, see [23].

*Remark 7*

While, for the sake of simplicity, the results in the present work have been presented under the assumption of vanishing disturbances, modifying the robust predictive control design to account for non-vanishing disturbances is relatively straightforward. The line of reasoning is as follows: let  $d_1$  denote the neighbourhood of the origin that the closed-loop state is required to converge to, then

pick a  $d_2 < d_1$ , and pick parameters in the control design of Equation (20) to achieve convergence to the  $d_2$  neighbourhood of the origin under continuous implementation. This, in turn implies that under continuous implementation of the robust predictive controller (i.e. with  $\Delta = 0$ ), one can still achieve convergence to the same neighbourhood of the origin. Finally, compute  $\Delta^*$ , such that under sample and hold implementation, with sampling time less than  $\Delta^*$ , convergence to the neighbourhood  $d_1$  is achieved. The reasoning can also be understood as follows: under vanishing disturbances, given an acceptable ‘loss’ in convergence due to discrete implementation (i.e. instead of requiring convergence to the origin, to require convergence to a neighbourhood of the origin), one can come up with a bound on the implement and hold time such that the convergence to the desired neighbourhood of the origin is achieved. Similarly, under non-vanishing disturbances with discrete implementation, the ‘loss’ in convergence to the origin is due to two factors, one due to the non-vanishing disturbance, and the other due to discrete implementation, however, both can be made as small as desired *via* appropriate choice of parameters.

4.2. Handling uncertain switching times and uncertainty in modes of operation

Consider now the switched uncertain system of Equation (1), subject to uncertainty in the switching time. Due to the presence of uncertainty in the constituent mode dynamics, constraints requiring the state to converge to a target mode’s stability region by the earliest possible time of switching, computed using the nominal process model (without the uncertainty term) no longer ensure stability for the uncertain switched system. Theorem 4 presents a controller designed to account for the uncertainty in the switching times as well as the constituent modes of operation. The key idea is to enforce tighter constraints using the nominal model so that the stability constraints continue to be satisfied for the uncertain system.

Consider again the nonlinear switched system of Equation (1), with a prescribed switching sequence defined by  $\mathcal{T}_{k,in} = \{t_{k_1^{in}}, t_{k_2^{in}}, \dots\}$  and  $\mathcal{T}_{k,out} = \{t_{k_1^{out}}, t_{k_2^{out}}, \dots\}$ , with  $t_{k_r^{in},min} \leq t_{k_r^{in}} \leq t_{k_r^{in},max}$  and  $t_{k_r^{out},min} \leq t_{k_r^{out}} \leq t_{k_r^{out},max}$ . For each mode of the switched system, a robust predictive controller of the form of Equations (17)–(21) has been designed and an estimate of the stability region generated. The control problem is formulated as the one of designing a Lyapunov-based predictive controller that guides the closed-loop system trajectory in a way that the schedule described by the switching times is followed in the presence of uncertainty in the switching times and stability of the closed-loop system is achieved. A predictive control algorithm that addresses this problem is presented in Theorem 4.

Theorem 4

Consider the constrained nonlinear system of Equation (1), the control Lyapunov functions  $V_k$ ,  $k = 1, \dots, p$ , and the stability region estimates  $\Omega_k$ ,  $k = 1, \dots, p$  under continuous implementation of the bounded controller of Equation (20) with fixed  $\rho_k > 0$ ,  $k = 1, \dots, p$ . Let  $0 < T_{design} < \infty$  be a design parameter. Let the system be evolving in mode  $k$ , scheduled to switch to mode  $m$  between  $t_{k_r^{out},min}$  and  $t_{k_r^{out},max}$ . Consider the following optimization problem:

$$P(x, t) : \min\{J(x, t, u_k(\cdot)) | u_k(\cdot) \in S_k\} \tag{22}$$

$$J(x, t, u_k(\cdot)) = \int_t^{t+T} [\|x^u(s; x, t)\|_Q^2 + \|u_k(s)\|_R^2] ds \tag{23}$$

where  $T$  is the prediction horizon given by  $T = t_{k_r, \max}^{\text{out}} - t$ , if  $t_{k_r, \max}^{\text{out}} < \infty$  and  $T = T_{\text{design}}$  if  $t_{k_r, \max}^{\text{out}} = \infty$ , subject to the following constraints:

$$\dot{x} = f_k(x) + G_k(x)u_k \tag{24}$$

$$L_{G_k} V_k(x)u(t) \leq L_{G_k} V_k u_{\text{bc}}(x(t)) \tag{25}$$

And if  $t_{k_r, \max}^{\text{out}} < \infty$  then

$$V_m(x(t_\tau)) \leq \left\{ \begin{array}{l} V_m(x(t_{m_{j-1}}^{\text{in}})) - \varepsilon_1^{**}, \quad j > 1, \quad V_m(x(t_{m_{j-1}}^{\text{in}})) > \delta'_m \\ \delta'_m - \varepsilon_2^{**}, \quad j > 1, \quad V_m(x(t_{m_{j-1}}^{\text{in}})) \leq \delta'_m \\ c_m^{\text{max}} - \varepsilon_3^{**}, \quad j = 1 \end{array} \right\} \tag{26}$$

$\forall \tau \in [t_{k_r, \min}^{\text{out}}, t_{k_r, \max}^{\text{out}}]$ . Then, given a positive real number  $d^{\text{max}}$ , there exist positive real numbers  $\Delta^*$  and  $\delta'_k, k = 1, \dots, m, \varepsilon_1^{**}, \varepsilon_2^{**}$  and  $\varepsilon_3^{**}$  such that if the optimization problem of Equations (22)–(26) is feasible at all times, the minimizing control is applied to the system over the interval  $[t, t + \Delta_{k_r}]$ , where  $\Delta_{k_r} \in (0, \Delta^*]$  with  $t_{k_r, \min}^{\text{out}} = t_{k_r, \min}^{\text{in}} + l_{k_r, \min} \Delta_{k_r}$ ,  $t_{k_r, \max}^{\text{out}} = t_{k_r, \max}^{\text{in}} + l_{k_r} \Delta_{k_r}$  and  $t_{k_r, \max}^{\text{out}} = t_{k_r, \max}^{\text{in}} + l_{k_r, \max} \Delta_{k_r}$  for some integers  $l_{k_r, \min}, l_{k_r}, l_{k_r, \max} > 0$  and the procedure is repeated, then,  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq d^{\text{max}}$ .

*Proof*

In contrast to Theorem 2, the proof of this theorem does not follow only from the assumption of feasibility of the constraints of Equations (25)–(26) at all times, but relies also on the fact that the difference between the system without uncertainty and that with bounded uncertainty can be made as small as desired *via* reducing the time over which the system is observed/predicted. This allows for showing that the satisfaction of the tighter constraints of Equation (26), enforced *via* using the model of Equation (24) (note that this model does not consider uncertainty) ensures satisfaction of the stability constraints for the true system with uncertainty.

To this end, consider a time  $t_{k_r, \max}^{\text{out}} - \Delta_{k_r}$ , for which  $x(t_{k_r, \max}^{\text{out}} - \Delta_{k_r}) \in \Omega_k$  (with  $x(t_{k_r, \min}^{\text{in}}) \in \Omega_k$ , this follows from Theorem 3) and the two systems

$$\dot{x}_1 = f_k(x_1) + G_k(x_1)u_k \tag{27}$$

and

$$\dot{x}_2 = f_k(x_2(t)) + G_k(x_2(t))u_k(t) + W_k(x_2)\theta_k(t) \tag{28}$$

with  $x_1(t_{k_r, \max}^{\text{out}} - \Delta_{k_r}) = x_2(t_{k_r, \max}^{\text{out}} - \Delta_{k_r})$ . Since the control action that is computed satisfies the constraint of Equation (26), therefore, depending on  $j$  and  $V_m(x(t_{m_{j-1}}^{\text{in}}))$ , we have that  $V_m(x_1(t_{k_r, \max}^{\text{out}})) \leq V_m(x(t_{m_{j-1}}^{\text{in}})) - \varepsilon_1^{**}$ , or  $V_m(x_1(t_{k_r, \max}^{\text{out}})) \leq \delta'_m - \varepsilon_2^{**}$  or  $V_m(x_1(t_{k_r, \max}^{\text{out}})) \leq c_m^{\text{max}} - \varepsilon_2^{**}$ . From the continuity of the functions  $V_k(\cdot)$ , there exists a positive real number  $\delta_x$ , such that if  $\|x_2 - x_1\| \leq \delta_x$  then  $V_m(x_1) \leq V_m(x(t_{m_{j-1}}^{\text{in}})) - \varepsilon_1^{**}$  implies  $V_m(x_2) \leq V_m(x(t_{m_{j-1}}^{\text{in}})) - \varepsilon_1^*$ ,  $V_m(x_1) \leq \delta'_m - \varepsilon_2^{**}$  implies  $V_m(x_2) \leq \delta'_m - \varepsilon_2^*$  and  $V_m(x_1) \leq c_m^{\text{max}} - \varepsilon_2^{**}$  implies  $V_m(x_2) \leq c_m^{\text{max}} - \varepsilon_2^*$  where  $\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$  are positive real numbers. From the continuity of solutions for Equations (27) and (28) and that the difference between the two right-hand sides is a bounded function for all  $x \in \Omega_k$ , it follows that given a positive real number  $\delta_x$ , if  $x_1(t_{k_r, \max}^{\text{out}} - \Delta_{k_r}) = x_2(t_{k_r, \max}^{\text{out}} - \Delta_{k_r})$ , then there exists a positive real number  $\Delta_k^*$

such that if  $\Delta_{k_r} \leq \Delta_k^*$  then  $\|x_2(t_{k_r^{\text{out}}}) - x_1(t_{k_r^{\text{out}}})\| \leq \delta_x$ . In summary, such a choice of  $\Delta \leq \Delta_k^*$  ensures that the MLF criteria as well as the stability region criteria are satisfied. The rest of the proof follows along similar lines as parts 1 and 2 of the proof of Theorem 2. This completes the proof of Theorem 4.  $\square$

*Remark 8*

Note that as opposed to the case without uncertainty in the constituent modes of operation, the transition constraints that preserve stability while satisfying the switching schedule are tighter to accommodate the fact that the model of Equation (24) used for the prediction does not include the uncertainty term. Note that the cause of the difference between the predicted and the true systems in satisfaction of the MLF stability constraints is the effect of the uncertainty only over the implement and hold time and not over the entire prediction horizon (due to the updating of the process state at the end of every implement and hold interval). While larger values of  $\varepsilon_i^{**}$  would allow for greater implementation and hold times it may negatively impact the assumption of continued feasibility of the tightened constraint and motivate a smaller ‘tightening’ of the constraints (smaller values of  $\varepsilon_i^{**}$ , which, as a natural trade-off would necessitate smaller implementation and hold times). The satisfaction of the tightened constraints, however, ensure satisfaction of the stability and transition constraints (as in Equation (16)) for the real uncertain system.

## 5. APPLICATION TO A CHEMICAL PROCESS EXAMPLE

We consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form  $A \xrightarrow{k} B$  takes place. The operation schedule requires switching between two available inlet streams consisting of pure A at flow rates  $F_1, F_2$ , concentrations  $C_{A1}, C_{A2}$ , and temperatures  $T_{A1}, T_{A2}$ , respectively. For each mode of operation, the mathematical model for the process takes the form

$$\begin{aligned} \dot{C}_A &= \frac{F_\sigma}{V}(C_{A\sigma} - C_A) - k_0 e^{(-E/RT_R)} C_A \\ \dot{T}_R &= \frac{F_\sigma}{V}(T_{A\sigma} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{(-E/RT_R)} C_A + \frac{Q_\sigma}{\rho c_p V} \end{aligned} \quad (29)$$

where  $C_A$  denotes the concentration of the species A,  $T_R$  denotes the temperature of the reactor,  $Q_\sigma$  is the heat removed from the reactor,  $V$  is the volume of the reactor,  $k_0, E, \Delta H$  are the pre-exponential constant, the activation energy, and the enthalpy of the reaction,  $c_p$  and  $\rho$ , are the heat capacity and fluid density in the reactor and  $\sigma(t) \in \{1, 2\}$  is the discrete variable. The values of all process parameters can be found in Table I. The control objective is to stabilize the reactor at the unstable equilibrium point  $(C_A^s, T_R^s) = (0.57 \text{ kmol/m}^3, 395.3 \text{ K})$  using the rate of heat input,  $Q_\sigma$ , and change in inlet concentration of species A,  $\Delta C_{A\sigma} = C_{A\sigma} - C_{A\sigma_s}$  as manipulated inputs with constraints:  $|Q_\sigma| \leq 0.5 \text{ KJ/s}$  and  $|\Delta C_{A\sigma}| \leq 0.5 \text{ kmol/m}^3$ ,  $\sigma = 1, 2$ . For both the modes, we considered quadratic Lyapunov functions of the form  $V(x) = x' P_k x$  where  $x = (C_A - C_A^s, T_R - T_R^s)$ , with

$$P_1 = \begin{bmatrix} 7.39 & 0.32 \\ 0.32 & 0.016 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 33.24 & 1.32 \\ 1.32 & 0.06 \end{bmatrix}$$

Table I. Process parameters and steady state values.

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$V = 0.1 \text{ m}^3$
$R = 8.314 \text{ kJ/kmol K}$
$C_{A1s} = 0.79 \text{ kmol/m}^3$
$C_{A2s} = 1.0 \text{ kmol/m}^3$
$T_{A1} = 352.6 \text{ K}$
$T_{A2} = 310.0 \text{ K}$
$Q_{1s} = 0.0 \text{ kJ/s}$
$Q_{2s} = 0.0 \text{ kJ/s}$
$\Delta H = -4.78 \times 10^4 \text{ kJ/kmol}$
$k_0 = 1.2 \times 10^9 \text{ s}^{-1}$
$E = 8.314 \times 10^4 \text{ kJ/kmol}$
$c_p = 0.239 \text{ kJ/kg K}$
$\rho = 1000.0 \text{ kg/m}^3$
$F_1 = 3.34 \times 10^{-3} \text{ m}^3/\text{s}$
$F_2 = 1.67 \times 10^{-3} \text{ m}^3/\text{s}$
$T_{Rs} = 395.33 \text{ K}$
$C_{As} = 0.57 \text{ kmol/m}^3$

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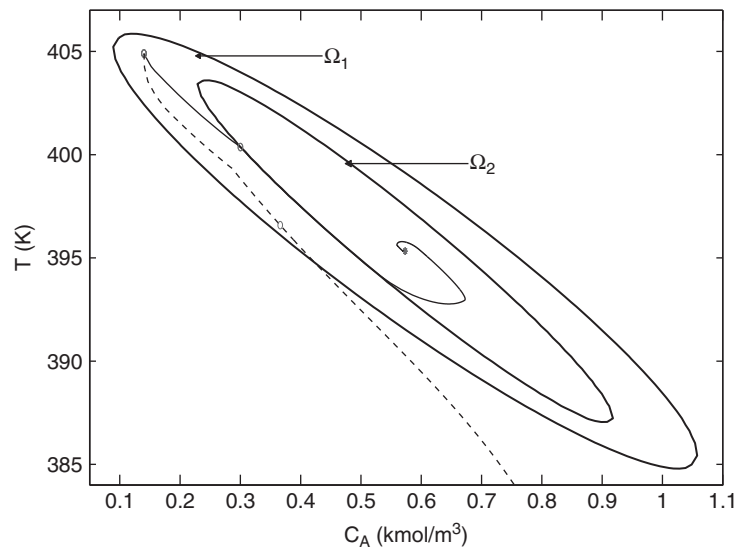


Figure 1. Closed-loop state trajectory without accounting for the uncertainty in the switching schedule (dashed line) and under the predictive controller of Equations (11)–(12) (solid line). The circles mark the state variables at the time of the switch.

and used these in the Lyapunov-based controller design to compute the stability regions for the two modes,  $\Omega_1$  and  $\Omega_2$ , shown in Figure 1. The matrices  $P_k$  were computed by solving a Riccati inequality using the linearized system matrices. The computation of the stability region (using Equations (9)–(10)), however, was done using the nonlinear system dynamics. The parameters in the objective function of Equation (6) are chosen as  $Q = qI$ , with  $q = 1$ , and  $R = rI$ , with

$r = 1.0$ . The constrained nonlinear optimization problem is solved using the MATLAB subroutine `fmincon`, and the set of ordinary differential equations (ODEs) is integrated using the MATLAB solver `ode45`.

In the simulations, we first demonstrate the implementation of the predictive controller formulation to preserve stability subject to uncertainty in the switching schedule. To this end, we consider an initial condition that belongs to the stability region of the predictive controller for mode 1. Starting from the initial condition  $(C_A, T_R) = (0.14 \text{ kmol/m}^3, 404.9 \text{ K})$ , we choose a schedule involving a switch from inlet stream 1 (mode 1) to inlet stream 2 (mode 2). It is only known that the stream 2 may become available between  $t = 0.1$  and  $0.2$  min. We first implement the predictive controller that does not account for the uncertainty in the switching schedule and simply drives the state of the system to the stability region by  $t = 0.2$  min. The state of the closed-loop system moves towards the desired steady state (as seen from the dashed line in Figure 1); however, when the system switches to mode 2 at  $t = 0.1$  min, the system state has not yet been driven inside the stability region of mode 2 (marked by the  $\circ$  in Figure 1). Switching to the predictive controller for mode 2 does not yield a feasible solution. If the bounded controller for mode 2 is implemented, the resulting control action is not able to stabilize the closed-loop system (dashed lines in Figures 1 and 2). In contrast, if the predictive controller of Equations (11)–(15) is implemented, it drives the

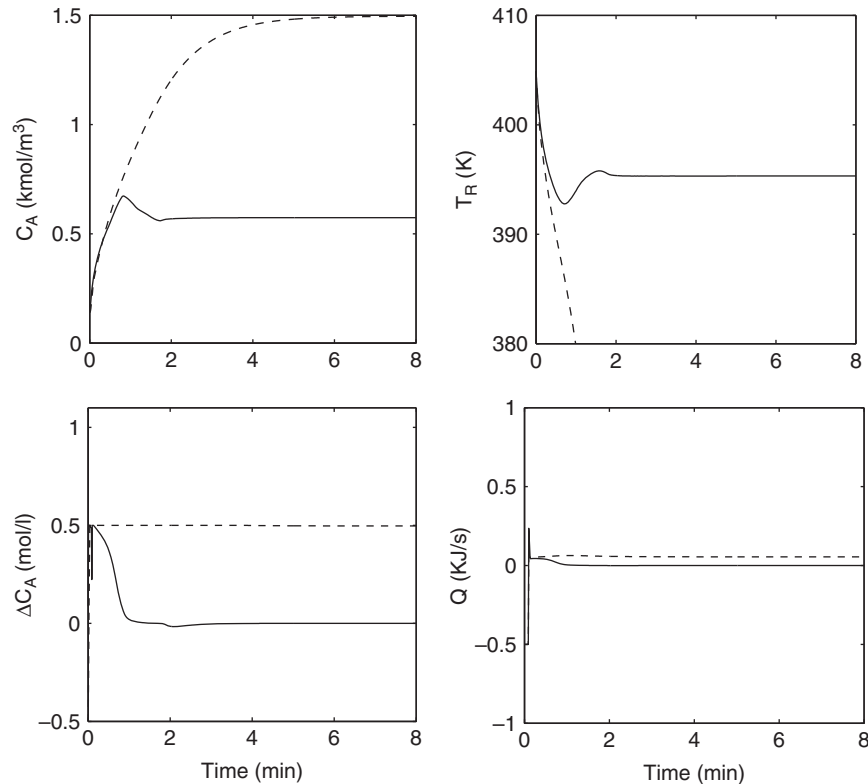


Figure 2. Closed-loop state and input profiles without accounting for the uncertainty in the switching schedule (dashed lines) and under the predictive controller of Equations (11)–(12) (solid lines).

state inside the stability region of mode 2 by the earliest time at which a switch may take place (i.e. at  $t = 0.1$  min). From this point onwards, after the switch takes place, the predictive controller for mode 2 is able to stabilize the closed-loop system (solid lines in Figures 1 and 2).

We next demonstrate the application of the controller of Equations (22)–(23) with the uncertainty appearing in the dynamics of the constituent modes. We focus on demonstrating the required ‘tightening’ of the constraints expressed in Equation (26) to enable satisfaction of the constraints for the uncertain nonlinear system, while only using the nominal model of the system in the predictive controller formulation. To this end, we again consider the system of Equation (29) with  $\theta_1(t) = T_{A0} - T_{A0s}$ ,  $\theta_2(t) = \Delta H - \Delta H_n$ , where the subscript  $s$  denotes the steady-state value and  $\Delta H_n$  denotes the nominal value of the heat of reaction. In the simulation runs,  $\theta_1(t) = \theta_0 \sin(3t)$ , where  $\theta_0 = 0.08T_{A0s}$  and  $\theta_2(t) = -0.5(-\Delta H_n)$ , with manipulated input constraints as  $|Q_\sigma| \leq 92$  kJ/s and  $|\Delta C_{A\sigma}| \leq 0.5$  kmol/m<sup>3</sup>,  $\sigma = 1, 2$ . As for the case without uncertainty, we considered quadratic Lyapunov functions of the form  $V(x) = x' P_k x$  where  $x = (C_A - C_A^s, T_R - T_R^s)$ , with

$$P_1 = \begin{bmatrix} 2.13 & 0.04 \\ 0.04 & 0.014 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 3.0486 & 0.0433 \\ 0.0433 & 0.0014 \end{bmatrix}$$

Starting from the initial condition  $(C_A, T_R) = (0.17$  kmol/m<sup>3</sup>, 400.25 K), we choose a schedule involving an early switch from inlet stream 1 (mode 1) to inlet stream 2 (mode 2) at  $t = 1.2$  s, and, for the sake of comparison, initially run the simulations in the absence of uncertainty. The controller computes a control action requiring the state to reside in the stability region of mode 2 (as predicted using the nominal model). As seen from the dotted line in Figure 3 (and the inset),

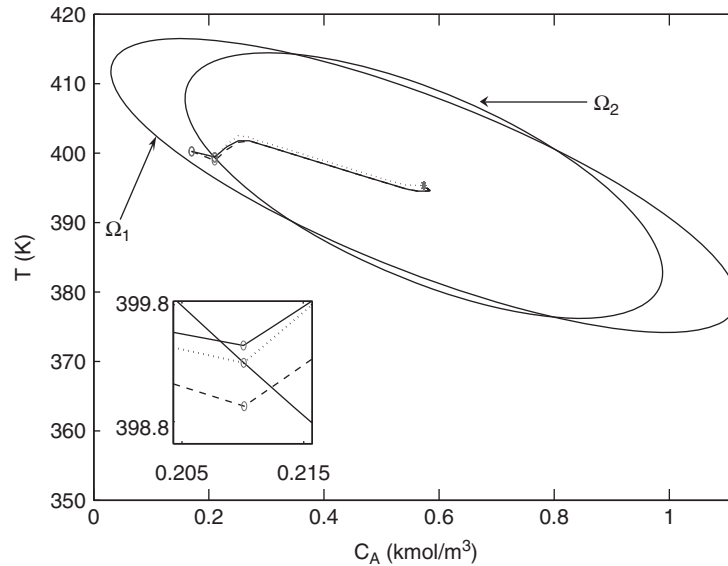


Figure 3. Closed-loop state trajectory without accounting for the uncertainty in the constituent dynamics in the absence (dotted line) and presence (dashed line) of uncertainty and under the controller of Equations (22)–(23) (solid line) in the presence of uncertainty. The circles mark the state variables at the time of the switch.



the closed-loop state trajectory of the system does reside in the stability region of mode 2 at the time of the switch because in the absence of uncertainty, the predicted behaviour of the system (using the nominal model) and the true evolution of the system are identical. However, in the presence of uncertainty, while the controller again computes a control action that requires the state to reside in the stability region of mode 2 (as predicted using the nominal model), the state of the system at the time of the switch does not reside in the stability region of mode 2 (dashed line in Figure 3). This happens because the predictive controller computes a control action that would have driven the closed-loop state inside  $\Omega_2$  in the absence of uncertainty. After the switch, the controller for mode 2, however, is able to stabilize the closed-loop system underscoring the fact that while the controller provides stability guarantees for all initial conditions in  $\Omega_2$ , states outside  $\Omega_2$  may also be stabilized using the robust predictive controller (albeit without *a priori* guarantees). Finally, as seen by the solid line in Figure 3, when the controller of Equations (22)–(23) is implemented and the constraint of Equation (26) is imposed in the predictive controller, the predictive controller implements a more ‘aggressive’ control action (solid lines in the insets in Figure 4) that drives the system state sufficiently inside  $\Omega_2$  (as predicted using the nominal model) so that the state of the uncertain nonlinear system also resides within  $\Omega_2$  at the time of the switch. The robust predictive controller for mode 2 successfully stabilizes the closed-loop system after the switch.

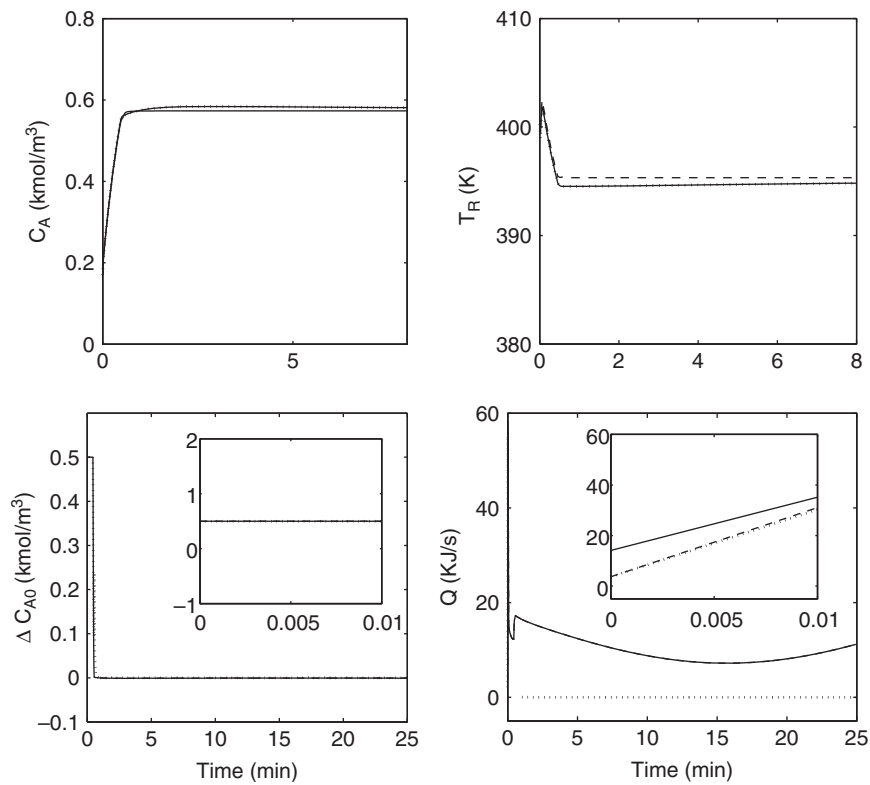


Figure 4. Closed-loop state and input profiles without accounting for the uncertainty in the constituent dynamics in the absence (dotted lines) and presence (dashed lines) of uncertainty and under the controller of Equations (22)–(23) (solid lines) in the presence of uncertainty.

## 6. CONCLUSIONS

In this work, the problem of developing a robust predictive control framework for the constrained stabilization of switched nonlinear systems was considered. The switched nonlinear system transits between the constituent modes at prescribed switching times subject to uncertainty in the constituent mode dynamics as well as in the switching times. We designed Lyapunov-based predictive controllers for each of the constituent modes in which the switched system operates, and incorporated constraints in the predictive controller design which upon satisfaction ensure stability of the switched closed-loop system while accounting for the presence of uncertainty in the switching schedule and the dynamics of the constituent modes. Simulation results, through application to a chemical process example, demonstrated the efficacy of the proposed predictive controller.

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