NONLINEAR CONTROL OF TWO-TIME-SCALE AND DISTRIBUTED PARAMETER SYSTEMS

Volume I

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GRADUATE SCHOOL

“στη Μαρινα”

“στο Δημητρη”

“στους γονεις μου”

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ABSTRACT

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All chemical engineering processes are inherently nonlinear. In addition to nonlinear nature, many chemical engineering processes are characterized by multiple-time-scale behavior and spatial variations. The mathematical models of chemical engineering processes are typically obtained from the dynamic conservation equations and consist of systems of nonlinear Ordinary Differential Equations (ODEs) and nonlinear Partial Differential Equations (PDEs). However, no mathematical model can precisely predict the dynamic behavior of a real process; there is always uncertainty. Model uncertainty is typically due to disturbances and unknown process parameters.

In the last decade, significant progress has been made towards the development of nonlinear control methods for systems of nonlinear ODEs. Differential geometry has proven to be a natural framework for the analysis and control of such systems. Within this framework, basic control problems, including the modification of the input/output behavior, the elimination of measurable disturbances, and the attenuation of unmeasured disturbances and unknown parameters have been successfully addressed. Although geometric control methods may lead to satisfactory control quality in nonlinear processes without time-scale multiplicity, they usually lead to poor performance in processes where multiple-time-scale behavior is present. Furthermore, few results are available for the synthesis of nonlinear controllers for nonlinear PDE systems. The need to develop control methods for nonlinear two-time-scale ODE systems and nonlinear PDE systems has been well-recognized both by industry and academia.

Motivated by the above, this doctoral thesis presents a general framework for the synthesis of nonlinear controllers for nonlinear two-time-scale ODE systems and nonlinear PDE systems that systematically addresses the problems of modification of the input/output behavior, elimination of measurable disturbances, and attenuation of unmeasured disturbances and unknown parameters. The proposed control algorithms are applied to industrially important chemical processes.

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Chapter 1

Introduction

All chemical engineering processes are inherently nonlinear. Nonlinearities arise from complex reaction mechanisms, Arrhenius dependence of reaction rates on temperature, empirical correlations, etc. In addition to nonlinear nature, many chemical engineering processes are characterized by multiple-time-scale behavior and spatial variations. Multiple-time-scale behavior typically arises from the presence of significantly different residence times and large mass/thermal capacitances, while spatial variations are due to the underlying convective, diffusive and dispersive phenomena. The mathematical models of chemical processes are typically obtained from the dynamic conservation equations and consist of systems of nonlinear Ordinary Differential Equations (ODEs) and nonlinear Partial Differential Equations (PDEs).

Motivated by the inherently nonlinear nature of physical and chemical processes, in the last decade, a flourishing research activity has emerged towards the development of feedback control methods for nonlinear ODE systems. Differential geometry has proven to be a natural framework for the analysis and control of such systems. Within this framework, key aspects of nonlinear dynamics and important control-theoretic properties have been well-understood. Furthermore, basic control problems, including the modification of the input/output behavior, the elimination of measurable disturbances, and the attenuation of unmeasured disturbances and unknown parameters have been successfully addressed. Implementations of the developed control algorithms on industrial processes have been also reported. Excellent surveys of both theoretical and application papers in this area can be found in [Isi89, NvdS90, KA91]. Although geometric control methods may lead to satisfactory control quality in nonlinear processes without time-scale multiplicity, they usually lead to poor performance in processes where multiple-time-scale behavior is present. Specifically, it is well-established that a direct application of these methods to two-time-scale systems may lead to controller ill-conditioning or even to closed-loop instability [KKO86].

Two-time-scale ODE systems have been primarily studied within the mathematical framework of singular perturbations. Within this framework, stability issues and geometric properties of nonlinear two-time-scale systems have been analyzed and optimal control algorithms have been proposed [SK84, Fen79, KKO86]. However, a controller synthesis framework for nonlinear two-time-scale ODE systems that addresses the basic problems of modification of the input/output behavior, elimination of measurable disturbances, and attenuation of unmeasured disturbances and unknown parameters is still lacking.

On the other hand, the conventional approach to the control of PDEs involves the discretization of the original PDE model in space followed by the application of control methods for ODEs. Although this approach may lead to satisfactory control quality in processes with mild spatial variations, it usually leads to poor performance in processes where the spatially varying nature is very strong. This realization motivated research on the development of distributed control algorithms for linear PDEs that directly account for their spatially varying nature. The literature on distributed control of linear PDEs is really extensive. Excellent surveys of both theoretical and application papers on this topic can be found in [Ray78, Bal82, Keu93]. However, the range of applicability and the efficiency of linear control methods are significantly restricted by the presence of severe nonlinearities in chemical processes.

PDEs are typically classified in two broad categories: a) hyperbolic PDEs which adequately describe convection-reaction processes, and b) parabolic PDEs which naturally model diffusion-convection-reaction processes. The distinct feature of first-order hyperbolic PDEs is that all the eigenmodes of the spatial differential operator contain the same or nearly the same amount of energy, and thus an infinite number of modes is required to accurately describe their dynamic behavior. This property prohibits the application of modal decomposition techniques to derive ODE models that approximately describe the dynamics of the PDE system and suggests addressing the control problem on the basis of the infinite dimensional model. Following this approach, distributed control algorithms have been developed employing optimal control and sliding-mode control. However, there does not exist a framework for the synthesis of nonlinear distributed controllers for quasi-linear hyperbolic PDE systems that addresses the problems of modification of the input/output behavior, elimination of measurable disturbances, and attenuation of unmeasured disturbances and unknown parameters.

In contrast to hyperbolic PDEs, parabolic PDEs are characterized by a finite-number of dominant modes, which implies that their dynamic behavior can be approximately described by ODE systems that can be used for controller design. Motivated by this, the standard approach to the control of parabolic PDEs involves the application of Galerkin's method to the PDE system to derive ODE systems that approximate the PDE system, which are subsequently used as the basis for controller synthesis. The main disadvantage of this approach is that the number of modes that should be retained to derive an ODE model that yields the desired degree of approximation may be very large, leading to high dimensionality of the controller, and thus, to implementation problems.

This doctoral thesis presents a general framework for the synthesis of nonlinear controllers for nonlinear two-time-scale ODE systems, and nonlinear hyperbolic and parabolic PDE systems that systematically addresses the problems of modification of the input/output behavior, elimination of measurable disturbances, and attenuation of unmeasured disturbances and unknown parameters. The proposed control algorithms are applied to industrially important chemical processes.

The first volume of the thesis is organized as follows. Chapter 2 addresses the problem of synthesizing well-conditioned feedback controllers for two-time-scale nonlinear ODE systems that modify the input/output behavior in a desired way. Singular perturbation methods are used to derive separate reduced-order models which describe the fast and slow dynamics of the original system. These models are subsequently used for controller synthesis using differential geometric methods. The proposed control method is applied to a biochemical reactor and a cascade of two autocatalytic reactors. Chapter 3 addresses the problem of elimination of the effect of measurable disturbances on the output in two-time-scale nonlinear ODE systems. The problem is addressed through appropriate incorporation of measurements of disturbances (feedforward action) in the feedback controllers derived in chapter 2. The resulting feedforward/feedback controller is applied to a catalytic reactor. In chapter 4, the problem of attenuation of the effect of unmeasured disturbances and unknown parameters on the output in two-time-scale nonlinear ODE systems is addressed. Nonlinear controllers are synthesized through combination of differential geometric methods and Lyapunov techniques that utilize bounds on the size of the uncertain variables. The method is applied to a chemical reactor. In chapter 5, a control method for multi-input multi-output two-time-scale ODE systems with uncertain variables is developed and applied to a fluidized catalytic cracking reactor. The proofs of the results presented in chapters 2, 3, 4 and 5 are given in appendices A, B, C and D, respectively.

The second volume of the thesis is organized as follows. Chapter 6 addresses the problem of synthesizing nonlinear distributed output feedback controllers for quasi-linear first-order hyperbolic PDE systems that induce a desired input/output behavior. The controllers are synthesized on the basis of the PDE model following a geometric approach. The proposed control method is applied to a plug-flow reactor. Chapter 7 deals with the problem of attenuation of the effect of unmeasured disturbances and unknown parameters on the output in hyperbolic PDE systems. Lyapunov techniques for infinite-dimensional systems are employed for the synthesis of distributed controllers that utilize bounds on the size of the uncertain variables to attenuate their effect on the output. The developed control method is applied to a fixed-bed reactor. Chapter 8 presents a methodology for the synthesis of nonlinear low-dimensional output feedback controllers for quasi-linear parabolic PDE systems that induce a desired input/output behavior. The controllers are synthesized on the basis of low-dimensional ODE models that accurately reproduce the solutions of the PDE system, derived through combination of nonlinear Galerkin's method and approximate inertial manifolds. The proposed control method is applied to a packed-bed reactor. Chapter 9 summarizes the main contributions of this work and proposes directions for future research. Finally, the proofs of the results presented in chapters 6, 7 and 8 are given in appendices E, F and G, respectively.

Chapter 2

Feedback Control of Nonlinear Two-Time-Scale Systems

2.1 Introduction

Most physical and chemical systems are inherently nonlinear and are characterized by the copresence of dynamical phenomena occurring in multiple-time-scales. Representative examples of nonlinear multiple-time-scale systems include catalytic CSTRs [CA84, Den86], reaction networks [BB91], fluidized catalytic cracking reactors [Geo77, MG87]), biochemical reactors [HTA67, BO87, BD90], high-purity distillation columns [LR91], electrical circuits [Kha92], electromechanical networks [Cho82], DC motor models [KKO86], flexible mechanical systems [DC93] etc..

Singular perturbation theory provides a natural framework for the analysis and control of multiple-time-scale systems [KKO86, Kha92]. Singular perturbation methods allow decomposing a multiple-time-scale system into separate reduced-order systems that evolve in different time-scales, and inferring its asymptotic properties from the knowledge of the behavior of the reduced-order systems [KES76]. Within this framework, stability issues [SK84] and geometric properties [Fen79, MK88] of nonlinear two-time-scale systems have been studied and feedback control algorithms have been developed using optimal control [KKO86], the integral manifold approach [SO87, BPMNC93], and sliding mode techniques [Hec91].

On the other hand, differential geometry has proven to be a natural framework for the analysis and control of nonlinear systems. Within this framework, several nonlinear control problems have been successfully addressed (see e.g. [Isi89, NvdS90, KA91]). However, most of the developed nonlinear control methods do not take explicitly into account possible time-scale multiplicities, and thus, may lead to ill-conditioned controllers, significant performance deterioration or even destabilization of the closed-loop system in the presence of such time-scale multiplicity ([KKO86]). The combination of singular perturbation and differential geometric methods for the solution of the problem of exact state-space linearization of a class of nonlinear two-time-scale systems was proposed by [Kho90].

This chapter addresses the state feedback control problem for two-time-scale nonlinear systems modeled within the mathematical framework of singular perturbations. The objective is to synthesize well-conditioned static state feedback laws that induce a well-characterized input/output behavior in the closed-loop system. The control laws are synthesized employing combination of singular perturbation and geometric methods. Conditions that guarantee the stability of the closed-loop system under the developed control laws are also derived. The developed control methodology is applied to two nonlinear chemical processes with time-scale multiplicity.

2.2 Preliminaries

We will consider two-time-scale nonlinear systems, modeled in singularly perturbed form, with the following state-space description:

$$\begin{array}{rcl} \dot{x} & = & f_1(x) + Q_1(x)z + g_1(x)u \\ \epsilon\dot{z} & = & f_2(x) + Q_2(x)z + g_2(x)u \\ y & = & h(x) \end{array} \tag{2.1}$$

where $x \in X \subset \mathbb{R}^n$ and $z \in Z \subset \mathbb{R}^p$ denote vectors of state variables, with X and Z open and connected sets, $u \in \mathbb{R}$ denotes the input, $y \in \mathbb{R}$ denotes the controlled output, ϵ is a small positive parameter which can be interpreted as the speed ratio of the slow versus the fast dynamical phenomena of the system. Furthermore, $f_1(x)$, $g_1(x)$, $f_2(x)$, $g_2(x)$ are analytic vector fields, $Q_1(x)$ and $Q_2(x)$ are analytic matrices of dimensions $n \times p$ and $p \times p$ respectively, and $h(x)$ is an analytic scalar function.

Modeling a two-time-scale process in a singularly perturbed form involves defining the singular perturbation parameter ϵ, taking into account the physicochemical characteristics of the process, so that the separation of the fast and slow variables becomes explicit, with ϵ multiplying the time derivatives of the fast variables z. ϵ usually represents small process parameters or the reciprocal of large process parameters (e.g. small/large masses, capacitances). The reader may also refer to [KKO86] for further discussion on this issue. Referring to the specific singularly perturbed system of Eq.2.1, we note that the parameter ϵ appears only in the left-hand side (multiplying the time-derivative $\dot{z}$), while the fast variable z enters in a linear fashion. The first assumption is made for notational simplicity and can be readily relaxed (see remark 2.7), while the second assumption is consistent with the fact that in many chemical processes the main nonlinearities are associated with the slow variables and also allows explicitness and analytical insight in the theoretical development.

The explicit separation of the slow and fast dynamics in the system of Eq.2.1, owing to the presence of the small parameter ϵ that multiplies the derivative of the state vector z allows decomposing it into separate reduced-order systems evolving in separate time-scales ([KES76]). In particular, assuming that the system of Eq.2.1 is in standard form i.e.. the matrix $Q_2(x)$ is nonsingular uniformly in $x \in X$. and setting $\epsilon = 0$, the system of Eq.2.1 takes the form:

$$\dot{x} = f_1(x) + Q_1(x)z_s + g_1(x)u \tag{2.2}$$

$$f_2(x) + Q_2(x)z_s + g_2(x)u = 0 \tag{2.3}$$

where z_s denotes a quasi-steady-state for z. The invertibility of the matrix $Q_2(x)$ guarantees that the system of algebraic equations (Eq.2.3) admits a unique solution for z_s, of the form:

$$z_s = -[Q_2(x)]^{-1}[f_2(x) + g_2(x)u] \tag{2.4}$$

Substituting Eq.2.4 into Eq.2.2 the following *reduced system* or *slow subsystem* is obtained:

$$\begin{array}{rcl} \dot{x} & = & F(x) + G(x)u \\ y^s & = & h(x) \end{array} \tag{2.5}$$

where y^s denotes output associated with the slow subsystem and

$$\begin{array}{rcl} F(x) & = & f_1(x) - Q_1(x)[Q_2(x)]^{-1}f_2(x) \\ G(x) & = & g_1(x) - Q_1(x)[Q_2(x)]^{-1}g_2(x) \end{array} \tag{2.6}$$

Note that the input u appears in an affine fashion in the system of Eq.3.5 because of the linearity in z in the original system. To obtain a representation of the system which describes the fast dynamics of the system of Eq.2.1. we define a fast time-scale:

$$\tau = \frac{t}{\epsilon} \tag{2.7}$$

In this new time-scale the original system takes the form:

$$\frac{dx}{d\tau} = \epsilon[f_1(x) + Q_1(x)z + g_1(x)u] \tag{2.8}$$

$$\frac{dz}{d\tau} = f_2(x) + Q_2(x)z + g_2(x)u \tag{2.9}$$

Setting ϵ equal to zero, the following *fast subsystem* is obtained:

$$\frac{dz}{d\tau} = f_2(x) + Q_2(x)z + g_2(x)u \tag{2.10}$$

where x can be considered approximately equal to its initial value $x(0)$. The following assumption states a stabilizability requirement on the system of Eq.2.10.

Assumption 2.1: *The pair $[Q_2(x)\ g_2(x)]$ is stabilizable, in the sense that there exists an analytic covector field $k^T(x)$ such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$.*

We will now review the standard definition of relative order for nonlinear systems described by Eq.2.5, the definition of order of magnitude, and a standard stability and closeness of solutions result (Tikhonov's theorem) for singularly perturbed systems, which will be used in our development.

Definition 2.1 : *Referring to the nonlinear system of Eq.2.5, the relative order of y^s with respect to u is defined as the smallest integer r for which*

$$L_G L_F^{r-1} h(x) \not\equiv 0 \tag{2.11}$$

or $r = \infty$ if such an integer does not exist.

Throughout the chapter, it will be assumed that Eq.2.11 holds for all $x \in X$.

Definition 2.2 ([Kha92]): $\delta(\epsilon) = O(\epsilon)$ if there exist positive constants k and c such that:

$$|\delta(\epsilon)| \leq k|\epsilon| \ , \ \forall\ |\epsilon| < c \tag{2.12}$$

Theorem 2.1 ([Tik48]): *If the slow and fast subsystems of Eqs.2.5-2.10, respectively, are locally exponentially stable, then there exists a positive real number ϵ^* such that if $\epsilon \in (0, \epsilon^*]$, then the system of Eq.2.1 is locally exponentially stable and its* *solutions $x(t), z(t)$ for all $t \in [0, \infty)$ satisfy:*

$$\begin{aligned} x(t) &= x_s(t) + O(\epsilon) \\ z(t) &= z_s(t) + \eta_f(\frac{t}{\epsilon}) + O(\epsilon) \end{aligned} \tag{2.13}$$

where $x_s(t)$ is the solution of the system of Eq.2.5 and $\eta_f(\frac{t}{\epsilon})$ is the solution of:

$$\frac{d\eta_f}{d\tau} = Q_2(x)\eta_f \tag{2.14}$$

2.3 Methodological aspects

A direct application of state feedback synthesis methods based on nonlinear inversion to systems of the form of Eq.2.1 may result in the derivation of ill-conditioned control laws, i.e., control laws that become singular as $\epsilon \rightarrow 0$, and/or closed-loop instability (for a detailed discussion, see remarks 2.4 and 2.5). Furthermore, such an approach may result in feedback laws that depend on the fast state vector z and lead to destabilization of the fast dynamics of the system ([KKO86]). Motivated by the above considerations, in this paper, we will address the problem of synthesizing *well-conditioned* static state feedback laws that guarantee exponential stability of the closed-loop system and ensure that the output of the closed-loop system satisfies a relation of the form:

$$y(t) = y^s(t) + O(\epsilon)\ , \ t \geq 0 \tag{2.15}$$

with $y^s(t)$ being the output of the closed-loop reduced system, where a prespecified input/output response is enforced. These requirements will be obtained for sufficiently small ϵ.

The control problem will be initially addressed for systems in standard form, that is, systems for which the matrix $Q_2(x)$ is invertible uniformly in $x \in X$, and then for systems in nonstandard form, that is, systems for which the matrix $Q_2(x)$ is singular for some $x \in X$. We will establish that, for systems in standard form, the synthesis of the requisite state feedback law can be performed on the basis of the open-loop system of Eq.2.1, while for systems in nonstandard form, it has to be preceded by the feedback regularization of the fast dynamics.

2.4 Control of two-time-scale nonlinear systems in standard form

In this section, we will address a state feedback synthesis problem for systems of the form of Eq.2.1 in standard form, with possibly unstable fast dynamics. Motivated by the possible instability of the fast dynamics and the affine appearance of the fast state z in the model of Eq.2.1, we will initially consider well-conditioned static state feedback laws of the form:

$$u = \tilde{u} + k^T(x)z \tag{2.16}$$

where $\tilde{u}$ is an auxiliary input and $k^T(x)$ is an analytic vector field in $\mathbb{R}^p$, to stabilize the fast dynamics of the closed-loop system. Substitution of the control law of Eq.2.16 into the system of Eq.2.1 yields:

$$\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)k^T(x)]z + g_1(x)\tilde{u} \\ \epsilon\dot{z} &= f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\tilde{u} \end{aligned} \tag{2.17}$$

One can immediately observe that the control law of Eq.2.16 preserves the two-time-scale nature of the system, and the linearity with respect to the state z and the auxiliary input $\tilde{u}$. Furthermore, performing a standard two-time-scale decomposition, the fast subsystem is given by:

$$\frac{dz}{d\tau} = f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\tilde{u} \tag{2.18}$$

while the slow subsystem takes the form:

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\tilde{u} \\ y^s &= h(x) \end{aligned} \tag{2.19}$$

where

$$\begin{aligned}\tilde{F}(x) &= f_1(x) - [Q_1(x) + g_1(x)k^T(x)][Q_2(x) + g_2(x)k^T(x)]^{-1}f_2(x) \\ \tilde{G}(x) &= g_1(x) - [Q_1(x) + g_1(x)k^T(x)][Q_2(x) + g_2(x)k^T(x)]^{-1}g_2(x)\end{aligned} \tag{2.20}$$

It is clear that the z-dependent state feedback law of Eq.2.16 allows modifying the stability characteristics of the fast dynamics of the original system by choosing the gain $k^T(x)$ such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$. Moreover, one can immediately observe that the reduced system of Eq.2.19 depends explicitly on the nonlinear feedback gain $k^T(x)$. Therefore, before we proceed with the solution of the control problem in the slow time-scale, we will establish that the state feedback law of Eq.2.16 preserves the relative order r of the output y^s. The main result is given in proposition 2.1 that follows. The proof of the proposition can be found in the appendix A.

Proposition 2.1: *Consider the two-time-scale system of Eq.2.1, assumed to be in standard form, for which assumption 2.1 holds. Then, under a static state feedback law of the form of Eq.2.16, such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$, the relative order of y^s with respect to $\tilde{u}$ in the reduced system of Eq.2.19 is equal to r.*

In order to enforce control objectives in the closed-loop slow subsystem, we will now seek well-conditioned static state feedback laws of the form:

$$\tilde{u} = p(x) + q(x)v \tag{2.21}$$

where $p(x)$, $q(x)$ are scalar fields with $q(x) \neq 0$ for all $x \in X$, and v is a reference input, that enforce output tracking and guarantee stability of the closed-loop reduced system. Under a control law of the form of Eq.2.21 the closed-loop system is given by:

$$\begin{aligned}\dot{x} &= [f_1(x) + g_1(x)p(x)] + [Q_1(x) + g_1(x)k^T(x)]z + g_1(x)q(x)v \\ \epsilon\dot{z} &= [f_2(x) + g_2(x)p(x)] + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)q(x)v \\ y &= h(x)\end{aligned} \tag{2.22}$$

Employing a two-time-scale decomposition for the system of Eq.2.22, the closed-loop fast subsystem is given by:

$$\frac{dz}{d\tau} = [f_2(x) + g_2(x)p(x)] + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)q(x)v \qquad (2.23)$$

and the closed-loop slow subsystem takes the form:

$$\begin{aligned} \dot{x} &= [\tilde{F}(x) + \tilde{G}(x)p(x)] + \tilde{G}(x)q(x)v \\ y^s &= h(x) \end{aligned} \qquad (2.24)$$

Proposition 2.2 that follows allows specifying the order of the requested input/output response in the closed-loop reduced system.

Proposition 2.2: *Consider the two-time-scale system of Eq.2.17, where the matrix $Q_2(x)+g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$. Then, under a static state feedback control law of the form of Eq.2.21, the relative order of y^s with respect to v in the closed-loop reduced system of Eq.2.24 is equal to r.*

Proof: The proof of the proposition involves the application of the two-time-scale decomposition procedure to the resulting closed-loop system, and the standard argument for nonlinear systems of the form of Eq.2.5 under static state feedback of the form of Eq.2.21 (see e.g. [Isi89]). △

The result of proposition 2.2 allows requesting an input/output behavior of relative order r in the closed-loop reduced system. For simplicity, a linear minimal-order input/output behavior will be postulated, of the form:

$$\beta_r \frac{d^r y^s}{dt^r} + \cdots + \beta_1 \frac{dy^s}{dt} + \beta_0 y^s = v \qquad (2.25)$$

where $\beta_0, \cdots, \beta_r$ are adjustable parameters, which can be chosen to guarantee input/output stability and to enforce desired performance specifications in the closed-loop reduced system.

We are now in a position to give an explicit synthesis formula for the state feedback controller of Eqs.2.16-2.21 that enforces the requested control objectives in the closed-loop system. Theorem 2.2 that follows provides the main result of this section (the proof of the theorem can be found in the appendix A).

Theorem 2.2: *Consider the two-time-scale system of Eq.2.1, assumed to be in standard form, for which assumption 2.1 holds. Then the static state feedback law:*

$$\begin{aligned} u &= [1 + k^T(x)[Q_2(x)]^{-1}g_2(x)] \left[\beta_r L_G L_F^{r-1} h(x)\right]^{-1} \left\{v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right\} \\ &\quad + k^T(x)[Q_2(x)]^{-1} f_2(x) + k^T(x) z \end{aligned} \tag{2.26}$$

where the feedback gain $k^T(x)$ is such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$,

a) guarantees local exponential stability of the fast dynamics of the closed-loop system.

b) ensures that the output of the closed-loop system satisfies a relation of the form :

$$y(t) = y^s(t) + O(\epsilon),\ t \geq 0 \tag{2.27}$$

for ϵ sufficiently small, with $y^s(t)$ being the solution of Eq.2.25.

Remark 2.1: The state feedback law of Eq.2.26 can be equivalently written as

$$u = \left[\beta_r L_G L_F^{r-1} h(x)\right]^{-1} \left\{v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right\} + k^T(x)(z - \xi) \tag{2.28}$$

where

$$\xi = -[Q_2(x)]^{-1} \left\{ f_2(x) + g_2(x)[\beta_r L_G L_F^{r-1} h(x)]^{-1} \left\{v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right\}\right\} \tag{2.29}$$

denotes a quasi-steady-state for the fast dynamics of the closed-loop system. It is clear that the above control law consists of two separate components: the component, $k^T(x)(z - \xi)$, which acts in the fast time-scale and is responsible for the stabilization of the fast dynamics of the system, and the component, $\left[\beta_r L_G L_F^{r-1} h(x)\right]^{-1} \left\{v - \sum_{k=0}^{r} \beta_k L_F^k h(x)\right\}$, which acts in the slow time-scale and induces the desired input/output behavior in the closed-loop reduced system. Control laws that address control objectives in different time-scales are called composite control laws ([Isi89]). The above analysis of the structure of the controller of theorem 2.2

suggests that the two components of the controller can be synthesized independently, on the basis of the open-loop system of Eq.2.1.

Remark 2.2: The fact that the component $k^T(x)(z-\xi)$ acts in the fast time-scale where the state vector x can be considered approximately equal to its initial value $x(0)$ allows designing the feedback gain $k^T(x)$ using standard control methods, such as pole placement, optimal control, etc. (for details see e.g. [KKO86]).

Remark 2.3: In the case of systems of the form of Eq.2.1 with stable fast dynamics, i.e. assuming that the eigenvalues of the matrix $Q_2(x)$ lie in the open left-half of the complex plane uniformly in $x \in X$, the controller of theorem 2.2 does not need to utilize feedback of the fast state vector z to stabilize the fast dynamics and thus, simplifies to:

$$u = \left[\beta_r L_G L_F^{r-1} h(x)\right]^{-1} \left\{ v - \sum_{k=0}^{r} \beta_k L_F^k h(x) \right\} \tag{2.30}$$

Remark 2.4: Referring to systems of the form of Eq.2.1 with stable fast dynamics, let ν, π denote the smallest integers for which $\dfrac{\partial y^{(\nu)}}{\partial z} \not\equiv 0$, $\dfrac{\partial y^{(\pi)}}{\partial z} \not\equiv 0$. A direct differentiation of the output of the system of Eq.2.1 with respect to time yields then the following expressions:

$$\begin{array}{rcl} y & = & h(x) \\ y^{(1)} & = & L_{f_1} h(x) \\ \vdots & & \\ y^{(\pi-1)} & = & L_{f_1}^{\pi-1} h(x) \\ y^{(\pi)} & = & L_{f_1}^{\pi} h(x) + \psi_0(x,z) \\ y^{(\pi+1)} & = & L_{f_1}^{\pi+1} h(x) + \psi_1(x,z) \\ \vdots & & \\ y^{(\nu-1)} & = & L_{f_1}^{\nu-1} h(x) + \psi_{\nu-\pi-1}(x,z) \\ y^{(\nu)} & = & L_{f_1}^{\nu} h(x) + \psi_{\nu-\pi}(x,z) + c(x,z)u \end{array} \tag{2.31}$$

where $\psi_i(x,z)$, $i = 0, \cdots, \nu-\pi$, are scalar functions whose specific form is omitted for brevity. Whenever $\nu < \pi$, the functions $\psi_i(x,z)$ are identically equal to zero and $c(x,z) = L_{g_1} L_{f_1}^{\nu-1} h(x)$. In this case, the expressions for the derivatives of y of

Eq.2.31 are independent of the fast state vector z. Furthermore, the vector fields $F(x)$ and $G(x)$ defined in Eq.2.6 satisfy the relations: $L_G L_F^{r-1} h(x) = L_{g_1} L_{f_1}^{\nu-1} h(x)$ and $L_F^k h(x) = L_{f_1}^k h(x)$, $k = 1, \cdots, r$. Clearly, the condition $\nu = r$ holds in this case, and the control law of Eq.2.30 induces an input/output behavior of the form:

$$\beta_\nu \frac{d^\nu y}{dt^\nu} + \cdots + \beta_1 \frac{dy}{dt} + \beta_0 y = v \tag{2.32}$$

in the closed-loop $full - order$ system. On the other hand, on the basis of Eq.2.31, it is clear that whenever $\nu = \pi$, a state feedback law that induces the input/output behavior of Eq.2.32 in the closed-loop full-order system requires feedback of the fast state vector z, and thus, may destabilize the fast dynamics of the system, while if $\nu > \pi$ such a control law will also be ill-conditioned.

Remark 2.5: In this remark, we will address the implications of the instability of the fast dynamics on the stability of the zero dynamics of systems of the form of Eq.1, for which $\nu \leq \pi$. As noted in remark 2.4, whenever $\nu > \pi$, an inversion-based state feedback law is ill-conditioned, and thus, such an analysis is not particularly meaningful for this case. Referring to the system of Eq.2.1 with $\nu \leq \pi$, there exists (see e.g. [Isi89]) a coordinate transformation:

$$(\zeta, z) = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-\nu} \\ \zeta_1 \\ \vdots \\ \zeta_\nu \\ z \end{bmatrix} = T(x, z) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\nu}(x) \\ h(x) \\ L_{f_1} h(x) \\ \vdots \\ L_{f_1}^{\nu-1} h(x) \\ z \end{bmatrix} \tag{2.33}$$

where $\phi_i(x)$, $i = 1, \cdots, n-\nu$ are scalar fields, such that the system of Eq.2.1 takes the form:

$$
\begin{aligned}
\dot{\eta} &= L_{f_1}\phi(\eta,\zeta) + \sum_{\kappa=1}^{p} L_{Q_{1\kappa}}\phi(\eta,\zeta)z_k \\
\dot{\zeta}_1 &= \zeta_2 \\
&\vdots \\
\dot{\zeta}_{\nu-1} &= \zeta_\nu \\
\dot{\zeta}_\nu &= L_{f_1}^{\nu}h(\eta,\zeta) + L_{Q_1}L_{f_1}^{\nu-1}h(\eta,\zeta)z + L_{g_1}L_{f_1}^{\nu-1}h(\eta,\zeta)u \\
\epsilon\dot{z} &= f_2(\eta,\zeta) + Q_2(\eta,\zeta)z + g_2(\eta,\zeta)u \\
y &= \zeta_1
\end{aligned}
\tag{2.34}
$$

where $\phi(x) = [\phi_1(x) \ \cdots \ \phi_{n-\nu}(x)]^T$, the (ζ,η)-dependence in the right-hand-side of Eq.2.34 implies the evaluation of the functions at $(x,z) = T^{-1}(\eta,\zeta,z)$, and $L_{f_1}\phi = [L_{f_1}\phi_1 \ \cdots \ L_{f_1}\phi_{n-\nu}]^T$, $L_{Q_{1\kappa}}\phi = [L_{Q_{1\kappa}}\phi_1 \ \cdots \ L_{Q_{1\kappa}}\phi_{n-\nu}]^T$, $L_{Q_1}L_{f_1}^{\nu-1}h = [L_{Q_{11}}L_{f_1}^{\nu-1}h \ \cdots \ L_{Q_{1p}}L_{f_1}^{\nu-1}h]$, with Q_{1k} being the $k-th$ column vector of the matrix Q_1. Whenever $\nu < \pi$, the dynamical system:

$$
\begin{aligned}
\dot{\eta} &= L_{f_1}\phi(\eta,0) + \sum_{\kappa=1}^{p} L_{Q_{1\kappa}}\phi(\eta,0)z_k \\
\epsilon\dot{z} &= f_2(\eta,0) + Q_2(\eta,0)z - g_2(\eta,0)\frac{L_{f_1}^{\nu}h(\eta,0)}{L_{g_1}L_{f_1}^{\nu-1}h(\eta,0)}
\end{aligned}
\tag{2.35}
$$

is the zero dynamics of the system of Eq.2.1 (see e.g., [Isi89]). One can immediately observe that the modes of the zero dynamics contain the modes of the fast dynamics, and thus, instability of the fast dynamics implies instability of the zero dynamics of the original system. Systems that possess a zero dynamics with a two-time-scale nature and unstable fast dynamics are called slightly nonminimum-phase (see e.g. [SHK89]). For such systems, one can immediately see that inversion-based state feedback laws lead to internal instability of the closed-loop system. On the other hand, under the state feedback law of Eq.2.26, the fast dynamics of the closed-loop system possess a locally exponentially stable equilibrium manifold, while the internal stability of the closed-loop system depends on the stability of the zero dynamics of the *open − loop reduced system* (see section 2.5). Furthermore, whenever $\nu = \pi$, the zero dynamics of the system of Eq.2.1 in the coordinates of Eq.2.33 takes the form:

$$\begin{aligned}\dot{\eta} &= L_{f_1}\phi(\eta,0) + \sum_{\kappa=1}^{p} L_{Q_{1\kappa}}\phi(\eta,0)z_k \\ \epsilon\dot{z} &= f_2(\eta,0) + [Q_2(\eta,0) - \frac{L_{Q_1}L_{f_1}^{\nu-1}h(\eta,0)}{L_{g_1}L_{f_1}^{\nu-1}h(\eta,0)}]z - g_2(\eta,0)\frac{L_{f_1}^{\nu}h(\eta,0)}{L_{g_1}L_{f_1}^{\nu-1}h(\eta,0)}\end{aligned} \tag{2.36}$$

and thus, the system of Eq.2.1 may or may not be slightly non-minimum phase, depending on its specific structure.

2.5 Control of two-time-scale nonlinear systems in nonstandard form

In this section, we will address a state feedback controller synthesis problem for two-time-scale nonlinear systems of Eq.2.1 in nonstandard form, i.e., systems for which the matrix $Q_2(x)$ is singular. The immediate implication of the non-invertibility of the matrix $Q_2(x)$ is the lack of a well-defined quasi-steady-state for the fast state vector z ([Kha89]). Therefore, an application of the two-time-scale decomposition procedure to the system of Eq.2.1 will result in the derivation of a singular reduced system. Referring to this reduced system, it can be easily seen that the concept of relative order is not well-defined, and the results of propositions 2.1 and 2.2 that allowed the formulation and solution of the controller synthesis problems for two-time-scale systems in standard form do not hold.

The previous considerations imply that any attempt for the synthesis of state feedback laws for this class of systems, requires the regularization of the fast dynamics (in the sense of inducing a well-defined quasi-steady-state for the fast state vector z) through appropriate feedback of the fast state vector z. Motivated by this, in what follows, we will initially employ appropriate feedback of the state vector z to induce an exponentially stable quasi-steady-state for the fast dynamics, and we will subsequently formulate and solve the synthesis problem on the basis of the resulting two-time-scale system.

In particular, we will initially consider a well-conditioned static state feedback law of the form:

$$u = \hat{u} + k^T(x)z \tag{2.37}$$

where $k^T(x)$ is an analytic vector field in $\mathbb{R}^p$ and $\hat{u}$ is an auxiliary input, to regularize the fast dynamics. Under the control law of Eq.2.37 the system of Eq.2.1 takes the form:

$$\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)k^T(x)]z + g_1(x)\hat{u} \\ \epsilon\dot{z} &= f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\hat{u} \end{aligned} \tag{2.38}$$

Performing a two-time-scale decomposition, the corresponding fast subsystem takes the form:

$$\frac{dz}{d\tau} = f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\hat{u} \tag{2.39}$$

while the corresponding slow subsystem is given by:

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\hat{u} \\ \hat{y}^s &= h(x) \end{aligned} \tag{2.40}$$

where the vector fields $\tilde{F}(x)$, $\tilde{G}(x)$ are given in Eq.2.20. It is clear that the z-dependent state feedback law of Eq.2.37 allows us to regularize the fast dynamics of the system by choosing $k^T(x)$ in such a manner so that the matrix $Q_2(x)+g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$. A necessary and sufficient condition for the regularization of the fast dynamics is $rank[Q_2(x)\ g_2(x)] = m$, uniformly in $x \in X$ ([Kha89]). Note that this condition is not an additional requirement, since it is directly implied by the stabilizability property of the pair $[Q_2(x)\ g_2(x)]$.

It is now possible to formulate and solve a controller synthesis problem on the basis of the system of Eq.2.38. In particular, we will seek a well-conditioned static state feedback law of the form:

$$\hat{u} = \tilde{p}(x) + \tilde{q}(x)v \tag{2.41}$$

where $\tilde{p}(x)$, $\tilde{q}(x)$ are scalar fields, with $\tilde{q}(x) \neq 0$ for all $x \in X$, and v is the external reference input. Referring to the system of Eq.2.40, let $\hat{r}$ denote the relative order of the output y^s with respect to the auxiliary input $\hat{u}$. It is then straightforward to show that the result of proposition 2.2 holds for the two-time-scale system of Eq.2.38 under the state feedback law of Eq.2.41. This allows postulating the following input/output behavior:

$$\beta_{\tilde{r}} \frac{d^{\tilde{r}} \hat{y}^s}{dt^{\tilde{r}}} + \cdots + \beta_1 \frac{d\hat{y}^s}{dt} + \beta_0 \hat{y}^s = v \tag{2.42}$$

in the closed-loop reduced system. Theorem 2.3 that follows summarizes the main result of this section. The proof of the theorem can be found in appendix A.

Theorem 2.3: *Consider the two-time-scale system of Eq.2.1, assumed to be in non-standard form, for which assumption 2.1 holds. Then the static state feedback law:*

$$u = \left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1} \left\{ v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) \right\} + k^T(x) z \tag{2.43}$$

where the feedback gain $k^T(x)$ is such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$,

a) guarantees local exponential stability of the fast dynamics of the closed-loop system, and

b) ensures that the output of the closed-loop system satisfies a relation of the form :

$$y(t) = \hat{y}^s(t) + O(\epsilon) \, , \, t \geq 0 \tag{2.44}$$

for ϵ sufficiently small, with $\hat{y}^s(t)$ being the solution of Eq.2.42.

Remark 2.6: The result of theorem 2.3 reveals a fundamental difference in the nature of the control problem between systems in standard and nonstandard form. In particular, in the case of systems in nonstandard form, the formulation (relative order of the requested response) and solution of the control problem in the slow time-scale are based on the reduced system of Eq.2.40, and thus explicitly depend on the gain $k^T(x)$ used to regularize the fast dynamics, as opposed to systems in standard form where the control problem in the slow time-scale is independent of the choice of $k^T(x)$.

Remark 2.7: The control algorithms of theorems 2.2 and 2.3 can be directly applied to two-time-scale systems with ϵ-dependent right-hand side, with the following state-space description:

$$\begin{aligned}
\dot{x} &= f_1(x, \epsilon z, \epsilon) + Q_1(x, \epsilon z, \epsilon)z + g_1(x, \epsilon z, \epsilon)u \\
\epsilon\dot{z} &= R(x, z)[f_2(x, \epsilon z, \epsilon) + Q_2(x, \epsilon z, \epsilon)z + g_2(x, \epsilon z, \epsilon)u] \\
y &= h(x)
\end{aligned} \tag{2.45}$$

where $R(x, z)$ is a diagonal matrix of dimension $p \times p$, which is positive definite for all $x \in X$, $z \in Z$. This is possible because i) the stabilizability requirement of assumption 2.1 suffices to ensure that the fast subsystem of the system of Eq.2.45 can be made exponentially stable, and ii) in the case of non-singular $Q_2(x, \epsilon z, \epsilon)$, the open-loop slow subsystem of the system of Eq.2.45 is the same as the one of Eq.2.5 (which ensures the applicability of theorem 2.2), while in the case of singular $Q_2(x, \epsilon z, \epsilon)$, the slow subsystem obtained after the regularization of the fast dynamics is the same as the one of Eq.2.40 (which ensures the applicability of theorem 2.3).

2.6 Conditions for closed-loop stability

In this section, we will address the stability of the unforced ($v = 0$) closed-loop full-order system under the state feedback laws of theorems 2.2 and 2.3. Initially, we will consider two-time-scale systems of the form of Eq.2.1 in standard form and assume that: 1) the eigenvalues of the matrix $Q_2(x) + g_2(x)k^T(x)$ lie in the open left-half of the complex plane uniformly in $x \in X$, 2) the roots of the polynomial $\beta_0 + \beta_1 s + \cdots + \beta_r s^r = 0$ lie in the open left-half of the complex plane, 3) the open-loop reduced system of Eq.2.5 is minimum-phase, i.e. its zero dynamics is locally exponentially stable. Condition 1 guarantees that fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold. Moreover, it is straightforward to show that conditions 2 and 3 guarantee the local exponential stability of the unforced closed-loop reduced system, under the controller of theorem 2.2. The local exponential stability of the closed-loop reduced-order systems implies that the unforced closed-loop full-order system is locally exponentially stable. for a sufficiently small ϵ (see theorem 2.1).

In the case of systems in nonstandard form. one can use similar arguments to show that the local exponential stability of the unforced closed-loop full-order system. for ϵ sufficiently small. is guaranteed if condition 1 holds. the roots of the polynomial $\beta_0 + \beta_1 s + \cdots + \beta_{\bar{r}} s^{\bar{r}} = 0$ lie in the open left-half of the complex plane. and the zero dynamics of the reduced system of Eq.2.40 is locally exponentially stable.

2.7 Simulation studies

2.7.1 Application to a biochemical reactor

Consider the biochemical continuous stirred tank reactor shown in Figure 2.1. where the following enzymatic reaction ([HTA67]) takes place:

$$S + E = C \rightarrow P + E \tag{2.46}$$

where S. E. C and P. denote the substrate. the enzyme. the complex and the product respectively. The inlet stream F_1 consists of substrate of concentration C_{S0} and temperature T_{S0}. while the inlet stream F_2 consists of enzyme of concentration C_{E0} and temperature T_{E0}. Under the following assumptions:

- Perfect mixing in the reactor
- Uniform temperature in the reactor
- Constant volume of the liquid in the reactor
- Constant density and heat capacity of the reacting liquid

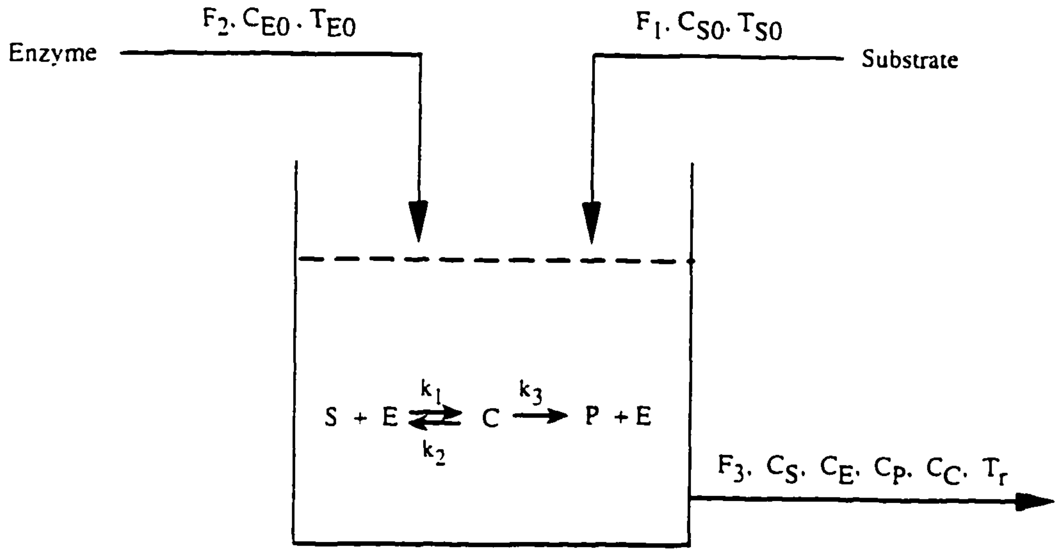

Figure 2.1: A biochemical continuous stirred tank reactor.

the material and energy balances that describe the dynamical behavior of the bioreactor take the following form:

$$
\begin{aligned}
V_r\frac{dC_P}{dt} &= -F_3C_P + k_{30}e^{\frac{-E_3}{RT_r}}C_CV_r \\
V_r\frac{dC_S}{dt} &= F_1C_{S0} - F_3C_S - k_{10}e^{\frac{-E_1}{RT_r}}C_SC_EV_r + k_{20}e^{\frac{-E_2}{RT_r}}C_CV_r \\
V_r\frac{dC_E}{dt} &= F_2C_{E0} - F_3C_E - k_{10}e^{\frac{-E_1}{RT_r}}C_SC_EV_r + k_{20}e^{\frac{-E_2}{RT_r}}C_CV_r + k_{30}e^{\frac{-E_3}{RT_r}}C_CV_r \\
V_r\frac{dC_C}{dt} &= -F_3C_C + k_{10}e^{\frac{-E_1}{RT_r}}C_SC_EV_r - k_{20}e^{\frac{-E_2}{RT_r}}C_CV_r - k_{30}e^{\frac{-E_3}{RT_r}}C_CV_r \\
V_r\frac{dT_r}{dt} &= F_1T_{S0} + F_2T_{E0} - F_3T_r + \frac{Q}{\rho_m c_{pm}} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{\frac{-E_1}{RT_r}}C_SC_EV_r \\
&\quad + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{\frac{-E_2}{RT_r}}C_CV_r + \frac{(-\Delta H_{r_3})}{\rho_m c_{pm}}k_{30}e^{\frac{-E_3}{RT_r}}C_CV_r
\end{aligned}
\tag{2.47}
$$

where C_S, C_E, C_C, C_P denote the concentrations of the species S, E, C, P, and C_{S0} and C_{E0} denote the inlet concentrations of the species E and S, T_r and V_r denote the temperature and the volume of the reactor, F_3 denotes the outlet flow rate, and k_{10}, k_{20}, k_{30}, E_1, E_2, E_3, ΔH_1, ΔH_2, ΔH_3 denote the pre-exponential constants, the activation energies, and the enthalpies of the three reactions. The control objective is the regulation of the concentration of the substrate C_S by manipulating the heat input to the reactor Q. In this type of bioreactions, the inlet concentration of the enzyme is usually much smaller than the one of the substrate ([HTA67]). Defining the parameter ϵ as:

$$
\epsilon = \frac{C_{E0}}{C_{S0}}
\tag{2.48}
$$

and setting:

$$
u = Q - Q_s,\quad x_1 = \frac{C_P}{C_{S0}},\quad x_2 = \frac{C_S}{C_{S0}},\quad x_3 = \frac{C_C}{C_{S0}},\quad x_4 = T_r,\quad z = \frac{C_E}{C_{E0}},\quad y = x_2
\tag{2.49}
$$

the original set of equations can be put in the form of Eq.2.1 with:

$$f_1(x) = \begin{bmatrix} \dfrac{-F_3}{V_r}x_1 + k_{30}e^{\frac{-E_3}{Rx_4}}x_3 \\ \dfrac{F_1}{V_r} - \dfrac{F_3}{V_r}x_2 + k_{20}e^{\frac{-E_2}{Rx_4}}x_3 \\ -\dfrac{F_3}{V_r}x_3 - k_{20}e^{\frac{-E_2}{Rx_4}}x_3 - k_{30}e^{\frac{-E_3}{Rx_4}}x_3 \\ \dfrac{F_1}{V_r}T_{S0} + \dfrac{F_2}{V_r}T_{E0} - \dfrac{F_3}{V_r}x_4 + \dfrac{Q_s}{\rho_m c_{pm} V_r} + \dfrac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{\frac{-E_2}{Rx_4}}x_3 C_{S0} \\ +\dfrac{(-\Delta H_{r_3})}{\rho_m c_{pm}}k_{30}e^{\frac{-E_3}{Rx_4}}x_3 C_{S0} \end{bmatrix}$$

$$Q_1(x) = \begin{bmatrix} 0 \\ -k_{10}e^{\frac{-E_1}{Rx_4}}x_2 C_{E0} \\ k_{10}e^{\frac{-E_1}{Rx_4}}x_2 C_{E0} \\ \dfrac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{\frac{-E_1}{Rx_4}}x_2 C_{E0} C_{S0} \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{\rho_m c_{pm} V_r} \end{bmatrix}$$

$$f_2(x) = \begin{bmatrix} \dfrac{F_2}{V_r}\epsilon + k_{20}e^{\frac{-E_2}{Rx_4}}x_3 + k_{30}e^{\frac{-E_3}{Rx_4}}x_3 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \end{bmatrix}$$

$$Q_2(x) = \begin{bmatrix} \dfrac{-F_3}{V_r}\epsilon - k_{10}e^{\frac{-E_1}{Rx_4}}C_{E0}x_2 \end{bmatrix}, \quad h(x) = \begin{bmatrix} x_2 \end{bmatrix}$$

It can be easily seen that $Q_2(x)$ is nonsingular and thus the two-time-scale system is in standard form. Moreover, the matrix $Q_2(x)$ is Hurwitz since x_2 is always positive.

The values of the system parameters and the corresponding steady-state values of the system variables are given in Table 2.1. It was verified that these conditions correspond to a stable equilibrium point, while the zero dynamics of the reduced system are exponentially stable. Since the fast dynamics of the system is stable, the

C_{S0}	=	12.92	$kmol\ m^{-3}$
C_{E0}	=	11.91	$kmol\ m^{-3}$
T_{E0}	=	310.0	K
T_{S0}	=	329.24	K
F_1	=	2.1	$m^3\ min^{-1}$
F_2	=	0.1	$m^3\ min^{-1}$
F	=	2.2	$m^3\ min^{-1}$
V_r	=	1.0	m^3
R	=	1.987	$kcal\ kmol^{-1}\ K^{-1}$
c_{pm}	=	0.231	$kcal\ kg^{-1}\ K^{-1}$
ρ_m	=	900.0	$kg\ m^{-3}$
k_{10}	=	3.36×10^6	$m^3\ kmol^{-1}\ min^{-1}$
k_{20}	=	1.80×10^6	min^{-1}
k_{30}	=	5.79×10^7	min^{-1}
E_1	=	8.0×10^3	$kcal\ kmol^{-1}$
E_2	=	9.0×10^3	$kcal\ kmol^{-1}$
E_3	=	10.0×10^3	$kcal\ kmol^{-1}$
ΔH_1	=	1.0×10^3	$kcal\ kmol^{-1}$
ΔH_2	=	1.0×10^3	$kcal\ kmol^{-1}$
ΔH_3	=	5.0×10^3	$kcal\ kmol^{-1}$
C_{C_s}	=	0.491	$kmol\ m^{-3}$
C_{E_s}	=	0.05	$kmol\ m^{-3}$
C_{S_s}	=	11.2	$kmol\ m^{-3}$
C_{P_s}	=	0.67	$kmol\ m^{-3}$
T_r	=	300.0	K
Q_s	=	-3.8×10^3	$kcal\ min^{-1}$

Table 2.1: Process parameters and steady-state values

controller of Eq.2.30 was employed in the simulations. Setting $\epsilon = 0$. the open-loop reduced system of Eq.2.5 can be easily obtained with:

$$F(x) = \begin{bmatrix} \frac{-F_3}{V_r}x_1 + k_{30}e^{\frac{-E_3}{Rx_4}}x_3 \\ \frac{F_1}{V_r}C_{S0} - \frac{F_3}{V_r}x_2 - k_{30}e^{\frac{-E_3}{Rx_4}}x_3 \\ -\frac{F_3}{V_r}x_3 \\ \frac{F_1}{V_r}T_{S0} + \frac{F_2}{V_r}T_{E0} - \frac{F_3}{V_r}x_4 + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{\frac{-E_2}{Rx_4}}C_{S0}x_3 + \frac{(-\Delta H_{r_3})}{\rho_m c_{pm}} \\ \bullet k_{30}e^{\frac{-E_3}{Rx_4}}C_{S0}x_3 + \frac{(-\Delta H_{r_1})C_{S0}}{\rho_m c_{pm}}[k_{20}e^{\frac{-E_2}{Rx_4}}x_3 + k_{30}e^{\frac{-E_3}{Rx_4}}x_3] + \frac{Q_s}{\rho c_{pm}V_r} \end{bmatrix}$$

$$G(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\rho_m c_{pm} V_r} \end{bmatrix}$$

It can be easily verified that the relative order is $r = 2$. The controller was tuned to give an overdamped response for changes in the reference input with time constant and damping factor:

$$\tau = 3.88\ min \quad , \quad \zeta = 1.93$$

through the following choice of the controller parameters:

$$\beta_0 = 1.0 \quad , \quad \beta_1 = 15.0\ min \quad , \quad \beta_2 = 15.0\ min^2$$

Several simulations were performed to evaluate the performance of the controller. In the first simulation run, a 0.8 mol/lt increase in the reference input value was imposed at time $t = 1.0\ min$. Figure 2.2 shows the corresponding output and input profiles. One can see that the controller regulates the output to the new value of the reference input. Figure 2.3 shows the profile of the concentration of the enzyme under the quasi-

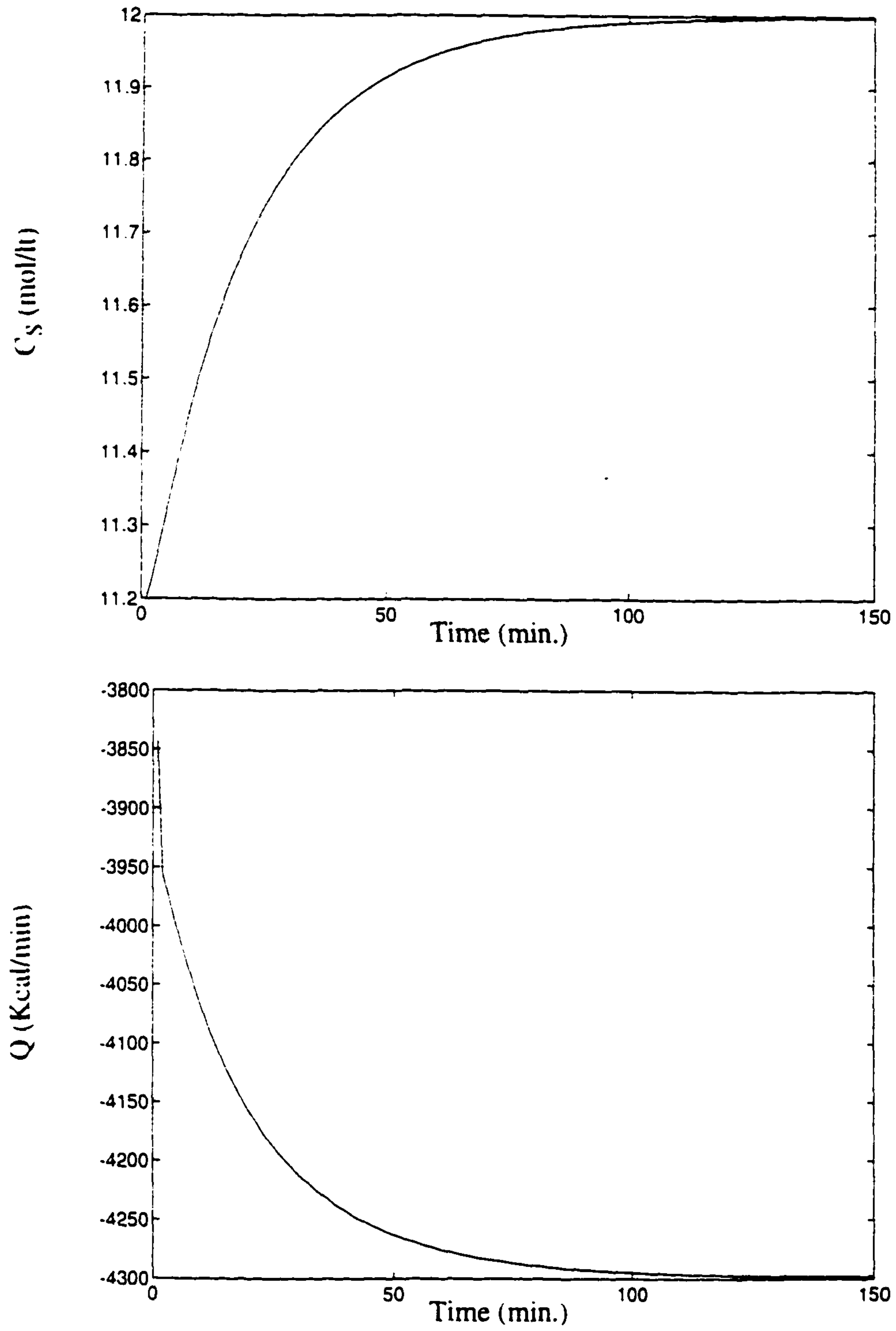

Figure 2.2: Output and input profiles for a positive change in the reference input.

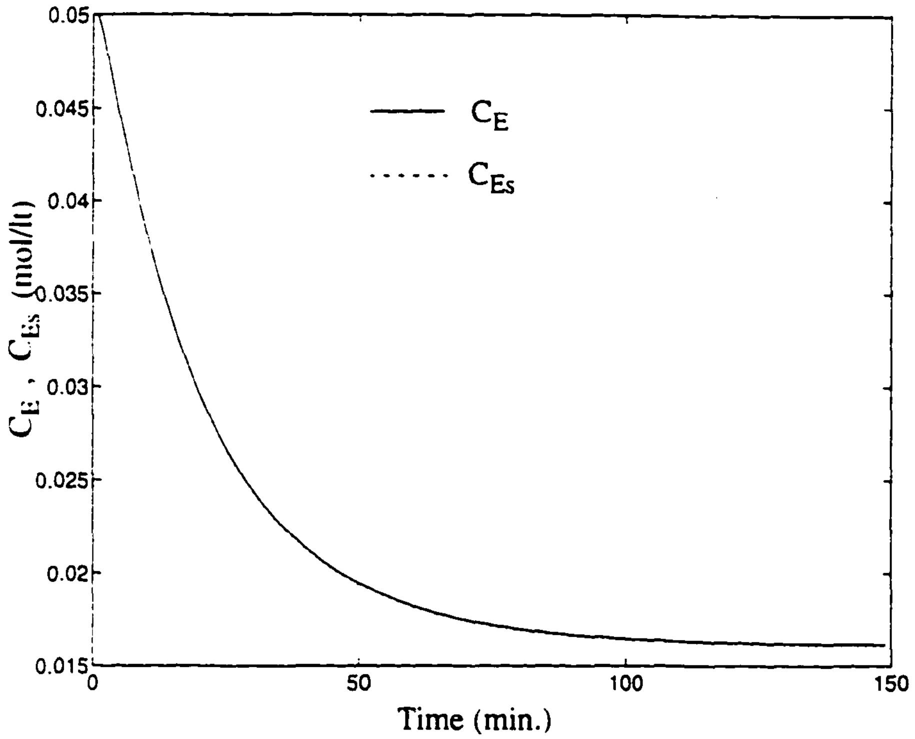

Figure 2.3 Profile of the concentration of the enzyme under the quasi-steady-state assumption (dotted line). Comparison with the actual one (solid line).

steady-state assumption. C_{Es}, and the profile of the concentration of the enzyme in the reactor, C_E. One can observe that the two profiles almost coincide which implies that the dynamics of the concentration of the enzyme are indeed negligible. In the second simulation run, a 0.8 mol/lt decrease in the value of the reference input was imposed at time $t = 1.0\ min$. Figure 2.4 shows the corresponding output and input profiles. The controller regulates the output to the new value of the reference input.

Remark 2.8: A standard input/output linearizing controller synthesized on the basis of the full-order system yields:

$$\begin{aligned} u \;=\; & [\beta_2 L_{g_1} L_{f_1} h(x)]^{-1} \\ & \Big\{ y - \beta_0 h(x) - \beta_1 [L_{f_1} h(x) + L_{Q_1} h(x) z] - \beta_2 [L_{f_1}^2 h(x) + L_{Q_1} L_{f_1} h(x) z \\ & + (L_{f_1} L_{Q_1} h(x) + L_{Q_1}^2 h(x) + L_{g_1} L_{Q_1} h(x)) z + \frac{L_{Q_1} h(x)(f_2(x) + Q_2(x) z)}{\epsilon}] \Big\} \end{aligned} \tag{2.50}$$

which becomes singular as $\epsilon \rightarrow 0$. Moreover, the implementation of this feedback law requires measurements of the concentration of the enzyme in the reactor, which are difficult to obtain in practice.

2.7.2 Application to a cascade of two continuous stirred tank reactors

Consider the cascade of the two continuous stirred tank reactors shown in Figure 2.5, where the following autocatalytic reaction ([GS90]) takes place:

$$A + B \rightarrow 2B \tag{2.51}$$

where A is a reactant, B is the autocatalytic species, followed by the zero-order side-reaction

$$B \rightarrow C \tag{2.52}$$

where C is the undesired product. The species A is assumed to be in excess in the two reactors, while the inlet streams consist of autocatalytic species B of concentration C_{B0}. Under the following assumptions:

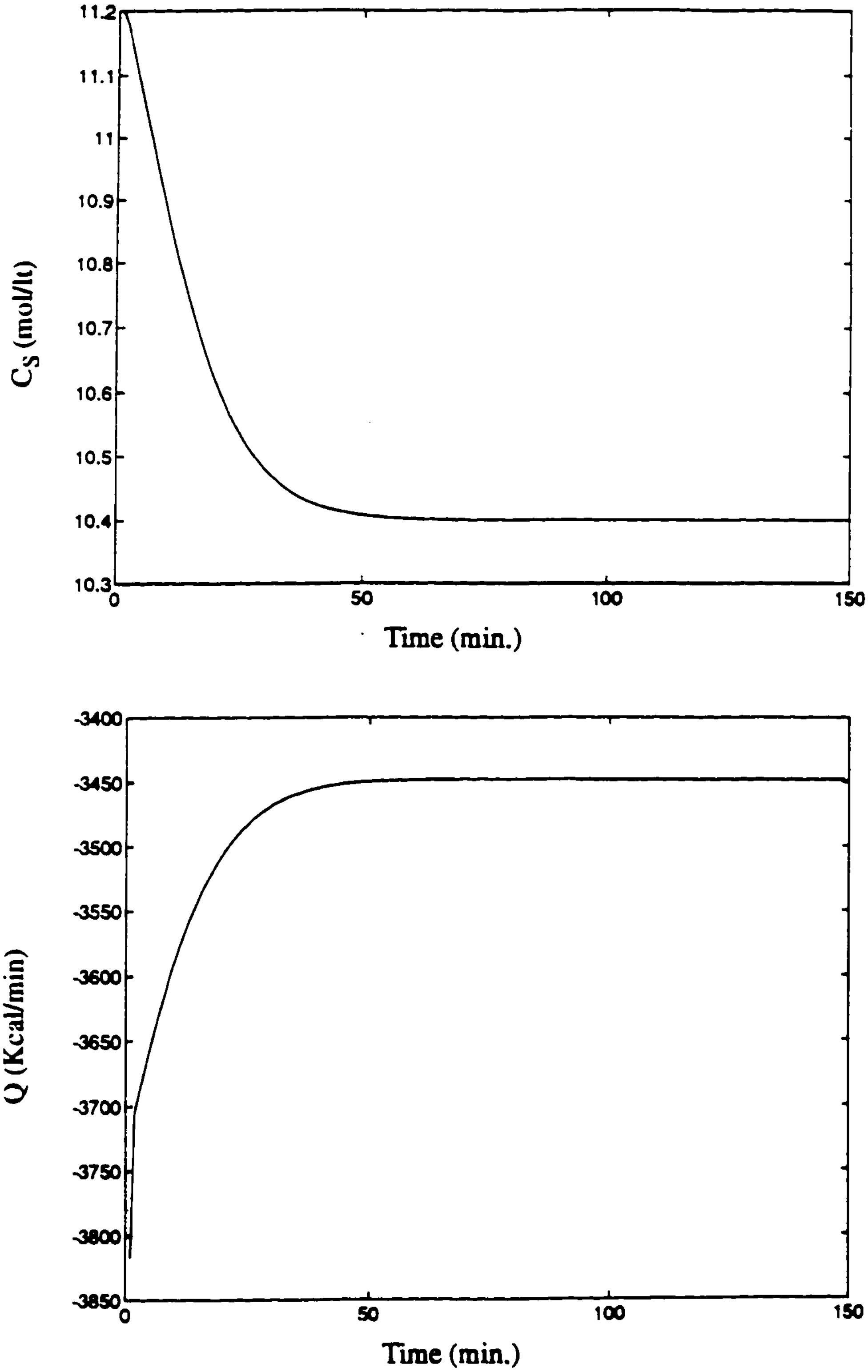

Figure 2.4: Output and input profiles for a negative change in the reference input.

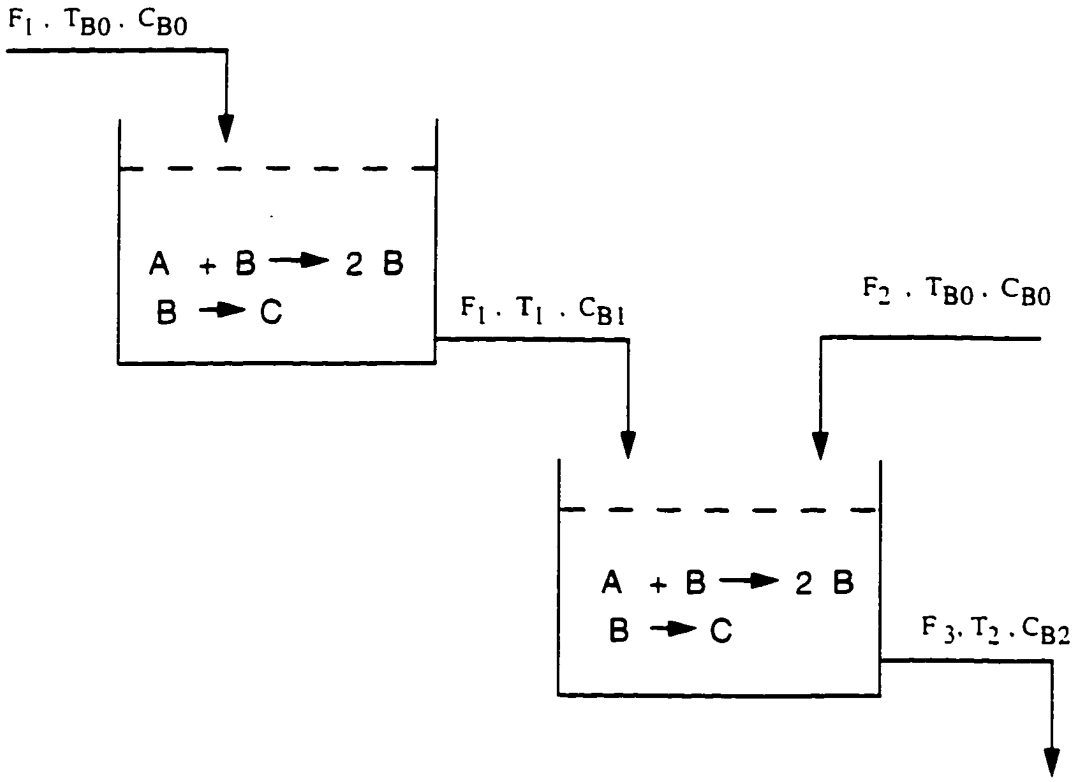

Figure 2.5: A cascade of two continuous stirred tank reactors

- Perfect mixing in the reactor
- Uniform temperature in the reactor
- Constant volume of the liquid in the reactor
- Constant density and heat capacity of the reacting liquid

the material and energy balances that describe the dynamical behavior of the system take the following form:

$$\begin{aligned}
V_1\frac{dC_{B1}}{dt} &= F_1C_{B0} - F_1C_{B1} + k_{10}e^{\frac{-E_1}{RT_1}}C_{B1}V_1 - k_0e^{\frac{-E_0}{RT_1}}V_1 \\
\frac{dT_1}{dt} &= \frac{F_1}{V_1}(T_{B0} - T_1) + \frac{\bar{Q}_1}{\rho_m c_{pm} V_1} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{\frac{-E_1}{RT_1}}C_{B1} \\
&\quad + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}}k_0e^{\frac{-E_0}{RT_1}} \\
V_2\frac{dC_{B2}}{dt} &= F_1C_{B1} + F_2C_{B0} - F_3C_{B2} + k_{10}e^{\frac{-E_1}{RT_2}}C_{B2}V_2 - k_0e^{\frac{-E_0}{RT_2}}V_2 \\
\frac{dT_2}{dt} &= \frac{F_2}{V_2}T_{B0} + \frac{F_1}{V_2}T_1 - \frac{F_3}{V_2}T_2 + \frac{\bar{Q}_2}{\rho_m c_{pm} V_2} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{\frac{-E_1}{RT_2}}C_{B2} \\
&\quad + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}}k_0e^{\frac{-E_0}{RT_2}}
\end{aligned} \tag{2.53}$$

where C_{B1}, T_1 and C_{B2}, T_2 denote temperature and the concentration of the autocatalytic species in the first and second reactor. C_{B0} and T_{B0} denote the inlet temperature and concentration of the species B. F_3 is the outlet flowrate in the second reactor. $\bar{Q}_1$, $\bar{Q}_2$ denote the heat inputs to the reactors. k_{10}, k_0, E_1, E_0, ΔH_1, ΔH_0 denote the pre-exponential constants, the activation energies, and the enthalpies of the two reactions.

The control objective is the regulation of the concentration of the autocatalytic species B in the second reactor by manipulating the inlet concentration C_{B0}. In order to decrease the effect of the side-reaction, i.e., minimize the production of the species

C, the liquid holdup of the first reactor is smaller than the liquid hold up of the second reactor ([Lev72]). Defining the parameter ϵ as:

$$\epsilon = \frac{V_1}{V_2} \tag{2.54}$$

and setting:

$$u = C_{B0} - C_{B0s},\ x_1 = T_1,\ x_2 = C_{B2},\ x_3 = T_2,\ z_1 = C_{B1},\ y = x_2 \tag{2.55}$$

the original set of equations can be put in the form of Eq.2.1 with:

$$f_1(x) = \begin{bmatrix} \frac{F_1}{V_1}(T_{B0} - x_1) + \frac{\bar{Q}_1}{\rho_m c_{pm} V_1} + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}} k_0 e^{\frac{-E_0}{Rx_1}} \\ F_2 C_{B0s} - F_3 x_2 + k_{10} e^{\frac{-E_1}{Rx_3}} x_2 V_2 - k_0 e^{\frac{-E_0}{Rx_3}} V_2 \\ \frac{F_2}{V_2} T_{B0} + \frac{F_1}{V_2} x_1 - \frac{F_3}{V_2} x_3 + \frac{\bar{Q}_2}{\rho_m c_{pm} V_2} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{\frac{-E_1}{Rx_3}} x_2 \\ + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}} k_0 e^{\frac{-E_0}{Rx_3}} \end{bmatrix}$$

$$Q_1(x) = \begin{bmatrix} \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{\frac{-E_1}{Rx_1}} \\ F_1 \\ 0 \end{bmatrix},\ g_1(x) = \begin{bmatrix} 0 \\ F_2 \\ 0 \end{bmatrix}$$

$$f_2(x) = \begin{bmatrix} \frac{F_1}{V_2} C_{B0s} - \frac{k_0 e^{\frac{-E_0}{Rx_1}} V_1}{V_2} \end{bmatrix},\ Q_2(x) = \begin{bmatrix} -\frac{F_1}{V_2} + \frac{k_{10} e^{\frac{-E_1}{Rx_1}} V_1}{V_2} \end{bmatrix}$$

$$g_2(x) = \begin{bmatrix} \frac{F_1}{V_2} \end{bmatrix},\ h(x) = \begin{bmatrix} x_2 \end{bmatrix}$$

The values of the system parameters and the corresponding steady-state values of the system variables are given in Table 2.2. One can easily see that for these operating conditions the matrix $Q_2(x)$ is invertible, and the fast dynamics of the system

V_1	=	0.2	m^3
V_2	=	1.0	m^3
R	=	1.987	$kcal\ kmol^{-1}\ K^{-1}$
C_{B0_s}	=	2.0	$kmol\ m^{-3}$
T_{B0}	=	305.0	K
c_{pm}	=	0.231	$kcal\ kg^{-1}\ K^{-1}$
ρ_m	=	900.0	$kg\ m^{-3}$
Q_1	=	1.1×10^5	$kcal\ min^{-1}$
Q_2	=	2.2×10^4	$kcal\ min^{-1}$
ΔH_0	=	5.4×10^3	$kcal\ kmol^{-1}$
ΔH_1	=	10.67×10^3	$kcal\ kmol^{-1}$
k_0	=	4.71×10^6	$kmol\ m^{-3} min^{-1}$
k_{10}	=	1.08×10^7	min^{-1}
E_0	=	8.0×10^3	$kcal\ kmol^{-1}$
E_1	=	9.0×10^3	$kcal\ kmol^{-1}$
F_1	=	0.2	$m^3\ min^{-1}$
F_2	=	2.0	$m^3\ min^{-1}$
C_{B1_s}	=	2.5	$kmol\ m^{-3}$
T_{1_s}	=	300.0	K
C_{B2_s}	=	3.125	$kmol\ m^{-3}$
T_{2_s}	=	300.0	K

Table 2.2: Process parameters and steady-state values

are unstable. Therefore, the controller of theorem 2.2 was employed in the simulations. Moreover, it was verified that the zero dynamics of the reduced system are exponentially stable.

Setting $\epsilon = 0$, the representation of the open-loop reduced system of the form of Eq.2.5 can be easily obtained with:

$$F(x) = \begin{bmatrix} \frac{F_1}{V_1}(T_{B0} - x_1) + \frac{\bar{Q}_1}{\rho_m c_{pm} V_1} + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}} k_0 e^{\frac{-E_0}{Rx_1}} + \\ \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{\frac{-E_1}{Rx_1}} \\ \bullet(F_1 - k_{10} e^{\frac{-E_1}{Rx_1}} V_1)^{-1}(F_1 C_{B0s} - k_0 e^{\frac{-E_0}{Rx_1}} V_1) \\ F_2 C_{B0s} - F_3 x_2 + k_{10} e^{\frac{-E_1}{Rx_3}} x_2 V_2 - k_0 e^{\frac{-E_0}{Rx_3}} V_2 + F_1 \\ \bullet(F_1 - k_{10} e^{\frac{-E_1}{Rx_1}} V_1)^{-1}(F_1 C_{B0s} - k_0 e^{\frac{-E_0}{Rx_1}} V_1) \\ \frac{F_2}{V_2} T_{B0} + \frac{F_1}{V_2} x_1 - \frac{F_3}{V_2} x_3 + \frac{\bar{Q}_2}{\rho_m c_{pm} V_2} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{\frac{-E_1}{Rx_3}} x_2 \\ + \frac{(-\Delta H_{r_0})}{\rho_m c_{pm}} k_0 e^{\frac{-E_0}{Rx_3}} \end{bmatrix}$$

$$G(x) = \begin{bmatrix} \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{\frac{-E_1}{Rx_1}} (F_1 - k_{10} e^{\frac{-E_1}{Rx_1}} V_1)^{-1} F_1 \\ F_2 + F_1 (F_1 - k_{10} e^{\frac{-E_1}{Rx_1}} V_1)^{-1} F_1 \\ 0 \end{bmatrix}$$

It can be easily verified that the relative order is $r = 1$ and the controller of theorem 2.3 takes the form:

$$\begin{aligned} u &= [1 + k^T(x)[Q_2(x)]^{-1} g_2(x)]\, [\beta_1 L_G h(x)]^{-1} \left\{ v - \sum_{k=0}^{1} \beta_k L_F^k h(x) \right\} \\ &\quad + k^T(x)[Q_2(x)]^{-1} f_2(x) + k^T(x) z_1 \end{aligned} \qquad (2.56)$$

The nonlinear gain $k^T(x)$ was chosen as:

$$k^T(x) \;=\; -\frac{3.0}{F_1}\,[-F_1 + k_{10}e^{\frac{-E_1}{Rx_1}}V_1] \tag{2.57}$$

to place the eigenvalue of the matrix $Q_2(x) + g_2(x)k^T(x)$ in the open left-half of the complex plane. The time constant for the input/output response was chosen to be $\tau = 15\ min$, through the following choice of the controller parameters:

$$\beta_0 = 1.0\quad ,\quad \beta_1 = 15.0\ min$$

Several simulations were performed to evaluate the performance of the controller. In the first simulation run, a 0.8 mol/lt increase in the value of the reference input was imposed at time $t = 0$. Figure 2.6 shows the output and the input profiles. Clearly, the controller drives the output of the system to the new value of the reference input, while stabilizing the fast dynamics of the system. Figure 2.7 shows the profiles of the two components of the control law identified in remark 2.1 for this particular simulation run. More specifically, U_1 denotes the component:

$$U_1 \;=\; [\beta_1 L_G h(x)]^{-1}\left\{v - \sum_{k=0}^{1}\beta_k L_F^k h(x)\right\} \tag{2.58}$$

which acts in the slow time-scale and induces the requested input/output behavior in the closed-loop reduced system, while U_2 denotes the component:

$$U_2 \;=\; k^T(x)(z_1 - \xi_1) \tag{2.59}$$

where ξ_1 is defined in Eq.2.29, which acts in the fast time-scale and stabilizes the fast dynamics of the system. Clearly the behavior of U_1 and U_2 conforms with the theoretical predictions. More specifically, the component U_1 acts for all times in order to drive the output of the system to the new value of the reference input, while the component U_2 approaches zero quickly.

In the second simulation run, a 0.3 mol/lt decrease in the value of the reference input was imposed at time $t = 0$. The corresponding output and input profiles are

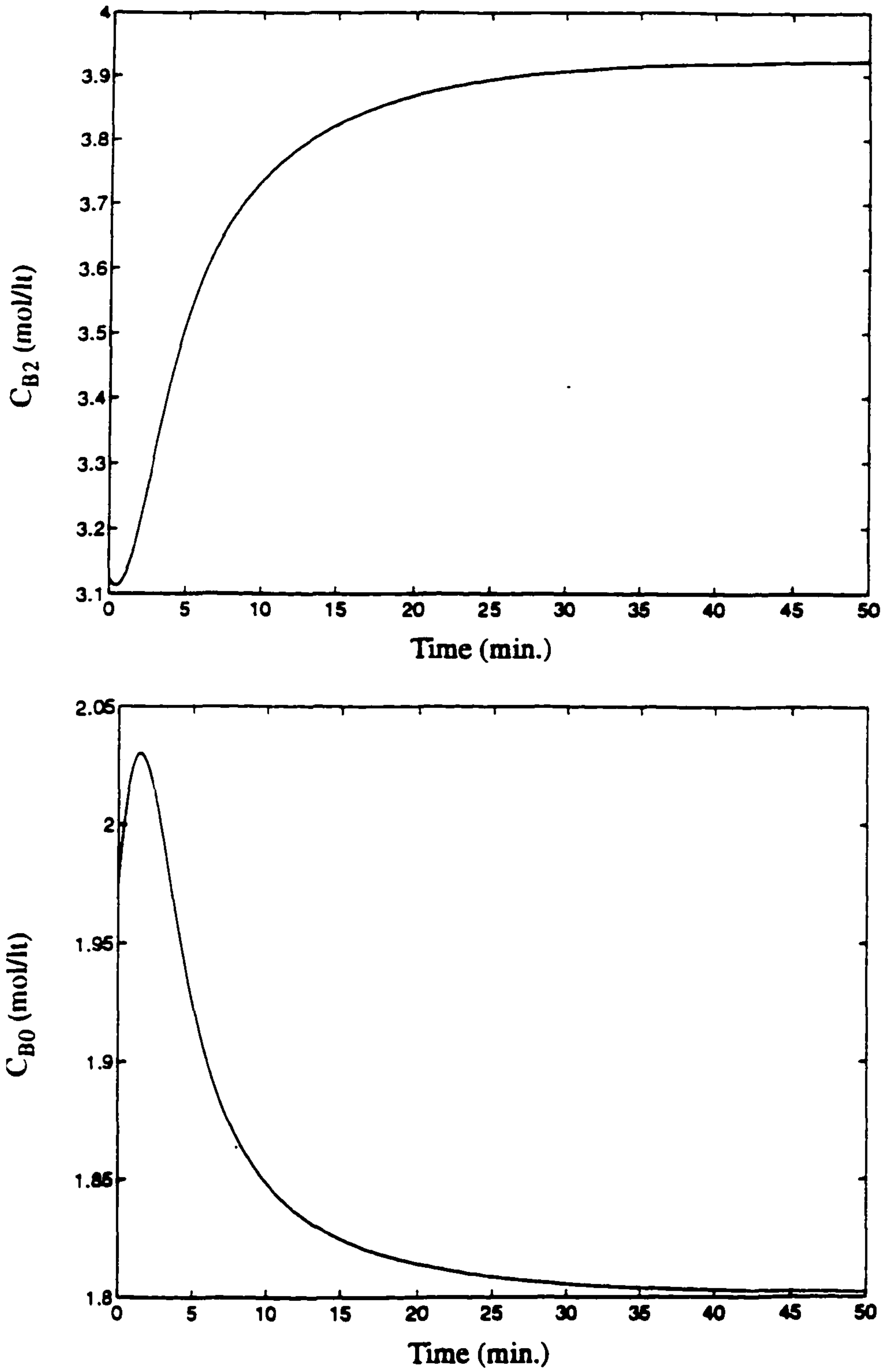

Figure 2.6: Output and input profiles for positive change in the reference input.

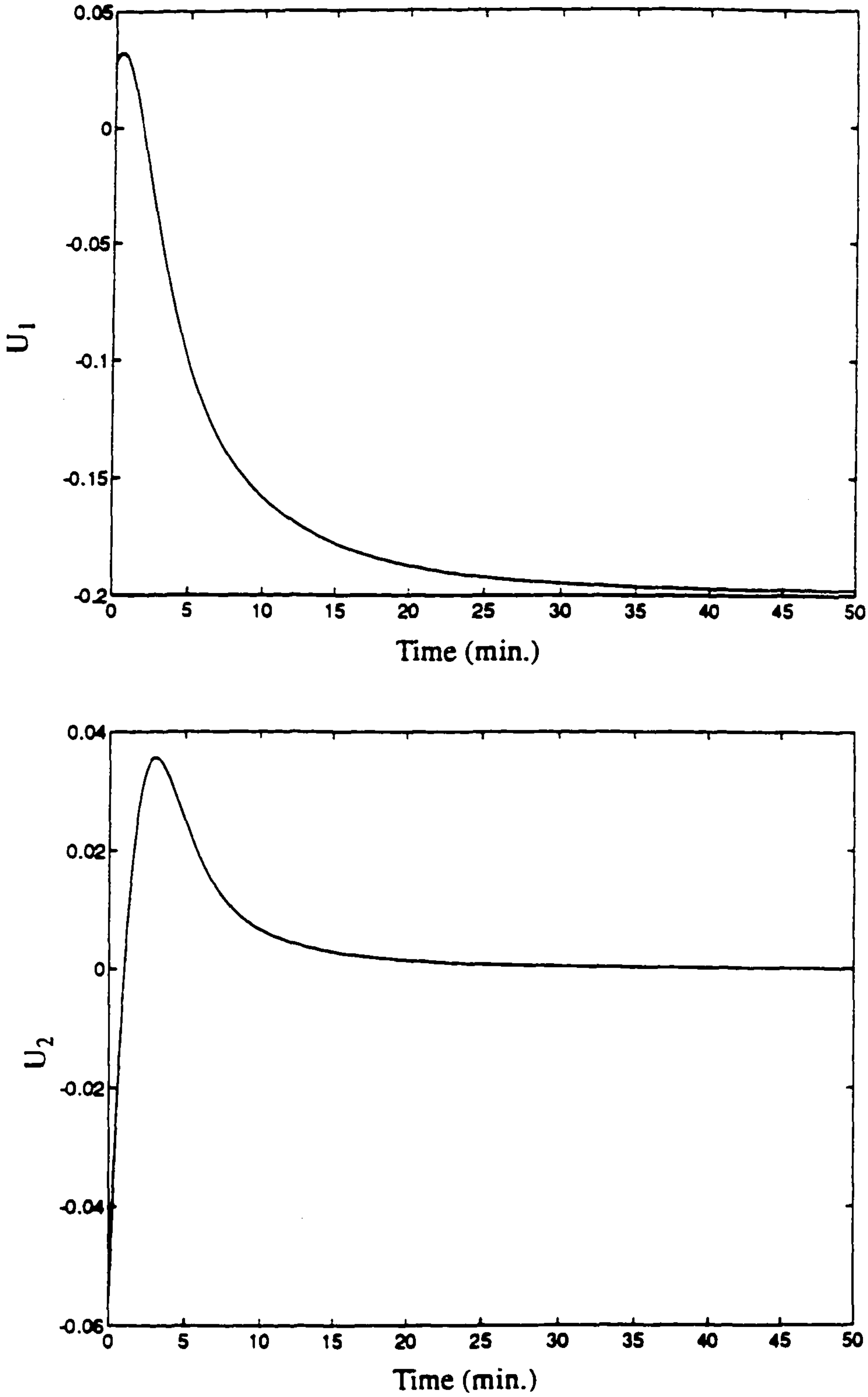

Figure 2.7: Fast (U_2) and slow (U_1) components of the input.

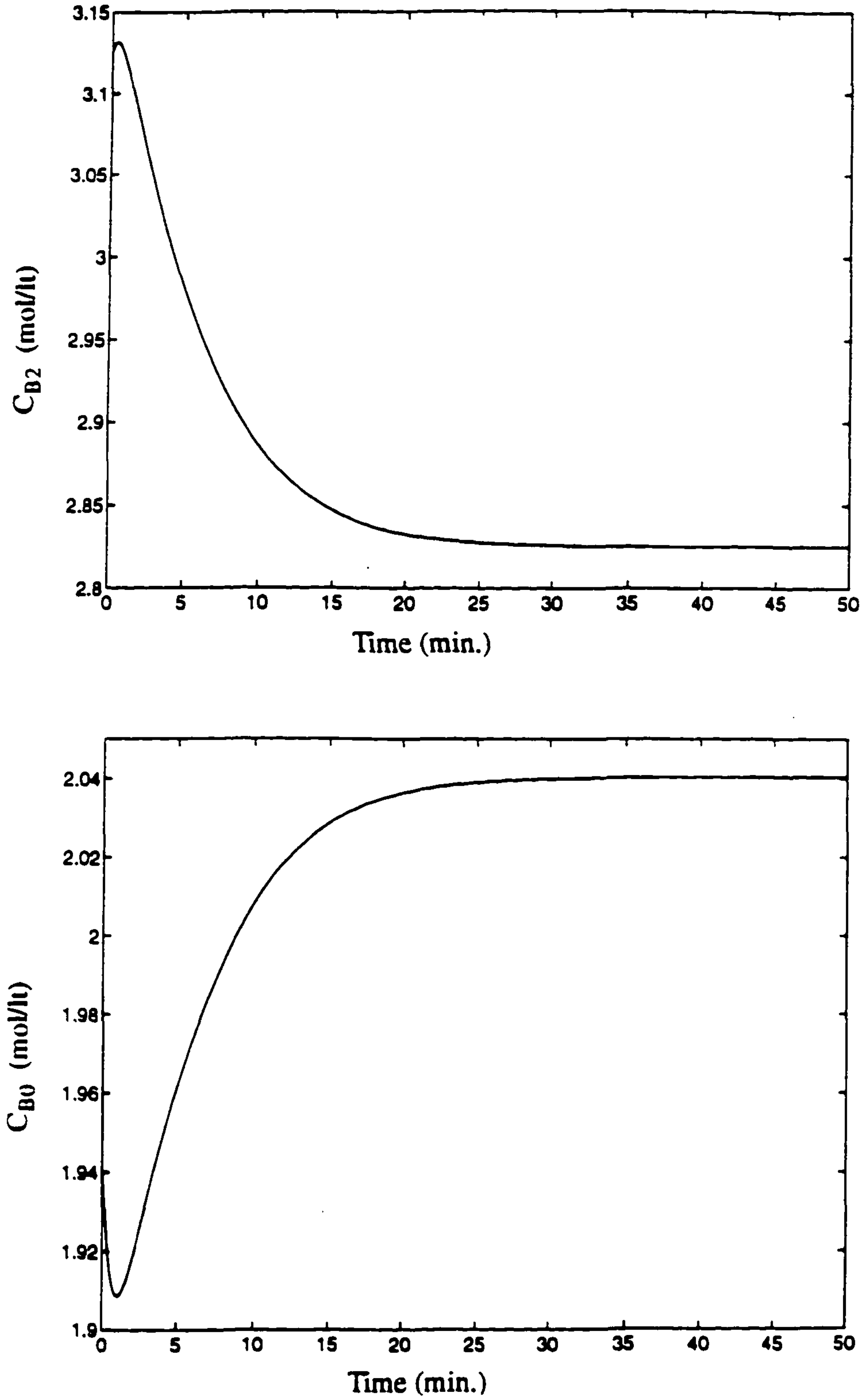

Figure 2.8: Output and input profiles for a negative change in the reference input.

shown in Figure 2.8. The controller regulates the output to the new reference input value, while stabilizing the fast dynamics of the system. Finally, an input/output linearizing control law, which was synthesized on the basis of the original system, was also employed in the simulations. A 0.3 mol/lt decrease in the reference input value was imposed at time $t = 0$. Figure 2.9. shows the profile of the fast state z_1 which corresponds to the concentration of the species B in the first reactor, and the corresponding input. It is clear that the input/output linearizing controller leads to closed-loop instability, which is expected, since, as can be easily verified following the development of remark 2.5, the process is slightly non-minimum phase.

2.8 Conclusions

In this chapter, we addressed and solved the problem of synthesizing well-conditioned static state feedback laws for a class of two-time-scale nonlinear systems, whose fast dynamics may be unstable or singular. The derived control laws guarantee exponential stability of the fast dynamics and induce a well-characterized input/output behavior in the closed-loop reduced system. It was established that if the zero dynamics of the reduced system is exponentially stable, then the closed-loop system is exponentially stable and the discrepancy between the output of the closed-loop full-order system and the output of the closed-loop reduced system is of order ϵ, for sufficiently small values of ϵ. The proposed control methodology was successfully applied to two representative nonlinear chemical processes with time-scale multiplicity and its superiority with respect to nonlinear control methods that neglect the presence of time-scale multiplicity was documented through simulations.

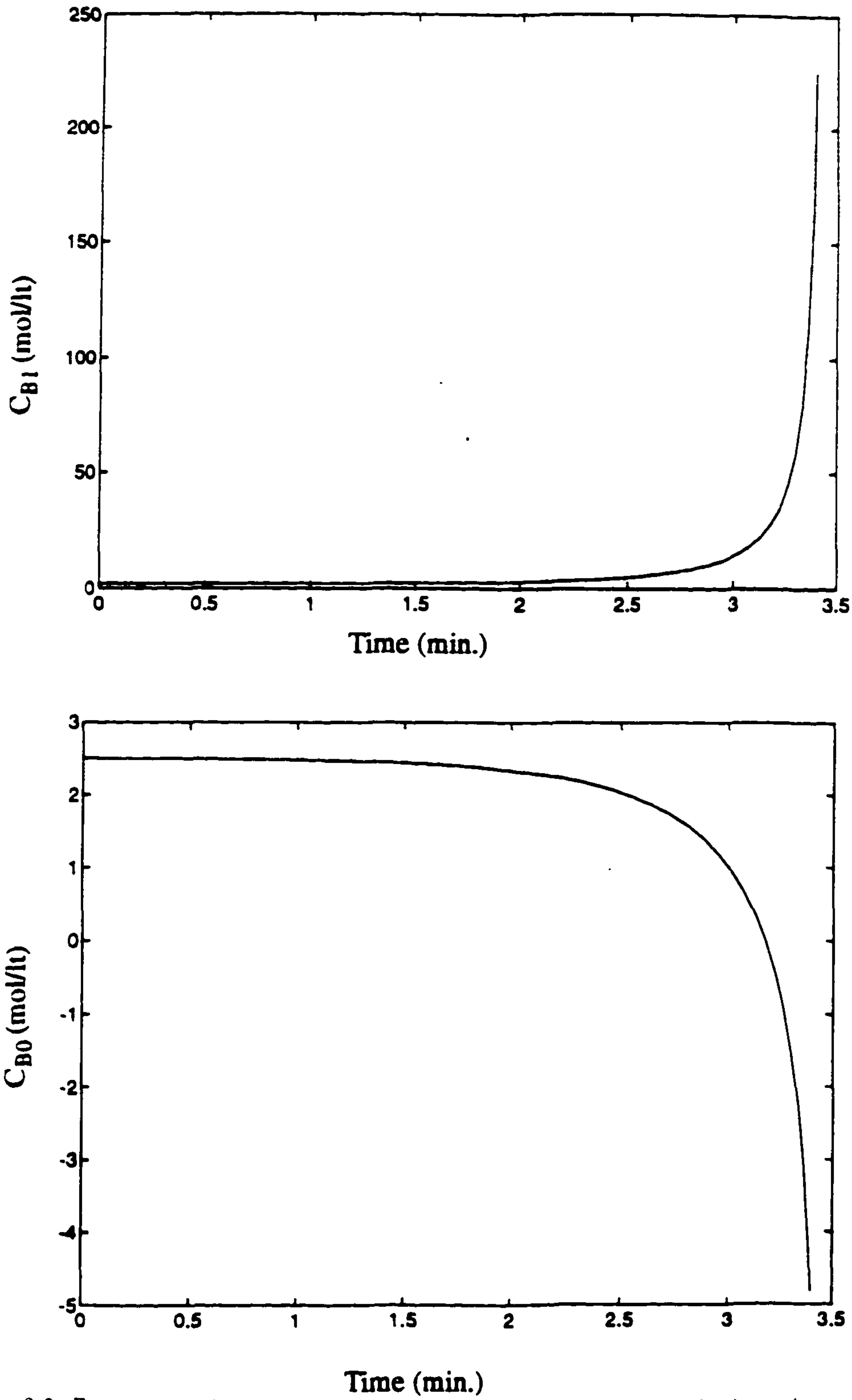

Figure 2.9: Fast state and input profiles for reference input tracking, under input/output linearizing controller.

Notation

Roman Letters

C_{B0} = inlet concentration of the autocatalytic species
C_{B1} = concentration of the autocatalytic species in the first reactor
C_{B2} = concentration of the autocatalytic species in the second reactor
C_C = concentration of the complex in the reactor
C_{E0} = inlet concentration of the enzyme
C_E = concentration of the enzyme in the reactor
C_P = concentration of the product in the reactor
C_{S0} = inlet concentration of the substrate
C_S = concentration of the substrate in the reactor
c_{pm} = heat capacity of the reacting mixture
E_0, E_1, E_2, E_3 = activation energies
$F, \tilde{F}, f_1, f_2$ = vector fields
F_1, F_2, F_3 = flow rates
$G, \tilde{G}, g_1, g_2$ = vector fields associated with the input
h = output scalar function
k^T = covector field
$k_0, k_{10}, k_{20}, k_{30}$ = pre-exponential constants
Q_1 = matrix of dimension $n \times p$ associated with the slow state vector x
Q_2 = matrix of dimension $p \times p$ associated with the fast state vector z
$Q, \bar{Q}_1, \bar{Q}_2$ = heat inputs to the reactors
$r, \hat{r}$ = relative orders in the reduced systems
T_{B0} = inlet temperature of the autocatalytic species
T_{B1} = temperature in the first reactor
T_{B2} = temperature in the second reactor
T_{E0} = inlet temperature of the enzyme
T_r = temperature in the bioreactor
T_{S0} = inlet temperature of the substrate
t = time
u = input
$\tilde{u}, \hat{u}$ = auxiliary inputs
V_r, V_1, V_2 = volumes of the liquid holdup in the reactors
v = reference input
x = vector of the slow state variables
y = output
z = vector of the fast state variables

Greek Letters

β_k = adjustable parameters
$\Delta H_{r_0}, \Delta H_{r_1}, \Delta H_{r_2}, \Delta H_{r_3}$ = enthalpy of the reactions
ϵ = singular perturbation parameter
ζ = state vector in normal form coordinates
η = state vector in normal form coordinates
$\bar{\eta}$ = auxiliary variable
$\xi, \tilde{\xi}$ = equilibrium manifolds for the fast dynamics in the closed-loop
ν = integer associated with the input in the full-order system
π = integer associated with the state vector z in the full-order system
ρ_m = density of the reacting mixture
ϕ_i = auxiliary functions

Math Symbols

$L_f h$	=	Lie derivative of a scalar field h with respect to the vector f
$L_f^k h$	=	k-th order Lie derivative
$L_g L_f^{k-1} h$	=	mixed Lie derivative
$\mathbb{R}$	=	real line
$\mathbb{R}^i$	=	i−dimensional Euclidean space
$\in$	=	belongs to
T	=	transpose
$\lvert \cdot \rvert$	=	standard Euclidean norm

Chapter 3

Compensation of Measurable Disturbances for Nonlinear Two-Time-Scale Systems

3.1 Introduction

Exogenous measurable disturbances which may vary arbitrarily with time are present in all practical applications. It is well-known that the presence of disturbances, if not taken into account in the controller design, may cause significant degradation of the nominal performance and in some instances closed-loop instability. Motivated by this, the problem of complete elimination of the effect of disturbances on the outputs (known as disturbance decoupling) for standard nonlinear systems has been studied extensively. Geometric conditions for the solvability of this problem via static state feedback [IKGGM81] and static feedforward/static state feedback [MG83] have been derived, while the solution of this problem through dynamic feedforward/static state feedback [DK93] has also been obtained. Other available results on the treatment of disturbance inputs deal with the use of high-gain feedback to achieve almost disturbance decoupling (e.g., [MRvdS88]), and the use of feedforward compensation in the context of exact state-space linearization [CA88]. However, a direct application of the aforementioned control methods to multiple-time-scale systems may lead to controller ill-conditioning and/or instability of the closed-loop system due to possible slightly non-minimum phase behavior (see remark 2.5).

Motivated by the above realization, the problem of rejection of disturbances for linear two-time-scale systems has been addressed using H^{∞} control methods [KC92, PB93a], stochastic control methods (see [KKO86] and the references therein), and controller design via Lyapunov functions [CGG93]. For nonlinear two-time-scale systems with stable fast dynamics, combination of singular perturbation theory and adaptive control schemes [TKMK89] has been proposed for this purpose. For nonlinear two-time-scale systems with unstable fast dynamics, an alternative approach that utilizes combination of singular perturbations and Lyapunov's direct method has been proposed in [Kho89].

In this chapter, a broad class of two-time-scale nonlinear systems modeled within the singular perturbation framework, with measurable time-varying disturbances, is considered; these include both systems in standard and nonstandard form. For these systems, we synthesize well-conditioned control laws that utilize feedback of the full state vector and feedforward compensation of disturbances to exponentially stabilize the fast dynamics and enforce a prespecified input/output behavior independently of the disturbances in the closed-loop slow subsystem. Singular perturbation methods are employed to establish that the discrepancy between the output of the closed-loop full-order system and the output of the closed-loop slow subsystem is proportional to the value of the singular perturbation parameter. Key differences in the nature of the control problem between systems in standard and nonstandard form are identified and discussed. The stability of the closed-loop system is analyzed using Lyapunov functions and precise conditions that guarantee boundedness of the trajectories of the closed-loop system, for sufficiently small values of the singular perturbation parameter, are derived. Finally, the developed methodology is applied to a catalytic continuous stirred tank reactor modeled as a singularly perturbed system in nonstandard form.

3.2 Preliminaries

We will consider two-time-scale nonlinear systems with the following state-space representation:

$$\begin{aligned} \dot{x} &= f_1(x) + Q_1(x)z + g_1(x)u + W_1(x)d(t) \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)u + W_2(x)d(t) \\ y &= h(x) \end{aligned} \tag{3.1}$$

where $x \in X \subset \mathbb{R}^n$ and $z \in Z \subset \mathbb{R}^p$ denote vectors of state variables, with X and Z open and connected sets which contain the equilibrium point of interest, $u \in \mathbb{R}$ denotes the manipulated input, $d = [d_1(t) \cdots d_q(t)] \in \mathbb{R}^q$ denotes the vector of disturbance inputs which are assumed to be measurable and sufficiently smooth functions of time, and $y \in \mathbb{R}$ denotes the controlled output, and ϵ is a small positive parameter. Furthermore, $f_1(x)$, $f_2(x)$, $g_1(x)$, $g_2(x)$, are analytic vector fields, $Q_1(x)$, $Q_2(x)$ and $W_1(x)$, $W_2(x)$, are analytic matrices of dimensions $n \times p$, $p \times p$ and $n \times q$, $p \times q$ respectively, and $h(x)$ is an analytic scalar function. In what follows, for simplicity, we will suppress the time-dependence in the notation of the disturbance input vector $d(t)$.

Assuming that the system of Eq.3.1 is in standard form i.e., the matrix $Q_2(x)$ is nonsingular uniformly in $x \in X$, and setting $\epsilon = 0$, the system of Eq.3.1 takes the form:

$$\dot{x} = f_1(x) + Q_1(x)z_s + g_1(x)u + W_1(x)d \tag{3.2}$$

$$f_2(x) + Q_2(x)z_s + g_2(x)u + W_2(x)d = 0 \tag{3.3}$$

where z_s denotes a quasi-steady-state for z. The invertibility of the matrix $Q_2(x)$ guarantees that the system of algebraic equations (Eq.3.3) admits a unique solution for z_s, of the form:

$$z_s = -[Q_2(x)]^{-1}[f_2(x) + g_2(x)u + W_2(x)d] \tag{3.4}$$

Substituting Eq.3.4 into Eq.3.2 the following *reduced system* or *slow subsystem* is obtained:

$$\begin{aligned} \dot{x} &= F(x) + G(x)u + W(x)d \\ y^s &= h(x) \end{aligned} \tag{3.5}$$

where y^s denotes output associated with the slow subsystem and

$$\begin{aligned} F(x) &= f_1(x) - Q_1(x)[Q_2(x)]^{-1}f_2(x) \\ G(x) &= g_1(x) - Q_1(x)[Q_2(x)]^{-1}g_2(x) \\ W(x) &= W_1(x) - Q_1(x)[Q_2(x)]^{-1}W_2(x) \end{aligned} \tag{3.6}$$

Note that the input u and the disturbance input vector d appear in an affine and separable fashion in the system of Eq.3.5 because of the linearity in z in the original system. To obtain a representation of the system which describes the fast dynamics of the system of Eq.3.1, we define a fast time-scale:

$$\tau = \frac{t}{\epsilon} \tag{3.7}$$

In this new time-scale the original system takes the form:

$$\frac{dx}{d\tau} = \epsilon[f_1(x) + Q_1(x)z + g_1(x)u + W_1(x)d] \tag{3.8}$$

$$\frac{dz}{d\tau} = f_2(x) + Q_2(x)z + g_2(x)u + W_2(x)d \tag{3.9}$$

Setting ϵ equal to zero, the following *fast subsystem* is obtained:

$$\frac{dz}{d\tau} = f_2(x) + Q_2(x)z + g_2(x)u + W_2(x)d \tag{3.10}$$

where x can be considered approximately equal to its initial value $x(0)$ and d constant. The affine appearance of the fast state z in the system of Eq.3.10 implies that its bounded-input bounded-state stability depends exclusively on the eigenstructure of the matrix $Q_2(x)$. For the open-loop fast subsystem of Eq.3.10, we assume that the stabilizability requirement of assumption 2.1 holds.

In closing this section, we will review the concept of relative order of the output y^s with respect to the disturbance input vector d for a system of the form of Eq.3.5, which will be used in our development.

Definition 3.1 : *Referring to the nonlinear system of Eq.3.5, the relative order of the output y^s with respect to the disturbance input vector d is defined as the smallest integer ρ for which*

$$[L_{W^1} L_F^{\rho-1} h(x) \cdots L_{W^q} L_F^{\rho-1} h(x)] \not\equiv [0 \cdots 0] \tag{3.11}$$

where W^κ denote the $\kappa - th$ column vector of the matrix W, or $\rho = \infty$ if such an integer does not exist.

3.3 Formulation of the control problem

We will address and solve the problem of synthesizing well-conditioned control laws that preserve the two-time-scale nature of the open-loop system, with the following objectives for the closed-loop system:

1. Boundedness of the trajectories
2. The output of the closed-loop system satisfies a relation of the form:

$$y(t) \;\; = \;\; y^s(t) + O(\epsilon)\;,\; t \geq 0 \tag{3.12}$$

   with $y^s(t)$ being the output of the closed-loop reduced system, where a prespecified input/output response is enforced independently of the disturbances.

The above requirements will be enforced in the closed-loop system for sufficiently small values of the singular perturbation parameter ϵ. The control laws will utilize feedback of the state z to stabilize the fast dynamics and feedback of the state x combined with feedforward compensation of disturbances to enforce the desired behavior on the output.

The above-specified control problem will be initially addressed for systems in standard form and then for systems in nonstandard form. For systems in standard form, it will be established that the requisite control law can be synthesized utilizing exclusively information concerning the original system of Eq.3.1, while for systems in nonstandard form it has to be preceded by the feedback regularization of the fast dynamics.

3.4 Disturbance compensation for systems in standard form

In this section, we will consider systems of the form of Eq.3.1 in standard form, with possibly unstable fast dynamics, i.e., systems for which some of the eigenvalues of the matrix $Q_2(x)$ may lie in the open right-half of the complex plane for some $x \in X$. The instability of the fast dynamics dictates the need to employ feedback of the state z. In particular, motivated by the affine appearance of the fast state z in the model of Eq.3.1, let us initially consider control laws of the form:

$$u = \tilde{u} + k^T(x)z \tag{3.13}$$

where $k^T(x)$ is in $\mathbb{R}^p$, and $\tilde{u}$ is an auxiliary input, to achieve stabilization of the fast dynamics of the system. Under a control law of the form of Eq.3.13, the two-time-scale system of Eq.3.1 takes the form:

$$\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)k^T(x)]z + g_1(x)\tilde{u} + W_1(x)d \\ \epsilon\dot{z} &= f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\tilde{u} + W_2(x)d \end{aligned} \tag{3.14}$$

One can immediately observe that a control law of the form of Eq.3.13 preserves the two-time-scale nature of the process, and the linearity with respect to the state z, the auxiliary input $\tilde{u}$, and the disturbance vector d. Furthermore, performing a standard two-time-scale decomposition, one can easily show that the fast subsystem is given by:

$$\frac{dz}{d\tau} = f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\tilde{u} + W_2(x)d \tag{3.15}$$

while the reduced system takes the form:

$$\begin{aligned}\dot{x} &= \tilde{F}(x) + \tilde{G}(x)\tilde{u} + \tilde{W}(x)d \\ y^s &= h(x)\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\tilde{F}(x) &= f_1(x) - [Q_1(x) + g_1(x)k^T(x)][Q_2(x) + g_2(x)k^T(x)]^{-1}f_2(x) \\ \tilde{G}(x) &= g_1(x) - [Q_1(x) + g_1(x)k^T(x)][Q_2(x) + g_2(x)k^T(x)]^{-1}g_2(x) \\ \tilde{W}(x) &= W_1(x) - [Q_1(x) + g_1(x)k^T(x)][Q_2(x) + g_2(x)k^T(x)]^{-1}W_2(x)\end{aligned} \tag{3.17}$$

Clearly, the z-dependent control law of Eq.3.13 allows stabilizing the fast dynamics of the system by choosing $k^T(x)$ such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$ (assumption 2.1). Proposition 3.1 that follows establishes conditions for the invariance of relative orders r, ρ, under the control law of Eq.3.13. The proof of the proposition can be found in the appendix B.

Proposition 3.1: *Consider the two-time-scale system of Eq.3.14, assumed to be in standard form. Then, referring to the reduced system of Eq.3.16:*

a) the relative order of y^s with respect to the auxiliary input $\tilde{u}$ is equal to r, and

b) the relative order of y^s with respect to the disturbance input vector d is equal to ρ, if $\rho \leq r$.

In order to enforce the desired behavior on the output, let us now consider feedforward/state feedback laws of the form:

$$\tilde{u} = p(x) + q(x)v + Q\left(x, d, d^{(1)}, \cdots\right) \tag{3.18}$$

where $p(x)$, $q(x)$ are analytic scalar functions, with $q(x) \neq 0$ uniformly in $x \in X$, v is the reference input, and Q is a smooth algebraic function which is nonsingular under nominal conditions $(x_0, 0, \cdots)$ and well-defined and finite for all possible smooth functions of time d. Given the asymptotic stability of the fast dynamics of the system of Eq.3.14, the control law of Eq.3.18 uses feedback of the slow state vector x only, to avoid destabilization of the fast dynamics, while dynamic feedforward terms are allowed in order to achieve complete elimination of the effect of d on y in the closed-loop reduced system.

Substitution of the control law of Eq.3.18 into the system of Eq.3.1 yields the following closed-loop system :

$$\begin{aligned}
\dot{x} &= [f_1(x) + g_1(x)p(x)] + [Q_1(x) + g_2(x)k^T(x)]z + g_1(x)q(x)v \\
&\quad + g_1(x)Q\left(x, d, d^{(1)} \cdots\right) + W_1(x)d \\
\epsilon\dot{z} &= [f_2(x) + g_2(x)p(x)] + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)q(x)v \\
&\quad + g_2(x)Q\left(x, d, d^{(1)} \cdots\right) + W_2(x)d \\
y &= h(x)
\end{aligned} \tag{3.19}$$

On the basis of Eq.3.19, it is clear that the control law of Eq.3.18 does not modify the two-time-scale nature of the system, and the linearity with respect to the state vector z and the reference input v. Employing a two-time-scale decomposition for the system of Eq.3.19, it can be easily shown that the closed-loop fast subsystem is given by:

$$\begin{aligned}
\frac{dz}{d\tau} &= [f_2(x) + g_2(x)p(x)] + [Q_2(x) + g_2(x)k^T(x)]z \\
&\quad + g_2(x)[q(x)v + Q\left(x, d, d^{(1)} \cdots\right)] + W_2(x)d
\end{aligned} \tag{3.20}$$

and the closed-loop reduced system takes the form:

$$\begin{aligned}
\dot{x} &= [\tilde{F}(x) + \tilde{G}(x)p(x)] + \tilde{G}(x)[q(x)v + Q\left(x, d, d^{(1)} \cdots\right)] + \tilde{W}(x)d \\
y^s &= h(x)
\end{aligned} \tag{3.21}$$

where $\tilde{F}(x)$, $\tilde{G}(x)$ and $\tilde{W}(x)$ are defined in Eq.3.17. Referring to Eq.3.20, one can immediately see that the feedforward/state feedback law of Eq.3.18 preserves the stability characteristics of the fast dynamics of the original system. Proposition 3.2 that follows will allow formulating the controller synthesis problem.

Proposition 3.2: *Consider the two-time-scale system of Eq.3.14, assumed to be in standard form. Then, a feedforward/static state feedback control law of the form of Eq.3.18 preserves the relative order r, in the sense that the relative order of the output y^s with respect to the reference input v in the closed-loop reduced system of Eq.3.21 is equal to r.*

Proof: The proof of the proposition involves the application of the two-time-scale decomposition procedure to the resulting closed-loop system, and the standard argument for nonlinear systems of the form of Eq.3.21 under feedforward/state feedback of the form of Eq.3.18 (see e.g. [DK93]). △

The result of proposition 3.2 suggests requesting an input/output response of relative order r in the closed-loop reduced system. For simplicity, the following linear input/output response will be postulated:

$$\beta_r \frac{d^r y^s}{dt^r} + \cdots + \beta_1 \frac{dy^s}{dt} + \beta_0 y^s = v \tag{3.22}$$

where $\beta_0, \cdots, \beta_r$ are adjustable parameters which can be chosen to guarantee input/output stability in the closed-loop reduced system and enforce desired performance characteristics.

Theorem 3.1 that follows summarizes the main result of this section. The proof of the theorem can be found in appendix B.

Theorem 3.1: *Consider the two-time-scale nonlinear system of Eq.3.1, assumed to be in standard form. Consider also the reduced system of Eq.3.5, and assume that $\rho < r$. Then, the conditions:*

$$L_G \Phi_\ell (x, d) \equiv 0, \quad \ell = 0, 1, \cdots, r - \rho - 1 \tag{3.23}$$

where

$$\Phi_\ell (x, d) = \sum_{\mu=0}^{\ell} L_F^{\ell-\mu} \left(\sum_{\kappa=1}^{q} d_\kappa(t) L_{W^\kappa} + \frac{\partial}{\partial t} \right) \left(L_F + \sum_{\kappa=1}^{q} d_\kappa(t) L_{W^\kappa} + \frac{\partial}{\partial t} \right)^{\mu} L_F^{\rho-1} h(x) \tag{3.24}$$

are necessary and sufficient in order for a control law of the form of Eq.3.18 to induce an input/output behavior of the form of Eq.3.22 in the closed-loop reduced system.

If these conditions are satisfied and assumption 2.1 also holds, the control law :

$$\begin{aligned} u \;=\; & [1 + k^T(x)[Q_2(x)]^{-1}g_2(x)]\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1} \\ & \bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x) - \sum_{k=\rho}^{r}\beta_k\Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\} \\ & + k^T(x)[Q_2(x)]^{-1}[f_2(x) + W_2(x)d] + k^T(x)z \end{aligned} \tag{3.25}$$

where the feedback gain $k^T(x)$ is such that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$,

a) guarantees exponential stability of the fast dynamics of the closed-loop system.

b) ensures that the output of the closed-loop system satisfies a relation of the form :

$$y(t) \;=\; y^s(t) + O(\epsilon)\,,\; t \geq 0 \tag{3.26}$$

for ϵ sufficiently small, with $y^s(t)$ being the solution of Eq.3.22.

Remark 3.1: The form of the functions defined in Eq.3.24 and the pattern of the conditions of Eq.3.23 can be found in [DK93]. From the result of the theorem, we note that the verification of the solvability conditions of Eq.3.23 and the evaluation of the functions Φ_ℓ, $\ell = 0, 1, \cdots, r - \rho - 1$ are independent of the selection of the feedback gain $k^T(x)$ used for the stabilization of the fast dynamics, and thus, can be performed on the basis of the open-loop reduced system of Eq.3.5. This fact is an immediate implication of the results of propositions 3.1 and 3.2.

Remark 3.2: The feedforward/state feedback law of Eq.3.25 can be decomposed into two separate components: the component, $k^T(x)(z - \xi)$, where ξ denotes a control-dependent quasi-steady-state for the fast dynamics of the closed-loop system, defined as follows:

$$\begin{aligned} \xi \;=\; & -[Q_2(x)]^{-1}\left\{f_2(x) + g_2(x)[\beta_r L_G L_F^{r-1}h(x)]^{-1}\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right.\right. \\ & \left.\left. - \sum_{k=\rho}^{r}\beta_k\Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\} + W_2(x)d\right\} \end{aligned} \tag{3.27}$$

which acts in the fast time-scale and stabilizes the fast dynamics of the system, and the component

$$\left[\beta_r L_G L_F^{r-1} h(x)\right]^{-1} \left\{ v - \sum_{k=0}^{r} \beta_k L_F^k h(x) - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right) \right\} \tag{3.28}$$

which acts in the slow time-scale and addresses the posed synthesis problem, i.e., induces the desired input/output behavior in the closed-loop reduced system independently of the disturbances.

Remark 3.3: In the case of two-time-scale systems of the form of Eq.3.1 with stable fast dynamics, i.e., when the matrix $Q_2(x)$ is Hurwitz uniformly in x, there is no need to employ feedback of the fast variable to stabilize the fast dynamics and the feedforward/feedback law of Eq.3.25 reduces to:

$$u = \left[\beta_r L_G L_F^{r-1} h(x)\right]^{-1} \left\{ v - \sum_{k=0}^{r} \beta_k L_F^k h(x) - \sum_{k=\rho}^{r} \beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right) \right\} \tag{3.29}$$

Finally, motivated by the available results on disturbance decoupling for standard nonlinear systems of the form of Eq.3.5 via static state feedback [IKGGM81] of the form:

$$u = p(x) + q(x)v \tag{3.30}$$

we will now provide necessary and sufficient conditions for the solvability of the control problem formulated in section 3.3, via well-conditioned static state feedback laws of the form:

$$u \quad = \quad p(x) + q(x)v + k^T(x)z \tag{3.31}$$

The main result is given in proposition 3.3 that follows (the proof is included in the appendix B).

Proposition 3.3: *Consider the two-time-scale system of Eq.3.1 in standard form, for which assumption 2.1 holds. Consider also the reduced system of Eq.3.5. Then, the* *conditions i) $r < \rho$, and ii) $k^T(x)[Q_2(x)]^{-1}W_2(x) \equiv \bar{0}$, where $\bar{0}$ is the zero covector of dimension q, are necessary and sufficient in order for a well-conditioned static state feedback law of the form of Eq.3.31 to achieve approximate decoupling of the effect of d on y (in the sense made precise in requirement 2 at section 3.3) with stabilization of the fast dynamics in the closed-loop system.*

Remark 3.4: Condition (i) of the proposition is expected to hold for the solvability of this problem, since it is necessary and sufficient for the solvability of the disturbance decoupling problem via static state feedback of the form of Eq.3.30 (e.g., [Isi89]) and the approximate disturbance decoupling problem for two-time-scale systems of Eq.3.1 with exponentially stable fast dynamics, via the same class of control laws (see [LR91]). Condition (ii) of the proposition guarantees that the relative order of the output y^s with respect to the disturbance input vector d in the reduced system of Eq.3.16, say $\tilde{\rho}$, is $\tilde{\rho} = \rho > r$.

3.5 Disturbance compensation for systems in nonstandard form

In this section, we will consider two-time-scale nonlinear systems of Eq.3.1 in non-standard form, i.e., systems for which the matrix $Q_2(x)$ is singular for some $x \in X$. The direct consequence of the singularity of the matrix $Q_2(x)$ is the absence of a well-defined quasi-steady-state for the fast state variables z [Kha89], and thus, the lack of a well-defined open-loop reduced system. Therefore, the results of propositions 3.1 and 3.2 that allowed the verification of the solvability conditions of Eq.3.23 and the synthesis of the controller of Eq.3.25 on the basis of the open-loop reduced system of Eq.3.5 cannot be recovered. Motivated by this, we will follow a two-step procedure for the synthesis of a feedforward/state feedback law that solves the posed problem. In the first step, appropriate feedback of the state vector z will be employed to regularize the fast dynamics, in the sense of inducing an exponentially stable quasi-steady-state for the fast dynamics. In the second step, we will formulate and solve the synthesis problem on the basis of the resulting two-time-scale system.

More specifically, we will initially consider a control law of the form:

$$u = \hat{u} + k^T(x)z \tag{3.32}$$

where $k^T(x)$ is a vector field in $\mathbb{R}^p$, and $\hat{u}$ is an auxiliary input, to achieve regularization of the fast dynamics. Under the control law of Eq.3.32 the system of Eq.3.1 takes the form:

$$\begin{aligned} \dot{x} &= f_1(x) + [Q_1(x) + g_1(x)k^T(x)]z + g_1(x)\hat{u} + W_1(x)d \\ \epsilon\dot{z} &= f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\hat{u} + W_2(x)d \end{aligned} \tag{3.33}$$

Performing a two-time-scale decomposition, the corresponding fast subsystem takes the form:

$$\frac{dz}{d\tau} = f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\hat{u} + W_2(x)d \tag{3.34}$$

while the corresponding slow subsystem is given by:

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\hat{u} + \tilde{W}(x)d \\ \hat{y}^s &= h(x) \end{aligned} \tag{3.35}$$

where the vector fields $\tilde{F}(x)$, $\tilde{G}(x)$ and $\tilde{W}(x)$ are given in Eq.3.17. It is clear that the z-dependent control law of Eq.3.32 allows us to regularize the fast dynamics of the system by choosing $k^T(x)$ in such a manner so that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$. It is now possible to formulate and solve a controller synthesis problem on the basis of the system of Eq.3.33. More specifically, we will seek a feedforward/state feedback law of the form:

$$\hat{u} = \tilde{p}(x) + \tilde{q}(x)v + \tilde{Q}\left(x, d, d^{(1)}, \cdots\right) \tag{3.36}$$

where $\tilde{p}(x)$, $\tilde{q}(x)$ are scalar fields, with $\tilde{q}(x) \neq 0$ for all $x \in X$, $\tilde{Q}$ is a smooth algebraic nonsingular function, and v is the reference input. Referring to the system of Eq.3.35, let $\hat{r}$ and $\hat{\rho}$ denote the relative orders of the output y with respect to the auxiliary input $\hat{u}$ and the disturbance input vector d, and let the following input/output behavior:

$$\beta_{\hat{r}} \frac{d^{\hat{r}} \hat{y}^s}{dt^{\hat{r}}} + \cdots + \beta_1 \frac{d\hat{y}^s}{dt} + \beta_0 \hat{y}^s = v \tag{3.37}$$

be postulated in the closed-loop reduced system.

Theorem 3.2 that follows summarizes the main result of this section. The proof of the theorem can be found in appendix B.

Theorem 3.2: *Consider the two-time-scale nonlinear system of Eq.3.1, assumed to be in nonstandard form. Consider also the nonlinear system of Eq.3.35, and assume that $\hat{\rho} < \hat{r}$. Then, the conditions:*

$$L_{\hat{G}} \tilde{\Phi}_\ell (x, d) \equiv 0, \quad \ell = 0, 1, \cdots, \hat{r} - \hat{\rho} - 1 \tag{3.38}$$

where

$$\tilde{\Phi}_\ell (x, d) = \sum_{\mu=0}^{\ell} L_{\hat{F}}^{\ell-\mu} \left(\sum_{\kappa=1}^{q} d_\kappa(t) L_{\hat{W}^\kappa} + \frac{\partial}{\partial t} \right) \left(L_{\hat{F}} + \sum_{\kappa=1}^{q} d_\kappa(t) L_{\hat{W}^\kappa} + \frac{\partial}{\partial t} \right)^{\mu} L_{\hat{F}}^{\hat{\rho}-1} h(x) \tag{3.39}$$

are necessary and sufficient in order for a control law of the form of Eq.3.36 to induce the input/output behavior of the form of Eq.3.37 in the closed-loop reduced system. If these conditions are satisfied and assumption 2.1 also holds, the control law:

$$\begin{aligned} u \;=\; & \left[\beta_{\hat{r}} L_{\hat{G}} L_{\hat{F}}^{\hat{r}-1} h(x)\right]^{-1} \left\{ v - \sum_{k=0}^{\hat{r}} \beta_k L_{\hat{F}}^{k} h(x) - \sum_{k=\hat{\rho}}^{\hat{r}} \beta_k \tilde{\Phi}_{k-\hat{\rho}} \left(x, d, d^{(1)}, \cdots, d^{(k-\hat{\rho})}\right) \right\} \\ & + k^T(x) z \end{aligned} \tag{3.40}$$

where the feedback gain $k^T(x)$ is such that the matrix $Q_2(x) + g_2(x) k^T(x)$ is Hurwitz uniformly in $x \in X$,

a) guarantees exponential stability of the fast dynamics of the closed-loop system,

b) ensures that the output of the closed-loop system satisfies a relation of the form :

$$y(t) \;=\; \hat{y}^s(t) + O(\epsilon), \; t \geq 0 \tag{3.41}$$

for ϵ sufficiently small, with $\hat{y}^s(t)$ being the solution of Eq.3.37.

Remark 3.5: The result of theorem 3.3 reveals a fundamental difference in the nature of the control problem between two-time-scale systems in standard and nonstandard form. In particular, in the case of systems in standard form the solvability conditions of the problem can be checked in the open-loop reduced system of Eq.3.5 since they are independent of the feedback gain $k^T(x)$ used to stabilize the fast dynamics. On the other hand, in the case of systems in nonstandard form the solvability conditions should be checked in the reduced system of Eq.3.35, and thus depend explicitly on $k^T(x)$ used to regularize the fast dynamics. These considerations imply that for systems in nonstandard form, it is possible, depending on the structure of the system under consideration, to select the feedback gain $k^T(x)$ to guarantee stabilization of the fast dynamics as well as to ensure that the solvability conditions are satisfied.

Remark 3.6: In the case of two-time-scale systems of the form of Eq.3.1 in non-standard form, one can show that the condition $\hat{r} < \hat{\rho}$ is necessary and sufficient for achieving approximate decoupling of the effect of the disturbances on the output of the closed-loop full-order system via well-conditioned static state feedback of the form of Eq.3.31. Notice that this condition depends explicitly on the feedback gain $k^T(x)$ utilized to regularize the fast dynamics, because of the lack of an analogue of proposition 3.1 in the case of two-time-scale systems in nonstandard form.

Remark 3.7: The control algorithms of theorems 3.1 and 3.2 can be directly applied to two-time-scale systems with ϵ-dependent right-hand side, with the following state-space description:

$$\begin{array}{rcl} \dot{x} & = & f_1(x,\epsilon z,\epsilon) + Q_1(x,\epsilon z,\epsilon)z + g_1(x,\epsilon z,\epsilon)u + W_1(x,\epsilon z,\epsilon)d \\ \epsilon\dot{z} & = & R(x,z)[f_2(x,\epsilon z,\epsilon) + Q_2(x,\epsilon z,\epsilon)z + g_2(x,\epsilon z,\epsilon)u + W_2(x,\epsilon z,\epsilon)d] \\ y & = & h(x) \end{array} \quad (3.42)$$

where $R(x,z)$ is a diagonal matrix of dimension $p \times p$, which is positive definite for all $x \in X$, $z \in Z$. The reasons for which this is possible were stated in remark 2.7.

3.6 Stability analysis of the closed-loop system

In this section, the stability of the closed-loop full-order system under the derived control laws will be analyzed. Our objective is to specify sufficient conditions and provide an explicit formula for the calculation of the upper bound on ϵ, such that the states of the closed-loop system are bounded. The presence of time-varying disturbances in the model of the system does not allow utilizing standard stability results for two-time-scale systems [SK84, Kha92] (which are concerned with asymptotic stability) and requires further analysis of the closed-loop system to show boundedness of the states. Theorem 3.3 that follows states the main stability result for two-time-scale systems in standard form, under the control law of Eq.3.25 (the proof is given in appendix B).

Theorem 3.3: *Consider the two-time-scale nonlinear system with the state-space representation of Eq.3.1, assumed to be in standard form. Then, if $d(t)$ and its derivatives up to order $r - \rho + 1$ are sufficiently small and the following conditions hold:*

1. *The roots of the polynomial*

$$\beta_0 + \beta_1 s + \cdots + \beta_r s^r = 0 \tag{3.43}$$

   lie in the open left-half of the complex plane.

2. *The unforced zero dynamics of the open-loop reduced system of Eq.3.5 is exponentially stable.*

there exists an ϵ^* such that the trajectories of the closed-loop system under the controller of theorem 3.1 remain bounded for all $\epsilon \in (0, \epsilon^*]$.

The result of theorem 3.3 also holds for two-time-scale systems in nonstandard form.

with the second condition imposed on the reduced system of Eq.3.35. The proof of this result is completely analogous to the one of theorem 3.3 and thus is omitted for brevity. Finally, we note that the stability result of theorem 3.3 holds even in the presence of sufficiently small constant parametric uncertainty and unmeasured disturbances (see the proof of the theorem for the detailed justification).

3.7 Application to a catalytic continuous stirred tank reactor

In this section, the proposed control methodology will be applied to a representative chemical process with time-scale multiplicity. Consider the catalytic continuous stirred tank reactor shown in Figure 3.1, where a homogeneous reaction $A \rightarrow B$ and a catalytic reaction $A \rightarrow C$ take place. The first reaction leads to the generation of the side-product B, while the second reaction leads to the production of the desired product C. The inlet stream F_1 consists of pure species A of concentration C_{A0}, and temperature T_{A0}. Under the following assumptions:

- Perfect mixing in the reactor
- Uniform temperature in the homogeneous and the catalytic phase
- Constant density and heat capacity of the reacting liquid and the catalyst
- The outlet flow rate of the reactor is constant

the process dynamic model consists of the following set of material and energy balances:

- Reactor mass balance:

$$\frac{dV_r}{dt} = F_1 - F_2 \tag{3.44}$$

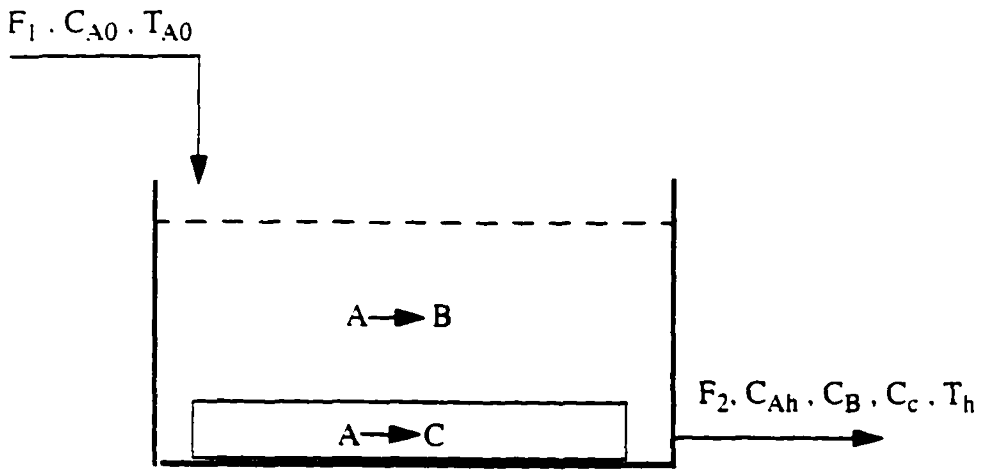

Figure 3.1: A catalytic continuous stirred tank reactor.

- Mole balance for the species A (homogeneous phase):

$$\frac{dC_{A_h}}{dt} = \frac{1}{V_r}\left(F_1(C_{A0} - C_{A_h}) - k_h \exp(\frac{-E_h}{RT_h})V_r - K_c A_c (C_{A_h} - C_{A_c})\right) \quad (3.45)$$

- Reactor energy balance (homogeneous phase):

$$\begin{aligned} \rho_h c_{p_h} \frac{dT_h}{dt} &= \frac{1}{V_r}\Big(\rho_h c_{p_h} F_1 (T_{A0} - T_h) + (-\Delta H_h) k_h \exp(\frac{-E_h}{RT_h}) V_r \\ &\quad - U_w A_w (T_h - T_w) - U_c A_c (T_h - T_c)\Big) \end{aligned} \quad (3.46)$$

- Mole balance for the species A (catalytic phase):

$$\frac{dC_{A_c}}{dt} = \frac{K_c A_c}{V_c}(C_{A_h} - C_{A_c}) - k_c \exp(\frac{-E_c}{RT_c}) C_{A_c} \quad (3.47)$$

- Reactor energy balance (catalytic phase):

$$\rho_c c_{p_c} \frac{dT_c}{dt} = \frac{U_c A_c}{V_c}(T_h - T_c) + (-\Delta H_c) k_c \exp(\frac{-E_c}{RT_c}) C_{A_c} \quad (3.48)$$

where F_1, F_2 denote the inlet and outlet flow rates, V_r, V_c, denote the volumes of the homogeneous and catalytic phase, C_{A_h}, T_h and C_{A_c}, T_c denote the concentration and temperature of the species A in the homogeneous phase and catalytic phase, k_h, k_c, E_h, E_c, ΔH_h, ΔH_c denote the pre-exponential factors, the activation energies and the enthalpies of the two reactions, K_c and U_c, U_w denote mass and heat transfer coefficients, and A_w, A_c denote the surface of the wall and the catalyst.

The control objective is the regulation of the temperature of the catalyst by manipulating the inlet flow rate F_1, in order to maintain the generation of the product species C at the desired level. The inlet concentration and temperature of the species A, C_{A0} and T_{A0}, respectively, as well as the wall temperature T_w are assumed to be the measurable disturbances. The values of the system parameters and the corresponding steady-state values of the system variables are given in Table 3.1. It was verified that these conditions correspond to a critically stable equilibrium point (the holdup of the homogeneous phase acts as an integrator). The process exhibits two-time-scale behavior owing to the large heat capacity of the catalytic phase (i.e. the

F_2	=	500.0	$lt\ min^{-1}$
c_{ph}	=	0.231	$kcal\ kg^{-1}\ K^{-1}$
c_{pc}	=	2.31	$kcal\ kg^{-1}\ K^{-1}$
ρ_h	=	0.9	$kg\ lt$
ρ_c	=	90.0	$kg\ lt$
$K_c A_c$	=	1618.0	$lt\ min^{-1}$
$U_c A_c$	=	6667.0	$kcal\ min^{-1}\ K^{-1}$
$U_w A_r$	=	340.0	$kcal\ min^{-1}\ K^{-1}$
R	=	1.987	$kcal\ kmol^{-1}\ K^{-1}$
k_h	=	164.68	$lt\ mol^{-1} min^{-1}$
E_h	=	8.0×10^3	$kcal\ kg^{-1}$
k_c	=	2000.0	min^{-1}
E_c	=	9.0×10^3	$kcal\ kg^{-1}$
ΔH_h	=	69.2006	$kcal\ kmol^{-1}$
ΔH_c	=	−99.0781	$kcal\ kmol^{-1}$
V_c	=	145.1	lt
V_s	=	1000.0	lt
C_{Ah}	=	5.0	$mol\ lt^{-1}$
T_h	=	690	K
C_{Ac}	=	3.75	$mol\ lt^{-1}$
T_c	=	720	K
F_{1s}	=	500	$lt\ min^{-1}$
C_{A0s}	=	10.0	$mol\ lt^{-1}$
T_{A0s}	=	305.0	K
T_{ws}	=	310.0	K

Table 3.1: Process parameters and steady-state values

term $\rho_c c_{p_c}$ that multiplies the time-derivative of the catalyst temperature is large). This implies that $V_r, C_{A_h}, T_h, C_{A_c}$ are the fast process variables, while T_c is the slow process variable (this fact was also validated through open-loop simulations). It was also verified that the process exhibits slightly non-minimum phase behavior, i.e., its zero dynamics exhibits a two-time-scale behavior with critically stable fast dynamics (the fast process modes are part of the zero dynamics modes). In order to obtain a singularly perturbed representation of the process, where the partition to slow and fast variables is consistent with the dynamic behavior of the process, the parameter ϵ is defined as:

$$\epsilon = \frac{1}{\rho_c c_{p_c}} = 0.048 \frac{K\,lt}{Kcal} \tag{3.49}$$

Setting:

$$x_1 = T_c,\;\; z_1 = V_r,\;\; z_2 = C_{A_h},\;\; z_3 = T_h,\;\; z_4 = C_{A_c},\;\; u = F_1 - F_{1s}$$

$$d_1 = C_{A0} - C_{A0s},\;\; d_2 = T_{A0} - T_{A0s},\;\; d_3 = T_w - T_{ws},\;\; y = x_1,\;\; \bar{t} = \frac{t}{\rho_c c_{p_c}}$$

the original set of equations can be put in the following singularly perturbed form (where the time-derivatives are taken with respect to $\bar{t}$):

$$\begin{aligned}
\dot{x}_1 &= \frac{U_c A_c}{V_c}(z_3 - x_1) + (-\Delta H_c) k_c \exp(\frac{-E_c}{R x_1}) z_4 \\
\epsilon \dot{z}_1 &= u \\
\epsilon \dot{z}_2 &= \frac{1}{z_1}\Big(F_{1s} C_{A0s} + -k_h \exp(\frac{-E_h}{R z_3}) z_1 + (-F_{1s} - K_c A_c) z_2 + K_c A_c z_3 \\
&\quad + (C_{A0} - z_2) u + F_{1s} d_1\Big) \\
\epsilon \dot{z}_3 &= \frac{1}{z_1}\Big(F_{1s}(T_{A0} - z_3) - \frac{U_w A_w}{\rho_h c_{p_h}}(z_3 - T_{ws}) - \frac{U_c A_c}{\rho_h c_{p_h}}(z_3 - x_1) \\
&\quad + \frac{(-\Delta H_h)}{\rho_h c_{p_h}} k_h \exp(\frac{-E_h}{R z_3}) z_1 + (T_{A0} - z_3) u + F_{1s} d_2 + \frac{U_w A_w}{\rho_h c_{p_h}} d_3\Big) \\
\epsilon \dot{z}_4 &= \frac{K_c A_c}{V_c} z_2 + (-\frac{K_c A_c}{V_c} - k_c \exp(\frac{-E_c}{R x_1})) z_4
\end{aligned} \tag{3.50}$$

From the structure of the differential equation of z_1 in the above system, it is clear that fast dynamics of the process are singular i.e. there exists no well-defined quasi-steady-state for the fast state vector z (this is an implication of the assumption that the outlet flow rate is constant). Since the process is in nonstandard form, the two-step procedure of section 3.5 will be followed for the synthesis of the controller. In the first step, the regularizing feedback law of the form:

$$u = \hat{u} - z_1 \tag{3.51}$$

was used to transform the original two-time-scale system into a new one in standard form with exponentially stable fast dynamics. Under this preliminary feedback the original system takes the form (where the time-derivatives are taken with respect to $\bar{t}$):

$$\begin{aligned}
\dot{x}_1 &= \frac{U_c A_c}{V_c}(z_3 - x_1) + (-\Delta H_c) k_c \exp(\frac{-E_c}{R x_1}) z_4 \\
\epsilon \dot{z}_1 &= -z_1 + \hat{u} \\
\epsilon \dot{z}_2 &= \frac{1}{z_1}\left(F_{1s} C_{A0s} + \left(C_{A0} - z_2 - k_h \exp(\frac{-E_h}{R z_3})\right) z_1 + (-F_{1s} - K_c A_c) z_2\right. \\
&\quad \left. + K_c A_c z_3 + (C_{A0} - z_2)\hat{u} + F_{1s} d_1\right) \\
\epsilon \dot{z}_3 &= \frac{1}{z_1}\left(F_{1s}(T_{A0} - z_3) - \frac{U_w A_w}{\rho_h c_{p_h}}(z_3 - T_{ws}) - \frac{U_c A_c}{\rho_h c_{p_h}}(z_3 - x_1) + (T_{A0} - z_3)\hat{u}\right. \\
&\quad \left. + \left(T_{A0} - z_3 + \frac{(-\Delta H_h) k_h}{\rho_h c_{p_h}} \exp(\frac{-E_h}{R z_3})\right) z_1 + F_{1s} d_2 + \frac{U_w A_w}{\rho_h c_{p_h}} d_3\right) \\
\epsilon \dot{z}_4 &= \frac{K_c A_c}{V_c} z_2 + \left(-\frac{K_c A_c}{V_c} - k_c \exp(\frac{-E_c}{R x_1})\right) z_4
\end{aligned} \tag{3.52}$$

Setting $\epsilon = 0$ in the above system and using the fact that for a physically meaningful problem $z_1 = V_r > 0$, the following set of differential and algebraic equations can be obtained:

$$
\begin{aligned}
\dot{x}_1 &= \frac{U_cA_c}{V_c}(z_{3s} - x_1) + (-\Delta H_c)k_c \exp(\frac{-E_c}{Rx_1})z_{4s} \\
0 &= F_{1s}C_{A0s} + \left(C_{A0} - z_{2s} - k_h \exp(\frac{-E_h}{Rz_{3s}})\right)\hat{u} + (-F_{1s} - K_cA_c)z_{2s} \\
&\quad + K_cA_cz_{3s} + (C_{A0} - z_{2s})\hat{u} + F_{1s}d_1 \\
0 &= F_{1s}(T_{A0} - z_{3s}) - \frac{U_wA_w}{\rho_h c_{p_h}}(z_{3s} - T_{ws}) - \frac{U_cA_c}{\rho_h c_{p_h}}(z_{3s} - x_1) \\
&\quad + \left(\frac{(-\Delta H_h)k_h}{\rho_h c_{p_h}}\exp(\frac{-E_h}{Rz_{3s}})\right)\hat{u} + F_{1s}d_2 + \frac{U_wA_w}{\rho_h c_{p_h}}d_3 \\
0 &= \frac{K_cA_c}{V_c}z_{2s} + \left(-\frac{K_cA_c}{V_c} - k_c \exp(\frac{-E_c}{Rx_1})\right)z_{4s}
\end{aligned}
\tag{3.53}
$$

Owing to the nonlinear appearance of the algebraic variable z_{3s} in the above model, the equilibrium manifold of the fast dynamics is of the form $z_s = \tilde{g}(x_1, \hat{u}, d_1, d_2, d_3)$, where $\tilde{g}$ is a smooth vector function. The slow subsystem takes then the form:

$$
\begin{aligned}
\dot{x}_1 &= \frac{U_cA_c}{V_c}(\tilde{g}_3(x_1, \hat{u}, d_1, d_2, d_3) - x_1) + \\
&\quad (-\Delta H_c)k_c \exp(\frac{-E_c}{Rx_1})\tilde{g}_4(x_1, \hat{u}, d_1, d_2, d_3) =: \mathcal{D}(x_1, \hat{u}, d_1, d_2, d_3)
\end{aligned}
\tag{3.54}
$$

where $\tilde{g}_3 = z_{3s}$, $\tilde{g}_4 = z_{4s}$, and $\mathcal{D}(x_1, \hat{u}, d_1, d_2, d_3)$ is a nonlinear function. On the basis of this system, it is clear that the relative orders are $\hat{r} = 1$, $\hat{\rho} = 1$. Due to the nonlinear dependence of the auxiliary input $\hat{u}$ and the disturbances d_1, d_2, d_3 in the system of Eq.3.54, the synthesis formula of theorem 3.2 could not be readily used. Instead, the auxiliary input $\hat{u}$ was computed numerically by solving the nonlinear equation $\mathcal{D}(x_1, \hat{u}, d_1, d_2, d_3) = \dfrac{v - \beta_0 x_1}{\beta_1}$ for $\hat{u}$, so that the following response:

$$
\beta_1 \dot{y}^s + \beta_0 y^s = v \tag{3.55}
$$

is enforced in the closed-loop reduced system, where the parameters β_0, β_1 were chosen to be:

$$
\beta_0 = 1.0 \quad , \quad \beta_1 = 1.87 \ min
$$

Several simulations were performed to evaluate the performance of the controller.

In the first set of simulation runs, we addressed the capability of the controller to maintain the output of the system at the operating steady-state in the presence of the following step disturbances $d_1 = -0.8\ mol/lt$, $d_2 = -5.0\ K$, $d_3 = 5.0\ K$, imposed at $t = 0\ min$. The corresponding output and input profiles are shown in Figure 3.2; clearly, the controller guarantees stability of the fast dynamics and regulates the output at the steady-state, attenuating the effect of the disturbances. In the next set of simulation runs, we evaluated the reference input tracking capabilities of the controller in the presence of the above constant disturbances. A 10 K increase in the value of the reference input was imposed at time $t = 0\ min$. Figure 3.3 displays the output and input profiles. The performance of the controller is excellent, guaranteeing the stability of the fast dynamics, regulating the output at the new reference input value and compensating for the effect of the disturbances. For the sake of comparison, we implemented the controller without measurements of the disturbances. Figure 3.4 depicts the output and input profiles. One can immediately see that the controller cannot attenuate the effect of the disturbance leading to offset. A feedforward/state feedback controller [DK93] synthesized on the basis of the original two-time-scale system was also employed in the simulations. Figure 3.5 shows the output profile and the profile of the volume of the reactor. Clearly, this controller leads to instability of the closed-loop system because the process exhibits a slightly non-minimum phase behavior.

In the next set of simulation runs, we evaluated the capability of the controller to keep the output of the system at the operating steady-state in the presence of time-varying disturbances. In particular, the following disturbances were imposed at $t = 0\ min$:

$$d_1 = -0.8 + 0.5sin(\frac{2\pi}{T}t)\ mol/lt\ ,\ d_2 = -5.0 + sin(\frac{2\pi}{T}t)\ K\ ,\ d_3 = 5.0 + sin(\frac{2\pi}{T}t)\ K$$

where $T = 0.2\ min$. Figure 3.6 shows the output and input profiles. Clearly, the

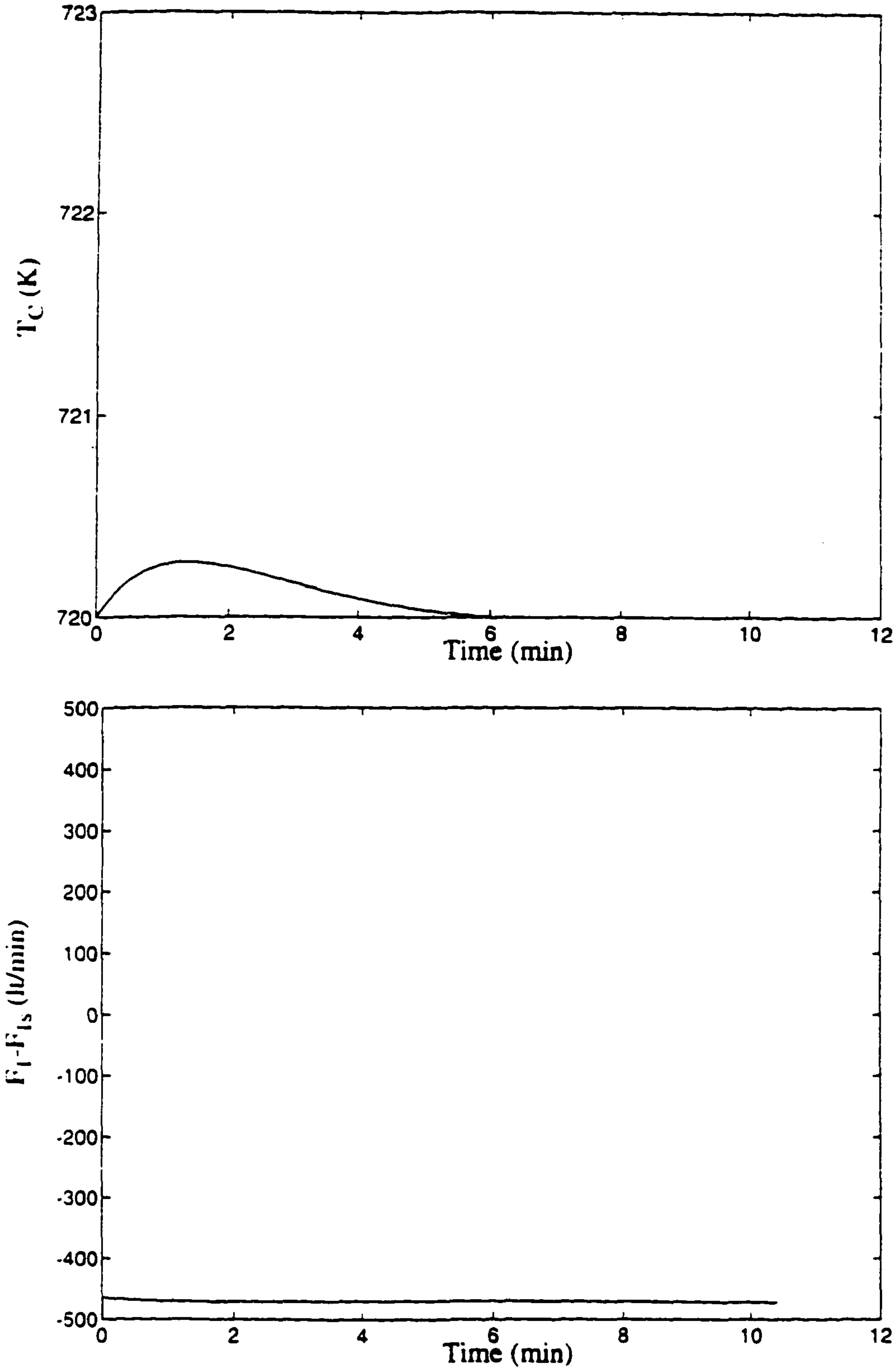

Figure 3.2: Output and input profiles for regulation in the presence of constant disturbances.

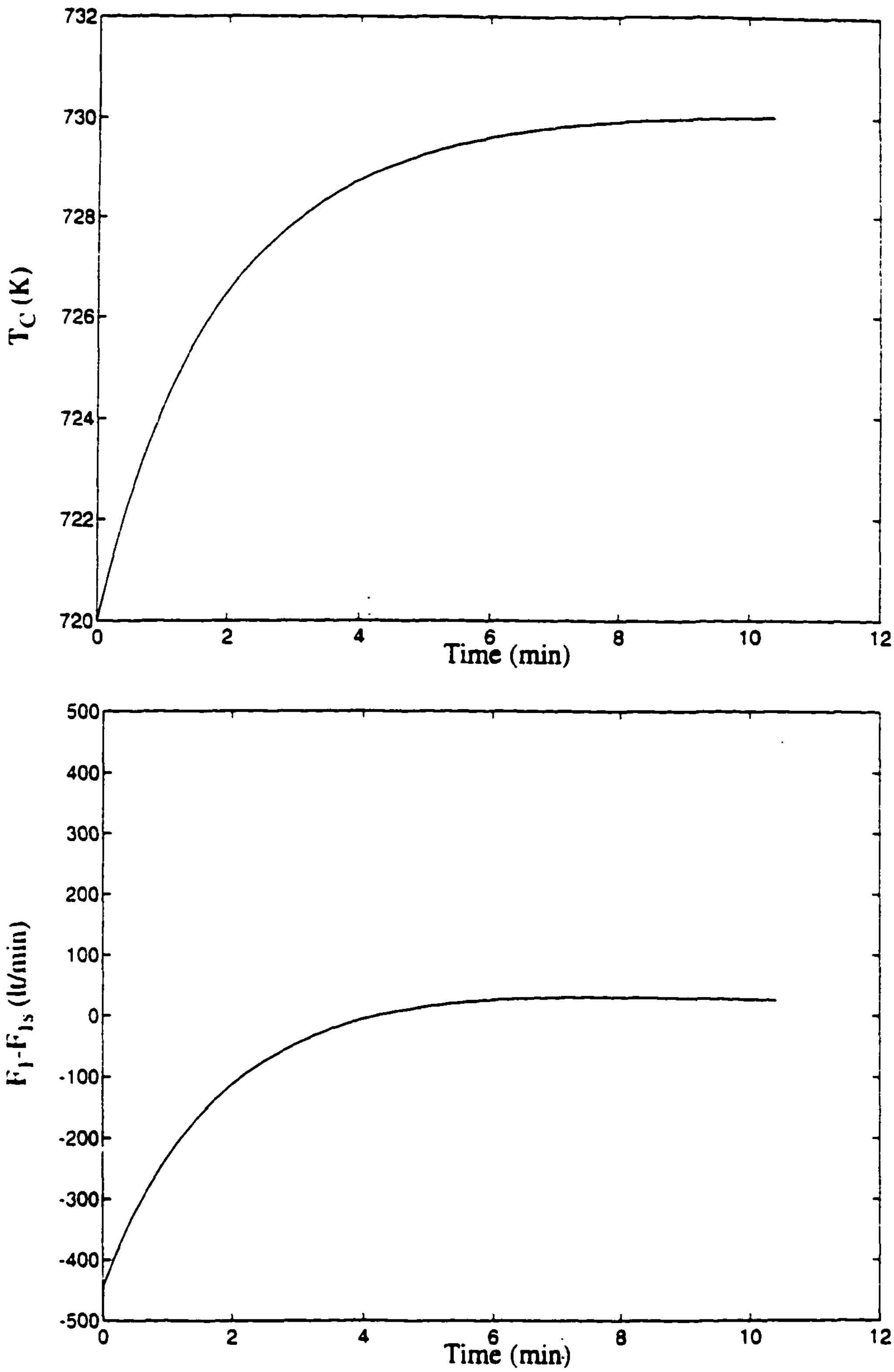

Figure 3.3: Output and input profiles for reference input tracking in the presence of constant disturbances.

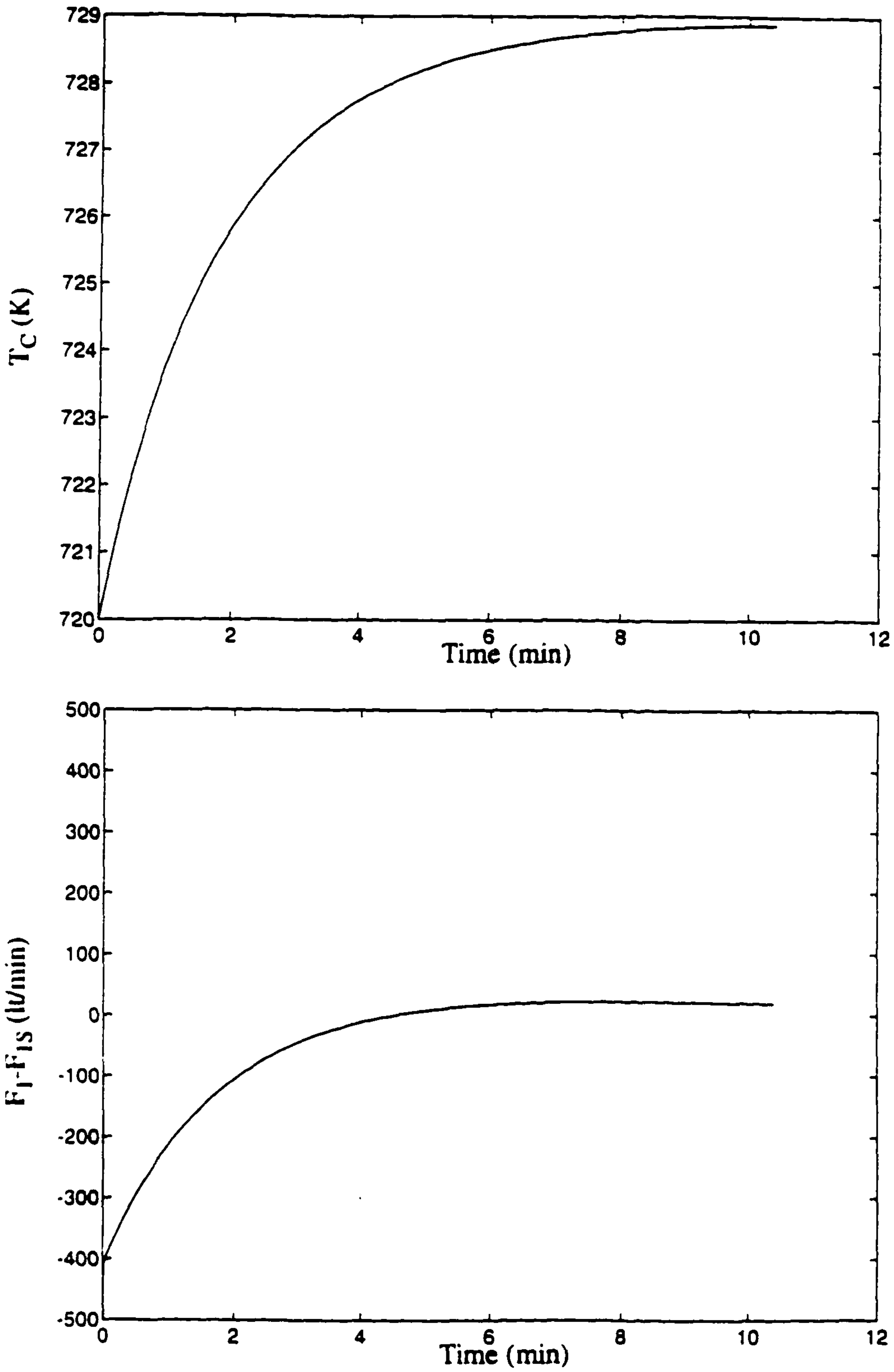

Figure 3.4: Output and input profiles for reference input tracking in the presence of constant disturbances (pure feedback controller).

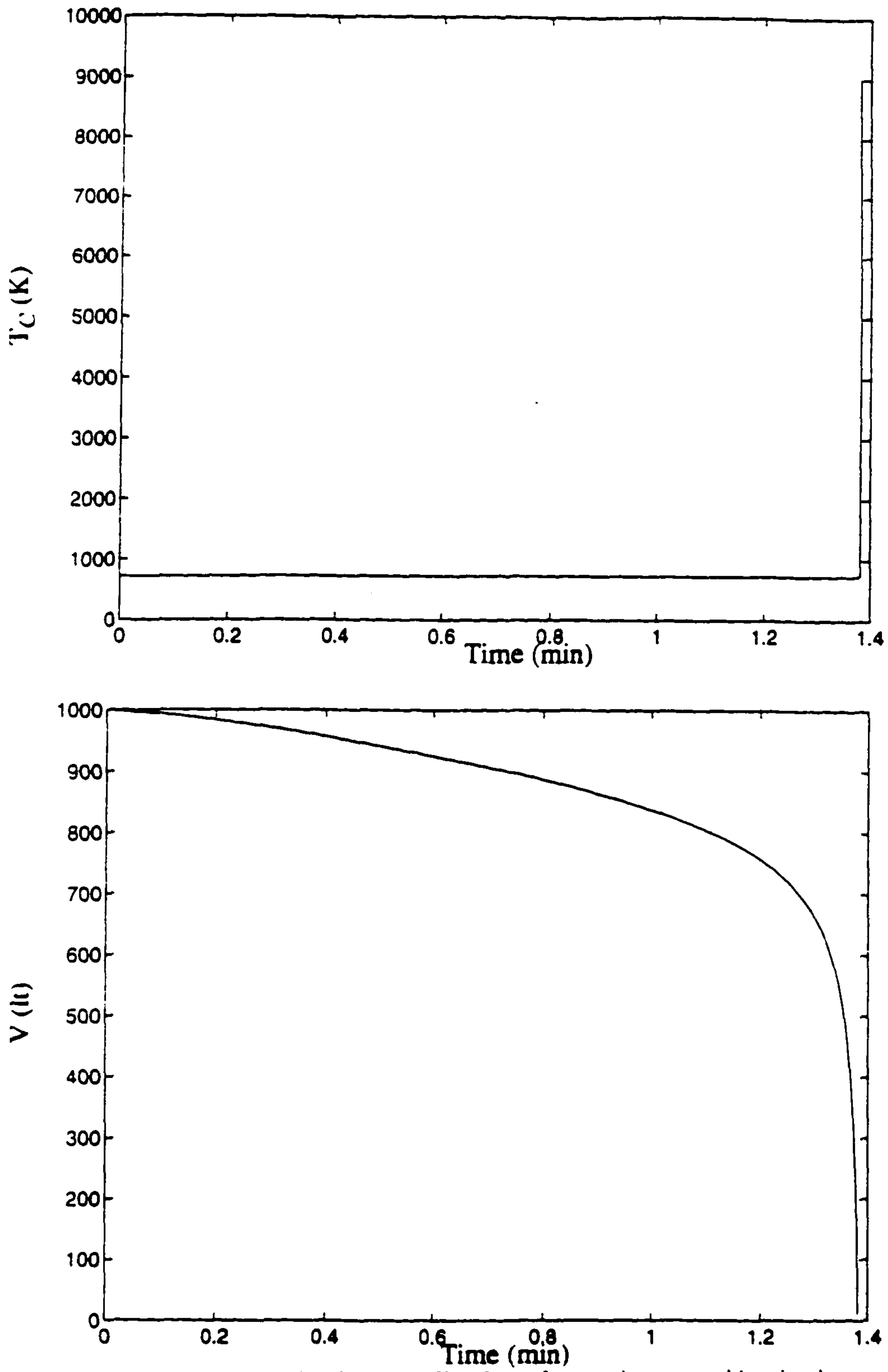

Figure 3.5: Output and volume profiles for reference input tracking in the presence of constant disturbances (feedforward/feedback input/output linearizing controller).

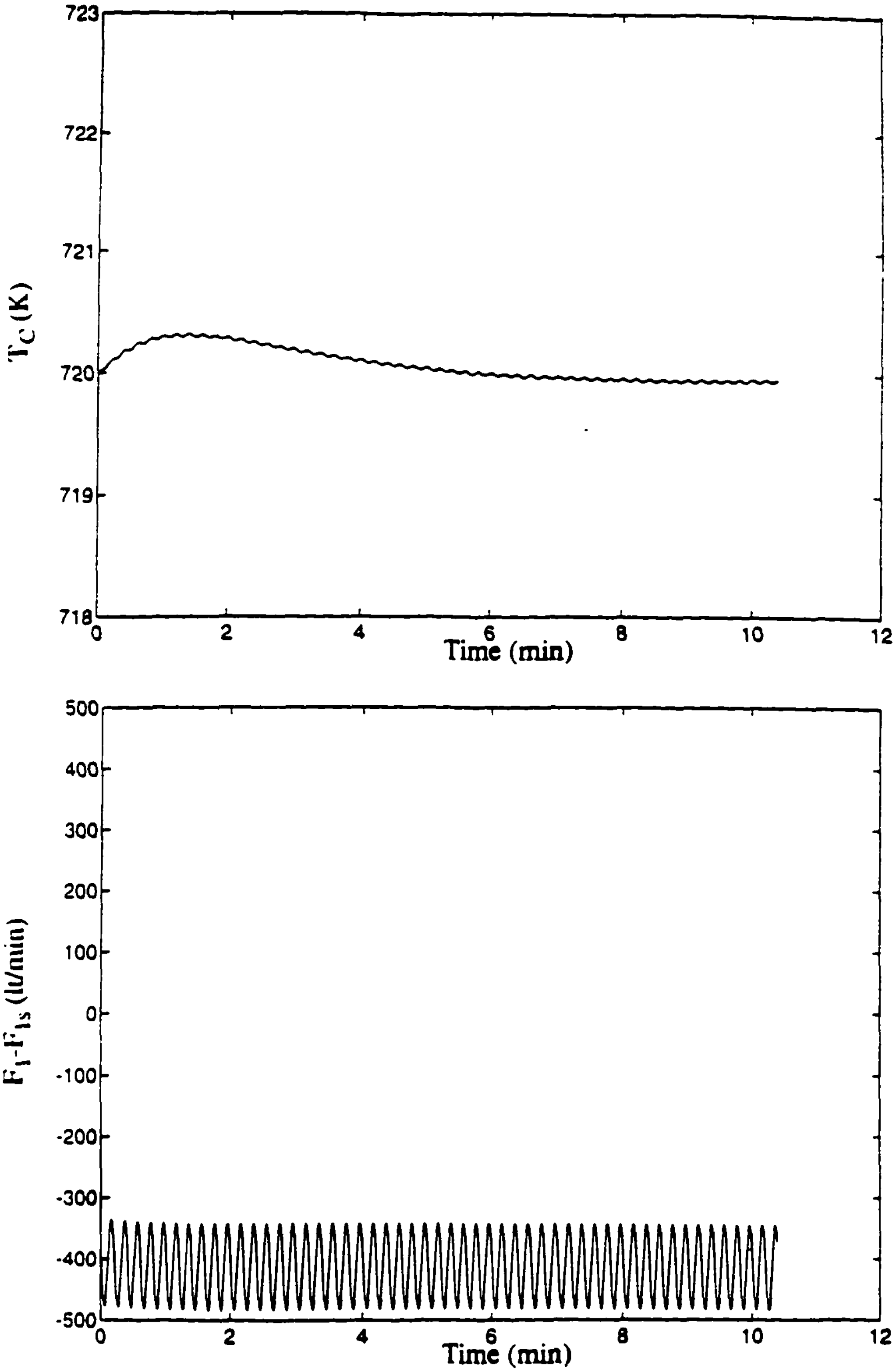

Figure 3.6: Output and input profiles for regulation in the presence of time-varying disturbances.

controller performance is very satisfactory guaranteeing the stability of the fast dynamics. regulating the output at the steady-state and attenuating the effect of the disturbances. Finally. the servo behavior of the controller was evaluated in the presence of the above time-varying disturbances. Figure 3.7 illustrates the resulting output and input profiles. As expected. the controller stabilizes the fast dynamics. while regulating the output at the new reference input value and attenuating the effect of the disturbances on the output.

Remark 3.8: Note that the implementation of the controller requires measurements of the volume of the reactor and the temperature of the catalytic phase for the feedback component. and measurements of the three disturbances for the feedforward component. It is important to compare these requirements with the ones of control methods that do not take explicitly into account the multiple-time-scale behavior exhibited by the process under consideration. For example. let us consider a feedforward/state feedback controller synthesized on the basis of the full-order system [DK93]. The explicit form of the controller takes the form:

$$
\begin{aligned}
u \;=\; & [\beta_2 L_{g_1} L_{f_1} h(x)]^{-1} \\
& \Bigg\{ y - \beta_0 h(x) - \beta_1 [L_{f_1} h(x) + L_{Q_1} h(x) z + \sum_{\kappa=1}^{3} L_{W_1^\kappa} L_{f_1} h(x) d_\kappa(t)] \\
& - \beta_2 [L_{f_1}^2 h(x) + L_{Q_1} L_{f_1} h(x) z + (L_{f_1} L_{Q_1} h(x) + L_{Q_1}^2 h(x) + L_{g_1} L_{Q_1} h(x)) z \\
& + \frac{L_{Q_1} h(x)(f_2(x) + Q_2(x) z + W_2(x) d)}{\epsilon}] \Bigg\}
\end{aligned}
\tag{3.56}
$$

which becomes singular as $\epsilon \rightarrow 0$. It can be also easily verified that the implementation of this control law requires measurements of the full state vector of the process (feedback component) as well as measurements of the three disturbance inputs (feedforward component). This includes measurements of concentrations of the species A both in homogeneous and catalytic phase. which are difficult to obtain in practice. $\triangle$

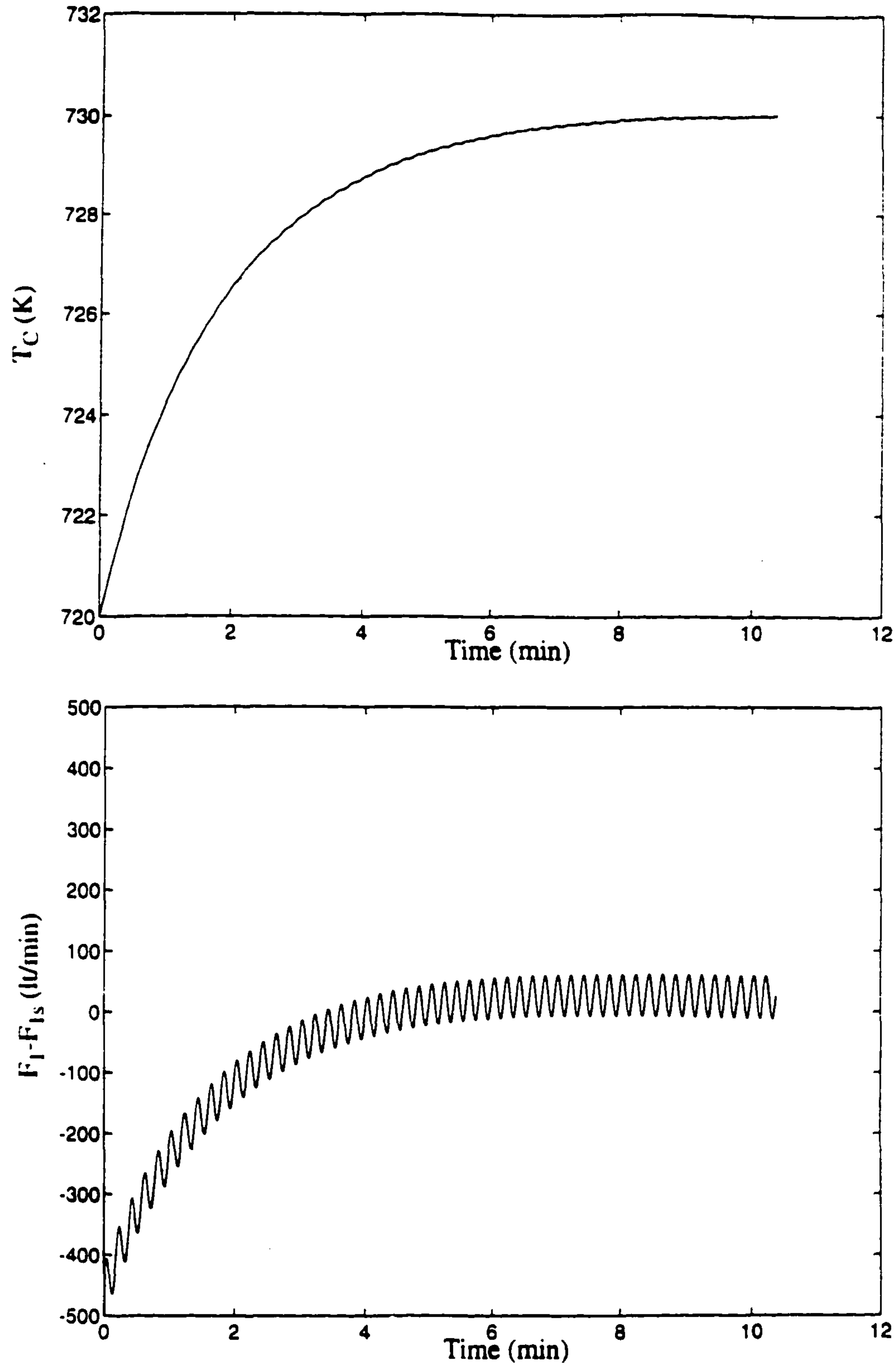

Figure 3.7: Output and input profiles for reference input tracking in the presence of time-varying disturbances.

From the results of the simulation study and the observations of remark 3.8, it is obvious that the consideration of the two-time-scale nature of the process in question at the modeling and control level, allows simplifying the synthesis and implementation of the control system as well as controlling effectively the process.

3.8 Conclusions

In this chapter, a class of nonlinear two-time-scale control systems with external time-varying disturbances was considered. Systems in both standard and nonstandard form were studied. For such systems, we synthesized well-conditioned feedforward/state feedback laws that guarantee exponential stability of the fast dynamics and induce a prespecified input/output response in the closed-loop system independently of the disturbances in the limit as $\epsilon \to 0$. Utilizing singular perturbation methods, we established that the discrepancy between the output of the closed-loop full-order system and the output of closed-loop reduced system is of $O(\epsilon)$, for sufficiently small values of the singular perturbation parameter. We also identified and discussed fundamental differences in the nature of the control problem between systems in standard and nonstandard form. Lyapunov's direct method was used to study the stability properties of the closed-loop system and derive precise conditions that guarantee boundedness of the trajectories. Finally, the derived control methodology was successfully implemented on a catalytic reactor, modeled by a singularly perturbed system in nonstandard form. Comparison with an inversion-based control method, which does not take into account the time-scale multiplicity exhibited by the process, established that the developed control methodology yields significantly superior performance.

Notation

Roman Letters

A_c = surface of the catalyst
A_r = surface of the reactor
C_{A0} = inlet concentration of the species A
C_{Ah} = concentration of the species A in the homogeneous phase
C_{Ac} = concentration of the species A in the catalytic phase
c_{ph} = heat capacity of the homogeneous phase
c_{pc} = heat capacity of the catalytic phase
E_h, E_c = activation energies
d = disturbance input vector
$F, \tilde{F}, f_1, f_2$ = vector fields
F_1 = inlet flow rate
F_2 = outlet flow rate
$G, \tilde{G}, g_1, g_2$ = vector fields associated with the manipulated input
h = output scalar function
K_c = mass transfer coefficient
k^T = covector field
k_h, k_c = pre-exponential constants
L = Lyapunov function
Q_1, Q_2 = matrices associated with the fast state vector z
$r, \hat{r}$ = relative orders with respect to the input in the reduced system
S = matrix
T_{A0} = inlet temperature of the species A
T_h = temperature of the homogeneous phase
T_c = temperature of the catalytic phase
T_w = temperature of the wall
t = time
U_c, U_w = heat transfer coefficients
u = input
$\hat{u}, \check{u}$ = auxiliary inputs
V = Lyapunov function
V_r = volume of the homogeneous phase
V_c = volume of the catalytic phase
$W, \tilde{W}, W_1, W_2$ = matrices associated with the disturbance inputs
x = vector of the slow state variables
y = output
y^s = output associated with slow subsystem
z = vector of fast state variables

Greek Letters

β_k = adjustable parameters
$\Delta H_h, \Delta H_c$ = enthalpy of the reactions
ϵ = singular perturbation parameter
ζ, ζ^s = state vectors in normal form coordinates
η = state vector in normal form coordinates
η_f = fast state vector
$\xi, \hat{\xi}$: = equilibrium steady-states for the fast dynamics in the closed-loop system
$\rho, \hat{\rho}, \bar{\rho}$ = relative order with respect to the disturbance vector in reduced system
ρ_h = density of the homogeneous phase
ρ_c = density of the catalytic phase
$\Phi_i, \tilde{\Phi}_i$ = auxiliary functions
Φ_ζ, Φ_η = matrices of smooth functions
$\psi, \Psi, \bar{\Psi}$ = auxiliary functions
Ω = Lyapunov function
$\bar{\Omega}$ = compact set

Math Symbols

$L_f h$	=	Lie derivative of a scalar field h with respect to the vector field f
$L_f^k h$	=	k-th order Lie derivative
$L_g L_f^{k-1} h$	=	mixed Lie derivative
$\mathbb{R}$	=	real line
$\mathbb{R}^i$	=	i-dimensional Euclidean space
$\in$	=	belongs to
T	=	transpose
$\lvert\cdot\rvert$	=	standard Euclidean norm
$\bar{0}$	=	zero covector of dimension q

Chapter 4

Robust Control of Nonlinear Two-Time-Scale Systems

4.1 Introduction

All practical control systems must be robust with respect to uncertainty. Uncertain time-varying variables arise naturally in chemical engineering applications from unknown or partially known process parameters (e.g. heat transfer coefficients, kinetic constants, etc.) and unmeasured disturbance inputs (e.g. concentration of inlet streams). It is well-established that controllers that guarantee offsetless output tracking and stability in the nominal closed-loop system, may lead to poor transient performance, offset and even closed-loop instability in the presence of uncertain variables. The traditional approach followed to ensure asymptotic offsetless rejection of uncertain variables is to incorporate integral action in the controller. This approach is adequate in the case of *constant* uncertain variables but it may lead to severe failure in the presence of *time-varying* uncertainty.

Motivated by this, the design of robust controllers that are capable of coping with uncertain time-varying variables has received considerable attention in the past.

Research initially focused on the development of robust control methods for linear systems in the frequency-domain (see [MZ89] for example). including H_∞. μ-synthesis. etc.. More recently. the state-space counterparts of the H_∞ frequency-domain results have been developed (see for example [DGKF93]). This motivated research on the extension of H_∞ control methods to certain classes of uncertain nonlinear systems. In this direction. theoretical results have been derived (e.g. [vdS92. PB93b]). but their practical applicability is still in question because the explicit construction of the controllers requires the analytical solution of nonlinear partial differential equations. The interested reader may refer to [ARG94] for a critical analysis on the applicability of nonlinear H^∞ control methods to engineering applications. Recently. adaptive control schemes [SI89. TKKS91] have been proposed. The main disadvantage of this approach is that the uncertainty is assumed to be time-invariant. which significantly restricts the practical applicability of these methods.

An alternative approach for the synthesis of robust control systems for linear/ nonlinear uncertain systems is based on Lyapunov's direct method. The basic idea is to design a controller. utilizing the knowledge of certain bounding functions on the size of the uncertainty. so that the time-derivative of an appropriate Lyapunov function calculated along the trajectories of the uncertain closed-loop system is negative definite as long as the state of the system is larger than a constant which can be made arbitrarily small by suitable choice of controller parameters. This guarantees that the ultimate discrepancy between the output of the closed-loop system and the reference input is arbitrarily small. This idea was originally proposed in [CL81] and was further explored for robust controller design (see e.g.. the review papers [Cor93. Lei93]). For single-input single-output input-output linearizable nonlinear systems. combination of this approach with geometric control schemes was studied in [KP88. AC92. RT93].

For uncertain two-time-scale nonlinear systems with stable fast dynamics. an approach that utilizes a combination of singular perturbation methods and adaptive control schemes has been proposed by [TKMK89, KKM92]. For the same class of systems, an alternative approach that utilizes a combination of singular perturbation theory and controller design via Lyapunov functions to derive robust controllers has been developed by [Cor87, Kho89]. For linear singularly perturbed systems with time-varying uncertainties for which the fast dynamics may be unstable, a class of nonlinear composite controllers, synthesized using Lyapunov's direct method, was recently proposed by [CGG93]. For linear systems, other available results concern the use of H^∞ control methods [KC92, PB93a] and stochastic control methods (see [KKO86] and the references therein) for robust controller design.

In this chapter, we consider a broad class of uncertain singularly perturbed nonlinear systems. The uncertainty is allowed to be time-varying. We assume that the fast subsystem is stabilizable and the slow subsystem is input/output linearizable with ISS inverse dynamics. For such systems, we synthesize continuous, possibly time-varying, state feedback controllers that guarantee boundedness of the trajectories of the closed-loop system and achieve arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output, for initial conditions and uncertainties in an arbitrarily large compact set, as long as the singular perturbation parameter is sufficiently small. Our main result is based on a recent result concerning the robustness of the input-to-state stability property with respect to uniformly globally asymptotically stable singular perturbations, developed by [CT95], which is reviewed at the end of appendix C (see also [CT96]) and uses calculations similar to those used in the derivation of the ISS nonlinear small gain theorem established by [ZPTP95] (see also [Tee96]). The explicit construction of the controller requires the knowledge of upper bounds on the size of uncertainty. Key differences in the nature of the matching condition of our method, between singularly perturbed systems in standard and nonstandard form are also identified and discussed. In the case of two-time-scale systems with stable fast dynamics, our main result establishes a fundamental robustness property of a controller synthesized on the basis of the low-dimensional slow model, with respect to both the parametric uncertainty and the fast (unmodeled) dynamics. The developed control methodology is applied to a typical chemical reactor example with time-scale multiplicity and uncertainties, and its performance and robustness characteristics are evaluated through simulations.

4.2 Notation and facts

- $|\cdot|$ denotes the standard Euclidean norm, $sgn(\cdot)$ denotes the sign function, and Id denotes the identity function.
- For any measurable (with respect to the Lebesgue measure) function $\theta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$, $||\theta||$ denotes ess.sup.$|\theta(t)|$, $t \geq 0$.
- A function $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be positive definite if $W(x)$ is positive for all nonzero x and is zero at zero.
- A function $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be proper if $W(x)$ tends to $+\infty$ as $|x|$ tends to $+\infty$.
- A function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class K if it is continuous, increasing and is zero at zero. It is of class K_∞, if in addition, it is proper.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.
- For any function γ of class K_∞, its inverse function is well defined and is again of class K_∞.
- A matrix $A(x)$ of dimension $n \times n$ is said to be Hurwitz uniformly in $x \in \mathbb{R}^n$ if there exists a positive real number c such that $Re[\lambda_i(A(x))] \leq -c$, $i = 1, \cdots, n$

for all $x \in \mathbb{R}^n$, where λ_i denotes the i-th eigenvalue of the matrix.

- Let A and B be two matrices of dimensions $n_1 \times n_2$ and $n_2 \times n_1$ respectively, and let $I_{n_1 x n_1}$, $I_{n_2 x n_2}$ also denote the identity matrices of dimensions $n_1 \times n_1$, $n_2 \times n_2$. Then, the following identity holds:

$$det(I_{n_1 x n_1} + AB) = det(I_{n_2 x n_2} + BA) \tag{4.1}$$

- Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ be two vectors and $A \in \mathbb{R}^{n \times n}$ be a matrix, then if $\sigma_{max}\{A\}$, $\sigma_{min}\{A\}$ denote the maximum and minimum singular values of A, the following relations hold:

$$x^T Ay \leq \sigma_{max}\{A\}|x||y|, \quad -x^T Ay \leq -\sigma_{min}\{A\}|x||y| \tag{4.2}$$

4.3 Preliminaries

We will consider uncertain singularly perturbed nonlinear systems with the following state-space description:

$$\begin{aligned} \dot{x} &= f_1(x,\theta(t)) + Q_1(x,\theta(t))z + g_1(x,\theta(t))u \\ \epsilon\dot{z} &= f_2(x,\theta(t)) + Q_2(x,\theta(t))z + g_2(x,\theta(t))u \\ y &= h(x) \end{aligned} \tag{4.3}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ denote vectors of state variables, $u \in \mathbb{R}$ denotes the input, $\theta \in \mathbb{R}^q$ denotes the vector of the uncertain (possibly time-varying) variables, $y \in \mathbb{R}$ denotes the output (to be controlled), and ϵ is a small parameter which can be interpreted as the speed ratio of the slow versus the fast dynamical phenomena of the system. The vector of uncertain variables θ may include time-varying parametric uncertainties and/or unmeasured exogenous disturbances. The vector functions $f_1(x,\theta)$, $f_2(x,\theta)$, $g_1(x,\theta)$ and $g_2(x,\theta)$ are sufficiently smooth, $Q_1(x,\theta)$ and $Q_2(x,\theta)$ are sufficiently smooth matrices of appropriate dimensions, and $h(x)$ is a sufficiently smooth scalar function. Our assumptions will impose among other things that the output y and the input u in the system of Eq.4.3 are in deviation variables, and the rate of change of the uncertainty, $\dot{\theta}$, is bounded.

Setting $\epsilon = 0$ and assuming that the matrix $Q_2(x, \theta)$ is invertible uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ (this assumption will be removed later), the system of Eq.4.3 takes the form:

$$\dot{x} = f_1(x, \theta) + Q_1(x, \theta) z_s + g_1(x, \theta) u \tag{4.4}$$

$$f_2(x, \theta) + Q_2(x, \theta) z_s + g_2(x, \theta) u = 0 \tag{4.5}$$

where z_s denotes a quasi-steady-state for the fast state vector z. The fact that the matrix $Q_2(x, \theta)$ is invertible implies that the algebraic equation (4.5) admits a unique solution of the form:

$$z_s = -[Q_2(x, \theta)]^{-1} [f_2(x, \theta) + g_2(x, \theta) u] \tag{4.6}$$

Note that the quasi-steady-state z_s is an explicit function of the uncertainty vector θ which implies that the exact value of the vector z_s is not known. Substituting Eq.4.6 into Eq.4.4, the following dynamical system is obtained:

$$\begin{aligned} \dot{x} &= F(x, \theta) + G(x, \theta) u \\ y &= h(x) \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} F(x, \theta) &= f_1(x, \theta) - Q_1(x, \theta)[Q_2(x, \theta)]^{-1} f_2(x, \theta) \\ G(x, \theta) &= g_1(x, \theta) - Q_1(x, \theta)[Q_2(x, \theta)]^{-1} g_2(x, \theta) \end{aligned} \tag{4.8}$$

The dynamical system of Eq.4.7 is called the *reduced system* or *slow subsystem*. Note that the control u appears linearly in the above system which is an implication of the linearity in z in the original system.

The inherent two-time-scale behavior of the system of Eq.4.3 can be analyzed by defining a fast time-scale:

$$\tau = \frac{t}{\epsilon} \tag{4.9}$$

In this new time-scale the original system takes the form:

$$\frac{dx}{d\tau} = \epsilon [f_1(x, \theta) + Q_1(x, \theta) z + g_1(x, \theta) u] \tag{4.10}$$

$$\frac{dz}{d\tau} = f_2(x,\theta) + Q_2(x,\theta)z + g_2(x,\theta)u \tag{4.11}$$

Setting ϵ equal to zero, the following system is obtained:

$$\frac{dz}{d\tau} = f_2(x,\theta) + Q_2(x,\theta)z + g_2(x,\theta)u \tag{4.12}$$

where x can be considered equal to its initial value $x(0)$ and θ can be viewed as constant. In what follows, we will refer to the dynamical system of Eq.4.12 as the *fast subsystem*.

In this work, we are dealing with systems of the form of Eq.4.3 for which the matrix $Q_2(x,\theta)$ may be singular or possess eigenvalues in the right-half of the complex plane for some $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. The following assumption states our stabilizability requirement on the open-loop fast subsystem of Eq.4.12.

Assumption 4.1: *The pair $[Q_2(x,\theta)\ g_2(x,\theta)]$ is stabilizable uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, in the sense that there exists a co-vector field $k^T(x)$ such that the matrix $Q_2(x,\theta) + g_2(x,\theta)k^T(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.*

In closing this section, we define what is meant by input-to-state stability for a system of the form of Eq.4.7.

Definition 4.1 [Son89]: *The system in Eq.4.7 (with $u \equiv 0$) is said to be ISS with respect to θ if there exist a function β of class KL and a function γ of class K such that for each $x_o \in \mathbb{R}^n$ and for each measurable, essentially bounded input $\theta(\cdot)$ on $[0,\infty)$ the solution of Eq.4.7 with $x(0) = x_o$ exists for each $t \geq 0$ and satisfies*

$$|x(t)| \leq \beta(|x(0)|,t) + \gamma(||\theta||), \ \forall t \geq 0 \tag{4.13}$$

4.4 Problem statement-Approach

In this chapter, we address and solve the problem of synthesizing well-conditioned continuous, possibly time-varying, static state feedback controllers that preserve the two-time-scale nature of the original system, with the following objectives for the closed-loop system:

- Boundedness of the trajectories of the closed-loop system
- Arbitrary degree of asymptotic attenuation of the effect of the uncertainty on the output
- Robust output tracking for changes in the reference input

The above requirements will be obtained for ϵ sufficiently small.

The approach followed for the synthesis of the controller is essentially a two-step one. In the first step, we take advantage of assumption 4.1 to design a preliminary control law that transforms the original singularly perturbed system of Eq.4.3 into a new singularly perturbed system in standard form with globally exponentially stable fast dynamics. In the second step, we synthesize a control law that uses feedback of the slow state vector x only and utilizes the knowledge of upper bounds on the size of uncertainty to achieve robust output tracking with arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output. It is established that the resulting controller enforces the requested three properties in the closed-loop system, provided that the singular perturbation parameter ϵ is sufficiently small.

4.5 Robust controller synthesis

In this section, we will develop a solution to the state feedback control problem, specified in the previous section, for uncertain singularly perturbed nonlinear systems of the form of Eq.4.3 for which the matrix $Q_2(x,\theta)$ may be singular or possess eigenvalues which lie in the right half of the complex plane, for some $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Motivated by the affine appearance of the fast variable z in the model of Eq.4.3, we will initially consider control laws of the form:

$$u = \tilde{u} + k^T(x)z \tag{4.14}$$

where $k^T(x)$ is a vector field on $\mathbb{R}^n$ and $\tilde{u}$ is an auxiliary input. The control law of Eq.4.14 allows stabilizing the fast dynamics of the system of Eq.4.3 by choosing appropriately the gain $k^T(x)$ (see assumption 4.1).

Under a control law of the form of Eq.4.14 the system of Eq.4.3 takes the form:

$$\begin{aligned} \dot{x} &= f_1(x,\theta) + [Q_1(x,\theta) + g_1(x,\theta)k^T(x)]z + g_1(x,\theta)\tilde{u} \\ \epsilon\dot{z} &= f_2(x,\theta) + [Q_2(x,\theta) + g_2(x,\theta)k^T(x)]z + g_2(x,\theta)\tilde{u} \end{aligned} \tag{4.15}$$

One can immediately observe that a control law of the form of Eq.4.14 preserves the two-time-scale nature of the original system, and the linearity with respect to the state z and the auxiliary input $\tilde{u}$. Furthermore, performing a standard two-time-scale decomposition, one can easily show that the fast subsystem is given by:

$$\frac{dz}{d\tau} = f_2(x,\theta) + [Q_2(x,\theta) + g_2(x,\theta)k^T(x)]z + g_2(x,\theta)\tilde{u} \tag{4.16}$$

Clearly, the control law of Eq.4.14 guarantees the stability of the fast dynamics if the gain $k^T(x)$ is chosen in such a manner so that the matrix $Q_2(x,\theta) + g_2(x,\theta)k^T(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Note that this is always possible because of assumption 4.1.

The reduced system corresponding to the singularly perturbed system of Eq.4.15 takes the form:

$$\begin{aligned} \dot{x} &= \hat{F}(x,\theta) + \hat{G}(x,\theta)\tilde{u} \\ y &= h(x) \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \hat{F}(x,\theta) &= f_1(x,\theta) - [Q_1(x,\theta) + g_1(x,\theta)k^T(x)][Q_2(x,\theta) + g_2(x,\theta)k^T(x)]^{-1}f_2(x,\theta) \\ \hat{G}(x,\theta) &= g_1(x,\theta) - [Q_1(x,\theta) + g_1(x,\theta)k^T(x)][Q_2(x,\theta) + g_2(x,\theta)k^T(x)]^{-1}g_2(x,\theta) \end{aligned} \tag{4.18}$$

The usual approach followed for the design of stabilizing controllers for uncertain nonlinear systems of the form of Eq.4.17 involves two steps: first, the derivation of bounds (in terms of continuous functions) for the uncertainty: second, the design of a robust controller for the system of Eq.4.17 via Lyapunov's direct method, utilizing the bounding functions for the uncertainty. This approach was originally proposed by [CL81], and was further developed by [Kho89]. However, there are certain disadvantages associated with this approach. First, the actual construction of the controller requires the existence and knowledge of a nonlinear Lyapunov function for the nominal system. Second, there are certain matching conditions that the uncertainty vector θ has to satisfy for the applicability of this method. From the aforementioned difficulties, it is clear that the first one poses fundamental limitations on the practical applicability of this method, while the second one may or may not pose restrictions depending on the specific structure of the system under consideration. Motivated by the above considerations, research efforts have focused on the design of robust controllers using Lyapunov's direct method for uncertain systems of the form of Eq.4.17 with weaker matching conditions (e.g., [Qu93]).

In what follows, we consider the synthesis of robust nonlinear control laws for systems of the form of Eq.4.17. The central differences with the above-mentioned robust controller design methods are that the synthesis of the controller does not require the existence of a nonlinear Lyapunov function for the nominal system, while the matching condition required for the application of this theory is different than the standard one employed in [CL81].

We now proceed with the design of the controller. Motivated by the requirement of output tracking with attenuation of the effect of the uncertainty on the output, we initially assume that there exists a coordinate transformation that renders the system of Eq.4.17 in a partially linear form. Our requirement is precisely formulated in assumption 4.2 that follows:

Assumption 4.2 : *The vector of uncertain variables $\theta(t)$ is absolutely continuous,* *and there exists an integer $\tilde{r}$ and a set of coordinates:*

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{\tilde{r}} \\ \eta_1 \\ \vdots \\ \eta_{n-\tilde{r}} \end{bmatrix} = \mathcal{X}(x,\theta) = \begin{bmatrix} h(x) \\ L_{\tilde{F}}h(x) \\ \vdots \\ L_{\tilde{F}}^{\tilde{r}-1}h(x) \\ \chi_1(x,\theta) \\ \vdots \\ \chi_{n-\tilde{r}}(x,\theta) \end{bmatrix} \tag{4.19}$$

where $\chi_1(x,\theta),\cdots,\chi_{n-\tilde{r}}(x,\theta)$ are scalar functions such that the reduced system of Eq.4.17 takes the form:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{\tilde{r}-1} &= \zeta_{\tilde{r}} \\ \dot{\zeta}_{\tilde{r}} &= L_{\tilde{F}}^{\tilde{r}}h(\mathcal{X}^{-1}(\zeta,\eta,\theta)) + L_{\tilde{G}}L_{\tilde{F}}^{\tilde{r}-1}h(\mathcal{X}^{-1}(\zeta,\eta,\theta))\tilde{u} \\ \dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\tilde{r}} &= \Psi_{n-\tilde{r}}(\zeta,\eta,\theta,\dot{\theta}) \\ y &= \zeta_1 \end{aligned} \tag{4.20}$$

where $L_{\tilde{G}}L_{\tilde{F}}^{\tilde{r}-1}h(x) \neq 0$ for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Moreover, for each $\theta \in \mathbb{R}^q$, the states ζ, η are bounded if and only if the state x is bounded.

Remark 4.1 : We note that assumption 4.2 implicitly includes the matching condition of our theory. Notice that this condition is different than the standard one which restricts the uncertainty vector θ to lie in the span of the vector field $\tilde{G}(x,\theta)$ [TKMK89, CGG93]. Specifically, it allows handling a larger class of uncertain inputs than the standard one, but requires posing a stronger stability requirement on the inverse dynamics of the system of Eq.4.20 (see assumption 4.3). The motivation for considering this matching condition is given by the fact that it is satisfied by a large number of practical applications, including the chemical reactor example of section 4.7, where the standard matching condition does not hold in the slow subsystem.

The following assumption poses our stability requirement on the inverse dynamics of the reduced system of Eq.4.17.

Assumption 4.3 : *The dynamical system:*

$$\begin{aligned} \dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\tilde{r}} &= \Psi_{n-\tilde{r}}(\zeta,\eta,\theta,\dot{\theta}) \end{aligned} \tag{4.21}$$

is ISS with respect to $\zeta,\theta,\dot{\theta}$.

Remark 4.2: We note that although, from a theoretical point of view, assumptions 4.2 and 4.3 limit the class of systems for which our methodology is applicable, they are satisfied by the majority of chemical processes of practical interest. Furthermore, we remark that even in the case where these assumptions hold around a reasonably large neighborhood of the equilibrium point, but not necessarily globally, the application of our methodology is still possible (see the chemical reactor example of section 4.7).

Defining the variable $\tilde{\delta}(x,\theta)$ as:

$$\tilde{\delta}(x,\theta) = [\tilde{F}(x,\theta) - \tilde{F}_{nom}(x)] \tag{4.22}$$

where $\tilde{F}_{nom}(x)$ denotes the vector field resulting from $\tilde{F}(x,\theta)$ by setting θ equal to its nominal value θ_0 and using the fact that $L_{\tilde{F}}^{k-1}h(x) = L_{\tilde{F}_{nom}}^{k-1}h(x)$, $k = 1,\ldots,\tilde{r}$ (which follows from the structure of Eq.4.20), the system of Eq.4.20 can be written as:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{\tilde{r}-1} &= \zeta_{\tilde{r}} \\ \dot{\zeta}_{\tilde{r}} &= L_{\tilde{F}_{nom}}^{\tilde{r}}h(\mathcal{X}^{-1}(\zeta,\eta,\theta)) + L_{\tilde{G}}L_{\tilde{F}}^{\tilde{r}-1}h(\mathcal{X}^{-1}(\zeta,\eta,\theta))\tilde{u} + L_{\tilde{\delta}}L_{\tilde{F}}^{\tilde{r}-1}h(\mathcal{X}^{-1}(\zeta,\eta,\theta)) \\ \dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\tilde{r}} &= \Psi_{n-\tilde{r}}(\zeta,\eta,\theta,\dot{\theta}) \end{aligned} \tag{4.23}$$

where $\zeta_k = L_{\tilde{F}_{nom}}^{k-1}h(x)$, $k = 1,\ldots,\tilde{r}$.

In most practical applications, there exists partial information about the uncertain terms of the process model. Information of this kind may result from physical considerations, preliminary simulations, experimental data, etc. In what follows, we will quantify possible knowledge about the uncertainty by assuming the existence of known state-dependent, possibly time-varying, bounds that capture the size of uncertainty for all times. Assumption 4.4 that follows formalizes this requirement:

Assumption 4.4: *The term $L_{\bar{G}}L_{\bar{F}}^{\bar{r}-1}h(x)$ has known constant sign for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and there exist known functions $\bar{c}_1(x,t)$, $\bar{c}_2(x,t)$ such that the following conditions hold:*

$$\begin{aligned} |L_{\bar{\xi}}L_{\bar{F}}^{\bar{r}-1}h(x)| &\leq \bar{c}_1(x,t) \\ 0 < \bar{c}_2(x,t) &\leq |L_{\bar{G}}L_{\bar{F}}^{\bar{r}-1}h(x)| \end{aligned} \tag{4.24}$$

for all $\theta \in \mathbb{R}^q$.

Remark 4.3 : We can assume, without loss of generality, that (θ, x) restricted in compact sets imply $\bar{c}_1(x,t)$ is bounded and $\bar{c}_2(x,t)$ is bounded away from zero.

We are now in position to proceed with the design of the auxiliary input $\bar{u}$ to achieve the asymptotic attenuation of the effect of the uncertainty on the output. In particular, we consider static state feedback laws of the form:

$$\bar{u} = r_1(x,t)[p(x) + q(x)\bar{v} + r_2(x,t)] \tag{4.25}$$

where $p(x)$, $r_1(x,t)$, $r_2(x,t)$ are scalar functions, $q(x)$ is a row vector function, with $r_1(x,t)q(x) \neq 0$ for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$, and $\bar{v} = [v \; v^{(1)} \; \cdots \; v^{(\bar{r})}]^T$, where $v^{(k)}$ denotes the $k-th$ time derivative of the external reference input v, which is assumed to be a sufficiently smooth function of time.

The control law of Eq.4.25 consists of three components. The component $r_1(x,t)$ which is used to account for the effect of uncertainty that enters the system coupled with the input, the component $p(x)+q(x)\bar{v}$ which is responsible for the output tracking and stabilization of the nominal reduced system of Eq.4.17, and the component $r_2(x,t)$ which is responsible for the attenuation of the effect of the uncertainty on the output in the closed-loop reduced system.

We are now in position to state the main result of this chapter in the form of a theorem.

Theorem 4.1: *Consider the uncertain singularly perturbed nonlinear system of Eq.4.3, for which assumptions 4.1, 4.2, 4.3, and 4.4 hold, under the static state feedback law:*

$$u = sgn[L_{\bar{G}}L_{\bar{F}}^{\bar{r}-1}h(x)]\,[\bar{c}_2(x,t)]^{-1}\left\{\sum_{k=1}^{\bar{r}}\frac{\beta_k}{\beta_{\bar{r}}}(v^{(k)} - L_{\bar{F}_{nom}}^{k}h(x)) + \sum_{k=1}^{\bar{r}}\frac{\beta_k}{\beta_{\bar{r}}}(v^{(k-1)} - L_{\bar{F}_{nom}}^{k-1}h(x)) - 2[\bar{c}_1(x,t) + |\sum_{k=1}^{\bar{r}}\frac{\beta_k}{\beta_{\bar{r}}}(v^{(k)} - L_{\bar{F}_{nom}}^{k}h(x))|]w(x,\phi)\right\} + k^T(x)z \tag{4.26}$$

where the feedback gain $k^T(x)$ is such that the matrix $Q_2(x,\theta)+g_2(x,\theta)k^T(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, β_k are parameters chosen so that the polynomial $s^{\bar{r}-1} + \frac{\beta_{\bar{r}-1}}{\beta_{\bar{r}}}s^{\bar{r}-2} + \cdots + \frac{\beta_2}{\beta_{\bar{r}}}s + \frac{\beta_1}{\beta_{\bar{r}}} = 0$ is Hurwitz, and the scalar function $w(x,\phi)$ is given by:

$$w(x,\phi) = \frac{\sum_{k=1}^{\bar{r}}\frac{\beta_k}{\beta_{\bar{r}}}(L_{\bar{F}_{nom}}^{k-1}h(x) - v^{(k-1)})}{|\sum_{k=1}^{\bar{r}}\frac{\beta_k}{\beta_{\bar{r}}}(L_{\bar{F}_{nom}}^{k-1}h(x) - v^{(k-1)})| + \phi} \tag{4.27}$$

where ϕ is an adjustable parameter. Then, for each set of positive real numbers $\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d$, there exists $\phi^* > 0$ and for each $\phi \in (0,\phi^*]$, there exists $\epsilon^*(\phi) > 0$, such that if $\phi \in (0,\phi^*]$, $\epsilon \in (0,\epsilon^*(\phi)]$ and $|x(0)| \le \delta_x$, $|z(0)| \le \delta_z$, $||\theta|| \le \delta_\theta$, $||\dot{\theta}|| \le \delta_{\dot{\theta}}$, $||\bar{v}|| \le \delta_{\bar{v}}$, the output of the closed-loop system satisfies a relation of the form:

$$\limsup_{t\to\infty}|y(t) - v(t)| \le d \tag{4.28}$$

Remark 4.4 : The feedback gain $k^T(x)$ can be designed using standard linear control methods, such as pole placement, optimal control, etc. (for details see [KKO86]).

Remark 4.5 : Referring to the controller of Eq.4.26, one can easily observe that is comprised of two pieces: the term $k^T(x)z$ which is responsible for the stabilization of the fast dynamics of the closed-loop system and the term:

$$
\begin{aligned}
a(x, \bar{v}, t) \;:=\; & sgn[L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)] \, [\tilde{c}_2(x, t)]^{-1} \left\{ \sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} (v^{(k)} - L_{\tilde{F}_{nom}}^{k} h(x)) + \sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} (v^{(k-1)} \right. \\
& \left. - L_{\tilde{F}_{nom}}^{k-1} h(x)) - 2[\tilde{c}_1(x, t) + |\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} (v^{(k)} - L_{\tilde{F}_{nom}}^{k} h(x))|] w(x, \phi) \right\}
\end{aligned} \tag{4.29}
$$

which guarantees boundedness of the trajectories of the closed-loop reduced system and output tracking with arbitrary degree of asymptotic attenuation of the effect of θ on y.

Remark 4.6: Note that the result of theorem 4.1 holds for arbitrarily large initial conditions, uncertainties and their derivatives, and does not impose any kind of interconnection or growth conditions on the nonlinearities of the system.

Remark 4.7: In practical applications, the value of the singular perturbation parameter is typically fixed by the process, say ϵ_p, and thus there is a limit on how small the ultimate bound d can be chosen. For example, the proof of theorem 4.1 can be used to calculate ϕ^* from the desired $(\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d)$ and, in turn, the value ϵ^* for $\phi \leq \phi^*$. If this ϵ^* is less than ϵ_p, then d may need to be readjusted (increased) so that $\epsilon^* \geq \epsilon_p$. Of course, if ϵ_p is too large, there may be no value of d that works. On the other hand, the value of ϵ^*, calculated from the proof of the theorem, is typically conservative, and so it can be useful to check the appropriateness of ϕ^* (and d) through computer simulations.

Remark 4.8: Theorem 4.1 provides a bound for the discrepancy between the output and the reference input that holds asymptotically. A characterization of the transient performance of the controller of Eq.4.26 through a bound for the quantity $||y(t) - v(t)||$ in terms of the norm of the initial condition and the ultimate bound d is established in the proof of the theorem (consider Eq.C.26 with $||\tilde{e}_{\tilde{r}}||$ estimated as $|\tilde{e}_r(0)| + 2\phi$).

4.6 A note on the matching condition

In our development so far, we considered singularly perturbed systems of the form of Eq.4.3 for which the matrix $Q_2(x, \theta)$ could be singular for some $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. To develop a solution to the state feedback control problem we essentially followed a two-step procedure. In the first step, we employed a preliminary control law of the form of Eq.4.14 to derive a two-time-scale system in standard form with globally exponentially stable fast dynamics. In the second step, we specified the explicit form of the auxiliary input $\tilde{u}$ to achieve arbitrary degree of asymptotic attenuation of the effect of the uncertainty on the output. A direct implication of this approach is that the coordinate change of assumption 4.2 (which implicitly includes the matching condition of our methodology) depends explicitly on the feedback gain $k^T(x)$ employed to stabilize the fast dynamics. Therefore, it may be possible, depending on the structure of the system under consideration, to choose the gain $k^T(x)$ to stabilize the fast dynamics as well as to ensure that the matching condition of our theory is satisfied.

In this section, our objective is to show that in the case of singularly perturbed systems of the form of Eq.4.3 in standard form i.e., systems for which the matrix $Q_2(x, \theta)$ is invertible uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, with possibly unstable fast dynamics, the matching condition of this methodology is independent of the selection of the gain $k^T(x)$ employed to stabilize the fast dynamics, and therefore it can be verified on the basis of the open-loop reduced system Eq.4.7.

In order to simplify the statement of our result, we make the following definitions. Referring to the system of Eq.4.7, let r denote an integer for which $\frac{\partial y^{(k)}}{\partial u} \equiv 0$, if $k = 1, \ldots, r-1$, for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and $\frac{\partial y^{(r)}}{\partial u} \neq 0$, for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and let s denote the smallest integer for which $\frac{\partial y^{(s)}}{\partial \theta} \not\equiv 0$, for some $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. Similarly, referring to the system of Eq.4.17, let $\tilde{r}$ denote an integer for which $\frac{\partial y^{(k)}}{\partial u} \equiv 0$, if

$k = 1, \ldots, \tilde{r}-1$, for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and $\frac{\partial y^{(\tilde{r})}}{\partial u} \neq 0$, for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and let $\tilde{s}$ denote the smallest integer for which $\frac{\partial y^{(\tilde{s})}}{\partial \theta} \not\equiv 0$, for some $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$. The main result of this section is summarized in proposition 4.1 that follows.

Proposition 4.1: *Consider the uncertain singularly perturbed nonlinear system of Eq.4.3, for which the matrix $Q_2(x,\theta)$ is invertible uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ and assumption 4.1 holds. Then, if r is well-defined and the condition $r \leq s$ holds for the system of Eq.4.7 and the feedback gain $k^T(x)$ is chosen so that the matrix $Q_2(x,\theta) + g_2(x,\theta)k^T(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, $\tilde{r}$ is well-defined and the condition $r = \tilde{r} \leq \tilde{s}$ holds for the system of Eq.4.17.*

From the result of proposition 4.1, it is clear that for systems in standard form, the verification of the condition $r \leq s$ on the basis of the open-loop reduced system of Eq.4.17 is sufficient for the satisfaction of the matching condition of the developed robust control methodology.

4.7 Application to a nonisothermal continuous stirred tank reactor

Two parallel irreversible reactions of the form:

$$A \xrightarrow{k_1} B, \quad A \xrightarrow{k_2} C \tag{4.30}$$

take place in a continuous stirred tank reactor which is shown in Figure 4.1. The desired product is the species B, while species C is the side product. The inlet stream consists of pure A at flow rate F, concentration C_{A0} and temperature T_{A0}. The reactions are endothermic and a heating jacket is used to heat the reactor. Fluid is added to the jacket at a flow rate F_j and an inlet temperature T_{j0}. The rate of

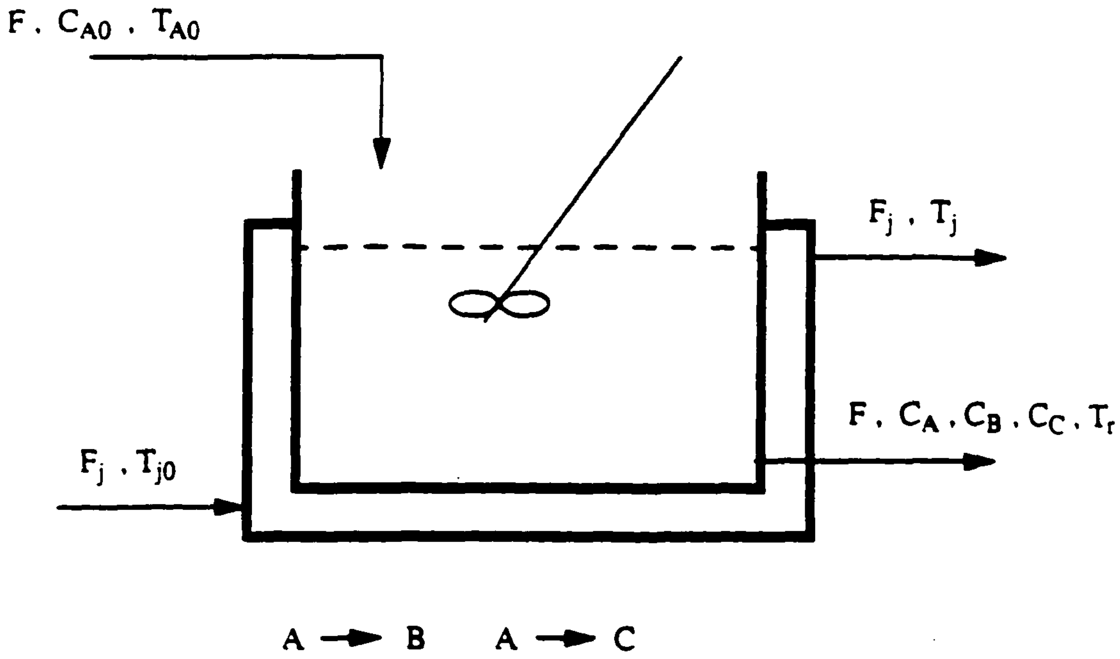

Figure 4.1: A nonisothermal continuous stirred tank reactor.

reactions are assumed to be of the following form:

$$r_1 = -k_{10}\exp(\frac{-E_1}{RT_r})C_A$$
$$r_2 = -k_{20}\exp(\frac{-E_2}{RT_r})C_A$$

Under the following assumptions:

- Perfect mixing in the reactor
- Uniform temperature in the reactor
- Constant volume of the liquid in the reactor
- Constant density and heat capacity of the reacting liquid
- Constant density and heat capacity of the fluid in the jacket

the process model consists of the following set of material and energy balances:

- Reactor Mass Balance for the species A:

$$V_r\frac{dC_A}{dt} = F_r(C_{A0} - C_A) - k_{10}e^{\frac{-E_1}{RT_r}}C_AV_r - k_{20}e^{\frac{-E_2}{RT_r}}C_AV_r \tag{4.31}$$

- Reactor Mass Balance for the species B:

$$V_r\frac{dC_B}{dt} = -F_rC_B + k_{10}e^{\frac{-E_1}{RT_r}}C_AV_r \tag{4.32}$$

- Reactor Energy Balance:

$$V_r\frac{dT_r}{dt} = F_r(T_{A0} - T_r) + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{\frac{-E_1}{RT_r}}C_AV_r + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{\frac{-E_2}{RT_r}}C_AV_r + \frac{UA_r}{\rho_m c_{pm}}(T_j - T_r) \tag{4.33}$$

- Jacket Energy Balance:

$$V_j\frac{dT_j}{dt} = F_jT_{j0} - F_jT_j + \frac{UA_r}{\rho_j c_{pj}}(T_r - T_j) \tag{4.34}$$

where C_A and C_B denote the concentrations of the species A and B. T_r and T_j denote the temperatures of the reactor and the jacket. V_r and V_j denote the volumes of the reactor and the jacket. k_{10}, k_{20}, E_1, E_2, ΔH_1, ΔH_2 denote the pre-exponential constants, the activation energies, and the enthalpies of the two reactions. c_{pm}, c_{pj} and ρ_m, ρ_j denote the heat capacities and densities of the reactor and the jacket, respectively, and U denotes the heat transfer coefficient.

Typically, the volume of the reactor V_r and the volume of the jacket V_j are significantly different, thus, the parameter ϵ can be defined as the following ratio:

$$\epsilon = \frac{V_j}{V_r} \tag{4.35}$$

The control objective is the regulation of the concentration of the species B in the reactor, in order to maintain the production of B at the desired level, by manipulating the inlet temperature of the fluid in the jacket. The inlet concentration and temperature of the species A, C_{A0} and T_{A0}, respectively, are assumed to be the main uncertainties.

Defining:

$$x_1 = C_A, \quad x_2 = C_B, \quad x_3 = T_r, \quad z_1 = T_j, \quad u = T_{j0} - T_{j0s} \tag{4.36}$$

$$\theta_1 = C_{AO} - C_{AOs}, \quad \theta_2 = T_{AO} - T_{AOs}, \quad y = C_B \tag{4.37}$$

where the subscript s denotes the steady-state values, one obtains the representation of the form of Eq.4.3 with:

$$f_1(x,\theta) = \begin{bmatrix} F_r(C_{A0s} - x_1) + F_r\theta_1 - k_{10}e^{\frac{-E_1}{Rx_3}} x_1 V_r - k_{20}e^{\frac{-E_2}{Rx_3}} x_1 V_r \\ -F_r x_2 + k_{10}e^{\frac{-E_1}{Rx_3}} x_1 V_r \\ F_r(T_{A0s} - x_3) + F_r\theta_2 + F_r\frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10}e^{\frac{-E_1}{Rx_3}} x_1 V_r \\ +\frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20}e^{\frac{-E_2}{Rx_3}} x_1 V_r - \frac{UA_r}{\rho_m c_{pm}} x_3 \end{bmatrix}$$

$$Q_1(x,\theta) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{UA_r}{\rho_m c_{pm}} \end{bmatrix},\ g_1(x,\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ f_2(x,\theta) = \begin{bmatrix} F_j T_{j0s} + \dfrac{UA_r}{\rho_j c_j} x_3 \end{bmatrix}$$

$$Q_2(x,\theta) = \begin{bmatrix} -(F_j + \dfrac{UA_r}{\rho_j c_j}) \end{bmatrix},\ g_2(x,\theta) = \begin{bmatrix} F_j \end{bmatrix},\ h(x) = \begin{bmatrix} x_2 \end{bmatrix}$$

One can easily verify that the system is in standard form with globally exponentially stable fast dynamics. Setting $\epsilon = 0$, the representation of the open-loop reduced system can be easily obtained with:

$$F(x,\theta) = \begin{bmatrix} F_r(C_{A0s} - x_1) + F_r\theta_1 - k_{10}e^{\frac{-E_1}{Rx_3}} x_1 V_r - k_{20}e^{\frac{-E_2}{Rx_3}} x_1 V_r \\ -F_r x_2 + k_{10}e^{\frac{-E_1}{Rx_3}} x_1 V_r \\ F_r(T_{A0s} - x_3) + F_r\theta_2 + \dfrac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10}e^{\frac{-E_1}{Rx_3}} x_1 V_r \\ +\dfrac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20}e^{\frac{-E_2}{Rx_3}} x_1 V_r + \dfrac{UA}{\rho_m c_{pm}} \dfrac{1}{F_j + \dfrac{UA}{\rho_j c_j}} [\dfrac{UA}{\rho_j c_j} T_r + F_j T_{j0s}] \end{bmatrix}$$

$$G(x,\theta) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{UA}{\rho_m c_{pm}} \dfrac{1}{F_j + \dfrac{UA}{\rho_j c_j}} F_j \end{bmatrix},\ h(x) = \begin{bmatrix} x_2 \end{bmatrix}$$

It can be easily verified that a local version of assumption 4.2 holds (note also that for this particular example we do not need to assume that (θ_1, θ_2) are absolutely continuous).

The values of the process parameters and the corresponding steady-state values of the process variables are given in Table 4.1. It was verified that these conditions correspond to a stable equilibrium point. We will now verify that a local version of

V_r	=	1.00	m^3
V_j	=	0.08	m^3
A_r	=	6.0	m^2
U	=	1000.0	$kcal\ hr^{-1}\ m^{-2}\ K^{-1}$
R	=	1.987	$kcal\ kmol^{-1}\ K^{-1}$
C_{A0s}	=	3.75	$kmol\ m^{-3}$
T_{A0s}	=	310.0	K
T_{j0s}	=	357.5	K
ΔH_1	=	5.4×10^4	$kcal\ kmol^{-1}$
ΔH_2	=	50.67×10^4	$kcal\ kmol^{-1}$
k_{10}	=	3.36×10^6	hr^{-1}
k_{20}	=	7.21×10^6	hr^{-1}
E_1	=	8.0×10^4	$kcal\ kg^{-1}$
E_2	=	9.0×10^4	$kcal\ kg^{-1}$
c_{pm}	=	0.231	$kcal\ kg^{-1}\ K^{-1}$
c_{pj}	=	0.2	$kcal\ kg^{-1}\ K^{-1}$
ρ_m	=	900.0	$kg\ m^{-3}$
ρ_j	=	800.0	$kg\ m^{-3}$
F	=	3.0	$m^3\ hr^{-1}$
F_j	=	20.0	$m^3\ hr^{-1}$
C_{As}	=	1.125	$kmol\ m^{-3}$
C_{Bs}	=	1.875	$kmol\ m^{-3}$
T_{rs}	=	300.0	K
T_{js}	=	320.0	K

Table 4.1: Process parameters and steady-state values

assumption 4.3 also holds. To this end, consider the coordinate transformation:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} h(x) \\ L_F h(x) \\ \chi(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -F_r x_2 + k_{10} e^{\frac{-E_1}{Rx_3}} x_1 V_r \\ x_1 - C_{A0s} \end{bmatrix} \tag{4.38}$$

In the new coordinates the η_1 dynamics takes the form:

$$\dot{\eta}_1 = -F_r\eta_1 + F_r\theta_1 - (\zeta_2 + F_r\zeta_1) - k_{20}\left(\frac{\zeta_2 + F_r\zeta_1}{k_{10}(\eta_1 + C_{A0s})V_r}\right)^{\frac{E_2}{E_1}} (\eta_1 + C_{A0s})V_r \tag{4.39}$$

For a meaningful problem η_1 should always be greater than zero (notice that η_1 represents the concentration of the species A in the reactor). Setting $\zeta_1 = \zeta_2 = \theta_1 = 0$ the above equation reduces to:

$$\dot{\eta}_1 = -F_r\eta_1 \tag{4.40}$$

From the above, it follows that the system of Eq.4.39 possesses a local input-to-state stability property with respect to θ and ζ, and thus assumption 4.3 holds locally. Moreover, assumption 4.4 takes the form :

$$|L_{\bar{\delta}} L_F h(x)| \leq \bar{c}_1(x,t) = [k_{10} e^{\frac{-E_1}{Rx_3}} V_r F_r |\theta_1| + k_{10} e^{\frac{-E_1}{Rx_3}} \frac{E_1}{Rx_3^2} V_r F_r |\theta_2|] \tag{4.41}$$

where $|\theta_1|$, $|\theta_2|$ denote the upper bounds on the size of uncertain variables.

We now proceed with the design of the controller. Motivated by physical arguments, we assume that the external reference input is constant, and thus, the controller of theorem 4.1 takes the form:

$$\begin{aligned} u = {} & [L_{G_{nom}} L_{F_{nom}} h(x)]^{-1} \left\{ -\sum_{k=1}^{2} \frac{\beta_k}{\beta_2} L_{F_{nom}}^k h(x) + \sum_{k=1}^{2} \frac{\beta_k}{\beta_2} (v^{(k-1)} - L_{F_{nom}}^{k-1} h(x)) \right. \\ & \left. -2[\bar{c}_1(x,t)| + |\sum_{k=1}^{2} \frac{\beta_k}{\beta_r} L_{F_{nom}}^k h(x)|]w(x,\phi) \right\} \end{aligned} \tag{4.42}$$

where the scalar function $w(x, \phi)$ is given by:

$$w(x, \phi) = \frac{[\frac{\beta_1}{\beta_2}(x_2 - v) + L_{F_{nom}}h(x)]}{|\frac{\beta_1}{\beta_2}(x_2 - v) + L_{F_{nom}}h(x)| + \phi} \tag{4.43}$$

Taking into account the time-constants of the open-loop reduced system, the following values were chosen for the parameters β_k:

$$\beta_0 = 1.0 \quad , \quad \beta_1 = 5.0 \quad , \quad \beta_2 = 10.0$$

The controller of Eq.4.42 was implemented with $\phi = 0.01$ to guarantee that the output of the system satisfies a relation of the form:

$$\limsup_{t \to \infty} |y - v| \leq 0.02 \tag{4.44}$$

for a value of the singular perturbation parameter ϵ equal to 0.08.

Simulations were performed to evaluate the performance and robustness properties of the controller. The uncertainties that were considered were:

$$\theta_1 = \theta_{10} sin(t) \ , \ \theta_2 = \theta_{20} sin(t) \tag{4.45}$$

where $\theta_{10} = 0.3 \ mol/lt$ and $\theta_{10} = 3.0 \ K$. The upper bounds on the uncertainty were assumed to be: $|\theta_1| = 0.3$, $|\theta_2| = 3.0$. In all simulation runs, it was verified that the states of the system stay bounded.

The first simulation run addressed the regulatory characteristics of the controller in the presence of uncertainties. Figure 4.2 shows the controlled output and the open-loop response of the process as well as the profile of the input. One can immediately observe that the effect of uncertainties on the output has been significantly reduced and the output of the process remains very close to the nominal value of the reference input as predicted by the theory.

The second simulation run addressed the reference input tracking capabilities of the controller in the presence of uncertainties. A $0.8 mol/lt$ (almost 43 %) decrease

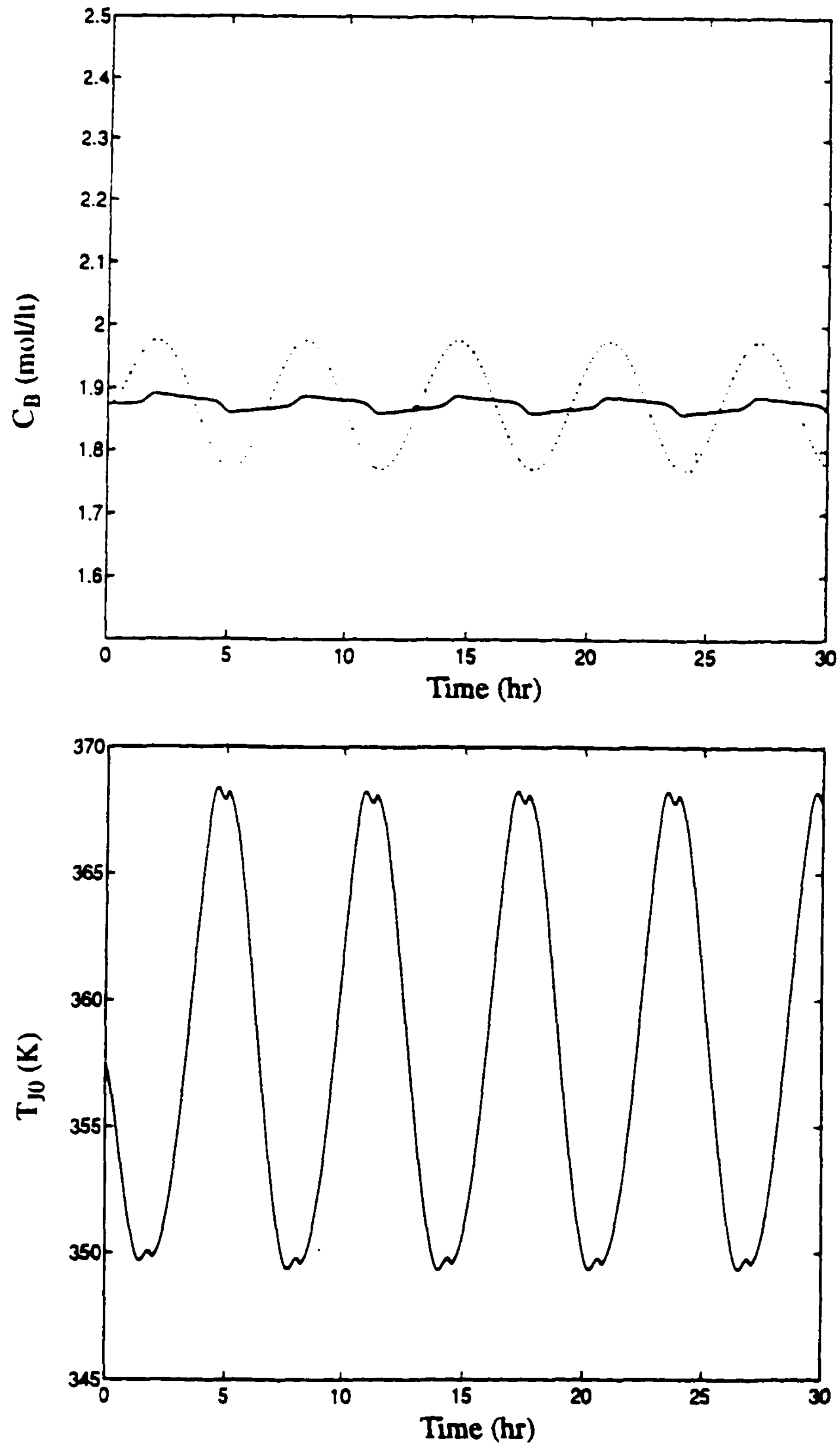

Figure 4.2: Open-loop (dotted) and closed-loop (solid) output profiles and input profile in the presence of uncertainty ($\phi = 0.01$, $\epsilon = 0.08$).

in the reference input value (v = 1.075) was imposed at time t=0. The output and input profiles are shown in Figure 4.3. It is clear that the controller achieves the requested degree of asymptotic attenuation of the effect of the uncertainty on the output and regulates the output to the new reference input value despite the presence of uncertainties.

In the final run, we considered the performance of the controller for $o = 0.001$ under the conditions used in the previous simulation. Figure 4.4 depicts the corresponding output and input profiles. As expected, the degree of attenuation of the uncertainty increases (compare with figure 4.3), while the output of the system is driven to the new reference input value.

4.8 Conclusions

In this chapter, we considered a broad class of singularly perturbed systems with time-varying uncertainties, whose fast dynamics may be unstable or singular. The uncertainty was allowed to be time-varying. It was assumed that the boundary layer system is stabilizable and the reduced system is input/output linearizable with ISS inverse dynamics. For such systems, we proposed a class of well-conditioned control laws that guarantee boundedness of the trajectories of the closed-loop system and achieve arbitrary degree of asymptotic attenuation of the effect of uncertainty on the output, for initial conditions and uncertainties in an arbitrarily large compact set, as long as the singular perturbation parameter is sufficiently small. Fundamental differences in the nature of the matching condition between systems in standard and nonstandard form were identified and discussed. Finally, the derived controller was successfully applied to a nonisothermal continuous stirred tank reactor, with fast jacket dynamics and uncertainties in the concentration and temperature of the inlet stream, modeled by a singularly perturbed system in standard form.

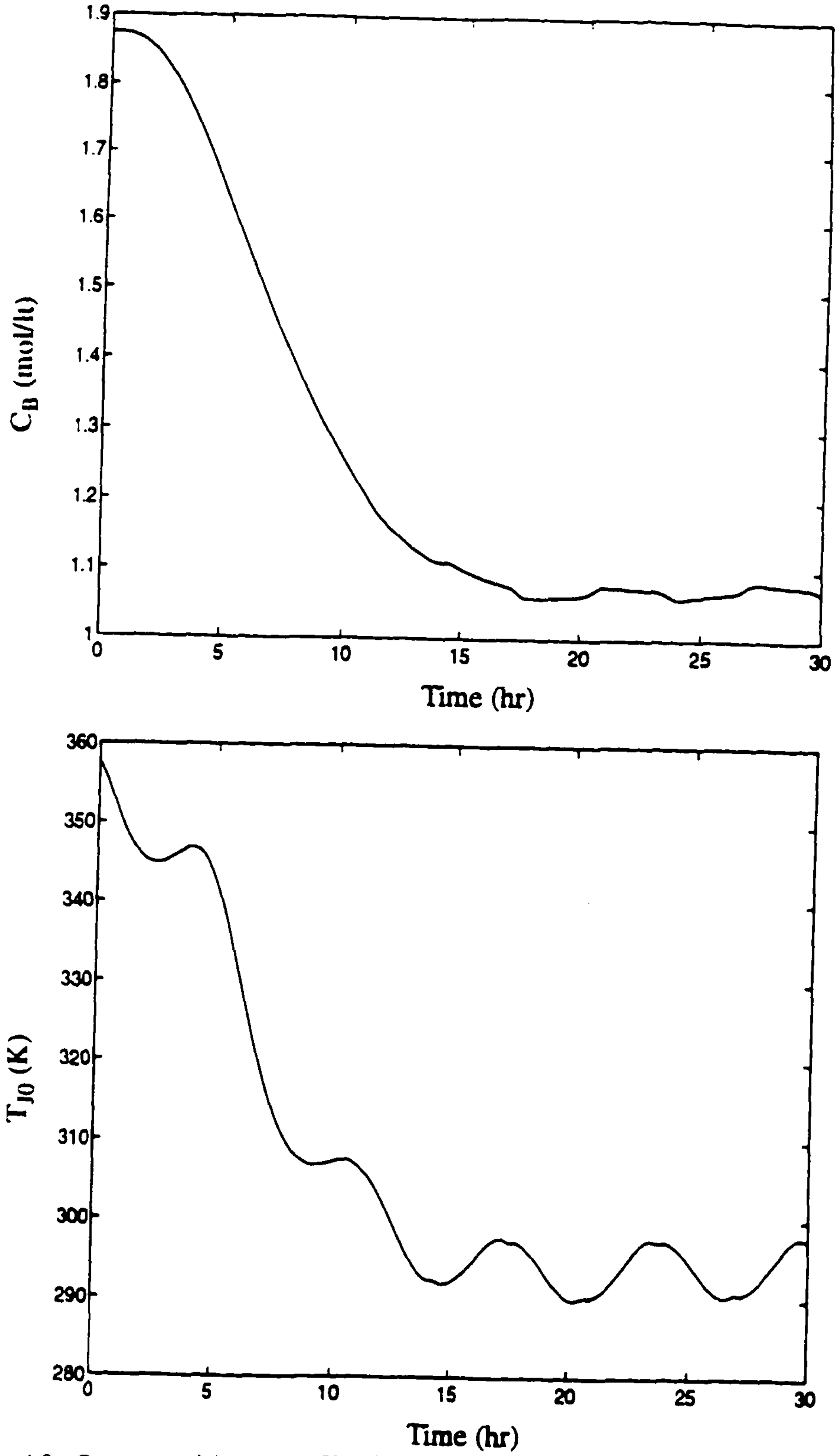

Figure 4.3: Output and input profiles for reference input tracking in the presence of uncertainty ($\phi = 0.01$, $\epsilon = 0.08$).

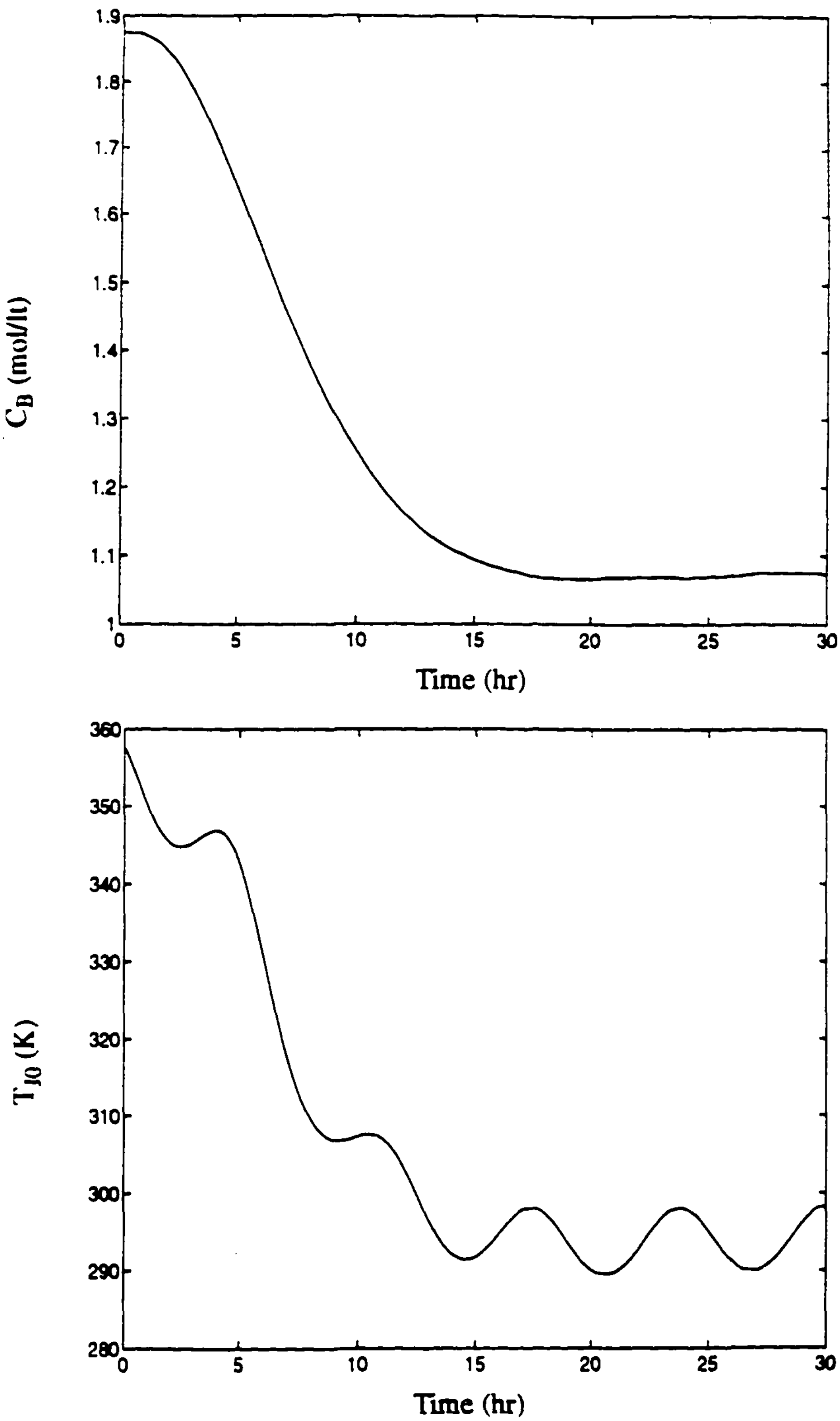

Figure 4.4: Output and input profiles for reference input tracking in the presence of uncertainty ($\alpha = 0.001$, $\epsilon = 0.08$).

Notation

Roman Letters

c_{pm} = heat capacity of the reacting mixture
c_{pj} = heat capacity of the fluid in the jacket
$d, d_{\bar{\epsilon}_{\tilde{r}}}, d_{\bar{\epsilon}}, d_{\bar{\eta}}$ = positive real numbers
E_1, E_2 = activation energies
$F, \tilde{F}, \tilde{F}_{nom}, f_1, f_2$ = vector fields
F_r, F_j = flow rates
$G, \tilde{G}, g_1, g_2$ = vector fields associated with the input
h = output scalar field
k^T = vector field
$\bar{k}, \tilde{k}$ = positive real numbers
k_{10}, k_{20} = pre-exponential constants
Q_1 = matrix of dimension $n \times p$ associated with the slow state vector x
Q_2 = matrix of dimension $p \times p$ associated with the fast state vector z
$r, \tilde{r}$ = integers associated with the input $\tilde{u}$ in the reduced systems
$s, \tilde{s}$ = integers associated with the variable θ in the reduced systems
t = time
u = input
$\tilde{u}$ = auxiliary input
V_j, V_r = volumes of the liquid holdup in the jacket and reactor
v = external reference input
x = vector of the slow state variables
y = output
z = vector of the fast state variables

Greek Letters

β_k = adjustable parameters
$\delta, \delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{\epsilon}_{\tilde{r}}}, \delta_{\bar{\eta}}, \delta_{\bar{\epsilon}}, \delta_\eta, \tilde{\delta}_{\bar{\epsilon}_{\tilde{r}}}, \tilde{\delta}_{\bar{\eta}}$ = positive real numbers
$\Delta H_{r_1}, \Delta H_{r_2}$ = enthalpy of the reactions
$\epsilon, \epsilon^{\bar{\epsilon}_{\tilde{r}}}, \epsilon^{\bar{\eta}}, \epsilon^{\bar{\epsilon}}$ = singular perturbation parameters
ζ = state vector
η = state vector
ρ_j = density of the reacting mixture
ρ_m = density of the fluid in the jacket
ϕ = adjustable parameter

Math Symbols

$L_f h$	=	Lie derivative of a scalar field h with respect to the vector field f
$L_f^k h$	=	k-th order Lie derivative
$L_g L_f^{k-1} h$	=	mixed Lie derivative
$\mathbb{R}$	=	real line
$\mathbb{R}^i$	=	$i-$dimensional Euclidean space
$\in$	=	belongs to
T	=	transpose

Chapter 5

Robust Control of Multivariable Nonlinear Two-Time-Scale Systems

5.1 Introduction

In this chapter, we extend the robust output tracking control methodology proposed in the previous chapter to multi-input multi-output two-time-scale nonlinear systems with time-varying uncertainty, for which the fast dynamics may be unstable or singular. The objective is to provide an explicit formula of a multivariable state feedback controller that guarantees boundedness of the trajectories and an arbitrarily small ultimate discrepancy between the outputs and the reference inputs in the closed-loop system by a suitable selection of controller parameters. The resulting controller is a continuous function of the state of the system, and its construction requires the knowledge of appropriate bounding functions on the size of the uncertain terms. The performance and robustness properties of the developed robust controller design method are evaluated through simulations in an fluidized catalytic cracking reactor, modeled in the standard singularly perturbed form, with unknown heat of the combustion reaction.

5.2 Preliminaries

We consider multi-input multi-output two-time-scale nonlinear systems with the following state-space representation:

$$\begin{array}{rcl} \dot{x} & = & f_1(x,\theta(t)) + Q_1(x,\theta(t))z + G_1(x,\theta(t))u \\ \epsilon\dot{z} & = & f_2(x,\theta(t)) + Q_2(x,\theta(t))z + G_2(x,\theta(t))u \\ y_i & = & h_i(x),\ \ i = 1,\dots,m \end{array} \tag{5.1}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ denote vectors of state variables, $u = [u_1 \ \cdots \ u_m]^T \in \mathbb{R}^m$ denotes the vector of manipulated inputs, $\theta(t) = [\theta_1(t) \ \cdots \ \theta_q(t)] \in \mathbb{R}^q$ denotes the vector of uncertain variables, $y_i \in \mathbb{R}$, denotes the i-th controlled output, and ϵ is a small positive parameter, which quantifies the degree of coupling between the fast and slow dynamical phenomena of the system. $f_1(x,\theta(t))$, $f_2(x,\theta(t))$, are sufficiently smooth vector functions, $Q_1(x,\theta(t))$, $Q_2(x,\theta(t))$, $G_1(x,\theta(t))$, $G_2(x,\theta(t))$, are sufficiently smooth matrices of appropriate dimensions, and $h_i(x)$, $i = 1,\dots,m$ are sufficiently smooth scalar fields. The notation and some definitions used in the paper are given in the appendix.

The fact that ϵ multiplies the time-derivative of the state z allows decomposing the system of Eq.5.1 into separate lower-order systems evolving in different time-scales [KKO86]. Defining a fast time-scale, $\tau = \frac{t}{\epsilon}$, and setting ϵ equal to zero, the following *fast subsystem* is obtained:

$$\frac{dz}{d\tau} = f_2(x,\theta) + Q_2(x,\theta)z + G_2(x,\theta)u \tag{5.2}$$

where x can be considered equal to its initial value $x(0)$ and θ can be thought of as constant. Assumption 5.1 states a stabilizability requirement on the fast subsystem.

Assumption 5.1: *The pair* $[Q_2(x,\theta) \ G_2(x,\theta)]$ *is stabilizable uniformly in* $x \in \mathbb{R}^n$,

$\theta \in \mathbb{R}^q$, in the sense that there exists a sufficiently smooth matrix $K(x)$ such that the matrix $Q_2(x,\theta) + G_2(x,\theta)K(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.

Assuming that the matrix $Q_2(x,\theta)$ is invertible uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ and setting $\epsilon = 0$, the following representation for the equilibrium manifold of the fast dynamics of the system of Eq.5.1 is obtained:

$$z_s = -[Q_2(x,\theta)]^{-1}[f_2(x,\theta) + G_2(x,\theta)u] \tag{5.3}$$

where z_s denotes a quasi-steady-state for the fast state vector z. Utilizing the above equation, the *reduced system* or *slow subsystem* takes the form:

$$\begin{array}{rcl} \dot{x} & = & F(x,\theta) + G(x,\theta)u \\ y_i^s & = & h_i(x), \;\; i = 1,\ldots,m \end{array} \tag{5.4}$$

where the superscript s in y_i^s denotes that this output is associated with the slow subsystem and

$$\begin{array}{rcl} F(x,\theta) & = & f_1(x,\theta) - Q_1(x,\theta)[Q_2(x,\theta)]^{-1}f_2(x,\theta) \\ G(x,\theta) & = & G_1(x,\theta) - Q_1(x,\theta)[Q_2(x,\theta]^{-1}G_2(x,\theta) \end{array} \tag{5.5}$$

5.3 Robust controller design

5.3.1 Control objectives

In this work, we will address the problem of synthesizing well-conditioned (i.e. independent of the singular perturbation parameter ϵ) robust static state feedback laws of the form:

$$u \;\; = \;\; \mathcal{R}(x,v_i^{(k)}) + K(x)z \tag{5.6}$$

where $\mathcal{R}(x,v_i^{(k)})$ is a vector function, $v_i^{(k)}$ denotes the $k-th$ time derivative of the external reference input v_i, which is assumed to be a smooth function of time and $K(x)$ is a sufficiently smooth matrix, which preserve the two-time-scale property of the open-loop system of Eq.5.1 and enforce the following objectives in the closed-loop system:

1. Boundedness of the trajectories

2. Output tracking of external reference inputs with arbitrary degree of asymptotic attenuation of the effect of uncertainty on the outputs

The above requirements will be enforced in the closed-loop system for sufficiently small values of the singular perturbation parameter ϵ.

5.3.2 Assumptions

In the general case where $Q_2(x,\theta)$ is singular or unstable, a preliminary control law of the form:

$$u = \tilde{u} + K(x)z \tag{5.7}$$

where $\tilde{u}$ denotes an auxiliary input and $K(x)$ is a sufficiently smooth matrix of dimension $m \times p$ can be used to stabilize the fast dynamics, and thus induce a well-defined stable quasi-steady-state for the states z. More specifically, substitution of the control law of Eq.5.7 into the singularly perturbed system of Eq.5.1 yields:

$$\begin{aligned} \dot{x} &= f_1(x,\theta) + [Q_1(x,\theta) + G_1(x,\theta)K(x)]z + G_1(x,\theta)\tilde{u} \\ \epsilon\dot{z} &= f_2(x,\theta) + [Q_2(x,\theta) + G_2(x,\theta)K(x)]z + G_2(x,\theta)\tilde{u} \\ y_i &= h_i(x), \quad i = 1,\dots,m \end{aligned} \tag{5.8}$$

From assumption 5.1, $K(x)$ can be chosen so that the matrix $Q_2(x,\theta)+G_2(x,\theta)K(x)$ is Hurwitz, uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, and thus the fast dynamics of the above system is exponentially stable. The reduced system takes then the form:

$$\begin{aligned} \dot{x} &= \bar{F}(x,\theta) + \bar{G}(x,\theta)\tilde{u} \\ \bar{y}_i^s &= h_i(x), \ i = 1,\dots,m \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \bar{F}(x,\theta) &= f_1(x,\theta) - A(x,\theta)f_2(x,\theta) \\ \bar{G}(x,\theta) &= G_1(x,\theta) - A(x,\theta)G_2(x,\theta) \end{aligned} \tag{5.10}$$

and $A(x,\theta) = [Q_1(x,\theta) + G_1(x,\theta)K(x)][Q_2(x,\theta) + G_2(x,\theta)K(x)]^{-1}$. Referring to the system of Eq.5.9 we will state three assumptions that will allow us to proceed with the robust controller design.

Assumption 5.2 : *Referring to the system of Eq.5.9. there exist a set of integers $(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_m)$ and a coordinate transformation $(\zeta, \eta) = T(x, \theta)$ such that the representation of the system, in the coordinates (ζ, η), takes the form:*

$$
\begin{aligned}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\
&\vdots \\
\dot{\zeta}_{\tilde{r}_1-1}^{(1)} &= \zeta_{\tilde{r}_1}^{(1)} \\
\dot{\zeta}_{\tilde{r}_1}^{(1)} &= L_{\tilde{F}}^{\tilde{r}_1} h_1(T^{-1}(\zeta, \eta, \theta)) + \sum_{j=1}^{m} L_{\tilde{G}^j} L_{\tilde{F}}^{\tilde{r}_1-1} h_1(T^{-1}(\zeta, \eta, \theta)) \tilde{u}_j \\
&\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} \\
&\vdots \\
\dot{\zeta}_{\tilde{r}_m-1}^{(m)} &= \zeta_{\tilde{r}_m}^{(m)} \\
\dot{\zeta}_{\tilde{r}_m}^{(m)} &= L_{\tilde{F}}^{\tilde{r}_m} h_m(T^{-1}(\zeta, \eta, \theta)) + \sum_{j=1}^{m} L_{\tilde{G}^j} L_{\tilde{F}}^{\tilde{r}_m-1} h_m(T^{-1}(\eta, \zeta, \theta)) \tilde{u}_j \\
\dot{\eta}_1 &= \Psi_1(\zeta, \eta, \theta, \dot{\theta}) \\
&\vdots \\
\dot{\eta}_{n-\sum_i \tilde{r}_i} &= \Psi_{n-\sum_i \tilde{r}_i}(\zeta, \eta, \theta, \dot{\theta}) \\
\tilde{y}_i^s &= \zeta_1^{(i)}, \; i = 1, \ldots, m
\end{aligned}
\tag{5.11}
$$

where $x = T^{-1}(\zeta, \eta, \theta)$, $\zeta = [\zeta^{(1)} \cdots \zeta^{(m)}]^T$, $\eta = [\eta_1 \cdots \eta_{n-\sum_i \tilde{r}_i}]^T$, and $\tilde{G}^j$ denotes the j-th column vector of the matrix $\tilde{G}(x, \theta)$. Moreover, for each $\theta \in \mathbb{R}^q$, the states ζ, η are bounded if and only if the state x is bounded.

Assumption 5.2 incorporates the matching condition of our methodology, namely that a direct time-differentiation of the output y_i up to order $\tilde{r}_i - 1$ yields a set of equations which are independent of the vector of uncertain variables θ. We note that, from an application point of view, this matching condition is less restrictive from the standard one imposed in [CL81], which restricts $\theta(t)$ to enter the system in the same differential equations as u. The consequence of this generalization is that the η–subsystem depends on $\dot{\theta}(t)$, which implies that we need to assume that $\dot{\theta}(t)$ exists and is bounded (cf. theorem 5.1). Referring to the system of Eq.5.11, we will assume, for simplicity, that the matrix:

$$\tilde{C}(x,\theta) = \begin{bmatrix} L_{\tilde{G}^1} L_{\tilde{F}}^{\tilde{r}_1-1} h_1(x,\theta) & \cdots & L_{\tilde{G}^m} L_{\tilde{F}}^{\tilde{r}_1-1} h_1(x,\theta) \\ \vdots & \cdots & \vdots \\ L_{\tilde{G}^1} L_{\tilde{F}}^{\tilde{r}_m-1} h_m(x,\theta) & \cdots & L_{\tilde{G}^m} L_{\tilde{F}}^{\tilde{r}_m-1} h_m(x,\theta) \end{bmatrix} \quad (5.12)$$

is nonsingular uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ (see remark 5.6 for a discussion on how to handle the case of singular $\tilde{C}(x,\theta)$).

Assumption 5.3 : *The dynamical system:*

$$\begin{aligned} \dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\sum_i \tilde{r}_i} &= \Psi_{n-\sum_i \tilde{r}_i}(\zeta,\eta,\theta,\dot{\theta}) \end{aligned} \quad (5.13)$$

is input-to-state stable with respect to $\zeta,\theta,\dot{\theta}$.

The above assumption is more restrictive compared to the standard one of asymptotic stability of the zero dynamics without inputs [CL81], however it is necessary in order to prove that the states of the closed-loop system are bounded for initial conditions, uncertainty and rate of change of uncertainty arbitrarily large (see also [TP95]). Furthermore, this assumption could be relaxed to local asymptotic stability of the $\eta-$subsystem with $(\zeta,\theta,\dot{\theta})$ identically equal to zero, leading to local results (see remark 5.2).

We will now quantify possible knowledge about the uncertainty by assuming the existence of known bounding functions that capture the size of the uncertain terms in the system of Eq.5.11. To this end, we define the variable $\tilde{\delta}(x,\theta)$ as:

$$\tilde{\delta}(x,\theta) = [\tilde{F}(x,\theta) - \tilde{F}_{nom}(x)] \quad (5.14)$$

where $\tilde{F}_{nom}(x)$ denotes the vector field resulting from $\tilde{F}(x,\theta)$ by setting θ equal to its nominal value θ_0. Using the fact that $L_{\tilde{F}}^{k-1} h_i(x) = L_{\tilde{F}_{nom}}^{k-1} h_i(x)$, $i = 1,\ldots,m$,

$k = 1, \ldots, \bar{r}_i$. Eq.5.11 can be written as follows:

$$
\begin{aligned}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\
&\vdots \\
\dot{\zeta}_{\bar{r}_1-1}^{(1)} &= \zeta_{\bar{r}_1}^{(1)} \\
\dot{\zeta}_{\bar{r}_1}^{(1)} &= L_{\tilde{F}_{nom}}^{\bar{r}_1} h_1(T^{-1}(\zeta,\eta,\theta)) + \sum_{j=1}^{m} L_{\tilde{G}_j} L_{\tilde{F}}^{\bar{r}_1-1} h_1(T^{-1}(\zeta,\eta,\theta))\tilde{u}_j \\
&\quad + L_{\tilde{\xi}} L_{\tilde{F}}^{\bar{r}_1-1} h_1(T^{-1}(\zeta,\eta,\theta)) \\
&\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} \\
&\vdots \\
\dot{\zeta}_{\bar{r}_m-1}^{(m)} &= \zeta_{\bar{r}_m}^{(m)} \\
\dot{\zeta}_{\bar{r}_m}^{(m)} &= L_{\tilde{F}_{nom}}^{\bar{r}_m} h_m(T^{-1}(\zeta,\eta,\theta)) + \sum_{j=1}^{m} L_{\tilde{G}_j} L_{\tilde{F}}^{\bar{r}_m-1} h_m(T^{-1}(\eta,\zeta,\theta))\tilde{u}_j \\
&\quad + L_{\tilde{\xi}} L_{\tilde{F}}^{\bar{r}_m-1} h_1(T^{-1}(\zeta,\eta,\theta)) \\
\dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) \\
&\vdots \\
\dot{\eta}_{n-\sum_i \bar{r}_i} &= \Psi_{n-\sum_i \bar{r}_i}(\zeta,\eta,\theta,\dot{\theta}) \\
\tilde{y}_i^s &= \zeta_1^{(i)}, \quad i = 1,\ldots,m
\end{aligned}
\tag{5.15}
$$

where $\zeta_k^{(i)} = L_{\tilde{F}_{nom}}^{k-1} h_1(T^{-1}(x), i = 1,\ldots,m, k = 1,\ldots,\bar{r}_i$.

Assumption 5.4: *The terms $L_{\tilde{G}_j} L_{\tilde{F}}^{\bar{r}_i-1} h_i(x)$, $i \in [1,m]$, have the same known constant sign, sgn_j, for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, for each $j \in [1,m]$, (some of these terms may be zero, but the requirement that $\tilde{C}(x,\theta)$ is nonsingular has to be satisfied), and there exist known functions $\tilde{c}_1(x,t)$, $\tilde{c}_{2j}(x,t)$, such that the following conditions hold:*

$$
\begin{aligned}
|[L_{\tilde{\xi}} L_{\tilde{F}}^{\bar{r}_1-1} h_1(x) \ \cdots \ L_{\tilde{\xi}} L_{\tilde{F}}^{\bar{r}_m-1} h_m(x)]^T| &\leq \tilde{c}_1(x,t) \\
0 < \tilde{c}_{2j}(x,t) &\leq \min_{i\in[1,m]} |L_{\tilde{G}_j} L_{\tilde{F}}^{\bar{r}_i-1} h_i(x)|
\end{aligned}
\tag{5.16}
$$

and there exists a known positive real number b such that the $m \times m$ matrix $M(x,\theta,t) = \tilde{C}(x,\theta)\Delta(x,t)$, where $\Delta(x,t)$ is an $m \times m$ diagonal matrix whose (i,i)-th element is of the form $\frac{sgn_i}{\tilde{c}_{2i}}$, satisfies the property $1 \leq \sigma_{min}\{M(x,\theta,t)\} \leq \sigma_{max}\{M(x,\theta,t)\} \leq b$, for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, $t \geq 0$.

Assumption 5.4 requires the existence of a nonlinear function that captures the size of the additive uncertain terms in the system of Eq.5.11. This requirement is standard in all Lyapunov-based robust control methods (see [CL81, KP88, AC92] for example). However, in contrast to [CL81, KP88, AC92], we do not impose any growth conditions on the nonlinear function $\bar{c}_1(x,t)$, such as quadratic growth, etc.. Assumption 5.4 also requires the existence of a nonlinear function $\bar{c}_{2j}(x,t)$ for each manipulated input u_j that lower bounds the m-coefficients $(L_{\tilde{G}j}L_{\tilde{F}}^{\bar{r}_1-1}h_1(x),\ldots,L_{\tilde{G}j}L_{\tilde{F}}^{\bar{r}_m-1}h_m(x))$ (note that this requirement does not have to be satisfied for the coefficients that are zero). This requirement is less restrictive than the ones used in [CL81, KP88, AC92], where an upper bound (typically small) on the size of the uncertain terms that enter the system coupled with the manipulated inputs is assumed to exist.

5.3.3 Main result

We are now in position to proceed with the design of the auxiliary input $\tilde{u}$ to achieve the aforementioned control objectives in the closed-loop reduced system. Specifically, motivated by the requirement of output tracking, and the presence of additive and multiplicative uncertainties in the systems of Eq.5.11, we consider static state feedback laws of the form:

$$\tilde{u} \quad = \quad R_1(x,t)[p(x)+Q(x,v_i^{(k)})+R_2(x,t)] \tag{5.17}$$

where $p(x)$, $R_2(x,t)$ are vector functions, $R_1(x,t)$ is a matrix, $Q(x,v_i^{(k)})$ is a column vector, and $v_i^{(k)}$ denotes the $k-th$ time derivative of the external reference input v^i, which is assumed to be a smooth function of time. The motivation for considering control laws of the form of Eq.5.17 is provided by the requirement of stabilization with output tracking in the nominal reduced system (term $p(x)+Q(x,v_i^{(k)})$), and the fact that the system in Eq.5.9 includes uncertainties entering coupled with the input (term $R_1(x,t)$) as well as additive ones (term $R_2(x,t)$).

Theorem 5.1 that follows provides an explicit formula of the controller that solves the control problem formulated in the previous section (the proof is given in the appendix D). To simplify the statement of the theorem, we set $\bar{v}_i = [v_i \; v_i^{(1)} \; \cdots \; v_i^{(\bar{r}_i)}]^T$ and $\bar{v} = [\bar{v}_1^T \; \bar{v}_2^T \; \cdots \; \bar{v}_m^T]^T$.

Theorem 5.1: *Consider the uncertain singularly perturbed nonlinear system of Eq.5.1, for which assumptions 5.1, 5.2, 5.3, and 5.4 hold, and the static state feedback law:*

$$
\begin{aligned}
u \;=\; & K(x)z + \Delta(x,t)\left\{\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}(v_i^{(k)} - L_{\bar{F}_{nom}}^{k}h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}(v_i^{(k-1)}\right. \\
& \left. - L_{\bar{F}_{nom}}^{k-1}h_i(x)) - (2+b)[\bar{c}_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}(v_i^{(k)} - L_{\bar{F}_{nom}}^{k}h_i(x)) + |]w(x,\phi)\right\}
\end{aligned}
\tag{5.18}
$$

where the feedback matrix $K(x)$ is such that the matrix $Q_2(x,\theta) + G_2(x,\theta)K(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, $\frac{\beta_{ik}}{\beta_{i\bar{r}_i}} = [\frac{\beta_{ik}^1}{\beta_{i\bar{r}_i}^1} \; \cdots \; \frac{\beta_{ik}^m}{\beta_{i\bar{r}_i}^m}]^T$ are column vectors of parameters chosen so that the roots of the equation $det(B(s)) = 0$, where $B(s)$ is an $m \times m$ matrix, whose (i,j)-th element is of the form $\sum_{k=1}^{\bar{r}_i}\frac{\beta_{jk}^i}{\beta_{j\bar{r}_i}^i}s^{k-1}$, lie in the open left-half of the complex plane, and the vector function $w(x,\phi)$ is given by:

$$
w(x,\phi) \;=\; \frac{\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i-1}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}(L_{\bar{F}_{nom}}^{k-1}h_i(x) - v_i^{(k-1)})}{\displaystyle|\sum_{i=1}^{m}\sum_{k=1}^{\bar{r}_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}(L_{\bar{F}_{nom}}^{k-1}h_i(x) - v_i^{(k-1)})| + \phi}
\tag{5.19}
$$

where ϕ is an adjustable parameter. Then, for each set of positive real numbers $\delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d$, there exists ϕ^* and for each $\phi \leq \phi^*$, there exists $\epsilon^*(\phi)$, such that if $\phi \leq \phi^*$, $\epsilon \leq \epsilon^*(\phi)$ and $|x(0)| \leq \delta_x$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$,

a) the state of the closed-loop system is bounded, and

b) the outputs of the closed-loop system satisfy:

$$
\limsup_{t\to\infty}|y_i - v_i| \leq d, \quad i = 1,\ldots,m \tag{5.20}
$$

Remark 5.1: Referring to the result of theorem 5.1, we note that the dependence of the upper bound on the singular perturbation parameter, ϵ^*, on the adjustable parameter, ϕ, is due to the presence of ϕ on the dynamical system that describes the fast dynamics of the closed-loop system.

Remark 5.2: Theorem 5.1 establishes a semi-global type result (the initial conditions, uncertainty and rate of change of uncertainty can be in arbitrarily large compact sets) and does not impose any kind of interconnection or growth conditions on the nonlinearities of the system. We note that if the coordinate transformation of assumption 5.2 holds locally and the unforced zero dynamics:

$$\begin{aligned} \dot{\eta}_1 &= \Psi_1(\zeta, 0, 0, 0) \\ &\vdots \\ \dot{\eta}_{n-\sum_i \tilde{r}_i} &= \Psi_{n-\sum_i \tilde{r}_i}(\zeta, 0, 0, 0) \end{aligned} \tag{5.21}$$

is locally exponentially stable, then the result of theorem 5.1 holds locally (i.e. for sufficiently small initial conditions, uncertainty and rate of change of uncertainty).

Remark 5.3: Referring to the control law of Eq.5.18, one can show that if the matrix $\tilde{C}(x,\theta)$ in the system of Eq.5.9 is independent of θ, then the following simplifications can be made: $\Delta(x,t) = \{\tilde{C}(x)\}^{-1}$, $b = 0$, and $[\bar{c}_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x))|] = \bar{c}_1(x,t)$.

Remark 5.4: Referring to the control law of Eq.5.18, we note that the explicit expression of the term which is responsible for the attenuation of the effect of additive uncertainties is nonlinear not only is the x-coordinates but also in the (ζ,η) coordinates (see Eqs.C.10-C.11). This is in agreement with the robust control methods [CL81, KP88] and contrast to the robust control methodology proposed in [AC92], where the expression of the term $R_2(x,t)$ is linear in (ζ,η) coordinates, and is a consequence of the fact that we allow $\bar{c}_1(x,t)$ to be a general nonlinear function.

Remark 5.5: The singular perturbation formulation provides a natural setting for addressing robustness with respect to unmodeled dynamics [KKO86]. In particular, if the open-loop fast subsystem of Eq.5.2 is exponentially stable uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, the controller of Eq.5.18 can be simplified to:

$$
\begin{aligned}
u \;=\; & \Delta(x,t)\left\{\sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k)} - L_{F_{nom}}^{k}h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{ir_i}}(v_i^{(k-1)} - L_{F_{nom}}^{k-1}h_i(x))\right.\\
& \left. -(2+b)[c_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{r_i}\frac{\beta_{ik}}{\beta_{i\bar{r}_i}}(v_i^{(k)} - L_{F_{nom}}^{k}h_i(x)) + |]w(x,\phi)\right\}
\end{aligned}
\tag{5.22}
$$

which can be explicitly synthesized utilizing information about the open-loop reduced system of Eq.5.4. In this case, the result of theorem 5.1 establishes a fundamental robustness property of the controller of Eq.5.22, with respect to uniformly exponentially stable unmodeled dynamics, provided that they are sufficiently fast. This robustness property is very important in many practical applications, where fast dynamics, such as sensor and actuator dynamics, are usually neglected in the controller design.

Remark 5.6: Whenever the characteristic matrix $\tilde{C}(x,\theta)$ of the reduced system of Eq.5.9 is singular, one can utilize available dynamic extension algorithms (see for example [Isi89]) to design a dynamic state feedback law of the general form:

$$
\begin{aligned}
\dot{\xi} &= a(x,\xi) + b(x,\xi)\bar{v}\\
\tilde{u} &= c(x,\xi) + d(x,\xi)\bar{v}
\end{aligned}
\tag{5.23}
$$

where $a(x,\xi), b(x,\xi), c(x,\xi), d(x,\xi)$ are vector functions of appropriate dimensions, $\bar{v} \in \mathbb{R}^m$ denotes a vector of auxiliary inputs, such that the augmented system

$$
\begin{aligned}
\dot{\xi} &= a(x,\xi) + b(x,\xi)\bar{v}\\
\dot{x} &= \tilde{F}(x,\theta) + \tilde{G}(x,\theta)c(x,\xi) + \tilde{G}(x,\theta)d(x,\xi)\bar{v}\\
\tilde{y}_i^s &= h_i(x),\; i = 1,\ldots,m
\end{aligned}
\tag{5.24}
$$

with $\tilde{y}^s \in \mathbb{R}^m$ as output vector and $\bar{v} \in \mathbb{R}^m$ as input vector possesses a nonsingular characteristic matrix. Then, the robust control methodology of this paper can be applied to the system of Eq.5.24.

5.4 Application to a fluidized catalytic cracker

In this section, we illustrate the implementation of the developed control methodology on the industrially significant process of a fluidized catalytic cracker (FCC) shown in Figure 5.1. The FCC unit consists of a cracking reactor, where the desired reactions include cracking of high boiling gas oil fractions into lighter hydrocarbons (e.g. gasoline) and the undesired ones include carbon formation reactions, and a regenerator, where the carbon removal reactions take place. For a detailed discussion on the features of the FCC unit, the reader may refer to [Den86, MRBG93], while applications of linear control methods to the process can be found for example in [MG87, HMS95]. Under the following standard modeling assumptions:

- Well-mixed reactive catalyst in the reactor
- Small-size catalyst particles
- Constant solid holdup in reactor and regenerator
- Uniform and constant pressure in reactor and regenerator

the process dynamic model takes the form [Den86]:

$$\begin{aligned}
V_{ra}\frac{dC_{cat}}{dt} &= -60F_{rc}C_{cat} + 50R_{cf} \\
V_{ra}\frac{dC_{sc}}{dt} &= -60F_{rc}(C_{rc} - C_{sc}) + 50R_{cf} \\
V_{ra}\frac{dT_{ra}}{dt} &= 60F_{rc}(T_{rg} - T_{ra}) + 0.875\frac{S_f}{S_c}D_{tf}R_{tf}(T_{fp} - T_{ra}) \\
&\quad +0.875\frac{(-\Delta H_{fv})}{S_c}D_{tf}R_{tf} + 0.5\frac{(-\Delta H_{cr})}{S_c}R_{oc} \qquad (5.25) \\
V_{rg}\frac{dC_{rc}}{dt} &= 60F_{rc}(C_{sc} - C_{rc}) - 50R_{cb} \\
V_{rg}\frac{dT_{rg}}{dt} &= 60F_{rc}(T_{ra} - T_{rg}) + 0.5\frac{S_a}{S_c}R_{ai}(T_{ai} - T_{rg}) + 0.5(-\frac{\Delta H_{rg}}{S_c})R_{cb}
\end{aligned}$$

where C_{cat}, C_{sc}, C_{rc} denote the concentrations of catalytic carbon on spent catalyst, the total carbon on spent catalyst, and carbon on regenerated catalyst, T_{ra}, T_{rg} denote

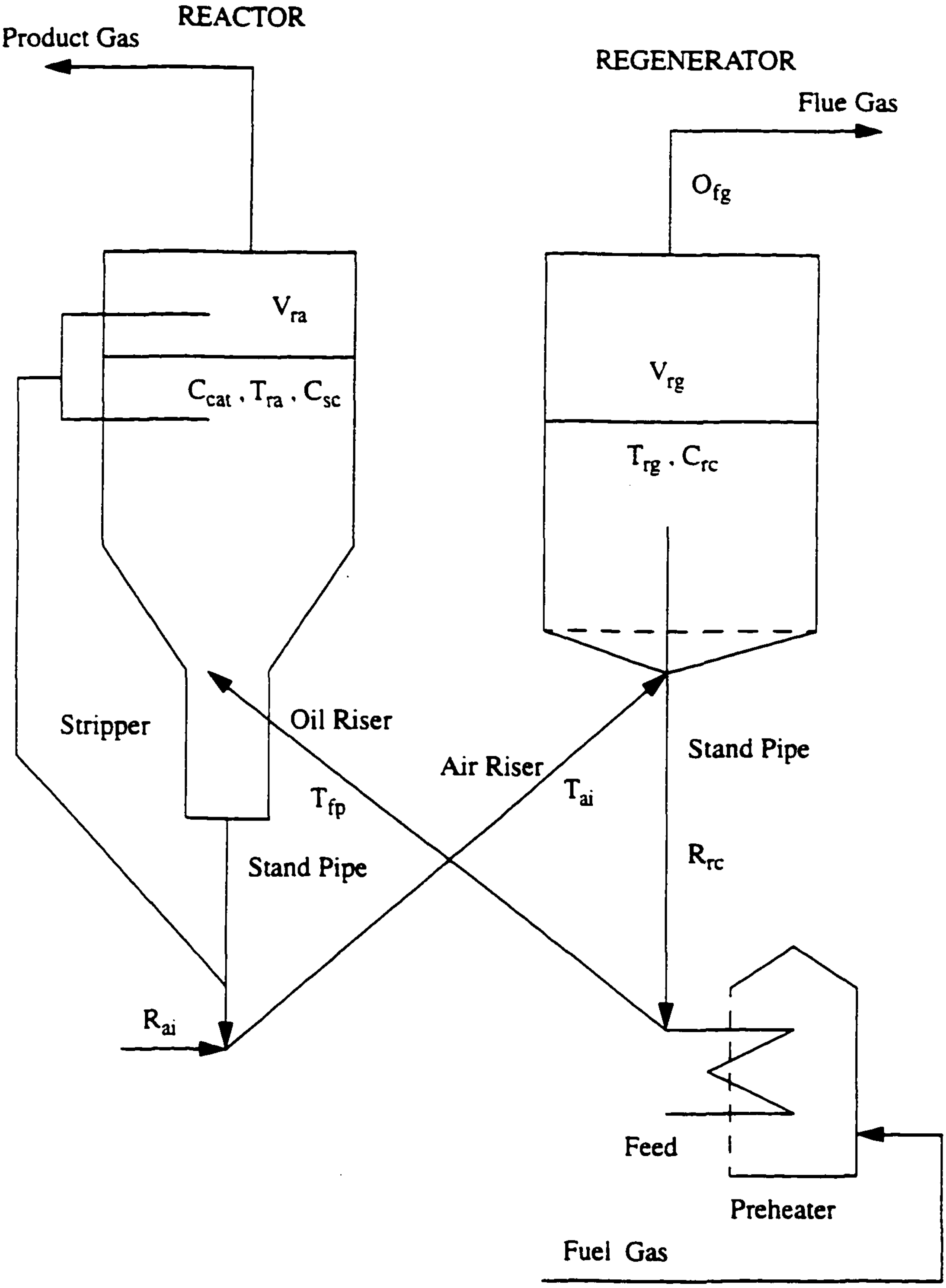

Figure 5.1: A fluidized catalytic cracker.

the temperatures in the reactor and the regenerator. D_{tf} is the density of total feed. V_{ra}, V_{rg} denote the holdup of the reactor and the regenerator. $\Delta H_{rg}, \Delta H_{cr}$ are heat of reactions. ΔH_{fv} is the heat of feed vaporization. F_{rc} denotes the flow rate of catalyst from reactor to regenerator. S_a, S_c, S_f denote specific heats. T_{fp}, T_{ai} denote the inlet temperatures of the feed in the reactor and the air in the regenerator, and R_{cf}, R_{oc}, R_{cb} denote reaction rates. The analytic expressions for the reaction rates R_{cf}, R_{oc}, R_{cb} are [Den86]:

$$\begin{aligned}
R_{cf} &= \frac{k_{cc} V_{ra} P_{ra}}{C_{cat} C_{rc}^{0.06}} exp\left\{\frac{-E_{cc}}{R(T_{ra}+460.0)}\right\} \\
R_{oc} &= \frac{k_{cr} V_{ra} P_{ra} R_{tf} D_{tf} K_{cr}}{R_{tf} + V_{ra} P_{ra} K_{cr}} \\
K_{cr} &= \frac{k_{cr}}{C_{cat} C_{rc}^{0.15}} exp\left\{\frac{-E_{cr}}{R(T_{ra}+460.0)}\right\} \\
R_{cb} &= \frac{R_{ai}(21 - O_{fg})}{200} \\
O_{fg} &= 21 exp\left\{\frac{-\frac{V_{rg} P_{rg}}{R_{ai}}}{\frac{10^6}{4.76 R_{ai}^2} + \frac{100}{k_{or}} exp\left\{\frac{-E_{or}}{R(T_{rg}+460.0)} C_{rc}\right\}}\right\}
\end{aligned} \tag{5.26}$$

where k_{cc}, k_{cr}, k_{or} are pre-exponential kinetic rate constants, E_{cc}, E_{cr}, E_{or} are activation energies. O_{fg} is the oxygen in flue gas, R_{tf} is the total feed rate, and R_{ai} is the air rate. The values of the process parameters and the corresponding steady-state values are given in table 5.1. The process exhibits a two-time-scale behavior because the residence time in the reactor is smaller than the one in the regenerator [Den86]. This implies that although the operator focus is on the reactor, the control problem must focus on the regenerator which is the process that essentially determines the dynamic response of the entire FCC unit. To this end, the control objective is the regulation of the temperature in the regenerator, T_{rg}, and the concentration of the carbon on the regenerated catalyst, C_{rc}, by manipulating the inlet temperatures, T_{fp} and T_{ai}. The uncertain variable for the system is taken to be the heat of combustion

E_{cc}	=	18000.0	$Btu\ lb^{-1}\ mole^{-1}$
E_{cr}	=	27000.0	$Btu\ lb^{-1}\ mole^{-1}$
E_{oc}	=	63000.0	$Btu\ lb^{-1}\ mole^{-1}$
k_{cc}	=	8.59	$Mlb\ hr^{-1}\ psia^{-1} ton^{-1} (wt\%)^{-1.06}$
k_{cr}	=	11600	$Mbbl\ day^{-1}\ psia^{-1} ton^{-1} (wt\%)^{-1.15}$
k_{or}	=	3.15×10^{10}	$Mlb\ hr^{-1}\ psia^{-1} ton^{-1}$
O_{fg}	=	0.2	$mole\%$
V_{rg}	=	60.0	ton
V_{ra}	=	200.0	ton
T_{fps}	=	744.0	F
T_{as}	=	175.0	F
P_{rg}	=	25.0	$psia$
P_{ra}	=	40.0	$psia$
ΔH_{fv}	=	60.0	$Btu\ lb^{-1}$
ΔH_{cr}	=	77.2	$Btu\ lb^{-1}$
ΔH_{rg}	=	10700.0	$Btu\ lb^{-1}$
S_a	=	0.3	$Btu\ lb^{-1} F^{-1}$
S_c	=	0.7	$Btu\ lb^{-1} F^{-1}$
R_{rc}	=	40.0	$ton\ min^{-1}$
R_{tf}	=	100.0	$Mbbl/day$
D_{tf}	=	7.0	$lb\ gal^{-1}$
R_{as}	=	400.0	$Mlb\ min^{-1}$
C_{cat}	=	0.875	$wt\%$
C_{sc}	=	1.475	$wt\%$
C_{rc}	=	0.6	$wt\%$
T_{ra}	=	930.0	F
T_{rg}	=	1155.0	F

Table 5.1: Process parameters and steady-state values

in the regeneration, which is also assumed to be time-varying. Defining the singular perturbation parameter ϵ as:

$$\epsilon = \frac{V_{ra}}{V_{rg}} \tag{5.27}$$

and setting $x = [x_1 \; x_2]^T = [C_{rc} \; T_{rg}]^T$, $z = [z_1 \; z_2 \; z_3]^T = [C_{cat} \; C_{sc} \; T_{ra}]^T$, $u = [u_1 \; u_2]^T = [T_{fp} \; T_{ai}]^T$, $y = [y_1 \; y_2]^T = [C_{rc} \; T_{rg}]^T$, $\theta = [\Delta h_{rg}]$, the system of Eq.5.25 can be put into the standard singularly perturbed form:

$$\begin{aligned}
\frac{dx_1}{dt} &= f_{11}(x_1, x_2) + Q_{11}z_2 \\
\frac{dx_2}{dt} &= f_{12}(x_1, x_2, \theta) + Q_{12}z_3 + g_{12}u_2 \\
\epsilon\frac{dz_1}{dt} &= Q_{21}(z_1, z_2, z_3, x_1) \\
\epsilon\frac{dz_2}{dt} &= f_{22}(x_1) + Q_{22}(z_1, z_2, z_3, x_1) \\
\epsilon\frac{dz_3}{dt} &= f_{23}(x_2) + Q_{23}(z_1, z_2, z_3, x_1) + g_{23}u_1
\end{aligned} \tag{5.28}$$

where $f_{11}, Q_{11}, f_{12}, Q_{12}, g_{12}, Q_{21}, f_{22}, Q_{22}, f_{23}, Q_{23}, g_{23}$ are functions whose specific form is omitted for brevity. It was verified that the system of Eq.5.28 possesses an exponentially stable equilibrium manifold for the fast dynamics, which implies that it is not needed to utilize a preliminary feedback law of the form of Eq.5.7 to stabilize the fast dynamics. Setting $\epsilon = 0$, the equilibrium manifold of the fast dynamics can be calculated analytically and is of the form: $z_s = g(x_1, x_2, u_1)$, where g is a smooth vector function (note that the input u_1 enters this algebraic equation in a nonlinear fashion, due to the nonlinear appearance of the fast state z in the system of Eq.5.28). The reduced system can then be found to be of the form:

$$\begin{aligned}
\frac{dx_1}{dt} &= F_1(x_1, x_2) + G_{11}(x_1, x_2, u_1) \\
\frac{dx_2}{dt} &= F_2(x_1, x_2, \theta) + G_{21}(x_1, x_2, u_1) + G_{22}u_2
\end{aligned} \tag{5.29}$$

with $F_1, G_{11}, F_2, G_{21}, G_{22}$ appropriately defined (their exact expressions are omitted for brevity). The system of Eq.5.29 is already in the form of Eq.5.11, with u appearing in a nonlinear fashion, and the condition $r_1 + r_2 = 1 + 1 = 2$ holds, which implies that this system does not possess zero dynamics. Furthermore, assumption 5.4 is also satisfied and the function $\bar{c}_1(x)$ takes the form:

$$\bar{c}_1(x) = F_2(x_1, x_2, |\theta|) \tag{5.30}$$

where $|\theta|$ denotes the upper bound on the size of the uncertain variable, which is assumed to be constant. Clearly, the assumptions required for the application of the result of the theorem are satisfied. Referring to the system of Eq.5.29, we also note that the uncertain variable θ does not appear coupled with the input variables u_1, u_2 and that the differential equation that describes the evolution of x_1 is independent of θ and u_2. The former fact allows using the simplifications discussed in remark 5.3 in the controller formula, while the latter fact suggests that there is no need for uncertainty compensation term to be included in the formula that calculates the control action for the manipulated input u_1. Note that the synthesis formula of Eq.5.18 cannot be readily used due to the nonlinear appearance of the input u_1 in the system of Eq.5.29. To resolve this problem, we first considered the algebraic equation $G_{11}(x_1, x_2, u_1) = \alpha$ and derived its solution in terms of u_1, i.e., $u_1 = \bar{G}_{11}(x_1, x_2, \alpha)$. Then, the necessary controller was found to be:

$$\begin{aligned}
u_1 &= \bar{G}_{11}(x_1, x_2, \frac{1}{\beta_{11}}(v_1 - x_1) - F_1(x_1, x_2)) \\
u_2 &= \frac{1}{G_{22}}\left\{\frac{1}{\beta_{21}}(v_2 - x_2) - [F_2(x_1, x_2, \theta_0)\right. \\
&\quad \left. + G_{21}(x_1, x_2, \bar{G}_{11}(x_1, x_2, \frac{1}{\beta_{11}}(v_1 - x_1) - F_1(x_1, x_2)) - 2\bar{c}_1(x)\frac{x_2 - v_2}{|x_2 - v_2| + \phi}\right\}
\end{aligned} \tag{5.31}$$

For the above robust controller, we note that its practical implementation requires measurements of only two of the states of the process (concentration of carbon on regenerated catalyst and temperature of the regenerator). A slowly-varying uncertainty was considered expressed by a sinusoidal function of the form:

$$\theta = 180.0 sin(0.13 * t) \tag{5.32}$$

The upper bound on the uncertainty was taken to be $|\theta| = 180.0$. From theorem 5.1, it is clear that there exists a trade-off between the upper bound on the value of the singular perturbation parameter ϵ^* and the level of asymptotic attenuation d that can be achieved. In the application in question, the value of the singular perturbation parameter is fixed by the design of the process, i.e. $\epsilon_p = 0.3$, and thus there exists a lower bound on the selection of the level d. We performed a set of computer simulations (for the regulation problem) to calculate ϕ^* for certain values of d, and, in turn, the value of ϵ^* for $\phi \leq \phi^*$. The following set of parameters were found to give an $\epsilon^* \leq \epsilon_p$ and used in the simulations :

$$\beta_{11} = 0.1, \quad \beta_{21} = 0.02, \quad \phi = 0.5 \tag{5.33}$$

to achieve an ultimate degree of attenuation $d = 0.01$, for a value of the singular perturbation parameter $\epsilon = 0.3$.

Two representative simulation runs are reported. In both runs, the process was initially ($t = 0.0\ hr$) assumed to be at steady-state. In the first simulation run, we tested the regulatory capabilities of the controller. Figure 5.2 shows the closed-loop output profiles, while Figure 5.3 displays the corresponding manipulated input profiles. Clearly the controller regulates the output at the operating steady-state compensating for the effect of uncertainty and satisfying the requirements $\limsup_{t\to\infty} |y_1 - v_1| \leq 0.01$, $\limsup_{t\to\infty} |y_2 - v_2| \leq 0.01$. For the sake of comparison, we also implemented the same controller without the term which is responsible for the compensation of uncertainty, i.e. a decoupling input/output linearizing controller for the nominal open-loop reduced system. The output profiles for this simulation run are shown in Figure 5.4. One can immediately see how strong is the effect of the uncertainty on the outputs of the process, leading to poor transient performance and offset. In the next simulation run, we tested the output tracking capabilities of the controller. A 25.0 F increase in the value of the output y_2 was imposed at

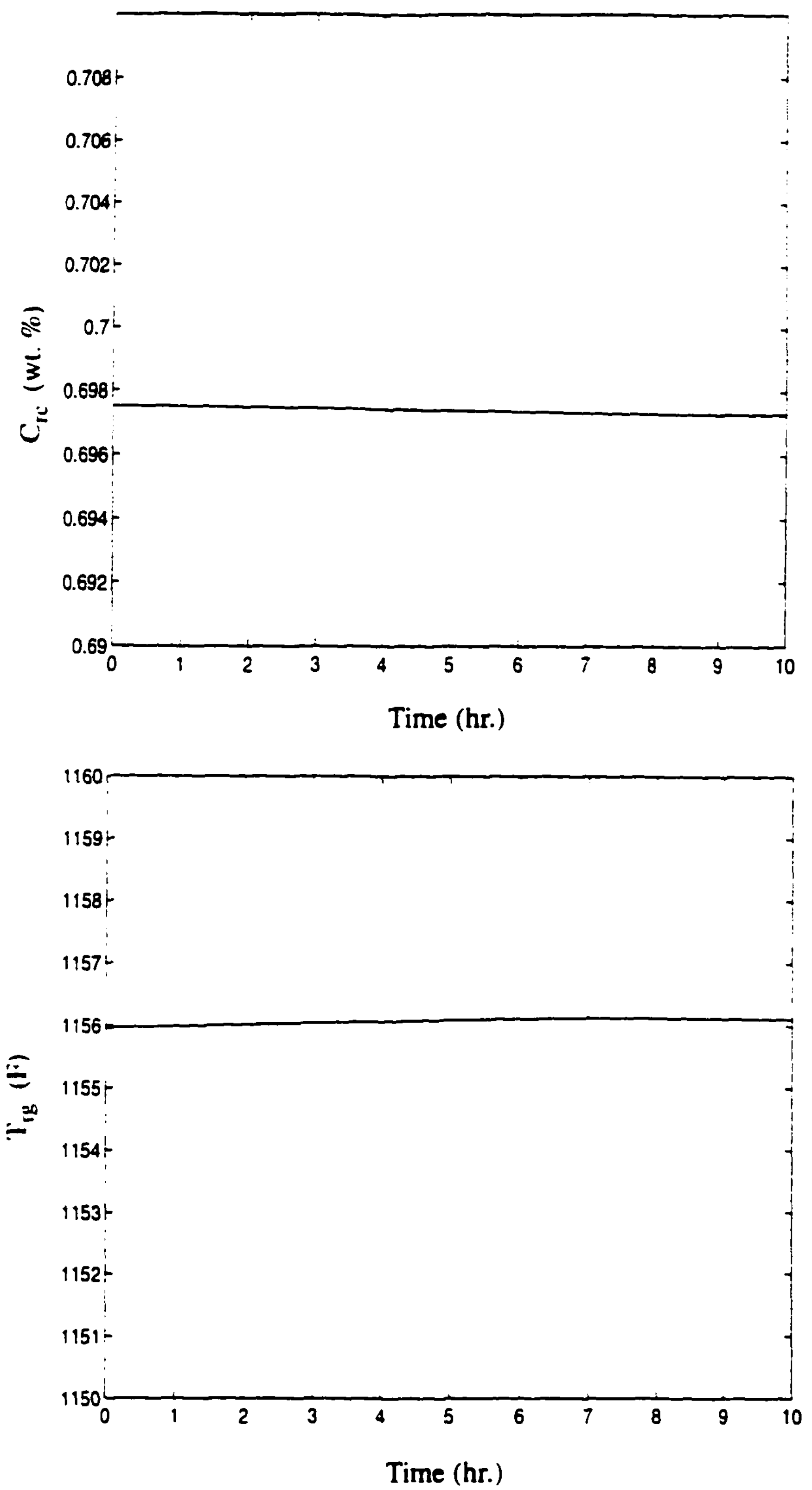

Figure 5.2: Closed-loop output profiles for regulation.

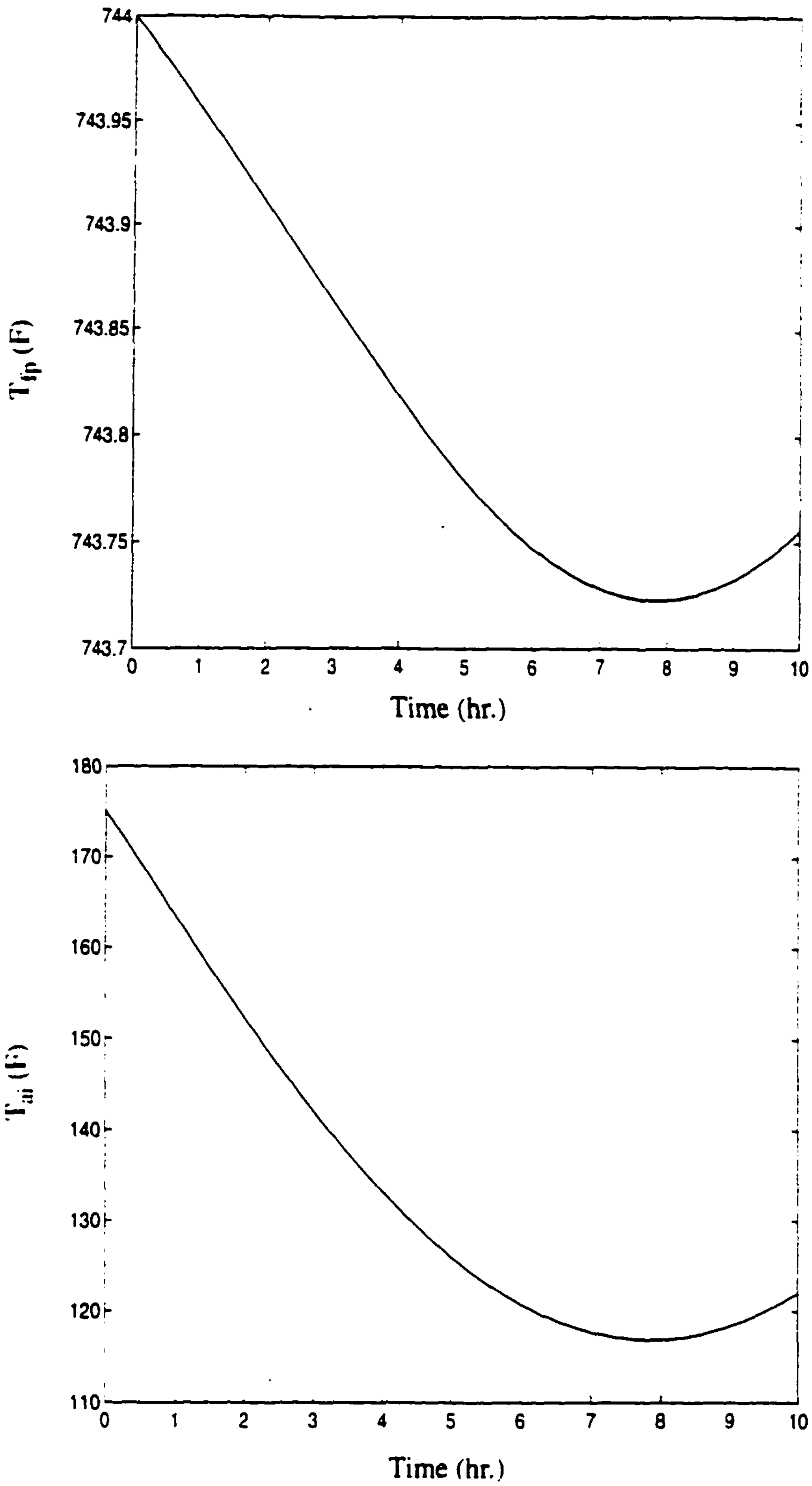

Figure 5.3: Closed-loop input profiles for regulation.

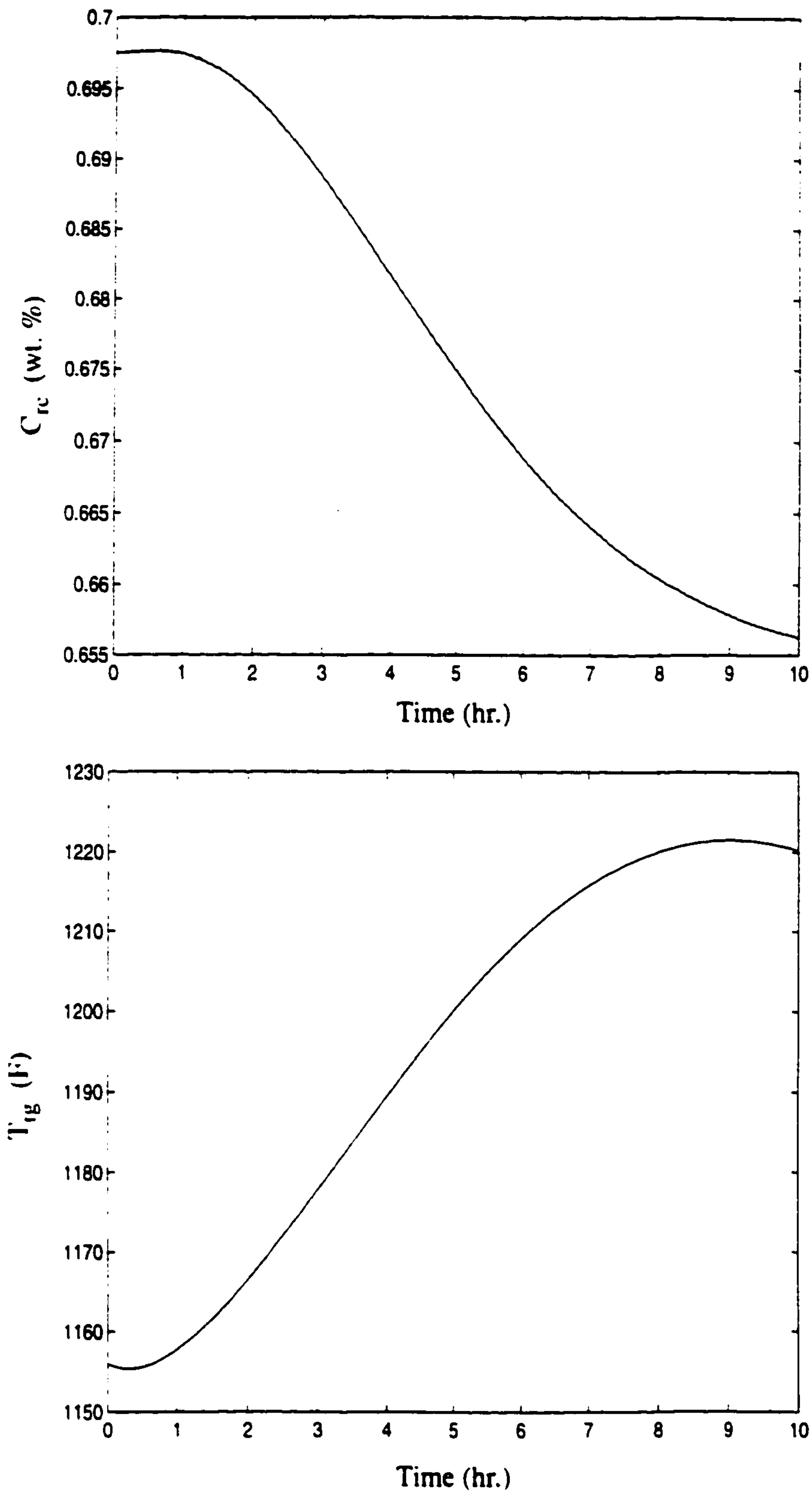

Figure 5.4: Closed-loop output profiles for regulation (no uncertainty compensation).

time $t = 0.0\ hr$. The output profiles are depicted in Figure 5.5 and the profiles of the corresponding manipulated inputs are given in Figure 5.6 It is clear that the controller drives the output y_2 to its new reference input value, achieving the requirement $\limsup_{t\to\infty} |y_2 - v_2| \leq 0.01$. It can be also observed that the output y_1 stays very close to its reference input value (i.e. the requirement $\limsup_{t\to\infty} |y_1 - v_1| \leq 0.01$ is satisfied). Finally, Figure 5.7 shows the closed-loop output for this simulation run in the case of implementing the decoupling input/output linearizing controller to the process. It is clear that this controller cannot attenuate the effect of the uncertainty yielding unacceptable performance. From the results of the simulation study, we conclude that the proposed methodology is a powerful tool for the synthesis of nonlinear controllers that compensate for the effect of uncertainty.

5.5 Conclusions

In this chapter, we proposed a robust multivariable controller design methodology for a broad class of multi-input multi-output two-time-scale nonlinear processes with explicit time-scale separation, modeled within the mathematical framework of singular perturbations. The proposed controller guarantees boundedness of the state and asymptotic output tracking with arbitrary degree of attenuation of the effect of the uncertainty on the output by a suitable choice of controller parameters, as long as the singular perturbation parameter is sufficiently small. The method was applied to fluidized catalytic cracking reactor with unknown heat of the combustion reaction and its performance was successfully tested through computer simulations.

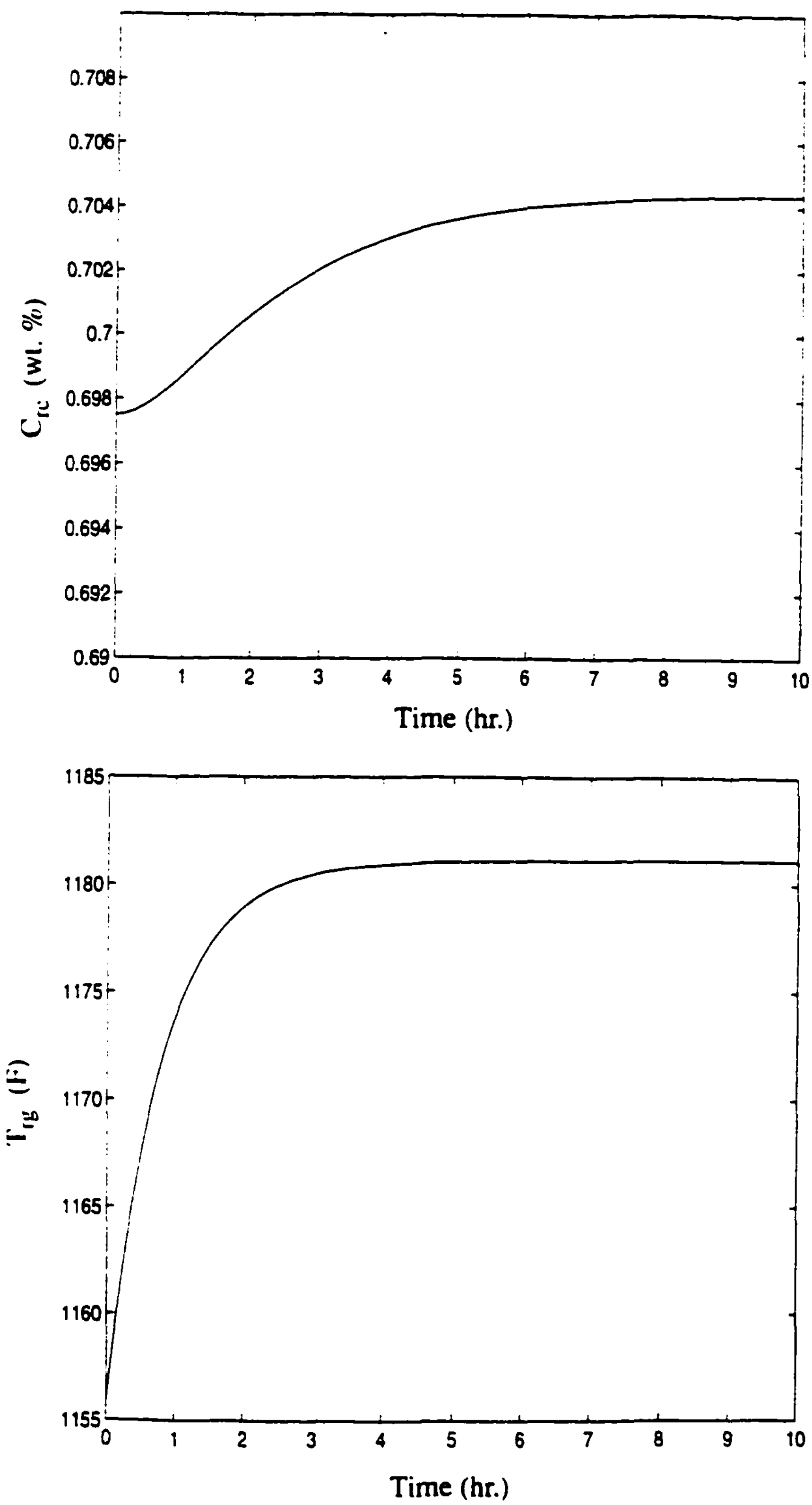

Figure 5.5: Closed-loop output profiles for reference input tracking.

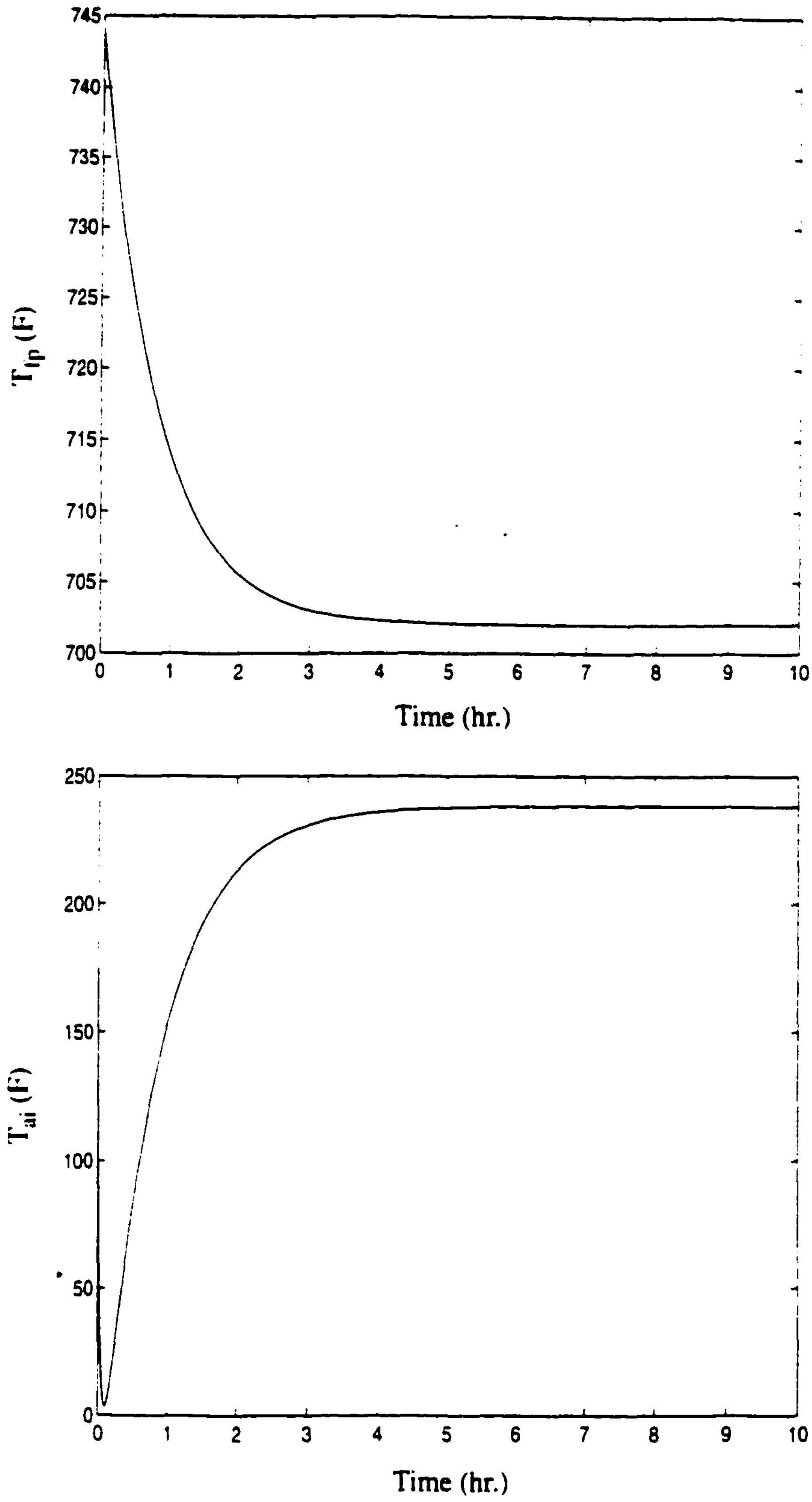

Figure 5.6: Closed-loop input profiles for reference input tracking.

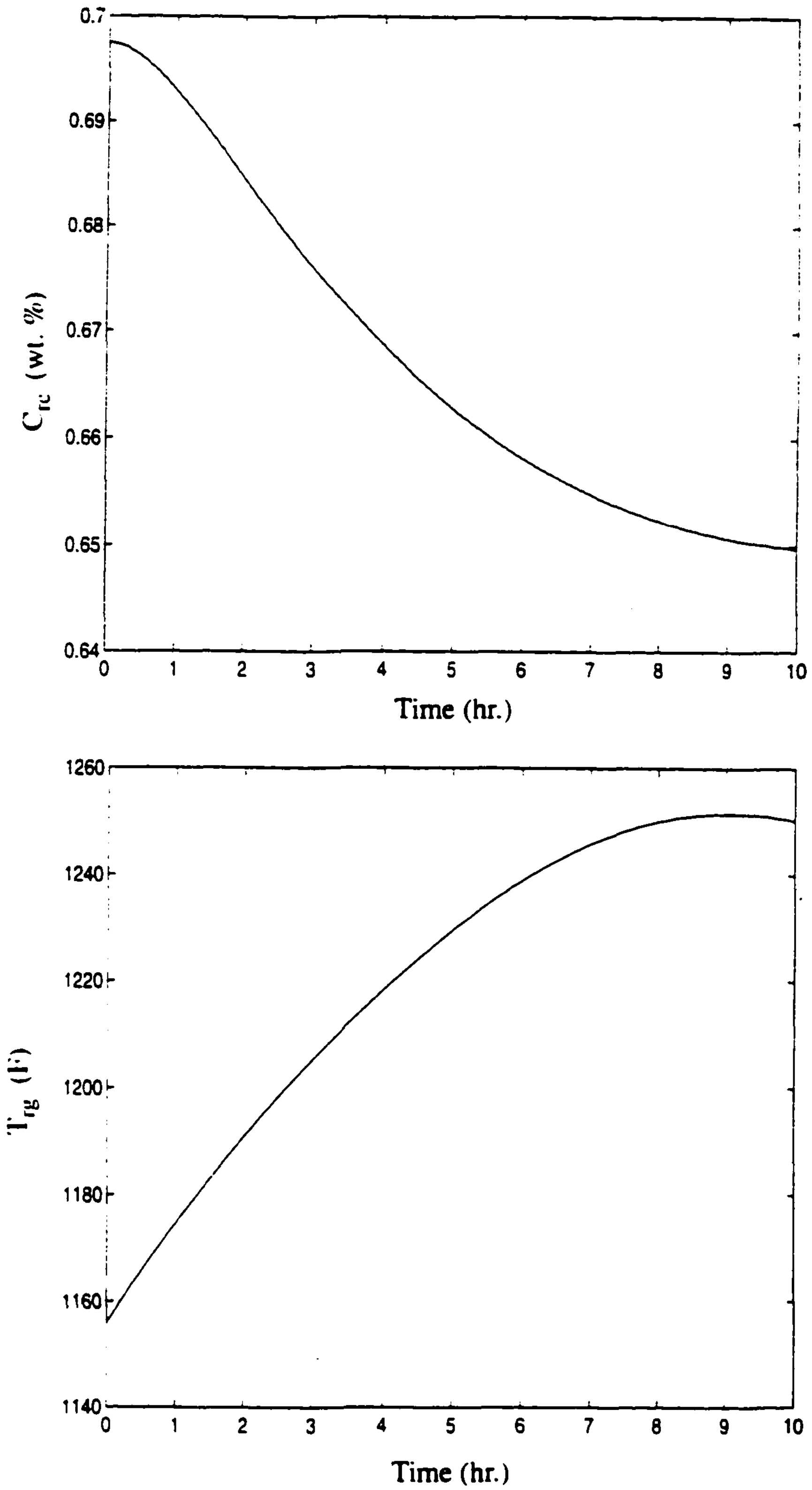

Figure 5.7 Closed-loop output profiles for reference input tracking (no uncertainty compensation)

Notation

Roman Letters

C_{cat} = concentration of catalytic carbon on spent catalyst
C_{sc} = concentration of total carbon on spent catalyst
C_{rc} = concentration of carbon on regenerated catalyst
D_{tf} = density of total feed
$d, d_{\bar{e}_{\bar{r}}}, d_{\bar{e}}, d_{\bar{\eta}}$ = positive real numbers
E_{cc}, E_{cr}, E_{or} = activation energies
$F, \bar{F}, \tilde{F}_{nom}, f_1, f_2$ = vector fields
F_{rc} = flow rate of catalyst from reactor to regenerator
$G, \bar{G}, G_1, G_2$ = vector fields associated with the inputs
h_i = i-th output scalar field
$K(x)$ = sufficiently smooth matrix
k_{cc}, k_{cr}, k_{or} = pre-exponential kinetic constants
O_{fg} = oxygen in flue gas
Q_1 = matrix of dimension $n \times p$ associated with the slow state vector x
Q_2 = matrix of dimension $p \times p$ associated with the fast state vector z
R_{ai} = air rate
R_{cb}, R_{cf}, R_{oc} = reaction rates
R_{tf} = total feed rate
$r_i, \bar{r}_i$ = integers associated with the input vector $\bar{u}$ in the reduced systems
S_a, S_c, S_f = specific heats
T_{fp}, T_{ai} = temperatures of the feed in the reactor and the air in the regenerator
T_{ra}, T_{rg} = temperatures in reactor and regenerator
t = time
$u, \bar{v}$ = manipulated input vectors
$\bar{u}$ = auxiliary input vector
V_{ra}, V_r = reactor and regenerator catalyst holdups
v = external reference input vector
x = vector of the slow state variables
y_i = i-th output
z = vector of the fast state variables

Greek Letters

β_{ik} = adjustable parameters
$\delta, \delta_x, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{e}_{\bar{r}}}, \delta_{\bar{\eta}}, \delta_{\bar{e}}, \delta_\eta, \tilde{\delta}_{\bar{e}_{\bar{r}}}, \tilde{\delta}_{\bar{\eta}}$ = positive real numbers
$\Delta H_{rg}, \Delta H_{cr}$ = heat of the reactions
ΔH_{fv} = heat of feed vaporization
$\epsilon, \epsilon^{\bar{e}_{\bar{r}}}, \epsilon^{\bar{\eta}}, \epsilon^{\bar{e}}$ = singular perturbation parameters
ζ = state vector
η = state vector
ϕ = adjustable parameter

Math Symbols

$\lvert\cdot\rvert$	=	standard Euclidean norm
$sgn(\cdot)$	=	sign function
Id	=	identity function
$L_f h$	=	Lie derivative of a scalar field h with respect to the vector field f
$L_f^k h$	=	k-th order Lie derivative
$L_g L_f^{k-1} h$	=	mixed Lie derivative
$\mathbb{R}$	=	real line
$\mathbb{R}^i$	=	i–dimensional Euclidean space
$\in$	=	belongs to
T	=	transpose

Appendices

Appendix A

Proofs of Chapter 2

Proof of proposition 2.1:

It is straightforward to show that the static state feedback law of Eq.2.16, on the quasi-steady-state $z_s = -[Q_2(x)]^{-1}[f_2(x) + g_2(x)]u$, takes the form:

$$u = [1 + k^T(x)Q_2^{-1}(x)g_2(x)]^{-1}[\tilde{u} - k^T(x)Q_2^{-1}(x)f_2(x)] \tag{A.1}$$

where the scalar function $1 + k^T(x)Q_2^{-1}(x)g_2(x)$ is nonzero uniformly in $x \in X$. Under the state feedback law of Eq.A.1, the closed-loop reduced system takes the form:

$$\begin{aligned} \dot{x} &= F(x) + G(x)[1 + k^T(x)Q_2^{-1}(x)g_2(x)]^{-1}[\tilde{u} - k^T(x)Q_2^{-1}(x)f_2(x)] \\ y^s &= h(x) \end{aligned} \tag{A.2}$$

where the vector fields $F(x)$ and $G(x)$ are given by Eq.2.6. For the above system, one can directly show that the relative order of the output y with respect to the auxiliary input $\tilde{u}$ is exactly equal to r. It remains to be proved that the closed-loop reduced systems of Eqs.2.19 and A.2 are identical. Lemma 2.1 that follows states this result.

Lemma 2.1: *Consider the two-time-scale system of the form Eq.2.1 in standard form, subject to the static state feedback law of Eq.2.16, and assume that the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$. Then, the closed-loop reduced systems of Eqs.2.19 and A.2, derived by commuting the order of the operations: i)* *closing the feedback loop, and ii) setting $\epsilon = 0$, are identical.*

The proof of the lemma can be obtained along the lines of the proof which was given in the case of two-time-scale linear systems under linear static state feedback laws (see e.g. [KKO86]), and will be omitted for brevity. $\triangle$

Proof of theorem 2.2:

Under the control law of Eq.2.26 the closed-loop system takes the form:

$$\begin{aligned}
\dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\left\{[1 + k^T(x)[Q_2(x)]^{-1}g_2(x)]\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\
&\quad \left.\bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right\}\right\} + g_1(x)\left\{k^T(x)[Q_2(x)]^{-1}f_2(x) + k^T(x)z\right\} \\
\epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)\left\{[1 + k^T(x)[Q_2(x)]^{-1}g_2(x)]\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\
&\quad \left.\bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right\}\right\} + g_2(x)\left\{k^T(x)[Q_2(x)]^{-1}f_2(x) + k^T(x)z\right\} \\
y &= h(x)
\end{aligned} \tag{A.3}$$

Using the set of variables ξ, defined in Eq.2.29 the closed-loop system of Eq.A.3 takes the form:

$$\begin{aligned}
\dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\
&\quad \left.\bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right\} + k^T(x)(z - \xi)\right\} \\
\epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\
&\quad \left.\bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right\} + k^T(x)(z - \xi)\right\}
\end{aligned} \tag{A.4}$$

The properties of the above two-time-scale system can be analyzed by performing a standard two-time-scale decomposition. More specifically, introducing a new set of variables $\bar{\eta}$, defined as $\bar{\eta} = z - \xi$, one can easily show that the closed-loop fast subsystem is given by:

$$\frac{d\bar{\eta}}{d\tau} = [Q_2(x) + g_2(x)k^T(x)]\bar{\eta} \tag{A.5}$$

Since the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$ by the appropriate choice of the feedback gain $k^T(x)$, the fast dynamics of the closed-loop system possesses an exponentially stable equilibrium manifold of the form:

$$\xi = -[Q_2(x)]^{-1}\left[f_2(x) + g_2(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right\}\right\}\right] \tag{A.6}$$

Furthermore, the closed-loop reduced system takes the form:

$$\begin{aligned} \dot{x} &= [f_1(x) - Q_1(x)(Q_2(x))^{-1}f_2(x)] + [g_1(x) - Q_1(x)(Q_2(x))^{-1}g_2(x)] \\ &\quad \bullet \left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{v - \sum_{k=1}^{r}\beta_k L_F^k h(x)\right\}\right\} \\ y^s &= h(x) \end{aligned} \tag{A.7}$$

Calculating the derivatives of the output on the basis of Eq.A.7 yields the following expressions:

$$\begin{aligned} y^s &= h(x) \\ y^{s(1)} &= L_F h(x) \\ &\vdots \\ y^{s(r-1)} &= L_F^{r-1}h(x) \\ y^{s(r)} &= L_F^r h(x) + L_G L_F^{r-1}h(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right\}\right\} \end{aligned} \tag{A.8}$$

Substituting the above expressions to Eq.2.25, it is straightforward to verify that the desired input/output behavior is indeed enforced in the closed-loop reduced system.

We will now establish that the relation of Eq.2.27 holds for the output of the closed-loop full-order system. To this end, let us denote by $x^f(t), x^s(t)$, the solutions of the x-subsystem of Eq.A.3 and of Eq.A.7, respectively. Based on the stability results of section 2.6, the closed-loop reduced system of Eq.A.7 is locally exponentially stable. Furthermore, as established earlier in this proof, the fast subsystem of Eq.A.5 is exponentially stable, uniformly in $x \in X$. Utilizing these two stability properties, it can be shown using theorem 2.1, that under consistent initialization of the states x_i^s, x_i^f, i.e., $x_i^s(0) = x_i^f(0)$, $i = 1, \cdots, n$, the following estimate holds for the solutions of these systems:

$$x^f(t) = x^s(t) + O(\epsilon)\ ,\ t \geq 0 \tag{A.9}$$

for ϵ sufficiently small. Then, the analyticity of the scalar field $h(x)$ and the boundedness of the trajectories $x^f(t), x^s(t)$ for all times, directly imply that the following estimate holds for the output y of the closed-loop full-order system:

$$y(t) \;=\; y^s(t) + O(\epsilon)\,,\; t \geq 0 \tag{A.10}$$

where $y(t)$, $y^s(t)$ denote the outputs of the systems of Eq.A.3 and Eq.A.7, respectively. This completes the proof of theorem 2.2. $\triangle$

Proof of theorem 2.3:

Under the control law of Eq.2.43 the closed-loop system takes the form:

$$\begin{aligned}
\dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x)\right\}\right\} \\
&\quad + g_1(x) k^T(x) z \\
\epsilon \dot{z} &= f_2(x) + Q_2(x)z + g_2(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x)\right\}\right\} \\
&\quad + g_2(x) k^T(x) z \\
y &= h(x)
\end{aligned} \tag{A.11}$$

Employing a standard two-time-scale decomposition, it is straightforward to show that the closed-loop fast subsystem takes the form:

$$\begin{aligned}
\frac{dz}{d\tau} &= f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x) \\
&\quad \bullet \left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x)\right\}\right\}
\end{aligned} \tag{A.12}$$

Clearly, the fast dynamics of the closed-loop system possesses an exponentially stable equilibrium manifold of the form:

$$\begin{aligned}
\xi &= -[Q_2(x) + g_2(x)k^T(x)]^{-1}[f_2(x) + g_2(x) \\
&\quad \bullet \left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x)\right\}\right\}]
\end{aligned} \tag{A.13}$$

since the matrix $Q_2(x)+g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$ by the appropriate choice of the feedback gain $k^T(x)$.

Furthermore, the closed-loop reduced system takes the form:

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=1}^{\tilde{r}} \beta_k L_{\tilde{F}}^{k} h(x)\right\}\right\} \\ \hat{y}^s &= h(x) \end{aligned} \tag{A.14}$$

It is then straightforward to show by a direct calculation of the derivatives of the output y on the basis of Eq.A.14, that the requested input/ouput behavior is indeed enforced in the closed-loop reduced system. Finally, using the same arguments as in theorem 2.2, it can be shown that the relation of Eq.2.44 holds for the output of the closed-loop system of Eq.A.11. △

Appendix B

Proofs of Chapter 3

Proof of proposition 3.1:

It is straightforward to show that the control law of Eq.3.13, on the quasi-steady-state takes the form:

$$u = [1 + k^T(x)Q_2^{-1}(x)g_2(x)]^{-1}[\tilde{u} - k^T(x)Q_2^{-1}(x)(f_2(x) + W_2(x)d)] \qquad \text{(B.1)}$$

where the scalar function $1 + k^T(x)Q_2^{-1}(x)g_2(x)$ is nonzero uniformly in $x \in X$. Under the state feedback law of Eq.B.1, the reduced system takes the form:

$$\begin{aligned} \dot{x} &= F(x) + G(x)[1 + k^T(x)Q_2^{-1}(x)g_2(x)]^{-1}[\tilde{u} \\ &\quad -k^T(x)Q_2^{-1}(x)(f_2(x) + W_2(x)d)] + W(x)d \qquad \text{(B.2)} \\ y^s &= h(x) \end{aligned}$$

where the vector fields $F(x)$ and $G(x)$ are given by Eq.3.6. For the system of Eq.B.2, $\tilde{\rho}$ denotes the relative order of the output y^s with respect to the disturbance input vector d. For this system, one can directly show that the relative order of the output y with respect to the auxiliary input $\tilde{u}$ is exactly equal to r. On the other hand, if $\rho \leq r$ a direct differentiation of the output of the system of Eq.B.2 yields the following expressions:

$$
\begin{array}{rcl}
y^s & = & h(x) \\
y^{s(1)} & = & L_F h(x) \\
\vdots & & \\
y^{s(\tilde{\rho}-1)} & = & L_F^{\tilde{\rho}-1} h(x) \\
y^{s(\tilde{\rho})} & = & L_F^{\tilde{\rho}} h(x) + \Phi_0(x, d) \\
y^{s(\tilde{\rho}+1)} & = & L_F^{\tilde{\rho}+1} h(x) + \Phi_1(x, d, d^{(1)}) \\
\vdots & & \\
y^{s(r-1)} & = & L_F^{r-1} h(x) + \Phi_{r-\tilde{\rho}-1}(x, d, d^{(1)}, \cdots, d^{(r-\tilde{\rho}-1)}) \\
y^{s(r)} & = & L_F^r h(x) + L_G L_F^{r-1} h(x)[1 + k^T(x) Q_2^{-1}(x) g_2(x)]^{-1} \\
& & \bullet[\tilde{u} - k^T(x) Q_2^{-1}(x)(f_2(x) + W_2(x)d)] + \Phi_{r-\tilde{\rho}}(x, d, d^{(1)}, \cdots, d^{(r-\tilde{\rho})})
\end{array}
\tag{B.3}
$$

On the basis of Eq.B.3, it is clear that $\tilde{\rho} = \rho$, whenever $\rho \leq r$. It remains to be proved that the closed-loop reduced systems of Eqs.3.16 and B.2 are identical. Lemma 3.1 that follows establishes this result.

Lemma 3.1: *Consider the two-time-scale system of the form Eq.3.1 in standard form, subject to the control law of Eq.3.13, and assume that the matrix* $Q_2(x) + g_2(x)k^T(x)$ *is invertible uniformly in* $x \in X$. *Then, the closed-loop reduced systems of Eqs.3.16 and B.2, derived by commuting the order of the operations: i) closing the feedback loop, and ii) setting* $\epsilon = 0$, *are identical.*

Lemma 3.1 generalizes a standard result for two-time-scale linear systems under linear static state feedback laws (see e.g., [CGG93]) to nonlinear two-time-scale systems under feedback laws of the form of Eq.3.13. The proof of the lemma can be readily obtained along the lines of the one given for the linear case [CGG93] and will be omitted for brevity. △

Proof of theorem 3.1:

Part 1: Using the result of lemma 3.1, the reduced system of Eq.3.16 can be equivalently written in the form of Eq.B.2. Substitution of the control law of Eq.3.18 to the reduced system of Eq.B.2 yields then the following closed-loop reduced system:

$$\begin{aligned}\dot{x} &= F(x) + G(x)[1 + k^T(x)Q_2^{-1}(x)g_2(x)]^{-1}[p(x) + q(x)v + Q\left(x, d, d^{(1)}, \cdots\right) \\ &\quad -k^T(x)Q_2^{-1}(x)(f_2(x) + W_2(x)d)] + W(x)d \\ y^s &= h(x)\end{aligned} \tag{B.4}$$

On the basis of Eq.B.4 and following [DK93], it can then be shown that the input/output behavior of the form of Eq.3.22 can be enforced in the above closed-loop reduced system if and only if the conditions of Eq.3.23 are satisfied.

Part 2: Under the control law of Eq.3.25 the closed-loop system takes the form:

$$\begin{aligned}\dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\left\{[1 + k^T(x)[Q_2(x)]^{-1}g_2(x)]\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\ &\quad \left.\bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x) - \sum_{k=\rho}^{r}\beta_k\Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\}\right\} \\ &\quad +g_1(x)\left\{k^T(x)[Q_2(x)]^{-1}[f_2(x) + W_2(x)d] + k^T(x)z\right\} + W_1(x)d \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)\left\{[1 + k^T(x)[Q_2(x)]^{-1}g_2(x)]\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\ &\quad \left.\bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x) - \sum_{k=\rho}^{r}\beta_k\Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\}\right\} \\ &\quad +g_2(x)\left\{k^T(x)[Q_2(x)]^{-1}[f_2(x) + W_2(x)d] + k^T(x)z\right\} + W_2(x)d\end{aligned} \tag{B.5}$$

Defining a set of auxiliary variables ξ, as follows:

$$\begin{aligned}\xi &= -[Q_2(x)]^{-1}\left\{f_2(x) + g_2(x)[\beta_r L_G L_F^{r-1}h(x)]^{-1}\right. \\ &\quad \left.\bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x) - \sum_{k=\rho}^{r}\beta_k\Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\} + W_2(x)d\right\}\end{aligned} \tag{B.6}$$

the closed-loop system of Eq.B.5 takes the form:

$$
\begin{aligned}
\dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\
&\quad \left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x) - \sum_{k=\rho}^{r}\beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\} \\
&\quad \left. + k^T(x)(z-\xi)\right\} + W_2(x)d \\
\epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\right. \\
&\quad \bullet\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x) - \sum_{k=\rho}^{r}\beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\} \\
&\quad \left. + k^T(x)(z-\xi)\right\} + W_2(x)d
\end{aligned}
\tag{B.7}
$$

The properties of the above two-time-scale system can be analyzed by performing a standard two-time-scale decomposition. More specifically, introducing a new set of variables η, defined as $\eta_f = z - \xi$, one can easily show that the closed-loop fast subsystem is given by:

$$
\frac{d\eta_f}{d\tau} = [Q_2(x) + g_2(x)k^T(x)]\eta_f \tag{B.8}
$$

Since the matrix $Q_2(x)+g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$ by the appropriate choice of the feedback gain $k^T(x)$, the fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold of the form:

$$
\eta_f = 0 \tag{B.9}
$$

Furthermore, the closed-loop reduced system takes the form:

$$
\begin{aligned}
\dot{x} &= F(x) + G(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right.\right. \\
&\quad \left.\left. - \sum_{k=\rho}^{r}\beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\}\right\} + W(x)d =: \mathcal{F}(x, v, \bar{d}) \\
y^s &= h(x)
\end{aligned}
\tag{B.10}
$$

where $\bar{d} = [d \; d^{(1)} \; \cdots d^{(r-\rho)}]^T$ denotes an extended disturbance vector. From part 1 of the proof, we have that the output of the above closed-loop reduced system is completely independent of d and the input/output response of Eq.3.22 is also enforced.

Furthermore, from proposition 3.2, the relative order of y^s with respect to v is equal to r, and thus there exists a coordinate transformation of the form:

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_r \\ \eta \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_{\mathcal{F}}h(x) \\ \vdots \\ L_{\mathcal{F}}^{r-1}h(x) \\ \psi(x) \end{bmatrix} \tag{B.11}$$

where $\eta \in \mathbb{R}^{n-r}$ and $\psi(x)$ is a smooth vector function, such that the closed-loop reduced system of Eq.B.10 takes the form:

$$\begin{aligned} \dot{\zeta} &= A\zeta + bv \\ \dot{\eta} &= \Psi(\zeta, \eta, \bar{d}) \\ y^s &= \zeta_1 \end{aligned} \tag{B.12}$$

where A is Hurwitz matrix of dimension $r \times r$, $b = [0 \ \cdots 0 \ 1] \in \mathbb{R}^{r\times 1}$ is a column vector, and $\Psi(\zeta, \eta, \bar{d})$ is a vector of smooth functions. Now, consider the following form of the closed-loop full-order system:

$$\begin{aligned} \dot{x} &= F(x) + G(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right.\right. \\ &\quad \left.\left. -\sum_{k=\rho}^{r}\beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\}\right\} \\ &\quad + W(x)d + [Q_1(x) + g_1(x)k^T(x)](z - \xi) \\ \epsilon\dot{z} &= [Q_2(x) + g_2(x)k^T(x)](z - \xi) \end{aligned} \tag{B.13}$$

or in terms of the coordinates (ζ, η, z):

$$\begin{aligned} \dot{\zeta} &= A\zeta + bv + \Phi_\zeta(\zeta, \eta)\eta_f \\ \dot{\eta} &= \Psi(\zeta, \eta, \bar{d}) + \Phi_\eta(\zeta, \eta)\eta_f \\ \epsilon\dot{z} &= [Q_2(\zeta, \eta) + g_2(\zeta, \eta)k^T(\zeta, \eta)]\eta_f \\ y &= \zeta_1 \end{aligned} \tag{B.14}$$

where Φ_ζ, Φ_η are matrices of smooth functions. From the stability result of theorem 3.3, we have that if the hypotheses of the theorem are satisfied, there exists an $\epsilon^* > 0$, such that if $\epsilon \in (0, \epsilon^*]$, the states (ζ, η, z) of the closed-loop full-order system of Eq.B.14 are bounded. Let $\epsilon \in (0, \epsilon^*]$ and consider the singularly perturbed system, resulting from the states (ζ, z) of the system of Eq.B.14:

$$\begin{aligned} \dot{\zeta} &= A\zeta + bv + \Phi_\zeta(\zeta, \eta)\eta_f \\ \epsilon\dot{z} &= [Q_2(\zeta, \eta) + g_2(\zeta, \eta)k^T(\zeta, \eta)]\eta_f \\ y &= \zeta_1 \end{aligned} \tag{B.15}$$

where η can be thought of as a bounded input. Performing a two-time-scale decomposition on the above system, it can be easily seen that the fast subsystem is exponentially stable uniformly in $x \in X$, and the reduced system takes the form:

$$\begin{aligned} \dot{\zeta}^s &= A\zeta^s + bv \\ y^s &= \zeta_1^s \end{aligned} \tag{B.16}$$

where the superscript s denotes state of reduced system, which is also exponentially stable. Utilizing these two stability properties, it can be shown using theorem 2.1 that under consistent initialization of the states ζ_i^s, ζ_i, i.e., $\zeta_i^s(0) = \zeta_i(0)$, $i = 1, \cdots, r$, there exists an $\epsilon^{**} \in (0, \epsilon^*]$, such that if $\epsilon \in (0, \epsilon^{**}]$, the following estimate holds for the solutions of systems of Eqs.B.15-B.16:

$$\zeta(t) = \zeta^s(t) + O(\epsilon)\ ,\ t \geq 0 \tag{B.17}$$

From the above estimate, Eq.3.26 follows directly. The proof of the theorem is complete. $\triangle$

Proof of proposition 3.3 :

Necessity. Consider the two-time-scale system of Eq.3.1, assumed to be in standard form. Consider also its corresponding reduced system (Eq.3.5) and assume that $\rho \leq r$. Under the control law of Eq.3.31, the two-time-scale system of Eq.3.1 yields:

$$\begin{aligned} \dot{x} &= [f_1(x) + g_1(x)p(x)] + [Q_1(x) + g_1(x)k^T(x)]z + g_1(x)q(x)v + W_1(x)d \\ \epsilon\dot{z} &= [f_2(x) + g_2(x)p(x)] + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)q(x)v + W_2(x)d \end{aligned} \tag{B.18}$$

One can now easily verify that the fast subsystem is exponentially stable, subject to appropriate choice of $k^T(x)$. Moreover, the closed-loop reduced system takes the form:

$$\begin{aligned} \dot{x} &= [\tilde{F}(x) + \tilde{G}(x)p(x)] + \tilde{G}(x)q(x)v + \tilde{W}(x)d \\ y^s &= h(x) \end{aligned} \tag{B.19}$$

For the above system, one can conclude, utilizing the results of propositions 3.1 and 3.2, that the relative orders of the pairs (y^s, v) and (y^s, d) are equal to r and ρ, respectively, for which by assumption $\rho \leq r$. But for nonlinear systems of the form of Eq.3.5, it is well-established (e.g., [Isi89]) that the necessary and sufficient condition for achieving exact disturbance decoupling via static state feedback laws of the form $u = p(x) + q(x)v$ is, $r < \rho$, which establishes the necessity of this condition. Finally, if the condition $r < \rho$ holds, but the condition $k^T(x)[Q_2(x)]^{-1}W_2(x) \equiv \bar{0}$ does not hold, i.e., for some $x \in X$ $k^T(x)[Q_2(x)]^{-1}W_2(x) \neq \bar{0}$, we have from Eq.B.3 that for the reduced system of Eq.3.16 $\tilde{\rho} = r < \rho$, and thus a static state feedback law of the form of Eq.3.31 does not suffice to completely eliminate the effect of d on y^s in the closed-loop reduced system of Eq.B.19, which shows the necessity of this condition as well.

Sufficiency : Referring to the reduced system of Eq.3.5, assume that $r < \rho$ and $k^T(x)[Q_2(x)]^{-1}W_2(x) \equiv \bar{0}$. Then, it is straightforward to show that the control law of theorem 3.1 takes the form:

$$\begin{aligned} u \;=\; & [1 + k^T(x)[Q_2(x)]^{-1}g_2(x)]\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{v - \sum_{k=0}^{r}\beta_k L_F^k h(x)\right\} \\ & + k^T(x)[Q_2(x)]^{-1}f_2(x) + k^T(x)z \end{aligned} \quad \text{(B.20)}$$

which clearly belongs in the class of control laws described by Eq.3.31 and does achieve stabilization of the fast dynamics with approximate decoupling of the effect of d on y. △

Proof of theorem 3.2:

Part 1: Consider the closed-loop reduced system:

$$\begin{aligned} \dot{x} &= \tilde{F}(x) + \tilde{G}(x)\tilde{p}(x) + \tilde{G}(x)\tilde{q}(x)v + \tilde{G}(x)\tilde{Q}\left(x, d, d^{(1)}, \cdots\right) + \tilde{W}(x)d \\ y^s &= h(x) \end{aligned} \quad \text{(B.21)}$$

Following [DK93], it can be shown that the input/output behavior of the form of Eq.3.37 is enforced in the above closed-loop reduced system if and only if the conditions of Eq.3.38 are satisfied.

Part 2: Under the control law of Eq.3.40 the closed-loop system takes the form:

$$\begin{aligned}\dot{x} &= f_1(x) + Q_1(x)z + g_1(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x)\right.\right. \\ &\left.\left. -\sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}\left(x, d, d^{(1)}, \cdots, d^{(k-\tilde{\rho})}\right)\right\}\right\} + g_1(x)k^T(x)z + W_1(x)d \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x)\right.\right. \\ &\left.\left. -\sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}\left(x, d, d^{(1)}, \cdots, d^{(k-\tilde{\rho})}\right)\right\}\right\} + g_2(x)k^T(x)z + W_2(x)d\end{aligned} \tag{B.22}$$

Employing a standard two-time-scale decomposition, it is straightforward to show that the closed-loop fast subsystem takes the form:

$$\begin{aligned}\frac{dz}{d\tau} &= f_2(x) + [Q_2(x) + g_2(x)k^T(x)]z + g_2(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\right. \\ &\left.\left\{v - \sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) - \sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}\left(x, d, d^{(1)}, \cdots, d^{(k-\tilde{\rho})}\right)\right\}\right\} + W_2(x)d\end{aligned} \tag{B.23}$$

Clearly, the fast dynamics of the closed-loop system possess an exponentially stable equilibrium manifold of the form:

$$\begin{aligned}\xi &= -[Q_2(x) + g_2(x)k^T(x)]^{-1}\left[f_2(x) + g_2(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v\right.\right.\right. \\ &\left.\left.\left. -\sum_{k=0}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x) - \sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}\left(x, d, d^{(1)}, \cdots, d^{(k-\tilde{\rho})}\right)\right\}\right\} + W_2(x)d\right]\end{aligned} \tag{B.24}$$

since the matrix $Q_2(x)+g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$, by the appropriate choice of the feedback gain $k^T(x)$.

Furthermore, the closed-loop reduced system takes the form:

$$\begin{aligned}\dot{x} &= \tilde{F}(x) + \tilde{G}(x)\left\{\left[\beta_{\tilde{r}} L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)\right]^{-1}\left\{v - \sum_{k=1}^{\tilde{r}} \beta_k L_{\tilde{F}}^k h(x)\right.\right. \\ &\left.\left. -\sum_{k=\tilde{\rho}}^{\tilde{r}} \beta_k \tilde{\Phi}_{k-\tilde{\rho}}\left(x, d, d^{(1)}, \cdots, d^{(k-\tilde{\rho})}\right)\right\}\right\} + \tilde{W}(x)d \\ \hat{y}^s &= h(x)\end{aligned} \tag{B.25}$$

Finally, using the same arguments as in theorem 1, it can be shown that the controller of Eq.3.40 enforces the input/output response of Eq.3.37 in the closed-loop system in the limit as $\epsilon \rightarrow 0$. △

Proof of theorem 3.3:

The proof of the theorem will be derived utilizing a Lyapunov function which is obtained as the composition of two Lyapunov functions for the fast and slow closed-loop subsystems (see also, [SK84, Kha92]) and a Lyapunov argument used in [CT95].

The closed-loop system of Eq.B.13, with $v = 0$, in terms of the coordinates x, η_f takes the following form:

$$\begin{aligned}
\dot{x} &= F(x) + G(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{-\sum_{k=0}^{r}\beta_k L_F^k h(x)\right.\right. \\
&\quad \left.\left. -\sum_{k=\rho}^{r}\beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\}\right\} + W(x)d + [Q_1(x) + g_1(x)k^T(x)]\eta_f \\
\epsilon\dot{\eta}_f &= [Q_2(x) + g_2(x)k^T(x)]\eta_f + \epsilon\left(\frac{\partial \xi}{\partial x}\dot{x} + \frac{\partial \xi}{\partial \bar{d}}\dot{\bar{d}}\right)
\end{aligned} \tag{B.26}$$

Under conditions 1 and 2, it can be shown, following (e.g., [Isi89]) that the closed-loop reduced system, with $d = 0$, that is:

$$\dot{x} = F(x) + G(x)\left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{-\sum_{k=0}^{r}\beta_k L_F^k h(x)\right\}\right\} =: \mathcal{F}_1(x) \tag{B.27}$$

is exponentially stable. Using theorem 4.5. in [Kha92], we have that there exists a smooth Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(a_1, a_2, a_3, a_4, a_5)$, such that the following conditions hold:

$$\begin{aligned}
& a_1|x|^2 \leq V(x) \leq a_2|x|^2 \\
& \dot{V}(x) = \frac{\partial V}{\partial x}\mathcal{F}_1(x) \leq -a_3|x|^2 \\
& \left|\frac{\partial V}{\partial x}\right| \leq a_4|x|
\end{aligned} \tag{B.28}$$

for all $x \in X$ that satisfy $|x| \leq a_5$. Consider the closed-loop reduced system with $d(t) \neq 0$:

$$\begin{aligned}
\dot{x} &= \mathcal{F}_1(x) + \left\{\left[\beta_r L_G L_F^{r-1}h(x)\right]^{-1}\left\{-\sum_{k=\rho}^{r}\beta_k \Phi_{k-\rho}\left(x, d, d^{(1)}, \cdots, d^{(k-\rho)}\right)\right\}\right\} \\
&\quad + W(x)d =: \mathcal{F}_1(x) + \mathcal{Q}(x, \bar{d})
\end{aligned} \tag{B.29}$$

Observing that the term $\mathcal{Q}(x, \bar{d})$ vanishes if $\bar{d}(t) \equiv 0$, and using the third part of Eq.B.28 and $|x| \leq a_5$, a direct computation of the time-derivative of the function $V(x)$ along the trajectories of the above system yields:

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x}[\mathcal{F}_1(x) + \mathcal{Q}(x, \bar{d})] \\ &= \frac{\partial V}{\partial x}(\mathcal{F}_1(x)) + \frac{\partial V}{\partial x}(\mathcal{Q}(x, \bar{d})) \\ &\leq -a_3|x|^2 + \frac{\partial V}{\partial x}(\mathcal{Q}(x, \bar{d})) \\ &\leq -a_3|x|^2 + a_4|x|a_6|\bar{d}| \end{aligned} \tag{B.30}$$

where a_6 is positive real number. Since $d(t)$ and its derivatives are assumed to be sufficiently small, there exists a positive real number μ_1 that satisfies $\mu_1 \leq \frac{a_7 a_3}{a_4 a_6}$, where $a_7 < a_5$, such that $|\bar{d}| \leq \mu_1$. Whenever this condition holds, an application of theorem 4.10 in [Kha92] gives that the state of the system of Eq.B.29 is bounded, for all $x \in X$ that satisfy $|x| \leq a_5$. Let $|\bar{d}| \leq \mu_1$ and calculate the time-derivative of $V(x)$ along the trajectories of the x-subsystem of Eq.B.26:

$$\dot{V}(x) \leq -a_3|x|^2 + a_4|x|a_6|\bar{d}| + a_8|x||\eta_f| \tag{B.31}$$

where a_8 is a positive real number.

Since the matrix $Q_2(x) + g_2(x)k^T(x)$ is Hurwitz uniformly in $x \in X$, the fast subsystem:

$$\epsilon\dot{\eta}_f = [Q_2(x) + g_2(x)k^T(x)]\eta_f \tag{B.32}$$

is exponentially stable. Using lemma 5.7 in [Kha92], there exists a smooth Lyapunov function $\Omega : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ of the form:

$$\Omega(x, \eta_f) = \eta_f^T S(x)\eta_f \tag{B.33}$$

and a set of positive real numbers $(b_1, b_2, b_3, b_4, b_5, b_6)$, such that the following conditions hold:

$$\begin{aligned} & b_1|\eta_f|^2 \le \Omega(x,\eta_f) \le b_2|\eta_f|^2 \\ & \dot{\Omega}(x,\eta_f) = \frac{\partial \Omega}{\partial x}[Q_2(x) + g_2(x)k^T(x)]\eta_f \le -b_3|\eta_f|^2 \\ & |\frac{\partial \Omega}{\partial \eta_f}| \le b_4|\eta_f|, \ |\frac{\partial \Omega}{\partial x}| \le b_5|\eta_f|^2 \end{aligned} \tag{B.34}$$

for all η_f that satisfy $|\eta_f| \le b_6$. Observe that the time-derivative of the state vector x can be bounded by the following quantity:

$$|\dot{x}| \le w_1 + w_2|x| + w_3|x|\ |\eta_f| \tag{B.35}$$

where w_1, w_2, w_3 are positive constants, for all $x \in X$ that satisfy $|x| \le a_5$. Furthermore, note that since d and its derivatives up to order $r - \rho + 1$ are sufficiently small, there exists a positive real number (possibly small) μ_2 such that $|\dot{\bar{d}}| \le \mu_2$. Using this fact and the estimates $|\frac{\partial \xi}{\partial x}| \le l_3$ and $|\frac{\partial \xi}{\partial \bar{d}}| \le l_4|x|$, where l_3, l_4 are positive real numbers, we have:

$$\begin{aligned} |\frac{\partial \xi}{\partial x}\dot{x} + \frac{\partial \xi}{\partial \bar{d}}\dot{\bar{d}}| & \le l_3(w_1 + w_2|x| + w_3|x|\ |\eta_f|) + \mu_2 l_4|x| \\ & \le l_3 w_1 + (l_3 w_2 + \mu_2 l_4)|x| + l_3 w_3|x|\ |\eta_f| \end{aligned} \tag{B.36}$$

Computing the time-derivative of the function $\Omega(x,\eta_f)$ along the trajectories of the η_f-subsystem of Eq.B.26 and using the inequality of Eq.B.35, we get:

$$\begin{aligned} \dot{\Omega}(x,\eta_f) & = \frac{\partial \Omega}{\partial \eta_f}\dot{\eta}_f + \frac{\partial \Omega}{\partial x}\dot{x} \\ & \le -\frac{b_3}{\epsilon}|\eta_f|^2 + b_4|\eta_f|(l_3 w_1 + (l_3 w_2 + \mu_2 l_4)|x| + l_3 w_3|x|\ |\eta_f|) \\ & \quad + b_5|\eta_f|^2(w_1 + w_2|x| + w_3|x|\ |\eta_f|) \\ & \le -(\frac{b_3}{\epsilon} - b_5 w_1)|\eta_f|^2 + (b_5 w_2 + b_4 l_3 w_3)|x||\eta_f|^2 + b_5 w_3|x|\ |\eta_f|^3 \\ & \quad + b_4 l_3 w_1|\eta_f| + b_4(l_3 w_2 + \mu_2 l_4)|x||\eta_f| \end{aligned} \tag{B.37}$$

Consider now the smooth function [SK84] $L : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\ge 0}$:

$$L(x,\eta_f) = V(x) + \Omega(x,\eta_f) \tag{B.38}$$

as Lyapunov function candidate for the system of Eq.B.26. From Eq.B.28 and Eq.B.34, we have that $L(x,\eta_f)$ is positive definite and proper (tends to $+\infty$ as $|x| \to$

∞, or $|\eta_f| \to \infty$), with respect to its arguments. To establish boundedness of the trajectories, we will follow an approach similar to the one in [CT95]. To this end, we will specify a region in the state-space where the analysis will be performed. First, let l_5 be some positive real number and define $\omega_4 := a_2(a_7)^2 + l_5$. Now, define:

$$\omega_5 = \max_{\left\{ (x,\eta_f) \in \mathbb{R}^n \times \mathbb{R}^p : |x| \le a_5 \ , \ |\eta_f| \le b_6 \right\}} L(x,\eta_f) \ . \tag{B.39}$$

Let $\omega_6 > \max\{\omega_4, \omega_5\}$ and define the set $\bar{\Omega}$ as follows:

$$\bar{\Omega} = \left\{ \begin{array}{l} (x,\eta_f,\bar{d},\dot{\bar{d}}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q : \\ \qquad |\bar{d}| \le \mu_1 \ , \ |\dot{\bar{d}}| \le \mu_2 \ , \ \omega_4 \le L(x,\eta_f) \le \omega_6 \end{array} \right\} . \tag{B.40}$$

The set $\bar{\Omega}$ is compact. Computing the time-derivative of L along the trajectories of this system, the following expression can be easily obtained:

$$\dot{L}(x,\eta_f) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial \Omega}{\partial x}\dot{x} + \frac{\partial \Omega}{\partial \eta_f}\dot{\eta}_f \tag{B.41}$$

Using the inequalities of Eqs.B.31 and B.37, the following bound for $\dot{L}$ can be computed:

$$\begin{aligned} \dot{L}(x,\eta_f) \ \le \ & -a_3|x|^2 + a_4|x|a_6|\bar{d}| + a_8|x||\eta_f| - (\frac{b_3}{\epsilon} - b_5w_1)|\eta_f|^2 \\ & +(b_5w_2 + b_4l_3w_3)|x||\eta_f|^2 + b_5w_3|x|\,|\eta_f|^3 + b_4l_3w_1|\eta_f| \\ & +b_4(l_3w_2 + \mu_2 l_4)|x||\eta_f| \end{aligned} \tag{B.42}$$

Using the fact that $|\eta_f| \le b_6$, we get:

$$\dot{L}(x,\eta_f) \ \le \ -a_3|x|^2 + a_4a_6|x||\bar{d}| - (\frac{b_3}{\epsilon} - b_5w_1)|\eta_f|^2 + l_6|x||\eta_f| + b_4l_3w_1|\eta_f| \tag{B.43}$$

where $l_6 = a_8 + b_6b_5w_2 + b_6b_4l_3w_3 + b_6^2b_5w_3 + b_4l_3w_2 + b_4\mu_2l_4$. We will now use a three-step argument to show that there exists a positive real number ϵ^* such that if $\epsilon \in (0,\epsilon^*]$, then $\dot{L}$ is negative for all $(x,\eta_f,\bar{d},\dot{\bar{d}}) \in \bar{\Omega}$.

Step 1: $\eta_f = 0$. From the definition of the set $\bar{\Omega}$, $(x,\eta_f,\bar{d},\dot{\bar{d}}) \in \bar{\Omega} \cap \{(x,\eta_f,\bar{d},\dot{\bar{d}}) : \eta_f = 0\}$ implies that $L(x,\eta_f) = V(x) \ge \omega_4$, which in turn implies that $|x| \ge$

$(\frac{a_2(a_7)^2+l_5}{a_2})^{\frac{1}{2}} \geq a_7$. Using Eq.B.43, we have that on $\bar{\Omega} \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}})$:

$$\dot{L} = \dot{V} \leq -a_3|x|^2 + a_4 a_6 |x||\bar{d}| < 0 \tag{B.44}$$

Step 2: We will now show that there exists $l_7 > 0$, such that for all $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega} \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}}) : |\eta_f| \leq l_7\}$, $\dot{L}$ is negative. In this case, $\dot{L}$ takes the form:

$$\dot{L} = -a_3|x|^2 + a_4 a_6 |x||\bar{d}| - \frac{b_3}{\epsilon}|\eta_f|^2 + \bar{\Psi}(x, \eta_f) \tag{B.45}$$

where $\bar{\Psi}(x, \eta_f)$ is a continuous scalar function which satisfies $\bar{\Psi}(x, 0) = 0$. Since $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \Delta \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}}) : \eta_f = 0\}$ implies that $|x| \geq (\frac{a_2(a_7)^2+l_5}{a_2})^{\frac{1}{2}}$ and since L is continuous, there exists $l_8 > 0$ such that $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega} \cap \{(x, \eta_f, \bar{d}, \dot{\bar{d}}) : |\eta_f| \leq l_8\}$ implies $|x| \geq a_7$. So, from the previous step, we already have that the sum of the first two terms in Eq.B.45 is negative. Moreover, the sum of the third and the fourth term is nonpositive. Then, using the properties of $\bar{\Psi}$, there exists $l_7 \leq l_8$ such that $\dot{L}$ is negative.

Step 3: $|\eta_f| \geq l_7$. Using Eq.B.43, and the various bounds for $(x, \bar{d}, \eta_f)$, we have:

$$\begin{aligned} \dot{L}(x, \eta_f) &\leq -a_3|x|^2 + a_4 a_6 |x||\bar{d}| - (\frac{b_3}{\epsilon} - b_5 w_1)|\eta_f|^2 + l_6|x||\eta_f| + b_4 l_3 w_1 |\eta_f| \\ &\leq -a_3|x|^2 - \frac{b_3}{\epsilon} l_7^2 + l_9 \end{aligned} \tag{B.46}$$

where $l_9 = a_4 a_6 a_5 \mu_1 + b_5 w_1 b_6^2 + l_6 a_5 b_6 + b_4 l_3 w_1 b_6$. From Eq.B.46, it is clear that $\dot{L}$ is negative, for all $(x, \eta_f, \bar{d}, \dot{\bar{d}}) \in \bar{\Omega}$, provided that:

$$\epsilon \leq \frac{b_3 l_7^2}{l_9} =: \epsilon^* \tag{B.47}$$

Using a claim analogous to the one in [CT95], it can be shown that $L(x, \eta_f)$ is bounded for all $t \geq 0$. Since, it is also proper with respect to (x, η_f), the boundedness of the trajectories follows directly, for all $\epsilon \in (0, \epsilon^*]$. This completes the proof of the theorem.

$\triangle$

Appendix C

Proofs of Chapter 4

Proof of theorem 4.1: The proof of the theorem consists of three main parts. In the first part, the global exponential stability of the closed-loop fast subsystem is established. In the second part, the closed-loop reduced system is analyzed using Lyapunov techniques to derive ISS inequalities that capture the evolution of the states. Then, a direct application of the result of theorem 2 developed in [CT95], (which is briefly summarized at the end of this appendix) is made to establish that these ISS inequalities continue to hold up to an arbitrarily small offset, for arbitrarily large initial conditions. In the third part, the resulting ISS inequalities are studied, using techniques similar to those used in [ZPTP95], to show boundedness of the trajectories and establish the inequality of Eq.4.28. All the above results will be obtained for sufficiently small values of ϕ and ϵ.

Part 1: In this part of the proof, we will establish that the closed-loop fast subsystem is globally exponentially stable. Under the control law of Eq.4.26 the closed-loop system takes the form:

$$\dot{x} = \tilde{F}(x,\theta) + \tilde{G}(x,\theta)a(x,\bar{v},t) + [Q_1(x,\theta) + g_1(x,\theta)k^T(x)][z - C(x,\theta,\phi,\bar{v})] \tag{C.1}$$

$$\epsilon\dot{z} = [Q_2(x,\theta) + g_2(x,\theta)k^T(x)][z - C(x,\theta,\phi,\bar{v})] \tag{C.2}$$

where:

$$C(x,\theta,\phi,\bar{v}) = -[Q_2(x,\theta)+g_2(x,\theta)k^T(x)]^{-1}[f_2(x,\theta)+g_2(x,\theta)a(x,\bar{v},t)] \quad (C.3)$$

It is clear that the closed-loop fast subsystem:

$$\frac{dz}{d\tau} = [Q_2(x,\theta)+g_2(x,\theta)k^T(x)][z-C(x,\theta,\phi,\bar{v})] \quad (C.4)$$

possesses a globally exponentially stable equilibrium manifold of the form of Eq.C.3, since the matrix $Q_2(x,\theta)+g_2(x,\theta)k^T(x)$ is Hurwitz by the appropriate choice of the feedback gain $k^T(x)$.

Part 2: To proceed with the rest of the proof, we consider the representation of the closed-loop in terms of the ζ,η,z coordinates. For ease of notation, we set $e_z = (z-C(x,\theta,\phi,\bar{v}))$, $x=\mathcal{X}^{-1}(\zeta,\eta,\theta)$. We then have:

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 + e_z\bar{\Psi}_1(\zeta,\eta,\theta,\bar{v}) \\
&\vdots \\
\dot{\zeta}_{\bar{r}-1} &= \zeta_{\bar{r}} + e_z\bar{\Psi}_{\bar{r}-1}(\zeta,\eta,\theta,\bar{v}) \\
\dot{\zeta}_{\bar{r}} &= L_{\tilde{F}_{nom}}^{\bar{r}}h(x) + L_{\tilde{\xi}}L_{\tilde{F}}^{\bar{r}-1}h(x) + e_z\bar{\Psi}_{\bar{r}}(\zeta,\eta,\theta,\bar{v}) + L_{\tilde{G}}L_{\tilde{F}}^{\bar{r}-1}h(x)a(x,\bar{v},t) \\
\dot{\eta}_1 &= \Psi_1(\zeta,\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_{\bar{r}+1}(\zeta,\eta,\theta,\bar{v}) \\
&\vdots \\
\dot{\eta}_{n-\bar{r}} &= \Psi_{n-\bar{r}}(\zeta,\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_n(\zeta,\eta,\theta,\bar{v}) \\
\epsilon\dot{z} &= [Q_2(x,\theta)+g_2(x,\theta)k^T(x)]e_z
\end{aligned} \quad (C.5)$$

where $\bar{\Psi}_i$, $i=1,\cdots,n$ are Lipschitz functions of their arguments. Introducing, the variables $e_i = \zeta_i - v^{(i-1)}$, $i=1,\cdots,\bar{r}$, $\tilde{e}_{\bar{r}} = e_{\bar{r}} + \sum_{k=1}^{\bar{r}-1}\frac{\beta_k}{\beta_{\bar{r}}}e_k$, and the notation $\tilde{v} =$

$[v \; v^{(1)} \; \cdots \; v^{(\tilde{r}-1)}]^T$, the system of Eq.C.5 takes the form:

$$\begin{aligned}
\dot{e}_1 &= e_2 + \epsilon_z \bar{\Psi}_1(\bar{e}, \tilde{e}_{\tilde{r}}, \tilde{v}, \eta, \theta) \\
&\vdots \\
\dot{e}_{\tilde{r}-1} &= -\sum_{k=1}^{\tilde{r}-1} \frac{\beta_k}{\beta_{\tilde{r}}} e_k + \tilde{e}_{\tilde{r}} + e_z \bar{\Psi}_{\tilde{r}-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \tilde{v}, \eta, \theta) \\
\dot{\tilde{e}}_{\tilde{r}} &= e_z \bar{\Psi}_{\tilde{r}}(\bar{e}, \tilde{e}_{\tilde{r}}, \tilde{v}, \eta, \theta) + L_{\tilde{F}_{nom}}^{\tilde{r}} h(x) - v^{(\tilde{r})} + L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(x) \\
&\quad + \sum_{k=1}^{\tilde{r}-1} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1} + L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x) a(x, \bar{v}, t) \qquad \text{(C.6)} \\
\dot{\eta}_1 &= \Psi_1(\bar{e}, \tilde{e}_{\tilde{r}}, \tilde{v}, \eta, \theta, \dot{\theta}) + e_z \bar{\Psi}_{\tilde{r}+1}(\bar{e}, \tilde{e}_{\tilde{r}}, \tilde{v}, \eta, \theta) \\
&\vdots \\
\dot{\eta}_{n-\tilde{r}} &= \Psi_{n-\tilde{r}}(\bar{e}, \tilde{e}_{\tilde{r}}, \tilde{v}, \eta, \theta, \dot{\theta}) + e_z \bar{\Psi}_n(\bar{e}, \tilde{e}_{\tilde{r}}, \tilde{v}, \eta, \theta) \\
\epsilon \dot{z} &= [Q_2(x, \theta) + g_2(x, \theta) k^T(x)] e_z
\end{aligned}$$

Set $\bar{e} = [e_1 \; e_2 \; \cdots \; e_{\tilde{r}-1}]^T$, $\bar{\eta} = [\bar{e}^T \; \eta^T]^T$. We now follow a two-step procedure to establish ISS inequalities for the states $\bar{e}, \tilde{e}_{\tilde{r}}, \bar{\eta}$ of the system of Eq.C.6. Specifically, in the first step, we derive these inequalities in the absence of singular perturbations, i.e. $\epsilon_z = 0$. In the second step, we establish, using the result of theorem C1 given at the end of this appendix, that these properties also hold up to an arbitrarily small offset, for initial conditions and uncertainties in an arbitrarily large compact set, as long as ϵ is sufficiently small.

Step 1 : Initially, we note that a straightforward Lyapunov function argument can be used to show that there exist positive real numbers, $k_1, a, \gamma_{\tilde{e}_{\tilde{r}}}$ such that the following ISS inequality holds for the reduced $\bar{e}$ subsystem:

$$|\bar{e}(t)| \leq k_1 e^{-at} |\bar{e}(0)| + \gamma_{\tilde{e}_{\tilde{r}}} ||\tilde{e}_{\tilde{r}}|| \qquad \text{(C.7)}$$

In what follows, our objective is to show that the state $\tilde{e}_{\tilde{r}}$ of the reduced system of Eq.C.6 possesses an ISS property with respect to $\tilde{e}_{\tilde{r}}, \bar{e}, \bar{v}, \eta, \theta$, and moreover the gain function saturates at ϕ. Let us consider the system comprised of the state $\tilde{e}_{\tilde{r}}$ of the system of Eq.C.6:

$$\dot{\tilde{e}}_{\tilde{r}} = L_{\tilde{F}_{nom}}^{\tilde{r}} h(x) - v^{(\tilde{r})} + L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(x) + \sum_{k=1}^{\tilde{r}-1} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1} + L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x) a(x, \bar{v}, t) \quad (C.8)$$

To establish that the state $\tilde{e}_{\tilde{r}}$ of the above system satisfies an ISS property with respect to $\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta$, we consider the following smooth function $V : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$:

$$V = \frac{1}{2}\tilde{e}_{\tilde{r}}^2 \quad (C.9)$$

Defining $e_{\tilde{r}+1} = L_{\tilde{F}_{nom}}^{\tilde{r}} h(x) - v^{(\tilde{r})}$, $M = |L_{\tilde{G}} L_{\tilde{F}}^{\tilde{r}-1} h(x)| [\bar{c}_2(x,t)]^{-1}$, and computing the time-derivative of V along the trajectory of the system of Eq.C.8, we get:

$$\begin{aligned} \dot{V} = \tilde{e}_{\tilde{r}} \Bigg[& \sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1} + L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(x) + M \Bigg\{ -\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1} - \tilde{e}_{\tilde{r}} \\ & -2[\bar{c}_1(x,t) + |\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1}|] w(x,\phi) \Bigg\} \Bigg] \end{aligned} \quad (C.10)$$

Furthermore, it is straightforward to show that the representation of the function $w(x, \phi)$ in terms of the variable $\tilde{e}_{\tilde{r}}$ is given by:

$$w(\tilde{e}_{\tilde{r}}, \phi) = \frac{\tilde{e}_{\tilde{r}}}{|\tilde{e}_{\tilde{r}}| + \phi} \quad (C.11)$$

Substituting Eqs.C.11 into Eq.C.10, we have:

$$\begin{aligned} \dot{V} \leq\; & M\tilde{e}_{\tilde{r}} \Bigg\{ -\tilde{e}_{\tilde{r}} - 2\left[\bar{c}_1(x,t) + |\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1}|\right] \frac{\tilde{e}_{\tilde{r}}}{|\tilde{e}_{\tilde{r}}| + \phi} \\ & + (M^{-1} - 1) \sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1} + L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(x) \Bigg\} \\ \leq\; & M \Bigg\{ -\tilde{e}_{\tilde{r}}^2 - 2\left[\bar{c}_1(x,t) + |\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1}|\right] \frac{\tilde{e}_{\tilde{r}}^2}{|\tilde{e}_{\tilde{r}}| + \phi} \\ & + |\tilde{e}_{\tilde{r}}| |\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1}| + |\tilde{e}_{\tilde{r}}| \, |L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(x)| \Bigg\} \\ \leq\; & M \Bigg\{ -\tilde{e}_{\tilde{r}}^2 - \left[\bar{c}_1(x,t) + |\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1}|\right] \frac{\tilde{e}_{\tilde{r}}^2}{|\tilde{e}_{\tilde{r}}| + \phi} \\ & + \phi \left[|L_{\tilde{\delta}} L_{\tilde{F}}^{\tilde{r}-1} h(x)| + |\sum_{k=1}^{\tilde{r}} \frac{\beta_k}{\beta_{\tilde{r}}} e_{k+1}| \right] \frac{|\tilde{e}_{\tilde{r}}|}{|\tilde{e}_{\tilde{r}}| + \phi} \Bigg\} \end{aligned} \quad (C.12)$$

From the last inequality and the fact that $M \geq 1$ (this follows from the definition of M and assumption 4.4), it follows from assumption 4.3 that whenever $|\tilde{e}_{\tilde{r}}| \geq \phi$, the time-derivative of the Lyapunov function satisfies $\dot{V} \leq -\tilde{e}_{\bar{r}}^2$. This fact implies that the ultimate bound on the state $\tilde{e}_{\bar{r}}$ of the system of Eq.C.8 depends only on the parameter ϕ and is independent of the states $\bar{e}, \eta$ and the vector of uncertain variables θ.

Let us now study the time-derivative of V for $|\tilde{e}_{\bar{r}}| < \phi$. To simplify the notation, we set $\mathcal{U} = [\tilde{e}_{\bar{r}} \; \bar{\eta}^T \; \theta^T \; \bar{v}^T]^T$. Then, Eq.C.12 can be written as:

$$\begin{aligned} \dot{V} &\leq M\left\{-\tilde{e}_{\bar{r}}^2 + |\tilde{e}_{\bar{r}}|[|L_{\tilde{g}}L_{\tilde{F}}^{\bar{r}-1}h(x)| + |\sum_{k=1}^{\bar{r}}\frac{\beta_k}{\beta_{\bar{r}}}e_{k+1}|]\right\} \\ &\leq -\tilde{e}_{\bar{r}}^2 + |\tilde{e}_{\bar{r}}|\rho(|\mathcal{U}|) \end{aligned} \tag{C.13}$$

where ρ is a class K_∞ function. Summarizing, we have that $\dot{V}$ satisfies the following properties:

$$\dot{V} \leq -\frac{|\tilde{e}_{\bar{r}}|^2}{2}, \quad |\tilde{e}_{\bar{r}}| \geq \min\{\phi, 2\rho(|\mathcal{U}|)\} =: \tilde{\gamma}_{\mathcal{U}}(|\mathcal{U}|) \tag{C.14}$$

From the above equation, a direct application of the result of theorem 4.10 reported in [Kha92], can be performed to conclude that the following ISS inequality holds for the state $\tilde{e}_{\bar{r}}$ of the system of Eq.C.8:

$$|\tilde{e}_{\bar{r}}(t)| \leq e^{-0.5t}|\tilde{e}_{\bar{r}}(0)| + \tilde{\gamma}_{\mathcal{U}}(||\mathcal{U}||) \tag{C.15}$$

Before we proceed with the rest of this step, we will assume that $\phi \in (0, \phi^*]$, where :

$$\phi^* = \frac{d}{1 + 2\gamma_{\tilde{e}_{\bar{r}}}} \tag{C.16}$$

Consider now the following reduced system:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_{\bar{r}-1} &= -\sum_{k=1}^{\bar{r}-1}\frac{\beta_k}{\beta_{\bar{r}}}e_k + \tilde{e}_{\bar{r}} \\ \dot{\eta}_1 &= \Psi_1(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta, \dot{\theta}) \\ &\vdots \\ \dot{\eta}_{n-\bar{r}} &= \Psi_{n-\bar{r}}(\bar{e}, \tilde{e}_{\bar{r}}, \bar{v}, \eta, \theta, \dot{\theta}) \end{aligned} \tag{C.17}$$

From assumption 4.3, we have that the η state of the above system possesses an ISS property with respect to $\bar{e}, \tilde{e}_{\bar{r}}, \theta, \dot{\theta}$. We also note that the system of Eq.C.17 is a cascaded interconnection of two ISS subsystems and thus, the state $\bar{\eta} = [\bar{e}^T \ \eta^T]^T$ possesses an ISS property with respect to $\tilde{e}_{\bar{r}}, \theta, \dot{\theta}$ [Son89, ZPTP95]. To apply the result of theorem C1 (appendix C), in step 2 of the proof, we need the existence of a converse function for the system in Eq.C.17. The existence of a converse function for this system can be derived utilizing the converse theorem developed in [SW95]. In particular, we have that there exists a converse function for the system in Eq.C.17 and the existence of this function implies that there exist a function $\beta_{\bar{\eta}}$ of class KL and a function $\bar{\gamma}_{\bar{\eta}}$ of class K such that the following ISS inequality holds for the state $\bar{\eta}$:

$$\begin{aligned} |\bar{\eta}(t)| &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\bar{\eta}}(||[\tilde{e}_{\bar{r}} \ \theta^T \ \dot{\theta}^T]^T||) \\ &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(||\tilde{e}_{\bar{r}}||) + \bar{\gamma}_{\theta}(||\theta||) + \bar{\gamma}_{\dot{\theta}}(||\dot{\theta}||) \end{aligned} \tag{C.18}$$

where $\bar{\gamma}_{\tilde{e}_{\bar{r}}}, \bar{\gamma}_{\theta}, \bar{\gamma}_{\dot{\theta}}$ are class K functions respectively, defined as $\bar{\gamma}_{\tilde{e}_{\bar{r}}}(s) = \bar{\gamma}_{\theta}(s) = \bar{\gamma}_{\dot{\theta}}(s) = \bar{\gamma}_{\bar{\eta}}(3s)$.

Step 2: We will now utilize the result of theorem C1 (appendix C) to establish that the ISS inequalities of Eqs.C.15-C.18 continue to hold up to an arbitrarily small offset, for the states $\tilde{e}_{\bar{r}}, \bar{\eta}$ of the singularly perturbed system of Eq.C.6. In order to proceed with the application of this result, we define a set of positive real numbers $(\delta_{\tilde{e}_{\bar{r}}}, \delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, \tilde{\delta}_{\bar{\eta}}, \tilde{\delta}_{\tilde{e}_{\bar{r}}}, d_{\tilde{e}_{\bar{r}}}, d_{\bar{\eta}})$, where $\delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}$ were specified in the statement of the theorem, $d_{\bar{\eta}}$ is an arbitrary positive real number,

$$\begin{aligned} \delta_{\tilde{e}_{\bar{r}}} &\geq \max_{|x| \leq \delta_x, |\bar{v}| \leq \delta_v} \left\{ \left| \sum_{k=1}^{\bar{r}} \frac{\beta_k}{\beta_{\bar{r}}} (v^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1} h(x)) \right| \right\} \\ \delta_{\bar{\eta}} &\geq \max_{|x| \leq \delta_x, |\bar{v}| \leq \delta_v, |\theta| \leq \delta_\theta} \left\{ \sum_{k=1}^{\bar{r}-1} |(v^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1} h(x))| + \sum_{\nu=1}^{n-\bar{r}} |\chi_\nu(x, \theta)| \right\} \end{aligned} \tag{C.19}$$

where $\chi_\nu(x, \theta)$, $\nu = 1, \ldots, n - \bar{r}$ are scalar fields defined in assumption 4.2, $\tilde{\delta}_{\tilde{e}_{\bar{r}}} > \delta_{\tilde{e}_{\bar{r}}} + 2\phi$, $\tilde{\delta}_{\bar{\eta}} > D^{\bar{\eta}} + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(\tilde{\delta}_{\tilde{e}_{\bar{r}}})$, with $D^{\bar{\eta}} := \beta_{\bar{\eta}}(\delta_{\bar{\eta}}, 0) + \bar{\gamma}_\theta(\delta_\theta) + \bar{\gamma}_{\dot{\theta}}(\delta_{\dot{\theta}}) + d_{\bar{\eta}}$, and $d_{\tilde{e}_{\bar{r}}} = \phi < d$ (without any loss of generality). The reason for these choices will become clear subsequently.

Let us now consider the singularly perturbed system comprised of the states $(\tilde{e}_{\bar{r}}, z)$ of the system of Eq.C.6. For this system, the application of theorem C1 (appendix C) is possible because this system is in standard form, possesses a uniformly globally exponentially stable fast subsystem, and the reduced system of Eq.C.8 is input-to-state stable with respect to $\mathcal{U}$ (Eq.C.15). Then, there is an $\epsilon^{\tilde{e}_{\bar{r}}}(\phi) > 0$ such that if $\epsilon \in (0, \epsilon^{\tilde{e}_{\bar{r}}}(\phi)]$ and $|\tilde{e}_{\bar{r}}(0)| \leq \delta_{\tilde{e}_{\bar{r}}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\bar{v}|| \leq \delta_{\bar{v}}$, $||\bar{\eta}|| \leq \tilde{\delta}_{\bar{\eta}}$, $||\tilde{e}_{\bar{r}}|| \leq \tilde{\delta}_{\tilde{e}_{\bar{r}}}$, then

$$|\tilde{e}_{\bar{r}}(t)| \leq e^{-0.5t}|\tilde{e}_{\bar{r}}(0)| + \tilde{\gamma}_{\mathcal{U}}(||\mathcal{U}||) + \phi \tag{C.20}$$

Notice also that the explicit dependence of the upper bound on the singular perturbation parameter $\epsilon^{\tilde{e}_{\bar{r}}}$ on ϕ is due to the appearance of parameter ϕ in the right-hand-side of the differential equation which describes the closed-loop fast dynamics.

It can be also verified that the result of theorem C1 reported at the end of this appendix is also applicable to the singularly perturbed system comprised of the states $(\tilde{e}, \eta, z)$ of the system of Eq.C.6, with the same converse function which exists for the system in Eq.C.17 and its resulting $(\beta_{\bar{\eta}}, \bar{\gamma}_{\bar{\eta}})$ (see also the discussion at the end of step 1). So we have that if $\epsilon \in (0, \epsilon^{\bar{\eta}}(\phi)]$ and $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$, $||\tilde{e}_{\bar{r}}|| \leq \tilde{\delta}_{\tilde{e}_{\bar{r}}}$, then

$$\begin{aligned} |\bar{\eta}(t)| &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\bar{\eta}}(||[\tilde{e}_{\bar{r}}\ \theta^T\ \dot{\theta}^T]^T||) + d_{\bar{\eta}} \\ &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(||\tilde{e}_{\bar{r}}||) + \bar{\gamma}_\theta(||\theta||) + \bar{\gamma}_{\dot{\theta}}(||\dot{\theta}||) + d_{\bar{\eta}} \end{aligned} \tag{C.21}$$

For the same reasons discussed above, the upper bound $\epsilon^{\bar{\eta}}$ depends on ϕ.

Part 3 : In this part of the proof, we will show that given the set of positive real numbers $\delta_{\tilde{e}_{\bar{r}}}, \delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d$ (already specified) and with ϕ^* defined as in Eq.C.16 and $\phi \in (0, \phi^*]$, there exists $\epsilon^*(\phi) \in (0, \epsilon^{\bar{\eta}}(\phi)]$, such that if $\epsilon \in (0, \epsilon^*(\phi)]$ and $|\tilde{e}_{\bar{r}}(0)| \leq \delta_{\tilde{e}_{\bar{r}}}$, $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$ the output of the closed-loop system of Eq.C.2 satisfies the relation of Eq.4.28.

To establish this result, we will analyze the behavior of the dynamical system comprised of the states $\tilde{e}_{\tilde{r}}, \bar{\eta}$ of the system of Eq.C.6, for which the inequalities of Eqs.C.20-C.21 hold, using calculations similar to those used in [ZPTP95]. In the first step, we will use a contradiction argument to show that if $\phi \in (0, \phi^*]$ the evolution of the states $\tilde{e}_{\tilde{r}}, \bar{\eta}$, starting from initial conditions that satisfy $|\tilde{e}_{\tilde{r}}(0)| \leq \delta_{\tilde{e}_{\tilde{r}}}$, $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$ and for θ, $\dot{\theta}$, $\bar{v}$ such that $||\theta|| \leq \delta_{\theta}$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$ satisfies the following inequalities:

$$|\tilde{e}_{\tilde{r}}(t)| \leq \tilde{\delta}_{\tilde{e}_{\tilde{r}}}, \quad |\bar{\eta}(t)| \leq \tilde{\delta}_{\bar{\eta}} \tag{C.22}$$

for all times. In the second step, we will analyze the asymptotic behavior of the system comprised of the states $\tilde{e}, \tilde{e}_{\tilde{r}}$ of Eq.C.6 and establish that the inequality of Eq.4.28 holds.

Step 1 : We will proceed by contradiction. Let T be the smallest time such that there is a δ so that $t \in (T, T+\delta)$ implies either $|\tilde{e}_{\tilde{r}}(t)| > \tilde{\delta}_{\tilde{e}_{\tilde{r}}}$ or $|\bar{\eta}(t)| > \tilde{\delta}_{\bar{\eta}}$. Then, for each $t \in [0, T]$ the conditions of Eq.C.22 hold. Consider the functions $\tilde{e}_{\tilde{r}}^T(t)$, $\bar{\eta}^T(t)$ defined as follows:

$$\tilde{e}_{\tilde{r}}^T(t) = \left\{ \begin{array}{ll} \tilde{e}_{\tilde{r}}(t) & t \in [0, T] \\ 0 & t \in (T, \infty) \end{array} \right\}, \quad \bar{\eta}^T(t) = \left\{ \begin{array}{ll} \bar{\eta}(t) & t \in [0, T] \\ 0 & t \in (T, \infty) \end{array} \right\} \tag{C.23}$$

From the fact that $||\theta|| \leq \delta_{\theta}$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, we have that:

$$\sup_{0 \leq t \leq T} \left(\beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\theta}(||\theta||) + \bar{\gamma}_{\dot{\theta}}(||\dot{\theta}||) + d_{\bar{\eta}}\right) \leq \beta_{\bar{\eta}}(\delta_{\bar{\eta}}, 0) + \bar{\gamma}_{\theta}(\delta_{\theta}) + \bar{\gamma}_{\dot{\theta}}(\delta_{\dot{\theta}}) + d_{\bar{\eta}} =: D^{\bar{\eta}} \tag{C.24}$$

$$\sup_{0 \leq t \leq T} \left(e^{-0.5t}|\tilde{e}_{\tilde{r}}(0)| + \bar{\gamma}_{\mathcal{U}}(||\mathcal{U}_T||) + \phi\right) \leq \delta_{\tilde{e}_{\tilde{r}}} + \phi + \phi \leq \delta_{\tilde{e}_{\tilde{r}}} + 2\phi$$

Combining the above inequalities with Eq.C.20, and also combining Eq.C.24 with Eq.C.21, we have that for each $\phi \in (0, \phi^*]$, $\epsilon \in (0, \epsilon^{\bar{\eta}}(\phi)]$, and for all $t \geq 0$:

$$\begin{aligned} ||\tilde{e}_{\tilde{r}}^T|| &\leq \delta_{\tilde{e}_{\tilde{r}}} + 2\phi < \tilde{\delta}_{\tilde{e}_{\tilde{r}}} \\ ||\bar{\eta}^T|| &\leq D^{\bar{\eta}} + \bar{\gamma}_{\tilde{e}_{\tilde{r}}}(||\tilde{e}_{\tilde{r}}^T||) \leq D^{\bar{\eta}} + \bar{\gamma}_{\tilde{e}_{\tilde{r}}}(\tilde{\delta}_{\tilde{e}_{\tilde{r}}}) < \tilde{\delta}_{\bar{\eta}} \end{aligned} \tag{C.25}$$

By continuity, we have that there exist some positive real number $\bar{k}$ such that

$||\tilde{e}_{\bar{r}}^{T+\bar{k}}(t)|| \leq \delta_{\tilde{e}_{\bar{r}}}$ and $||\bar{\eta}^{T+\bar{k}}(t)|| \leq \tilde{\delta}_{\bar{\eta}}$, $\forall\ t \in [0, T+\bar{k}]$. This contradicts the definition of T. Hence, Eq.C.22 holds $\forall\ t \geq 0$.

Step 2 : In this step, we will analyze the asymptotic properties of the interconnection consisting of the states $\bar{e}, \tilde{e}_{\bar{r}}$ of the system of Eq.C.6, in order to recover the inequality of Eq.4.28. We initially proceed with a direct application of theorem C1 reported at the end of this appendix to the interconnection comprised of the states $(\bar{e}, z)$ of the system of Eq.C.6. To this end, we consider the set of positive real numbers $(\delta_{\bar{e}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d_{\bar{e}})$, where $\delta_{\bar{e}} \geq \max_{|x|\leq\delta_x, |\bar{v}|\leq\delta_{\bar{v}}}\{\sum_{k=1}^{\bar{r}-1}|(v^{(k-1)} - L_{\bar{F}_{nom}}^{k-1}h(x))|\}$, $\delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}$ were specified in the theorem, and $d_{\bar{e}} = \phi$, without any loss of generality. Then, there is an $\epsilon^{\bar{e}}(\phi) \in (0, \epsilon^{\bar{\eta}}(\phi)]$ such that if $\epsilon \in (0, \epsilon^{\bar{e}}(\phi)]$ and $|\bar{e}(0)| \leq b$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$, then:

$$|\bar{e}(t)| \leq k_1 e^{-at}|\bar{e}(0)| + \gamma_{\tilde{e}_{\bar{r}}}||\tilde{e}_{\bar{r}}|| + \phi \tag{C.26}$$

In step 1, we established that the trajectories of the closed-loop system state bounded for all times, for $\phi \in (0, \phi^*]$ and $\epsilon \in (0, \epsilon^{\bar{e}}(\phi)]$.

Claim : The following inequality holds:

$$\limsup_{t\to\infty} |\bar{e}(t)| \leq \gamma_{\tilde{e}_{\bar{r}}}(\limsup_{t\to\infty} |\tilde{e}_{\bar{r}}(t)|) + \phi \tag{C.27}$$

Proof : First, we note that:

$$||\bar{e}(t)|| \leq k_1\delta_{\bar{e}} + \gamma_{\tilde{e}_{\bar{r}}}\tilde{\delta}_{\tilde{e}_{\bar{r}}} + \phi =: b \tag{C.28}$$

The fact that the inequality of Eq.C.26 holds for every initial time t_0 [CT95], yields that $\forall\ t \geq t_0$:

$$|\bar{e}(t)| \leq k_1 e^{-a(t-t_0)}|\bar{e}(t_0)| + \gamma_{\tilde{e}_{\bar{r}}} \sup_{\tau\geq t_0} |\tilde{e}_{\bar{r}}(\tau)| + \phi \tag{C.29}$$

Pick $t_0 = \frac{t}{2}$. We then have:

$$|\bar{e}(t)| \leq k_1 e^{-a(\frac{t}{2})}(b) + \gamma_{\tilde{e}_{\bar{r}}} \sup_{\tau\geq \frac{t}{2}} |\tilde{e}_{\bar{r}}(\tau)| + \phi \tag{C.30}$$

Taking the limsup of both sides, the result of the claim can be obtained. $\triangle$

Furthermore we have that (see Eq.48):

$$\limsup_{t\to\infty} |\tilde{e}_{\bar{r}}(t)| \leq \phi + \phi \leq 2\phi \tag{C.31}$$

Combining the inequalities of Eqs.C.27-C.31, the following bound can be written for $\bar{e}$:

$$\limsup_{t\to\infty} |\bar{e}(t)| \leq \gamma_{\tilde{e}_{\bar{r}}}(2\phi) + \phi \leq d \tag{C.32}$$

We then have that for each $\phi \in (0, \phi^*]$ and $\epsilon \in (0, \epsilon^{\bar{e}}(\phi)]$:

$$\limsup_{t\to\infty} |y(t) - v(t)| = \limsup_{t\to\infty} |\bar{e}_1(t)| \leq d \tag{C.33}$$

Summarizing the above inequality and all the other results of the theorem are obtained for $\phi \in (0, \phi^*]$ and $\epsilon^*(\phi) = \epsilon^{\bar{e}}(\phi)$. The proof of the theorem is complete. $\triangle$

Proof of proposition 4.1: Initially, a direct application of the identity of Eq.4.1 yields:

$$\begin{aligned} det(Q_2(x,\theta) + g_2(x,\theta)k^T(x)) &= det(Q_2(x,\theta))det(I_{p\times p} + Q_2^{-1}(x,\theta)g_2(x,\theta)k^T(x)) \\ &= det(Q_2(x,\theta))det(1 + k^T(x)Q_2^{-1}(x,\theta)g_2(x,\theta)) \end{aligned} \tag{C.34}$$

From the hypothesis of invertibility of the matrix $Q_2(x,\theta)$ uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ and the above calculation, it follows directly that the matrix $Q_2(x,\theta) + g_2(x,\theta)k^T(x)$ is invertible uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ if and only if the scalar $1 + k^T(x)Q_2^{-1}(x,\theta)g_2(x,\theta)$ is nonzero uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.

It is now straightforward to show that the control law of Eq.4.14, calculated on the quasi-steady-state of Eq.4.6 takes the form:

$$u = [1 + k^T(x)Q_2^{-1}(x,\theta)g_2(x,\theta)]^{-1}[\tilde{u} - k^T(x)Q_2^{-1}(x,\theta)f_2(x,\theta)] \tag{C.35}$$

Substitution of the state feedback law of Eq.C.35 into the open-loop reduced system of Eq.4.7 yields the following reduced system:

$$\begin{array}{rcl} \dot{x} & = & F(x,\theta) + G(x,\theta)[1 + k^T(x)Q_2^{-1}(x,\theta)g_2(x,\theta)]^{-1}[\tilde{u} - k^T(x)Q_2^{-1}(x,\theta)f_2(x,\theta)] \\ y & = & h(x) \end{array} \tag{C.36}$$

where the vector functions $F(x,\theta)$ and $G(x,\theta)$ are given by Eq.4.7. Now, it can be easily verified that the reduced system of Eq.C.36 is identical to the reduced system of Eq.4.17. Note that these systems were derived by commuting the operations *setting* $\epsilon = 0$ and *closing the feedback loop*, (see also [KKO86, CGG93], for an analogous result in the case of linear singularly perturbed systems).

On the other hand, it is clear from the structure of the system of Eq.C.36, and the hypothesis $r \leq s$ that a direct differentiation of the output of the system of Eq.C.36 up to order r yields the following expressions:

$$\begin{array}{rcl} y & = & h(x) \\ y^{(1)} & = & L_F h(x) \\ y^{(2)} & = & L_F^2 h(x) \\ \vdots & & \\ y^{(r-1)} & = & L_F^{r-1} h(x) \\ y^{(r)} & = & L_F^r h(x) + L_G L_F^{r-1} h(x)[1 + k^T(x)Q_2^{-1}(x,\theta)g_2(x,\theta)]^{-1} \\ & & [\tilde{u} - k^T(x)Q_2^{-1}(x,\theta)f_2(x,\theta)] \end{array} \tag{C.37}$$

On the basis of the above expressions, it is clear that for the reduced system of Eq.C.36, (and therefore for the reduced system of Eq.4.17), $r = \tilde{r} \leq \tilde{s}$. This completes the proof of the proposition. $\triangle$

Review of theorem 2 in [CT95]: We recall a result developed in [CT95], (see also [CT96]). Consider the singularly perturbed system:

$$\begin{array}{rcl} \dot{x} & = & f(x,z,\theta(t),\epsilon) \\ \epsilon\dot{z} & = & g(x,z,\theta(t),\epsilon) \end{array} \tag{C.38}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $\theta \in \mathbb{R}^q$. The functions f and g are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times [0,\bar{\epsilon})$, for $\bar{\epsilon}$ sufficiently small.

Assumption C1: *The algebraic equation $g(x, z_s, \theta) = 0$ possesses a unique root:*

$$z_s = h(x, \theta) \tag{C.39}$$

with the properties that $h : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^p$ and its partial derivatives $\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial \theta}\right)$ are locally Lipschitz and $h(0,0) = 0$.

Consider also the corresponding slow and fast subsystems:

$$\dot{x} = f(x, h(x, \theta), \theta, 0) \tag{C.40}$$

$$\frac{dy}{d\tau} = g(x, h(x, \theta) + y, \theta, 0) \tag{C.41}$$

where $y = z - h(x, \theta)$. The assumptions that follow state our stability requirements on the slow and fast subsystems.

Assumption C2: *The reduced system in Eq.C.40 is input-to-state stable with the class-KL function β_x and the class-K function γ_x: more precisely there exist a smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and class-K_∞ functions a_1, a_2, a_3, a_4 such that:*

$$a_1(|x|) \leq V(x) \leq a_2(|x|) \tag{C.42}$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x, h(x, \theta), \theta, 0) \leq a_4(|\theta|) - a_3(|x|) \tag{C.43}$$

for all $\theta \in \mathbb{R}^q$ and all $x \in \mathbb{R}^n$, hold and the existence of such a V implies that Eq.4.13 holds with $\beta = \beta_x$ and $\gamma = \gamma_x$.

Assumption C3: *The equilibrium $y = 0$ of the boundary layer system in Eq.C.41 is globally asymptotically stable, uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$.*

The main result is as follows:

Theorem C1 : *Consider the singularly perturbed system in Eq.C.38 for which assumption C1 holds and let $y = z - h(x, \theta)$. If $\theta(t)$ is absolutely continuous and assumptions C2 and C3 hold, then there exists a function β_y of class KL such that,* *for each set of positive real numbers* $(\delta_x, \delta_y, \delta_\theta, \delta_{\dot\theta}, d_x, d_y)$*. there is an* $\epsilon^* > 0$ *such that if* $\epsilon \in (0, \epsilon^*]$ *and* $|x(0)| \leq \delta_x$, $|y(0)| \leq \delta_y$, $||\theta|| \leq \delta_\theta$, $||\dot\theta|| \leq \delta_{\dot\theta}$, *then*

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_x(||\theta||) + d_x \tag{C.44}$$

$$|y(\tau)| \leq \beta_y(|y(0)|, \tau) + d_y \tag{C.45}$$

where β_x *and* γ_x *are the functions defined in assumption C2.*

Appendix D

Proofs of Chapter 5

Proof of theorem 5.1:

The proof of the theorem is conceptually analogous (although notationally more involved) to the one given in Appendix C for single-input single-output systems. In order to stress the issues associated with the multivariable nature of the problem and avoid repetitions, we will refer without proof to some results established in the proof of theorem 4.1. The proof consists of three parts: initially, the global exponential stability of the fast dynamics of the closed-loop system is established; then, the closed-loop reduced system is analyzed using Lyapunov techniques to derive bounds that capture the evolution of the states in terms of the initial conditions and the inputs, and a direct application of a result of theorem C1 (appendix C) is made to establish that these bounds continue to hold up to an arbitrarily small offset, for the singularly perturbed system. Finally, the resulting bounds are utilized, using techniques (small gain theorem type calculations) similar to those used in [Tee96] and [ZPTP95], to show boundedness of the trajectories and establish the inequality of Eq.5.20. All the above results will be obtained for sufficiently small values of ϕ and ϵ.

Part 1: In this part of the proof, we will establish that the closed-loop fast subsystem is globally exponentially stable. Under the control law of Eq.5.18 the closed-loop system takes the form:

$$\begin{aligned}\dot{x} &= \tilde{F}(x,\theta) + \tilde{G}(x,\theta)\left\{\Delta(x,t)\left\{\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^{k}h_i(x))\right.\right.\\ &\quad +\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1}h_i(x))\\ &\quad \left.\left.-(2+b)[\tilde{c}_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^{k}h_i(x))|]w(x,\phi)\right\}\right\}\\ &\quad +[Q_1(x,\theta) + G_1(x,\theta)K(x)][z - C(x,\theta,\phi,v_i^{(k)})]\end{aligned} \tag{D.1}$$

$$\epsilon\dot{z} = [Q_2(x,\theta) + G_2(x,\theta)K(x)][z - C(x,\theta,\phi,v_i^{(k)})] \tag{D.2}$$

where:

$$\begin{aligned}C(x,\theta,\phi,v_i^{(k)}) &= -[Q_2(x,\theta) + G_2(x,\theta)K(x)]^{-1}[f_2(x,\theta) + G_2(x,\theta)\{\Delta(x,t)\\ &\quad \left\{\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^{k}h_i(x)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1}h_i(x))\right.\\ &\quad \left.\left.-(2+b)[\tilde{c}_1(x,t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}(v_i^{(k)} - L_{\tilde{F}_{nom}}^{k}h_i(x))|]w(x,\phi)\right\}\right\}]\end{aligned} \tag{D.3}$$

It is clear that if the matrix $K(x)$ is chosen so that the matrix $Q_2(x,\theta)+G_2(x,\theta)K(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$ the closed-loop fast subsystem:

$$\frac{dz}{d\tau} = [Q_2(x,\theta) + G_2(x,\theta)K(x)][z - C(x,\theta,\phi,v_i^{(k)})] \tag{D.4}$$

possesses a globally exponentially stable equilibrium manifold of the form of Eq.D.3.

Part 2: To simplify the notation, set $\tilde{C}_i(x,\theta) = [L_{\tilde{G}^1}L_{\tilde{F}}^{\tilde{r}_i-1}h_i(T^{-1}(\zeta,\eta,\theta)) \cdots L_{\tilde{G}^m}L_{\tilde{F}}^{\tilde{r}_i-1}h_i(T^{-1}(\zeta,\eta,\theta))]$, $i = 1,\ldots,m$, $\epsilon_z = (z - C(x,\theta,\phi,v_i^{(k)}))$. Observing the similarity in the structure of the system of Eq.D.1 and the x-subsystem of Eq.5.9 and using assumption 5.2, the representation of the closed-loop system of Eqs.D.1-D.2 in (ζ, η, z) coordinates takes the form:

$$
\begin{aligned}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} + \epsilon_z \bar{\Psi}_1^{(1)}(\zeta, \eta, \theta, v_i^{(k)}) \\
&\vdots \\
\dot{\zeta}_{\tilde{r}_1-1}^{(1)} &= \zeta_{\tilde{r}_1}^{(1)} + \epsilon_z \bar{\Psi}_{\tilde{r}_1}^{(1)}(\zeta, \eta, \theta, v_i^{(k)}) \\
\dot{\zeta}_{\tilde{r}_1}^{(1)} &= L_{\tilde{F}}^{\tilde{r}_1} h_1(T^{-1}(\zeta, \eta, \theta)) + \epsilon_z \bar{\Psi}_{\tilde{r}_1}^{(1)}(\zeta, \eta, \theta, v_i^{(k)}) + \tilde{C}_1(x, \theta) \\
&\left\{ \Delta(x, t) \left\{ \sum_{i=1}^{m} \sum_{k=1}^{\tilde{r}_i} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} (v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x)) \right.\right. \\
&+ \sum_{i=1}^{m} \sum_{k=1}^{\tilde{r}_i} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} (v_i^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1} h_i(x)) \\
&\left.\left. -(2+b)[\tilde{c}_1(x, t) + |\sum_{i=1}^{m} \sum_{k=1}^{\tilde{r}_i} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} (v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x))|]w(x, \phi) \right\} \right\} \\
&\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} + \epsilon_z \bar{\Psi}_1^{(m)}(\zeta, \eta, \theta, v_i^{(k)}) \\
&\vdots \\
\dot{\zeta}_{\tilde{r}_m-1}^{(m)} &= \zeta_{\tilde{r}_m}^{(m)} + \epsilon_z \bar{\Psi}_{\tilde{r}_m-1}^{(m)}(\zeta, \eta, \theta, v_i^{(k)}) \\
\dot{\zeta}_{\tilde{r}_m}^{(m)} &= L_{\tilde{F}}^{\tilde{r}_m} h_m(T^{-1}(\zeta, \eta, \theta)) + \epsilon_z \bar{\Psi}_{\tilde{r}_m}^{(m)}(\zeta, \eta, \theta, v_i^{(k)}) + \tilde{C}_m(x, \theta) \{\Delta(x, t) \\
&\left\{ \sum_{i=1}^{m} \sum_{k=1}^{\tilde{r}_i} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} (v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x)) + \sum_{i=1}^{m} \sum_{k=1}^{\tilde{r}_i} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} (v_i^{(k-1)} - L_{\tilde{F}_{nom}}^{k-1} h_i(x)) \right. \\
&\left. -(2+b)[\tilde{c}_1(x, t) + |\sum_{i=1}^{m} \sum_{k=1}^{\tilde{r}_i} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} (v_i^{(k)} - L_{\tilde{F}_{nom}}^{k} h_i(x))|]w(x, \phi) \right\} \} \\
\dot{\eta}_1 &= \Psi_1(\eta, \zeta, \theta, \dot{\theta}) + \epsilon_z \bar{\Psi}_{\sum_i \tilde{r}_i + 1}(\zeta, \eta, \theta, v_i^{(k)}) \\
&\vdots \\
\dot{\eta}_{n-\sum_i \tilde{r}_i} &= \Psi_{n-\sum_i \tilde{r}_i}(\zeta, \eta, \theta, \dot{\theta}) + \epsilon_z \bar{\Psi}_n(\zeta, \eta, \theta, v_i^{(k)}) \\
\epsilon \dot{z} &= [Q_2(T^{-1}(\zeta, \eta, \theta), \theta) + G_2(T^{-1}(\zeta, \eta, \theta), \theta) K(T^{-1}(\zeta, \eta, \theta))] \epsilon_z \\
y_i &= \zeta_1^{(i)}, \quad i = 1, \dots, m
\end{aligned}
\tag{D.5}
$$

where $\bar{\Psi}_k^{(i)}$, $i = 1, \dots, m$, $k = 1, \dots, \tilde{r}_i$, and $\bar{\Psi}_{\sum_i \tilde{r}_i + 1}, \dots, \bar{\Psi}_n$, are Lipschitz functions of their arguments.

Introducing, the variables $e_k^{(i)} = \zeta_k^{(i)} - v_i^{(k-1)}$, $i = 1, \cdots, m$, $k = 1, \dots, \tilde{r}_i$, $\tilde{e}_{\tilde{r}_i}^{(i)} = \sum_{i=1}^{m} e_{\tilde{r}_i}^{(i)} + \sum_{i=1}^{m} \sum_{k=1}^{\tilde{r}_i - 1} \frac{\beta_{ik}}{\beta_{i\tilde{r}_i}} e_k^{(i)}$, and the notation $\bar{e}^{(i)} = [e_1^{(i)} \; e_2^{(i)} \; \cdots \; e_{\tilde{r}_i-1}^{(i)}]^T$,

$\bar{e} = [e^{(1)T}\ e^{(2)T}\ \cdots\ e^{(m)T}]^T$, $\bar{\eta} = [\bar{e}^T\ \eta^T]$, $\tilde{e}_{\tilde{r}} = [\tilde{e}_{\tilde{r}_i}^{(1)}\ \tilde{e}_{\tilde{r}_i}^{(2)}\ \cdots\ \tilde{e}_{\tilde{r}_i}^{(m)}]^T$, $v_i = [v_i^{(0)}\ v_i^{(1)}\ \cdots\ v_i^{(\tilde{r}_i-1)}]^T$, $\tilde{v} = [v_1^T\ v_2^T\ \cdots\ v_m^T]^T$, $M_i(x,\theta,t) = \tilde{C}_i(x,\theta)\Delta(x,t)$, the system of Eq.D.5 takes the form:

$$
\begin{aligned}
\dot{e}_1^{(1)} &= e_2^{(1)} + e_z\bar{\Psi}_1^{(1)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)\\
&\vdots\\
\dot{e}_{\tilde{r}_1-1}^{(1)} &= -\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i-1}\frac{\beta_{ik}^1}{\beta_{i\tilde{r}_i}^1}e_k^{(i)} + \sum_{i=1}^{m}\tilde{e}_{\tilde{r}_1}^{(1)} + e_z\bar{\Psi}_{\tilde{r}_1}^{(1)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)\\
\dot{\tilde{e}}_{\tilde{r}_1}^{(1)} &= e_z\bar{\Psi}_{\tilde{r}_1}^{(1)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) + L_{\tilde{g}}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}^{(1)}}{\beta_{i\tilde{r}_i}^{(1)}}e_{k+1}^{(i)}\\
&\quad +M_1(x,\theta,t)\left\{-\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} - \tilde{e}_{\tilde{r}} - (2+b)[\tilde{c}_1((\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t)\right.\\
&\quad \left.+|\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]w(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)),\phi)\right\}\\
&\vdots\\
\dot{e}_1^{(m)} &= e_2^{(m)} + e_z\bar{\Psi}_1^{(m)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)\\
&\vdots\\
\dot{e}_{\tilde{r}_m-1}^{(m)} &= e_{\tilde{r}_m}^{(m)} + e_z\bar{\Psi}_{\tilde{r}_m-1}^{(m)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)\\
\dot{e}_{\tilde{r}_m}^{(m)} &= e_z\bar{\Psi}_{\tilde{r}_m}^{(m)}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta) + L_{\tilde{g}}L_{\tilde{F}}^{\tilde{r}_m-1}h_m(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}^{(m)}}{\beta_{i\tilde{r}_i}^{(m)}}e_{k+1}^{(i)}\\
&\quad +M_m(x,\theta,t)\left\{-\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} - \tilde{e}_{\tilde{r}} - (2+b)[\tilde{c}_1((\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t)\right.\\
&\quad \left.+|\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]w(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)),\phi)\right\}\\
\dot{\eta}_1 &= \Psi_1(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_{\sum_i\tilde{r}_i+1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)\\
&\vdots\\
\dot{\eta}_{n-\sum_i\tilde{r}_i} &= \Psi_{n-\sum_i\tilde{r}_i}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta,\dot{\theta}) + e_z\bar{\Psi}_n(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)\\
\epsilon\dot{z} &= [Q_2(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)) + G_2(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta))K(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta))]e_z\\
y_i &= \zeta_1^{(i)},\quad i=1,\ldots,m
\end{aligned}
\tag{D.6}
$$

We will proceed with a two-step procedure to establish the ISS bounds that capture the evolution of the states $\bar{e}, \tilde{e}_{\tilde{r}}, \bar{\eta}$ of the above system. In particular, we will initially obtain these bounds in the absence of singular perturbations (i.e. when the singular perturbation parameter ϵ is equal to zero). Once these bounds are derived, we will apply the result of theorem C1 (appendix C) to establish that these bounds continue to hold up to an arbitrarily small offset, for initial conditions and uncertainties in an arbitrarily large compact set, provided that ϵ is sufficiently small.

Step 1 : First, note that the linear structure of $\bar{e}$ subsystem of the reduced system of Eq.D.6 and the fact that it is exponentially stable when $\tilde{e}_{\tilde{r}} \equiv 0$ allows using a direct Lyapunov function argument to show that there exist positive real numbers, $k_1, a, \gamma_{\tilde{e}_{\tilde{r}}}$ such that the following ISS bound holds for the reduced $\bar{e}$ subsystem:

$$|\bar{e}(t)| \leq k_1 e^{-at}|\bar{e}(0)| + \gamma_{\tilde{e}_{\tilde{r}}}||\tilde{e}_{\tilde{r}}|| \tag{D.7}$$

In the rest of this step, we will show show that the controller of Eq.5.18 ensures that the subsystem consisting of the state vector $\tilde{e}_{\tilde{r}} = [\tilde{e}_{\tilde{r}_1}^{(1)} \; \tilde{e}_{\tilde{r}_2}^{(2)} \; \cdots \; \tilde{e}_{\tilde{r}_m}^{(m)}]^T$ of the reduced system of Eq.D.6 possesses an ISS property with respect to $\tilde{e}_{\tilde{r}}, \bar{e}, \bar{v}, \eta, \theta$, and moreover the gain function saturates at ϕ. To this end consider the following singularly perturbed system:

$$\begin{aligned}
\dot{\tilde{e}}_{\tilde{r}} &= e_z\bar{\Psi}_{\tilde{r}}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta) + L_{\tilde{g}}L_{\tilde{F}}^{\tilde{r}-1}h(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} \\
&\quad + M(x, \theta, t)\left\{-\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} - \tilde{e}_{\tilde{r}}\right. \\
&\quad \left. -(2+b)[\bar{c}_1((\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta), t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]w(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta)), \phi)\right\} \\
\epsilon\dot{z} &= [Q_2(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta), \theta) + G_2(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta), \theta)K(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta))]e_z
\end{aligned} \tag{D.8}$$

where $\bar{\Psi}_{\tilde{r}}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta) = [\bar{\Psi}_{\tilde{r}_1}^{(1)}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta) \; \cdots \; \bar{\Psi}_{\tilde{r}_m}^{(m)}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta)]^T$,
$v^{(\tilde{r})} = [v_1^{(\tilde{r}_1)} \; \cdots \; v_m^{(\tilde{r}_m)}]^T$,
$L_{\tilde{g}}L_{\tilde{F}}^{\tilde{r}-1}h(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta)) = [L_{\tilde{g}}L_{\tilde{F}}^{\tilde{r}_1-1}h_1(T^{-1}(\bar{e}, \tilde{e}_{\tilde{r}}, \bar{v}, \eta, \theta)) \; \cdots$

$L_{\bar{\delta}}L_{\bar{F}}^{\tilde{r}_m-1}h_m(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta))]^T$. From part 1, we have that the fast dynamics of the above system is globally exponentially stable. Consider the reduced system corresponding to the singularly perturbed system of Eq.D.8:

$$\begin{aligned}\dot{\tilde{e}}_{\tilde{r}} &= L_{\bar{\delta}}L_{\bar{F}}^{\tilde{r}-1}h(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} + M\left\{-\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} - \tilde{e}_{\tilde{r}}\right. \\ &\quad \left. -(2+b)[\tilde{c}_1((\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]w(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)),\phi)\right\}\end{aligned} \tag{D.9}$$

To establish that the above system is ISS with respect to $\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta$, we use the following smooth function $V:\mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$:

$$V = \frac{1}{2}\tilde{e}_{\tilde{r}}^2 \tag{D.10}$$

Calculating the time-derivative of V along the trajectory of the system of Eq.D.9, we have:

$$\begin{aligned}\dot{V} &= \tilde{e}_{\tilde{r}}^T\left[L_{\bar{\delta}}L_{\bar{F}}^{\tilde{r}-1}h(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)) + \sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} + M\left\{-\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} - \tilde{e}_{\tilde{r}}\right.\right. \\ &\quad \left.\left. -(2+b)[\tilde{c}_1((\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]w(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta)),\phi)\right\}\right]\end{aligned} \tag{D.11}$$

Furthermore, it is straightforward to show that the representation of the vector function $w(x,\phi)$ in terms of the vector $\tilde{e}_{\tilde{r}}$ is given by:

$$w(\tilde{e}_{\tilde{r}},\phi) = \frac{\tilde{e}_{\tilde{r}}}{|\tilde{e}_{\tilde{r}}| + \phi} \tag{D.12}$$

Substituting Eqs.D.12 into Eq.D.11 and using the fact that $1 \leq \sigma_{min}\{M(x,\theta,t)\} \leq$

$\sigma_{max}\{M(x,\theta,t)\} \leq b$ and the inequalities of Eq.4.2. we have:

$$\begin{aligned}
\dot{V} &\leq \tilde{e}_{\tilde{r}}^T\Big\{-M(x,\theta,t)\tilde{e}_{\tilde{r}} - (2+b)M(x,\theta,t)[\bar{c}_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) \\
&\quad +|\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]\frac{\tilde{e}_{\tilde{r}}}{|\tilde{e}_{\tilde{r}}|+\phi} \\
&\quad + (M(x,\theta,t)-I_{m\times m})\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)} + L_{\bar{g}}L_{\bar{F}}^{\tilde{r}-1}h(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta))\Big\} \\
&\leq \Big\{-\tilde{e}_{\tilde{r}}^2 - (2+b)[\bar{c}_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]\frac{\tilde{e}_{\tilde{r}}^2}{|\tilde{e}_{\tilde{r}}|+\phi} \\
&\quad +(1+b)(|\tilde{e}_{\tilde{r}}|(|\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}| + \bar{c}_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t))\Big\} \\
&\leq \Big\{-\tilde{e}_{\tilde{r}}^2 - [\bar{c}_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]\frac{\tilde{e}_{\tilde{r}}^2}{|\tilde{e}_{\tilde{r}}|+\phi} \\
&\quad +\phi(1+b)[\bar{c}_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]\frac{\tilde{e}_{\tilde{r}}}{|\tilde{e}_{\tilde{r}}|+\phi}\Big\}
\end{aligned} \tag{D.13}$$

From the last inequality, it follows directly that if $|\tilde{e}_{\tilde{r}}| \geq \phi(1+b)$, the time-derivative of the Lyapunov function satisfies $\dot{V} \leq -\tilde{e}_{\tilde{r}}^2$. This fact implies that the ultimate bound on the state $\tilde{e}_{\tilde{r}}$ of the system of Eq.D.9 depends only on the parameter ϕ and is independent of the states $\bar{e}, \eta$.

We will now analyze the time-derivative of V for $|\tilde{e}_{\tilde{r}}| < \phi(1+b)$. For ease of notation, we set $\mathcal{U} = [\tilde{e}_{\tilde{r}}\ \bar{\eta}^T\ \theta^T\ \bar{v}^T]^T$. Then, Eq.D.13 can be written as:

$$\begin{aligned}
\dot{V} &\leq \Big\{-\tilde{e}_{\tilde{r}}^2 + (1+b)|\tilde{e}_{\tilde{r}}|[\bar{c}_1(T^{-1}(\bar{e},\tilde{e}_{\tilde{r}},\tilde{v},\eta,\theta),t) + |\sum_{i=1}^{m}\sum_{k=1}^{\tilde{r}_i}\frac{\beta_{ik}}{\beta_{i\tilde{r}_i}}e_{k+1}^{(i)}|]\Big\} \\
&\leq -\tilde{e}_{\tilde{r}}^2 + (1+b)|\tilde{e}_{\tilde{r}}|\rho(|\mathcal{U}|)
\end{aligned} \tag{D.14}$$

where ρ is a class K_∞ function. Summarizing, we have that $\dot{V}$ satisfies the following properties:

$$\dot{V} \leq -\frac{|\tilde{e}_{\tilde{r}}|^2}{2}, \quad |\tilde{e}_{\tilde{r}}| \geq \min\{\phi(1+b), (2(1+b)\rho(|\mathcal{U}|))\} =: \gamma_{\mathcal{U}}(|\mathcal{U}|) \tag{D.15}$$

Using the result of theorem 4.10 in [Kha89], we get that the following ISS bound holds for the state $\tilde{e}_{\bar{r}}$ of the system of Eq.D.9:

$$|\tilde{e}_{\bar{r}}(t)| \leq e^{-0.5t}|\tilde{e}_{\bar{r}}(0)| + \tilde{\gamma}_{\mathcal{U}}(||\mathcal{U}||) \tag{D.16}$$

Before we proceed with the rest of this step, we will assume that $\phi \in (0, \phi^*]$, where :

$$\phi^* = \frac{d}{(1+b)(1+2\gamma_{\tilde{e}_{\bar{r}}})} \tag{D.17}$$

Consider the singularly perturbed system consisting of the states $(\bar{e}, \eta, z)$ of the system of Eq.D.6. For this system, it can be shown that its fast dynamics are globally exponentially stable and the η-subsystem of the reduced system is ISS with respect to $\bar{e}, \tilde{e}_{\bar{r}}, \theta, \dot{\theta}$. Furthermore, the state $\bar{\eta} = [\bar{e}^T \ \eta^T]^T$ of this system possesses an ISS property with respect to $\tilde{e}_{\bar{r}}, \theta, \dot{\theta}$ [Son89, ZPTP95]. Utilizing the converse theorem developed in [SW95], we have that there exists a converse function for the system comprised of the states $(\bar{e}, \eta)$ and the existence of this function implies that there exist a function $\beta_{\bar{\eta}}$ of class KL and a function $\bar{\gamma}_s$ of class K such that the following ISS inequality holds for the state $\bar{\eta}$:

$$\begin{aligned} |\bar{\eta}(t)| &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_s(||[\tilde{e}_{\bar{r}} \ \theta^T \ \dot{\theta}^T]^T||) \\ &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(||\tilde{e}_{\bar{r}}||) + \bar{\gamma}_{\theta}(||\theta||) + \bar{\gamma}_{\dot{\theta}}(||\dot{\theta}||) \end{aligned} \tag{D.18}$$

where $\bar{\gamma}_{\tilde{e}_{\bar{r}}}, \bar{\gamma}_{\theta}, \bar{\gamma}_{\dot{\theta}}$ are class K functions respectively, defined as $\bar{\gamma}_{\tilde{e}_{\bar{r}}}(s) = \bar{\gamma}_{\theta}(s) = \bar{\gamma}_{\dot{\theta}}(s) = \bar{\gamma}_s(3s)$.

Step 2: We will now utilize the result of of theorem C1 (appendix C) to establish that the ISS inequalities of Eqs.D.16-D.18 continue to hold up to an arbitrarily small offset, for the states $\tilde{e}_{\bar{r}}, \bar{\eta}$ of the singularly perturbed system of Eq.D.6. Following the proof of theorem 4.1, it can be shown that given the set of positive real numbers $(\delta_{\tilde{e}_{\bar{r}}}, \delta_{\bar{\eta}}, \delta_z, \delta_{\theta}, \delta_{\dot{\theta}}, \delta_{\bar{v}}, \dot{\delta}_{\bar{\eta}}, \dot{\delta}_{\tilde{e}_{\bar{r}}}, d_{\tilde{e}_{\bar{r}}}, d_{\bar{\eta}})$ (which can be specified from the data of the theorem, for details see the proof of theorem 4.1), that there is an $\epsilon^{\tilde{e}_{\bar{r}}}(\phi)$ such that if $\epsilon \in (0, \epsilon^{\tilde{e}_{\bar{r}}}(\phi)]$ and $|\tilde{e}_{\bar{r}}(0)| \leq \delta_{\tilde{e}_{\bar{r}}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_{\theta}$, $||\bar{v}|| \leq \delta_{\bar{v}}$, $||\bar{\eta}|| \leq \dot{\delta}_{\bar{\eta}}$, $||\tilde{e}_{\bar{r}}|| \leq \dot{\delta}_{\tilde{e}_{\bar{r}}}$, then

$$|\tilde{e}_{\bar{r}}(t)| \leq e^{-0.5t}|\tilde{e}_{\bar{r}}(0)| + \tilde{\gamma}_{\mathcal{U}}(||\mathcal{U}||) + \phi \tag{D.19}$$

Furthermore, it can be also shown using the result of theorem C1 (appendix C) that the singularly perturbed system comprised of $\bar{e}, \eta, z$, with the same converse function which exists for the reduced system comprised of the states $\bar{e}, \eta$ and its resulting $(\beta_{\bar{\eta}}, \bar{\gamma}_s)$. Thus we have that if $\epsilon \in (0, \epsilon^{\bar{\eta}}(\phi)]$ and $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$, $||\tilde{e}_{\bar{r}}|| \leq \tilde{\delta}_{\tilde{e}_{\bar{r}}}$, then

$$\begin{aligned} |\bar{\eta}(t)| &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_s(||[\tilde{e}_{\bar{r}}\ \theta^T\ \dot{\theta}^T]^T||) + d_{\bar{\eta}} \\ &\leq \beta_{\bar{\eta}}(|\bar{\eta}(0)|, t) + \bar{\gamma}_{\tilde{e}_{\bar{r}}}(||\tilde{e}_{\bar{r}}||) + \bar{\gamma}_\theta(||\theta||) + \bar{\gamma}_{\dot{\theta}}(||\dot{\theta}||) + d_{\bar{\eta}} \end{aligned} \tag{D.20}$$

Part 3 : The proof of the theorem can be completed by showing that for any given set of positive real numbers $\delta_{\tilde{e}_{\bar{r}}}, \delta_{\bar{\eta}}, \delta_z, \delta_\theta, \delta_{\dot{\theta}}, \delta_{\bar{v}}, d$ (already specified) and with ϕ^* defined as in Eq.D.17, there exists $\epsilon^*(\phi) \in (0, \epsilon^{\bar{\eta}}(\phi)]$, such that if $\epsilon \in (0, \epsilon^*(\phi)]$ and $|\tilde{e}_{\bar{r}}(0)| \leq \delta_{\tilde{e}_{\bar{r}}}$, $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$, $|z(0)| \leq \delta_z$, $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$ the output of the closed-loop system of Eq.D.2 satisfies the relation of Eq.5.20, for each $\phi \in (0, \phi^*]$.

This result can be established by analyzing the behavior of the dynamical system comprised of the states $\tilde{e}_{\bar{r}}, \bar{\eta}$ of the system of Eq.D.6, for which the inequalities of Eqs.D.19-D.20 hold, using calculations similar to those used in [Tee96] and [ZPTP95]. First, a contradiction argument can be used to show that if $\phi \in (0, \phi^*]$ the evolution of the states $\tilde{e}_{\bar{r}}, \bar{\eta}$, starting from initial conditions that satisfy $|\tilde{e}_{\bar{r}}(0)| \leq \delta_{\tilde{e}_{\bar{r}}}$, $|\bar{\eta}(0)| \leq \delta_{\bar{\eta}}$ and for θ, $\dot{\theta}$, $\bar{v}$ such that $||\theta|| \leq \delta_\theta$, $||\dot{\theta}|| \leq \delta_{\dot{\theta}}$, $||\bar{v}|| \leq \delta_{\bar{v}}$ satisfies the following inequalities:

$$|\tilde{e}_{\bar{r}}(t)| \leq \tilde{\delta}_{\tilde{e}_{\bar{r}}}, \quad |\bar{\eta}(t)| \leq \tilde{\delta}_{\bar{\eta}} \tag{D.21}$$

for all times. Second, the asymptotic behavior of the system comprised of the states $\bar{e}, \tilde{e}_{\bar{r}}$ of Eq.D.6 is analyzed to establish that the inequality of Eq.5.20 holds. These calculations are similar to the ones in the proof of theorem 4.1 and are omitted for reasons of brevity. △

NONLINEAR CONTROL OF TWO-TIME-SCALE AND DISTRIBUTED PARAMETER SYSTEMS

Volume II

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Chapter 6

Feedback Control of Hyperbolic PDE Systems

6.1 Introduction

Chemical engineering processes are inherently nonlinear and very frequently involve state variables that change in both time and space. Representative examples of processes with significant spatial variations include plug-flow reactors [Ray81], countercurrent absorbers-reactors [RAA86], fixed- and fluidized-bed reactors [SF70, Ray81, GAA77], etc.. The mathematical models of these processes are typically derived from the dynamic conservation equations and consist of nonlinear PDEs.

The conventional approach for the control of PDE systems is based on the spatial discretization of the PDE model followed by the controller design on the basis of the resulting (linear or nonlinear) ordinary differential equation (ODE) model (see e.g., [SJC80, DBTM92, PWE92]). However, there are certain well-known disadvantages associated with this approach. For example, fundamental control-theoretic properties, like controllability and observability, which should depend only on the location of sensors and actuators, may also depend on the discretization method and the number and location of discretization points [Ray81]. Moreover, neglecting the infinite dimensional nature of the original system may lead to erroneous conclusions concerning the stability properties of the open-loop and/or the closed-loop system. Furthermore, in processes where the spatially distributed nature is very strong, due to the underlying convection and diffusion phenomena, such an approach limits the controller performance, and may lead to unacceptable control quality.

Motivated by the above considerations, significant research efforts have focused on the development of control methods for PDE systems that directly account for their spatially distributed nature. Excellent surveys of theoretical as well as application papers on this topic can be found in [Bal82, Keu93, Las95, Ray78]. Initially, systems of linear PDEs were considered, for which key system and control-theoretic properties (e.g., existence and uniqueness of solutions, stability, controllability and observability) were well-understood [CP78]. The well-known classification of PDE systems to hyperbolic, parabolic and elliptic [Smo83], according to the properties of the spatial differential operator, essentially determined the approach followed for the solution of the control problem. Thus, for parabolic PDE systems (e.g., diffusion-reaction processes), the fact that the system dynamics is practically determined by a finite number of modes, motivated the use of modal decomposition techniques to derive ODE models that capture the dominant dynamics of the system (see for example the papers [Cur82, GAA77, HP92, GR95] and chapter 8 of this thesis for a detailed treatment on control of parabolic PDEs); the controller design problem was then addressed using methods for linear ODE systems. On the other hand, the distinct feature of hyperbolic PDEs (convection-reaction processes) is that all the eigenmodes of the spatial differential operator contain the same or nearly the same amount of energy, and thus an infinite number of modes is required to accurately describe their dynamic behavior. This feature prohibits the application of modal decomposition techniques to derive reduced-order ODE models that approximately describe the dynamics of the PDE system and suggests addressing the control problem on the basis of the infinite-dimensional model itself. Motivated by this, alternative approaches have been followed, using mainly optimal control methods (e.g., [Wan66b, Lo73, Bal86]) on the basis of the original infinite dimensional system, or ODE control methods on the basis of equivalent ODE realizations obtained by the method of characteristics [Ray81].

Recently, considerable attention has focused on the understanding of system-theoretic properties and the dynamical behavior of nonlinear PDEs, by treating them as evolution equations in appropriate infinite dimensional spaces (see e.g., [Tem88, BKJ91]). Yet, available results on the control of systems of nonlinear PDEs are rather sparse, with the exception of optimal control approaches (see e.g., [Bal91, BK91, KI92]). Due to their practical relevance and importance, quasi-linear PDE systems have attracted particular research interest. For quasi-linear parabolic PDE systems, an approach that utilizes combination of eigenfunction expansion techniques and nonlinear control schemes was proposed in [CC92]. For processes modeled by a single first-order quasi-linear hyperbolic PDE, an approach based on combination of the method of characteristics and sliding mode techniques was proposed in [SR89]. This method was further developed in [HP95] to account for possible discontinuous behavior of the control action and was applied to a heat exchanger. An alternative approach to the control of quasi-linear hyperbolic PDE systems is based on Lyapunov's direct method [Wan64, Wan66a]. The basic idea in this approach is to design a controller so that the time-derivative of an appropriate Lyapunov functional calculated along the trajectories of the closed-loop system is negative definite, which ensures that the closed-loop system is asymptotically stable. This approach was followed by Alonso and Ydstie in [AY95], where concepts from thermodynamics were employed to construct a Lyapunov functional candidate, which was used to derive conditions that guarantee asymptotic stability of the closed-loop system under boundary proportional-integral-derivative control.

In this chapter, we address the feedback control problem for systems described by quasi-linear first-order hyperbolic PDEs, for which the manipulated input, the controlled output and the measured output are distributed in space. Systems of this form arise naturally in chemical engineering as models of transport-reaction processes, whenever convective mechanisms dominate over diffusive and dispersive ones (see the book [RAA86] for representative examples). For such systems, our objective is to synthesize nonlinear distributed output feedback controllers that enforce output tracking and guarantee stability in the closed-loop system.

The present chapter is structured as follows: after reviewing the necessary preliminaries, a concept of characteristic index between the controlled output and the manipulated input (which can be thought of as the analogue of relative order) is introduced and used for the synthesis of distributed state feedback controllers that induce output tracking in the closed-loop system. A notion of zero-output constraint dynamics for hyperbolic PDEs is introduced and used to derive precise conditions that guarantee the stability of the closed-loop system. Then, output feedback controllers are synthesized through combination of appropriate distributed state observers with the developed state feedback controllers. Theoretical analogies between the proposed approach and available feedback control methods for the stabilization of linear hyperbolic PDEs are pointed out. Controller implementation issues are also discussed. Finally, the application of the developed control method is illustrated through a nonisothermal plug-flow reactor example modeled by a system of three quasi-linear hyperbolic PDEs.

6.2 First-order hyperbolic PDE systems

6.2.1 Preliminaries

We consider systems of quasi-linear first-order partial differential equations in one spatial dimension with the following state-space representation:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)u \\ y &= h(x), \quad q = p(x) \end{aligned} \tag{6.1}$$

subject to the boundary condition:

$$C_1 x(a,t) + C_2 x(b,t) = R(t) \tag{6.2}$$

and the initial condition:

$$x(z,0) = x_0(z) \tag{6.3}$$

where $x(z,t) = [x_1(z,t) \ \cdots \ x_n(z,t)]^T$ denotes the vector of state variables, $x(z,t) \in \mathcal{H}^n[(a,b),\mathbb{R}^n]$, with $\mathcal{H}^n$ being the infinite dimensional Hilbert space of n-dimensional vector functions defined on the interval $[a,b]$ whose spatial derivatives up to n-th order are square integrable, $z \in [a,b] \subset \mathbb{R}$ and $t \in [0,\infty)$, denote position and time respectively, $u(z,t)$ denotes the manipulated variable, $y(z,t)$ denotes the controlled variable, and $q(z,t)$ denotes the measured variable. $A(x)$ is a sufficiently smooth matrix, $f(x)$ and $g(x)$ are sufficiently smooth vector functions, $h(x), p(x)$ are sufficiently smooth scalar functions, $R(t)$ is a column vector which is assumed to be a sufficiently smooth function of time, $x_0(z) \in \mathcal{H}[(a,b),\mathbb{R}^n]$, with $[(a,b),\mathbb{R}^n]$ being the Hilbert space of n-dimensional vector functions defined on the interval $[a,b]$ which are square integrable, and C_1, C_2 are constant matrices of dimension $n \times n$.

The model of Eq.6.1 describes the majority of convection-reaction processes arising in chemical engineering [RAA86] and constitutes a natural generalization of *linear* PDE models (see Eq.6.9 below), considered in [PA70, Ray81] in the context of linear distributed state estimation and control. The distributed and affine appearance of the manipulated variable u is typical in most practical applications (see e.g., [PA70, GR95, Ray81]), where the jacket temperature is usually selected as the manipulated variable (see subsection 6.7.4 below for a detailed discussion on how the jacket temperature is manipulated in practice). Furthermore, the possibility that the system of Eq.6.1 may admit boundary conditions at two separate points (e.g., counter-current processes) is captured by the boundary condition of Eq.6.2.

Depending on the eigenvalues of the matrix $A(x)$, the system of Eq.6.1 can be hyperbolic, parabolic or elliptic [Smo83]. Assumption 6.1 that follows ensures that the system of Eq.6.1 has a well-defined solution and specifies the class of systems considered in this chapter.

Assumption 6.1: *The matrix $A(x)$ is real symmetric and its eigenvalues satisfy:*

$$\lambda_1(x) \leq \cdots \leq \lambda_k(x) < 0 < \lambda_{k+1}(x) \leq \cdots \leq \lambda_n(x) \tag{6.4}$$

for all $x \in \mathcal{H}^n[(a,b), \mathbb{R}^n]$.

Typical examples where Assumption 6.1 is satisfied include heat exchangers, plug-flow reactors and countercurrent absorbers-reactors where the matrix $A(x)$ is constant and diagonal and its elements are the fluid velocities, as well as chromatography of two interacting solutes where $A(x)$ is a full matrix [RAA86]. Systems of the form of Eq.6.1 for which the eigenvalues of the matrix A are real and distinct are said to be *hyperbolic*, while systems for which some of the eigenvalues of the matrix A are identically equal are said to be *weakly hyperbolic* [Smo83].

6.2.2 Specification of the control problem

Consider the system of quasi-linear PDEs of the form of Eq.6.1, for which the manipulated variable $u(z,t)$, the measured variable $q(z,t)$, and the controlled variable $y(z,t)$ are distributed in space. Let's assume that for the control of the variable y, there exists a finite number of control actuators, l, and the same number of measurement sensors; clearly, it is not possible to control the variable $y(z,t)$ at all positions. Therefore, it is meaningful to formulate the control problem as the one of controlling $y(z,t)$ at a finite number of spatial intervals. In particular, referring to the single spatial interval $[z_i, z_{i+1}]$, we suppose that the manipulated input is $\bar{u}^i(t)$, with $\bar{u}^i \in \mathbb{R}$, and the measured output is $\bar{q}^i(t)$, with $\bar{q}^i \in \mathbb{R}$, while the controlled output is $\bar{y}^i(t)$, with $\bar{y}^i \in \mathbb{R}$, such that the following relations hold:

$$u(z,t) = b^i(z)\bar{u}^i(t), \quad \bar{y}^i(t) = C^i y(z,t), \quad \bar{q}^i(t) = Q^i p(z,t), \quad z_i \leq z \leq z_{i+1} \tag{6.5}$$

where $b^i(z)$ is a known smooth function of z, and C^i, Q^i are bounded linear operators, mapping $\mathcal{H}^n$ into $\mathbb{R}$. Figure 6.1 shows a pictorial representation of this formulation in the case of a prototype example. From a practical point of view, the function $b^i(z)$ describes how the control action $\bar{u}^i(t)$ is distributed in the spatial interval $[z_i, z_{i+1}]$, while the operator Q^i determines the structure of the sensor in the same spatial interval. Whenever the control action enters the system at a single point z_0 (e.g. lateral flow injections), with $z_0 \in [z_i, z_{i+1}]$ (i.e. point actuation), the function $b^i(z)$ is taken to be nonzero in a finite space interval of the form $[z_0 - \epsilon, z_0 + \epsilon]$, where ϵ is a small positive real number, and zero elsewhere in $[z_i, z_{i+1}]$. Similarly, in the case of a point sensor acting at z_0, the operator Q^i is assumed to act in $[z_0 - \epsilon, z_0 + \epsilon]$, and considered to be zero elsewhere. The operator C^i depends on the desired performance specifications and in the majority of practical applications (see e.g., [Ray81, HP92]), is of the following form:

$$\bar{y}^i(t) \;=\; C^i h(x) = \int_{z_i}^{z_{i+1}} c^i(z) h(x(z,t))dz \tag{6.6}$$

where $c^i(z)$ is a known smooth function of z. For simplicity, the functions $b^i(z), c^i(z)$, $i = 1, \ldots, l$ will be assumed to be normalized in the interval $[a,b]$, i.e., $\sum_{i=1}^{l}\int_{z_i}^{z_{i+1}} b^i(z)dz = \sum_{i=1}^{l}\int_{z_i}^{z_{i+1}} c^i(z)dz = 1$. Using the relations of Eqs.6.5-6.6, the system

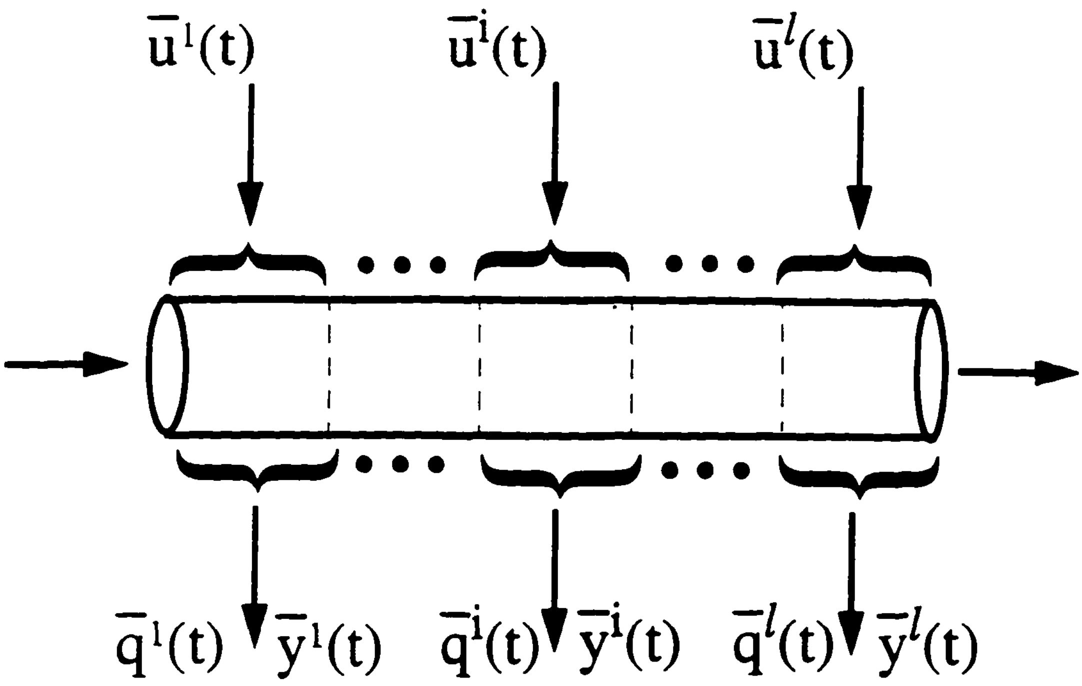


Figure 6.1: Control problem specification in the case of a prototype example.

of Eq.6.1 takes the form:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\bar{u} \\
\bar{y} &= \mathcal{C}h(x), \quad \bar{q} = \mathcal{Q}p(x) \\
C_1x(a,t) + C_2x(b,t) &= R(t)
\end{aligned}
\tag{6.7}
$$

where

$$
\begin{aligned}
\bar{u} &= [\bar{u}^1 \cdots \bar{u}^l]^T \\
\bar{y} &= [\bar{y}^1 \cdots \bar{y}^l]^T \\
b(z) &= [(H(z-z_1) - H(z-z_2))\,b^1(z) \cdots (H(z-z_l) - H(z-z_{l+1}))\,b^l(z)] \\
\mathcal{C} &= [(H(z-z_1) - H(z-z_2))\mathcal{C}^1 \cdots (H(z-z_l) - H(z-z_{l+1}))\,\mathcal{C}^l]^T \\
\mathcal{Q} &= [(H(z-z_1) - H(z-z_2))\,\mathcal{Q}^1 \cdots (H(z-z_l) - H(z-z_{l+1}))\,\mathcal{Q}^l]^T
\end{aligned}
\tag{6.8}
$$

with $H(\cdot)$ being the standard Heaviside function.

Referring to the system of Eq.6.7, we note that by setting $A(x) = A$, $f(x) = Bx$, $g(x) = w$, $h(x) = kx$, $p(x) = px$ where B is a matrix and k, w, p are vectors of appropriate dimensions, it reduces to the following system of linear first-order hyperbolic PDEs:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx + wb(z)\bar{u} \\
\bar{y} &= \mathcal{C}kx, \quad \bar{q} = \mathcal{Q}px
\end{aligned}
\tag{6.9}
$$

subject to the boundary condition of Eq.6.2 and the initial condition of Eq.6.3.

The following example will be used throughout the chapter to illustrate the various aspects of our methodology.

Example : Consider a steam-jacketed tubular heat exchanger [Ray81]. The dynamic model of the process is of the form:

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= -v_l\frac{\partial T}{\partial z} - aT + aT_j \\
T(0,t) &= R(t)
\end{aligned}
\tag{6.10}
$$

where $T(z,t)$ denotes the temperature of the reactor, $z \in [0,1]$, $T_j(z,t)$ denotes the jacket temperature, v_l denotes the fluid velocity in the exchanger and a is a positive constant. Considering T_j as the manipulated variable, and T as the controlled and measured variable, the above model can be put in the form of Eq.6.1:

$$\begin{aligned} \frac{\partial x}{\partial t} &= -v_l \frac{\partial x}{\partial z} - ax + au \\ y &= x, \quad \bar{q} = x \\ x(0,t) &= R(t) \end{aligned} \tag{6.11}$$

Consider the case where there exists one actuator with distribution function $b(z) = 1$, the controlled output is assumed to be $\bar{y}(t) = \int_0^1 y(z,t)dz$, and there is a point sensor acting at $z = 0.5$. Utilizing these relations, the system of Eq.6.11 takes the form:

$$\begin{aligned} \frac{\partial x}{\partial t} &= -v_l \frac{\partial x}{\partial z} - ax + a\bar{u}(t) \\ \bar{y}(t) &= \int_0^1 x(z,t)dz, \quad \bar{q} = \int_{0.5-\epsilon}^{0.5+\epsilon} x(z,t)dz \end{aligned} \tag{6.12}$$

$\triangle$

6.2.3 Review of system-theoretic properties

The objective of this subsection is to review basic system-theoretic properties of systems of first-order hyperbolic PDEs which will be used in the subsequent sections. For more details on these subjects, the reader may refer to [CP78, Rus78]. We will start with the definitions of the inner product and the norm, with respect to which the notion of exponential stability for the systems under consideration will be defined.

- Let ω_1, ω_2 be two elements of $\mathcal{H}([a,b];\mathbb{R}^n)$. Then, the inner product and the norm, in $\mathcal{H}([a,b];\mathbb{R}^n)$, are defined as follows:

$$\begin{aligned} (\omega_1, \omega_2) &= \int_a^b (\omega_1(z), \omega_2(z))_{\mathbb{R}^n} dz \\ ||\omega_1||_2 &= (\omega_1, \omega_1)^{\frac{1}{2}} \end{aligned} \tag{6.13}$$

  where the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in the Euclidean space $\mathbb{R}^n$.

Referring to the linear system of Eq.6.9. for which assumption 6.1 holds. it is well-established [Rus78] that the operator

$$\mathcal{L}x \;=\; A\frac{\partial x}{\partial z} + Bx \qquad (6.14)$$

defined on the domain in $\mathcal{H}[(a,b);\mathbb{R}^n]$ consisting of functions $x \in \mathcal{H}^1[(a,b);\mathbb{R}^n]$ which satisfy the boundary condition of Eq.6.2, generates a strongly continuous semigroup $U(t)$ of bounded linear operators on $\mathcal{H}[(a,b);\mathbb{R}^n]$. This fact implies the existence. uniqueness, and continuity of solutions for the system of Eq.6.9. In particular, the generalized solution of this system is given by:

$$x \;=\; U(t)x_0 + \int_0^t U(t-\tau)wb\bar{u}(\tau)d\tau + C(t)R \qquad (6.15)$$

where $C(t)$ is a bounded linear operator for each t mapping $\mathcal{H}[(0,t);\mathbb{R}^n]$ into $\mathcal{H}[(a,b);\mathbb{R}^n]$. The notion of semigroup can be thought of as an analog of the notion of the state transition matrix used for linear finite-dimensional systems. As can be easily seen from Eq.6.15, $U(t)$ evolves the initial condition x_0 forward in time. From general semigroup theory [Fri76], it is known that $U(t)$ satisfies the following growth property:

$$||U(t)||_2 \;\leq\; Ke^{at}, \;\; t \geq 0 \qquad (6.16)$$

where $K \geq 1$, a is the largest real part of the eigenvalues of the operator $\mathcal{L}$, and an estimate of K, a can be obtained utilizing the Hiller-Yoshida theorem [Fri76]. Whenever the parameter a is strictly negative, we will say that the operator of Eq.6.14 generates an exponentially stable semigroup $U(t)$. We note that although there exist many stability concepts for PDEs (e.g., weak (asymptotic) stability [Fri76, Smo83]), we will focus, throughout the chapter, on exponential stability, because of its robustness to bounded perturbations, which is required in most practical applications, where there is always some uncertainty associated with the process model. The aforementioned concepts allow stating precisely a standard (see also [PA70, Bal86]) detectability requirement for the system of Eq.6.9, which will be exploited in section 6 for the design of distributed state observers.

Assumption 6.2: *The pair $[\mathcal{Q}p\ \mathcal{L}]$, is detectable, i.e., there exist a bounded linear operator $\mathcal{P}$, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$, such that the linear operator $\mathcal{L}_o = \mathcal{L} - \mathcal{P}\mathcal{Q}p$ generates an exponentially stable semigroup.*

The above detectability assumption does not impose any restrictions on the form of the operator $\mathcal{Q}$ and thus, on the structure of the sensors (e.g. distributed, point sensors).

In closing this subsection, motivated by the lack of general stability results for systems of quasi-linear PDEs, we will review a result that will allow characterizing the local stability properties of the quasi-linear system of Eq.6.7 on the basis of its corresponding linearized system. To this end, let's consider the linearization of the quasi-linear system of Eq.6.7:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A(z)\frac{\partial x}{\partial z} + B(z)x + w(z)b(z)\bar{u} \\ \bar{y} &= Ck(z)x, \quad \bar{q} = \mathcal{Q}p(z)x \end{aligned} \tag{6.17}$$

where $A(x) = A(x_s(z))$, $B(z) = \left(\dfrac{\partial f(x)}{\partial x}\right)_{x=x_s(z)}$, $w(z) = g(x_s(z))$, $k(z) = \left(\dfrac{\partial h(x)}{\partial x}\right)_{x=x_s(z)}$, $p(z) = \left(\dfrac{\partial p(x)}{\partial x}\right)_{x=x_s(z)}$, and $x_s(z)$ denotes some steady-state profile.

Proposition 6.1 ([Smo83], p.121) : *The system of Eq.6.7 (with $\bar{u} = 0$) for which assumption 6.1 holds, subject to the boundary condition of Eq.6.2, is locally exponentially stable if the operator of the linearized system of Eq.6.17:*

$$\bar{\mathcal{L}}x = A\frac{\partial x}{\partial z} + B(z)x \tag{6.18}$$

generates an exponentially stable semigroup.

The following remark provides conditions, which can be easily verified in practice, that guarantee the open-loop stability of hyperbolic PDE systems of the form of

Eq.6.9 (Eq.6.7).

Remark 6.1 : Consider the system of linear (quasi-linear) first-order PDEs of the form of Eq.6.9 (Eq.6.7) with $\bar{u} = 0$, and assume that the following conditions hold:

- $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 0$.
- $C_2 = 0$

In this case, it can be shown [Rus78] that the eigenvalues of the operator of Eq.6.18 (Eq.6.14) are of the following form:

$$a_\mu = -\infty + \mu\pi i \, , \; \mu = -\infty, \cdots, \infty \tag{6.19}$$

Thus, first-order PDE systems that satisfy the above conditions (physical examples include plug-flow reactors, co-current heat exchangers, etc.) possess eigenvalues which lie on a line crossing the real axis at $s = -\infty$, which, according to Eq.6.16, implies that they are exponentially stable.

6.2.4 Methodological framework

Motivated by the fact that control methods for quasi-linear distributed parameter systems should explicitly account for their nonlinear and spatially varying nature, our methodology entails the following two steps:

1. Synthesize distributed nonlinear state feedback controllers that enforce output tracking and derive conditions that guarantee the exponential stability of the closed-loop system.

2. Synthesize distributed nonlinear output feedback controllers through combination of the developed state feedback controllers with appropriate distributed state observers.

Motivated by the mathematical properties of these systems, the state feedback control problem is solved on the basis of the original PDE model, by following an approach conceptually similar to the one used for the synthesis of inversion-based controllers for ODE systems. In order to motivate the approach followed for the quasi-linear case and identify theoretical analogies between our approach and available results on feedback stabilization of linear hyperbolic PDEs, we will also present the development for the case of systems of linear PDEs of the form of Eq.6.9. The development for the case of quasi-linear systems will be performed by essentially generalizing the results developed for the linear case in a nonlinear context.

6.3 Characteristic index

In this section, we will introduce a concept of characteristic index between the output $\bar{y}$ and the input $\bar{u}$ for systems of the form of Eq.6.9, that will allow us to formulate and solve the state feedback control problem. To reveal the origin and illustrate the role of this concept, we consider the operation of differentiation of the output $\bar{y}^i$ of the system of Eq.6.9 with respect to time, which yields:

$$\begin{aligned}\bar{y}^i &= C^i k x \\ \frac{d\bar{y}^i}{dt} &= \frac{d}{dt}C^i k x = C^i k \frac{\partial x}{\partial t} \\ &= C^i k\left(A\frac{\partial}{\partial z} + B\right)x + C^i k w b^i(z)\bar{u}^i\end{aligned} \tag{6.20}$$

Now, if the scalar $C^i k w b^i(z)$ is nonzero, we will say that the characteristic index of $\bar{y}^i$ with respect to $\bar{u}^i$, denoted by σ^i, is equal to one. If $C^i k w b^i(z) = 0$, the characteristic index is greater than one, and from Eq.6.20 we have that:

$$\frac{d\bar{y}^i}{dt} = C^i k\left(A\frac{\partial}{\partial z} + B\right)x \tag{6.21}$$

Performing one more time-differentiation, we obtain:

$$\frac{d^2\bar{y}^i}{dt^2} = C^i k\left(A\frac{\partial}{\partial z}+B\right)^2 x + C^i k\left(A\frac{\partial}{\partial z}+B\right) wb^i(z)\bar{u}^i \tag{6.22}$$

In analogy with the above, if the scalar $C^i k\left(A\frac{\partial}{\partial z}+B\right) wb^i(z)$ is nonzero, the characteristic index is equal to two, while if $C^i k\left(A\frac{\partial}{\partial z}+B\right) wb^i(z) = 0$, the characteristic index is greater than two.

Generalizing the above development, one can give the definition of characteristic index for systems of the form of Eq.6.9.

Definition 6.1: *Referring to the system of linear first-order partial differential equations of the form of Eq.6.9, we define the characteristic index of the output $\bar{y}^i$ with respect to the input $\bar{u}^i$ as the smallest integer σ^i for which*

$$C^i k\left(A\frac{\partial}{\partial z}+B\right)^{\sigma^i-1} wb^i(z) \neq 0 \tag{6.23}$$

or $\sigma^i = \infty$ if such an integer does not exist.

Remark 6.2: According to definition 6.1, the characteristic index is the smallest order time-derivative of the output $\bar{y}^i$ which explicitly depends on the manipulated input $\bar{u}^i$. In this sense, it can be thought of as a natural generalization of the concept of relative order for the systems under consideration. For the case of linear ODE systems, the relative order can be interpreted as the difference in the degree of the denominator polynomial and numerator polynomial. Such an interpretation cannot be given for the concept of characteristic index because the frequency domain representation of the system of Eq.6.9 typically gives rise to transfer functions which involve complicated transcendental forms.

In analogy with the linear case, the following concept of characteristic index will be introduced for the quasi-linear PDE system of Eq.6.7.

Definition 6.2: *Referring to the system of quasi-linear first-order partial differential equations of the form of Eq.6.7, we define the characteristic index of the output $\bar{y}^i$ with respect to the input $\bar{u}^i$ as the smallest integer σ^i for which*

$$C^i L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\sigma^i - 1} h(x) b^i(z) \not\equiv 0 \qquad (6.24)$$

where a_j denotes the j-th column vector of the matrix $A(x)$, and L_{a_j}, L_f denote the standard Lie derivative notation, or $\sigma^i = \infty$ if such an integer does not exist.

Throughout the chapter, it will be assumed that Eq.6.24 holds for all $x \in \mathcal{H}^n$, $z \in [a, b]$.

From definitions 6.1 and 6.2, one can immediately see that the characteristic index σ^i depends on the structural properties of the process (the matrices A, B and the vectors w, k for the linear case, or the matrix $A(x)$ and the functions $f(x)$, $g(x)$, $h(x)$ for the quasi-linear case), as well as on the selection of the control system and objectives (the functions $b^i(z)$ and the output operators C^i). Note that in the control problem specification of subsection 6.2.2, we have implicitly assumed that C^i and $b^i(z)$ are chosen to act in the same spatial interval (collocated); in the case where C^i and $b^i(z)$ are chosen to act in different spatial intervals (noncollocated), it follows directly from Eqs.6.23-6.24 that the characteristic index $\sigma^i = \infty$, which implies that this selection leads to loss of controllability of the output $\bar{y}^i$ from the input $\bar{u}^i$.

In most practical applications, the selection of $(b^i(z), C^i)$ is typically consistent for all pairs $(\bar{y}^i, \bar{u}^i)$, in a sense which is made precise in the following assumption.

Assumption 6.3: *Referring to the system of first-order partial differential equations of the form of Eq.6.9 (Eq.6.7), $\sigma^1 = \sigma^2 = \cdots = \sigma^l = \sigma$.*

Given the above assumption, σ can be also thought of as the characteristic index between the output vector $\bar{y}$ and the input vector $\bar{u}$.

6.4 State feedback control

6.4.1 Linear systems

In this subsection, we focus on systems of linear first-order partial differential equations of the form of Eq.6.9 and address the problem of synthesizing a distributed state feedback controller that forces the output of the closed-loop system to track a reference input in a prespecified manner. More specifically, we consider distributed state feedback laws of the form:

$$\bar{u} = \mathcal{S}x + sv \tag{6.25}$$

where $\mathcal{S}$ is a linear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, s is an invertible diagonal matrix of functionals, and $v \in \mathbb{R}^l$ is the vector of reference inputs. The structure of the control law of Eq.6.25 is motivated by available results on stabilization of linear PDEs systems via distributed state feedback (e.g., [Wan66b, Bal86]) and the requirement of output tracking. Substituting the distributed state feedback law of Eq.6.25 into the system of Eq.6.9, the following closed-loop system is obtained:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx + wb(z)\mathcal{S}x + wb(z)sv \\ \bar{y} &= \mathcal{C}kx \end{aligned} \tag{6.26}$$

It is clear that feedback laws of the form of Eq.6.25 preserve the linearity with respect to the reference input vector v. We also note that the evolution of the linear PDE system of Eq.6.26 is governed by a strongly continuous semigroup of bounded linear operators, because $\mathcal{L}$ generates a strongly continuous semigroup and $b(z)\mathcal{S}x, b(z)sv$ are bounded, finite dimensional perturbations [Fri76], ensuring that the closed-loop system has a well-defined solution (see subsection 6.2.3). Proposition 6.2 that follows allows specifying the order of the input/output response in the closed-loop system.

Proposition 6.2: *Consider the system of partial differential equations of Eq.6.9 subject to the boundary condition of Eq.6.2, for which assumptions 6.1 and 6.3 hold.*

Then, a distributed state feedback control law of the form of Eq.6.25 preserves the characteristic index σ, in the sense that the characteristic index of $\bar{y}$ with respect to v in the closed-loop system of Eq.6.26 is equal to σ.

The fact that the characteristic index between the output $\bar{y}$ and the reference input v is equal to σ suggests requesting the following input/output response for the closed-loop system:

$$\gamma_\sigma \frac{d^\sigma \bar{y}}{dt^\sigma} + \cdots + \gamma_1 \frac{d\bar{y}}{dt} + \bar{y} = v \tag{6.27}$$

where $\gamma_1, \gamma_2, \cdots, \gamma_\sigma$ are adjustable parameters. These parameters can be chosen to guarantee input/output stability and enforce desired performance specifications in the closed-loop system. Referring to Eq.6.27, note that, motivated by physical arguments, we request, for each pair $(\bar{y}^i, v^i)$, $i = 1, \ldots, l$, an input/output response of order σ with the same transient characteristics (i.e. the parameters γ_k are chosen to be the same for each pair $(\bar{y}^i, v^i)$). This requirement can be readily relaxed if necessary to impose responses with different transient characteristics for the various pairs $(\bar{y}^i, v^i)$.

We are now in a position to state the main result of this subsection in the form of a theorem (the proof can be found in the appendix E).

Theorem 6.1: *Consider the system of linear partial differential equations of Eq.6.9 subject to the boundary condition of Eq.6.2, for which assumptions 6.1 and 6.3 hold. Then, the distributed state feedback law:*

$$\begin{aligned} \bar{u} \;=\; & \left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \\ & \bullet \left\{v - Ckx - \sum_{\nu=1}^{\sigma} \gamma_\nu Ck\left(A\frac{\partial}{\partial z} + B\right)^{\nu} x\right\} \end{aligned} \tag{6.28}$$

enforces the input/output response of Eq.6.27 in the closed-loop system.

Remark 6.3: Referring to the controller of Eq.6.28, it is clear that the calculation of the control action requires algebraic manipulations as well as differentiations and integrations in space, which is expected because of the distributed nature of the controller.

Remark 6.4: The distributed state feedback controller of Eq.6.28 was derived following an approach conceptually similar to the one employed for the synthesis of inversion-based controllers for ODE systems. We note that this is possible, because, for the system of Eq.6.9: a) the solution is well-defined (i.e. the evolution of the state is locally governed by a strongly continuous semigroup of bounded linear operators); b) the input/output spaces are finite dimensional; and c) the manipulated input, the measured output and the controlled output are distributed in space. These three requirements are standard in most control theories for PDE systems (e.g., [Bal86, Bal91]) and only the third one poses some practical limitations excluding processes where the manipulated input appears in the boundary.

Remark 6.5: The class of distributed state feedback laws of Eq.6.25 is a generalization of control laws of the form

$$\bar{u} \; = \; \mathcal{F}x \tag{6.29}$$

where $\mathcal{F}$ is a bounded linear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, which are used for the stabilization of linear PDEs. The usual approach followed for the design of the gain operator $\mathcal{F}$ utilizes optimal control methods (e.g., [Lo73, Ray81]).

Example (Cont'd) : In the case of the heat exchanger example introduced earlier, it can be easily verified that the characteristic index of the system of Eq.6.12 is equal to one. Therefore, a first-order input/output response is requested in the closed-loop system:

$$\gamma_1 \frac{d\bar{y}}{dt} + \bar{y} = v \tag{6.30}$$

Using the result of theorem 6.1, the appropriate control law that enforces this response

is:
$$\bar{u} = \frac{1}{\gamma_1}\left\{v - \int_0^1 x(z,t)dz - \gamma_1\int_0^1 \left(-v_l\frac{\partial x}{\partial z}(z,t) - ax(z,t)\right)dz\right\} \tag{6.31}$$

6.4.2 Quasi-linear systems

In this subsection, we consider systems of quasi-linear first order PDEs of the form of Eq.6.7 and control laws of the form:

$$\bar{u} \;=\; \bar{\mathcal{S}}(x) + \bar{s}(x)v \tag{6.32}$$

where $\bar{\mathcal{S}}(x)$ is a nonlinear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, $\bar{s}(x)$ is an invertible diagonal matrix of functionals, and $v \in \mathbb{R}^l$ is the vector of reference inputs. The class of control laws of Eq.6.32 is a natural generalization of the class of control laws considered for the case of linear systems (Eq.6.25). Under the control law of Eq.6.32, the closed-loop system takes the form:

$$\begin{aligned}\frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\bar{\mathcal{S}}(x) + g(x)b(z)\bar{s}(x)v \\ \bar{y} &= Ch(x)\end{aligned} \tag{6.33}$$

It is straightforward to show that the above system has locally a well-defined solution, and the counterpart of proposition 6.2 also holds, i.e., the characteristic index of the output $\bar{y}$ with respect to v in the closed-loop system of Eq.6.33 is equal to σ, which suggests seeking a linear input/output response of the form of Eq.6.27 in the closed-loop system. Theorem 6.2 that follows states the controller synthesis result for this case.

Theorem 6.2: *Consider the system of quasi-linear partial differential equations of Eq.6.7 subject to the boundary condition of Eq.6.2, for which assumptions 6.1 and 6.3* *hold. Then, the distributed state feedback law:*

$$\begin{aligned}\bar{u} \;=\; & \left[\gamma_\sigma C L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\sigma-1} h(x) b(z)\right]^{-1} \\ & \bullet \left\{ v - Ch(x) - \sum_{\nu=1}^{\sigma} \gamma_\nu C \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\nu} h(x) \right\}\end{aligned} \tag{6.34}$$

enforces the input/output response of Eq.6.27 in the closed-loop system.

Remark 6.6: Theorem 6.2 provides an analytical formula of a distributed nonlinear state feedback controller that enforces a linear input/output response in the closed-loop system. In this sense, the controller of Eq.6.34 can be viewed as the counterpart of input/output linearizing control laws for nonlinear ODE systems (see [KA91] and the references therein), in the case of infinite dimensional systems of the form of Eq.6.7.

6.5 Closed-loop stability

The goal of this section is to define a concept of zero dynamics and the associated notion of minimum-phaseness for systems of first-order hyperbolic PDEs of the form of Eq.6.9 (Eq.6.7), subject to the boundary condition of Eq.6.2; this will allow us to state conditions that guarantee exponential stability of the closed-loop system. We will initially define the concept of zero dynamics for the case of linear systems (the definition for the case of quasi-linear systems is completely similar and will be omitted for brevity). Our definition is analogous with the one given in [BGH94] (see also [Poh81]) for the case of linear parabolic PDE systems with boundary feedback control.

Definition 6.3: *The zero dynamics associated with the system of linear first-order partial differential equations of Eq.6.9 is the system obtained by constraining the out-*

put to zero, i.e., the system:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx - wb(z)\left[Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1}\left\{Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma} x\right\} \\
Ckx &\equiv 0 \\
C_1x(a,t) &+ C_2x(b,t) = R(t)
\end{aligned} \tag{6.35}$$

From definition 6.3, it is clear that the dynamical system which describes the zero dynamics is an infinite dimensional one. The concept of zero dynamics allows us to define a notion of minimum-phaseness for systems of the form of Eq.6.9. More specifically, if the zero dynamics is exponentially stable the system of Eq.6.9 is said to be minimum-phase, while if the zero dynamics is unstable the system of Eq.6.9 is said to be nonminimum-phase.

We have now introduced the necessary elements that will allow us to address the issue of closed-loop stability. Proposition 6.3 that follows provides conditions that guarantee the exponential stability of the closed-loop system (the proof can be found in the appendix E).

Proposition 6.3 : *Consider the system of Eq.6.9 for which assumptions 6.1 and 6.3 hold, under the controller of Eq.6.28. Then, the closed-loop system is exponentially stable (i.e., the differential operator of the closed-loop system generates an exponentially stable semigroup) if the following conditions are satisfied:*

1. *The roots of the equation*

$$1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0 \tag{6.36}$$

lie in the open left-half of the complex plane.

2. *The system of Eq.6.35 is exponentially stable.*

Remark 6.7: Referring to the above proposition, we note that the first condition addresses the input/output stability of the closed-loop system and the second condition addresses its internal stability. Note also that the first condition is associated with the stability of a *finite* number of poles, while the second condition concerns the stability of an *infinite* number of poles. This is expected since the input/output spaces are finite dimensional, while the state of the system evolves in infinite dimensions.

Remark 6.8: From the result of proposition 6.3, it follows that the controller of theorem 6.1 places a finite number of poles of the open-loop infinite/dimensional system of Eq.6.9 at prespecified (depending on the choice of parameters γ_k) locations, by essentially canceling an infinite number of poles, those included in the zero dynamics. Furthermore, the closed-loop system is exponentially stable if the zero dynamics of the original system is exponentially stable (condition 2 of proposition 6.3). This result is analogous to available results on stabilization of systems of linear PDEs of the form of Eq.6.9 with feedback of the form of Eq.6.29. Specifically, it is well-known (e.g., [Rus78, Bal86]) that control laws of the form of Eq.6.29 allow placing a finite number of open-loop poles at prespecified locations, while in addition guaranteeing the exponential stability of the closed-loop system, if the pair $[\mathcal{L}\ wb(z)]$ is stabilizable (i.e., the remaining infinite uncontrolled poles are in the open left-half of the complex plane).

Remark 6.9: From the result of proposition 6.3 and the discussion of remark 6.8, it is clear that the derivation of exponential stability results for the closed-loop system under the control law of Eq.6.25 (or the control law of Eq.6.29) requires that the open-loop system is minimum phase (or stabilizable). The a-priori verification of these properties can in principle be performed by utilizing spectral theory for operators in infinite-dimensions [Fri76, Poh81]. However, these calculations are difficult to perform in the majority of practical applications. In practice, the stabilizability and minimum-phase properties can be checked through simulations.

In closing this section, we address the issue of closed-loop stability for systems of quasi-linear PDEs. Proposition 6.4 that follows provides the counterpart of the result of proposition 6.3 for the case of quasi-linear systems.

Proposition 6.4: *Consider the system of Eq.6.7 for which assumptions 6.1 and 6.3 hold, under the controller of Eq.6.34. Then, the closed-loop system is locally exponentially stable (i.e., the differential operator of the linearized closed-loop system generates an exponentially stable semigroup) if the following conditions are satisfied:*

1. *The roots of the equation*

$$1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0 \tag{6.37}$$

   lie in the open left-half of the complex plane.

2. *The zero dynamics of the system of Eq.6.7 is locally exponentially stable.*

Remark 6.10: The exponential stability of the closed-loop system guarantees, in both the linear and the quasi-linear case, that in the presence of small modeling errors, the states of the closed-loop system will be bounded. Furthermore, since the input/output spaces of the closed-loop system are finite dimensional, and the controller of Eq.6.34 enforces a linear input/output dynamics between $\bar{y}$ and v, it is possible to implement a linear error feedback controller around the $(\bar{y} - v)$ loop to ensure asymptotic offsetless output tracking in the closed-loop system, in the presence of constant unknown model parameters and unmeasured disturbance inputs.

6.6 Output feedback control

In this section, we will consider the synthesis of distributed output feedback controllers for systems of the form of Eq.6.9 (Eq.6.7). The requisite controllers will be synthesized employing combination of the developed distributed state feedback controllers with distributed state observers. Analysis of the resulting closed-loop system will allow deriving precise conditions which guarantee that the requirements of exponential stability and output tracking are enforced in the closed-loop system.

The conventional approach followed for the design of state estimators for linear PDE systems is to discretize the system equations and then apply results from estimation theory for ODE systems (e.g., [SJC80]). It has been shown however that methods for state estimation that treat the full distributed parameter system lead to state observers that yield significantly superior performance [Ray81, CRC86]. In this direction, available results on state estimation for systems of first-order hyperbolic PDEs concern mainly the use of Kalman filtering theory for the design of distributed state observers [PA70, YSR74].

6.6.1 Linear systems

We consider state observers with the following general state-space description [PA70]:

$$\frac{\partial \eta}{\partial t} = A\frac{\partial \eta}{\partial z} + B\eta + wb(z)\bar{u} + \mathcal{P}(\bar{q} - \mathcal{Q}p\eta) \tag{6.38}$$

where $\mathcal{P}$ is a bounded linear operator, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$, that has to be designed so that the operator $\mathcal{L}_o = \mathcal{L} - \mathcal{P}\mathcal{Q}p$ generates an exponentially stable semigroup (note that this is possible by assumption 6.2). The system of Eq.6.38 consists of a replica of the process system and the term $\mathcal{P}(\bar{q} - \mathcal{Q}p\eta)$ used to enforce a fast decay of the discrepancy between the estimated and the actual values of the states of the system. In practice, the design of the operator $\mathcal{P}$ can be performed via a) simple pole placement in case where the output measurements are not corrupted by noise, b) Kalman filtering theory, in case where the output measurements are noisy.

Theorem 6.3 that follows provides a state-space realization of the output feedback controller resulting from the combination of the state observer of Eq.6.38 with the state feedback controller of Eq.6.28 (the proof is given in appendix E).

Theorem 6.3 : *Consider the system of linear partial differential equations of Eq.6.9 subject to the boundary condition of Eq.6.2, for which assumptions 6.1, 6.2, 6.3 and the conditions of proposition 6.3 hold. Consider also the linear bounded operator $\mathcal{P}$ designed such that the operator $\mathcal{L}_o = \mathcal{L} - \mathcal{P}\mathcal{Q}p$ generates an exponentially stable semigroup. Then, the distributed output feedback controller:*

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= A\frac{\partial \eta}{\partial z} + B\eta + wb(z)\left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \left\{v - Ck\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z}+B\right)^{\nu}\eta\right\} + \mathcal{P}(\bar{q} - \mathcal{Q}p\eta) \\
\bar{u} &= \left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \bullet\left\{v - Ck\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z}+B\right)^{\nu}\eta\right\}
\end{aligned} \tag{6.39}$$

a)guarantees exponential stability of the closed-loop system.
b)enforces the input/output response of Eq.6.27 in the closed-loop system if $x(z,0) = \eta(z,0)$.

Remark 6.11: In the case of open-loop stable systems, a more convenient way to reconstruct the state of the system is to consider the observer of Eq.6.38 with the operator $\mathcal{P}$ set identically equal to zero. This is motivated by the fact that the open-loop stability of the system guarantees the convergence of the estimated values to the actual ones with transient behavior depending on the location of the spectrum of the operator of Eq.6.14.

Remark 6.12: Available results on stabilization of systems of linear hyperbolic PDEs via distributed output feedback (e.g., [Ray81, Bal86]) concern the design of controllers with the following general state space description:

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= A\frac{\partial \eta}{\partial z} + [B + wb(z)\mathcal{F}]\eta + \mathcal{R}(\bar{q} - \mathcal{Q}p\eta) \\
\bar{u} &= \mathcal{F}\eta
\end{aligned} \tag{6.40}$$

where $\mathcal{R}$ is a bounded linear operator, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$. We note that the main similarity between a controller of the form of Eq.6.40 and the controller of Eq.6.39 is that both are infinite dimensional (because of the state observers utilized), while their main difference lies in the fact that the controller of Eq.6.39 guarantees exponential stability of the closed-loop system, if the open-loop system is minimum-phase and detectable, while a controller of the form of Eq.6.40 will exponentially stabilize the closed-loop system, if the open-loop system is jointly stabilizable/detectable [Bal86].

Example (Cont'd) : Referring to the linear PDE system of Eq.6.12, we note that $\lambda = -v_l < 0$, while $C_2 = 0$, and thus, the system is open-loop stable according to the result of remark 6.1. The open-loop stability of the system allows using an output feedback controller which consists of the distributed state feedback controller coupled with an open-loop observer. The appropriate controller takes the form:

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -v_l \frac{\partial \eta}{\partial z} - a\eta + a\frac{1}{\gamma_1}\left\{v - \int_0^1 \eta(z,t)dz - \gamma_1 \int_0^1 (-v_l \frac{\partial \eta}{\partial z}(z,t) - a\eta(z,t))dz\right\} \\ \bar{u} &= \frac{1}{\gamma_1}\left\{v - \int_0^1 \eta(z,t)dz - \gamma_1 \int_0^1 \left(-v_l \frac{\partial \eta}{\partial z}(z,t) - a\eta(z,t)\right) dz\right\} \end{aligned} \tag{6.41}$$

6.6.2 Quasi-linear systems

In this subsection, we consider the synthesis of output feedback controllers for systems of the form of Eq.6.7. Given the lack of general available results on state estimation of such systems, we will proceed with the design of a nonlinear state observer which guarantees local exponential convergence of the state estimates to the actual state values. In particular, the following state observer will be used to estimate the state vector of the system in space and time:

$$\frac{\partial \eta}{\partial t} = A(\eta)\frac{\partial \eta}{\partial z} + f(\eta) + g(\eta)b(z)\bar{u} + \mathcal{P}(\bar{q} - \mathcal{Q}p(\eta)) \tag{6.42}$$

where η denotes the observer state vector and $\mathcal{P}$ is a linear operator, mapping $\mathbb{R}^l$ into $\mathcal{H}^n$, designed on the basis of the linearization of the system of Eq.6.42 so that the eigenvalues of the operator $\bar{\mathcal{L}}_o = \bar{\mathcal{L}} - \mathcal{P}\mathcal{Q}p(z)$ lie in the left-half plane.

The state observer of Eq.6.42 can be coupled with the state feedback controller of Eq.6.34 to derive an output feedback controller that guarantees output tracking and closed-loop stability. The resulting controller is given in theorem 6.4 that follows (the proof is given in the appendix E).

Theorem 6.4: *Consider the system of quasi-linear partial differential equations of Eq.6.7 subject to the boundary condition of Eq.6.2, for which assumptions 6.1, 6.2, 6.3 and the conditions of proposition 6.4 hold. Consider also the bounded operator $\bar{\mathcal{P}}$ designed such that the operator $\bar{\mathcal{L}}_o = \bar{\mathcal{L}} - \bar{\mathcal{P}}\mathcal{Q}p(z)$ generates an exponentially stable semigroup. Then, the distributed output feedback controller:*

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= A(\eta)\frac{\partial \eta}{\partial z} + f(\eta) + g(\eta)b(z)\left[\gamma_\sigma C L_g \left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(\eta)b(z)\right]^{-1} \\
&\bullet \left\{v - Ch(\eta) - \sum_{\nu=1}^{\sigma}\gamma_\nu C\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(\eta)\right\} + \bar{\mathcal{P}}(\bar{q} - \mathcal{Q}p(\eta)) \\
\bar{u} &= \left[\gamma_\sigma C L_g \left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(\eta)b(z)\right]^{-1} \\
&\bullet \left\{v - Ch(\eta) - \sum_{\nu=1}^{\sigma}\gamma_\nu C\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(\eta)\right\}
\end{aligned} \tag{6.43}$$

a)guarantees local exponential stability of the closed-loop system,

b)enforces the input/output response of Eq.6.27 in the closed-loop system if $x(z,0) = \eta(z,0)$.

In analogy with the linear case, for open-loop stable systems, the operator $\bar{\mathcal{P}}$ can be taken to be identically equal to zero, since the local exponential stability of the open-loop system guarantees the local convergence of the estimated values to the actual values.

Remark 6.13: Note that in the case of imperfect initialization of the observer states (i.e., $\eta(z,0) \neq x(z,0)$), although a slight deterioration of the performance may occur, (i.e., the input/output response of Eq.6.27 will not be exactly imposed in the closed-loop system), the output feedback controllers of theorem 6.3 and 6.4 guarantee exponential stability and asymptotic output tracking in the closed-loop system.

Remark 6.14: The nonlinear distributed output feedback of Eq.6.43 is an infinite dimensional one, due to the infinite dimensional nature of the observer of Eq.6.42. Therefore, a finite-dimensional approximation of the controller has to be derived for on-line implementation. This task can be performed utilizing standard discretization techniques such as finite differences, orthogonal collocation, etc.. It is expected that some performance deterioration will occur in this case, depending on the discretization method used and the number and location of discretization points (see the chemical reactor application presented in the next section). We finally note that it is well-established (e.g., [Bal86]) that as the number of discretization points increases, the closed-loop system resulting from the PDE model plus an approximate finite-dimensional controller converges to the closed-loop system resulting from the PDE model plus the infinite-dimensional controller, guaranteeing the well-posedness of the approximate finite-dimensional controller.

6.7 Application to a nonisothermal plug-flow reactor

6.7.1 Process description

Consider the non-isothermal plug-flow reactor shown in Figure 6.2, where two first-

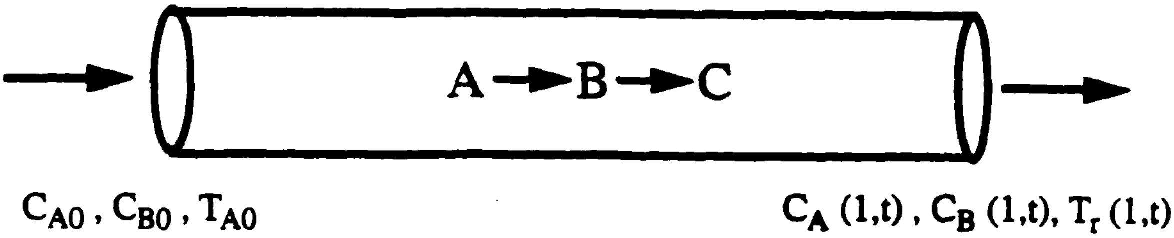


Figure 6.2: A nonisothermal plug-flow reactor.

order reactions in series take place:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

where A is the reactant species, B is the desired product, and C is an undesired product. The inlet stream consists of pure A of concentration C_{A0} and temperature T_{A0}. The reactions are endothermic and a jacket is used to heat the reactor. The reaction rate expressions are assumed to be of the following form:

$$\begin{aligned} r_1 &= -k_{10} \exp(\frac{-E_1}{RT_r}) C_A \\ r_2 &= -k_{20} \exp(\frac{-E_2}{RT_r}) C_B \end{aligned}$$

where k_{10}, k_{20} E_1, E_{20}, denote the pre-exponential constants and the activation energies of the reactions. Under the following assumptions:

- Perfect radial mixing in the reactor
- Constant volume of the liquid in the reactor
- Constant density and heat capacity of the reacting liquid
- Negligible diffusive and dispersive phenomena

the material and energy balances that describe the dynamical behavior of the process take the following form:

- Mole balance for the species A

$$\frac{\partial C_A}{\partial t} = -v_l \frac{\partial C_A}{\partial z} - k_{10} e^{\frac{-E_1}{RT_r}} C_A$$

- Mole balance for the species B

$$\frac{\partial C_B}{\partial t} = -v_l \frac{\partial C_B}{\partial z} + k_{10} e^{\frac{-E_1}{RT_r}} C_A - k_{20} e^{\frac{-E_2}{RT_r}} C_B$$

- Reactor energy balance

$$
\begin{aligned}
\frac{\partial T_r}{\partial t} &= -v_l \frac{\partial T_r}{\partial z} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{\frac{-E_1}{RT_r}} C_A + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} e^{\frac{-E_2}{RT_r}} C_B \\
&+ \frac{U_w}{\rho_m c_{pm} V_r}(T_j - T_r)
\end{aligned}
$$

subject to the following boundary conditions:

$$
C_A(0,t) = C_{A0}(t)\ ,\ C_B(0,t) = 0\ ,\ T_r(0,t) = T_{A0}(t)
$$

where C_A, C_B and denote the concentrations of the species A and B in the reactor. T_r denotes the temperature of the reactor, C_{A_S}, C_{B_S}, T_{r_S} denote the steady-state profiles for the state variables, ΔH_{r_1}, ΔH_{r_2} denote the enthalpies of the two reactions, ρ_m, c_{pm} denote the density and heat capacity of the fluid in the reactor, V_r denotes the volume of the reactor, U_w denotes the heat transfer coefficient, and T_j denotes the spatially uniform temperature in the jacket. The values used for process parameters are given in Table 6.1, while the corresponding steady-state profiles are shown in Figure 6.3. The control objective is the regulation of the concentration of the species B throughout the reactor by manipulating the jacket temperature T_j. We note that, in practice, T_j is usually manipulated indirectly through manipulation of the jacket inlet flow rate (this implementation issue is addressed in subsection 6.7.4).

Setting:

$$
u = T_j - T_{j_S},\ \ x_1 = C_A,\ \ x_2 = C_B,\ \ x_3 = T_r,\ \ y = C_B \tag{6.44}
$$

v_l	=	1.0	$m\ min^{-1}$
L	=	1.0	m
V_r	=	10.0	lt
E_1	=	2.0×10^4	$kcal\ kmol^{-1}$
E_2	=	5.0×10^4	$kcal\ kmol^{-1}$
k_{10}	=	5.0×10^{12}	min^{-1}
k_{20}	=	5.0×10^6	min^{-1}
R	=	1.987	$kcal\ kmol^{-1}\ K^{-1}$
ΔH_1	=	548000.1	$kcal\ kmol^{-1}$
ΔH_2	=	986000.1	$kcal\ kmol^{-1}$
c_{pm}	=	0.231	$kcal\ kg^{-1}\ K^{-1}$
ρ_m	=	0.09	$kg\ lt^{-1}$
c_{pj}	=	2.5	$kcal\ kg^{-1}\ K^{-1}$
ρ_j	=	1.44	$kg\ lt^{-1}$
U_w	=	2000.0	$kcal\ min^{-1}\ K^{-1}$
C_{A0}	=	4.0	$mol\ lt^{-1}$
C_{B0}	=	0.0	$mol\ lt^{-1}$
T_{A0}	=	320.0	K
T_{j0}	=	350.0	K
V_j^i	=	1.0	lt
τ_{jc}	=	0.001	min
F_{js}^1	=	28.87	$lt\ min^{-1}$
F_{js}^2	=	43.66	$lt\ min^{-1}$
F_{js}^3	=	59.63	$lt\ min^{-1}$
F_{js}^4	=	72.34	$lt\ min^{-1}$
F_{js}^5	=	77.23	$lt\ min^{-1}$

Table 6.1: Process parameters and steady-state values

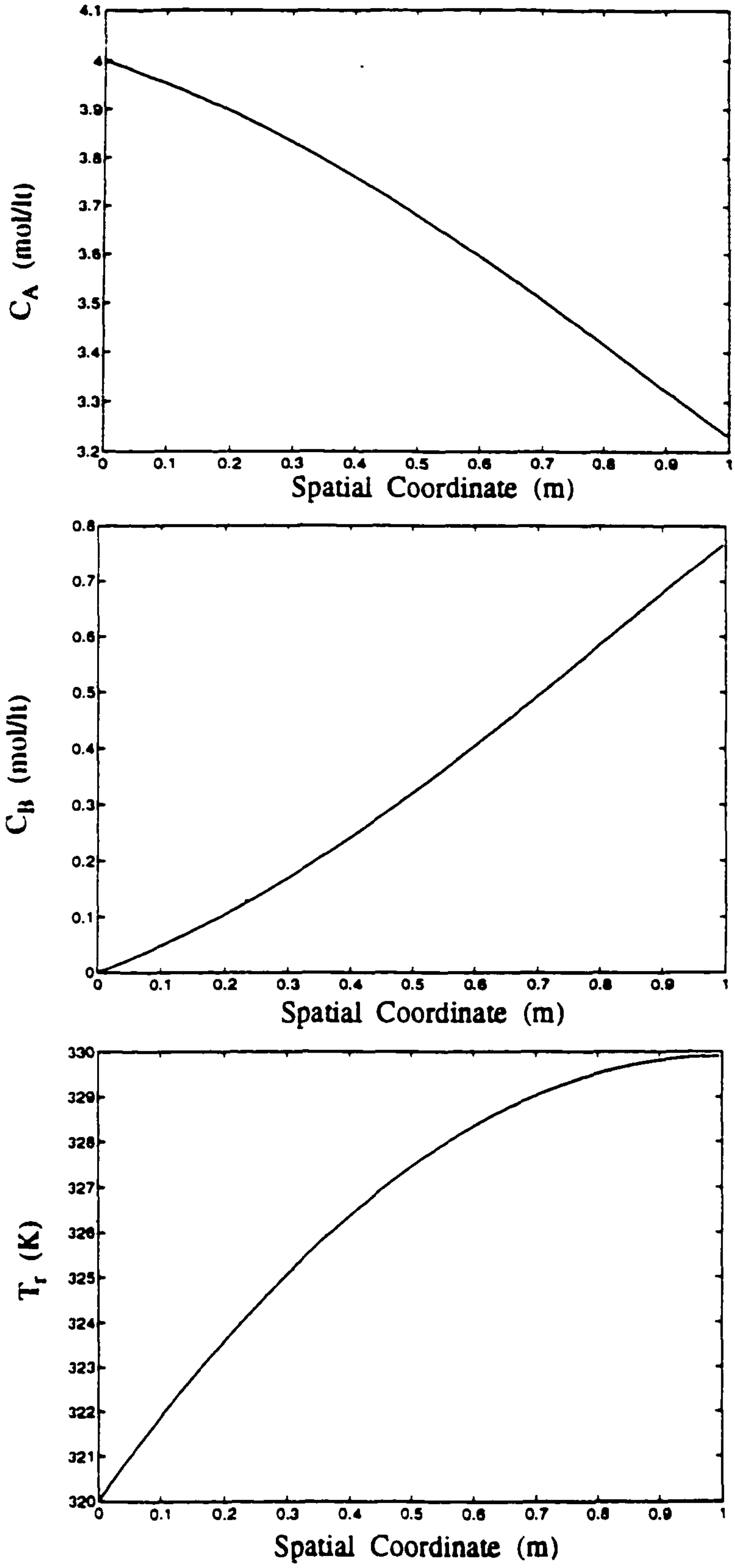


Figure 6.3: Steady-state profiles of reactor state variables.

the original set of equations can be put in the form of Eq.6.1 with:

$$f(x) = \begin{bmatrix} f_1(x_1,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} -k_{10}e^{\frac{-E_1}{Rx_3}}x_1 \\ k_{10}e^{\frac{-E_1}{Rx_3}}x_1 - k_{20}e^{\frac{-E_2}{Rx_3}}x_2 \\ \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{\frac{-E_1}{Rx_3}}x_1 + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{\frac{-E_2}{Rx_3}}x_2 \\ -\frac{U_w}{\rho_m c_{pm} V_r}(x_3 - T_{js}) \end{bmatrix}$$

$$A(x_1,x_2,x_3) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} -v_l & 0 & 0 \\ 0 & -v_l & 0 \\ 0 & 0 & -v_l \end{bmatrix}, \quad (6.45)$$

$$g(x) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{U_w}{\rho_m c_{pm} V_r} \end{bmatrix}, \; h(x) = \begin{bmatrix} x_2 \end{bmatrix} \quad (6.46)$$

It is clear that the matrix $A(x)$ is real symmetric and its eigenvalues satisfy Eq.6.4. Moreover, the eigenvalues of $A(x)$ are the same, which implies that the above system of quasi-linear PDEs is weakly hyperbolic.

6.7.2 Control problem formulation-Controller synthesis

In this subsection, we will proceed with the formulation and solution of the control problem. More specifically, it will be assumed that there are five control actuators which are characterized by a unity distribution function i.e., $b^i(z)\bar{u}^i = \bar{u}^i$ for all $i = 1, \cdots, 5$. The control actuators are taken to act over equispaced intervals, i.e.:

$$\bar{u}(t) = \begin{cases} \bar{u}^1(t) & , \; [0.0, 0.2] \\ \bar{u}^2(t) & , \; [0.2, 0.4] \\ \bar{u}^3(t) & , \; [0.4, 0.6] \\ \bar{u}^4(t) & , \; [0.6, 0.8] \\ \bar{u}^5(t) & , \; [0.8, 1.0] \end{cases} \quad (6.47)$$

The desired performance requirement is to control the averaging outputs:

$$\bar{y}(t) = \begin{cases} \bar{y}^1(t) = \int_{0.0}^{0.2} 5.0x_2(z,t)dz \\ \bar{y}^2(t) = \int_{0.2}^{0.4} 5.0x_2(z,t)dz \\ \bar{y}^3(t) = \int_{0.4}^{0.6} 5.0x_2(z,t)dz \\ \bar{y}^4(t) = \int_{0.6}^{0.8} 5.0x_2(z,t)dz \\ \bar{y}^5(t) = \int_{0.8}^{1.0} 5.0x_2(z,t)dz \end{cases} \tag{6.48}$$

Using the above relations, the model that will be used for the synthesis of the output feedback controller is given by:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)\sum_{\nu=1}^{5}\left(H(z-z_i) - H(z-z_{i+1})\right)\bar{u}^i \\ \bar{y}^i &= \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1}-z_i}x_2(z,t)dz \ , \ \ i = 1,\cdots,5 \end{aligned} \tag{6.49}$$

where the matrix $A(x)$ and the vector functions $f(x)$ and $g(x)$ are specified in Eq.6.46. A schematic of the reactor along with the control system is given in Figure 6.4. Referring to the system of Eq.6.49, its characteristic index can be calculated using the definition 6.2 introduced in section 6.3. In particular, we have that:

$$\begin{aligned} &C^iL_gh(x) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1}-z_i}L_gh(x) = 0 \ , \ \forall\, i = 1,\cdots,5 \\ &C^iL_g\left(\sum_{j=1}^{3}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1}-z_i}\frac{\partial f_2}{\partial x_3}g_3dz \not\equiv 0 \ , \ \forall\, i = 1,\cdots,5 \end{aligned} \tag{6.50}$$

Thus, the characteristic index of the system of quasi-linear PDEs of Eq.6.49 is equal to 2. This allows requesting the following second-order response in the closed-loop system between $\bar{y}^i$ and v^i, for all $i = 1,\cdots,5$:

$$\gamma_2\frac{d^2\bar{y}^i}{dt^2} + \gamma_1\frac{d\bar{y}^i}{dt} + \bar{y}^i = v^i \tag{6.51}$$

Moreover, the eigenvalues of the matrix $A(x)$ are negative and the boundary conditions are specified in a single point. Thus, the result of remark 6.1 applies directly yielding that the system is open-loop stable. Furthermore, it was also verified,

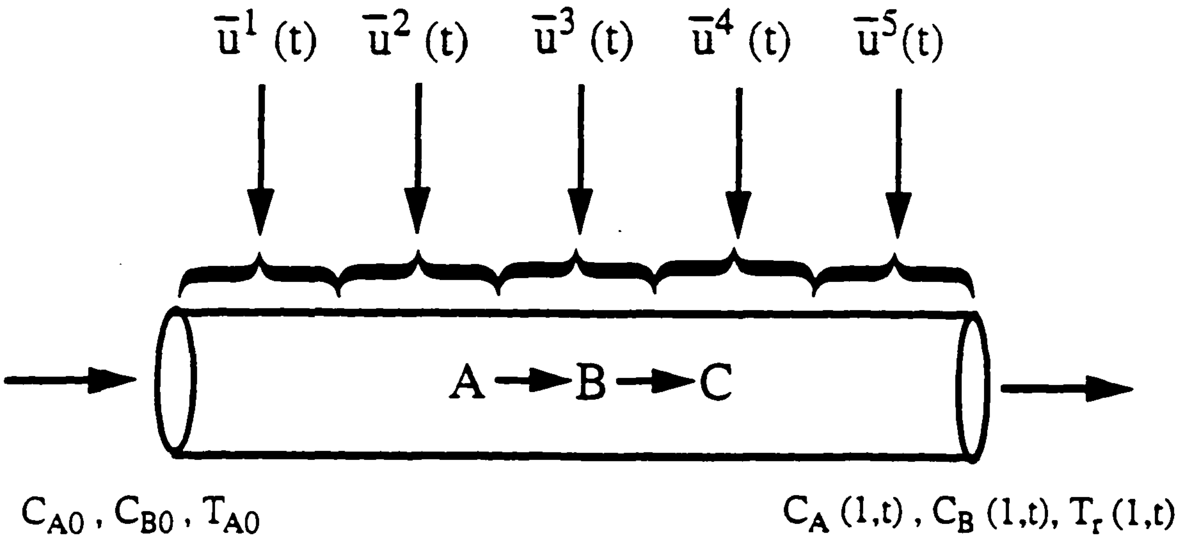


Figure 6.4: Specification of the control problem for the plug-flow reactor.

through simulations, that the process is minimum-phase. Therefore, the developed control method can be applied and the distributed output feedback controller of theorem 6.4, with $\mathcal{P} \equiv 0$, was employed in the simulations (note that due to open-loop stability of the process the controller does not use measured process outputs). The explicit form of the controller is as follows:

$$
\begin{aligned}
\frac{\partial \eta_1}{\partial t} &= -v\frac{\partial \eta_1}{\partial z} - k_{10}e^{\frac{-E_1}{R\eta_3}}\eta_1 \\
\frac{\partial \eta_2}{\partial t} &= -v\frac{\partial \eta_2}{\partial z} + k_{10}e^{\frac{-E_1}{R\eta_3}}\eta_1 - k_{20}e^{\frac{-E_2}{R\eta_3}}\eta_2 \\
\frac{\partial \eta_3}{\partial t} &= -v\frac{\partial \eta_3}{\partial z} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}}k_{10}e^{\frac{-E_1}{R\eta_3}}\eta_1 + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}}k_{20}e^{\frac{-E_2}{R\eta_3}}\eta_2 \\
&+\frac{U_w}{\rho_m c_{pm} V_r}(T_{w_s} - w_3) + \frac{U_w}{\rho_m c_{pm} V_r}\sum_{i=1}^{5}(H(z-z_i) - H(z-z_{i+1})) \\
&\bullet \left[\gamma_2 C^i L_g\left(\sum_{j=1}^{3}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)h(\eta)b^i(z)\right]^{-1} \\
&\bullet \left\{v^i - C^i h(\eta) - \gamma_1 C^i\left(\sum_{j=1}^{3}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)h(\eta)\right. \\
&\left. -\gamma_2 C^i\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^2 h(\eta)\right\} \\
\bar{u}^i &= \left[\gamma_2 C^i L_g\left(\sum_{j=1}^{3}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)h(\eta)b^i(z)\right]^{-1}\left\{v^i - C^i h(\eta)\right. \\
&\left. -\gamma_1 C^i\left(\sum_{j=1}^{3}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)h(\eta) - \gamma_2 C^i\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^2 h(\eta)\right\}
\end{aligned} \tag{6.52}
$$

where the analytical expressions of the terms included in the controller are as follows:

$$C^i L_g \left(\sum_{j=1}^{3} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f \right) h(\eta) b^i(z) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1} - z_i} \frac{\partial f_2}{\partial \eta_3} g_3 dz$$

$$C^i \left(\sum_{j=1}^{3} \frac{\partial \eta_j}{\partial z} L_{a_j} + L_f \right) h(\eta) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1} - z_i} \left(-v_l \frac{\partial \eta_2}{\partial z} + f_2(\eta_1, \eta_2, \eta_3) \right) dz$$

$$C^i L_f \left(\sum_{j=1}^{3} \frac{\partial \eta_j}{\partial z} L_{a_j} \right) h(\eta) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1} - z_i} (-v_l) \left(\frac{\partial f_2}{\partial \eta_1} \frac{\partial \eta_1}{\partial z} + \frac{\partial f_2}{\partial \eta_2} \frac{\partial \eta_2}{\partial z} + \frac{\partial f_2}{\partial \eta_3} \frac{\partial \eta_3}{\partial z} \right) dz$$

$$C^i \left(\sum_{j=1}^{3} \frac{\partial \eta_j}{\partial z} L_{a_j} \right) \left(\sum_{j=1}^{3} \frac{\partial \eta_j}{\partial z} L_{a_j} \right) h(\eta) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1} - z_i} v_l^2 \frac{\partial^2 \eta_2}{\partial z^2}$$

$$C^i \left(\sum_{j=1}^{3} \frac{\partial \eta_j}{\partial z} L_{a_j} \right) L_f h(\eta) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1} - z_i} (-v_l) \left(\frac{\partial f_2}{\partial \eta_1} \frac{\partial \eta_1}{\partial z} + \frac{\partial f_2}{\partial \eta_2} \frac{\partial \eta_2}{\partial z} + \frac{\partial f_2}{\partial \eta_3} \frac{\partial \eta_3}{\partial z} \right) dz$$

$$C^i L_f^2 h(\eta) = \int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1} - z_i} \left(\frac{\partial f_2}{\partial \eta_1} f_1(\eta_1, \eta_3) + \frac{\partial f_2}{\partial \eta_2} f_2(\eta_1, \eta_2, \eta_3) \right.$$
$$\left. + \frac{\partial f_2}{\partial \eta_3} f_3(\eta_1, \eta_2, \eta_3) \right) dz \tag{6.53}$$

The controller was tuned to give an overdamped response between the output $\bar{y}^i$ and the reference input v^i. In particular, the parameters γ_1 and γ_2 were chosen to be:

$$\gamma_1 = \quad 3.0 \; min, \quad \gamma_2 = \; 0.5 \; min^2$$

to achieve the following time constant and damping factor:

$$\tau \; = \; 0.707 \; min \quad , \quad \zeta \; = \; 2.12$$

6.7.3 Evaluation of controller performance

Several simulation runs were performed to evaluate the performance of the distributed output feedback controller of Eq.6.52. The method of finite differences was employed to derive a finite dimensional approximation of the output feedback controller of Eq.6.52, with a choice of 200 discretization points. In all the simulation runs, the process was initially assumed to be at steady-state.

In the fist simulation run, we addressed the reference input tracking capabilities of the controller. Initially, a 30% increase in the reference inputs v^i, $i = 1, \cdots, 5$ was imposed at time $t = 0.0\ min$. Figure 6.5 shows the corresponding output profiles. It is clear that the controller enforces the requested input/output response in the closed-loop system and regulates the output at the new reference input values. The corresponding input profiles for each control actuator are depicted in Figure 6.6. We observe that the control action, required by each actuator to drive the corresponding output to the new reference value, increases as we approach the outlet of the reactor. This is expected, because the amount of heat required to maintain the reaction rate that yields the necessary conversion increases along the length of the reactor. Figure 6.7 shows the evolution of the concentration of the species B throughout the reactor while Figure 6.8 shows the profile of the concentration of the species B at the outlet of the reactor. We observe that by using a finite number of control actuators, we achieve satisfactory control of the output variable C_B at all positions and times.

For the sake of comparison, we also consider the control of the reactor using a controller which was designed on the basis of a model resulting from discretization of the original PDE system in space. In particular, the method of finite differences was used to discretize the original PDE model of Eq.6.49 into a set of five (equal to the number of control actuators) ordinary differential equations in time. Subsequently, an input/output linearizing controller [KA91] was designed on the basis of the resulting ODE model. The corresponding output profiles are shown in Figure 6.9, while Figure 6.10, shows the profile of the concentration of the species B in the outlet of the reactor. It is clear that the controller leads to poor performance, because it does not explicitly take into account the spatially varying nature of the process.

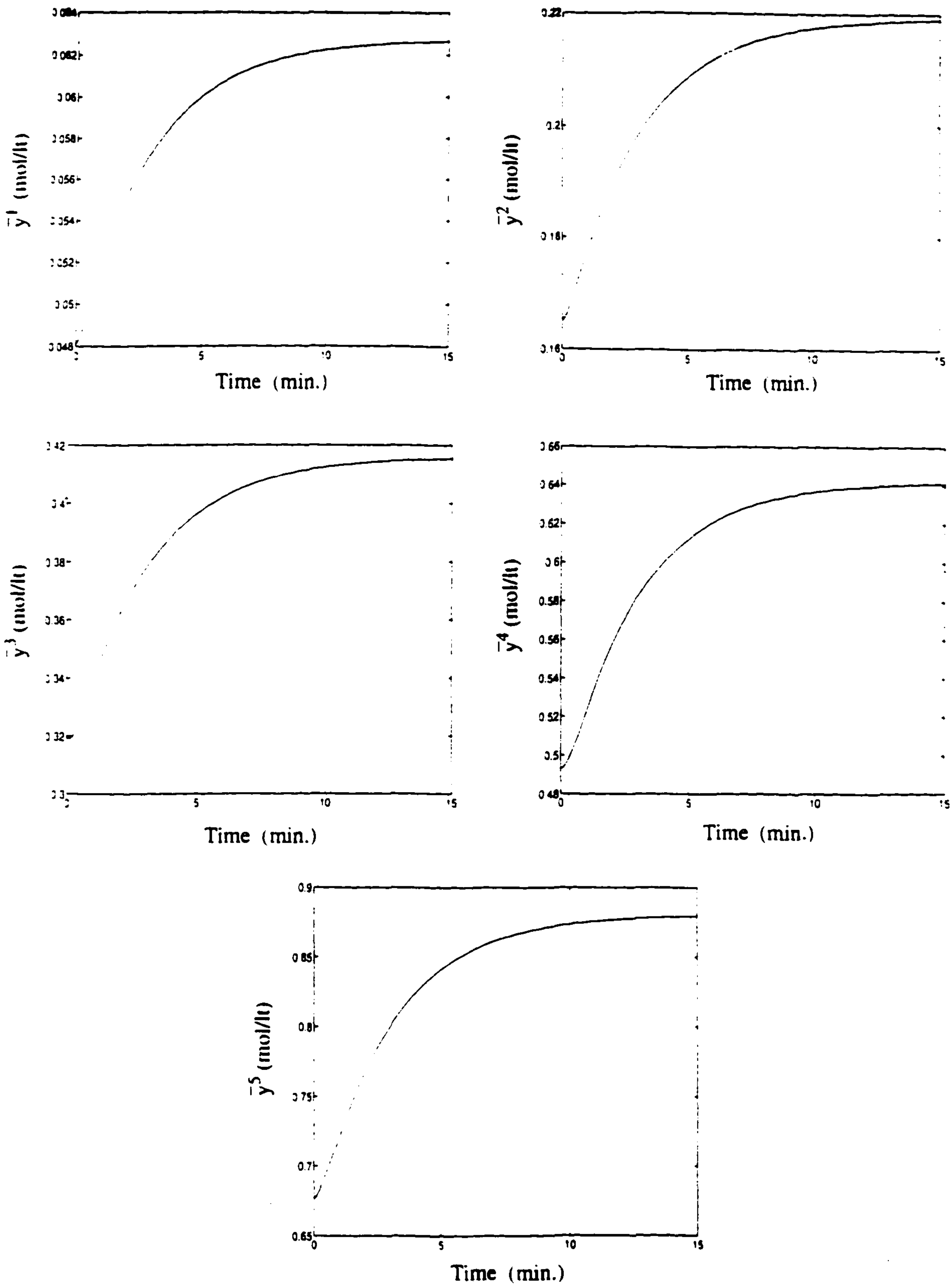


Figure 6.5: Closed-loop output profiles.

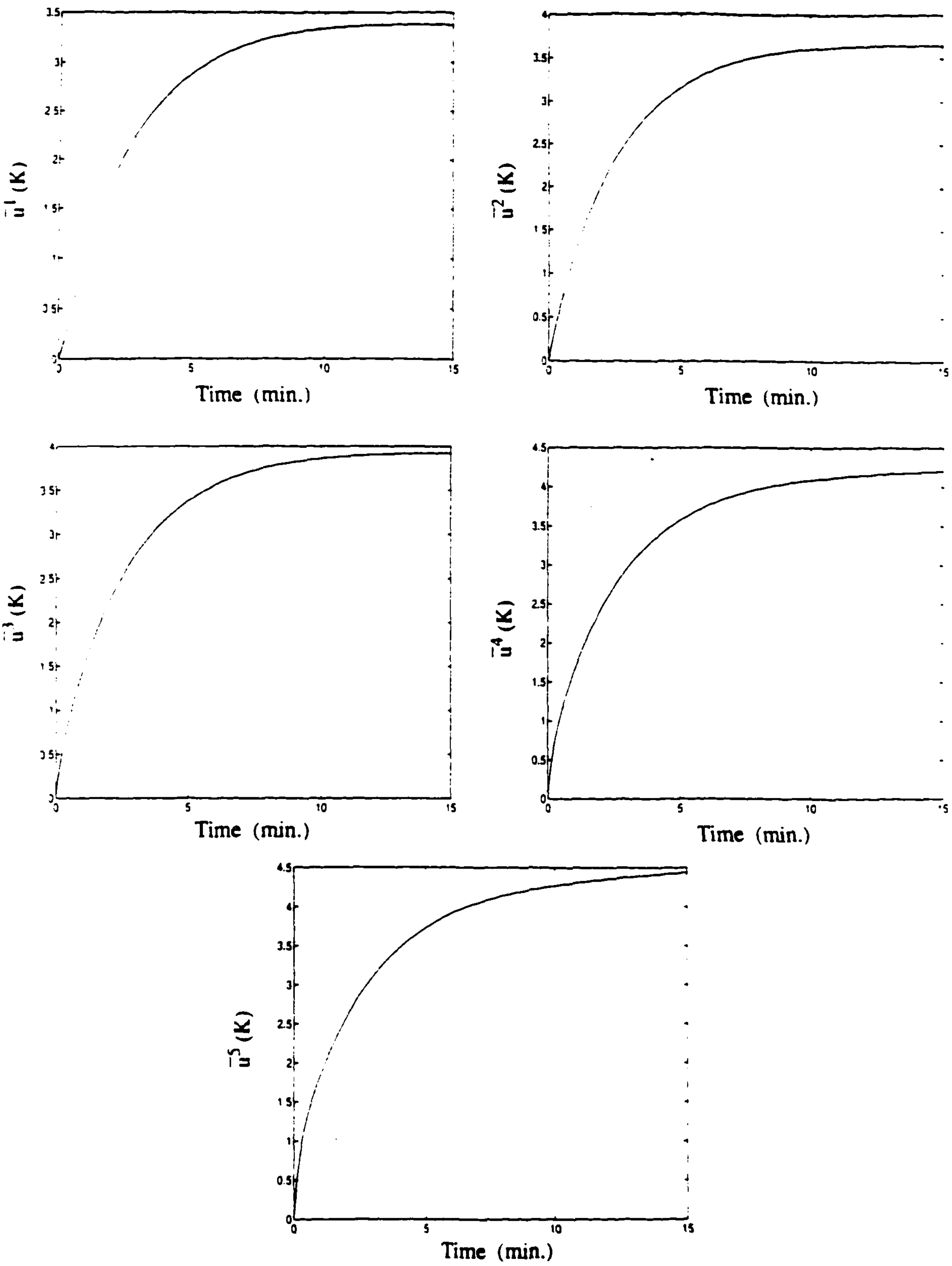


Figure 6.6: Manipulated input profiles.

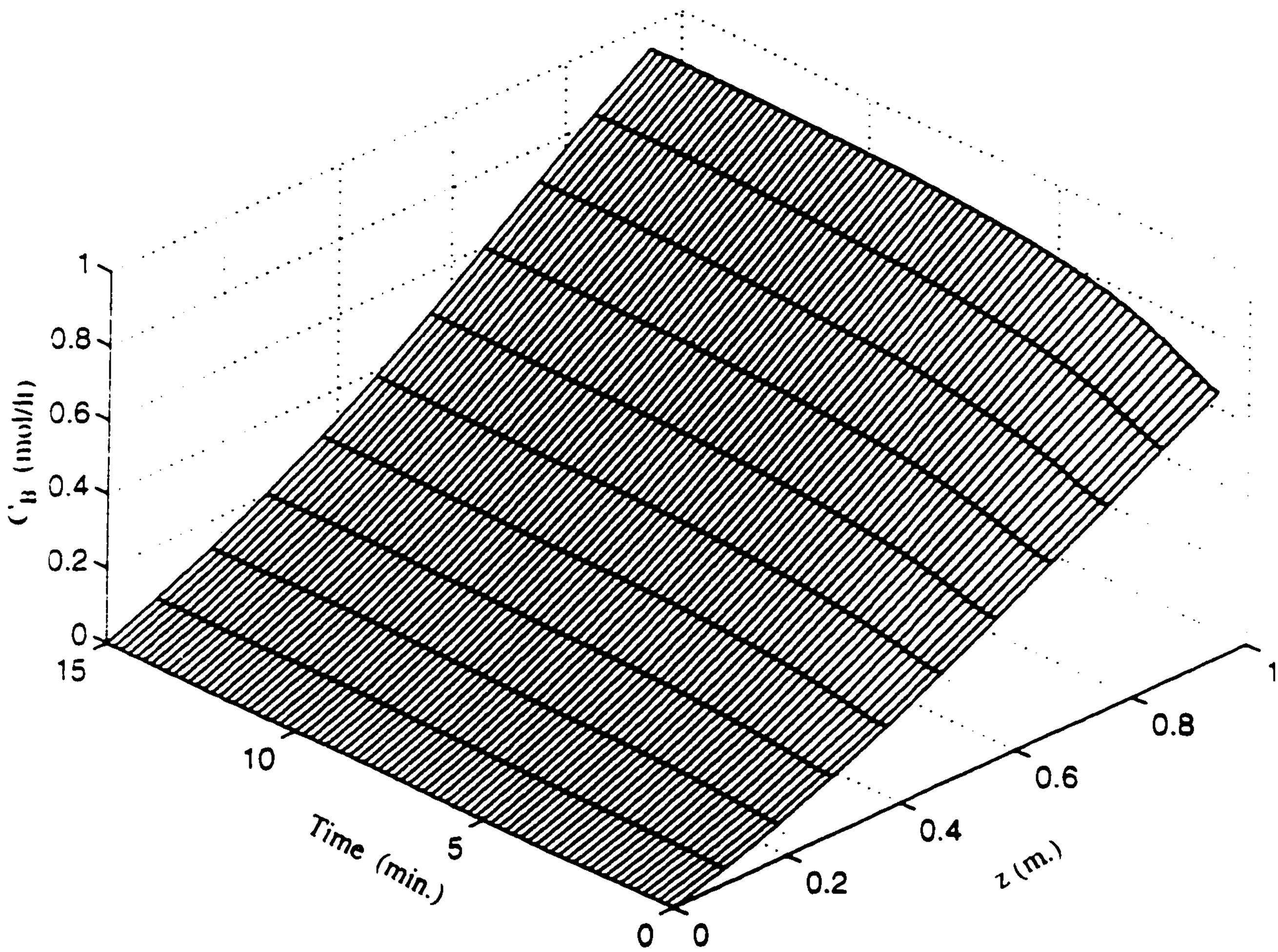


Figure 6.7: Profile of evolution of concentration of species B throughout the reactor.

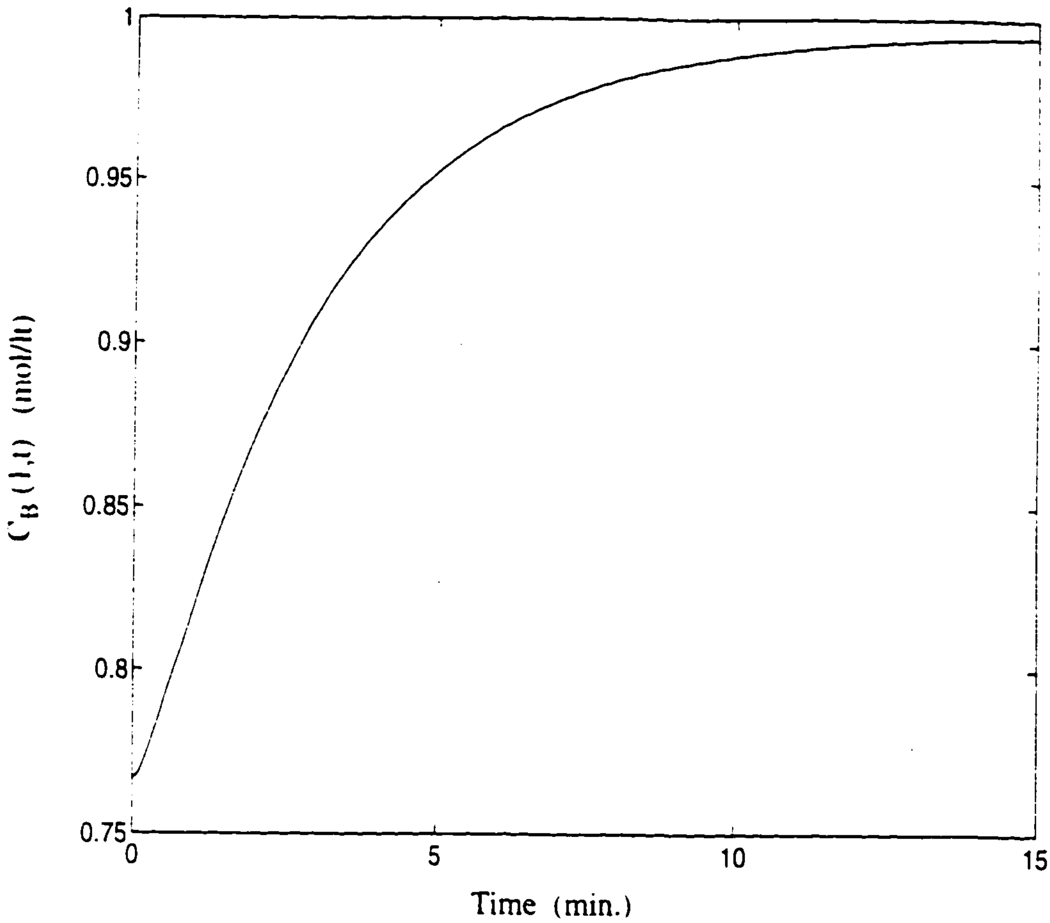


Figure 6.8: Profile of concentration of species B in the outlet of the reactor.

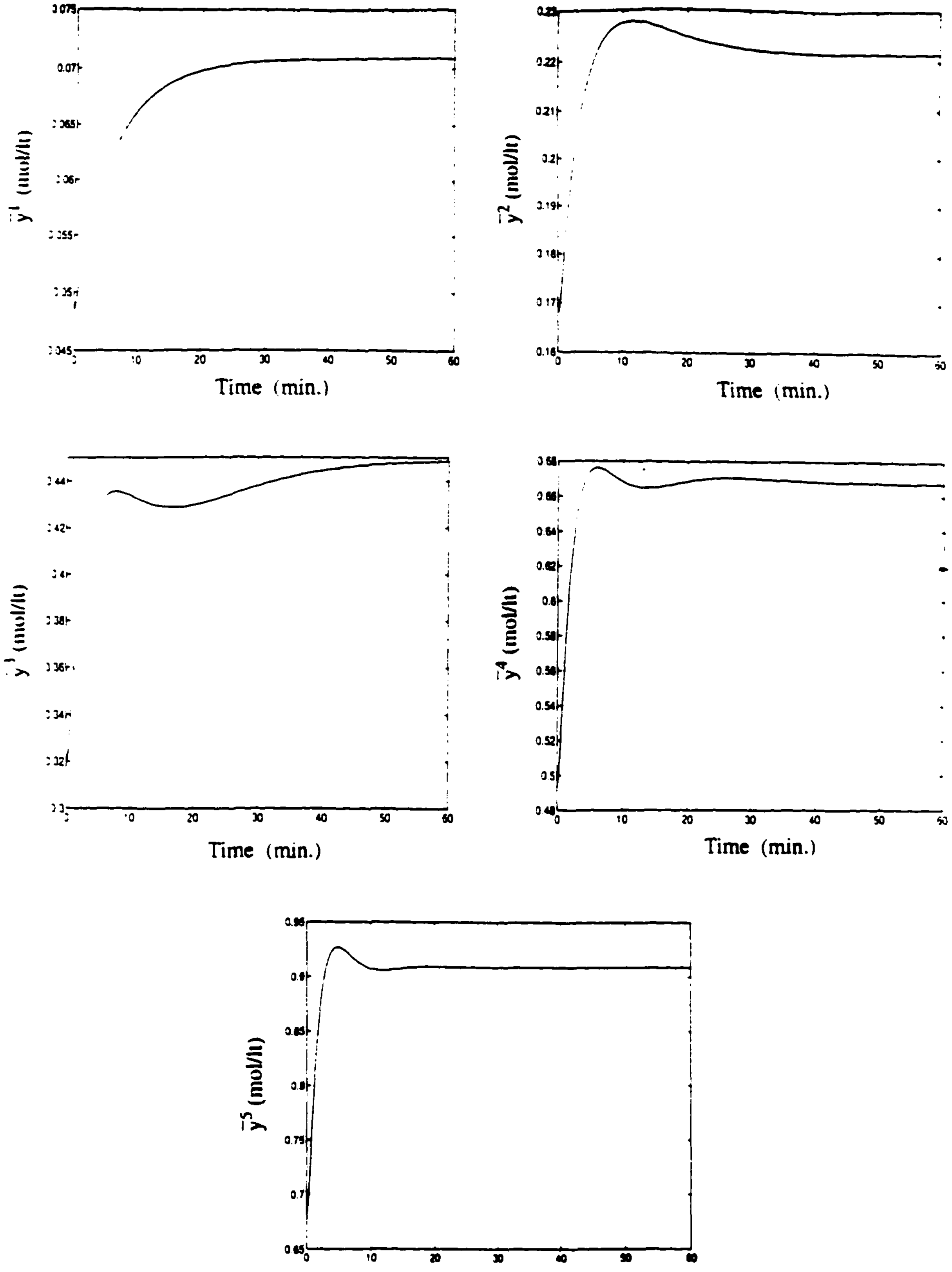


Figure 6.9: Closed-loop output profiles-discretisation based controller.

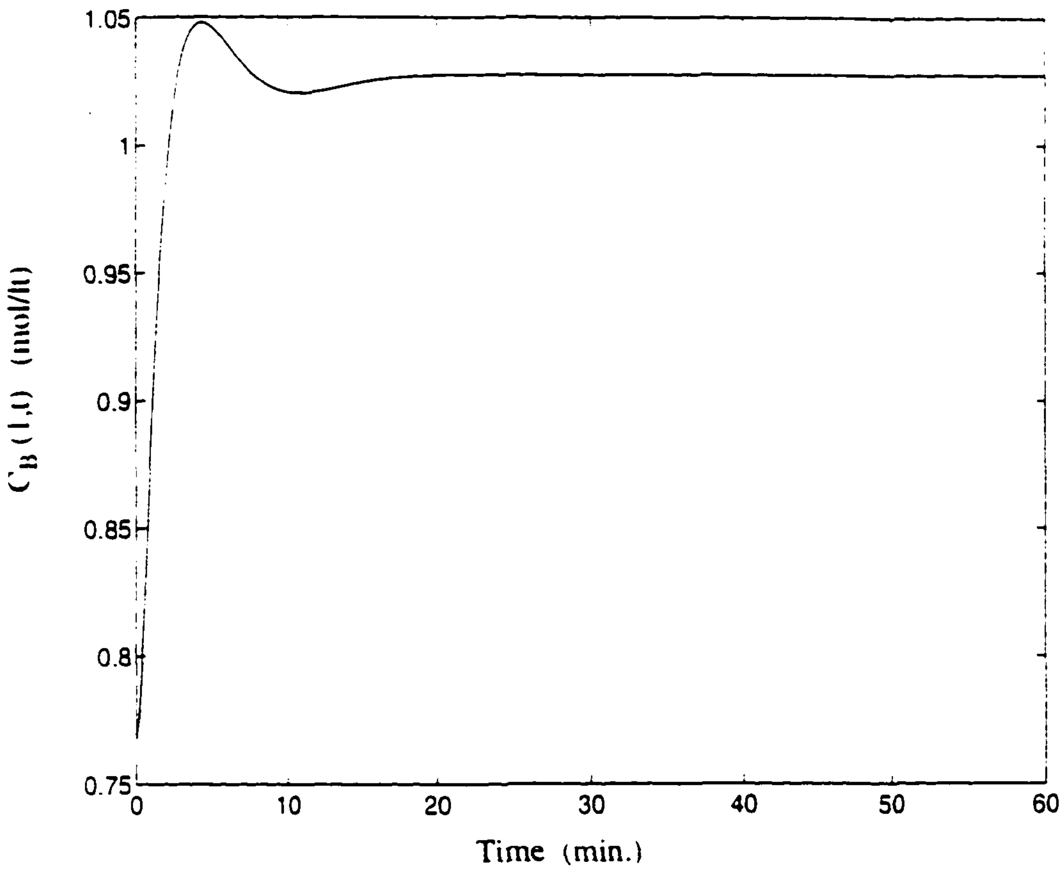


Figure 6.10: Profile of concentration of species B in the outlet of the reactor-discretization based controller.

6.7.4 Practical implementation issues

The distributed output feedback controller of Eq.6.52 assumes that the jacket temperature can be manipulated directly. In practice, the jacket temperature is usually manipulated indirectly through the jacket inlet flow rate. This can be achieved in a straightforward way by designing a controller to ensure that the jacket temperature obtains the values requested by the distributed controller of Eq.6.52.

Specifically, under the assumption of perfect mixing and constant volume, the dynamic model of the jacket takes the form:

$$\begin{aligned} \frac{dT_j^i}{dt} &= (T_{j0}^i - T_j^i)\frac{F_{js}^i}{V_j^i} + \frac{U_w}{\rho_j\, c_{pj}\, V_j^i}\left(\int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1}-z_i} T_r(z,t)dz - T_j^i\right) \\ &+ \frac{(T_{j0}^i - T_j^i)}{V_j^i} u_{fl}^i, \quad i = 1,\dots,5 \end{aligned} \tag{6.54}$$

where F_{js}^i is the steady-state jacket inlet flow rate, V_j^i is the jacket volume, ρ_j, c_{pj} are the density and heat capacity of the fluid in the jacket, T_{j0}^i is the temperature of the inlet stream to a jacket, and u_{fl}^i is the jacket inlet flow rate (chosen as the new manipulated input) in deviation variable form. Requesting a first-order response of the form:

$$\gamma_{jc}\frac{dT_j^i}{dt} + T_j^i = \bar{u}^i \tag{6.55}$$

where γ_{jc} is the time constant, the necessary controller takes the form:

$$u_{fl}^i = \frac{(\bar{u}^i - T_j^i)V_j^i - \gamma_{jc}\left[(T_{j0}^i - T_j^i)F_{js}^i + \dfrac{U_w}{\rho_j\, c_{pj}}\left(\displaystyle\int_{z_i}^{z_{i+1}} \frac{1}{z_{i+1}-z_i} T_r(z,t)dz - T_j^i\right)\right]}{\gamma_{jc}(T_{j0}^i - T_j^i)} \tag{6.56}$$

The parameter γ_{jc} should be chosen such that the response of Eq.6.55 is sufficiently fast compared to the response of Eq.6.51, while estimates of $T_r(z,t)$ can be obtained from the state observer of Eq.6.52.

The performance of the control scheme resulting from the combination of this controller with the distributed controller of Eq.6.52 was evaluated through simulations on the plug-flow reactor. The values used for the parameters and the steady-state values of the jacket inlet flow rate F^i_{js}, $i = 1, \ldots, 5$, are given in Table 6.1. Figure 6.11 shows the output profiles for the same 30 % increase in the value of the reference inputs as previously (the profiles for the jacket inlet flow rate are displayed in figure 6.12). Clearly, the performance of the control scheme is excellent, enforcing closed-loop output responses which are very close to the ones obtained by neglecting the jacket dynamics (compare with Figure 6.5).

6.8 Conclusions

In this chapter, we developed an output feedback control methodology for systems described by quasi-linear first-order hyperbolic partial differential equations, for which the manipulated input, the controlled output and the measurable output are distributed in space. The central idea of our approach is the combination of theory of partial differential equations and concepts from geometric control. Initially, a concept of characteristic index was introduced and used for the synthesis of distributed state feedback controllers that guarantee output tracking in the closed-loop system. Conditions that ensure exponential stability of the closed-loop system were derived. Analytical formulas of output feedback controllers were also derived through combination of suitable state observers with the developed distributed state feedback controllers. The proposed control methodology was implemented, through simulations, on a nonisothermal plug-flow reactor, modeled by three quasi-linear hyperbolic PDEs. Comparisons with a control method that involves discretization in space of the original PDE model and application of standard nonlinear control methods for ODE systems established that the new control method yields superior performance.

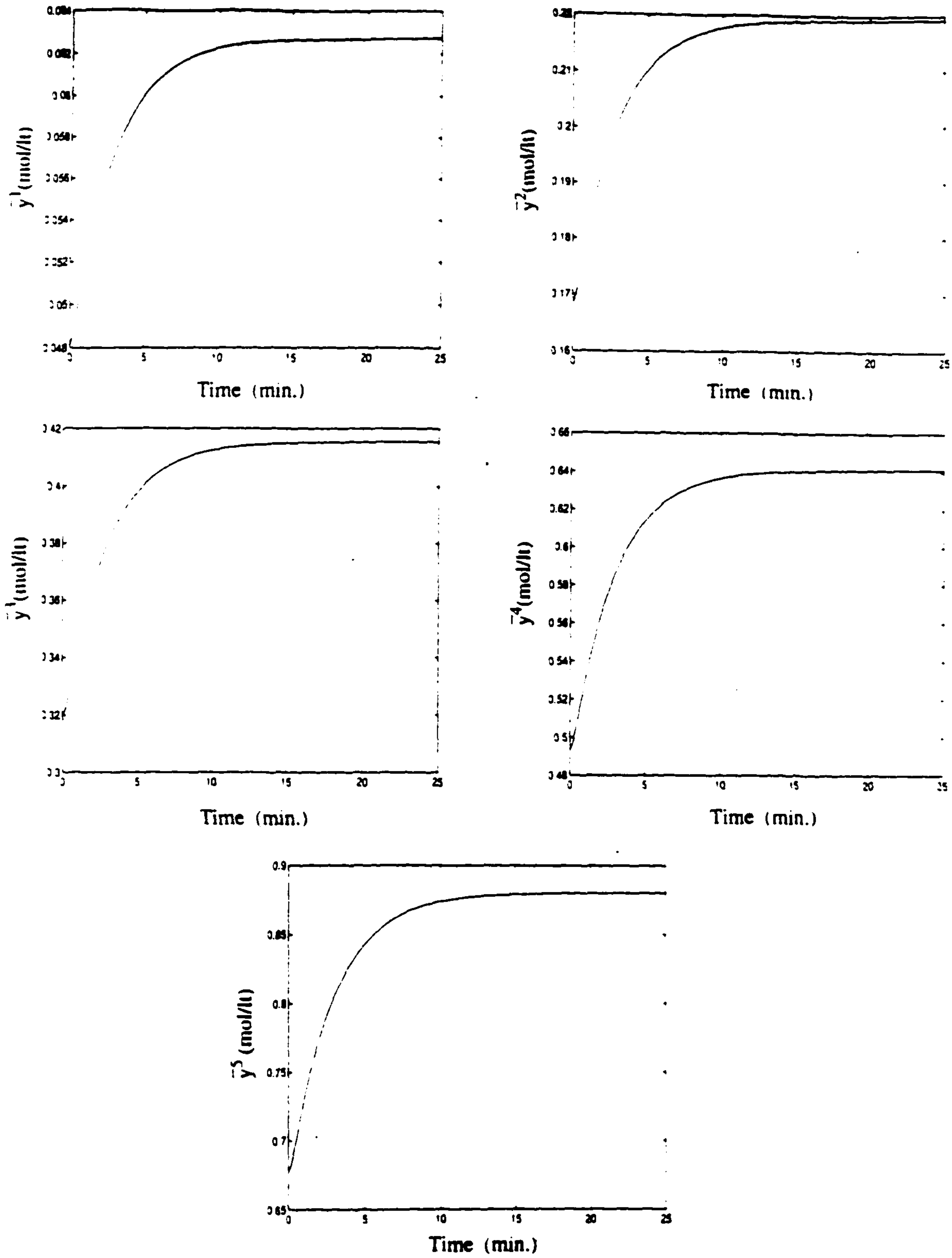


Figure 6.11: Closed-loop output profiles-the jacket inlet flow rates are used as the manipulated inputs.

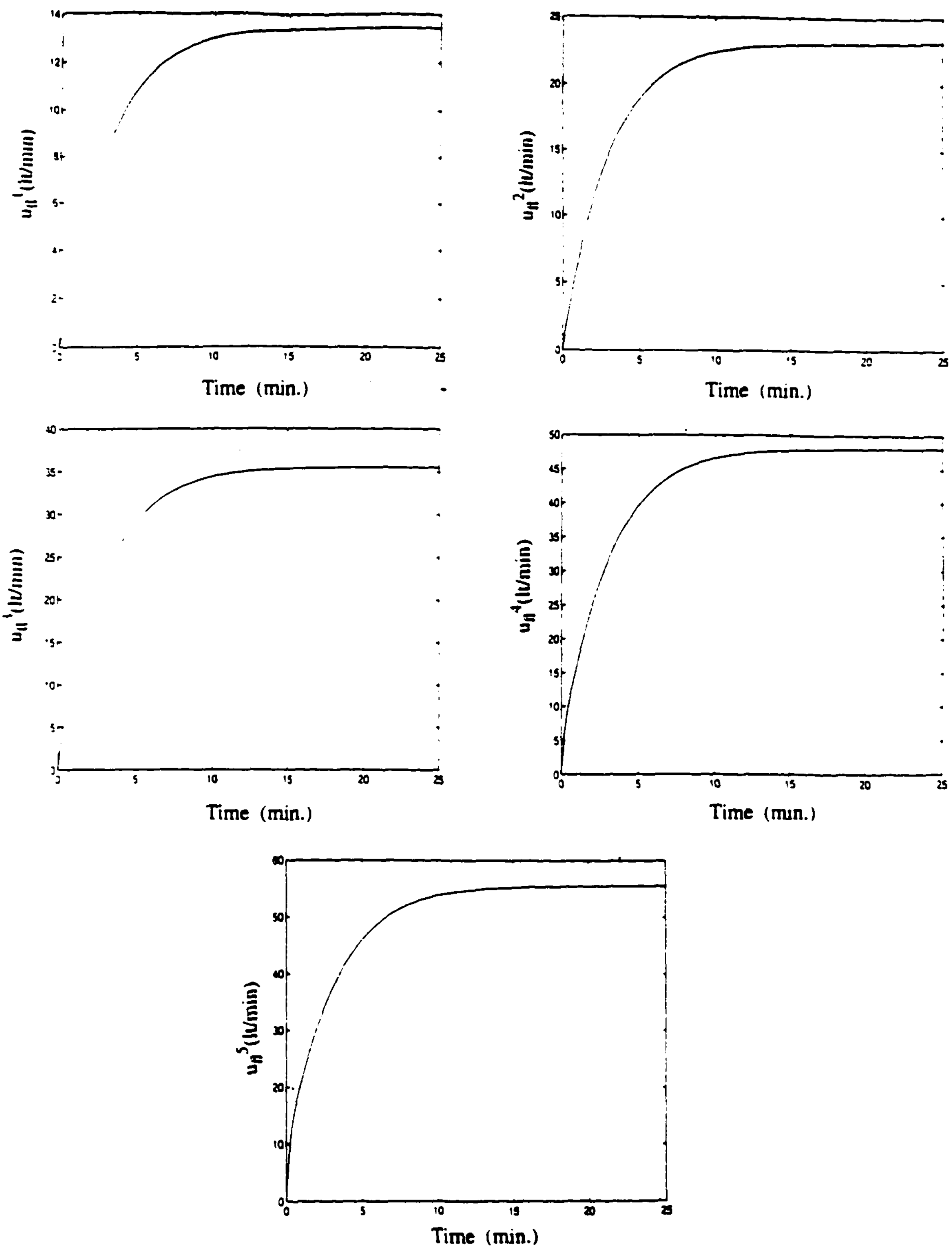


Figure 6.12: Profiles of the jacket inlet flow rates.

Notation

Roman Letters

$A, A(x), A(z)$	=	matrices
$b^i(z)$	=	distribution function of i-th actuator
$c^i(z)$	=	performance specification function for i-th output
C_{A0}	=	inlet concentration of the species A
C_1, C_2	=	constant matrices
C_A	=	concentration of the species A
C_B	=	concentration of the species B
c_{pm}	=	heat capacity of the reactor fluid
c_{pj}	=	heat capacity of the jacket fluid
E_1, E_2	=	activation energies
f	=	vector field
F	=	outlet flow rate
F_1	=	inlet flow rate
F_{js}^i	=	steady-state value for inlet flow rate to the i-th jacket
g	=	vector field associated with the manipulated variable
h	=	controlled output scalar field
k_1, k_2	=	pre-exponential constants
l	=	total number of control actuators
p	=	measurable output scalar field
T_{A0}	=	inlet temperature of the species A
T_{j0}^i	=	inlet temperature of the fluid in the jacket
T_j^i	=	temperature of the jacket
T_r	=	temperature of the reactor
T_w	=	temperature of the wall
t	=	time
U_w	=	heat transfer coefficient
u	=	manipulated variable
$\bar{u}$	=	manipulated input vector
$\bar{u}^i$	=	i-th manipulated input
V_j^i	=	volume of the jacket
V_r	=	volume of the reactor
v	=	reference input vector
v^i	=	reference input for the i-th actuator
v_l	=	fluid velocity
x	=	vector of state variables
y	=	controlled variable
$\bar{y}$	=	controlled output
z	=	spatial coordinate

Greek Letters

γ_k, γ_{jc}	=	adjustable parameters
$\Delta H_1, \Delta H_2$	=	enthalpy of the reactions
η	=	observer state vector
ρ_m	=	density of the fluid in the reactor
ρ_j	=	density of the fluid in the jacket
σ^i	=	characteristic index of $\bar{y}^i$ with respect to $\bar{u}^i$
σ	=	characteristic index of $\bar{y}$ with respect to $\bar{u}$

Math Symbols

$\mathcal{C}^i$	=	bounded linear operator
$\mathcal{H}$	=	infinite dimensional Hilbert space
$L_f h$	=	Lie derivative of a scalar field h with respect to the vector f
$L_f^k h$	=	k-th order Lie derivative
$L_g L_f^{k-1} h$	=	mixed Lie derivative
$\mathcal{P}$	=	bounded linear operator
$\bar{\mathcal{P}}$	=	bounded linear operator
$\mathcal{Q}$	=	bounded linear operator
$\mathcal{R}$	=	bounded linear operator
$\mathbb{R}$	=	real line
$\mathbb{R}^i$	=	$i-$dimensional Euclidean space
$\in$	=	belongs to
T	=	transpose
$\lvert \cdot \rvert$	=	standard Euclidean norm
$\lVert \cdot \rVert_2$	=	2-norm in $\mathcal{H}$
$(\cdot, \cdot)_{\mathbb{R}^n}$	=	inner product in $\mathbb{R}^n$

Chapter 7

Robust Control of Hyperbolic PDE Systems

7.1 Introduction

One of the most important theoretical and practical problems in control is the one of designing a controller that compensates for the presence of mismatch between the model used for controller design and the actual process model. Typical sources of model uncertainty include unknown or partially known time-varying process parameters. exogenous disturbances and unmodeled dynamics. It is well-known that the presence of uncertain variables and unmodeled dynamics. if not taken into account in the controller design. may lead to severe deterioration of the nominal closed-loop performance or even to closed-loop instability.

Research on robust control of PDEs with uncertain variables has been limited to linear systems. For linear parabolic PDEs, the problem of complete elimination of the effect of uncertain variables on the output via distributed state feedback (known as disturbance decoupling) was solved in [Cur84, Cur86]. Research on the problem of robust stabilization of linear PDE systems with uncertain variables led to the development of H^∞ control methods in the frequency-domain (e.g. [CG86, PO87, GX89]). The derivation of concrete relations between frequency domain and state-space concepts for a wide class of PDE systems in [JN88] motivated research on the development of the state-space counterparts of the H^∞ results for linear PDE systems (see [Keu93, BK91] for example). Within a state-space framework, an alternative approach for the design of controllers for linear PDE systems, that deals explicitly with time-invariant uncertain variables, involves the use of adaptive control methods [WB89, HB94, Dem94, Bal95].

On the other hand, the problem of robustness of control methods for PDE systems with respect to unmodeled dynamics is typically studied within the singular perturbation framework (e.g., [WB89]). This formulation involves synthesizing a controller on the basis of a reduced-order PDE model (which captures the dominant (slow) dynamics of the process) and deriving conditions under which the same controller stabilizes the actual closed-loop system (which includes the unmodeled dynamics). This approach was employed in [WB89] to establish robustness of a class of finite-dimensional adaptive controllers, which asymptotically stabilize a PDE system with time-invariant uncertain variables, with respect to unmodeled dynamics, provided that they are stable and sufficiently fast. Furthermore, the singular perturbation framework allows studying the control and order reduction of PDE systems with fast and slow dynamics. Soliman and Ray [SR79] utilized singular perturbation techniques to design well-conditioned state estimators for two-time-scale parabolic PDE systems, while Dochain and Bouaziz [DB93] recently used singular perturbations to reduce the parabolic PDE model that describes a fixed-bed reactor with strong diffusive phenomena to an ODE one.

In this chapter, we consider systems of first-order hyperbolic PDEs with uncertainty, for which the manipulated input and the controlled output are distributed in space. The objective is to develop a framework for the synthesis of distributed robust controllers that handle explicitly time-varying uncertain variables and unmodeled dynamics. For systems with uncertain variables, the problem of complete elimination of the effect of uncertainty on the output via distributed feedback is initially considered: a necessary and sufficient condition for its solvability as well as explicit controller synthesis formulas are derived. Then, a distributed robust controller is derived that guarantees boundedness of the state and achieves asymptotic output tracking with arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the output of the closed-loop system. The controller is designed constructively using Lyapunov's direct method and requires that there exist known bounding functions that capture the magnitude of the uncertain terms and a matching condition is satisfied. The problem of robustness with respect to unmodeled dynamics is then addressed within the context of control of two-time-scale systems modeled in singularly perturbed form. Initially, a robustness result of the bounded stability property of a reduced-order PDE model with respect to stable and fast dynamics is proved. This result is then used to establish that the controllers which are synthesized on the basis of a reduced-order slow model, and achieve uncertainty decoupling or uncertainty attenuation, continue to enforce these control objectives in the full-order closed-loop system, provided that the unmodeled dynamics are stable and sufficiently fast. The developed control method is tested through simulations on a nonisothermal fixed-bed reactor, where the reactant wave propagates through the bed with significantly larger speed than the heat wave, and the heat of reaction is unknown.

7.2 First-order hyperbolic PDEs

7.2.1 Description of hyperbolic PDEs with uncertain variables

We will focus on systems of quasi-linear hyperbolic first-order PDEs in one spatial variable, with the following state-space description:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\bar{u} + W(x)r(z)\theta(t) \\ \bar{y}_s &= Ch(x) \end{aligned} \tag{7.1}$$

subject to the boundary condition:

$$C_1 x(\alpha, t) + C_2 x(\beta, t) = R(t) \tag{7.2}$$

and the initial condition:

$$x(z, 0) = x_0(z) \tag{7.3}$$

In the above description, $x(z,t) = [x_1(z,t) \ \cdots \ x_n(z,t)]^T$ denotes the vector of state variables, $x(z,t) \in \mathcal{H}^n[(\alpha,\beta), \mathbb{R}^n]$, with $\mathcal{H}^n$ being the infinite-dimensional Hilbert space of n-dimensional vector functions defined on the interval $[\alpha, \beta]$ whose spatial derivatives up to n-th order are square integrable, and $z \in [\alpha, \beta] \subset \mathbb{R}$ and $t \in [0, \infty)$, denote position and time, respectively. $\bar{u} = [\bar{u}^1 \ \cdots \ \bar{u}^l]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, $\theta = [\theta_1 \ \cdots \theta_q] \in \mathbb{R}^q$ denotes the vector of uncertain variables, which may include uncertain process parameters or exogenous disturbances, $\bar{y}_s = [\bar{y}_s^1 \ \cdots \ \bar{y}_s^l]^T \in \mathbb{R}^l$ denotes the vector of controlled outputs. Figure 7.1 shows the location of the manipulated inputs and controlled outputs in the case of a prototype example. $A(x), W(x)$ are sufficiently smooth matrices of appropriate dimensions, $f(x)$ and $g(x)$ are sufficiently smooth vector functions, $h(x)$ is a sufficiently smooth scalar function, C_1, C_2 are constant matrices of appropriate dimensions, $R(t)$ is a time-varying column vector, and $x_0(z) \in \mathcal{H}[(\alpha,\beta), \mathbb{R}^n]$, with $\mathcal{H}[(\alpha,\beta), \mathbb{R}^n]$ being the Hilbert space of n-dimensional vector functions which are square integrable on the

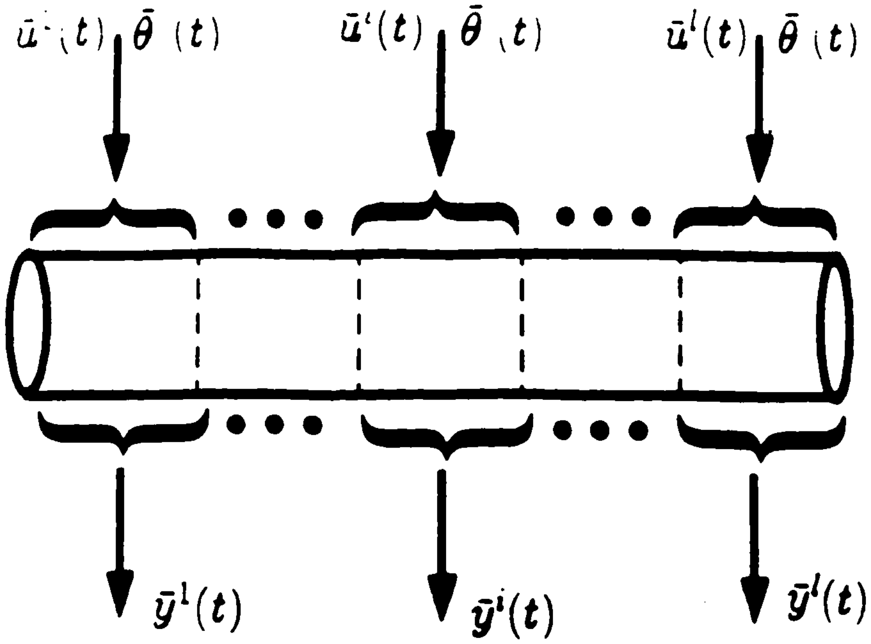


Figure 7.1: Specification of the control problem in a prototype example.

interval $[\alpha, \beta]$. $b(z)$ is a known smooth vector function of the form:

$$b(z) = [(H(z-z_1)-H(z-z_2))\,b^1(z) \;\cdots\; (H(z-z_l)-H(z-z_{l+1}))\,b^l(z)] \tag{7.4}$$

where H denotes the standard Heaviside function and $b^i(z)$ describes how the control action $\bar{u}^i(t)$ is distributed in the space interval $[z_i, z_{i+1}]$. $r(z)$ is a known matrix whose $(i,k)-th$ element is of the form $r_k^i(z)$, where the function $r_k^i(z)$ specifies the position of action of the uncertain variable θ_k on $[z_i, z_{i+1}]$. $\mathcal{C}$ is a bounded linear operator, mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, of the form:

$$\mathcal{C} = [(H(z-z_1)-H(z-z_2))\,\mathcal{C}^1 \;\cdots\; (H(z-z_l)-H(z-z_{l+1}))\,\mathcal{C}^l]^T \tag{7.5}$$

where the operator $\mathcal{C}^i$ depends on the desired performance specifications and in most practical applications is assumed to be of the form:

$$\bar{y}_s^i(t) = \mathcal{C}^i h(x) = \int_{z_i}^{z_{i+1}} c^i(z) h(x(z,t))dz \tag{7.6}$$

where $c^i(z)$ is a known smooth function. For simplicity, the functions $b^i(z), c^i(z), r_k^i(z)$, $i = 1, \ldots, l$, are assumed to be normalized in the interval $[\alpha, \beta]$, i.e., $\sum_{i=1}^{l}\int_{z_i}^{z_{i+1}} b^i(z)dz = \sum_{i=1}^{l}\int_{z_i}^{z_{i+1}} c^i(z)dz = \sum_{i=1}^{l}\int_{z_i}^{z_{i+1}} r_k^i(z)dz = 1$.

The quasi-linear hyperbolic PDE system of Eq.7.1, with $\theta(t) \equiv 0$, describes the majority of convection-reaction processes arising in chemical engineering [RAA86] and is a natural generalization of linear models considered in [Ray81, CD95] in the context of linear distributed state estimation and control. The assumption of affine and separable appearance of $\bar{u}$ and θ is a standard one in uncertainty decoupling and robust control studies for linear PDEs (e.g., [Cur84, Cur86]), and is satisfied in most practical applications, where the jacket temperature is usually chosen to be the manipulated input (see for example [Ray81, CD95]), and the heat of reactions, the pre-exponential constants, the temperature and concentrations of lateral inlet streams, etc. are typical uncertain variables.

7.2.2 Preliminaries

In this subsection, we review a basic stability result (converse Lyapunov theorem) for systems of the form of Eq.7.1 and introduce a concept of characteristic index between the output $\bar{y}^i$ and the uncertainty vector θ that will be used in our development.

Theorem 7.1 [Wan66a] (Converse Lyapunov theorem): *Consider the system of Eq.7.1, with* $(|\bar{u}| = |\theta| = 0)$, *and assume that the operator of Eq.6.14 generates a locally exponentially stable semigroup. Then, there exist a smooth functional* V : $\mathcal{H}^n \times [\alpha, \beta] \rightarrow \mathrm{IR}_{\geq 0}$ *of the form:*

$$V(t) = \int_a^b x^T q(z) x dz \tag{7.7}$$

where $q(z)$ *is a known positive definite function satisfying* $\sum_{i=1}^{l} \int_{z_i}^{z} \sum \int_{z_i}^{z} z) dz = 1$, *and a set of positive real numbers* a_1, a_2, a_3, a_4, a_5 *such that if* $||x||_2 \leq a_5$ *the following conditions hold:*

$$\begin{aligned}
& a_1 ||x||_2^2 \leq V(t) \leq a_2 ||x||_2^2 \\
& \frac{dV}{dt} = \frac{\partial V}{\partial x}\left[A(x)\frac{\partial x}{\partial z} + f(x)\right] \leq -a_3 ||x||_2^2 \\
& ||\frac{\partial V}{\partial x}||_2 \leq a_4 ||x||_2
\end{aligned} \tag{7.8}$$

Remark 7.1: Note that, for infinite-dimensional systems, stability with respect to one norm does not necessarily imply stability with respect to another norm. This difficulty is not encountered in finite-dimensional systems since all norms defined in a finite dimensional vector space are equivalent. In our case, we choose to study stability with respect to the L_2-norm since for hyperbolic systems it represents a measure of the total energy of the system at any time, and thus exponential stability with respect to this norm implies that the system's total energy tends to zero as $t \rightarrow \infty$ (i.e., since $V(t)$ is a quadratic function: $V(t) \rightarrow 0$ as $t \rightarrow \infty \Longrightarrow$ exponential stability).

We now define a concept of characteristic index between the output $\bar{y}^i$ and the uncertainty vector θ, for systems of the form of Eq.7.1 that will be used to express the solvability condition of the uncertainty decoupling problem via distributed state feedback and the matching condition in the robust uncertainty attenuation problem.

Definition 7.1: *Referring to the system of Eq.7.1, we define the characteristic index of the output $\bar{y}_s^i$ with respect to the vector of uncertainties θ as the smallest integer δ^i for which, there exists $k \in [1, q]$ such that:*

$$C^i L_{W_k} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\delta^i - 1} h(x) r_k^i(z) \neq 0 \tag{7.9}$$

where W_k denotes the k-th column vector of the matrix W, or $\delta^i = \infty$ if such an integer does not exist.

From the above definition, it follows that δ^i depends on the structure of the process (matrices $A(x), W(x)$ and functions $f(x)$, $h(x)$), as well as on the position of action of the uncertain variables θ (functions $r_k^i(z)$) and the function $c^i(z)$. In order to simplify the statement of our results, we define the characteristic index of the output vector $\bar{y}$ with respect to the vector of uncertainties θ as $\delta = \min\{\delta^1, \delta^2, \dots, \delta^l\}$.

7.3 Uncertainty decoupling

In this section, we consider systems of the form of Eq.7.1 and address the problem of synthesizing a distributed state feedback controller that stabilizes the closed-loop system and forces the output to track the external reference input in a prespecified manner for all times, independently of the uncertain variables. More specifically, we consider control laws of the form:

$$\bar{u} = \mathcal{S}(x) + s(x)v \tag{7.10}$$

where $\mathcal{S}(x)$ is a smooth nonlinear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, $s(x)$ is an invertible matrix of smooth functionals, and $v \in \mathbb{R}^l$ is the vector of external reference inputs. Under the control law of Eq.7.10, the closed-loop system takes the form:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\mathcal{S}(x) + g(x)b(z)s(x)v + W(x)r(z)\theta(t) \\ \bar{y}_s &= Ch(x) \end{aligned} \tag{7.11}$$

It is clear that feedback laws of the form of Eq.7.10 preserve the linearity with respect to the external reference input v. We also note that the evolution of the closed-loop system of Eq.7.11 is locally governed by a strongly continuous semigroup of bounded linear operators, because $b(z)\mathcal{S}(x), b(z)s(x), r(z)\theta(t)$ are bounded, finite dimensional perturbations, ensuring that the solution of the system of Eq.7.11 is well-defined. Proposition 7.1 that follows allows specifying the order of the input/output response in the nominal closed-loop system (the proof can be found in the appendix F).

Proposition 7.1: *Consider the system of Eq.7.1, for which assumption 6.1 holds, subject to the distributed state feedback law of Eq.7.10. Then, referring to the closed-loop system of Eq.7.11:*

a) the characteristic index of $\bar{y}_s$ with respect to the external reference input v is equal to σ, and

b) the characteristic index of $\bar{y}_s$ with respect to θ is equal to δ, if $\delta \leq \sigma$.

The fact that the characteristic index between the output $\bar{y}$ and the external reference input v is equal to σ suggests requesting the following input/output response for the closed-loop system:

$$\gamma_\sigma \frac{d^\sigma \bar{y}_s}{dt^\sigma} + \cdots + \gamma_1 \frac{d\bar{y}_s}{dt} + \bar{y}_s = v \tag{7.12}$$

where $\gamma_1, \gamma_2, \cdots, \gamma_\sigma$ are adjustable parameters which can be chosen to guarantee input/output stability in the closed-loop system. Referring to Eq.7.12, note that, motivated by physical arguments, we request, for each pair $(\bar{y}_s^i, v^i)$, $i = 1, \ldots, l$, an input/output response of order σ with the same transient characteristics (i.e. the parameters γ_k are chosen to be the same for each pair $(\bar{y}_s^i, v^i)$). The following theorem provides the main result of this section (the proof is given in the appendix F).

Theorem 7.2: *Consider the system of Eq.7.1 for which assumption 6.1 holds, and assume that: i) the roots of the equation* $1 + \gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0$ *lie in the open left-half of the complex plane, ii) the zero dynamics (Eq.6.35) is locally exponentially stable, and iii) there exist a positive real number* $\hat{\delta}$ *such that* $\max\{||\theta||, ||\dot{\theta}||\} \leq \hat{\delta}$. *Then, the condition* $\sigma < \delta$ *is necessary and sufficient in order for a distributed state feedback law of the form of Eq.7.10 to completely eliminate the effect of* θ *on* $\bar{y}_s$ *in the closed-loop system. Whenever this condition is satisfied, the control law:*

$$\begin{aligned} \bar{u} \;=\; & \left[\gamma_\sigma C L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\sigma-1} h(x) b(z)\right]^{-1} \\ & \bullet \left\{ v - Ch(x) - \sum_{\nu=1}^{\sigma} \gamma_\nu C \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f\right)^{\nu} h(x) \right\} \end{aligned} \tag{7.13}$$

a) guarantees boundedness of the state of the closed-loop system,

b) enforces the input/output response of Eq.7.12 in the closed-loop system.

Remark 7.2: Referring to the solvability condition of the problem, notice that it depends not only on the structural properties of the process but also on the shape of the actuator distribution functions $b(z)$, the position of action of the vector of uncertain variables (matrix function $r(z)$) and the performance specification functions $c(z)$, since σ and δ depend on $c(z), b(z)$ and $c(z), r(z)$, respectively. This implies that it is generally possible to select $(b(z), c(z))$ to allow achieving uncertainty decoupling via distributed state feedback (see illustrative example in remark 7.5 below).

Remark 7.3: The solvability condition of the problem and the distributed state feedback controller of Eq.7.13 were derived following an approach conceptually similar to the one used to solve the counterpart of this problems in the case of ODE systems (e.g. [DC95]). We note that this is possible, because, for the system of Eq.7.1: a) the input/output spaces are finite dimensional; and b) the input/output operators (functions $c^i(z), b^i(z)$) are bounded.

Remark 7.4: We note that the distributed state feedback controller of Eq.7.13 that solves the uncertainty decoupling problem is identical to the one derived in [CD95] which solves the output tracking problem via distributed state feedback, while the conditions i) and ii) of theorem 7.2, as proved in [CD95], ensure that the controller of Eq.7.13 exponentially stabilizes the nominal closed-loop system ($\theta(t) \equiv 0$).

Remark 7.5 (Illustrative example): Consider a plug-flow reactor with lateral feed, where a first-order reaction of the form $A \rightarrow B$ takes place. Assuming that the lateral flow rate is significantly smaller than the inlet flow rate (i.e. $v_l \gg F$, where v_l denotes the velocity of the inlet stream to the reactor and F denotes the lateral flow rate per unit volume), the dynamic model of the process is given by:

$$
\begin{aligned}
\frac{\partial C_A}{\partial t} &= -v_l \frac{\partial C_A}{\partial z} - kC_A + F(C_{Al} - C_A) \\
\frac{\partial T}{\partial t} &= -v_l \frac{\partial T}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}} kC_A + F(T_l - T) + \frac{U_w}{\rho_f c_{pf} V_r}(T_j - T) \\
& \qquad C_A(0,t) = C_{A0}, \;\; T(0,t) = T_0
\end{aligned}
\tag{7.14}
$$

where $C_A(z,t)$ and $T(z,t)$ denote the concentration of the species A and the temperature in the reactor, $z \in [0,1]$, $T_j(z,t)$ denotes the wall temperature, C_{Al}, T_l denote the concentration and the temperature of the lateral inlet stream, k denotes the reaction rate constant, ΔH denotes the enthalpy of the reaction, ρ_f, c_{pf} denote the density and heat capacity of the reacting liquid, U_w denotes the heat transfer coefficient, and V_r denotes the volume of the reactor. Set $x_1 = C_A$, $x_2 = T$, $u = T_j - T_{js}$, $\theta_1 = T_l - T_{ls}$, $\theta_2 = C_{Al} - C_{Als}$, $y = T$, $R_1(t) = C_{A0}(t)$ and $R_2(t) = T_0(t)$, where T_{js}, T_{ls}, C_{Als} denote steady-state values. Assume that for the control of the system there is available one control actuator with distribution function $b(z)$; let also $r_1(z)$ and $r_2(z)$ determine the position of action of θ_1 and θ_2, respectively, and $\bar{y}(t) = \int_0^1 c(z)x_2(z,t)dz$; then the system of Eq.7.14 can be written as:

$$
\begin{aligned}
\frac{\partial x_1}{\partial t} &= -v_l\frac{\partial x_1}{\partial z} - kx_1 + F(C_{Als} - x_1) + F\theta_2 \\
\frac{\partial x_2}{\partial t} &= -v_l\frac{\partial x_2}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}}kx_1 + \frac{U_w}{\rho_f c_{pf} V_r}(T_{js} - x_2) + F(T_{ls} - x_2) \\
&\quad + \frac{U_w}{\rho_f c_{pf} V_r} b(z)\bar{u} + Fr_1(z)\theta_1(t) \\
\bar{y}(t) &= \int_0^1 c(z)x_2(z,t)dz, \quad x_1(0,t) = R_1, \; x_1(0,t) = R_2
\end{aligned}
\tag{7.15}
$$

Performing a time-differentiation of the output $\bar{y}(t)$, we have:

$$
\begin{aligned}
\frac{d\bar{y}}{dt} &= \int_0^1 c(z)\left(-v_l\frac{\partial x_2}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}}kx_1 + \frac{U_w}{\rho_f c_{pf} V_r}(T_{js} - x_2) + F(T_{ls} - x_2)\right)dz \\
&\quad + \frac{U_w}{\rho_f c_{pf} V_r}\int_0^1 c(z)b(z)\bar{u}dz + F\int_0^1 c(z)r_1(z)\theta_1(t)dz
\end{aligned}
\tag{7.16}
$$

From the above equation and the result of theorem 7.2, it follows that if $(c(z), b(z)) = \int_0^1 c(z)b(z)dz \neq 0$ (in which case $\sigma = 1$), then the uncertainty decoupling problem via distributed state feedback is solvable as long as $\sigma < \delta$, that is when $(c(z), r_1(z)) = \int_0^1 c(z)r_1(z)dz = 0$. Whenever this condition holds, the necessary controller can be derived from the synthesis formula of Eq.7.13.

7.4 Robust control: uncertain variables

In this section, we consider systems of the form of Eq.7.1 for which the condition $\sigma < \delta$ is not necessarily satisfied and address the problem of synthesizing a distributed robust nonlinear controller of the form:

$$
\bar{u} = \bar{\mathcal{S}}(x) + \bar{s}(x)v + \bar{\mathcal{R}}(x,t) \tag{7.17}
$$

where $\bar{\mathcal{S}}(x)$ is a smooth nonlinear operator mapping $\mathcal{H}^n$ into $\mathbb{R}^l$, $\bar{s}(x)$ is an invertible matrix of smooth functionals, where $\bar{\mathcal{R}}(x,t)$ is a vector of nonlinear functionals, that guarantees closed-loop stability, enforces asymptotic output tracking, and achieves arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the output of the closed-loop system. The control law of Eq.7.17 consists of two components, the component $\bar{\mathcal{S}}(x) + \bar{s}(x)v$ which is used to enforce output tracking and stability in the nominal closed-loop system, and the component $\bar{\mathcal{R}}(x,t)$ which is used to asymptotically attenuate the effect of $\theta(t)$ on the output of the closed-loop system. The design of $\bar{\mathcal{S}}(x) + \bar{s}(x)v$ will be performed employing the geometric approach presented in the previous section, while the design of $\bar{\mathcal{R}}(x,t)$ will be performed constructively using Lyapunov's direct method. The central idea of Lyapunov-based controller design is to construct $\bar{\mathcal{R}}(x,t)$, using the knowledge of bounding functions that capture the size of the uncertain terms and assuming a certain path under which the uncertainty may affect the output, so that the ultimate discrepancy between the output of the closed-loop system and the reference input can be made arbitrarily small by a suitable selection of the controller parameters.

We will assume that there exists a known smooth (not necessarily small) vector function, which captures the magnitude of the vector of uncertain variables θ for all times. Information of this kind is usually available in practice, as a result of experimental data, preliminary simulations, etc. Assumption 7.1 states precisely our requirement.

Assumption 7.1: *There exists a known vector $\bar{\theta}(t) = [\bar{\theta}_1(t) \ \cdots \ \bar{\theta}_q(t)]$, whose elements are continuous positive definite functions defined on $t \in [0,\infty)$ such that:*

$$|\theta_k(t)| \leq \bar{\theta}_k(t), \quad k = 1,\ldots,q \tag{7.18}$$

Assumption 7.2 that follows characterizes precisely the class of systems of the form of Eq.7.1 for which our robust control methodology is applicable. It determines the path under which the vector of uncertain variables $\theta(t)$ may affect the output of the closed-loop system. More specifically, we assume that the vector of uncertain variables $\theta(t)$ enters the system in such a manner such that the expressions for the time-derivatives of the output $\bar{y}$ up to order $\sigma - 1$ are independent of θ. This assumption is significantly weaker than the solvability condition required for the uncertainty decoupling problem (cf. theorem 7.2) and is satisfied by many convection-reaction processes of practical interest (including the plug-flow and fixed-bed reactor examples studied in this chapter).

Assumption 7.2: $\sigma \leq \delta$.

Theorem 7.3 provides the main result of this section (the proof is given in the appendix F).

Theorem 7.3: *Consider the system of Eq.7.1, for which assumptions 6.1, 6.2, 7.1 and 7.2, and the conditions (i, ii, iii) of theorem 7.2 hold, under the distributed state feedback law of the form:*

$$\begin{aligned}
\bar{u} \;=\; & \left[\mathcal{C} L_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma - 1} h(x) b(z) \right]^{-1} \\
& \bullet \left\{ \gamma_0 \left[\frac{\gamma_1}{\gamma_\sigma} v - \sum_{\nu=1}^{\sigma-1} \frac{\gamma_\nu}{\gamma_\sigma} \mathcal{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\nu - 1} h(x) \right] \right. \\
& - \sum_{\nu=1}^{\sigma-1} \frac{\gamma_\nu}{\gamma_\sigma} \mathcal{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\nu} h(x) \\
& \left. - 2K(t)\Delta(x, \phi) \sum_{\nu=1}^{\sigma-1} \frac{\gamma_\nu}{\gamma_\sigma} \mathcal{C} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\nu - 1} h(x) - \frac{\gamma_1}{\gamma_\sigma} v \right\}
\end{aligned} \tag{7.19}$$

where ϕ, γ_k are adjustable parameters, and γ_k are chosen so that $\gamma_0 > 0$ and the polynomial $s^{\sigma-1} + \frac{\gamma_{\sigma-1}}{\gamma_\sigma} s^{\sigma-2} + \cdots + \frac{\gamma_2}{\gamma_\sigma} s + \frac{\gamma_1}{\gamma_\sigma} = 0$ is Hurwitz, $\Delta(x, \phi)$ is an $l \times l$ diagonal matrix whose $(i, i) - th$ is of the form

$$\Delta(x, \phi) = \left(\left| \sum_{\nu=1}^{\sigma-1} \frac{\gamma_\nu}{\gamma_\sigma} \mathcal{C}^i \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\nu - 1} h(x) - \frac{\gamma_1}{\gamma_\sigma} v^i \right| + \phi \right)^{-1}$$ *, and the*

time-varying gain $K(t)$ is of the form:

$$K(t) = \left[\sum_{k=1}^{q} CL_{W_k} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma - 1} h(x) r_k(z) \bar{\theta}_k(t) \right] . \tag{7.20}$$

Then, for each positive real number d, there exist a pair of positive real numbers $(\tilde{\delta}, \phi^*)$ such that

if $\max\{||x_0||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$ and $\phi \in (0, \phi^*]$, then:

a) the state of the closed-loop system is bounded, and

b) the output of the closed-loop system satisfies:

$$\lim_{t \to \infty} |\bar{y}^i - v^i| \leq d, \quad i = 1, \ldots, l \tag{7.21}$$

Remark 7.6: The calculation of the control action from the controllers of Eqs.7.13-7.19 requires algebraic manipulations as well as differentiations and integrations in space, which is expected because of their distributed nature.

Remark 7.7: The robust nonlinear controller of Eq.7.19 guarantees an arbitrary degree of asymptotic attenuation of the effect of a large class of uncertain *time − varying* variables on the output (those that satisfy assumption 7.2). In many practical applications, there exist unknown time-invariant process parameters which may not necessarily satisfy assumption 7.2. Since the manipulated input and controlled output spaces are finite-dimensional, the asymptotic rejection (in the sense that Eq.7.21 holds) of such constant uncertainties can be achieved by combining the controller of Eq.7.19 with an external linear error-feedback controller with integral action [DC95].

Remark 7.8: Comparing the uncertainty decoupling result of theorem 7.2 with the result of theorem 7.3 (Eq.7.21), we note that the latter is clearly applicable to a larger class of systems (those satisfying the condition $\sigma \leq \delta$), achieving, however, a weaker performance requirement. More specifically, in the uncertainty decoupling problem a well-characterized input/output response is enforced for all times independently of

$\theta(t)$, while in the case of robust uncertainty rejection the output error $(\bar{y}^i - v^i)$ is ensured to stay bounded and asymptotically approach arbitrarily close to zero (by appropriate choice of the parameter ϕ).

Remark 7.9 (Illustrative example (cont'd)): Whenever $(c(z), r_1(z)) \neq 0$ (which implies that $\delta = \sigma = 1$), the uncertainty decoupling problem via distributed state feedback is not solvable. However, since $\delta = \sigma$, we have that the matching condition of our robust control methodology is satisfied. Assuming that there exists an known upper bound $\bar{\theta}(t)$ on the size of the uncertain term $\theta(t)$, such that $|\theta| \leq \bar{\theta}(t)$, the control law of theorem 7.3 takes the form:

$$
\begin{aligned}
\bar{u} \; = \; & [\frac{U_w}{\rho_f c_{pf} V_r}\int_0^1 c(z)b(z)dz]^{-1}\left\{\gamma_0\left(v - \int_0^1 x_2 dz\right)\right. \\
& -\int_0^1 c(z)\left(-v_l\frac{\partial x_2}{\partial z} + \frac{(-\Delta H)}{\rho_f c_{pf}}kx_1 + \frac{U_w}{\rho_f c_{pf} V_r}(T_{js} - x_2) + F(T_{ls} - x_2)\right)dz \\
& \left. -2\bar{\theta}(t)\frac{\displaystyle\int_0^1 x_2(z,t)dz - v}{\displaystyle|\int_0^1 x_2(z,t)dz - v| + \varphi}\right\}
\end{aligned}
\tag{7.22}
$$

7.5 Two-time-scale hyperbolic PDE systems

In the rest of the theoretical part of this chapter, we will analyze the problem of robustness of the controllers of Eqs.7.13-7.19 with respect to stable unmodeled dynamics. This problem will be addressed within the broader context of control of two-time-scale hyperbolic PDE systems, modeled within the framework of singular perturbations. Such a formulation provides a natural setting for addressing robustness with respect to unmodeled dynamics. To this end, in this section, we will derive two general stability results for two-time-scale hyperbolic PDE systems which will be used in the next section, to derive conditions that guarantee robustness of the controllers of Eqs.7.13-7.19 to stable unmodeled dynamics.

7.5.1 Two-time-scale decomposition

We will focus on singularly perturbed hyperbolic PDE systems with the following state-space description:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A_{11}(x)\frac{\partial x}{\partial z} + A_{12}(x)\frac{\partial \eta}{\partial z} + f_1(x) + Q_1(x)\eta + g_1(x)b(z)\bar{u} + W_1(x)r(z)\theta(t) \\
\epsilon\frac{\partial \eta}{\partial t} &= A_{21}(x)\frac{\partial x}{\partial z} + A_{22}(x)\frac{\partial \eta}{\partial z} + f_2(x) + Q_2(x)\eta + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t) \\
\bar{y} &= Ch(x)
\end{aligned}
\tag{7.23}
$$

subject to the boundary conditions:

$$
\begin{aligned}
C_{11}x(a,t) + C_{12}x(b,t) &= R_1(t) \\
C_{21}\eta(a,t) + C_{22}\eta(b,t) &= R_2(t)
\end{aligned}
\tag{7.24}
$$

and the initial conditions:

$$
\begin{aligned}
x(z,0) &= x_0(z) \\
\eta(z,0) &= \eta_0(z)
\end{aligned}
\tag{7.25}
$$

where $x(z,t) = [x_1(z,t) \ \cdots \ x_n(z,t)]^T$, $\eta(z,t) = [\eta_1(z,t) \ \cdots \ \eta_p(z,t)]^T$, denote vectors of state variables, $x(z,t) \in \mathcal{H}^{(n)}[(\alpha,\beta),\mathbb{R}^n]$, $\eta(z,t) \in \mathcal{H}^{(n)}[(\alpha,\beta),\mathbb{R}^p]$, with $\mathcal{H}^{(i)}[(\alpha,\beta),\mathbb{R}^j]$ being the infinite-dimensional Hilbert space of j-dimensional vector functions defined on the interval $[\alpha,\beta]$ whose spatial derivatives up to i-th order are square integrable. $A_{11}(x), A_{12}(x), A_{21}(x), A_{22}(x)$ are sufficiently smooth matrices. $f_1(x), f_2(x), g_1(x), g_2(x)$ are sufficiently smooth vector functions. $Q_1(x), Q_2(x), W_1(x), W_2(x)$ are sufficiently smooth matrices. ϵ is a small parameter which quantifies the degree of coupling between the fast and slow modes of the system, and $R_1(t), R_2(t)$ are time-varying column vectors. The following assumption states that the two-time-scale system of Eq.7.23 is hyperbolic.

Assumption 7.3: *The $(n+p \times n+p)$ matrix:*

$$
\mathcal{A}(x) = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix}
\tag{7.26}
$$

is real symmetric, and its eigenvalues satisfy:

$$
\bar{\lambda}_1(x) \leq \cdots \leq \bar{\lambda}_k(x) < 0 < \bar{\lambda}_{k+1}(x) \leq \cdots \leq \bar{\lambda}_{n+p}(x)
\tag{7.27}
$$

for all $x \in \mathcal{H}^n[(\alpha,\beta),\mathbb{R}^n]$.

The time-scale multiplicity of the system of Eq.7.23 can be explicitly taken into account by decomposing it into separate reduced-order models associated with different time-scales. Defining $\mathcal{L}_{i1}x = A_{i1}(x)\frac{\partial x}{\partial z}+f_i(x)$, $\mathcal{L}_{2j}\eta = A_{1j}(x)\frac{\partial \eta}{\partial z}+Q_{1j}(x)\eta$, $i,j=1,2$, and setting $\epsilon = 0$, the PDE system of Eq.7.23, reduces to a system of coupled partial and ordinary differential equations of the form:

$$\begin{aligned}\frac{\partial x}{\partial t} &= \mathcal{L}_{11}x + \mathcal{L}_{12}\eta_s + g_1(x)b(z)\bar{u} + W_1(x)r(z)\theta(t)\\ 0 &= \mathcal{L}_{21}x + \mathcal{L}_{22}\eta_s + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t)\\ \bar{y}_s &= \mathcal{C}h(x)\end{aligned} \tag{7.28}$$

The solution of the ODE $\mathcal{L}_{21}x + \mathcal{L}_{22}\eta_s + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t) = 0$ subject to the boundary conditions of Eq.7.24 is of the form:

$$\eta_s = \mathcal{L}_{22}^{-1}(\mathcal{L}_{21}x + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t)) \tag{7.29}$$

The slow subsystem, which captures the slow dynamics of the system of Eq.7.23, takes the form:

$$\begin{aligned}\frac{\partial x}{\partial t} &= \mathcal{L}x + G(x)b(z)\bar{u} + W(x)r(z)\theta(t)\\ \bar{y}_s &= \mathcal{C}h(x)\end{aligned} \tag{7.30}$$

where $\mathcal{L}x = \mathcal{L}_{11}x - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}x$, $G(x) = g_1(x) - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}g_2(x)$, $W(x) = W_1(x) - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}W_2(x)$. Defining a fast time-scale $\tau = \frac{t}{\epsilon}$ and setting $\epsilon = 0$, the fast subsystem, which describes the fast dynamics of the system of Eq.7.23, takes the form:

$$\frac{\partial \eta}{\partial \tau} = \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\bar{u} + W_2(x)r(z)\theta(t) \tag{7.31}$$

where x,θ are independent of time.

7.5.2 Stability results

In this subsection, we will give two basic stability results for systems of the form of Eq.7.23 which are important on their own right and will also be used in the subsequent sections to prove a robustness property of the controllers of Eqs.7.13-7.19 with respect to exponentially stable unmodeled dynamics. We will begin with the statement of stability requirements on the fast and slow subsystems. In particular, we assume that the fast subsystem of Eq.7.31 and the unforced ($\bar{u} = \bar{\theta} = 0$) slow subsystem of Eq.7.30 are locally exponentially stable. Assumptions 7.4 and 7.5 that follow formalize these requirements.

Assumption 7.4: *The differential operator $\mathcal{L}_{22}\eta$ generates an exponentially stable strongly continuous semigroup $U_f(t)$ that satisfies:*

$$||U_f(t)||_2 \leq K_F e^{-a_f \frac{t}{\epsilon}}, \quad t \geq 0 \tag{7.32}$$

where $K_f \geq 1$, and a_f is some positive real number.

Assumption 7.5: *The differential operator $\mathcal{L}x$ generates a locally (i.e., for $x \in \mathcal{H}$ that satisfy $||x||_2 \leq \delta_x$, where δ_x is a positive real number) exponentially stable strongly continuous semigroup $U_s(t)$ that satisfies:*

$$||U_s(t)||_2 \leq K_s e^{-a_s t}, \quad t \geq 0 \tag{7.33}$$

where $K_s \geq 1$, and a_s is some positive real number.

From the above assumption and using Eq.6.15, the smoothness of $G(x)$ and $W(x)$, and the fact that for $||x||_2 \leq \delta_x$, there exists a pair of positive real numbers $\bar{M}_1, \bar{M}_2$ such that $||G(x)||_2 \leq \bar{M}_1$ and $||W(x)||_2 \leq \bar{M}_2$, we have that the following estimate holds for the system of Eq.7.30:

$$\begin{aligned} ||x||_2 &\leq K_s||x_0||_2 e^{-a_s t} + K_s \int_0^t e^{-a_s(t-\tau)}||G(x)b(z)||_2|\bar{u}(\tau)|d\tau \\ &\quad + K_s \int_0^t e^{-a_s(t-\tau)}||W(x)r(z)||_2|\theta(\tau)|d\tau \\ &\leq K_s||x_0||_2 e^{-a_s t} + M_1||\bar{u}|| + M_2||\theta|| \end{aligned} \tag{7.34}$$

where $M_1 = \dfrac{K_s \bar{M}_1 ||b(z)||_2}{a_s}$, $M_2 = \dfrac{K_s \bar{M}_2 ||r(z)||_2}{a_s}$, provided that the initial condition ($||x_0||_2$) and the inputs ($\bar{u}, \theta$) are sufficiently small to satisfy $K_s||x_0||_2 + M_1||\bar{u}|| +$

$M_2||\theta|| \leq \delta_x$. Theorem 7.4 provides the main stability result of this section (the proof is given in the appendix F).

Theorem 7.4: *Consider the system of Eq.7.23 with $\bar{u} \equiv 0$, for which assumptions 7.5, 7.6 and 7.7 hold, and define $\eta_f := \eta - \eta_s$. Then, there exist positive real numbers $(\bar{K}_s, \bar{a}_s, \bar{K}_f, \bar{a}_f)$, such that for each positive real number d, there exist positive real numbers $(\tilde{\delta}, \epsilon^*)$ such that if* $\max\{||x_0||_2, ||\eta_{f0}||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$ *and $\epsilon \in (0, \epsilon^*]$, then, for all $t \geq 0$:*

$$||x||_2 \leq \bar{K}_s||x_0||_2 e^{-\bar{a}_s t} + M_2||\theta|| + d \tag{7.35}$$

$$||\eta_f||_2 \leq \bar{K}_f||\eta_{f0}||_2 e^{-\bar{a}_f \frac{t}{\epsilon}} + d \tag{7.36}$$

Theorem 7.4 establishes a robustness result of the boundedness property that the slow subsystem of Eq.7.30 possesses, with respect to exponentially stable singular perturbations, provided that they are sufficiently fast. Theorem 7.5 that follows establishes, another conceptually important result, namely that the exponential stability property of a reduced system (consider the system of Eq.7.30 with $|\bar{u}| = |\theta(t)| \equiv 0$) is preserved in the presence of exponentially stable singular perturbations. The proof of this theorem in analogous to the one of theorem 7.4 and will be omitted for brevity.

Theorem 7.5: *Consider the system of Eq.7.23 with $|\bar{u}| = |\theta(t)| \equiv 0$, for which assumptions 7.3, 7.4 and 7.5 hold, and define $\eta_f := \eta - \eta_s$. Then, there exist positive real numbers $(\tilde{\delta}, \epsilon^*)$ such that if* $\max\{||x_0||_2, ||\eta_{f0}||_2\} \leq \tilde{\delta}$ *and $\epsilon \in (0, \epsilon^*]$, the system of Eq.7.23 is exponentially stable (i.e. Eqs.7.35-7.36 hold with $d = 0$).*

7.6 Robustness with respect to unmodeled dynamics

In this section, we utilize the general stability results of the previous section to establish robustness properties of the controllers of Eqs.7.13-7.19 with respect to unmodeled dynamics. In particular, we use the hyperbolic singularly perturbed system of

Eq.7.23 to represent the actual process model, and we assume that the model which is available for controller design is the slow system of Eq.7.30. Our objective is to establish that the controllers of Eqs.7.13-7.19, synthesized on the basis of the slow system, are robust to unmodeled dynamics in the sense that they enforce approximate uncertainty decoupling and robust disturbance rejection in the actual model of Eq.7.23, provided that the separation of the fast and slow modes is sufficiently large.

7.6.1 Robustness of uncertainty decoupling to unmodeled dynamics

In this subsection, we consider the uncertainty decoupling problem for the two-time-scale hyperbolic system of Eq.7.23 whose fast dynamics is stable (assumption 7.4). Substitution of the distributed state feedback law of Eq.7.10 into the system of Eq.7.23 yields:

$$\begin{aligned} \frac{\partial x}{\partial t} &= \mathcal{L}_{11}x + \mathcal{L}_{12}\eta + g_1(x)b(z)\mathcal{S}(x) + g_1(x)b(z)s(x)v + W_1(x)r(z)\theta(t) \\ \epsilon\frac{\partial \eta}{\partial t} &= \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\mathcal{S}(x) + g_2(x)b(z)s(x)v + W_2(x)r(z)\theta(t) \\ \bar{y} &= \mathcal{C}h(x) \end{aligned} \tag{7.37}$$

From the above representation of the closed-loop system, it is clear that the control law of Eq.7.10 preserves the two-time-scale property of the open-loop system and guarantees that the system of Eq.7.37 has a well-defined solution. Performing a two-time-scale decomposition on the system of Eq.7.37, the closed-loop fast subsystem takes the form:

$$\frac{\partial \eta}{\partial \tau} = \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)\mathcal{S}(x) + g_2(x)b(z)s(x)v + W_2(x)r(z)\theta(t) \tag{7.38}$$

where x, θ are independent of time. From the representation of the closed-loop fast subsystem it is clear that the control law of Eq.7.37 preserves the exponential stability property of the fast dynamics of the closed-loop system. The closed-loop slow system takes the form:

$$\begin{aligned}\frac{\partial x}{\partial t} &= \mathcal{L}x + G(x)b(z)\mathcal{S}(x) + G(x)b(z)s(x)v + W(x)r(z)\theta(t) \\ \bar{y}_s &= \mathcal{C}h(x)\end{aligned} \tag{7.39}$$

For the system of Eq.7.39 it is straightforward to show that the result of proposition 7.1 holds, which suggests requesting an input/output response of the form of Eq.7.12 in the closed-loop reduced system.

Theorem 7.6 that follows establishes a robustness property of the solvability condition of the uncertainty decoupling problem ($\sigma < \delta$) and the input/output response of Eq.7.12 with respect to stable, sufficiently fast, unmodeled dynamics (the proof of the theorem is given in the appendix F).

Theorem 7.6: *Consider the system of Eq.7.23 for which assumptions 7.3 and 7.4 hold, under the controller of Eq.7.13. Consider also the slow subsystem of Eq.7.30 and assume that the condition $\sigma < \delta$ and the stability conditions of theorem 7.2 hold. Then, for each positive real number d, there exist positive real numbers $(\tilde{\delta}, \epsilon^*)$ such that if* $\max\{||x_0||_2, ||\eta_{f0}||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$ *and $\epsilon \in (0, \epsilon^*]$, then:*

a) the state of the closed-loop system is bounded, and

b) the output of the closed-loop system satisfies for all $t > 0$:

$$\bar{y}^i(t) = \bar{y}^i_s(t) + d, \quad i = 1, \ldots, l \tag{7.40}$$

where $\bar{y}^i_s(t)$ are the solutions of Eq.7.12.

Remark 7.10: Referring to the above theorem, note that we do not impose any assumptions on the way the uncertain variables θ enter the actual process model of Eq.7.23 (i.e. the condition $\sigma < \delta$ has to be satisfied only in the slow subsystem of Eq.7.30). This is achieved at the expense of the controller of Eq.7.13, which is synthesized on the basis of the slow subsystem of Eq.7.30, enforcing a type of approximate uncertainty decoupling in the actual process model, in the sense that the discrepancy between the output of the actual closed-loop system and the output of the closed-loop reduced system (which is independent of θ) can be made smaller than a given positive number d (possibly small) for all times, provided that ϵ is sufficiently small (Eq.7.40).

7.6.2 Robust control: uncertain variables and unmodeled dynamics

In this subsection, we consider the robust uncertainty rejection problem for two-time-scale hyperbolic systems of the form of Eq.7.23 with stable fast dynamics. Under a control law of the form of Eq.7.17, the closed-loop system takes the form:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \mathcal{L}_{11}x + \mathcal{L}_{12}\eta + g_1(x)b(z)[\bar{S}(x) + \bar{s}(x)v + \bar{\mathcal{R}}(x,t)] + W_1(x)r(z)\theta(t) \\
\epsilon\frac{\partial \eta}{\partial t} &= \mathcal{L}_{21}x + \mathcal{L}_{22}\eta + g_2(x)b(z)[\bar{S}(x) + \bar{s}(x)v + \bar{\mathcal{R}}(x,t)] + W_2(x)r(z)\theta(t) \\
\bar{y} &= Ch(x)
\end{aligned} \tag{7.41}$$

The fast dynamics of the above system is exponentially stable and the reduced system takes the form:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \mathcal{L}x + G(x)b(z)[\bar{S}(x) + \bar{s}(x)v + \bar{\mathcal{R}}(x,t)] + W(x)r(z)\theta(t) \\
\bar{y}_s &= Ch(x)
\end{aligned} \tag{7.42}$$

Theorem 7.7 that follows establishes a robustness property of the controller of Eq.7.19 to sufficiently fast unmodeled dynamics (the proof is given in the appendix F).

Theorem 7.7: *Consider the system of Eq.7.23 for which assumptions 7.3 and 7.4 hold, under the controller of Eq.7.19. Consider also the slow subsystem of Eq.7.30 and suppose that assumption 7.2 and the stability conditions of theorem 7.2 hold. Then, for each positive real number d, there exist positive real numbers $(\tilde{\delta}, \phi^*)$ such that if* $\max\{||x_0||_2, ||\eta_{f0}||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$, $\phi \in (0, \phi^*]$, *there exists a positive real number $\epsilon^*(\phi)$ such that if*
$\max\{||x_0||_2, ||\eta_{f0}||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$, $\phi \in (0, \phi^*]$, *and* $\epsilon \in (0, \epsilon^*(\phi)]$, *then:*
a) the state of the closed-loop system is bounded, and

b) the output of the closed-loop system satisfies:

$$\lim_{t \to \infty} |\bar{y}^i - v^i| \leq d, \quad i = 1, \ldots, l \tag{7.43}$$

Remark 7.11: Regarding the result of the above theorem, a few observations are in order: *i*) no matching condition is imposed in the actual process model of Eq.7.23, *ii*) the dependence of ϵ^* on ϕ is due to the presence of ϕ on the closed-loop fast subsystem, and *iii*) a bound for the output error, for all times (and not only asymptotically as in Eq.7.43) can be obtained from the proof of the theorem in terms of the initial conditions and d.

Remark 7.12: Whenever the open-loop fast subsystem of Eq.7.31 is unstable, i.e. the operator $\mathcal{L}_{22}\eta$ generates an exponentially unstable semigroup, a preliminary distributed state feedback law of the form:

$$\bar{u} = \mathcal{F}\eta + \tilde{u} \tag{7.44}$$

can be used to stabilize the fast dynamics, under the assumption that the pair $[\mathcal{L}_{22} \; g_2(x)b(z)]$ is stabilizable, yielding thus a two-time-scale hyperbolic system for which assumption 7.7 holds. The design of the gain operator $\mathcal{F}$ can be performed using for example standard optimal control methods [Bal83].

7.7 Application to a fixed-bed reactor

Consider a fixed-bed reactor where an elementary reaction of the form $A \rightarrow B$ takes place, shown in Figure 7.2. The reaction is endothermic and a jacket is used to heat the reactor. Under the assumptions of perfect radial mixing, constant density and heat capacity of the reacting liquid, and negligible diffusive phenomena, a dynamic model of the process can be derived from material and energy balances and has the

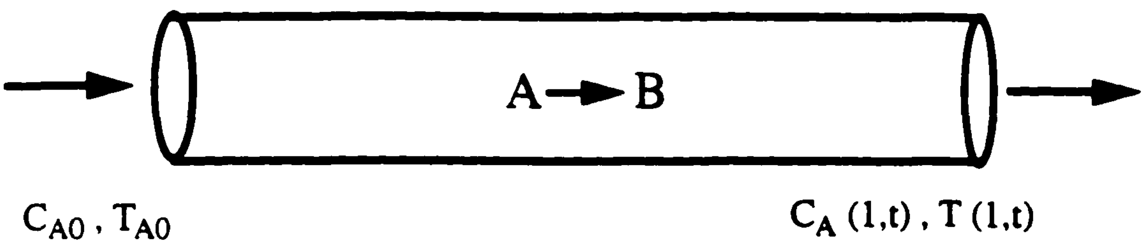


Figure 7.2: A nonisothermal fixed-bed reactor.

form:

$$
\begin{aligned}
\rho_b c_{pb}\frac{\partial T}{\partial t} &= -\rho_f c_{pf} v_l \frac{\partial T}{\partial z} + (-\Delta H) k_0 e^{-\frac{E}{RT}} C_A + \frac{U_w}{V_r}(T_j - T) \\
\bar{\epsilon}\frac{\partial C_A}{\partial t} &= -v_l \frac{\partial C_A}{\partial z} - k_0 e^{-\frac{E}{RT}} C_A
\end{aligned}
\tag{7.45}
$$

subject to the initial and boundary conditions:

$$
\begin{gathered}
C_A(0,t) = C_{A0}\,,\; C_A(z,0) = C_{A_s}(z) \\
T(0,t) = T_{A0}\,,\; T(z,0) = T_s(z)
\end{gathered}
$$

where C_A denotes the concentration of the species A, T denotes the temperature in the reactor, $\bar{\epsilon}$ denotes the reactor porosity, ρ_b, c_{pb} denote the density and heat capacity of the bed, ρ_f, c_{pf} denote the density and heat capacity of the fluid phase, v_l denotes the velocity of the fluid phase, U_w denotes the heat transfer coefficient, T_j denotes the spatially uniform temperature in the jacket, V_r denotes the volume of the reactor, $k_0, E, \Delta H$ denote the pre-exponential factor, the activation energy, and the enthalpy of the reaction, C_{A0}, T_{A0} denote the concentration and temperature of the inlet stream, and $C_{A_s}(z), T_s(z)$ denote the steady-state profiles for the concentration of the species A and temperature in the reactor.

The main feature of fixed-bed reactors is that the reactant wave propagates through the bed with a significantly larger speed than the heat wave, because the exchange of heat between the fluid and packing slows the thermal wave down [SF70]. Therefore, the system of Eq.7.45 possesses an inherent two-time-scale property, i.e., the concentration dynamics are much faster than the temperature dynamics (this fact was also verified through open-loop simulations). This implies that C_A is the fast variable, while T is the slow variable. In order to obtain a singularly perturbed representation of the process, where the partition to fast and slow variables is consistent with the dynamic characteristics of the process, the singular perturbation parameter ϵ was defined as:

$$
\epsilon = \frac{\bar{\epsilon}}{\rho_f c_{pf}} \tag{7.46}
$$

Setting $\bar{t} = \frac{t}{\rho_b c_{pb}}$, $x = T$, $\eta = C_A$, the system of Eq.7.45 can be written in the following singularly perturbed form:

$$\begin{aligned} \frac{\partial x}{\partial \bar{t}} &= -\rho_f c_{pf} v \frac{\partial x}{\partial z} + (-\Delta H) k_0 e^{-\frac{E}{Rx}} \eta + \frac{U_w}{V_r}(T_j - x) \\ \epsilon \frac{\partial \eta}{\partial \bar{t}} &= -v_l \frac{\partial \eta}{\partial z} - k_0 e^{-\frac{E}{Rx}} \eta \end{aligned} \tag{7.47}$$

The values of the process parameters are given in table 7.1. It was verified that they correspond to a stable steady-state for the open-loop system. The control problem considered is the one of controlling the temperature of the reactor (which is the variable that essentially determines the dynamics of the process) by manipulating the jacket temperature. Notice that T_j is chosen to be the manipulated input with the understanding that in practice its manipulation is achieved indirectly through manipulation of the jacket inlet flow rate (for more details see the discussion in remark 7.14). The enthalpy of the reaction is considered to be the main uncertain variable. Assuming that there is available one control actuator with distribution function $b(z) = 1$ and defining $\bar{u} = T_j - T_{js}$, $\bar{y} = \int_0^1 x dz$, $\theta = \Delta H - \Delta H_0$, we have from Eq.7.47:

$$\begin{aligned} \frac{\partial x}{\partial \bar{t}} &= -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 e^{-\frac{E}{Rx}} \eta + \frac{U_w}{V_r}(T_{js} - x) + \frac{U_w}{V_r}\bar{u} + k_0 exp^{-\frac{E}{Rx}} \eta\theta \\ \epsilon \frac{\partial \eta}{\partial \bar{t}} &= -v_l \frac{\partial \eta}{\partial z} - k_0 e^{-\frac{E}{Rx}} \eta \\ \bar{y} &= \int_0^1 x dz \end{aligned} \tag{7.48}$$

Performing a two-time-scale decomposition of the above system, the fast subsystem takes the form:

$$\frac{\partial \eta}{\partial \bar{\tau}} = -v_l \frac{\partial \eta}{\partial z} - k_0 e^{-\frac{E}{Rx}} \eta \tag{7.49}$$

where $\bar{\tau} = \frac{\bar{t}}{\epsilon}$ and the x in the above system depends only on the position z. From the system of Eq.7.49, it is clear that the fast dynamics of the system of Eq.7.48 are

v_l	=	30.0	$m\ hr^{-1}$
V_r	=	1.0	m^3
ϵ	=	0.01	
L	=	1.0	m
E	=	2.0×10^4	$kcal\ kmol^{-1}$
k_0	=	5.0×10^{12}	hr^{-1}
R	=	1.987	$kcal\ kmol^{-1}\ K^{-1}$
ΔH_0	=	35480.111	$kcal\ kmol^{-1}$
c_{pf}	=	0.0231	$kcal\ kg^{-1}\ K^{-1}$
ρ_f	=	90.0	$kg\ m^{-3}$
U_w	=	500.0	$kcal\ hr^{-1}\ K^{-1}$
C_{A0}	=	4.0	$kmol\ m^{-3}$
T_{A0}	=	320.0	K

Table 7.1: Process parameters and steady-state values

exponentially stable uniformly in x, and thus they can be neglected in the controller design. Setting $\epsilon = 0$, the model of Eq.7.48 reduces to a set of a partial and an ordinary differential equation of the form:

$$
\begin{aligned}
\frac{\partial x}{\partial \bar{t}} &= -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 e^{-\frac{E}{Rx}} \eta + \frac{U_w}{V_r}(T_{js} - x) + \frac{U_w}{V_r}\bar{u} + k_0 exp^{-\frac{E}{Rx}} \eta \theta \\
0 &= -v_l \frac{\partial \eta}{\partial z} - k_0 e^{-\frac{E}{Rx}} \eta
\end{aligned}
\qquad (7.50)
$$

The structure of the ordinary differential equation allows an analytic derivation of its solution subject to the boundary condition $\eta(0, t) = C_{A0}(t)$, which is of the form:

$$
\eta(z) = C_{A0} e^{\frac{k_0 \int_0^z \epsilon^{-\frac{E}{Rx}} dz}{v_l}} \qquad (7.51)
$$

Substituting Eq.7.51 into Eq.7.50, the following reduced system can be obtained:

$$
\begin{aligned}
\frac{\partial x}{\partial \bar{t}} &= -\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 e^{-\frac{E}{Rx}} C_{A0} exp^{\frac{k_0 \int_0^z exp^{-\frac{E}{Rx}} dz}{v_l}} \\
&\quad + \frac{U_w}{V_r}(T_{js} - x) + \frac{U_w}{V_r}\bar{u} + k_0 exp^{-\frac{E}{Rx}} C_{A0} exp^{\frac{k_0 \int_0^z exp^{-\frac{E}{Rx}} dz}{v_l}} \theta \\
\bar{y}_s &= \int_0^1 x dz
\end{aligned}
\qquad (7.52)
$$

The above system is clearly in the form of Eq.7.1 and a straightforward calculation of the time-derivative of the output yields that the characteristic indices of the output $\bar{y}$ with respect to the manipulated input $\bar{u}$ and the uncertain variable θ are $\sigma = \delta = 1$. This implies that it is not possible to decouple the effect of θ on $\bar{y}$ via distributed state feedback, and a robust controller of the form of Eq.7.19 should be synthesized to attenuate the effect of $\bar{\theta}$ on $\bar{y}$ (this is possible because the matching condition, assumption 7.4, holds for the system of Eq.7.52). Moreover, it was verified through simulations that the zero dynamics of the system of Eq.7.52 are locally exponentially stable. The explicit form of the controller of Eq.7.19 is:

$$\bar{u} = \frac{V_r}{U_w}\left\{\gamma_0\left(v - \int_0^1 x dz\right) - \int_0^1 (-\rho_f c_{pf} v_l \frac{\partial x}{\partial z} + (-\Delta H_0) k_0 exp^{-\frac{E}{Rx}} \right.$$

$$\left. \bullet C_{A0} exp^{\frac{k_0 \int_0^z exp^{-\frac{E}{Rx}} dz}{v_l}} + \frac{U_w}{V_r}(T_{js} - x))dz - K(t)\frac{\int_a^b x(z,t)dz - v}{|\int_a^b x(z,t)dz - v| + \phi}\right\} \tag{7.53}$$

where $K(t) = 2\bar{\theta}\int_0^1 \frac{k_0 V_r}{U_w} exp^{-\frac{E}{Rx}} C_{A0} exp^{\frac{k_0 \int_0^z exp^{-\frac{E}{Rx}} dz}{v_l}} dz$. A time-varying uncertainty was considered expressed by a sinusoidal function of the form:

$$\theta = 0.5(-\Delta H_0) sin(t) \tag{7.54}$$

The upper bound on the uncertainty was taken to be $\bar{\theta} = 0.5|(-\Delta H_0)|$. In this application, the value of the singular perturbation parameter is fixed i.e. $\epsilon = 0.01$. From theorem 7.6, it is clear that for a given value of ϵ^*, there exists a lower bound on the level of asymptotic attenuation d that can be achieved. We performed a set of computer simulations (for the regulation problem) to calculate ϕ^* for certain values of d, and, in turn, the value of ϵ^* for $\phi \leq \phi^*$. The following set of parameters were found to give an $\epsilon^* \leq 0.01$ and used in the simulations :

$$\gamma_0 = 0.2, \quad \phi = 0.1 \tag{7.55}$$

to achieve an ultimate degree of attenuation $d = 0.5$.

Two simulation runs were performed to test the regulatory, set-point tracking and uncertainty rejection capabilities of the controller. In both runs, the process was initially ($t = 0.0$ min) assumed to be at steady-state. In the first simulation run,

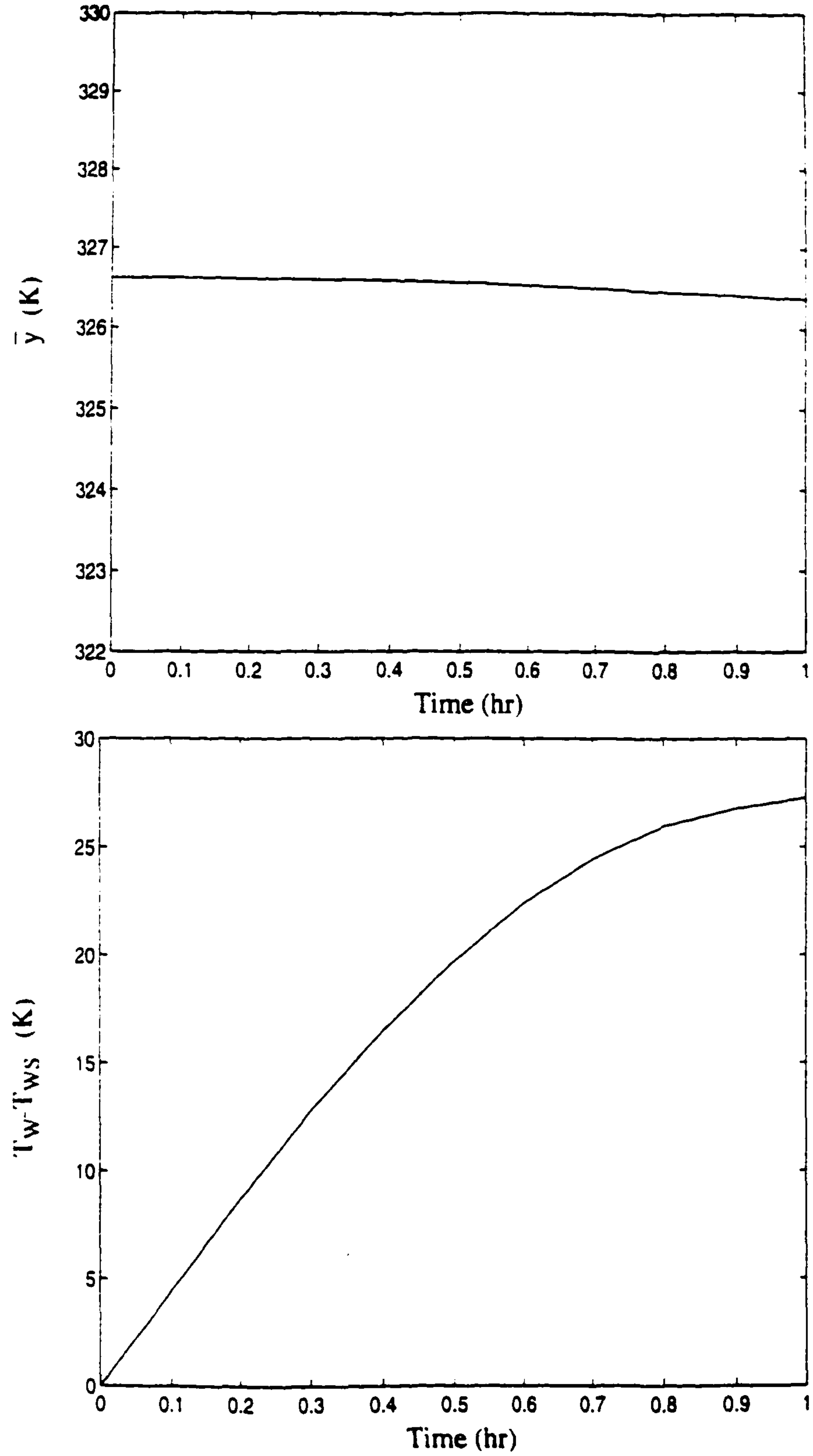


Figure 7.3: Closed-loop output and manipulated input profiles for regulation.

we tested the regulatory capabilities of the controller. Figure 7.3 shows the closed-loop output and manipulated input profiles, while Figure 7.4 displays the evolution of the temperature profile throughout the reactor. Clearly the controller regulates the output at the operating steady-state compensating for the effect of uncertainty and satisfying the requirements $\lim_{t \to \infty} |\bar{y} - v| \leq 0.5$. For the sake of comparison, we also implemented the same controller without the term which is responsible for the compensation of uncertainty. The output and manipulated input profiles for this simulation run are given in Figure 7.5. It is obvious that the controller cannot attenuate the effect of the uncertainty on the output of the process, leading to poor transient performance and offset. In the next simulation run, we tested the output tracking capabilities of the controller. A $6.6K$ increase in the value of the reference input was imposed at time $t = 0.0\ hr$. The output and manipulated input profiles are shown in figure 7.6. It is clear that the controller drives the output $\bar{y}$ to its new reference input value, achieving the requirement $\lim_{t \to \infty} |\bar{y} - v| \leq 0.5$. Figure 7.7 shows the output and manipulated profiles in the case of using the same controller without the term which is responsible for the compensation of uncertainty. Clearly, the performance is very poor. From the results of the simulation study, we conclude that there is a need to compensate for the effect of the uncertainty as well as that the proposed robust control methodology is a very efficient tool for this purpose.

Remark 7.13: The reduction in the dimensionality of the original model of the reactor using the fact that the process exhibits a two-time-scale property eliminates the need for using measurements of the concentration of the species A in the controller of Eq.7.53, which greatly facilitates its practical implementation.

Remark 7.14: Regarding the practical implementation of the robust distributed controller of Eq.7.53, we note that the manipulated variable T_j cannot be manipulated directly, but indirectly through manipulation of the jacket inlet flow rate. To

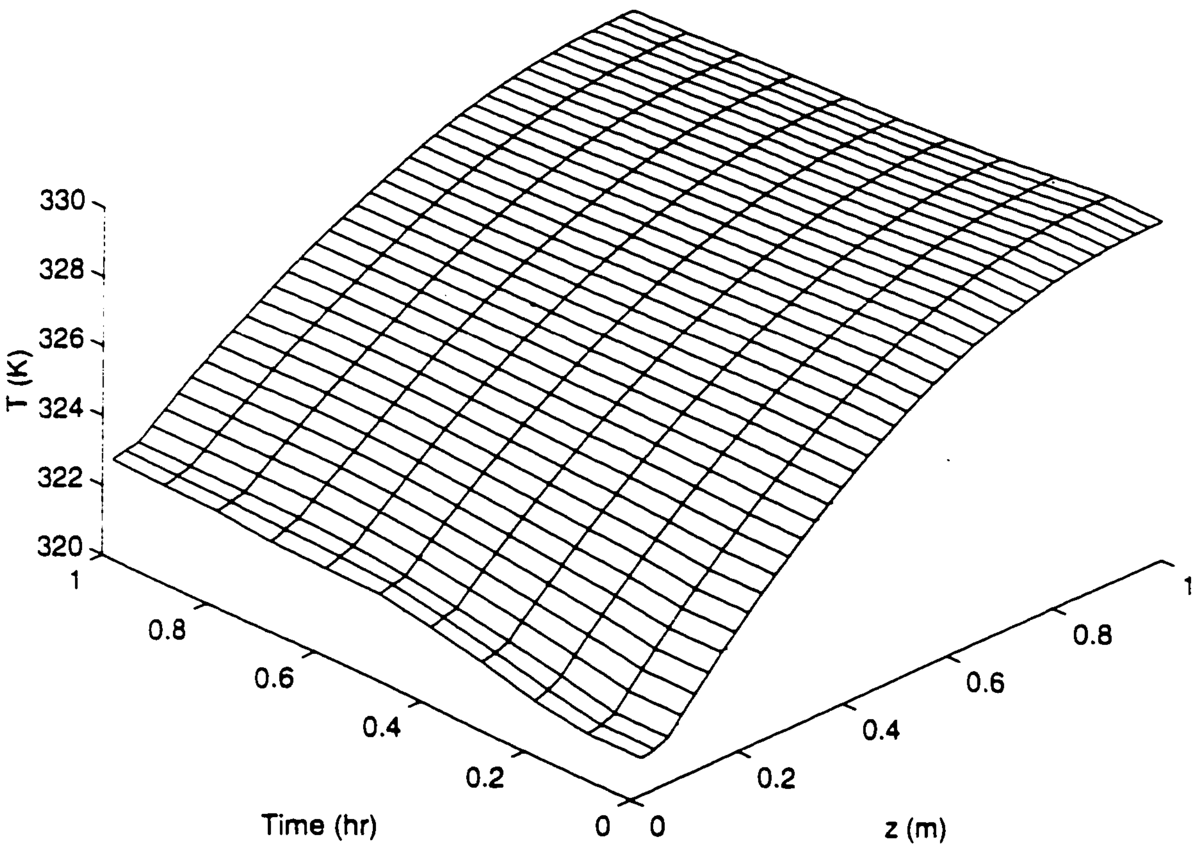


Figure 7.4: Profile of evolution of reactor temperature for regulation.

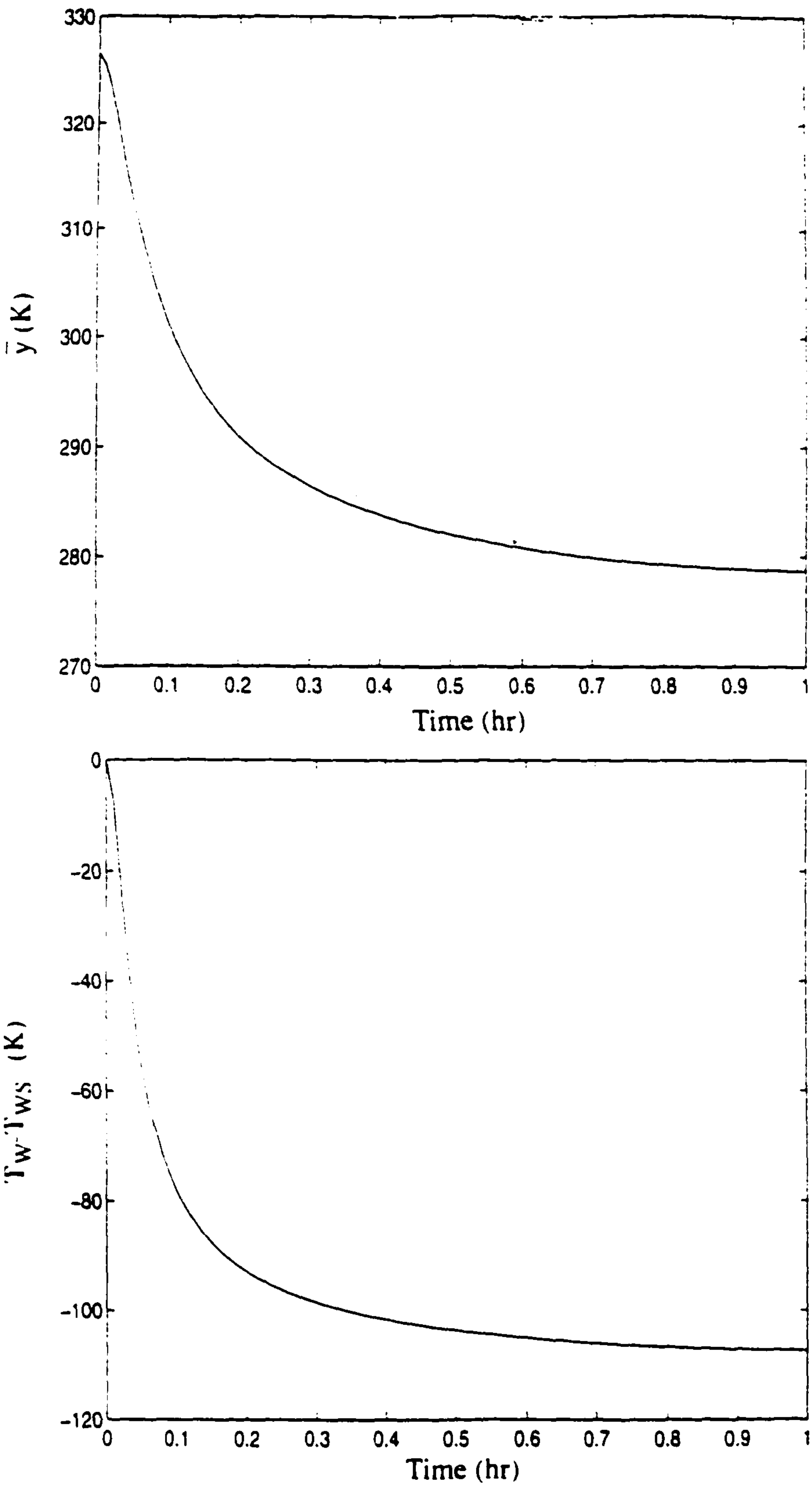


Figure 7.5. Closed-loop output and manipulated input profiles for regulation (no uncertainty comp-pensation)

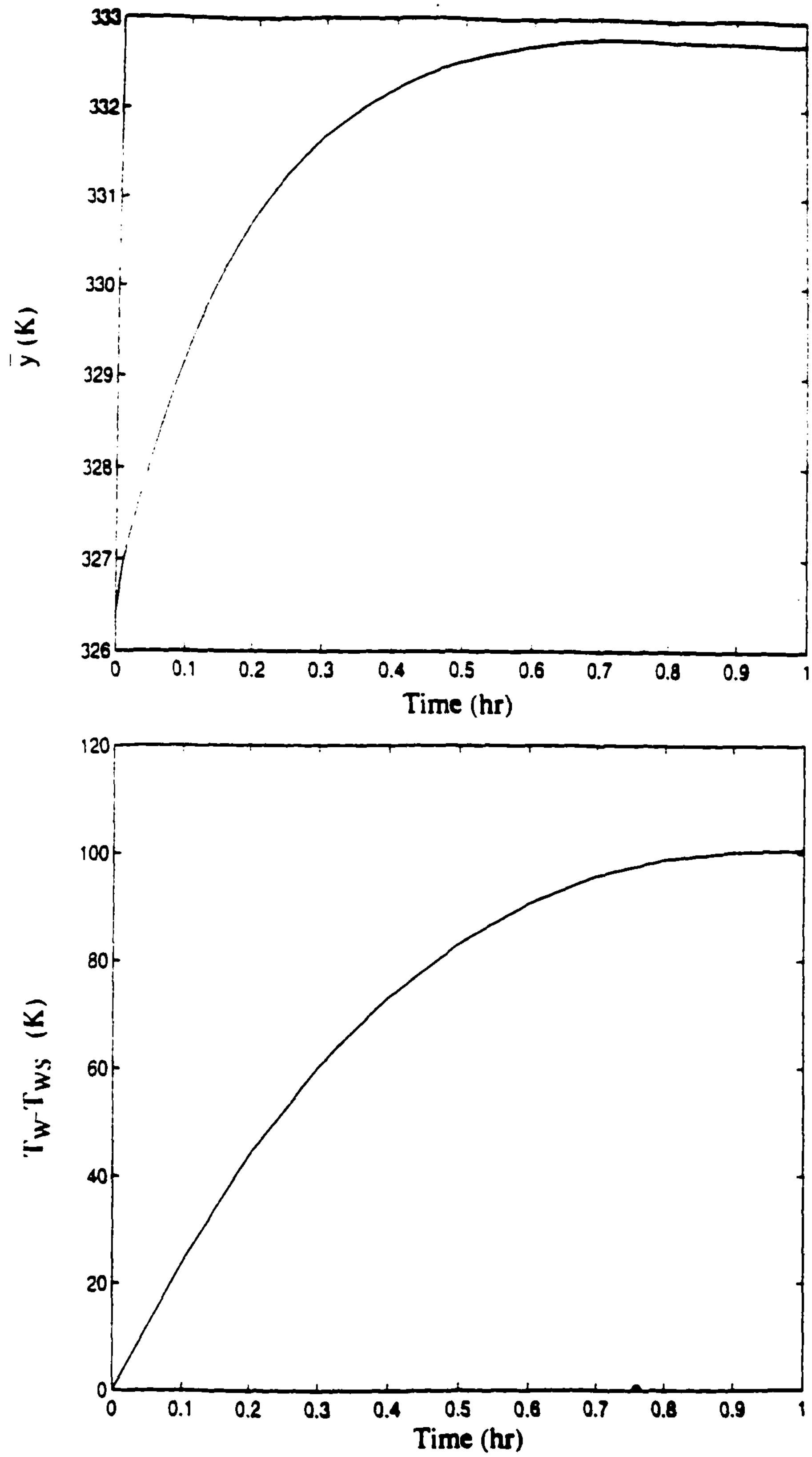
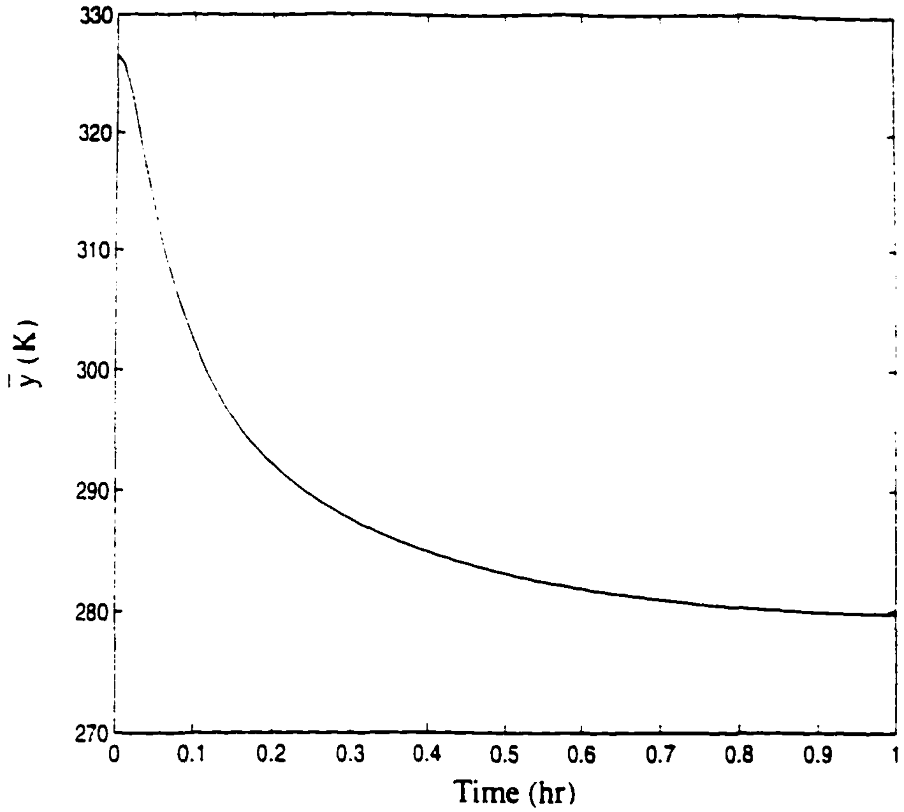


Figure 7.6: Closed-loop output and manipulated input profiles for reference input tracking.



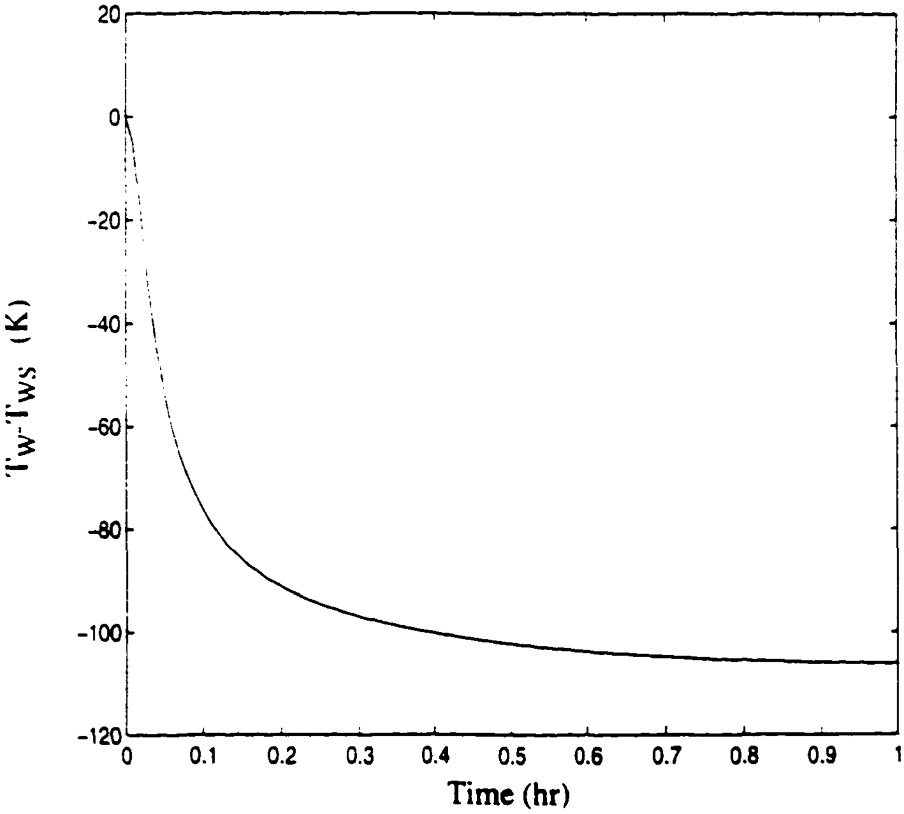


Figure 7.7: Closed-loop output and manipulated input profiles for reference input tracking (no uncertainty compensation).

this end, a controller should be designed based on an ODE model that describes the jacket dynamics, that operates in an internal loop to manipulate the jacket inlet flow rate to ensure that the jacket temperature obtains the values computed by the distributed robust controller (see subsection 6.7.4 for a detailed discussion on this issue). Of course, when such a controller is used, a slight deterioration of the closed-loop performance and robustness obtained under the assumption that T_j can be manipulated directly, will occur.

7.8 Conclusions

In this chapter, we considered systems of first-order hyperbolic PDEs with uncertainty, for which the manipulated input and the controlled output are distributed in space. Both uncertain variables and unmodeled dynamics were considered. In the case of uncertain variables, we initially derived a necessary and sufficient condition for the solvability of the problem of complete elimination of the effect of uncertainty on the output via distributed feedback as well as an explicit controller synthesis formula. Then, assuming that there exist known bounding functions that capture the magnitude of the uncertain terms and a matching condition is satisfied, we synthesized, using Lyapunov's direct method, a distributed robust controller that guarantees boundedness of the state and asymptotic output tracking with arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the output of the closed-loop system. In the presence of uncertain variables and unmodeled dynamics, we established that the proposed distributed controllers enforce approximate uncertainty decoupling or uncertainty attenuation in the closed-loop system as long as the unmodeled dynamics are stable and sufficiently fast. A nonisothermal fixed-bed reactor with unknown heat of the reaction was used to illustrate the application of the proposed control method. The fact that the reactant wave propagates through the bed faster than the heat wave was explicitly taken into account, and the original model of the process, which consists of two quasi-linear hyperbolic PDEs that describe the spatio-temporal evolution of the reactant concentration and the reactor temperature, was reduced to one quasi-linear hyperbolic PDEs that describes the evolution of the reactor temperature. A distributed robust controller was then designed on the basis of the reduced-order model to enforce output tracking and compensate for the effect of the uncertainty. Computer simulations were used to evaluate the performance and robustness properties of the controller.

Notation

Roman Letters

$A, A(x), A(z)$	=	matrices
$b^i(z)$	=	smooth function of z
$c^i(z)$	=	smooth function of z
C_{A0}	=	inlet concentration of the species A
C_{Al}	=	concentration of the species A in lateral stream
C_1, C_2	=	constant matrices
C_A	=	concentration of the species A
c_{pb}	=	heat capacity of the bed
c_{pf}	=	heat capacity of the fluid phase
E	=	activation energy
f, f_1, f_2	=	vector fields
F	=	lateral inlet flow rate
g, g_1, g_2	=	vector fields associated with the manipulated variable
h	=	controlled output scalar field
k_0	=	pre-exponential constant
l	=	total number of control actuators
Q_1, Q_2	=	sufficiently smooth matrices
T_{A0}	=	inlet temperature of the fluid in the reactor
T_{Al}	=	temperature of lateral stream
T	=	temperature of the reactor
T_j	=	temperature of the jacket
t	=	time
U_w	=	heat transfer coefficients
u	=	manipulated variable
$\bar{u}$	=	manipulated input vector
$\bar{u}^i$	=	i-th manipulated input
V_r	=	volume of the reactor
v	=	external reference input vector
v^i	=	external reference input for the i-th actuator
v_l	=	fluid velocity
W, W_1, W_2	=	sufficiently smooth matrices
x	=	vector of state variables
$\bar{y}, \bar{y}_s$	=	controlled outputs
z	=	spatial coordinate

Greek Letters

α	=	boundary of the spatial domain
β	=	boundary of the spatial domain
γ_k	=	adjustable parameters
ΔH	=	enthalpy of the reaction
η	=	observer state vector
δ^i	=	characteristic index of $\bar{y}^i$ with respect to $\bar{\theta}^i$
δ	=	characteristic index of $\bar{y}$ with respect to $\bar{\theta}$
θ	=	vector of uncertain variables
ρ_b	=	density of the bed
ρ_f	=	density of the fluid phase
σ^i	=	characteristic index of $\bar{y}^i$ with respect to $\bar{u}^i$
σ	=	characteristic index of $\bar{y}$ with respect to $\bar{u}$

Math Symbols

$\mathcal{C}^i$	=	bounded linear operator
$\mathcal{H}$	=	infinite dimensional Hilbert space
$L_f h$	=	Lie derivative of a scalar field h with respect to the vector field f
$L_f^k h$	=	k-th order Lie derivative
$L_g L_f^{k-1} h$	=	mixed Lie derivative
$\mathcal{S}, \bar{\mathcal{S}}, \bar{\mathcal{R}}$	=	nonlinear functionals
$\mathbb{R}$	=	real line
$\mathbb{R}_{\geq 0}$	=	positive real line
$\mathbb{R}^i$	=	$i-$dimensional Euclidean space
$\in$	=	belongs to
T	=	transpose
$\lvert \cdot \rvert$	=	standard Euclidean norm
$\lvert\lvert \cdot \rvert\rvert_2$	=	2-norm in $\mathcal{H}$
$(\cdot, \cdot)_{\mathbb{R}^n}$	=	inner product in $\mathbb{R}^n$
$\lvert\lvert \theta \rvert\rvert$	=	ess.sup.$\{\lvert\theta(t)\rvert, t \geq 0\}$, where $\theta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ is any measurable function
$u(t)$	=	signal defined on $[0, T)$
u_τ	=	signal defined on $[0, \infty)$, given by $u_\tau(t) = u(t)\ if\ t \in [0, \tau]$, and $u_\tau(t) = 0\ if\ t \in (\tau, \infty)$

Chapter 8

Feedback Control of Parabolic PDE Systems

8.1 Introduction

Parabolic PDE systems arise naturally as models of diffusion-convection-reaction processes [Ray81] and typically involve spatial differential operators whose eigenspectrum can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast one [Fri76, Bal79]. This implies that the dynamic behavior of such systems can be approximately described by finite-dimensional systems. Motivated by this, the standard approach to the control of parabolic PDEs involves the application of Galerkin's method to the PDE system to derive ODE systems that describe the dynamics of the dominant (slow) modes of the PDE system, which are subsequently used as the basis for the synthesis of finite-dimensional controllers [Bal79, Ray81]. However, there are two key controller implementation and closed-loop performance problems associated with this approach. First, the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to high dimensionality of the resulting controllers [GR95].

Second, there is a lack of a systematic way to characterize the discrepancy between the solutions of the PDE system and the approximate ODE system in finite time, which is essential for characterizing the transient performance of the closed-loop PDE system.

A natural framework to address the problem of deriving low-dimensional ODE systems that accurately reproduce the solutions of dissipative PDEs is based on the concept of inertial manifold (IM) (see [Tem88] and the references therein). An IM is a positively invariant, finite-dimensional Lipschitz manifold, which attracts every trajectory exponentially. The dynamics of a parabolic PDE system restricted on the inertial manifold is described by a set of ODEs called the inertial form. Hence, stability and bifurcation studies of the infinite-dimensional PDE system can be readily performed on the basis of the finite-dimensional inertial form [Tem88]. However, the explicit derivation of the inertial form requires the computation of the analytic form of the IM. Unfortunately, IMs have been proven to exist only for certain classes of PDEs (for example Kuramoto-Sivashinsky equation and some diffusion-reaction equations [Tem88]), and even then it is almost impossible to derive their analytic form. In order to overcome the problems associated with the existence and construction of IMs, the concept of approximate inertial manifold (AIM) has been introduced [FST89, FJK+89] and used for the derivation of ODE systems whose dynamic behavior approximates the one of the inertial form.

In the area of control of nonlinear parabolic PDE systems, few papers have appeared in the literature dealing with the application of IM for the synthesis of finite-dimensional controllers. In particular, in [SK95] the problem of stabilization of a parabolic PDE with boundary finite-dimensional feedback was studied; a standard observer-based controller augmented with a residual mode filter [Bal91] was used to induce an inertial manifold in the closed-loop system, and thus reduce the stabilization problem for the PDE system to a stabilization problem for the finite-dimensional inertial form. In [BGS95], the theory of inertial manifolds was utilized to determine the extent to which linear boundary proportional control influences the dynamic and steady-state response of the closed-loop system.

In this paper, we introduce a methodology for the synthesis of nonlinear finite-dimensional output feedback controllers for systems of quasi-linear parabolic PDEs. Singular perturbation methods are initially employed to establish that the discrepancy between the solutions of an ODE system of dimension equal to the number of slow modes, obtained through Galerkin's method, and the PDE system is proportional to the degree of separation of the fast and slow modes of the spatial operator. Then, a procedure, motivated by the theory of singular perturbations, is proposed for the construction of AIMs for the PDE system. The AIMs are used for the derivation of ODE systems of dimension equal to the number of slow modes, that yield solutions which are close, upto a desired accuracy, to the ones of the PDE system, for almost all times. These ODE systems are used as the basis for the synthesis of nonlinear output feedback controllers that guarantee stability and enforce the output of the closed-loop system to follow upto a desired accuracy, a prespecified response for almost all times. The proposed control methodology is implemented, through simulations, on a packed-bed reactor.

8.2 Preliminaries

We consider quasi-linear parabolic PDE systems of the form:

$$\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2} + wb(z)u + f(\bar{x}) \\ y^i &= \int_{z_i}^{z_{i+1}} c^i(z)k\bar{x}dz, \quad i = 1,\dots,l \end{aligned} \tag{8.1}$$

subject to the boundary conditions:

$$\begin{aligned} C_1\bar{x}(\alpha,t) + D_1\frac{\partial \bar{x}}{\partial z}(\alpha,t) &= R_1 \\ C_2\bar{x}(\beta,t) + D_2\frac{\partial \bar{x}}{\partial z}(\beta,t) &= R_2 \end{aligned} \tag{8.2}$$

and the initial condition:

$$\bar{x}(z,0) = \bar{x}_0(z) \tag{8.3}$$

where $\bar{x}(z,t) = [\bar{x}_1(z,t) \ \cdots \ \bar{x}_n(z,t)]^T$ denotes the vector of state variables, $[\alpha,\beta] \subset$ $\mathbb{R}$ is the domain of definition of the process, $z \in [\alpha,\beta]$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $u = [u^1 \ u^2 \ \cdots \ u^l]^T \in \mathbb{R}^l$ denotes the vector of manipulated inputs, and $y^i \in \mathbb{R}$ denotes a controlled output. $\frac{\partial \bar{x}}{\partial z}, \frac{\partial^2 \bar{x}}{\partial z^2}$ denote the first- and second-order spatial derivatives of $\bar{x}$, $f(\bar{x})$ is a vector function, w, k are constant vectors, A, B, C_1, D_1, C_2, D_2 are constant matrices, R_1, R_2 are column vectors, and $\bar{x}_0(z)$ is the initial condition. $b(z)$ is a known smooth vector function of z of the form $b(z) = [b^1(z) \ b^2(z) \ \cdots \ b^l(z)]$, where $b^i(z)$ describes how the control action $u^i(t)$ is distributed in the interval $[z_i, z_{i+1}] \subset [\alpha,\beta]$, and $c^i(z)$ is a known smooth function of z which is determined by the desired performance specifications in the interval $[z_i, z_{i+1}]$. Whenever the control action enters the system at a single point z_0, with $z_0 \in [z_i, z_{i+1}]$ (i.e. point actuation), the function $b^i(z)$ is taken to be nonzero in a finite space interval of the form $[z_0 - \epsilon, z_0 + \epsilon]$, where ϵ is a small positive real number, and zero elsewhere in $[z_i, z_{i+1}]$. Throughout the paper, we will use the order of magnitude notation $O(\epsilon)$. In particular, $\delta(\epsilon) = O(\epsilon)$ if there exist positive real numbers k_1 and k_2 such that: $|\delta(\epsilon)| \leq k_1|\epsilon| \, , \ \forall \, |\epsilon| < k_2$.

We formulate the system of Eqs.8.1-8.2-8.3 in a Hilbert space $\mathcal{H}([\alpha,\beta], \mathbb{R}^n)$, with $\mathcal{H}$ being the space of n-dimensional vector functions defined on $[\alpha,\beta]$ that satisfy the boundary condition of Eq.8.2, with inner product and norm:

$$\begin{aligned} (\omega_1,\omega_2) &= \int_\alpha^\beta (\omega_1(z),\omega_2(z))_{\mathbb{R}^n} dz \\ ||\omega_1||_2 &= (\omega_1,\omega_1)^{\frac{1}{2}} \end{aligned} \tag{8.4}$$

where ω_1, ω_2 are two elements of $\mathcal{H}([\alpha,\beta]; \mathbb{R}^n)$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function x on $\mathcal{H}([\alpha,\beta], \mathbb{R}^n)$ as

$$x(t) = \bar{x}(z,t), t > 0, \ \ z \in [\alpha,\beta], \tag{8.5}$$

the operator $\mathcal{A}$ in $\mathcal{H}([\alpha,\beta],\mathbb{R}^n)$ as:

$$\begin{aligned}
\mathcal{A}x &= A\frac{\partial \bar{x}}{\partial z} + B\frac{\partial^2 \bar{x}}{\partial z^2}. \\
x \in D(\mathcal{A}) &= \left\{ x \in \mathcal{H}([\alpha,\beta];\mathbb{R}^n);\ C_1x(\alpha) + D_1\frac{\partial x}{\partial z}(\alpha) = R_1, \right. \\
&\quad \left. C_2x(\beta) + D_2\frac{\partial x}{\partial z}(\beta) = R_2 \right\}
\end{aligned} \tag{8.6}$$

and the input and output operators as:

$$\mathcal{B}u = wbu, \quad \mathcal{C}x = (c, kx) \tag{8.7}$$

where $c = [c^1\ c^2\ \cdots c^l]$, the system of Eqs.8.1-8.2-8.3 takes the form:

$$\begin{aligned}
\dot{x} &= \mathcal{A}x + \mathcal{B}u + f(x) \\
y &= \mathcal{C}x \\
x(0) &= x_0
\end{aligned} \tag{8.8}$$

where $f(x(t)) = f(\bar{x}(z,t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear term $f(x)$ satisfies $f(0) = 0$ and is also locally Lipschitz continuous i.e., there exist positive real numbers a_0, K_0 such that for any $x_1, x_2 \in \mathcal{H}$ that satisfy $\max\{||x_1||_2, ||x_2||_2\} \leq a_0$, we have that:

$$||f(x_1) - f(x_2)||_2 \leq K_0||x_1 - x_2||_2 \tag{8.9}$$

For $\mathcal{A}$, the standard eigenvalue problem is of the form:

$$\mathcal{A}\phi_j = \lambda_j\phi_j, \quad j = 1, \ldots, \infty \tag{8.10}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction; the eigenspectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of $\mathcal{A}$, i.e. $\sigma(\mathcal{A}) = \{\lambda_1, \lambda_2, \ldots,\}$. Assumption 8.1 that follows states our hypotheses for the properties of $\sigma(\mathcal{A})$.

Assumption 8.1:

1. *$Re\ \lambda_1 \leq Re\ \lambda_2 \leq \cdots \leq Re\ \lambda_j \leq \cdots$, where $Re\ \lambda_j$ denotes the real part of λ_j.*

2. $\sigma(\mathcal{A})$ *can be partitioned as* $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) + \sigma_2(\mathcal{A})$, *where* $\sigma_1(\mathcal{A})$ *consists of the first* m *(with* m *finite) eigenvalues, i.e.* $\sigma_1(\mathcal{A}) = \{\lambda_1, \dots, \lambda_m\}$, *and* $O(\frac{Re\ \lambda_1}{Re\ \lambda_m}) = O(1)$.

3. $Re\ \lambda_{m+1} > 0$ *and* $|Re\ \lambda_1| << Re\ \lambda_{m+1}$.

Assumption 8.1 states that the eigenspectrum of $\mathcal{A}$ can be partitioned into a finite-dimensional part consisting of m slow eigenvalues, $\sigma_1(\mathcal{A}) = \{\lambda_1, \dots, \lambda_m\}$, and a stable infinite-dimensional part containing the remaining fast eigenvalues, $\sigma_2(\mathcal{A}) = \{\lambda_{m+1}, \dots, \}$, and that the separation between slow and fast eigenvalues of $\mathcal{A}$ is large. The assumption of finite number of unstable eigenvalues is always satisfied for parabolic PDEs [Fri76], while the existence of only a few dominant modes that capture the dominant dynamics of a parabolic PDE system is well-established for the majority of diffusion-convection-reaction processes (see for example the applications in the book [Ray81] and the packed-bed reactor example studied in [CD96f]).

Assumption 8.1 guarantees [Fri76] that $\mathcal{A}$ generates a strongly continuous semigroup of bounded linear operators $U(t)$ which implies that the generalized solution of the system of Eq.8.8 is given by:

$$x = U(t)x_0 + \int_0^t U(t-s)(\mathcal{B}u(s) + f(x(s)))ds \tag{8.11}$$

$U(t)$ satisfies the following growth property:

$$||U(t)||_2 \leq K_1 e^{a_1 t}, \ \forall t \geq 0 \tag{8.12}$$

where K_1, a_1 are positive real numbers, with $K_1 \geq 1$ and $a_1 \geq Re\ \lambda_1$. If a_1 is strictly negative, we will say that $\mathcal{A}$ generates an exponentially stable semigroup $U(t)$. Throughout the manuscript, we will focus on local exponential stability, and not on weak (asymptotic) stability [Fri76], because of its robustness to bounded perturbations (e.g. uncertain variables, disturbances), which are always present in most practical applications [Bal91].

We will now review the application of Galerkin's method to the system of Eq.8.8 to derive an approximate finite-dimensional system. Let $\mathcal{H}_s, \mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as $\mathcal{H}_s = span\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = span\{\phi_{m+1}, \phi_{m+2}, \ldots, \}$ (the existence of $\mathcal{H}_s$, $\mathcal{H}_f$ follows from assumption 1). Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq.8.8 can be decomposed as:

$$x = x_s + x_f = P_s x + P_f x \tag{8.13}$$

Applying P_s and P_f to the system of Eq.8.8 and using the above decomposition for x, the system of Eq.8.8 can be equivalently written in the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f) \\ \frac{\partial x_f}{\partial t} &= \mathcal{A}_f x_f + \mathcal{B}_f u + f_f(x_s, x_f) \\ y &= \mathcal{C} x_s + \mathcal{C} x_f \\ x_s(0) &= P_s x(0) = P_s x_0, \quad x_f(0) = P_f x(0) = P_f x_0 \end{aligned} \tag{8.14}$$

where $\mathcal{A}_s = P_s \mathcal{A} P_s$, $\mathcal{B}_s = P_s \mathcal{B}$, $f_s = P_s f$, $\mathcal{A}_f = P_f \mathcal{A} P_f$, $\mathcal{B}_f = P_f \mathcal{B}$ and $f_f = P_f f$ and the notation $\frac{\partial x_f}{\partial t}$ is used to denote that the state x_f belongs in an infinite-dimensional space. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = diag\{\lambda_j\}$, $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are Lipschitz vector functions, and $\mathcal{A}_f$ is an unbounded differential operator which generates a strongly continuous exponentially stable semigroup (following from part 3 of assumption 1 and the selection of $\mathcal{H}_s, \mathcal{H}_f$). Neglecting the fast modes, the following finite-dimensional system is derived:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, 0) \\ y_s &= \mathcal{C} x_s \end{aligned} \tag{8.15}$$

where the subscript s in y_s denotes that the output is associated with the slow system. The above system can be directly used for controller design employing standard control methods for ODEs [Bal79, Ray81, CC92].

Remark 8.1: We note that the large separation of slow and fast modes of the spatial operator in parabolic PDEs ensures that a controller which exponentially stabilizes the closed-loop ODE system, also stabilizes the closed-loop infinite-dimensional system [Bal79]. This is in contrast to the application of this approach to hyperbolic PDEs where the eigenmodes cluster along nearly vertical asymptotes in the complex plane and thus, the controller has to be modified to compensate for the destabilizing effect of the residual modes [Bal91].

Remark 8.2: Whenever the eigenfunctions ϕ_j of the operator $\mathcal{A}$ can not be calculated analytically, available identification schemes for the estimation of the eigenfunctions and the adjoint eigenfunctions of $\mathcal{A}$, (see for example [CC92]), that utilize numerical experiments, can be employed to approximately calculate them. In this case, the interconnection of Eq.8.14 will take the form

$$\begin{aligned}\frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{A}_{sf} x_f + \mathcal{B}_s u + f_s(x_s, x_f) \\ \frac{\partial x_f}{\partial t} &= \mathcal{A}_{fs} x_s + \mathcal{A}_f x_f + \mathcal{B}_f u + f_f(x_s, x_f) \\ y &= \mathcal{C} x_s + \mathcal{C} x_f\end{aligned} \tag{8.16}$$

where $\mathcal{A}_{sf} = P_s \mathcal{A} P_f, \mathcal{A}_{fs} = P_f \mathcal{A} P_s$ are bounded operators, and the terms $\mathcal{A}_{sf} x_f, \mathcal{A}_{fs} x_s$ represent the modeling errors resulting from the use of approximate eigenfunctions of $\mathcal{A}$ in the subspaces $\mathcal{H}_s$, $\mathcal{H}_f$ instead of the exact ones, and constitute interactions between the x_s and x_f subsystems.

8.3 Accuracy of ODE system derived via Galerkin's method

In this section, we establish, using singular perturbation methods, that if the infinite-dimensional x_f-subsystem of the system of Eq.8.14 is stable and the finite-dimensional system of Eq.8.15 is also stable, then the system of Eq.8.14 is stable and the discrepancy between the solution of the x_s-subsystem of the system of Eq.8.14 and the solution of the system of Eq.8.15, is proportional to the spectral separation of the slow and fast eigenvalues.

Defining $\epsilon = \frac{|Re\ \lambda_1|}{Re\ \lambda_{m+1}}$, the system of Eq.8.14 can be written in the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f) \\ \epsilon \frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon} x_f + \epsilon \mathcal{B}_f u + \epsilon f_f(x_s, x_f) \end{aligned} \tag{8.17}$$

where $\mathcal{A}_{f\epsilon}$ is an unbounded differential operator defined as $\mathcal{A}_{f\epsilon} = \epsilon \mathcal{A}_f$. Since $\epsilon \ll 1$ (following from assumption 8.1, part 3) and the operators $\mathcal{A}_s, \mathcal{A}_{f\epsilon}$ generate semigroups with growth rates which are of the same order of magnitude, the system of Eq.8.17 is in the standard singularly perturbed form, with x_s being the slow states and x_f being the fast states.

Introducing the fast time-scale $\tau = \frac{t}{\epsilon}$ and setting $\epsilon = 0$, we obtain the following infinite-dimensional fast subsystem from the system of Eq.8.17:

$$\frac{\partial x_f}{\partial \tau} = \mathcal{A}_{f\epsilon} x_f \tag{8.18}$$

From the fact that $Re\ \lambda_{m+1} > 0$ and the definition of ϵ, we have that the above system is globally exponentially stable. Setting $\epsilon = 0$ in the system of Eq.8.17 and using the fact that the inverse operator $\mathcal{A}_{f\epsilon}^{-1}$ exists and is also bounded (it follows from the fact that zero is in the resolvent of $\mathcal{A}_{f\epsilon}$), we have that:

$$x_f = 0 \tag{8.19}$$

and thus the finite-dimensional slow system takes the form:

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, 0) \tag{8.20}$$

We note that the above system is identical to the one obtained by applying the standard Galerkin's method to the system of Eq.8.8, keeping the first m ODEs and completely neglecting the x_f-subsystem. Assumption 8.2 that follows states a stability requirement on the system of Eq.8.20.

Assumption 8.2: *The finite-dimensional system of Eq.8.20 with $u(t) \equiv 0$ is exponentially stable i.e., there exist positive real numbers (K_2, a_2, a_4), with $a_4 > K_2 \geq 1$*

such that for all $x_s \in \mathcal{H}_s$ that satisfy $|x_s| \leq a_4$, the following bound holds:

$$|x_s(t)| \leq K_2 e^{-a_2 t}|x_s(0)|, \;\; \forall \, t \geq 0 \tag{8.21}$$

Proposition 8.1 that follows establishes that the solutions of the open-loop systems of Eqs.8.20-8.18, after a short finite time interval required for the trajectories of the system of Eq.8.17 to approach the quasi steady-state of Eq.8.19, consist an $O(\epsilon)$ approximation of the solutions of the open-loop system of Eq.8.17. The proof is given in the appendix G.

Proposition 8.1: *Consider the system of Eq.8.17 with $u(t) \equiv 0$ and suppose that assumptions 8.1 and 8.2 hold. Then, there exist positive real numbers μ_1, μ_2, ϵ^* such that if $|x_s(0)| \leq \mu_1$, $||x_f(0)||_2 \leq \mu_2$ and $\epsilon \in (0, \epsilon^*]$, then the solution $x_s(t), x_f(t)$ of the system of Eq.8.17 satisfies for all $t \in [0, \infty)$:*

$$\begin{aligned} x_s(t) &= \bar{x}_s(t) + O(\epsilon) \\ x_f(t) &= \bar{x}_f(\frac{t}{\epsilon}) + O(\epsilon) \end{aligned} \tag{8.22}$$

where $\bar{x}_s(t), \bar{x}_f(t)$ are the solutions of the slow and fast subsystems of Eqs.8.20-8.18 with $u(t) \equiv 0$, respectively.

Remark 8.3: The counterpart of the result of Proposition 8.1 in finite-dimensional spaces is well-known (Tikhonov's theorem, [Tik48]), while a similar result has also been established for linear infinite-dimensional systems [Bal84]. The main technical difference in establishing this result between linear and quasi-linear infinite-dimensional systems is that, for quasi-linear systems the proof is based on Lyapunov arguments, while for linear systems the proof is obtained using combination of estimates of the states, obtained from the application of variations of constants formula [Bal84]. This is a consequence of the fact that for quasi-linear systems it is not possible to derive a coordinate change that transforms the system of Eq.8.17 into a cascaded interconnection where the fast modes modes are decoupled from the slow modes, which allows to derive an exponentially decaying estimate, for sufficiently small ϵ, for the fast state, which is independent of the one of the slow state, and thus to prove the result through a direct combination of these estimates.

Remark 8.4: We note that it is possible, using standard results from center manifold theory for infinite-dimensional systems of the form of Eq.8.8 [Car81, p.118], to show that if the the system of Eq.8.20 is asymptotically stable, then the system of Eq.8.8 is also asymptotically stable and the discrepancy between the solution of the system of Eq.8.20 and the x_s-subsystem of the closed-loop full-order is asymptotically (as $t \rightarrow \infty$) proportional to ϵ. Although this result is important because it allows establishing asymptotic stability of the closed-loop infinite-dimensional system by performing a stability analysis on a low-order finite-dimensional system, it does not provide any information about the discrepancy between the solutions of these two systems for finite t.

8.4 Construction of minimal-order ODE systems via AIMs

In this section, we propose an approach originating from the theory of inertial manifolds for the construction of ODE systems of dimension m which yield solutions that are arbitrarily close (closer than $O(\epsilon)$) to the ones of the infinite-dimensional system of Eq.8.8, for almost all times. An inertial manifold $\mathcal{M}$ for the system of Eq.8.8 is a subset of $\mathcal{H}$, which satisfies the following properties [Tem88]: *i)* $\mathcal{M}$ is a finite-dimensional Lipschitz manifold, *ii)* $\mathcal{M}$ is a graph of a Lipschitz function $\Sigma(x_s, u, \epsilon)$ mapping $\mathcal{H}_s \times \mathbb{R}^l \times (0, \epsilon^*]$ into $\mathcal{H}_f$ and for every solution $x_s(t), x_f(t)$ of Eq.8.17 with $x_f(0) = \Sigma(x_s(0), u, \epsilon)$, then

$$x_f(t) \;=\; \Sigma(x_s(t), u, \epsilon), \quad \forall\, t \geq 0 \tag{8.23}$$

and *iii*) $\mathcal{M}$ attracts every trajectory exponentially. The evolution of the state x_f on $\mathcal{M}$ is given by Eq.8.23, while the evolution of the state x_s is given by the following finite-dimensional inertial form:

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, \Sigma(x_s, u, \epsilon)) \tag{8.24}$$

Assuming that $u(t)$ is smooth, differentiating Eq.8.23 and utilizing Eq.8.17, $\Sigma(x_s, u, \epsilon)$ can be computed as the solution of the following partial differential equation:

$$\epsilon\frac{\partial \Sigma}{\partial x_s}[\mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f)] + \epsilon\frac{\partial \Sigma}{\partial u}\dot{u} = \mathcal{A}_{f\epsilon} x_f + \epsilon \mathcal{B}_f u + \epsilon f_f(x_s, x_f) \tag{8.25}$$

which Σ has to satisfy for all $x_s \in \mathcal{H}_s$, $u \in \mathbb{R}^l$, $\epsilon \in (0, \epsilon^*]$. However, even for parabolic PDEs for which it is known that $\mathcal{M}$ exists, the derivation of an explicit analytic form of $\Sigma(x_s, u, \epsilon)$ is an extremely difficult (if not impossible) task.

Motivated by this, we will now propose a procedure, motivated by singular perturbations [KKO86], to compute approximations of $\Sigma(x_s, u, \epsilon)$ (approximate inertial manifolds) and approximations of the inertial form, of desired accuracy. To this end, consider an expansion of $\Sigma(x_s, u, \epsilon)$ and u in a power series in ϵ:

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots + \epsilon^k u_k + O(\epsilon^{k+1}) \\ \Sigma(x_s, u, \epsilon) &= \Sigma^0(x_s, u) + \epsilon\Sigma^1(x_s, u) + \epsilon^2\Sigma^2(x_s, u) + \cdots + \epsilon^k\Sigma^k(x_s, u) + O(\epsilon^{k+1}) \end{aligned} \tag{8.26}$$

where u_k, Σ^k are smooth functions. Substituting the expressions of Eq.8.26 into Eq.8.25, and equating terms of the same power in ϵ, one can obtain approximations of $\Sigma(x_s, u, \epsilon)$ up to a desired order. Substituting the expansion for $\Sigma(x_s, u, \epsilon)$ and u up to order k into Eq.8.24, the following approximation of the inertial form is obtained:

$$\begin{aligned} \frac{dx_s}{dt} = {} & \mathcal{A}_s x_s + \mathcal{B}_s\left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots + \epsilon^k u_k\right) \\ & + f_s(x_s, \Sigma^0(x_s, u) + \epsilon\Sigma^1(x_s, u) + \epsilon^2\Sigma^2(x_s, u) + \cdots + \epsilon^k\Sigma^k(x_s, u)) \end{aligned} \tag{8.27}$$

In order to characterize the discrepancy between the solution of the open-loop finite-dimensional system of Eq.8.27 and the solution of the x_s-subsystem of the open-loop infinite-dimensional system of Eq.8.17, we will impose a stability requirement on the system of Eq.8.27.

Assumption 8.3: *The finite-dimensional system of Eq.8.27 with $u(t) \equiv 0$ is exponentially stable i.e., there exist positive real numbers $(\bar{K}_2, \bar{a}_2, \bar{a}_4)$, with $\bar{a}_4 > \bar{K}_2 \geq 1$ such that for all $x_s \in \mathcal{H}_s$ that satisfy $|x_s| \leq \bar{a}_4$, the following bound holds:*

$$|x_s(t)| \leq \bar{K}_2 e^{-\bar{a}_2 t}|x_s(0)|, \ \forall\, t \geq 0 \tag{8.28}$$

Proposition 8.2 that follows establishes that the discrepancy between the solutions obtained from the open-loop system of Eq.8.27 and the expansion for $\Sigma(x_s, u, \epsilon)$ of Eq.8.30, and the solutions of the infinite-dimensional open-loop system of Eq.8.17 is of $O(\epsilon^{k+1})$, for almost all times. The proof is given in the appendix G.

Proposition 8.2: *Consider the system of Eq.8.17 with $u(t) \equiv 0$ and suppose that assumptions 8.1 and 8.3 hold. Then, there exist positive real numbers $\bar{\mu}_1, \bar{\mu}_2, \bar{\epsilon}^*$ such that if $|x_s(0)| \leq \bar{\mu}_1$, $||x_f(0)||_2 \leq \bar{\mu}_2$ and $\epsilon \in (0, \bar{\epsilon}^*]$, then the solution $x_s(t), x_f(t)$ of the system of Eq.8.17 satisfies for all $t \in [t_b, \infty)$:*

$$\begin{aligned} x_s(t) &= \tilde{x}_s(t) + O(\epsilon^{k+1}) \\ x_f(t) &= \tilde{x}_f(t) + O(\epsilon^{k+1}) \end{aligned} \tag{8.29}$$

where t_b is the time required for $x_f(t)$ to approach $\tilde{x}_f(t)$, $\tilde{x}_s(t)$ is the solution of Eq.8.27 with $u(t) \equiv 0$, and $\tilde{x}_f(t) = \epsilon\Sigma^1(\tilde{x}_s, 0) + \epsilon^2\Sigma^2(\tilde{x}_s, 0) + \cdots + \epsilon^k\Sigma^k(\tilde{x}_s, 0)$.

Remark 8.5: Following the proposed approximation procedure, we obtain that the $O(\epsilon)$ approximation of $\Sigma(x_s, 0, \epsilon)$ is $\Sigma^0(x_s, 0) = 0$ and the corresponding approximate inertial form is identical to the system of Eq.8.20 with $u(t) \equiv 0$. This system does not utilize any information about the structure of the fast subsystem, and thus, this system yields solutions which are only $O(\epsilon)$ close to the solutions of the open-loop system of Eq.8.8 (proposition 8.1). On the other hand, the $O(\epsilon^2)$ approximation of

$\Sigma(x_s, 0, \epsilon)$ can be shown to be of the form:

$$\Sigma(x_s, 0, \epsilon) = \Sigma^0(x_s, 0) + \epsilon\Sigma^1(x_s, 0) = \epsilon(\mathcal{A}_{f\epsilon})^{-1}[-f_f(x_s, 0)] \tag{8.30}$$

The corresponding open-loop approximate inertial form does utilize information about the structure of the fast subsystem, and thus allows to obtain solutions which are $O(\epsilon^2)$ close to the solutions of the open-loop system of Eq.8.8 (proposition 8.2).

Remark 8.6: The standard approach followed in the literature for the construction of AIMs for systems of the form of Eq.8.14 with $u(t) \equiv 0$ (see for example [BKJ91]) is to directly set $\dfrac{\partial x_f}{\partial t} \equiv 0$, solve the resulting algebraic equations for x_f and substitute the solution for x_f to the x_s–subsystem of Eq.8.14, to derive the following ODE system:

$$\frac{dx_s}{dt} = \mathcal{A}_s x_s + f_s(x_s, (\mathcal{A}_f)^{-1}[-f_f(x_s)]) \tag{8.31}$$

It is straightforward to show that the slow system of Eq.8.31 is identical to the one obtained by using the $O(\epsilon^2)$ approximation for $\Sigma(x_s, 0, \epsilon)$ for the construction of the approximate inertial form.

Remark 8.7: The expansion of u in a power series in ϵ is motivated by our intention to modify the synthesis of the feedback controller appropriately such that the output of the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form will be arbitrarily close to the output of the closed-loop PDE system for almost all times (see also remark 8.8).

8.5 Finite-dimensional control

In this section, we use the result of proposition 8.2 to establish that a nonlinear finite-dimensional output feedback controller that guarantees stability and enforces output tracking in the ODE system of Eq.8.27, exponentially stabilizes the closed-loop PDE system and ensures that the discrepancy between the output of the closed-loop ODE system and the output of the closed-loop PDE system is of $O(\epsilon^{k+1})$, provided that ϵ is sufficiently small.

The finite-dimensional output feedback controller, which achieves the desired objectives for the system of Eq.8.27, is constructed through a standard combination of a state feedback controller with a state observer. In particular, we consider a state feedback control law of the general form:

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \cdots + \epsilon^k u_k \\ &= p_0(x_s) + Q_0(x_s)v + \epsilon[p_1(x_s, \epsilon) + Q_1(x_s, \epsilon)v] + \cdots + \epsilon^k[p_k(x_s, \epsilon) + Q_k(x_s, \epsilon)v] \end{aligned} \tag{8.32}$$

where $p_0(x_s), \ldots, p_k(x_s)$ are smooth vector functions, $Q_0(x_s), \ldots, Q_k(x_s)$ are smooth matrices, and $v \in \mathbb{R}^l$ is the constant reference input vector (see remark 7 for a procedure for the synthesis of the control law i.e. the explicit computation of $[p_0(x_s), \ldots, p_k(x_s)\ Q_0(x_s), \ldots, Q_k(x_s)]$). The following m-dimensional state observer is also considered for the implementation of the state feedback law of Eq.8.32:

$$\begin{aligned} \frac{d\eta}{dt} &= \mathcal{A}_s\eta + \mathcal{B}_s\left(p_0(x_s) + Q_0(x_s)v + \epsilon[p_1(x_s, \epsilon) + Q_1(x_s, \epsilon)v] + \cdots + \right. \\ &\quad \left. \epsilon^k[p_k(x_s, \epsilon) + Q_k(x_s, \epsilon)v]\right) + f_s(\eta, \epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)) \\ &\quad + L(y - \left[C\eta + \mathcal{C}\{\epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)\}\right]) \end{aligned} \tag{8.33}$$

where $\eta \in \mathcal{H}_s$ denotes the observer state vector, and L is a matrix chosen so that the eigenvalues of the matrix $C_L = \mathcal{A}_s + \frac{\partial f_s}{\partial \eta}_{(\eta=\eta_s)}$
$-L\left[\mathcal{C}\eta_s + \mathcal{C}\{\frac{\partial}{\partial \eta}\left(\epsilon\Sigma^1(\eta, u(\eta)) + \epsilon^2\Sigma^2(\eta, u(\eta)) + \cdots + \epsilon^k\Sigma^k(\eta, u(\eta))\right)_{(\eta=\eta_s)}\}\right]$ lie in the open left-half of the complex plane, where η_s denotes the steady-state for the system of Eq.8.33. The finite-dimensional output feedback controller resulting from the combination of the state feedback controller of Eq.8.32 with the state observer of Eq.8.33

takes the form:

$$
\begin{aligned}
\frac{d\eta}{dt} &= \mathcal{A}_s\eta + \mathcal{B}_s\,(p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta,\epsilon) + Q_1(\eta,\epsilon)v] + \cdots + \\
&\quad \epsilon^k[p_k(\eta,\epsilon) + Q_k(\eta,\epsilon)v]\Big) + f_s(\eta,\epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) + \cdots + \epsilon^k\Sigma^k(\eta,u)) \\
&\quad + L(y - \Big[\mathcal{C}\eta + \mathcal{C}\{\Sigma^0(\eta,u) + \epsilon\Sigma^1(\eta,u) + \epsilon^2\Sigma^2(\eta,u) + \cdots + \epsilon^k\Sigma^k(\eta,u)\}\Big]) \\
u &= p_0(\eta) + Q_0(\eta)v + \epsilon[p_1(\eta,\epsilon) + Q_1(\eta,\epsilon)v] + \cdots + \epsilon^k[p_k(\eta,\epsilon) + Q_k(\eta,\epsilon)v]
\end{aligned}
\tag{8.34}
$$

We note that the above controller does not use feedback of the fast state vector x_f in order to avoid destabilization of the fast modes of the closed-loop system. Assumption 8.4 states the desired control objectives under the controller of Eq.8.34.

Assumption 8.4: *The finite-dimensional output feedback controller of the form of Eq.8.34 exponentially stabilizes the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form and ensures that its outputs $y_s^i(t)$, $i = 1,\ldots,l$, are the solutions of a known $l-$dimensional ODE system of the form $\phi(y_s^{i(r_i)}, y_s^{i(r_i-1)},\cdots,y_s^i,v) = 0$, where ϕ is a vector function and r_i is an integer.*

Theorem 8.1 provides a precise characterization of the stability and closed-loop transient performance enforced by the controller of Eq.8.34 in the closed-loop parabolic PDE system (the proof is given in the appendix G).

Theorem 8.1: *Consider the parabolic PDE system of Eq.8.8, for which assumptions 8.1 and 8.4 hold. Then, there exist positive real numbers $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\epsilon}^*$ such that if $|x_s(0)| \leq \tilde{\mu}_1$, $||x_f(0)||_2 \leq \tilde{\mu}_2$ and $\epsilon \in (0,\tilde{\epsilon}^*]$, then the controller of Eq.8.34:*

a) guarantees exponential stability of the closed-loop system, and

b) ensures that the outputs of the closed-loop system satisfy for all $t \in [t_b,\infty)$:

$$
y^i(t) = y_s^i(t) + O(\epsilon^{k+1}),\ i = 1,\ldots,l \tag{8.35}
$$

where $y_s^i(t)$ is the i-th output of the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form.

Remark 8.8: The construction of the state feedback law of Eq.8.32. to ensure that the control objectives stated in assumption 8.4 are enforced in the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form, can be performed following a sequential procedure. Specifically, the component $u_0 = p_0(x_s) + Q_0(x_s)v$ can be initially synthesized on the basis of the $O(\epsilon)$ approximation of the inertial form (Eq.8.20): then the component $u_1 = p_1(x_s, \epsilon) + Q_1(x_s, \epsilon)v$ can be synthesized on the basis of the $O(\epsilon^2)$ approximation of the inertial form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u_0 + \epsilon \mathcal{B}_s u_1 + f_s(x_s, \epsilon (A_f^\epsilon)^{-1}[-\mathcal{B}_f u_0 - f_f(x_s, 0)]) \\ &= : f_1(x_s, \epsilon) + g_1(x_s, \epsilon) u_1 \\ y_s &= \mathcal{C} x_s + \epsilon \mathcal{C}(\mathcal{A}_{f\epsilon})^{-1}[-\mathcal{B}_f u_0 - f_f(x_s, 0)] =: h_1(x_s, \epsilon) \end{aligned} \tag{8.36}$$

In general, at the k-th step, the component $u_k = p_k(x_s, \epsilon) + Q_k(x_s, \epsilon)v$ can be synthesized on the basis of the $O(\epsilon^k)$ approximation of the inertial form (Eq.8.27). The synthesis of $[p_\nu(x_s, \epsilon), Q_\nu(x_s, \epsilon)]$, $\nu = 0, \ldots, k$, can be performed, at each step, utilizing standard geometric control methods for nonlinear ODEs (see next section).

Remark 8.9: The implementation of the controller of Eq.8.34 requires to explicitly compute the vector function $\Sigma^k(\eta, u)$. However, $\Sigma^k(\eta, u)$ has an infinite-dimensional range and therefore cannot be implemented in practice. Instead a finite-dimensional approximation of $\Sigma^k(\eta, u)$, say $\Sigma_t^k(\eta, u)$, can be derived by keeping the first $\bar{m}$ elements of $\Sigma^k(\eta, u)$ and neglecting the remaining infinite ones. Clearly, as $\bar{m} \to \infty$, $\Sigma_t^k(\eta, u)$ approaches $\Sigma^k(\eta, u)$. This implies that by picking $\bar{m}$ to be sufficiently large, the controller of Eq.8.34 with $\Sigma_t^k(\eta, u)$ instead of $\Sigma^k(\eta, u)$ guarantees stability and enforces the requirement of Eq.8.35 in the closed-loop infinite-dimensional system.

8.6 Finite-dimensional controller synthesis

8.6.1 Preliminaries

In this section, we will synthesize a finite-dimensional output feedback controller for the system of Eq.8.8 on the basis of the system of Eq.8.27, using geometric control methods. To this end, we will initially review the concepts of relative order and characteristic matrix that will be used in the subsequent section to synthesize the controller. Referring to the system of Eq.8.20, we set, in order to simplify the notation, $\mathcal{A}_s x_s + f_s(x_s, 0) = f_0(x_s)$, $\mathcal{B}_s = g_0$, $\mathcal{C}x_s = h_0(x_s)$. The relative order of the output y_s^i with respect to the vector of manipulated inputs u is defined as the smallest integer r_i for which

$$\left[L_{g_0^1} L_{f_0}^{r_i-1} h_0^i(x_s) \; \cdots \; L_{g_0^l} L_{f_0}^{r_i-1} h_0^i(x_s)\right] \not\equiv [0 \; \cdots \; 0] \tag{8.37}$$

or $r_i = \infty$ if such an integer does not exist. Furthermore, the matrix:

$$C(x_s) = \begin{bmatrix} L_{g_0^1} L_{f_0}^{r_1-1} h_0^1(x_s) & \cdots & L_{g_0^l} L_{f_0}^{r_1-1} h_0^1(x_s) \\ L_{g_0^1} L_{f_0}^{r_2-1} h_0^2(x_s) & \cdots & L_{g_0^l} L_{f_0}^{r_2-1} h_0^2(x_s) \\ \vdots & & \\ L_{g_0^1} L_{f_0}^{r_l-1} h_0^l(x_s) & \cdots & L_{g_0^l} L_{f_0}^{r_l-1} h_0^l(x_s) \end{bmatrix} \tag{8.38}$$

is the characteristic matrix of the system. For simplicity, we will assume that $det(C(x_s)) \neq 0$.

8.6.2 Controller formula

Theorem 8.2 provides the synthesis formula of the output feedback controller and conditions that guarantee closed-loop stability in the case of considering a $O(\epsilon^2)$ approximation of the exact slow system for the synthesis of the controller. The derivation of synthesis formulas for higher-order approximations of the output feedback controller is notationally complicated, although conceptually straightforward, and thus will be omitted for reasons of brevity.

Theorem 8.2: *Consider the parabolic PDE system of Eq.8.8, for which assumptions 8.1 and 8.2 hold. Consider also the slow subsystem of Eq.8.36, and assume that its characteristic matrix $\tilde{C}(x_s,\epsilon)$ is invertible $\forall\ x_s \in \mathcal{H}_s$, $\epsilon \in [0,\epsilon^*]$. Suppose also that the following conditions hold:*

1. *The roots of the equation:*

$$det(B(s)) = 0 \tag{8.39}$$

where $B(s)$ is a $l \times l$ matrix, whose (i,j)-th element is of the form $\sum_{k=0}^{r_i} \beta^i_{jk} s^k$, lie in the open left-half of the complex plane.

2. *The unforced ($v \equiv 0$) zero dynamics of the system of Eq.8.36 is locally exponentially stable.*

Then, there exist constants μ_1, μ_2, ϵ^* such that if $|x_s(0)| \leq \mu_1$, $||x_f(0)||_2 \leq \mu_2$ and $\epsilon \in (0,\epsilon^*]$, then if $\eta(0) = x_s(0)$, the dynamic output feedback controller:

$$\begin{aligned}
\frac{d\eta}{dt} &= \mathcal{A}_s\eta + \mathcal{B}_s u(\eta) + f_s(\eta, \epsilon(\mathcal{A}_{f\epsilon})^{-1}[-\mathcal{B}_f u_0(\eta) - f_f(\eta,0)) \\
&\quad + L\left(y - [\mathcal{C}\eta + \epsilon(\mathcal{A}_{f\epsilon})^{-1}[-\mathcal{B}_f u_0(\eta) - f_f(\eta,0)]\right) \\
u &= u_0(\eta) + \epsilon u_1(\eta) := \{[\beta_{1r_1} \cdots \beta_{lr_l}]C(\eta)\}^{-1}\left\{v - \sum_{i=1}^{l}\sum_{k=0}^{r_i}\beta_{ik}L^k_{f_0}h^i_0(\eta)\right\} \\
&\quad + \epsilon\left\{[\beta_{1r_1} \cdots \beta_{lr_l}]\tilde{C}(\eta,\epsilon)\right\}^{-1}\left\{v - \sum_{i=1}^{l}\sum_{k=0}^{r_i}\beta_{ik}L^k_{f_1}h^i_1(\eta,\epsilon)\right\}
\end{aligned} \tag{8.40}$$

a) guarantees exponential stability of the closed-loop system, and

b) ensures that the outputs of the closed-loop system satisfy a relation of the form :

$$y^i(t) \;=\; y^i_s(t) + O(\epsilon^2)\ ,\ i = 1,\ldots,l\ ,\ t \geq t_b \tag{8.41}$$

where t_b is the time required for the off-manifold fast transients to decay to zero exponentially, and $y^i_s(t)$, $i = 1,\ldots,l$, is the solution of:

$$\sum_{i=1}^{l}\sum_{k=0}^{r_i}\beta_{ik}\frac{d^k y^i_s}{dt^k} = v \tag{8.42}$$

Remark 8.10: In the case of open-loop stable systems, a more convenient way to reconstruct the state of the system is to consider the observer of Eq.8.33 with the matrix L set identically equal to zero. This is motivated by the fact that the open-loop stability of the system guarantees the convergence of the estimated values to the actual ones with transient behavior depending on the location of the spectrum of the matrix $\bar{C}_L = \frac{\partial f_1}{\partial \eta_s}(\eta_s, \epsilon)$.

Remark 8.11: The exponential stability of the closed-loop system guarantees that in the presence of small errors in process parameters, the states of the closed-loop system will be bounded. Furthermore, since the input/output spaces of the closed-loop system are finite dimensional, and the controller of Eq.8.40 enforces a linear input/output dynamics between y and v, it is possible to implement a linear error feedback controller around the $(y - v)$ loop to ensure asymptotic offsetless output tracking in the closed-loop system, in the presence of constant unknown process parameters and unmeasured disturbance inputs.

Remark 8.12: Note that in the case of imperfect initialization of the observer states (i.e., $\eta(0) \neq x_s(0)$), although a slight deterioration of the performance may occur, (i.e., the requirement of Eq.8.41 will not be exactly imposed in the closed-loop system), the output feedback controller of theorem 8.1 guarantees exponential stability and asymptotic offsetless output tracking in the closed-loop system.

8.7 Application to a packed-bed reactor

We consider a non-isothermal packed-bed reactor shown in Figure 8.1, where an elementary endothermic reaction of the form $A \rightarrow B$ takes place. Under standard modeling assumptions, the dynamic model of the process expressed in dimensionless

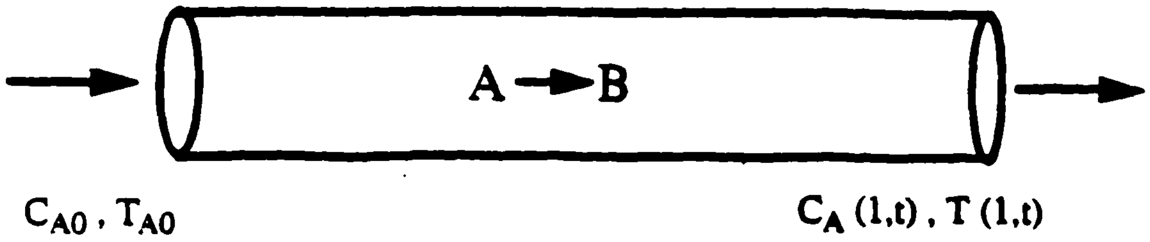


Figure 8.1: A nonisothermal packed-bed reactor.

variables takes the form [Ray81]:

$$\begin{aligned}
\frac{\partial \bar{x}_1}{\partial t} &= -\frac{\partial \bar{x}_1}{\partial z} + \frac{1}{Pe_T}\frac{\partial^2 \bar{x}_1}{\partial z^2} - B_T B_C exp^{\gamma \frac{\bar{x}_1}{1+\bar{x}_1}} \bar{x}_2 + \beta_T (u - \bar{x}_1) \\
\frac{\partial \bar{x}_2}{\partial t} &= -\frac{\partial \bar{x}_2}{\partial z} + \frac{1}{Pe_C}\frac{\partial^2 \bar{x}_2}{\partial z^2} - B_C exp^{\gamma \frac{\bar{x}_1}{1+\bar{x}_1}} \bar{x}_2
\end{aligned} \qquad (8.43)$$

subject to the boundary conditions:

$$z = 0, \quad Pe_T \bar{x}_1 = \frac{\partial \bar{x}_1}{\partial z}, \quad Pe_C(\bar{x}_2 - 1) = \frac{\partial \bar{x}_2}{\partial z}; \qquad z = 1, \quad \frac{\partial \bar{x}_1}{\partial z} = \frac{\partial \bar{x}_2}{\partial z} = 0 \qquad (8.44)$$

where $\bar{x}_1, \bar{x}_2$ denote dimensionless temperature and concentration in the reactor. Pe_T, Pe_C are the heat and mass Peclet numbers, B_T, B_C denote a dimensionless heat of reaction and a dimensionless pre-exponential factor, γ is a dimensionless activation energy, β_T is a dimensionless heat transfer coefficient, and u is a dimensionless jacket temperature which is assumed to be spatially uniform. The control objective is to control the temperature profile along the length of the reactor, i.e. the controlled output is of the form:

$$y(t) = \int_0^1 \bar{x}_1 dz \qquad (8.45)$$

The following typical values for the process parameters were used in our calculations:

$$Pe_T = 5.0, \quad Pe_C = 5.0, \quad B_C = 0.00001, \quad B_T = 1.0, \quad \beta_T = 15.62, \quad \gamma = 22.14 \qquad (8.46)$$

It was verified that the above process parameters correspond to a stable steady-state for the open-loop system. The process model of Eq.8.43 can be put in the form:

$$\begin{aligned}
\dot{x} &= \mathcal{A}x + \mathcal{B}u + f(x) \\
y &= \mathcal{C}x
\end{aligned} \qquad (8.47)$$

with

$$\mathcal{A}x = \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix} x = \begin{bmatrix} \frac{1}{Pe_T}\frac{\partial^2 \bar{x}_1}{\partial z^2} - \frac{\partial \bar{x}_1}{\partial z} & 0 \\ 0 & \frac{1}{Pe_C}\frac{\partial^2 \bar{x}_2}{\partial z^2} - \frac{\partial \bar{x}_2}{\partial z} \end{bmatrix} \qquad (8.48)$$

$$x \in D(\mathcal{A}) = \left\{ x \in \mathcal{H}^2([0,1];\mathbb{R}^2):\ Pe_T x_1(0) = \frac{\partial x_1}{\partial z}(0), \right. \\ \left. Pe_C(x_2(0)-1) = \frac{\partial x_2}{\partial z}(0);\ \frac{\partial x_1}{\partial z}(1) = \frac{\partial x_2}{\partial z}(1) = 0 \right\} \tag{8.49}$$

$$\mathcal{B}u = \begin{bmatrix} \beta_T \\ 0 \end{bmatrix} u, \quad \mathcal{C}x = (1, x) \tag{8.50}$$

The solution of the eigenvalue problem for the above spatial differential operator subject to the boundary conditions of Eq.8.44 can be obtained utilizing standard techniques from linear operator theory (see for example [Ray81]) and is of the form:

$$\begin{aligned} \lambda_{ij} &= \frac{\bar{a}_{ij}^2}{Pe} + \frac{Pe}{4}, \quad i = 1,2, \quad j = 1, \dots, \infty \\ \phi_{ij}(z) &= B_{ij} e^{Pe\frac{z}{2}} (cos(\bar{a}_{ij} z) + \frac{Pe}{2\bar{a}_{ij}} sin(\bar{a}_{ij} z)), \quad i = 1,2, \quad j = 1, \dots, \infty \\ \bar{\phi}_{ij}(z) &= e^{-Pe z} \phi_{ij}(z), \quad i = 1,2, \quad j = 1, \dots, \infty \end{aligned} \tag{8.51}$$

where $Pe = Pe_T = Pe_C$, and $\lambda_{ij}, \phi_{ij}, \bar{\phi}_{ij}$, denote the eigenvalues, eigenfunctions and adjoint eigenfunctions of L_i, respectively. $\bar{a}_{ij}, B_{ij}$ can be calculated from the following formulas:

$$\begin{aligned} tan(\bar{a}_{ij}) &= \frac{Pe\bar{a}_{ij}}{\bar{a}_{ij}^2 - (\frac{Pe}{2})^2}, \quad i = 1,2, \quad j = 1, \dots, \infty \\ B_{ij} &= \left\{ \int_0^1 \left(cos(\bar{a}_{ij} z) + \frac{Pe}{2\bar{a}_{ij}} sin(\bar{a}_{ij} z) \right)^2 dz \right\}^{-\frac{1}{2}}, \quad i = 1,2, \quad j = 1, \dots, \infty \end{aligned} \tag{8.52}$$

A direct computation of the first five eigenvalues of L yields $\lambda_{11} = \lambda_{21} = -1.94, \lambda_{12} = \lambda_{22} = -4.80, \lambda_{13} = \lambda_{23} = -10.42, \lambda_{14} = \lambda_{24} = -20.93$, and $\lambda_{15} = \lambda_{25} = -34.78$. These values indicate that the eigenspectrum of L exhibits a two-time-scale property and suggest considering the first two eigenmodes of each PDE as the slow ones and the remaining infinite eigenmodes as the fast ones. Defining $\mathcal{H}_s = span\{\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}\}$ and applying Galerkin's method to the system of Eq.8.47, a system of the form of Eq.8.14 is derived where $\mathcal{A}_s$ is a 4×4 diagonal matrix of the form

$\mathcal{A}_s = diag\{\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}\}$, and $\mathcal{A}_f$ is an exponentially stable differential operator whose smallest eigenvalue is $\lambda_{13} = \lambda_{23}$. Defining $\epsilon = \dfrac{\lambda_{11}}{\lambda_{13}} = \dfrac{\lambda_{21}}{\lambda_{23}}$ and following the procedure described in sections 5 and 6, we derived the $O(\epsilon)$ and $O(\epsilon^2)$ approximations of the exact slow system. The $O(\epsilon^2)$ approximation of $\Sigma(x_s, u, \epsilon)$ was constructed by retaining the first three of the fast modes for each PDE, and discarding the remaining infinite ones (this is because the use of more than three fast modes provides negligible improvement in the accuracy of the $O(\epsilon^2)$ approximation of the fourth-order model). The zero dynamics of the systems corresponding to the $O(\epsilon)$ and $O(\epsilon^2)$ approximations of the exact slow system were found to be exponentially stable. Therefore, the nonlinear finite-dimensional controller of Eq.8.40 with $\beta_1 = 0.2$, $\beta_0 = 1.0$ and $L = 0$ (this is possible because the open-loop process and thus the $O(\epsilon^2)$ approximation of the exact slow system are exponentially stable) was employed in the simulations. It was also established that the value of $\epsilon = 0.19$ ensures that the controller exponentially stabilizes the closed-loop parabolic PDE system as well.

Simulation runs were performed to illustrate the advantages obtained by using higher-order approximations of $\Sigma(x_s, u, \epsilon)$ in the construction of the slow subsystem of Eq.8.24 and evaluate the output tracking capabilities of the proposed control methodology. In particular, we compared the performance of the controller synthesized following the above procedure with that of a controller of the form of Eq.8.40 with $\epsilon = 0$ and $L = 0$ ($O(\epsilon)$ approximation). Figure 8.2 shows the output and manipulated input profiles for an 8.0% decrease in the value of the set-point (the new set-point value is $v = 1.11$). It is clear that the controller synthesized on the basis of the system which uses an $O(\epsilon^2)$ approximation for $\Sigma(x_s, u, \epsilon)$ provides an excellent performance driving the output (solid line) very close to the new set-point (note that as expected, $lim_{t\to\infty}|y - v| = O(\epsilon^2)$). On the other hand, the controller of the form of Eq.8.40 with $\epsilon = 0$ drives the output (dotted line) to a neighborhood of the set-point (note that $lim_{t\to\infty}|y - v| = O(\epsilon)$) leading to significant offset (compare with

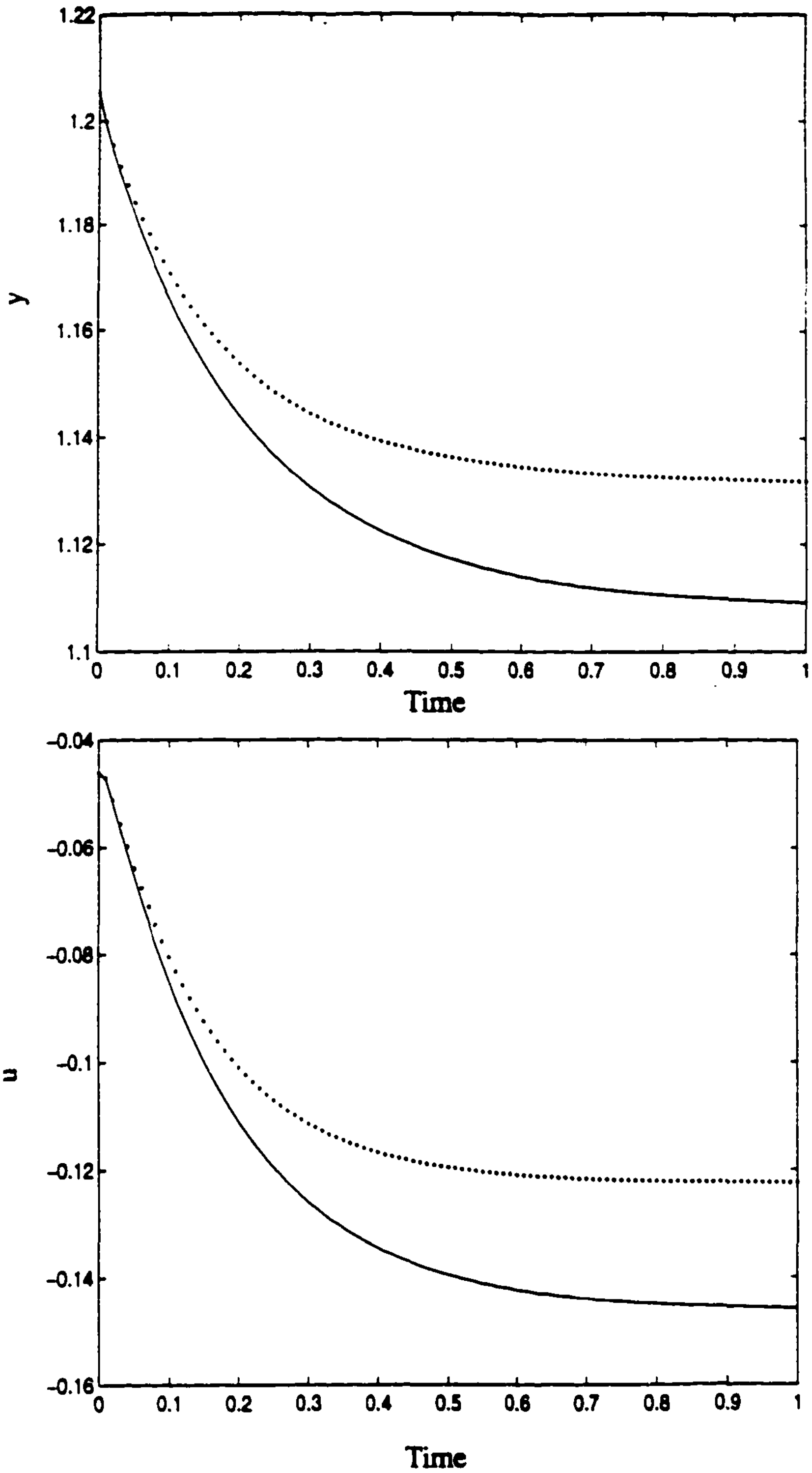


Figure 8.2: Comparison of output (top figure) and manipulated input (bottom figure) profiles of the closed-loop system, for an 8% decrease in the set point. The dotted lines correspond to a controller based on the slow ODE model with an $O(\epsilon)$ approximation for Σ, while the solid lines correspond to a controller based on the slow ODE model with an $O(\epsilon^2)$ approximation for Σ.

set-point value). Figure 8.3 displays the evolution of the dimensionless temperature of the reactor for the case of using an $O(\epsilon^2)$ approximation for $\Sigma(x_s, u, \epsilon)$. The controller achieves excellent performance, regulating the temperature at each point of the reactor to a new steady-state value, which is close to 8% lower than the one of the original steady-state. Finally, it was also verified through simulations that: *i*) the closed-loop output profile obtained by using a fourth-order system with an $O(\epsilon^2)$ approximation for $\Sigma(x_s, u, \epsilon)$ is comparable to the one achieved by using a tenth-order system with an $O(\epsilon)$ approximation for $\Sigma(x_s, u, \epsilon)$, and *ii*) the use of higher-order approximations for $\Sigma(x_s, u, \epsilon)$ and $u(x_s, \epsilon)$ (i.e. $O(\epsilon^3)$) in the construction of the fourth-order system provides minimal improvement. From the results of the simulation study, it is evident that the proposed methodology is a powerful tool for the synthesis of low-dimensional controllers which yield a desired closed-loop performance for diffusion-convection-reaction processes.

8.8 Conclusions

In this chapter, we developed a method for the synthesis of nonlinear output feedback controllers for systems of quasi-linear parabolic PDEs, for which the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast one. Combination of Galerkin's method with a novel procedure for the construction of AIMs was used, for the derivation of ODE systems of dimension equal to the number of slow modes, that yield solutions which are close, upto a desired accuracy, to the ones of the PDE system, for almost all times. These ODE systems were used as the basis for the synthesis of output feedback controllers that guarantee stability and enforce the output of the closed-loop system to follow upto a desired accuracy, a prespecified response for almost all times. The methodology was successfully applied to a packed-bed reactor.

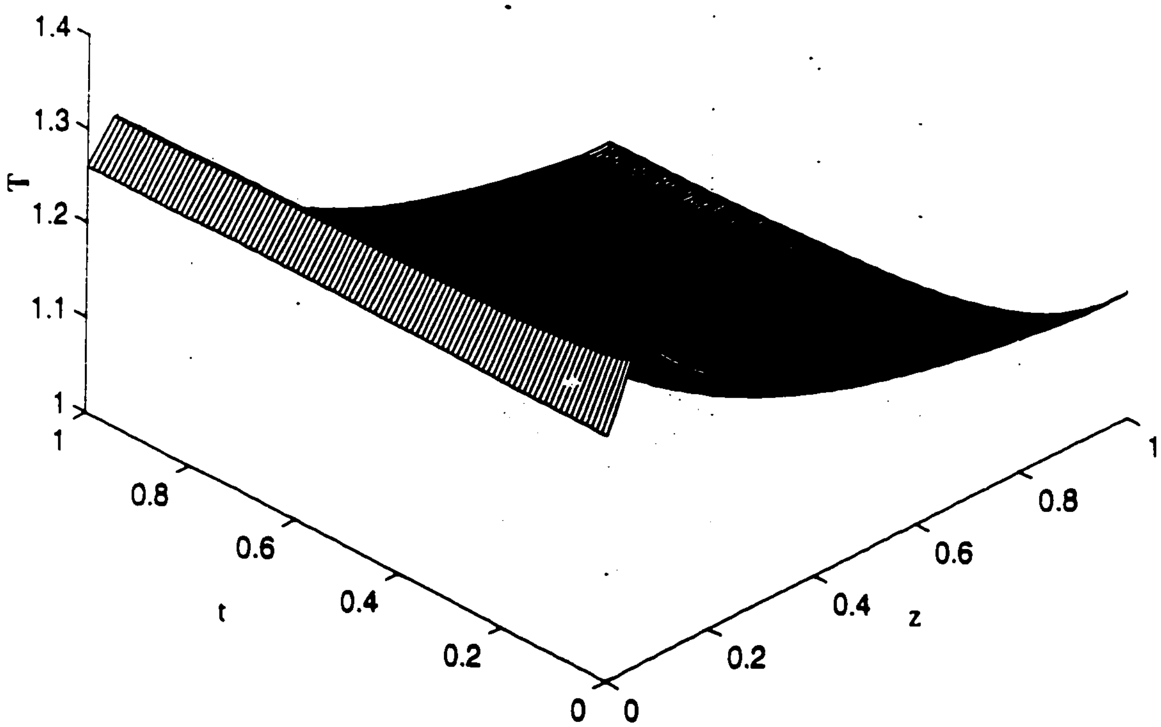


Figure 8.3: Profile of evolution of reactor temperature, for an 8% decrease in the reference input ($O(\epsilon^2)$ approximation for Σ).

Chapter 9

Conclusions and Future Research Directions

The present doctoral thesis has focused on the development of a framework, for the synthesis of nonlinear control systems for nonlinear two-time-scale ODE systems, and hyperbolic and parabolic PDE systems, that systematically addresses the problems of modification of the input/output behavior, elimination of measurable disturbances, and attenuation of unmeasured disturbances and unknown parameters. The proposed control algorithms were applied to industrially important chemical processes. Specifically, the main contributions of this thesis can be summarized as follows:

1. **Control of nonlinear two-time-scale ODE systems.** A methodology was developed for the synthesis of well-conditioned controllers for nonlinear two-time-scale ODE systems. Using combination of singular perturbation and differential geometric methods, well-conditioned feedback controllers were synthesized that guarantee closed-loop stability and enforce output tracking. The problem of elimination of the effect of measurable disturbances on the output was also addressed and solved through appropriate incorporation of feedforward compensation in the developed controllers.

2. **Control of nonlinear two-time-scale ODE systems with uncertain variables.** General methods were developed for the synthesis of controllers for nonlinear two-time-scale ODE systems with unmeasured disturbances and unknown parameters. Using combination of singular perturbations, geometric methods and Lyapunov techniques, well-conditioned robust feedback controllers were synthesized that:

   - guarantee stability of the closed-loop stability and enforce output tracking,
   - achieve arbitrary degree of asymptotic attenuation of the effect of uncertain variables on the outputs.

3. **Control of nonlinear hyperbolic PDE systems.** A methodology was developed for the synthesis of distributed output feedback controllers for nonlinear hyperbolic PDEs that addresses the issues of output tracking and closed-loop stability. The controllers are synthesized on the basis of the PDE model employing a geometric approach. This methodology was extended, within a Lyapunov-based framework, for the synthesis of distributed robust controllers for hyperbolic PDEs with uncertain variables.

4. **Control of nonlinear parabolic PDE systems.** The problem of synthesizing low-dimensional output feedback controllers for nonlinear parabolic PDEs that ensure output tracking with closed-loop stability was addressed and solved. The key step of the proposed solution is the derivation of low-dimensional ODE models, which accurately reproduce the dynamics of the PDE system, through combination of Galerkin's method and approximate inertial manifolds. These ODE models are subsequently used for controller design.

5. **Application to chemical process control.** The developed control methods were successfully applied to industrially important chemical processes. The methodologies for two-time-scale systems were applied to a biochemical reactor,

a catalytic reactor and a fluidized catalytic cracker, while the control algorithms for hyperbolic and parabolic PDEs were applied to a plug-flow reactor, a fixed-bed reactor, and a packed-bed reactor, respectively.

The results of my doctoral thesis provide the first step towards developing a framework for the analysis and controller synthesis for nonlinear chemical processes with time-scale multiplicity and strong distributed nature. Main unresolved theoretical and practical problems that need to be addressed include:

1. Control of two-time-scale processes with non-explicit time-scale separation. An attempt in this direction was recently made in the papers [KCD96b, KCD96a] where a systematic methodology was developed for modeling, via coordinate change, of a class of two-time-scale processes with non-explicit time-scale separation in standard singularly perturbed form. Computational singular perturbation are expected to play a key role in this effort.

2. The study of control problems in more general classes of nonlinear PDEs, such as models arising in fluid dynamics (e.g., Navier-Stokes equations) and pattern formation (they contain differential operators of fourth order in space, e.g., Kuramoto-Sivashinsky equation). It is well-known [FJK+89, BBKK96] that such systems exhibit low-dimensional dynamic behavior, which implies that the proposed framework for control of parabolic PDEs can be used, in principle, to address their control.

3. The development of control algorithms for nonlinear delay-differential equations, as well as DPS modeling disperse-phase processes (e.g., crystallizers, emulsion polymerization reactors), such as nonlinear integro-differential equation systems.

4. The development of robust control algorithms in order to deal with disturbances and parametric uncertainty, and the derivation of design guidelines for the de-

veloped control algorithms. that will allow handling of manipulated input constraints.

5. The control-relevant modeling and analysis (e.g., selection of manipulated inputs and measurements, as well as optimal locations for sensors and actuators), and the experimental implementation of the control algorithms to industrially important two-time-scale and distributed parameter systems.

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Appendices

Appendix E

Proofs of Chapter 6

The proofs of Theorem 6.2 and Proposition 6.4, concerning the case of systems of quasi-linear PDEs, are conceptually similar to the ones given for the linear counterparts of these results, and will therefore be omitted for brevity.

Proof of Proposition 6.2:

Consider the closed-loop system of Eq.6.26. Differentiating the output of this system with respect to time, we obtain the following set of expressions:

$$\begin{aligned}
\bar{y} &= Ckx \\
\frac{d\bar{y}}{dt} &= Ck\left(A\frac{\partial}{\partial z}+B+wb(z)S\right)x \\
\frac{d^2\bar{y}}{dt^2} &= Ck\left(A\frac{\partial}{\partial z}+B+wb(z)S\right)^2 x \\
&\vdots \\
\frac{d^{\bar{\sigma}}\bar{y}}{dt^{\bar{\sigma}}} &= Ck\left(A\frac{\partial}{\partial z}+B+wb(z)S\right)^{\bar{\sigma}} x + Ck\left(A\frac{\partial}{\partial z}+B+wb(z)S\right)^{\bar{\sigma}-1} wb(z)sv
\end{aligned} \tag{E.1}$$

It is sufficient to show that $\bar{\sigma}=\sigma$. Note that $Ck\left(A\frac{\partial}{\partial z}+B+wb(z)S\right) = Ck\left(A\frac{\partial}{\partial z}+B\right)+Ckwb(z)S = Ck\left(A\frac{\partial}{\partial z}+B\right)$, because $Ckwb(z)=0$. Similarly, it can be shown, by induction, that

$Ck\left(A\frac{\partial}{\partial z}+B+wb(z)S\right)^i = Ck\left(A\frac{\partial}{\partial z}+B\right)^i$, for $i = 1,\ldots,\sigma-1$. Using this simplification in the expressions for the time derivatives (Eq.E.1), it follows directly that:

$$\begin{aligned}
\bar{y} &= Ckx \\
\frac{d\bar{y}}{dt} &= Ck\left(A\frac{\partial}{\partial z}+B\right)x \\
\frac{d^2\bar{y}}{dt^2} &= Ck\left(A\frac{\partial}{\partial z}+B\right)^2 x \\
&\vdots \\
\frac{d^\sigma\bar{y}}{dt^\sigma} &= Ck\left(A\frac{\partial}{\partial z}+B\right)^\sigma x + Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)Sx \\
&\quad +Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)sv
\end{aligned} \tag{E.2}$$

This completes the proof of the proposition. △

Proof of Theorem 6.1:

Under the controller of Eq.6.28, the closed-loop system takes the form:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z}+Bx+wb(z)\left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \bullet\left\{v - Ckx - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z}+B\right)^\nu x\right\} \\
\bar{y} &= Ckx
\end{aligned} \tag{E.3}$$

From the result of proposition 6.2, it follows that a differentiation of the output of the system of Eq.E.3 yields the following expressions:

$$
\begin{aligned}
\bar{y} &= Ckx \\
\frac{d\bar{y}}{dt} &= Ck(A\frac{\partial}{\partial z} + B)x \\
\frac{d^2\bar{y}}{dt^2} &= Ck\left(A\frac{\partial}{\partial z} + B\right)^2 x \\
&\vdots \\
\frac{d^\sigma \bar{y}}{dt^\sigma} &= Ck\left(A\frac{\partial}{\partial z} + B\right)^\sigma x + Ck\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z) \\
&\bullet \left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \left\{v - Ckx - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z} + B\right)^\nu x\right\}
\end{aligned} \tag{E.4}
$$

Substituting the above relation into Eq.6.27, one can easily show that the result of the theorem holds. $\triangle$

Proof of Proposition 6.3:

Utilizing the expressions for the output derivatives of Eq.E.2, the closed-loop system of Eq.E.3 takes the form:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx + wb(z)\left[Cw\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} cb(z)\right]^{-1} \\
&\bullet \left\{\frac{1}{\gamma_\sigma}(v - \bar{y}) - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\frac{d^\nu \bar{y}}{dt^\nu} - Cw\left(A\frac{\partial}{\partial z} + B\right)^\sigma x\right\} \\
\bar{y} &= Ckx
\end{aligned} \tag{E.5}
$$

Defining the state vectors $\zeta_i = \frac{d^{i-1}\bar{y}}{dt^{i-1}}$, $i = 1, \cdots, \sigma$, the system of Eq.E.5 can be equivalently written in the form of the following interconnection:

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
&\vdots \\
\dot{\zeta}_{\sigma-1} &= \zeta_\sigma \\
\dot{\zeta}_\sigma &= -\frac{1}{\gamma_\sigma}\zeta_1 - \frac{\gamma_1}{\gamma_\sigma}\zeta_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_\sigma + \frac{1}{\gamma_\sigma}v \\
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx + wb(z)\left[Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \bullet \left\{\bar{\zeta} - Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma} x\right\}
\end{aligned} \tag{E.6}$$

where $\bar{\zeta} = \frac{1}{\gamma_\sigma}(v-\zeta_1) - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\zeta_{\nu+1}$. Condition 1 of the proposition guarantees that the ζ-subsystem of the above interconnection is exponentially stable, and thus the following condition holds:

$$|\bar{\zeta}| \leq K_1|\bar{\zeta}_0|e^{-a_1 t} \tag{E.7}$$

where $|\cdot|$ denotes the standard Eucledian norm, K_1, a_1 are positive real numbers, with $K_1 \geq 1$, and $\bar{\zeta}_0$ is the value of the variable $\bar{\zeta}$ at time $t = 0$, (i.e. $\bar{\zeta}_0 = \frac{1}{\gamma_\sigma}(v(0) - \zeta_{(1)0}) - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\zeta_{(\nu+1)0}$). From condition 2, we have that the differential operator of the system of Eq.6.35 generates an exponentially stable semigroup $\bar{U}$, i.e. $||\bar{U}||_2 \leq K_2 e^{-a_2 t}$, where K_2, a_2 are positive real numbers, with $K_2 \geq 1$. Utilizing Eq.6.15, the following estimate can be written for the state x of the system of Eq.E.5:

$$||x||_2 \leq K_2||x_0||_2 e^{-a_2 t} + K_2\int_0^t e^{-a_2(t-\tau)}||\bar{W}(z)||_2|\bar{\zeta}(\tau)|d\tau \tag{E.8}$$

where

$$\bar{W}(z) = wb(z)\left[Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1} wb(z)\right]^{-1} \tag{E.9}$$

Substituting Eq.E.7 into Eq.E.8, we have that:

$$\begin{aligned}
||x||_2 &\leq K_2||x_0||_2 e^{-a_2 t} + K_2 K_1\int_0^t e^{-a_2(t-\tau)}||\bar{W}(z)||_2|\bar{\zeta}_0|e^{-a_1\tau}d\tau \\
&= K_2||x_0||_2 e^{-a_2 t} + K_2 K_1||\bar{W}(z)||_2|\bar{\zeta}_0|e^{-a_2 t}\int_0^t e^{(a_2-a_1)\tau}d\tau
\end{aligned} \tag{E.10}$$

Let $\delta = a_2 - a_1$. If $\delta = 0$, $||x||_2 \leq K_2||x_0||_2 e^{-a_2 t} + K_2 K_1 ||\bar{W}(z)||_2 |\bar{\zeta}_0| e^{-a_2 t} t \leq K_2||x_0||_2 e^{-a_2 t} + K_2 K_1 ||\bar{W}(z)||_2 \frac{|\bar{\zeta}_0|}{a_2 - a_3} e^{-a_3 t}$, where $0 < a_3 < a_2$. Clearly, in this case, the closed-loop system is exponentially stable. Furthermore, If $\delta > 0$, $||x||_2 \leq K_2||x_0||_2 e^{-a_2 t} + K_2 K_1 ||\bar{W}(z)||_2 \frac{|\bar{\zeta}_0|}{\delta} e^{-a_1 t}(1 - e^{-(a_2 - a_1)t})$, while if $\delta < 0$, $||x||_2 \leq K_2||x_0||_2 e^{-a_2 t} + K_2 K_1 ||\bar{W}(z)||_2 \frac{|\bar{\zeta}_0|}{|\delta|} e^{-a_2 t}(1 - e^{(a_2 - a_1)t})$. In either case ($\delta > 0, \delta < 0$), we have that:

$$||x||_2 \;\leq\; K_2||x_0||_2 e^{-\bar{a}t} + K_2 K_1 ||\bar{W}(z)||_2 \frac{|\bar{\zeta}_0|}{|\delta|} e^{-\bar{a}t} \tag{E.11}$$

where $\bar{a} = \min\{a_1, a_2\}$. From Eqs.E.7-E.11, the exponential stability of the closed-loop system of Eq.E.5 follows directly. △

Proof of Theorem 6.3:

Part 1: Stability analysis. Substituting the controller of Eq.6.39 in the system of Eq.6.9, we have:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx + wb(z)\left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \left\{v - Ck\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z} + B\right)^{\nu}\eta\right\} \\
\frac{\partial \eta}{\partial t} &= A\frac{\partial \eta}{\partial z} + B\eta + wb(z)\left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \left\{v - Ck\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z} + B\right)^{\nu}\eta\right\} + \mathcal{P}(\bar{q} - \mathcal{Q}p\eta)
\end{aligned} \tag{E.12}$$

Introducing the error coordinate $\bar{e} = x - \eta$, the above closed-loop system can be written as:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= A\frac{\partial x}{\partial z} + Bx + wb(z)\left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z} + B\right)^{\sigma-1} wb(z)\right]^{-1} \\
&\quad \left\{v - Ckx - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z} + B\right)^{\nu}x\right\} + \bar{\mathcal{X}}\bar{e} \\
\frac{\partial \bar{e}}{\partial t} &= (\mathcal{L} - \mathcal{P}\mathcal{Q}p)\bar{e}
\end{aligned} \tag{E.13}$$

where

$$\bar{\mathcal{X}}\bar{e} \;=\; wb(z)\left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1}wb(z)\right]^{-1}\left\{Ck\bar{e}+\sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z}+B\right)^{\nu}\bar{e}\right\} \tag{E.14}$$

Because the operator $\mathcal{P}$ is designed such that the operator $\mathcal{L}-\mathcal{PQ}p$ generates an exponentially stable semigroup, the following estimate can be written for the evolution of the state $\bar{e}$ of the above system:

$$||\bar{e}||_2 \leq K_3||\bar{e}_0||_2 e^{-\bar{a}t} \tag{E.15}$$

where $\bar{e}_0$ denotes the vector of initial conditions and $K_3, \bar{a}$ are positive real numbers, with $K_3 \geq 1$. Furthermore, utilizing Eq.6.15 and the fact that the conditions of proposition 6.3 hold, the following inequality can be written for the state x of the system of Eq.E.13:

$$||x||_2 \leq K_4||x_0||_2 e^{-\hat{a}t} + K_4\int_0^t e^{-\hat{a}(t-\tau)}||\bar{\mathcal{X}}\bar{e}(\tau)||_2 d\tau \tag{E.16}$$

where $K_4, \hat{a}$ are positive real numbers, with $K_4 \geq 1$. Substituting the inequality of Eq.E.15 into Eq.E.16, and performing similar calculations as in the proof of proposition 6.3, it can be shown that the closed-loop system is exponentially stable.

Part 2: Input/output response. Under consistent initialization of the states x and η i.e., $x(z,0) = \eta(z,0)$, it follows that $\bar{e}(z,0) = 0$. From the dynamical system for $\bar{e}$, it is clear that if $\bar{e}(z,0) = 0$, then $\bar{e}(z,t) = 0$ for all $t \geq 0$. Thus, the system of Eq.E.13 reduces to:

$$\begin{aligned}\frac{\partial x}{\partial t} \;=\; & A\frac{\partial x}{\partial z}+Bx+wb(z)\left[\gamma_\sigma Ck\left(A\frac{\partial}{\partial z}+B\right)^{\sigma-1}wb(z)\right]^{-1} \\ & \left\{v-Ckx-\sum_{\nu=1}^{\sigma}\gamma_\nu Ck\left(A\frac{\partial}{\partial z}+B\right)^{\nu}x\right\}\end{aligned} \tag{E.17}$$

A direct application of theorem 6.1 completes the proof of the theorem. △

Proof of Theorem 6.4:

Part 1: Stability analysis. Substituting the controller of Eq.6.43 in the system of Eq.6.7, we have:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\left[\gamma_\sigma C L_g\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(\eta)b(z)\right]^{-1} \\
&\quad \left\{v - Ch(\eta) - \sum_{\nu=1}^{\sigma}\gamma_\nu C\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(\eta)\right\} \\
\frac{\partial \eta}{\partial t} &= A(\eta)\frac{\partial \eta}{\partial z} + f(\eta) + g(\eta)b(z)\left[\gamma_\sigma C L_g\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(\eta)b(z)\right]^{-1} \\
&\quad \left\{v - Ch(\eta) - \sum_{\nu=1}^{\sigma}\gamma_\nu C\left(\sum_{j=1}^{n}\frac{\partial \eta_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(\eta)\right\} + \bar{\mathcal{P}}(\bar{q} - \mathcal{Q}p(\eta))
\end{aligned}
\tag{E.18}
$$

In order to perform a local analysis of the stability properties of the above system, we consider its linearization:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A(z)\frac{\partial x}{\partial z} + B(z)x + w(z)b(z)\left[\gamma_\sigma Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1} \\
&\quad \left\{v - Ck(z)\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu}\eta\right\} + \bar{\mathcal{P}}(\bar{y} - Ck(z)\eta) \\
\frac{\partial \eta}{\partial t} &= A(z)\frac{\partial \eta}{\partial z} + B(z)\eta + w(z)b(z)\left[\gamma_\sigma Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1} \\
&\quad \left\{v - Ck(z)\eta - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu}\eta\right\} + \bar{\mathcal{P}}(\bar{q} - \mathcal{Q}p(z)\eta)
\end{aligned}
\tag{E.19}
$$

Introducing the error coordinate $\bar{e} = x - \eta$, the above closed-loop system can be written as:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A(z)\frac{\partial x}{\partial z} + B(z)x + w(z)b(z)\left[\gamma_\sigma Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1} \\
&\quad \left\{v - Ck(z)x - \sum_{\nu=1}^{\sigma}\gamma_\nu Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu} x\right\} + \tilde{\mathcal{X}}\bar{\epsilon} \\
\frac{\partial \bar{\epsilon}}{\partial t} &= (\bar{\mathcal{L}} - \bar{\mathcal{P}}\mathcal{Q}p(z))\bar{\epsilon}
\end{aligned}
\tag{E.20}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{X}}\bar{\epsilon} &= w(z)b(z)\left[\gamma_\sigma Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\sigma-1} w(z)b(z)\right]^{-1} \\
&\quad \left\{Ck(z)\bar{\epsilon} + \sum_{\nu=1}^{\sigma}\gamma_\nu Ck(z)\left(A(z)\frac{\partial}{\partial z} + B(z)\right)^{\nu} \bar{\epsilon}\right\}
\end{aligned}
\tag{E.21}
$$

From the conditions of the theorem, we have that the x-subsystem (with $\bar{\epsilon} = 0$) of the above interconnection is exponentially stable and the $\bar{\epsilon}$-subsystem is also exponentially stable, because the operator $\mathcal{P}$ is designed such that the operator $\bar{\mathcal{L}} - \bar{\mathcal{P}}\mathcal{Q}p(z)$ generates an exponentially stable semigroup. Following an approach analogous to the one used in the proof of proposition 6.3, one can show that the system of Eq.E.20 is exponentially stable. Utilizing the result of proposition 6.1, we have that the closed-loop system of Eq.E.18 is locally exponentially stable, because the linearized system of Eq.E.20 is exponentially stable.

Part 2: Input/output response. Under consistent initialization of the states x and η i.e., $x(z,0) = \eta(z,0)$, it follows that $\bar{\epsilon}(z,0) = 0$. From the dynamical system for $\bar{\epsilon}$, it is clear that if $\bar{\epsilon}(z,0) = 0$, then $\bar{\epsilon}(z,t) = 0$ for all $t \geq 0$. Thus, the system of Eq.E.20

reduces to:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\left[\gamma_\sigma C L_g \left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
& \left\{v - Ch(x) - \sum_{\nu=1}^{\sigma}\gamma_\nu C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(x)\right\} \\
\bar{y} &= Ch(x)
\end{aligned}
\tag{E.22}
$$

Since the characteristic index between $\bar{y}$ and v is σ, a differentiation of the output of the system of Eq.E.22 yields the following expressions:

$$
\begin{aligned}
\bar{y} &= Ch(x) \\
\frac{d\bar{y}}{dt} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right) h(x) \\
\frac{d^2\bar{y}}{dt^2} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right) h(x) \\
&\vdots \\
\frac{d^\sigma\bar{y}}{dt^\sigma} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} + CL_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z) \\
& \left[\gamma_\sigma C L_g \left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
& \left\{v - Ch(x) - \sum_{\nu=1}^{\sigma}\gamma_\nu C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\nu} h(x)\right\}
\end{aligned}
\tag{E.23}
$$

Substituting the above relation into Eq.6.27, one can easily show that the result of the theorem holds. $\triangle$

Appendix F

Proofs of Chapter 7

Proof of Proposition 7.1:

Part 7.1: The proof of the first part of the proposition is given in the proof of proposition 6.2.

Part 2: Referring to the open-loop system of Eq.7.1, let $\delta \leq \sigma$. We will first show that the result holds in the case $\delta < \sigma$. Consider the closed-loop system of Eq.7.11. From part one of the proposition, we have that the characteristic index of the output $\bar{y}$ with respect to v in the closed-loop system is equal to σ. From the definition of characteristic index of $\bar{y}$ with respect to θ, we have that the following relations hold for the open-loop system:

$$C^i L_{W_k} \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\mu - 1} h(x) r_k^i(z) \equiv 0, \;\; \forall \mu = 1, \ldots, \delta - 1, \;\; i = 1, \ldots, l \tag{F.1}$$

Differentiating the output of the closed-loop system with respect to time and using the relations of Eq.F.1, we get:

$$\begin{aligned}
\bar{y} &= Ch(x) \\
\frac{d\bar{y}}{dt} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
\frac{d^2\bar{y}}{dt^2} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
&\vdots \\
\frac{d^\delta\bar{y}}{dt^\delta} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\delta-1}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
&\quad + C\sum_{k=1}^{q}L_{W_k}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\delta-1}h(x)r_k(z)\theta_k(t)
\end{aligned} \tag{F.2}$$

From the above equations, it is clear that the result of the second part of the proposition holds if $\delta < \sigma$. The same argument can be used to show that the result is also true for the case $\delta = \sigma$. The details in this case are omitted for brevity. $\triangle$

Proof of Theorem 7.2:

Part 1 (uncertainty decoupling). Necessity. We will proceed by contradiction. Consider the system of Eq.7.1 and assume that $\delta \leq \sigma$. Referring to the closed-loop system of Eq.7.11, we have, from proposition 1, that the characteristic indices σ, δ are preserved, and the condition $\delta \leq \sigma$ holds. This fact implies that θ affects directly the δ-th time-derivative of $\bar{y}$ in the closed-loop system, and thus $\bar{y}$, which yields a contradiction.

Sufficiency. Referring to the system of Eq.7.1, suppose that $\sigma < \delta$. In this case, a time-differentiation of $\bar{y}$ up to σ-th order yields the following expressions:

$$\begin{aligned}
\bar{y} &= Ch(x) \\
\frac{d\bar{y}}{dt} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
\frac{d^2\bar{y}}{dt^2} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
&\vdots \\
\frac{d^\sigma\bar{y}}{dt^\sigma} &= C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1}\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)h(x) \\
&\quad +CL_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1}h(x)b(z)u
\end{aligned} \tag{F.3}$$

From the expression of the σ-th derivative of $\bar{y}$, it is clear that there exists a control law of the form of Eq.7.10 (e.g., the controller of Eq.7.13) which guarantees that $\bar{y}$ is independent of θ in the closed-loop system, for all times. Finally, one can easily show, utilizing the expressions of Eq.F.3, that the controller of Eq.7.13 enforces the input/output response of Eq.7.12 in the closed-loop system.

Part 2 (boundedness). First, we note that whenever $\theta(t) \equiv 0$, the conditions *i*) and *ii*) of the theorem guarantee that the nominal closed-loop system is locally exponentially stable (Proposition 6.3). Since the nominal closed-loop system is locally exponential stability, we have from theorem 6.1 that there exists a smooth Lyapunov functional $V : \mathcal{H}^n \times [\alpha, \beta] \rightarrow R$ of the form of Eq.7.7, and a set of positive real numbers a_1, a_2, a_3, a_4, a_5, such that the following properties hold:

$$\begin{aligned}
a_1||x||_2^2 \leq V(t) \leq a_2||x||_2^2 \\
\frac{dV}{dt} = \frac{\partial V}{\partial x}\bar{\mathcal{L}}(x) \leq -a_3||x||_2^2 \\
||\frac{\partial V}{\partial x}||_2 \leq a_4||x||_2
\end{aligned} \tag{F.4}$$

if $||x||_2 \leq a_5$ Whenever $(\theta(t) \neq 0)$ the closed-loop system, under the control law of

Eq.7.13, takes the form:

$$\frac{\partial x}{\partial t} = \bar{\mathcal{L}}(x) + W(x)r(z)\theta(t) \tag{F.5}$$

where $\bar{\mathcal{L}}(x)$ is a nonlinear operator. Computing the time-derivative of the functional $V : \mathcal{H}^n \times [\alpha, \beta] \rightarrow R$ along the trajectories of the uncertain closed-loop system of Eq.F.5, using Eq.F.4, and using the fact that if $||x||_2 \leq a_5$ there exists a positive real number a_6 such that $||W(x)||_2 \leq a_6$, we get:

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x}(\bar{\mathcal{L}}(x) + W(x)r(z)\theta(t)) \\ &\leq -a_3||x||_2^2 + a_4a_6||x||_2||r(z)||_2|\theta(t)| \end{aligned} \tag{F.6}$$

From the last inequality of the above equation, we have that if $|\theta(t)| \leq \dfrac{a_3a_5}{4a_4a_6||r(z)||_2} \leq \hat{\delta},\ \forall t \geq 0$, then

$$\frac{dV}{dt} \leq -\frac{a_3}{2}||x||_2^2 \tag{F.7}$$

if $\frac{a_5}{2} \leq ||x||_2 \leq a_5$. Using the result of theorem 4.10 reported in [Kha89], we have that $||x||_2$ is a bounded quantity, which implies that the state of the closed-loop system is bounded. $\triangle$

Proof of Theorem 7.3: First, we define the state vectors $\zeta_\nu^i = \dfrac{d^{\nu-1}\bar{y}^i}{dt^{\nu-1}}$, $i = 1, \cdots, l$, $\nu = 1, \cdots, \sigma$, $\zeta_\nu = [\zeta_\nu^1 \ \zeta_\nu^2 \ \cdots \zeta_\nu^l]^T$, $\tilde{\zeta}_1 = \zeta_1 - v$, $\tilde{\zeta}_\sigma^i = \zeta_\sigma^i + \dfrac{\gamma_1}{\gamma_\sigma}\tilde{\zeta}_1^i + \dfrac{\gamma_2}{\gamma_\sigma}\zeta_2^i + \cdots + \dfrac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_{\sigma-1}^i$, $\tilde{\zeta}_\sigma = [\tilde{\zeta}_\sigma^1 \ \tilde{\zeta}_\sigma^2 \ \cdots \tilde{\zeta}_\sigma^l]^T$ and for ease of notation, we also set:

$$\bar{s}(x) = \left[CL_g \left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial z} L_{a_j} + L_f \right)^{\sigma-1} h(x)b(z) \right]^{-1} \tag{F.8}$$

Using the above notation the controller of Eq.7.19 takes the form:

$$\bar{u} = \bar{s}(x) \left\{ -\gamma_0 \tilde{\zeta}_\sigma - \sum_{\nu=1}^{\sigma-1} \frac{\gamma_\nu}{\gamma_\sigma} \zeta_{\nu+1} - 2K(t)\Delta(\tilde{\zeta}_\sigma, \phi)\tilde{\zeta}_\sigma \right\} \tag{F.9}$$

where $\Delta(\tilde{\zeta}_\sigma, \phi)$ is an $l \times l$ diagonal matrix, whose $(i,i)-th$ element is of the form $(|\tilde{\zeta}_\sigma^i| + \phi)^{-1}$.

Part 1: Asymptotic output tracking. Using the definition for the state vectors ζ_ν, $\nu = 1,\cdots,\sigma$, and $\bar{\zeta}_1, \tilde{\zeta}_\sigma$, the closed-loop system

$$\begin{aligned} \frac{\partial x}{\partial t} &= A(x)\frac{\partial x}{\partial z} + f(x) + g(x)b(z)\bar{u} + W(x)r(z)\theta(t) \\ \bar{y} &= Ch(x) \end{aligned} \tag{F.10}$$

can be equivalently written in the following form:

$$\begin{aligned} \dot{\bar{\zeta}}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{\sigma-1} &= -\frac{\gamma_1}{\gamma_\sigma}\bar{\zeta}_1 - \frac{\gamma_2}{\gamma_\sigma}\zeta_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_{\sigma-1} + \tilde{\zeta}_\sigma \\ \dot{\tilde{\zeta}}_\sigma &= \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\dot{\zeta}_{\nu+1} - \gamma_0\tilde{\zeta}_\sigma - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\dot{\zeta}_{\nu+1} - 2K(t)\Delta(\tilde{\zeta}_\sigma,\phi)\tilde{\zeta}_\sigma + K(t) \\ \frac{\partial x}{\partial t} &= \mathcal{L}x + g(x)b(z)\bar{s}(x)\left\{-\gamma_0\tilde{\zeta}_\sigma - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\dot{\zeta}_{\nu+1} - 2K(t)\Delta(\tilde{\zeta}_\sigma,\phi)\tilde{\zeta}_\sigma\right\} \\ &\quad + W(x)r(z)\theta \end{aligned} \tag{F.11}$$

To establish the relation of Eq.7.21, we will first work with the $\tilde{\zeta}_\sigma$-subsystem of Eq.F.11 and establish a bound for the state of this system in terms of the initial condition and the parameter ϕ (in order to simplify the presentation we will focus on the i-th input/output pair). To this end, we consider the following smooth functions $V^i : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $i = 1,\ldots,l$:

$$V^i = \frac{1}{2}(\tilde{\zeta}_\sigma^i)^2 \tag{F.12}$$

to show that the time-derivative of V^i is negative definite outside of a region that includes the steady state in the space and this region can be made arbitrarily small by picking ϕ sufficiently small. Calculating the time-derivative of V^i along the trajectories of the $\tilde{\zeta}_\sigma^i$-subsystem of Eq.F.10, we have:

$$\begin{aligned}\frac{dV^i}{dt} &= \tilde{\zeta}_\sigma^i\left[-\gamma_0\tilde{\zeta}_\sigma^i - 2K(t)(|\tilde{\zeta}_\sigma^i|+\phi)^{-1}\tilde{\zeta}_\sigma^i + K(t)\right]\\ &\le -\gamma_0(\tilde{\zeta}_\sigma^i)^2 - 2K(t)(|\tilde{\zeta}_\sigma^i|+\phi)^{-1}(\tilde{\zeta}_\sigma^i)^2 + K(t)|\tilde{\zeta}_\sigma^i|\\ &\le -\gamma_0(\tilde{\zeta}_\sigma^i)^2 - 2K(t)(|\tilde{\zeta}_\sigma^i|+\phi)^{-1}(\tilde{\zeta}_\sigma^i)^2 + K(t)(|\tilde{\zeta}_\sigma^i|+\phi)^{-1}(\tilde{\zeta}_\sigma^i)^2 + |\tilde{\zeta}_\sigma^i|o\\ &\le -\gamma_0(\tilde{\zeta}_\sigma)^2 - K(t)(|\tilde{\zeta}_\sigma^i|+\phi)^{-1}|\tilde{\zeta}_\sigma|(|\tilde{\zeta}_\sigma^i|-\phi)\end{aligned} \qquad \text{(F.13)}$$

Clearly, if $|\tilde{\zeta}_\sigma^i| \ge \phi$, $\dot{V}^i$ is negative definite, which implies [Kha89, theorem 4.10] that there exist positive real numbers $\hat{K} \ge 1$, $\bar{a}$, γ, such that the following bound holds for the norm of the state of the $\tilde{\zeta}_\sigma^i$-subsystem:

$$|\tilde{\zeta}_\sigma^i| \;\le\; \hat{K}|(\tilde{\zeta}_\sigma^i)_0|e^{-\bar{a}t} + \gamma\phi \qquad \text{(F.14)}$$

where $(\tilde{\zeta}_\sigma^i)_0$ denotes the value of $\tilde{\zeta}_\sigma^i$ at time $t = 0$. Consider now the following subsystem:

$$\begin{aligned}\dot{\tilde{\zeta}}_1^i &= \zeta_2^i\\ &\vdots\\ \dot{\zeta}_{\sigma-1}^i &= -\frac{\gamma_1}{\gamma_\sigma}\tilde{\zeta}_1^i - \frac{\gamma_2}{\gamma_\sigma}\zeta_2^i - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_{\sigma-1}^i + \tilde{\zeta}_\sigma^i\end{aligned} \qquad \text{(F.15)}$$

where $\tilde{\zeta}_\sigma^i$ can be thought of as an external input. Since the parameters γ_k are chosen so that the polynomial $1+\gamma_1 s + \cdots + \gamma_\sigma s^\sigma = 0$ is Hurwitz, we have that there exists positive real numbers $K_\zeta, a_\zeta, \gamma_{\tilde{\zeta}_\sigma}$ such that the following bound can be written for the state vector $\zeta^i = [\tilde{\zeta}_1^i \; \zeta_2^i \; \cdots \; \zeta_{\sigma-1}^i]$ of the system of Eq.F.15:

$$|\zeta^i| \;\le\; K_\zeta|\zeta_0^i|e^{-a_\zeta t} + \gamma_{\tilde{\zeta}_\sigma}||\tilde{\zeta}_\sigma^i|| \qquad \text{(F.16)}$$

Taking the limit as $t \to \infty$, using the property that $\lim_{t\to\infty} sup_{t\ge 0}\{|\tilde{\zeta}_\sigma^i|\} = sup_{t\ge 0}\{\lim_{t\to\infty}|\tilde{\zeta}_\sigma^i|\}$, and using Eq.F.14 we have:

$$\lim_{t\to\infty}|\zeta^i| \le \gamma_{\tilde{\zeta}_\sigma}\gamma\phi \qquad \text{(F.17)}$$

Picking $\phi^* = \dfrac{d}{\gamma_{\tilde{\zeta}_\sigma}\gamma}$, the relation of Eq.7.21 follows directly from the fact $\lim_{t\to\infty}|\tilde{\zeta}_1^i| \le \lim_{t\to\infty}|\zeta^i|$.

Part 2: Boundedness of the state. The proof that the state is bounded, whenever the conditions of the theorem hold, can be obtained by using contradiction argument. It starts by assuming that there exists a maximal time T such that for all $t \in [0, T)$, the states $(x, \zeta, \tilde{\zeta}_\sigma)$ of the system of Eq.F.11 (note that T always exists because the system starts from bounded initial conditions). Then, the bounds that hold for the states $(x, \zeta, \tilde{\zeta}_\sigma)$ for $t \in [0, T)$ are derived and is shown that they continue to hold for $t \in [0, T + k_T]$, where k_T is some positive number. However, this contradicts the assumption that T is the maximal time in which the state is bounded, which implies that $T = \infty$, and thus the state of the closed-loop system is bounded for all times. $\triangle$

Proof of Theorem 7.4: Using L_{ij} as defined in subsection 7.7.1, the representation of the system of Eq.7.23, with $\bar{u} = 0$, in the coordinates (x, η_f) takes the form:

$$\begin{aligned} \frac{\partial x}{\partial t} &= \mathcal{L}x + W(x)r(z)\theta(t) + \mathcal{L}_{12}\eta_f \\ \epsilon\frac{\partial \eta_f}{\partial t} &= \mathcal{L}_{22}\eta_f + \epsilon(\frac{\partial z_s}{\partial x}\dot{x} + \frac{\partial z_s}{\partial \theta}\dot{\theta}) \end{aligned} \tag{F.18}$$

Since from assumption 7.6, the system:

$$\epsilon\frac{\partial \eta_f}{\partial t} = \mathcal{L}_{22}\eta_f \tag{F.19}$$

is exponentially stable, we have that if $(x, \theta, \dot{\theta})$ are bounded, then there exist a set of positive real numbers $K_f, \bar{a}_f, \gamma_{\eta_f}$ such that the following bound can be written for the state of the η_f-subsystem of Eq.F.18, for all $t \geq 0$:

$$||\eta_f||_2 \leq K_f||\eta_{f0}||_2 e^{-\bar{a}_f\frac{t}{\epsilon}} + \gamma_{\eta_f}\epsilon \tag{F.20}$$

Using assumption 7.5 and Eq.6.15, we have that the following bound holds for the state of the x-subsystem of Eq.F.18, for all $t \geq 0$:

$$\begin{aligned} ||x||_2 &\leq K_s||x_0||_2 e^{-a_s t} + K_s\int_0^t e^{-a_s(t-\tau)}||W(x)r(z)||_2|\theta(\tau)|d\tau \\ &\quad + K_s\int_0^t e^{-a_s(t-\tau)}||\mathcal{L}_{12}||_2||\eta_f||_2 d\tau \\ &\leq K_s||x_0||_2 e^{-a_s t} + M_2||\theta|| + M_3 \sup_{t\geq 0}\{||\eta_f||_2\} \end{aligned} \tag{F.21}$$

where $M_2 = \frac{K_s \tilde{M}_2 ||r(z)||_2}{a_s}$, and $M_3 = \frac{K_s ||\mathcal{L}_{12}||_2}{a_s}$, provided that the initial condition ($||x_0||_2$) and the inputs ($||\theta||, ||\dot{\theta}||, \sup\{||\eta_f||_2\}$) are sufficiently small. Note that since we do not know a-priori that the states (x, η_f) of the system of Eq.F.18 are bounded, we have to work with truncations and exploit causality in order to prove boundedness. Let $\tilde{\delta}$ be as given in the statement of the theorem, so that $\max\{||x_0||_2, ||\eta_{f_0}||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$, and let δ_x to be a positive real number that satisfies

$$\delta_x \quad > \quad K_s\tilde{\delta} + M_2\tilde{\delta} + d \tag{F.22}$$

where d is a positive real number specified in the statement of the theorem. Note that since $K_s \geq 1$, $\delta_x > \tilde{\delta}$, and using continuity with respect to initial conditions, we define $[0, T)$ to be the maximal interval in which $||x_t||_2 \leq \delta_x$, for all $t \in [0, T)$ and suppose that T is finite. We will now show by contradiction that $T = \infty$, provided that ϵ is sufficiently small.

First, notice that from Eq.F.19, we have that for all $t \in [0, T)$, $||\eta_{f_t}|| \leq K_f\tilde{\delta} + \gamma_\eta\epsilon$. Let ϵ_0 be a positive real number so that $\epsilon \in (0, \epsilon_0]$ and define $\delta_{\eta_f} := K_f\tilde{\delta} + \gamma_{\eta_f}\epsilon_0$. Note that $\delta_{\eta_f} > \tilde{\delta}$.

Lemma [CT95]: *Referring to the x-subsystem of Eq.F.18, let Eq.F.21 hold. Then, for each pair of positive real numbers $\bar{\delta}, \bar{d}$, there exists a positive real number ρ^* such that for each $\rho \in [0, \rho^*]$, if $\max\{||x_0||_2, ||\eta_f||_2, ||\theta||, ||\dot{\theta}||\} \leq \bar{\delta}$, then the solution of the x-subsystem of Eq.F.18 with $x(0) = x_0$ exists for each $t \geq 0$ and satisfies:*

$$||x||_2 \quad \leq \quad K_s||x_0||_2 e^{-a_s t} + M_2||\theta|| + M_3 \sup_{t\geq\rho}\{||\eta_f||_2\} + \bar{d} \tag{F.23}$$

Applying the result of the above lemma, we have that there exists a positive real number ρ (assume without loss of generality that $\rho < T$), such that if $\max\{||x_0||_2, ||\eta_f||, ||\theta||, ||\dot{\theta}||\} \leq \delta_{\eta_f}$, then the solution of the x-subsystem of Eq.F.18

with $x(0) = x_0$ exists for each $t \in [0, T)$ and satisfies:

$$||x||_2 \leq K_s||x_0||_2 e^{-a_s t} + M_2||\theta|| + M_3 \sup_{t \geq \rho}\{||\eta_f||_2\} + \frac{d}{2} \quad \text{(F.24)}$$

Substituting Eq.F.19 into Eq.F.24, if $\epsilon \in (0, \epsilon_0]$, we have for all $t \in [0, T)$:

$$||x||_2 \leq K_s||x_0||_2 e^{-a_s t} + M_2||\theta|| + \frac{d}{2} + M_3 K_f ||\eta_{f0}||_2 (e^{-\bar{a}_f \frac{\rho}{\epsilon}} + \gamma_{\eta_f}\epsilon) \quad \text{(F.25)}$$

From the fact that the last term of the above equation vanishes as $\epsilon \longrightarrow 0$, we have that there exists an $\epsilon_1 \in (0, \epsilon_0]$ such that if $\epsilon \in (0, \epsilon_1]$, then for all $t \in [0, T)$:

$$||x||_2 \leq K_s||x_0||_2 e^{-a_s t} + M_2||\theta|| + d \quad \text{(F.26)}$$

From the definition of δ_x, the assumption that T is finite and continuity of x, there must exist some positive real number k such that $sup_{t \in [0, T+k]}\{||x||_2\} < \delta_x$. This contradicts that T is maximal. Hence, $T = \infty$ and the inequality Eq.7.35 holds for all $t \geq 0$.

Finally, letting ϵ_3 be such that $\gamma_{\eta_f}\epsilon \leq d$ for all $\epsilon \in [0, \epsilon_3]$ it follows that both inequalities of Eqs.7.35-7.36 hold for $\epsilon \in (0, \epsilon^*]$ where $\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. $\triangle$

Proof of Theorem 7.6: Substituting the controller of Eq.7.13 into the system of Eq.7.23 and using the expression for the output derivatives of the closed-loop slow system of Eq.F.3, we have:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= \mathcal{L}x + g(x)b(z)\left[\gamma_\sigma C L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\quad \left\{\frac{1}{\gamma_\sigma}(v-\bar{y}) - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\frac{d^\nu \bar{y}}{dt^\nu} - C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\} \\
&\quad + W_1(x)r(z)\theta(t) + \mathcal{L}_{12}\eta_f \\
\epsilon\frac{\partial \eta_f}{\partial t} &= \mathcal{L}_{22}\eta_f + g_2(x)b(z)\left[\gamma_\sigma C L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\quad \left\{\frac{1}{\gamma_\sigma}(v-\bar{y}) - \sum_{\nu=1}^{\sigma-1}\frac{\gamma_\nu}{\gamma_\sigma}\frac{d^\nu \bar{y}}{dt^\nu} - C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\} \\
&\quad + W_2(x)r(z)\theta(t) + \epsilon\left(\frac{\partial z_s}{\partial x}\dot{x} + \frac{\partial z_s}{\partial \theta}\dot{\theta}\right) \\
\bar{y} &= Ch(x)
\end{aligned}
\tag{F.27}
$$

Defining the state vectors $\zeta_\nu = \dfrac{d^{\nu-1}\bar{y}}{dt^{\nu-1}}$, $\nu = 1,\cdots,\sigma$, the system of Eq.F.27 can be equivalently written in the following form:

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 + \sum_{k=1}^{p} CL_{\mathcal{L}_{12}^k} h(x)\eta_f \\
&\vdots \\
\dot{\zeta}_{\sigma-1} &= \zeta_\sigma + \sum_{k=1}^{p} CL_{\mathcal{L}_{12}^k}^{\sigma-2} h(x)\eta_f \\
\dot{\zeta}_\sigma &= -\frac{1}{\gamma_\sigma}\zeta_1 - \frac{\gamma_1}{\gamma_\sigma}\zeta_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_\sigma + \frac{1}{\gamma_\sigma}v + \sum_{k=1}^{p} CL_{\mathcal{L}_{12}^k}^{\sigma-1} h(x)\eta_f \\
\frac{\partial x}{\partial t} &= \mathcal{L}x + g(x)b(z)\left[\gamma_\sigma CL_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\bullet\left\{\bar{\zeta} - C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\} + \mathcal{L}_{12}\eta_f + W(x)r(z)\theta \\
\epsilon\frac{\partial \eta_f}{\partial t} &= \mathcal{L}_{22}\eta_f + g_2(x)b(z)\left[\gamma_\sigma CL_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\bullet\left\{\bar{\zeta} - C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\} + W_2(x)r(z)\theta(t) + \epsilon\left(\frac{\partial z_s}{\partial x}\dot{x} + \frac{\partial z_s}{\partial \theta}\dot{\theta}\right)
\end{aligned}
\tag{F.28}
$$

Performing a two-time-scale decomposition, it can be shown that the fast dynamics of the above system is locally exponentially stable, and the reduced system takes the form:

$$
\begin{aligned}
\dot{\zeta}_1^s &= \zeta_2^s \\
&\vdots \\
\dot{\zeta}_{\sigma-1}^s &= \zeta_\sigma^s \\
\dot{\zeta}_\sigma^s &= -\frac{1}{\gamma_\sigma}\zeta_1^s - \frac{\gamma_1}{\gamma_\sigma}\zeta_2^s - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\zeta_\sigma^s + \frac{1}{\gamma_\sigma}v \\
\frac{\partial x}{\partial t} &= \mathcal{L}x + g(x)b(z)\left[\gamma_\sigma C L_g \left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\quad \bullet \left\{\bar{\zeta} - C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\} + W(x)r(z)\theta
\end{aligned}
\tag{F.29}
$$

From theorem 7.2, we have that the state of the above system is bounded. Using the result of theorem 7.4, we have that the state of the closed-loop system of Eq.F.27 is bounded provided that the initial conditions, the uncertainty, the rate of change of uncertainty and the singular perturbation parameter are sufficiently small. Using the auxiliary variable $\tilde{\zeta}_\nu = \zeta_\nu - \zeta_\nu^s$, $\nu = 1,\ldots,\sigma$, the system of Eq.F.28 can be written as:

$$\begin{aligned}
\dot{\tilde{\zeta}}_1 &= \tilde{\zeta}_2 + \sum_{k=1}^{p} CL_{\mathcal{L}_{12}^k} h(x)\eta_f \\
&\vdots \\
\dot{\tilde{\zeta}}_{\sigma-1} &= \tilde{\zeta}_\sigma + \sum_{k=1}^{p} CL_{\mathcal{L}_{12}^k}^{\sigma-2} h(x)\eta_f \\
\dot{\tilde{\zeta}}_\sigma &= -\frac{1}{\gamma_\sigma}\tilde{\zeta}_1 - \frac{\gamma_1}{\gamma_\sigma}\tilde{\zeta}_2 - \cdots - \frac{\gamma_{\sigma-1}}{\gamma_\sigma}\tilde{\zeta}_\sigma + \frac{1}{\gamma_\sigma}v + \sum_{k=1}^{p} CL_{\mathcal{L}_{12}^k}^{\sigma-1} h(x)\eta_f \\
\frac{\partial x}{\partial t} &= \mathcal{L}x + g(x)b(z)\left[\gamma_\sigma C L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\quad \left\{\tilde{\zeta} + \tilde{\zeta}^s - C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\} + \mathcal{L}_{12}\eta_f + W(x)r(z)\theta \\
\epsilon\frac{\partial \eta_f}{\partial t} &= \mathcal{L}_{22}\eta_f + g_2(x)b(z)\left[\gamma_\sigma C L_g\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma-1} h(x)b(z)\right]^{-1} \\
&\quad \left\{\tilde{\zeta} + \tilde{\zeta}^s - C\left(\sum_{j=1}^{n}\frac{\partial x_j}{\partial z}L_{a_j} + L_f\right)^{\sigma} h(x)\right\} + W_2(x)r(z)\theta(t) \\
&\quad +\epsilon\left(\frac{\partial z_s}{\partial x}\dot{x} + \frac{\partial z_s}{\partial \theta}\dot{\theta}\right)
\end{aligned} \tag{F.30}$$

Using the fact that the state of the above system is bounded and the equality $\zeta_\nu(0) = \zeta_\nu^s(0)$, holds $\forall\ \nu = 1,\ldots,\sigma$, the following bounds can be written for the evolution of the states $\tilde{\zeta} = [\tilde{\zeta}_1^T\ \tilde{\zeta}_2^T\ \ldots\tilde{\zeta}_\sigma^T]^T$, η_f of the above system:

$$\begin{aligned}
|\tilde{\zeta}| &\leq M_4||\eta_f||_2 \\
||\eta_f||_2 &\leq K_f||\eta_{f0}||_2 e^{-\bar{a}_f\frac{t}{\epsilon}} + \gamma_{\eta_f}\epsilon
\end{aligned} \tag{F.31}$$

where M_4 is a positive real number. Combining the above inequalities, we have:

$$|\tilde{\zeta}| \leq M_4K_f||\eta_{f0}||_2 e^{-\bar{a}_f\frac{t}{\epsilon}} + M_4\gamma_{\eta_f}\epsilon \tag{F.32}$$

Since M_4, K_f, $||\eta_{f0}||_2$, γ_{η_f} are some finite numbers of order one, and the right hand side of the above inequality is a continuous function of ϵ which vanishes as $\epsilon \to 0$, we have that there exists an ϵ^* such that if $\epsilon \in (0, \epsilon^*]$, then $M_4 K_f ||\eta_{f0}||_2 e^{-\bar{a}_f \frac{t}{\epsilon}} + M_4 \gamma_{\eta_f} \epsilon \leq d$. $\forall\, t > 0$. Thus, $|\bar{y}^i(t) - \bar{y}^i_s(t)| = |\tilde{\zeta}^i_1(t)| \leq |\tilde{\zeta}_1(t)| \leq |\tilde{\zeta}(t)| \leq d$, $\forall\, t > 0$. $\triangle$

Proof of Theorem 7.7: The detailed presentation of the proof of the theorem is too lengthy and will be omitted for brevity. Instead, we will provide a brief outline of the proof. Initially, it can be shown, following analogous steps as in the proof of theorem 3, that the state of the closed-loop reduced system of Eq.7.42 is bounded and the there exist positive real numbers K_e, a_e, γ_e such that the following estimate holds for the i-th output error for all $t \geq 0$:

$$|\bar{y}^i - v^i| \quad \leq \quad K_e|(\bar{y}^i - v^i)_0|e^{-a_e t} + \gamma_e \phi \tag{F.33}$$

where $(\bar{y}^i - v^i)_0$ is the output error at time $t = 0$. Picking $\phi^* = \dfrac{d}{2\gamma_e}$, we have that for $\phi \in (0, \phi^*]$:

$$|\bar{y}^i - v^i| \quad \leq \quad K_e|(\bar{y}^i - v^i)_0|e^{-a_e t} + \frac{d}{2} \tag{F.34}$$

Now, referring to system of Eq.7.41 we have shown that satisfies the assumption of theorem 7.4 (the state of the uncertain closed-loop reduced system is bounded and the boundary layer is exponentially stable) and thus, the result of this theorem can be applied. This means that for each positive real number d, there exist positive real numbers $(\tilde{K}_e, \tilde{a}_e, \tilde{\gamma}_e, \tilde{\delta})$, such that if $\phi \in (0, \phi^*]$ there exists an $\epsilon^*(\phi)$ such that, if $max\{||x_0||_2, ||\eta_{f_0}||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$, $\phi \in (0, \phi^*]$ and $\epsilon \in (0, \epsilon^*(\phi)]$, the state of the closed-loop system of Eq.7.41 is bounded and the the following estimate holds for the output error for all $t \geq 0$:

$$|\bar{y}^i - v^i| \quad \leq \quad \tilde{K}_e|(\bar{y}^i - v^i)_0|e^{-\tilde{a}_e t} + \gamma_e \phi + \frac{d}{2} \tag{F.35}$$

Taking the limit as $t \to 0$ of the above inequality, we have that if $max\{||x_0||_2, ||\eta_{f_0}||_2, ||\theta||, ||\dot{\theta}||\} \leq \tilde{\delta}$, $\phi \in (0, \phi^*]$ and $\epsilon \in (0, \epsilon^*(\phi)]$, then:

$$\lim_{t \to \infty} |\bar{y}^i - v^i| \leq \gamma_e \phi + \frac{d}{2} \leq d \tag{F.36}$$

Appendix G

Proofs of Chapter 8

Proof of proposition 8.1:

The proof of the proposition will be obtained in two steps. In the first step, we will show that the system of Eq.8.17 is exponentially stable, provided that the initial conditions and ϵ are sufficiently small. In the second step, we will use the exponential stability property to prove closeness of solutions (Eq.8.22).

Exponential stability: First, the system of Eq.8.17 can be equivalently written as:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + f_s(x_s, 0) + [f_s(x_s, x_f) - f_s(x_s, 0)] \\ \epsilon \frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon} x_f + \epsilon f_f(x_s, x_f) \end{aligned} \tag{G.1}$$

Let μ_1^*, μ_2^* with $\mu_1^* \geq a_4$ be two positive real numbers such that if $|x_s| \leq \mu_1^*$ and $||x_f||_2 \leq \mu_2^*$, then there exist positive real numbers (k_1, k_2, k_3) such that

$$\begin{aligned} |f_s(x_s, x_f) - f_s(x_s, 0)| \leq k_1 ||x_f||_2 \\ ||f_f(x_s, x_f)||_2 \leq k_2 |x_s| + k_3 ||x_f||_2 \end{aligned} \tag{G.2}$$

Pick $\mu_1 < a_4 < \mu_1^*$ and $\mu_2 < \mu_2^*$. From assumption 8.3 and the converse Lyapunov theorem for finite-dimensional systems [Kha92, Theorem 4.10], we have that there exists a smooth Lyapunov function $V : \mathcal{H}_s \rightarrow \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(a_1, a_2, a_3, a_4, a_5)$, such that for all $x_s \in \mathcal{H}_s$ that satisfy $|x| \leq a_4$, the following conditions hold:

$$\begin{aligned} a_1|x_s|^2 &\le V(x_s) \le a_2|x_s|^2 \\ \dot{V}(x_s) &= \frac{\partial V}{\partial x_s}[\mathcal{A}_s x_s + f_s(x_s,0)] \le -a_3|x_s|^2 \\ |\frac{\partial V}{\partial x_s}| &\le a_5|x_s| \end{aligned} \tag{G.3}$$

From the global exponential stability property of the fast subsystem of Eq.8.18 (assumption 8.2), and the converse Lyapunov theorem for infinite-dimensional systems [Wan64, Wan66a], we have that there exists a Lyapunov functional $W : \mathcal{H}_f \to \mathbb{R}_{\ge 0}$ and a set of positive real numbers (b_1, b_2, b_3, b_4), such that for all $x_f \in \mathcal{H}_f$ the following conditions hold:

$$\begin{aligned} b_1||x_f||_2^2 &\le W(x_f) \le b_2||x_f||_2^2 \\ \dot{W}(x_f) &= \frac{1}{\epsilon}\frac{\partial W}{\partial x_f}\mathcal{A}_f^{\epsilon} x_f \le -\frac{b_3}{\epsilon}||x_f||_2^2 \\ ||\frac{\partial W}{\partial x_f}||_2 &\le b_4||x_f||_2 \end{aligned} \tag{G.4}$$

Consider now the smooth function $L : \mathcal{H}_s \times \mathcal{H}_f \to \mathbb{R}_{\ge 0}$:

$$L(x_s, x_f) \;=\; V(x_s) + W(x_f) \tag{G.5}$$

as Lyapunov function candidate for the system of Eq.G.1. From Eq.G.3 and Eq.G.4, we have that $L(x_s, x_f)$ is positive definite and proper (tends to $+\infty$ as $|x_s| \to \infty$, or $||x_f||_2 \to \infty$), with respect to its arguments. Computing the time-derivative of L along the trajectories of this system ,and using the bounds of Eq.G.3 and Eq.G.4 and the estimates of Eq.G.2, the following expressions can be easily obtained:

$$\begin{aligned}
\dot{L}(x_s, x_f) &= \frac{\partial V}{\partial x_s}\dot{x} + \frac{\partial W}{\partial x_f}\dot{x}_f + \\
&\leq \frac{\partial V}{\partial x_s}[\mathcal{A}_s x_s + f_s(x_s, 0)] + \frac{\partial V}{\partial x_s}[f_s(x_s, x_f) - f_s(x_s, 0)] + \frac{\partial W}{\partial x_f}\mathcal{A}^{\epsilon}_f x_f \\
&\quad + \frac{\partial W}{\partial x_f} f_f(x_s, x_f) \\
&\leq -a_3|x_s|^2 + a_5 k_1 |x_s| ||x_f||_2 - \frac{b_3}{\epsilon}||x_f||_2^2 + b_4||x_f||_2(k_2|x_s| + k_3||x_f||_2) \\
&\leq -a_3|x_s|^2 + (a_5 k_1 + b_4 k_2)|x_s| ||x_f||_2 - (\frac{b_3}{\epsilon} - b_4 k_3)||x_f||_2 \\
&\leq -\begin{bmatrix} |x_s| & ||x_f||_2 \end{bmatrix} \begin{bmatrix} a_3 & -\dfrac{a_5 k_1 + b_4 k_2}{2} \\ -\dfrac{a_5 k_1 + b_4 k_2}{2} & \dfrac{b_3}{\epsilon} - b_4 k_3 \end{bmatrix} \begin{bmatrix} |x_s| \\ ||x_f||_2 \end{bmatrix}
\end{aligned} \tag{G.6}$$

Defining $\epsilon_1 = \dfrac{a_3 b_3}{a_3 b_4 k_3 + \left(\dfrac{a_5 k_1 + b_4 k_2}{2}\right)^2}$, we have that if $\epsilon \in (0, \epsilon_1)$ then $\dot{L}(x_s, x_f) < 0$, which from the properties of L directly implies that the state of the system of Eq.G.1 is exponentially stable, i.e., there exists a positive real number σ such that:

$$\begin{bmatrix} |x_s| \\ ||x_f||_2 \end{bmatrix} \leq e^{-\sigma t} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \tag{G.7}$$

Closeness of solutions: First, we define the error coordinate $e_f(\tau) = x_f(\tau) - \bar{x}_f(\tau)$. Differentiating $e_f(\tau)$ with respect to τ, the following dynamical system can be obtained:

$$\frac{\partial e_f}{\partial \tau} = \mathcal{A}_{f\epsilon} e_f + \epsilon f_f(x_s, e_f + \bar{x}_f) \tag{G.8}$$

Referring to the above system with $\epsilon = 0$, we have from the properties of the unbounded operator $\mathcal{A}_{f\epsilon}$ (assumption 8.2) and the converse theorem of [Wan64, Wan66a], that there exists a Lyapunov functional $\bar{W} : \mathcal{H}_f \rightarrow \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$, such that for all $e_f \in \mathcal{H}_f$ the following conditions hold:

$$\begin{aligned}
&\bar{b}_1||e_f(\tau)||_2^2 \leq \bar{W}(e_f(\tau)) \leq \bar{b}_2||e_f(\tau)||_2^2 \\
&\frac{\partial \bar{W}}{\partial \tau} = \frac{\partial \bar{W}}{\partial e_f}\mathcal{A}_f^{\epsilon} e_f \leq -\bar{b}_3||e_f(\tau)||_2^2 \\
&||\frac{\partial \bar{W}}{\partial e_f}||_2 \leq \bar{b}_4||e_f(\tau)||_2
\end{aligned} \tag{G.9}$$

Computing the time-derivative of $\bar{W}(e_f)$ along the trajectories of the system of Eq.G.8 and using that $||f_f(x_s, e_f+\bar{x}_f)||_2 \leq k_4||e_f||_2+k_5$, where k_4, k_5 are positive real numbers (which follows from the fact that the states $(x_s, \bar{x}_f)$ are bounded), we have:

$$\begin{aligned}
\frac{\partial \bar{W}}{\partial \tau} &\leq -\bar{b}_3||e_f||_2^2 + \epsilon\bar{b}_4||e_f||_2(k_4||e_f||_2 + k_5) \\
&\leq -(\bar{b}_3 - \epsilon\bar{b}_4 k_4)||e_f||_2^2 + \epsilon\bar{b}_4 k_5||e_f||_2
\end{aligned} \tag{G.10}$$

Set $\epsilon_2 = \dfrac{\bar{b}_3}{\bar{b}_4 k_4}$ and $\epsilon^* = \min\{\epsilon_1, \epsilon_2\}$. From the above inequality, using theorem 4.10 in [Kha92], we have that if $\epsilon \in (0, \epsilon^*)$, the following bound holds for $||e_f(\tau)||_2$ for all $t \in [0, \infty)$:

$$||e_f(\tau)||_2 \leq K_3||e_f(0)||_2 e^{-a_3 \frac{t}{\epsilon}} + \epsilon\delta_{e_f} \tag{G.11}$$

where δ_{e_f} is a positive real number. From the above inequality and the fact that $||e_f(0)||_2 = 0$, the estimate $x_f(t) = \bar{x}_f(\frac{t}{\epsilon}) + O(\epsilon)$ follows directly.

Defining the error coordinate $e_s(t) = x_s(t) - \bar{x}_s(t)$ and differentiating $e_s(t)$ with respect to time, the following system can be obtained:

$$\frac{de_s}{dt} = \mathcal{A}_s e_s + f_s(\bar{x}_s + e_s, x_f) - f_s(\bar{x}_s) \tag{G.12}$$

The representation of the system of Eq.G.12 in the fast time-scale τ takes the form:

$$\frac{de_s}{d\tau} = \epsilon[\mathcal{A}_s e_s + f_s(\bar{x}_s + e_s, x_f) - f_s(\bar{x}_s)] \tag{G.13}$$

where $(e_s, \bar{x}_s)$ can be considered approximately constant, and thus using that $|e_s(0)| = 0$ and continuity of solutions for $e_s(\tau)$, the following bound can be written for $e_s(\tau)$ for all $\tau \in [0, \tau_b]$:

$$|e_s(\tau)| \leq \epsilon k_6 \tau_b, \ \forall \ \tau \in [0, \tau_b) \tag{G.14}$$

where k_6 is a positive real number and $\tau_b = \frac{t_b}{\epsilon} = O(1)$, with $t_b = O(\epsilon) > 0$ is the time required for $x_f(t)$ to approach $\bar{x}(t)$, i.e. $||x_f(t)||_2 \leq k_7\epsilon$ for $t \in [t_b, \infty)$, where k_7 is a positive real number. The system of Eq.G.12 with $\bar{x}_s(t) = x_f(t) \equiv 0$ is exponentially stable (assumption 8.3). Moreover, since $\bar{x}_s(t)$ decays exponentially, the system of Eq.G.12 is also exponentially stable if $x_f(t) \equiv 0$. This implies that for the system:

$$\frac{de_s}{dt} = \mathcal{A}_s e_s + f_s(\bar{x}_s + e_s, 0) - f_s(\bar{x}_s) \tag{G.15}$$

there exists a smooth Lyapunov function $\bar{V} : \mathcal{H}_s \rightarrow \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5)$, such that for all $e_s \in \mathcal{H}_s$ that satisfy $|e_s| \leq \bar{a}_4$ the following conditions hold:

$$\begin{aligned}
&\bar{a}_1|e_s|^2 \leq \bar{V}(e_s) \leq \bar{a}_2|e_s|^2 \\
&\dot{\bar{V}}(e_s) = \frac{\partial \bar{V}}{\partial e_s}[\mathcal{A}_s e_s + f_s(\bar{x}_s + e_s, 0) - f_s(\bar{x}_s)] \leq -\bar{a}_3|e_s|^2 \\
&|\frac{\partial \bar{V}}{\partial e_s}| \leq \bar{a}_5|e_s|
\end{aligned} \tag{G.16}$$

Computing the time-derivative of $\bar{V}(e_s)$ along the trajectories of the system of Eq.G.12 and using that for $t \in [t_b, \infty)$ $|f_s(\bar{x}_s + e_s, x_f) - f_s(\bar{x}_s + e_s, 0)| \leq k_8||x_f||_2 \leq k_7k_8\epsilon$, where k_8 is a positive real number (which follows from the fact that the states $(\bar{x}_s, x_f)$ are bounded), we have for all $t \in [t_b, \infty)$:

$$\dot{\bar{V}}(e_s) \leq -\bar{a}_3|e_s|^2 + k_7k_8\bar{a}_5|e_s|\epsilon \tag{G.17}$$

From the above inequality, using theorem 4.10 in [Kha92] and that $|e_s(t_b)| = O(\epsilon)$, we have that the following bound holds for $|e_s(t)|$ for all $t \in [t_b, \infty)$:

$$|e_s(t)| \leq \epsilon\delta_{e_s} \tag{G.18}$$

where δ_{e_s} is a positive real number. From inequalities of Eqs.G.14-G.18, the estimate $x_s(t) = \bar{x}_s(t) + O(\epsilon)$ for all $t \geq 0$ follows directly. △

Proof of proposition 8.2:

The proof of the proposition will be obtained following a two-step approach similar to the one used in the proof of proposition 8.1.

Exponential stability: This part of the proof of the proposition is completely analogous to the proof of exponential stability in the case of proposition 8.1. and thus. it will be omitted for brevity.

Closeness of solutions: From the first part of the proof $\epsilon \in (0, \bar{\epsilon}^*)$. Defining the error coordinate $\tilde{e}_s(t) = x_s(t) - \tilde{x}_s(t)$ and differentiating $\tilde{e}_s(t)$ with respect to time, the following system can be obtained:

$$\frac{d\tilde{e}_s}{dt} = \mathcal{A}_s\tilde{e}_s + f_s(\tilde{x}_s + \tilde{e}_s, x_f) - f_s(\tilde{x}_s, \tilde{x}_f) \tag{G.19}$$

where $\tilde{x}_f = \Sigma^0(\tilde{x}_s) + \epsilon\Sigma^1(\tilde{x}_s) + \epsilon^2\Sigma^2(\tilde{x}_s) + \cdots + \epsilon^k\Sigma^k(\tilde{x}_s)$. From assumption 8.3 and the fact that $\tilde{x}_s(t)$ decays exponentially to zero, we have that the system:

$$\frac{d\tilde{e}_s}{dt} = \mathcal{A}_s\tilde{e}_s + f_s(\tilde{x}_s + \tilde{e}_s, \tilde{x}_f) - f_s(\tilde{x}_s, \tilde{x}_f) \tag{G.20}$$

is exponentially stable, which implies that there exists a smooth Lyapunov function $\tilde{V} : \mathcal{H}_s \to \mathbb{R}_{\geq 0}$ and a set of positive real numbers $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5)$, such that for all $\tilde{e}_s \in \mathcal{H}_s$ that satisfy $|\tilde{e}_s| \leq \tilde{a}_4$, the following conditions hold:

$$\begin{aligned}
&\tilde{a}_1|\tilde{e}_s|^2 \leq \tilde{V}(\tilde{e}_s) \leq \tilde{a}_2|\tilde{e}_s|^2 \\
&\dot{\tilde{V}}(\tilde{e}_s) = \frac{\partial \tilde{V}}{\partial \tilde{e}_s}[\mathcal{A}_s\tilde{e}_s + f_s(\tilde{x}_s + \tilde{e}_s, \tilde{x}_f) - f_s(\tilde{x}_s, \tilde{x}_f)] \leq -\tilde{a}_3|\tilde{e}_s|^2 \\
&|\frac{\partial \tilde{V}}{\partial \tilde{e}_s}| \leq \tilde{a}_5|\tilde{e}_s|
\end{aligned} \tag{G.21}$$

Computing the time-derivative of $\tilde{V}(\tilde{e}_s)$ along the trajectories of the system of Eq.G.19 and using that for $t \in [0, \infty)$ $|f_s(\tilde{x}_s + \tilde{e}_s, x_f) - f_s(\tilde{x}_s + \tilde{e}_s, \tilde{x}_f)| \leq (\tilde{k}_1\epsilon^{k+1} + \tilde{k}_2 e^{\frac{\lambda_1 t}{\epsilon}})$, where $\tilde{k}_1, \tilde{k}_2$ are positive real numbers, we have for all $t \in [0, \infty)$:

$$\dot{\tilde{V}}(\tilde{e}_s) \leq -\tilde{a}_3|\tilde{e}_s|^2 + (\tilde{k}_1\epsilon^{k+1} + \tilde{k}_2 e^{\frac{\lambda_1 t}{\epsilon}})\tilde{a}_5|\tilde{e}_s| \tag{G.22}$$

From the above inequality, using theorem 4.10 in [Kha92] and the fact that $|\tilde{e}_s(0)| = 0$, we have that the following bound holds for $|\tilde{e}_s(t)|$ for all $t \in [0, \infty)$:

$$|\tilde{e}_s(t)| \leq \tilde{K}e^{\frac{\lambda_1 t}{\epsilon}} + \tilde{K}\epsilon^{k+1} \tag{G.23}$$

where $\bar{K}, \tilde{K}$ are positive real numbers. Since the term $\bar{K}\epsilon^{\frac{\lambda_1 t}{\epsilon}}$ vanishes outside the interval $[0, t_b]$ (where t_b is the time required for x_f to approach $\tilde{x}_f$), it follows from Eq.G.23 that for all $t \in [t_b, \infty)$:

$$|\tilde{e}_s(t)| \leq \tilde{K}\epsilon^{k+1} \tag{G.24}$$

From the above inequality, the estimate $x_s(t) = \tilde{x}_s(t) + O(\epsilon^{k+1})$, for $t \geq t_b$, follows directly. △

Proof of theorem 8.1:

Substituting the output feedback controller of Eq.8.34 into the system of Eq.8.17, we get:

$$\begin{aligned}
\frac{d\eta}{dt} &= \mathcal{A}_s\eta + f_s(\eta, \epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)) \\
&\quad + \mathcal{B}_s\,(p_0(\eta) + Q_0(x_s)v + \epsilon[p_1(\eta, \epsilon) + Q_1(\eta, \epsilon)v] \\
&\quad + \cdots + \epsilon^k[p_k(\eta, \epsilon) + Q_k(\eta, \epsilon)v]\Big) \\
&\quad + L(y - \Big[C\eta + C\epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)\Big]) \\
\frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s\,(p_0(\eta) + Q_0(x_s)v + \epsilon[p_1(\eta, \epsilon) + Q_1(\eta, \epsilon)v] \\
&\quad + \cdots + \epsilon^k[p_k(\eta, \epsilon) + Q_k(\eta, \epsilon)v]\Big) + f_s(x_s, x_f) \\
\epsilon\frac{\partial x_f}{\partial t} &= \mathcal{A}_{f\epsilon}x_f + \epsilon\mathcal{B}_f\,(p_0(\eta) + Q_0(x_s)v + \epsilon[p_1(\eta, \epsilon) + Q_1(\eta, \epsilon)v \\
&\quad + \cdots + \epsilon^k[p_k(\eta, \epsilon) + Q_k(\eta, \epsilon)v]\Big) + \epsilon f_f(x_s, x_f) \\
y^i &= Cx_s + Cx_f, \quad i = 1, \ldots, l
\end{aligned} \tag{G.25}$$

Performing a two-time-scale decomposition in the above system, the fast subsystem takes the form:

$$\frac{\partial x_f}{\partial \tau} = \mathcal{A}_{f\epsilon}x_f \tag{G.26}$$

which is exponentially stable. Furthermore, the $O(\epsilon^{k+1})$ approximation of the closed-loop inertial form is given by:

$$
\begin{aligned}
\frac{d\eta}{dt} &= \mathcal{A}_s\eta + f_s(\eta, \epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)) \\
&\quad + \mathcal{B}_s\,(p_0(\eta) + Q_0(x_s)v + \epsilon[p_1(\eta, \epsilon) + Q_1(\eta, \epsilon)v] \\
&\quad + \cdots + \epsilon^k[p_k(\eta, \epsilon) + Q_k(\eta, \epsilon)v]\Big) \\
&\quad + L(y - \Big[C\eta + C\Sigma^0(\eta, u) + \epsilon\Sigma^1(\eta, u) + \epsilon^2\Sigma^2(\eta, u) + \cdots + \epsilon^k\Sigma^k(\eta, u)\Big]) \\
\frac{dx_s}{dt} &= \mathcal{A}_s x_s + f_s(x_s, \epsilon\Sigma^1(x_s, u) + \epsilon^2\Sigma^2(x_s, u) + \cdots + \epsilon^k\Sigma^k(x_s, u)) \\
&\quad + \mathcal{B}_s\,(p_0(\eta) + Q_0(x_s)v + \epsilon[p_1(\eta, \epsilon) + Q_1(\eta, \epsilon)v] \\
&\quad + \cdots + \epsilon^k[p_k(\eta, \epsilon) + Q_k(\eta, \epsilon)v]\Big) \\
y_s^i &= Cx_s + C[\epsilon\Sigma^1(x_s, u) + \epsilon^2\Sigma^2(x_s, u) + \cdots + \epsilon^k\Sigma^k(x_s, u)], \quad i = 1, \ldots, l
\end{aligned}
\tag{G.27}
$$

Referring to the above closed-loop ODE system, assumption 8.4 yields that it is exponentially stable and the output y_s^i, $i = 1, \ldots, l$, changes in a prespecified manner. A direct application of the result of proposition 8.2 yields that there exist constants $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\epsilon}^*$ such that if $|x_s(0)| \leq \tilde{\mu}_1$, $||x_f(0)||_2 \leq \tilde{\mu}_2$ and $\epsilon \in (0, \tilde{\epsilon}^*]$, such that the closed-loop infinite-dimensional system is exponentially stable and the relation of Eq.8.35 holds. $\triangle$