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Predictive control of parabolic PDEs with state and control constraints

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SUMMARY

This work focuses on predictive control of linear parabolic partial differential equations (PDEs) with state and control constraints. Initially, the PDE is written as an infinite-dimensional system in an appropriate Hilbert space. Next, modal decomposition techniques are used to derive a finite-dimensional system that captures the dominant dynamics of the infinite-dimensional system, and express the infinite-dimensional state constraints in terms of the finite-dimensional system state constraints. A number of model predictive control (MPC) formulations, designed on the basis of different finite-dimensional approximations, are then presented and compared. The closed-loop stability properties of the infinite-dimensional system under the low order MPC controller designs are analysed, and sufficient conditions that guarantee stabilization and state constraint satisfaction for the infinite-dimensional system under the reduced order MPC formulations are derived. Other formulations are also presented which differ in the way the evolution of the fast eigenmodes is accounted for in the performance objective and state constraints. The impact of these differences on the ability of the predictive controller to enforce closed-loop stability and state constraints satisfaction in the infinite-dimensional system is analysed. Finally, the MPC formulations are applied through simulations to the problem of stabilizing the spatially-uniform unstable steady-state of a linear parabolic PDE subject to state and control constraints. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Transport-reaction processes are characterized by significant spatial variations due to the underlying diffusion and convection phenomena. The dynamic models of transport-reaction processes over finite spatial domains typically consist of parabolic partial differential equation (PDE) systems whose spatial differential operators are characterized by a spectrum that can be partitioned into a finite (possibly unstable) slow part and an infinite stable fast complement [1]. The traditional approach for control of linear/quasi-linear parabolic PDEs involves the application of spatial discretization techniques to the PDE system to derive systems of ordinary differential equations (ODEs) that accurately describe the dynamics of the dominant (slow) modes of the PDE system. The finite-dimensional systems are subsequently used as the basis for the synthesis of finite-dimensional controllers (e.g. see References [2–4]). A potential drawback of this approach, especially for quasi-linear parabolic PDEs, is that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to complex controller design and high dimensionality of the resulting controllers.

Motivated by these considerations, significant recent work has focused on the development of a general framework for the synthesis of low-order controllers for quasi-linear parabolic PDE systems (and other highly dissipative PDE systems that arise in the modelling of spatially-distributed systems) on the basis of low-order nonlinear ODE models derived through a combination of the Galerkin method (using analytical or empirical basis functions) with the concept of inertial manifolds [5]. Using these order reduction techniques, a number of control-relevant problems, such as nonlinear and robust controller design, dynamic optimization, and control under actuator saturation have been addressed for various classes of dissipative PDE systems (e.g. see References [6–10] and the book [5] for results and references in this area). In addition to these works, other recent studies on control of PDE systems include [11, 12]. The approaches proposed in the above works, however, do not address the issue of state constraints in the controller design. Operation of transport-reaction processes typically requires that the state of the closed-loop system be maintained within certain bounds to achieve acceptable performance. Examples include requiring the temperature of a tubular reactor not to exceed a certain value, and requiring the product concentration not to drop below some purity requirement. Handling both state and control constraints—the latter typically arising due to the finite capacity of control actuators—in the design of the controller, therefore, is an important consideration.

Model predictive control (MPC), also known as receding horizon control, is a popular control method for handling constraints (both on manipulated inputs and state variables) within an optimal control setting. In MPC, the control action is obtained by solving repeatedly, on-line, a finite horizon constrained open-loop optimal control problem. The popularity of this approach stems largely from its ability to handle, among other issues, multi-variable interactions, constraints on controls and states, and optimization requirements. Numerous research studies have investigated the properties of model predictive controllers and led to a plethora of MPC formulations that focus on a number of control-relevant issues, including issues of closed-loop stability, performance, implementation and constraint satisfaction (e.g. see References [13–16] for surveys of results and references in this area).

Most of the research in the area of predictive control, however, has focused on lumped-parameter processes modelled by ODE systems. Compared with lumped-parameter systems, the

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problem of designing predictive controllers for distributed parameter systems modelled by PDEs has received much less attention. Of the few results available on this problem, some have focused on analysing the receding horizon control problem on the basis of the infinite-dimensional system using control Lyapunov functionals (e.g. Reference [17]), while others have used spatial discretization techniques such as finite differences (e.g. Reference [18]) to derive approximate ODE models (of possibly high-order) for use within the MPC design, thus leading to computationally expensive model predictive control designs that are, in general, difficult to implement on-line.

Motivated by the above considerations, we focus in this work on the development of a framework for the design of predictive controllers for linear parabolic PDEs with state and control constraints. The rest of the paper is organized as follows. In Section 2, a class of parabolic PDEs is described and formulated as an infinite-dimensional system, and the predictive control problem is formulated on the basis of the infinite-dimensional system. Then, in Section 3, modal decomposition techniques are used to derive a finite-dimensional system that captures the dominant dynamics of the infinite-dimensional system and to express the state constraints for the infinite-dimensional system in terms of the finite-dimensional system state constraints. A number of MPC formulations, designed on the basis of different finite-dimensional approximations, are presented and compared. The closed-loop stability properties of the infinite-dimensional system under the low order MPC controller designs are analysed and sufficient conditions, which guarantee stabilization and state constraint satisfaction for the infinite-dimensional system under the reduced order MPC formulations, are derived. We also present other formulations which differ in the way the evolution of the fast eigenmodes is accounted for in the performance objective and state constraints. The implications of these differences, in terms of the ability of the predictive controller to enforce closed-loop stability and state constraints satisfaction in the infinite-dimensional system, is analysed. Finally, in Section 4, the MPC formulations are applied through simulations to the problem of stabilizing the spatially-uniform unstable steady-state of a linear parabolic PDE subject to state and control constraints.

2. PRELIMINARIES

2.1. Parabolic PDEs

To motivate the class of infinite-dimensional systems considered, we focus on a linear parabolic PDE, with distributed control, of the form

$$\frac{\partial \bar{x}}{\partial t} = b\frac{\partial^2 \bar{x}}{\partial z^2} + c\bar{x} + w\sum_{i=1}^{m} b_i(z)u_i \tag{1}$$

with the following boundary and initial conditions:

$$\bar{x}(0,t) = 0, \quad \bar{x}(\pi,t) = 0, \quad \bar{x}(z,0) = \bar{x}_0(z) \tag{2}$$

subject to the following input and state constraints:

$$u_i^{\min} \leqslant u_i \leqslant u_i^{\max}, \quad i = 1,\ldots,m \tag{3}$$

$$\chi^{\min} \leqslant \int_0^{\pi} r(z)\bar{x}(z,t)\,\mathrm{d}z \leqslant \chi^{\max} \tag{4}$$

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where $\bar{x}(z,t)$ denotes the state variable, $z \in [0,\pi]$ is the spatial co-ordinate, $t \in [0,\infty)$ is the time, $u_i \in \mathbb{R}$ denotes the ith constrained manipulated input; $u_i^{\min}$ and $u_i^{\max}$ are real numbers representing the lower and upper bounds on the ith input, respectively, and $\chi^{\min}$ and $\chi^{\max}$ are real numbers representing the lower and upper state constraints, respectively. The term $\partial^2\bar{x}/\partial z^2$ denotes the second-order spatial derivative of $\bar{x}$; b, c and w are constant coefficients with $b > 0$, and $\bar{x}_0(z)$ is a sufficiently smooth function of z. The function $b_i(z) \in L_2(0,\pi)$ is a known 'square integrable' function of z that describes how the control action, $u_i(t)$, is distributed in the spatial interval $[0,\pi]$. Whenever the control action is applied to the spatial domain at a single point z_a, with $z_a \in [0,\pi]$ (i.e. point actuation), the function $b_i(z)$ is taken to be non-zero in a finite spatial interval of the form $[z_a - \mu, z_a + \mu]$, where μ is a small positive real number, and zero elsewhere in $[0,\pi]$. In Equation (4), the function $r(z) \in L_2(0,\pi)$ is a 'state constraint distribution' function that describes how the state constraint is enforced in the spatial domain $[0,\pi]$. Note that we consider only integral constraints with square integrable state constraint distribution functions, and not pointwise or general state constraints distribution functions. Throughout the paper, the notation $|\cdot|$ will be used to denote the standard Euclidian norm in $\mathbb{R}^n$, while the notation $|\cdot|_Q$ will be used to denote the weighted norm defined by $|\hat{x}|_Q^2 = \hat{x}'Q\hat{x}$, where Q is a positive-definite matrix and $\hat{x}'$ denotes the transpose of $\hat{x}$. A function $\beta(r,s) : [0,a) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if, for each fixed $s \geqslant 0$, the mapping $\beta(r,s)$ is continuous, strictly increasing with respect to r and satisfies $\beta(0,s) = 0$, and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.

To proceed with the presentation of our results, we formulate the PDE of Equations (1)–(4) as an infinite-dimensional system in the state space $\mathcal{H} = L_2(0,\pi)$, with inner product and norm

$$(\omega_1,\omega_2) = \int_0^\pi \omega_1(z)\omega_2(z)\,\mathrm{d}z, \quad \|\omega_1\|_2 = (\omega_1,\omega_1)^{1/2} \tag{5}$$

where ω_1, ω_2 are two elements of $L_2(0,\pi)$.

Defining the state function $x(t)$ on the state-space $\mathcal{H}$ as

$$x(t) = \bar{x}(z,t), \quad t > 0, \quad 0 < z < \pi \tag{6}$$

the operator $\mathcal{A}$ as

$$\mathcal{A}\phi = b\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + c\phi, \quad 0 < z < \pi \tag{7}$$

where $\phi(z)$ is a smooth function on $(0,\pi)$ with $\phi(0) = 0$ and $\phi(\pi) = 0$, with the following dense domain:

$$\mathcal{D}(\mathcal{A}) = \left\{\phi(z) \in L_2(0,\pi) : \phi(z), \frac{\mathrm{d}\phi(z)}{\mathrm{d}z} \text{ are absolutely continuous} \right.$$
$$\left. \mathcal{A}\phi \in L_2(0,\pi), \phi(0) = 0 \text{ and } \phi(\pi) = 0\right\} \tag{8}$$

the input operator as

$$\mathcal{B}u = w\sum_{i=1}^{m} b_i(\cdot)u_i \tag{9}$$

and the state constraint as

$$\chi^{\min} \leqslant (r,x) \leqslant \chi^{\max} \tag{10}$$

the system of Equations (1)–(4) can be written as

$$\dot{x} = \mathscr{A}x + \mathscr{B}u, \quad x(0) = x_0 \tag{11}$$

$$u_i^{\min} \leqslant u_i(t) \leqslant u_i^{\max} \tag{12}$$

$$\chi^{\min} \leqslant (r, x) \leqslant \chi^{\max} \tag{13}$$

The spectrum of $\mathscr{A}$ can be obtained by solving the following eigenvalue problem:

$$\mathscr{A}\phi_j = b\frac{\mathrm{d}^2\phi_j}{\mathrm{d}z^2} + c\phi_j = \lambda_j\phi_j \tag{14}$$

subject to

$$\phi_j(0) = \phi_j(\pi) = 0 \tag{15}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction. A direct computation of the solution of the above eigenvalue problem yields

$$\lambda_j = c - bj^2, \quad \phi_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz), \quad j = 1, \ldots, \infty \tag{16}$$

The point spectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, is defined as the set of all eigenvalues of $\mathscr{A}$, i.e. $\sigma(\mathscr{A}) = \{\lambda_1, \lambda_2, \ldots\}$. From the expression for the eigenvalues, it is clear that all the eigenvalues of $\mathscr{A}$ are real, and that, for a given b and c, only a finite number of unstable eigenvalues exist, and the distance between any two consecutive eigenvalues (i.e. λ_j and λ_{j+1}) increases as j increases. Furthermore, $\sigma(\mathscr{A})$ can be partitioned as $\sigma(\mathscr{A}) = \sigma_1(\mathscr{A}) \cup \sigma_2(\mathscr{A})$, where $\sigma_1(\mathscr{A}) = \{\lambda_1, \ldots, \lambda_m\}$ contains the first m (with m finite) 'slow' eigenvalues (including all, if any, possibly unstable eigenvalues) and $\sigma_2(\mathscr{A}) = \{\lambda_{m+1}, \lambda_{m+2}, \ldots\}$ contains the remaining 'fast' stable eigenvalues. This implies that the dominant dynamics of the PDE can be described by a finite-dimensional system, and motivates the use of modal decomposition to derive a finite-dimensional system that captures the dominant (slow) dynamics of the PDE.

From the properties of $\mathscr{A}$ and its spectrum, it follows (Theorem 2.10 in Reference [1]) that $\mathscr{A}$ generates a strongly continuous $\mathscr{C}_0$-semigroup, $\mathscr{T}(t)$. Moreover, since $\mathscr{B}$ is a bounded operator, the system of Equation (11) has a mild solution (Theorem 2.31 in Reference [1]) of the form

$$x(t) = \mathscr{T}(t)x_0 + \int_0^t \mathscr{T}(t-\tau)\mathscr{B}u(\tau)\,\mathrm{d}\tau \tag{17}$$

2.2. *MPC formulation*

Referring to the system of Equation (11), we consider the problem of asymptotic stabilization of the origin, subject to the control constraints of Equation (12) and the state constraints of Equation (13). The problem will be addressed within the MPC framework (see Reference [16] for a review of various MPC algorithms for finite-dimensional systems) where the control, at state x and time t, is conventionally obtained by solving, repeatedly, a finite horizon constrained optimal control problem of the form

$$P(x, t) : \min\{J(x, t, u(\cdot)) | u(\cdot) \in S\} \tag{18}$$

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$$\text{s.t. } \dot{x}(\tau) = \mathscr{A}x(\tau) + \mathscr{B}u(\tau)$$

$$u(\tau) \in \mathscr{U}$$

$$\chi^{\min} \leqslant (r, x(\tau)) \leqslant \chi^{\max}, \quad \tau \in [t, t+T] \tag{19}$$

where $S = S(t, T)$ is the family of piecewise constant functions, with period Δ, mapping $[t, t+T]$ into $\mathscr{U} := \{u \in \mathbb{R}^m : u_i^{\min} \leqslant u_i \leqslant u_i^{\max},\ i = 1, \ldots, m\}$, and T is the specified horizon. A control $u(\cdot)$ in S is characterized by the sequence $u[k]$, where $u[k] := u(k\Delta)$, and satisfies $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. The performance index is given by

$$J(x, t, u(\cdot)) = \int_t^{t+T} [q\|x^u(\tau; x, t)\|_2^2 + |u(\tau)|_R^2]\,\mathrm{d}\tau + F(x(t+T)) \tag{20}$$

where q is a strictly positive real number, $x^u(\tau; x, t)$ denotes the solution of Equation (11), due to control u, with initial state x at time t, and $F(\cdot)$ denotes the terminal penalty. The minimizing control $u^0(\cdot) \in S$ is then applied to the system over the interval $[k\Delta, (k+1)\Delta]$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M(x) := u^0(t; x, t) \tag{21}$$

Since the predictive control problem of Equations (18)–(19) is formulated on the basis of the infinite-dimensional system, it leads to a predictive controller that is of higher-order and cannot be readily implemented in practice. To overcome this problem, we develop in the next section computationally efficient predictive control formulations that achieve stabilization of the system of Equation (11) subject to the control and state constraints of Equations (12)–(13).

Remark 1

It is well known that even in the case of finite-dimensional systems, the control law defined by Equations (18)–(21) is not necessarily stabilizing. For finite-dimensional systems, the issue of closed-loop stability is usually addressed by means of imposing suitable penalties and constraints on the state at the end of the optimization horizon (e.g. see References [14, 16] for surveys of different approaches).

3. PREDICTIVE CONTROL OF INFINITE-DIMENSIONAL SYSTEMS

In this section, we initially apply modal decomposition techniques to the system of Equation (11) to derive a finite-dimensional system that captures its dominant dynamics, and express the state constraints of Equation (13) in terms of constraints on the state of the finite-dimensional system. The finite-dimensional system is then used for the construction of a low-order predictive controller, and sufficient conditions guaranteeing stabilization and state constraints satisfaction for the infinite-dimensional closed-loop system are derived. Other MPC formulations, designed on the basis of appropriate finite-dimensional approximations and differing in the way the state constraints are handled within the optimization problem, are then presented and compared.

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3.1. Modal decomposition

Referring to the system of Equation (11), let $\mathscr{H}_s$ and $\mathscr{H}_f$ be modal subspaces of $\mathscr{A}$, defined as $\mathscr{H}_s = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathscr{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \ldots\}$ (the existence of $\mathscr{H}_s$, $\mathscr{H}_f$ follows from the properties of $\mathscr{A}$). Defining the orthogonal projection operators, P_s and P_f, such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Equation (11) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x \tag{22}$$

Applying P_s and P_f to the system of Equation (11) and using the above decomposition for x, the system of Equation (11) can be re-written in the following equivalent form:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \mathscr{A}_s x_s + \mathscr{B}_s u, \quad x_s(0) = P_s x(0) = P_s x_0$$

$$\frac{\mathrm{d}x_f}{\mathrm{d}t} = \mathscr{A}_f x_f + \mathscr{B}_f u, \quad x_f(0) = P_f x(0) = P_f x_0 \tag{23}$$

where $\mathscr{A}_s = P_s\mathscr{A}$, $\mathscr{B}_s = P_s\mathscr{B}$, $\mathscr{A}_f = P_f\mathscr{A}$, $\mathscr{B}_f = P_f\mathscr{B}$. In the above system, $\mathscr{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathscr{A}_s = \text{diag}\{\lambda_j\}$ (λ_j are possibly unstable eigenvalues of $\mathscr{A}_s$) and $\mathscr{A}_f$ is an unbounded differential operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathscr{H}_s$, $\mathscr{H}_f$). In the remainder of the paper, we will refer to the x_s- and x_f-subsystems in Equation (23) as the slow and fast subsystems, respectively. From the properties of $\mathscr{A}_s$ and $\mathscr{A}_f$ and the fact that $\mathscr{B}_s$ and $\mathscr{B}_f$ are bounded operators, it follows (Theorems 2.10 and 2.31 in Reference [1]) that there exist C_0-semigroups $\mathscr{T}_s$ and $\mathscr{T}_f$ such that the x_s- and x_f-subsystems of Equation (23) admit, on the interval $t \in [0, \infty)$, the following mild solutions:

$$x_s(t) = \mathscr{T}_s(t)x_s(0) + \int_0^t \mathscr{T}_s(t-\tau)\mathscr{B}_s u(\tau)\,\mathrm{d}\tau \tag{24}$$

$$x_f(t) = \mathscr{T}_f(t)x_f(0) + \int_0^t \mathscr{T}_f(t-\tau)\mathscr{B}_f u(\tau)\,\mathrm{d}\tau \tag{25}$$

Furthermore, since $\mathscr{A}_f$ is a stable operator, the spectrum of $\mathscr{A}_f$ satisfies $\sup\{\text{Re}\,\sigma(\mathscr{A}_f)\} < -\gamma$, for some $\gamma > |c - b(m+1)^2|$, and thus, $\mathscr{T}_f(t)$ satisfies ([1], p. 74)

$$\|\mathscr{T}_f(t)\|_2 \leqslant M_0 \mathrm{e}^{-\gamma t}, \quad t \geqslant 0 \tag{26}$$

for some $M_0 > 0$.

Remark 2
Note that while we use the PDE of Equations (1)–(4) to motivate and illustrate the development of the infinite-dimensional system of Equation (11), our subsequent results are not limited to single PDEs of the form of Equations (1)–(4). The results developed in this work apply to parabolic PDEs with possibly other types of boundary conditions (e.g. mixed boundary conditions), or other means of implementing control, such as boundary control [19] and also systems of parabolic PDEs, as long as they possess operators $\mathscr{A}$ that have a finite number of eigenvalues with positive real parts and the decomposition of Equations (22)–(23) can be written. These conditions can be shown to hold for all linear parabolic PDEs with self-adjoint

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operators that have only finitely many eigenvalues with positive real parts. In the domain of chemical process control, the class of linear parabolic PDEs with self-adjoint operators arises frequently from the linearization of first-principle models of diffusion–reaction processes. Note also, that while this work focusses on addressing the problem of ensuring state constraint satisfaction for the infinite dimensional system, and uses linear systems under the assumption of full-state feedback to illustrate the main idea, the proposed approach can also be used to address practical issues such as unavailability of state measurements [20] and nonlinearity [21].

3.2. MPC formulations: accounting for input and state constraints

We first present an MPC formulation, designed on the basis of the slow subsystem in Equation (23), that ensures stabilization of the infinite-dimensional system. The MPC law in this case is obtained by solving, in a receding horizon fashion, the following optimization problem:

$$\min_u \left[\int_t^{t+T} [q_s \|x_s(\tau)\|_2^2 + |u(\tau)|_R^2] \, d\tau + F(x_s(t+T)) \right] \tag{27}$$

$$\text{s.t.} \; \dot{x}_s(\tau) = \mathscr{A}_s x_s(\tau) + \mathscr{B}_s u(\tau)$$

$$u(\tau) \in \mathscr{U}$$

$$\chi^{\min} \leqslant (r, x_s(\tau)) \leqslant \chi^{\max}, \quad \tau \in [t, t+T]$$

$$x_s(t+T) = 0 \tag{28}$$

To proceed, we assume that the predictive control law of Equations (27)–(28), with a fixed horizon length T, is initially and successively feasible and achieves stabilization of the x_s-subsystem for all $x_s(0) \in \Omega_s \subset \mathscr{H}_s$. Note that the set Ω_s depends on the constraints on the states and inputs, the system dynamics and T (see remarks 5 and 6 for discussion on this issue). This assumption is precisely stated below.

Assumption 1

There exists a set $\Omega_s \subset \mathscr{H}_s$ such that for all $x_s(0) \in \Omega_s$, the steady-state solution $x_s(t) = 0$ of the closed-loop system of Equation (24) under the MPC law of Equations (27)–(28) is asymptotically stable in the sense that $x_s(t) \in \Omega_s$ for all $t \geqslant 0$ and satisfies $\|x_s(t)\|_2 \leqslant \beta(\|x_s(0)\|_2, t)$, where $\beta(\cdot,\cdot)$ is a class $\mathscr{K}\mathscr{L}$ function.

Note that the symbolic terminal endpoint constraint of Equation (28) is used as an example of a stability constraint, and the results of this work are not limited to this particular choice of stability constraint. Other approaches, that account for stability considerations via incorporating other form of constraints or penalties on the state variables can very well be used within the proposed approach. What is required, though, is for the predictive control formulation to be able to stabilize the closed-loop system and Assumption 1 above formalizes this requirement. Proposition 1 below establishes that the predictive control law of Equations (27)–(28), for which

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Assumption 1 holds, also achieves asymptotic stability of the closed-loop infinite-dimensional system.

Proposition 1

Consider the system of Equation (11) subject to the input and state constraints of Equations (12)–(13), under the predictive controller of Equations (27)–(28) for which Assumption 1 holds. If the initial condition of the infinite-dimensional system, $x(0)$, is such that $x_s(0) \in \Omega_s$, then $x(t) = 0$ is an asymptotically stable solution of the closed-loop infinite-dimensional system.

Proof of Proposition 1

We first note that the control law of Equations (27)–(28) is only a function of the state of the slow subsystem, $u = M(x_s)$, and satisfies $\lim_{t\to\infty} |u(t)| = 0$ (this follows from the assumption that $x(t) = 0$ is a steady-state solution of the closed-loop system of Equation (24) and Equations (27)–(28), Assumption 1). Using the decomposition of Equation (23), the closed-loop system of Equation (11) under the MPC law of Equations (27)–(28) can therefore be written as a cascaded system of the form

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \mathscr{A}_s x_s + \mathscr{B}_s M(x_s) \tag{29}$$

$$\frac{\mathrm{d}x_f}{\mathrm{d}t} = \mathscr{A}_f x_f + \mathscr{B}_f M(x_s) \tag{30}$$

Since $x_s(0) \in \Omega_s$, we have from Assumption 1 that $x_s(t)$ of Equation (29) satisfies $\|x_s(t)\| \leqslant \beta(\|x_s(0)\|_2, t)$ for all $t \geqslant 0$. We now show that the origin of the fast x_f-subsystem of Equation (30) is also asymptotically stable, and thus the origin of the infinite-dimensional closed-loop system of Equations (11)–(13) under the control of Equations (27)–(28) is asymptotically stable. To this end, we first note that since $\mathscr{A}_f$ is a stable operator, which generates a C_0-semigroup $\mathscr{T}_f$ that satisfies Equation (26), and since $\mathscr{B}_f$ is a bounded operator (boundedness of $\mathscr{B}_f$ follows from the fact that $b(z) \in L_2(0, \pi)$), by taking the 2-norm in space of both sides of Equation (25), the following bound for the state of the x_f-subsystem of Equation (30) can be written

$$\|x_f(t)\|_2 \leqslant K\mathrm{e}^{-\gamma t}\|x_f(0)\|_2 + \|\mathscr{B}_f\|_2 \left| \int_0^t M_0 \mathrm{e}^{-\gamma(t-\tau)} u(\tau)\, \mathrm{d}\tau \right| \tag{31}$$

where $K \geqslant M_0$ is a positive real number and $u(\tau) = M(x_s(\tau))$. Further, since $u(t)$ is bounded (i.e. $|u(t)| \leqslant \bar{u}$ where $\bar{u} = \max\{|u^{\max}|, |u^{\min}|\}$) and $\mathrm{e}^{-\gamma(t-\tau)} \geqslant 0$ for $0 \leqslant \tau \leqslant t$, Equation (31) can be written as

$$\|x_f(t)\|_2 \leqslant K\mathrm{e}^{-\gamma t}\|x_f(0)\|_2 + M_0\|\mathscr{B}_f\|_2 \sup_{0\leqslant\tau\leqslant t} |u(\tau)| \left| \int_0^t \mathrm{e}^{-\gamma(t-\tau)}\, \mathrm{d}\tau \right| \tag{32}$$

or

$$\|x_f(t)\|_2 \leqslant K\mathrm{e}^{-\gamma t}\|x_f(0)\|_2 + M_2 \sup_{0\leqslant\tau\leqslant t} |u(\tau)| \tag{33}$$

where $M_2 = M_0\|\mathscr{B}_f\|_2/\gamma$. To use Equation (33) to prove that $\lim_{t\to\infty} \|x_f(t)\|_2 = 0$, we need the following argument. First, we note that by taking the supremum over time of both sides of the

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inequality of Equation (33), we have

$$\sup_{t\geqslant 0} \|x_f(t)\|_2 \leqslant K\|x_f(0)\|_2 + M_2\bar{u} =: \bar{K} \tag{34}$$

The fact that the inequality of Equation (33) holds for every initial time, t_0, yields that for $\forall t \geqslant t_0$

$$\|x_f(t)\|_2 \leqslant K\mathrm{e}^{-\gamma(t-t_0)}\|x_f(t_0)\|_2 + M_2 \sup_{\tau\geqslant t_0} |u(\tau)| \tag{35}$$

Taking $t_0 = t/2$ and using the bound of Equation (34) in Equation (35), we have

$$\|x_f(t)\|_2 \leqslant K\mathrm{e}^{-\gamma(t/2)}\bar{K} + M_2 \sup_{\tau\geqslant t/2} |u(\tau)| \tag{36}$$

Taking the limit of both sides as $t \to \infty$ and using the fact that $\lim_{t\to\infty} \sup_{\tau\geqslant t} |u(\tau)| = 0$ (this follows directly from $\lim_{t\to\infty} |u(t)| = 0$), we finally have

$$\lim_{t\to\infty} \|x_f(t)\|_2 \leqslant \lim_{t\to\infty} (K\mathrm{e}^{-\gamma(t/2)}\bar{K}) + \lim_{t\to\infty} \left(M_2 \sup_{\tau\geqslant t/2} |u(\tau)| \right) = 0 \tag{37}$$

We therefore have that $\lim_{t\to\infty} \|x_f(t)\|_2 = 0$ and thus the infinite-dimensional closed-loop system is asymptotically stable. This completes the proof of Proposition 1. □

Remark 3

Note that the MPC formulation of Equations (27)–(28) is designed on the basis of the slow subsystem only. The evolution of the fast states is unaccounted for, whether in the cost function or in the state constraints. Therefore, while the resulting MPC controller enforces closed-loop stability for the infinite-dimensional system, there is no guarantee that the state constraints for the infinite-dimensional system will be satisfied for all times (note that satisfaction of $\chi^{\min} \leqslant (r, x_s) \leqslant \chi^{\max}$ does not guarantee that $\chi^{\min} \leqslant (r, x_s + x_f) = (r, x) \leqslant \chi^{\max}$). So, unlike the stabilization objective, which can be achieved independently of the fast subsystem, the additional objective of state constraints satisfaction requires that the MPC design accounts in some way for the contribution of the fast states to the evolution of the state of the infinite-dimensional system.

We now present a modification of the MPC controller of Equations (27)–(28), and give a sufficient condition on the initial condition of the fast subsystem such that the resulting control law, when applied to the infinite-dimensional system, achieves both stabilization and state constraint satisfaction. The key idea is to revise (shrink) the bounds on the slow states in the controller design to compensate for the contribution of the fast states to the evolution of the infinite-dimensional system state. By invoking input-to-state boundedness of the fast subsystem, we then derive appropriate bounds on the initial condition for the fast subsystem which ensure state constraint satisfaction for the infinite-dimensional closed-loop system. Proposition 2 below formalizes the input-to-state boundedness property, and Theorem 1 states the sufficient condition for state constraints satisfaction for the infinite-dimensional system. To this end, consider the MPC formulation of Equations (27)–(28) with the bounds $\chi^{\max}$ and $\chi^{\min}$ replaced by $S^{\max}$ and $S^{\min}$, respectively, as follows:

$$\min_u \left[\int_t^{t+T} [q_s\|x_s(\tau)\|_2^2 + |u(\tau)|_R^2]\, \mathrm{d}\tau + F(x_s(t+T)) \right] \tag{38}$$

$$\text{s.t. } \dot{x}_s(\tau) = \mathscr{A}_s x_s(\tau) + \mathscr{B}_s u(\tau)$$

$$u(\tau) \in \mathcal{U}$$

$$S^{\min} \leqslant (r, x_s(\tau)) \leqslant S^{\max}, \quad \tau \in [t, t+T]$$

$$x_s(t+T) = 0 \tag{39}$$

where $S^{\max} \leqslant \chi^{\max} - \alpha$ and $S^{\min} \geqslant \chi^{\min} + \alpha$, where α is a positive real number to be determined later. We denote by Ω_s' the set of initial conditions for which the predictive controller of Equations (38)–(39) satisfies the conditions of Assumption 1.

Proposition 2
Consider the system of Equation (11) and the input and state constraints of Equations (12)–(13). There exist positive real numbers M_1^* and M_2^* such that $|(r, x_f(t))| \leqslant M_1^* \|x_f(0)\|_2 + M_2^* \bar{u}$, for all $t \geqslant 0$, where $\bar{u} = \max\{|u^{\max}|, |u^{\min}|\}$.

Proof of Proposition 2
First, using Holder's inequality [22], we get

$$\left| \int_0^\pi r(z) f(z,\tau)\, \mathrm{d}z \right| \leqslant \left(\int_0^\pi r(z)^2\, \mathrm{d}z \right)^{1/2} \times \left(\int_0^\pi f(z,\tau)^2\, \mathrm{d}z \right)^{1/2} \tag{40}$$

Let $M_3 = (\int_0^\pi |r(z)|^2 \mathrm{d}z)^{1/2}$. Note that since $r(z) \in L_2(0,\pi)$, M_3 is a positive real number. Identifying the term on the left-hand side of the inequality with $|(r, x_f)|$, and the second term on the right-hand side with $\|x_f\|_2$, we have that

$$|(r, x_f)| \leqslant M_3 \|x_f\|_2 \tag{41}$$

From Equation (33), using the fact that $\sup_{0 \leqslant \tau \leqslant t} |u(t)| \leqslant \bar{u}$, it follows that

$$\|x_f(t)\|_2 \leqslant M_1 \|x_f(0)\|_2 + M_2 \bar{u} \tag{42}$$

where $M_1 = K$ and $M_2 = M_0 \|\mathcal{B}_f\|_2 / \gamma$. Substituting Equation (42) into Equation (41), we have

$$|(r, x_f(t))| \leqslant M_1^* \|x_f(0)\|_2 + M_2^* \bar{u} \tag{43}$$

where $M_1^* = M_1 M_3$ and $M_2^* = M_2 M_3$. This completes the proof of Proposition 2. □

Theorem 1
Consider the system of Equation (11), subject to the input and state constraints of Equations (12)–(13), under the predictive control law of Equations (38)–(39) for which Assumption 1 holds with $x_s(0) \in \Omega_s'$. Then, given that there exists a positive real number δ such that $\chi^{\max} - \chi^{\min} \geqslant 2(M_2^* \bar{u} + \delta)$, where M_2^*, $\bar{u}$ were defined in Proposition 2, there exist positive real numbers α and β such that if $x_s(0) \in \Omega_s'$ and $\|x_f(0)\|_2 \leqslant \beta$, then $x(t) = 0$ is an asymptotically stable solution of the closed-loop system under the controller of Equations (38)–(39) and $\chi^{\min} \leqslant (r, x(t)) \leqslant \chi^{\max}$ for all $t \geqslant 0$.

Proof of Theorem 1
Asymptotic stability of the closed-loop infinite-dimensional system under the control law of Equations (38)–(39) for which Assumption 1 holds with $x_s(0) \in \Omega_s'$ can be shown using arguments similar to the ones in the Proof of Proposition 1. We now consider the problem of constraint satisfaction. First, we assume that there exists a positive real number δ such that

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$\chi^{\max} - \chi^{\min} \geqslant 2(M_2^*\bar{u} + \delta)$, where M_2^*, $\bar{u}$ were defined in Proposition 2. We also note that

$$(r, x_s + x_f) = (r, x_s) + (r, x_f) \tag{44}$$

and pick $\alpha = M_2^*\bar{u} + \delta$ and $\beta = \delta/M_1^*$. By choosing $\|x_f(0)\|_2 \leqslant \beta$, we have from Equation (43) that $-M_1^*\beta - M_2^*\bar{u} \leqslant (r, x_f(t)) \leqslant M_1^*\beta + M_2^*\bar{u}$ (Proposition 2) which, upon substitution of $\beta = \delta/M_1^*$, yields $-\delta - M_2^*\bar{u} \leqslant (r, x_f(t)) \leqslant \delta + M_2^*\bar{u}$ and thus $-\alpha \leqslant (r, x_f(\tau)) \leqslant \alpha$.

Furthermore, satisfaction of the constraint $(r, x_s(\tau)) \leqslant S^{\max} \leqslant \chi^{\max} - \alpha$, together with Equation (44), implies that $(r, x(\tau)) \leqslant \chi^{\max} - \alpha + (r, x_f(t)) \leqslant \chi^{\max}$ since $(r, x_f(\tau)) \leqslant \alpha$. Similarly, satisfaction of the constraint $(r, x_s(\tau)) \geqslant S^{\min} \geqslant \chi^{\min} + \alpha$, together with Equation (44), implies that $(r, x(\tau)) \geqslant \chi^{\min} + \alpha + (r, x_f(\tau)) \geqslant \chi^{\min}$ since $(r, x_f(\tau)) \geqslant -\alpha$. This proves that for the above choices of α and β, the satisfaction of the constraint of $S^{\min} \leqslant (r, x_s) \leqslant S^{\max}$ implies $\chi^{\min} \leqslant (r, x) \leqslant \chi^{\max}$, which is the state constraint for the infinite-dimensional system. Note that the condition $\chi^{\max} - \chi^{\min} \geqslant 2(M_2^*\bar{u} + \delta)$ ensures that for this choice of α, we get $S^{\max} \geqslant S^{\min}$, which is necessary for the optimization problem to be feasible. This completes the Proof of Theorem 1. □

Remark 4

Note that the evolution of the fast subsystem is affected by both the initial condition and the control input, and therefore finding initial conditions starting from where the future evolution of x_f is guaranteed to remain within a certain range requires that the effect of the input on x_f not exceed this range. Towards this end, the assumption regarding the existence of a positive δ that satisfies $\chi^{\max} - \chi^{\min} \geqslant 2(M_2^*\bar{u} + \delta)$ ensures that the control input does not have such a strong influence that causes x_f to violate the given state constraints regardless of the initial condition. In some sense, the existence of δ ensures the needed compatibility between the effects of the input and the state constraints on the evolution of x_f.

Remark 5

Note that the conditions imposed on the initial condition, and the revision of the state constraints in the controller design, are 'worst case' corrections, and are therefore only sufficient conditions. It may happen, for instance, that the controller design of Theorem 1, when implemented in the closed-loop system may achieve infinite-dimensional state constraint satisfaction, even if the fast modes of the infinite-dimensional system do not satisfy the required condition ($\|x_f(0)\|_2 \leqslant \beta$).

Remark 6

Note that the set of initial conditions for which a given MPC formulation is guaranteed to be initially and successively feasible is in general a complex function of the inherent dynamics of the system $(\mathscr{A}, \mathscr{B})$, the constraints on the input and the states $(u^{\min}, u^{\max}, \chi^{\min}, \chi^{\max}, r(\cdot))$ and the controller parameters (T, Q, R). Within the MPC framework, it is in general difficult to come up with an explicit characterization of this set and/or compare, for instance, the sets Ω_s (of Proposition 1) and Ω_s' (of Theorem 1) or the sets we subsequently define. This, however, is not the focus of this work. What Theorem 1, and other formulations that follow, do is only provide sufficient conditions that need to be incorporated in the reduced-order MPC formulation, and once satisfied guarantee state constraints satisfaction and stabilization for the infinite-dimensional closed-loop system. Note also that this resolves a more fundamental issue of just being unable to implement control and enforce constraints using the infinite dimensional model. In practice, this also translates into computational cost savings; not through a reduction in the

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number of decision variables in the optimization problem (whether using the reduced-order model or the 'full' model, the number of decision variables depends on the number of manipulated inputs), but via a reduction in the number of 'states' that are in the model used for prediction, with the cost savings varying on a problem-by-problem basis.

Remark 7

The problem of implementing the predictive control approach in a way that provides an explicit characterization of the closed-loop stability region for finite-dimensional systems has been addressed via embedding the implementation of predictive control algorithms within the stability region of another controller (the hybrid predictive control approach, see Reference [23]) or via the design of predictive control algorithms in a way that they allow for an explicit characterization of their stability region [24]. A similar approach can be utilized to provide an explicit characterization for the closed-loop stability region for finite-dimensional input and state constrained systems. The use of the reduced-order MPC formulations allows the use of these approaches because the set Ω_s only pertains to the set of initial conditions for a finite number of state variables, and can be obtained using the approach in References [23, 24]. Note that the initial conditions for the fast states only need to satisfy a bound, and hence are also characterized by a finite (in this case, one) number of conditions.

3.3. Higher-order MPC formulations

Since the MPC formulation of Theorem 1 accounts only indirectly for the evolution of the fast states (by shrinking the slow state constraints and restricting the initial fast states), the formulation can be conservative in terms of restricting the set of initial conditions for which stabilization and state constraints satisfaction for the infinite-dimensional system are accomplished simultaneously. To alleviate this conservatism, we present in this subsection two MPC formulations which explicitly account for the evolution of the x_f-subsystem.

One way to account for the effect of the fast states on the state constraints of the infinite-dimensional system, is to incorporate the fast states explicitly into the state constraints equation. The control action in this case is computed by solving the following optimization problem:

$$\min_u \left[\int_t^{t+T} [q_s \|x_s(\tau)\|_2^2 + |u(\tau)|_R^2] \, d\tau + F(x_s(t+T)) \right] \tag{45}$$

$$\text{s.t. } \dot{x}_s(\tau) = \mathcal{A}_s x_s(\tau) + \mathcal{B}_s u(\tau)$$

$$\dot{x}_f(\tau) = \mathcal{A}_f x_f(\tau) + \mathcal{B}_f u(\tau)$$

$$u(\tau) \in \mathcal{U}$$

$$\chi^{\min} \leqslant (r, x_s(\tau) + x_f(\tau)) \leqslant \chi^{\max}$$

$$x_s(t+T) = 0, \quad \tau \in [t, t+T] \tag{46}$$

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Note that given any initial condition, for which the above formulation is initially and successively feasible, stabilization and state constraints satisfaction for the infinite-dimensional system are achieved. Stabilization of the infinite-dimensional closed-loop system under the formulation of Equations (45)–(46) can be proved using an argument similar to the one in the Proof of Proposition 1. The implementation of the above controller, however, requires computation of the fast mode dynamics, which can only be done approximately in practice. The key feature of this formulation is that it underscores the fact that even when using a sufficiently high number of modes to simulate the dynamics of the fast modes, the fast modes need not be part of the cost function, thereby keeping the computational requirement low.

Remark 8
A drawback of incorporating the fast states directly in the state constraints equation is that the set of initial conditions for which the optimization problem is feasible becomes infinite-dimensional, and therefore impossible to compute or even estimate. The realization that stability of the slow subsystem is sufficient to ensure stability of the infinite-dimensional system justifies the use of only the slow modes in the cost functional and the stability constraint, thereby substantially reducing the computational requirement. In practice, the evolution of fast modes can be accounted for in the state constraint equation by including a sufficiently high, but finite, number of fast modes.

To reduce some of the computational load associated with solving the x_f-subsystem in the formulation of Equations (45)–(46), we now present another MPC formulation that approximates the effect of the fast dynamics by exploiting the two time-scale separation between the slow and fast subsystems and deriving an approximate model that describes the evolution of the fast subsystem. We define $\varepsilon := |\mathrm{Re}\{\lambda_1\}|/|\mathrm{Re}\{\lambda_{m+1}\}|$ and multiply the x_f-subsystem of Equation (23) by ε to obtain the following system [5]:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \mathscr{A}_s x_s + \mathscr{B}_s u$$

$$\varepsilon \frac{\mathrm{d}x_f}{\mathrm{d}t} = \mathscr{A}_{f\varepsilon} x_f + \varepsilon \mathscr{B}_f u \tag{47}$$

where $\mathscr{A}_{f\varepsilon}$ is an unbounded differential operator defined as $\mathscr{A}_{f\varepsilon} = \varepsilon \mathscr{A}_f$. Introducing the fast time scale $\bar{\tau} = t/\varepsilon$, the system of Equation (47) takes the form

$$\frac{\mathrm{d}x_s}{\mathrm{d}\bar{\tau}} = \varepsilon(\mathscr{A}_s x_s + \mathscr{B}_s u)$$

$$\frac{\mathrm{d}x_f}{\mathrm{d}\bar{\tau}} = \mathscr{A}_{f\varepsilon} x_f + \varepsilon \mathscr{B}_f u \tag{48}$$

Setting $\varepsilon = 0$, we get

$$\frac{\mathrm{d}\bar{x}_s}{\mathrm{d}\bar{\tau}} = 0$$

$$\frac{\mathrm{d}\bar{x}_f}{\mathrm{d}\bar{\tau}} = \mathscr{A}_{f\varepsilon} \bar{x}_f \tag{49}$$

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From the properties of $\mathscr{A}_{f\varepsilon}$, we have that the solution of the x_f-subsystem of Equation (23) can be approximated by $\bar{x}_f(t) = e^{\mathscr{A}_f t}\bar{x}_f(0)$ (note that from the properties of $\mathscr{A}_f$, it follows that $\mathscr{T}_f(t) = e^{\mathscr{A}_f t}$, [25], p. 153). Based on this two time-scale analysis, we consider the following MPC formulation:

$$\min_u \left[\int_t^{t+T} [q_s\|x_s(\tau)\|_2^2 + |u(\tau)|_R^2]\,d\tau + F(x_s(t+T))\right] \tag{50}$$

$$\text{s.t. } \dot{x}_s(\tau) = \mathscr{A}_s x_s(\tau) + \mathscr{B}_s u(\tau)$$

$$u(\tau) \in \mathscr{U}$$

$$\bar{S}^{\min} \leqslant (r, x_s(\tau) + e^{\mathscr{A}_f(\tau - t)}x_f(t)) \leqslant \bar{S}^{\max} \tag{51}$$

where $\tau \in [t, t \in T]$, and $\bar{S}^{\min} = \chi^{\min} + M_2^*\bar{u}$ and $\bar{S}^{\max} = \chi^{\max} - M_2^*\bar{u}$, where M_2^*, $\bar{u}$ were defined in Proposition 2. We denote the set of initial conditions for which the predictive controller of Equations (50)–(51) achieves stabilization of the $x(t) = 0$ solution of the closed-loop infinite-dimensional systems by Ω'. Note that, unlike Theorem 1, the set of initial conditions now includes the slow as well as the fast states because the fast states appear explicitly—though 'approximately'—in the constraints in the optimization problem. State constraints satisfaction for the infinite-dimensional system is achieved by revising the state constraints in the controller formulation by the worst-case error (due to neglecting the effect of the input on the evolution of the fast modes) in the prediction of the fast state dynamics. We formalize this idea in the following theorem.

Theorem 2

Consider the system of Equation (11), the input and state constraints of Equations (12)–(13), under the control law of Equations (50)–(51). Then, if $x(0) \in \Omega'$, the $x(t) = 0$ is an asymptotically stable solution of the closed-loop system of Equation (11) and Equations (50)–(51), and $\chi^{\min} \leqslant (r, x(t)) \leqslant \chi^{\max}$ for all $t \geqslant 0$.

Proof of Theorem 2

The fact that the control law of Equations (50)–(51) achieves stabilization of the closed-loop infinite-dimensional system can be proved using an argument similar to the one in the Proof of Proposition 1. We focus on constraint satisfaction.

Satisfaction of the constraint $(r, x_s + e^{\mathscr{A}_f(\tau - t)}x_f(t)) \leqslant \bar{S}^{\max} = \chi^{\max} - M_2^*\bar{u}$ implies that $(r, x_s(\tau)) + (r, e^{\mathscr{A}_f(\tau - t)}x_f(t)) + M_2^*\bar{u} \leqslant \chi^{\max}$. Note that $(r, x_s(\tau)) + (r, x_f(\tau)) \leqslant (r, x_s(\tau)) + (r, e^{\mathscr{A}_f(\tau - t)} x_f(t)) + M_2^*\bar{u}$ (see Proof of Proposition 2). Hence, $(r, x_s(\tau)) + (r, e^{\mathscr{A}_f(\tau - t)}x_f(t)) + M_2^*\bar{u} \leqslant \chi^{\max}$ implies that $(r, x(\tau)) \leqslant \chi^{\max}$. Similarly, satisfaction of the constraint $(r, x_s(\tau) + e^{\mathscr{A}_f(\tau - t)} x_f(t)) \geqslant \bar{S}^{\min} = \chi^{\min} + M_2^*\bar{u}$ implies that $(r, x_s(\tau)) + (r, e^{\mathscr{A}_f(\tau - t)}x_f(t)) - M_2^*\bar{u} \geqslant \chi^{\min}$. Note, once again, that $(r, x_f(\tau)) \geqslant (r, e^{\mathscr{A}_f(\tau - f)}x_f(t)) - M_2^*\bar{u}$ (see Proof of Proposition 2). Therefore, $(r, x_s(\tau)) + (r, e^{\mathscr{A}_f(\tau - t)}x_f(t)) - M_2^*\bar{u} \geqslant \chi^{\min}$ implies that $(r, x(\tau)) \geqslant \chi^{\min}$. This completes the Proof of Theorem 2. □

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4. SIMULATION EXAMPLE

In this section, we demonstrate and compare, through computer simulations, the implementation of the various MPC formulations discussed in the previous section. To this end, we consider the parabolic PDE of Equation (1) with $b=1$, $c=1.66$, $w=2$ and two control actuators ($m=2$) with the following distribution functions $b_i(z)=1/2\mu$ for $z\in[z_{ai}-\mu, z_{ai}+\mu]$ and $b_i(z)=0$ elsewhere in $[0,\pi]$, where $\mu=0.005$ is a small positive real number and $z_{a1}=\pi/3$ and $z_{a2}=2\pi/3$. For these values, it was verified that the operating steady-state, $\bar{x}(z,t)=0$, is an unstable one. The control objective is to stabilize the state profile at the unstable zero steady-state by manipulating $u_i(t)$ subject to the input and state constraints of Equations (12)–(13) with $u_i^{\min}=-3$, $u^{\max}=3$, for $i=1,2$, $\chi^{\min}=-3.0$, $\chi^{\max}=3.0$. $r(z)$ is the state constraint distribution function, chosen to be $r(z)=1/\zeta$ for $z\in[z_c-v, z_c+v]$, with $\zeta=0.0036$, $v=0.0018$ and $z_c=1.156$, and zero elsewhere. The solution of the eigenvalue problem of the spatial differential operator of Equation (14) is

$$\lambda_j = 1.66 - j^2, \quad \phi_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz), \quad j=1,\ldots,\infty \tag{52}$$

For this system, we consider the first two eigenmodes to be the dominant ones. To simplify the presentation of the results, eigenmodes of the PDE of Equation (11) are considered as state variables. Specifically, using standard modal decomposition, we derive the following high-order ODE system that describes the temporal evolution of the first l eigenmodes:

$$\begin{aligned}\dot{a}_s(t) &= A_s a_s(t) + B_s u(t)\\ \dot{a}_f(t) &= A_f a_f(t) + B_f u(t)\end{aligned} \tag{53}$$

where $a_s(t)=[a_1(t)\ a_2(t)]'$, $a_f(t)=[a_3(t)\ a_4(t)\ \cdots\ a_l(t)]'$, $a_i(t)\in\mathbb{R}$ is the modal amplitude of the ith eigenmode, the notation a_s' denotes the transpose of a_s, l is chosen to be 50, $u(t)=[u_1\ (t)\ u_2(t)]'$, the matrices A_s and A_f are diagonal matrices, given by $A_s=\text{diag}\{\lambda_i\}$, for $i=1,2$ and $A_f=\text{diag}\{\lambda_i\}$, for $i=3,\ldots,l$. B_s and B_f are a 2×2 and $(l-2)\times m$ matrices, respectively, whose (i,j)th element is numerically calculated by taking inner product of $b_j(z)$ and $\phi_i(z)$, that is $B_{ij}=(b_j(z),\phi_i(z))$. Note that $\bar{x}(z,t)=\sum_{i=1}^{l} a_i(t)\phi_i(z)$, $x_s(t)=a_1(t)\phi_1+a_2(t)\phi_2$, $x_f(t)=\sum_{i=3}^{50} a_i(t)\phi_i$ and that $(x(t),\phi_i)=a_i(\phi_i,\phi_i)$. Using these projections, the state constraints of Equation (13) can be expressed as constraints on the modal states as follows:

$$\chi^{\min} \leqslant \int_0^{\pi} r(z)\left[\sum_{i=1}^{2} a_i(t)\phi_i(z) + \sum_{i=3}^{l} a_i(t)\phi_i(z)\right]\mathrm{d}z \leqslant \chi^{\max} \tag{54}$$

We now proceed with the design and implementation of the different predictive control formulations presented in the previous section. In the first scenario, we use the a_s-subsystem in Equation (53) as the basis for the predictive controller design (the a_f-subsystem is neglected). For this case, we consider an MPC formulation of the form of Equations (27)–(28) with the following objective function and constraints:

$$\min_u \left[\int_t^{t+T} [q_s|a_s(\tau)|^2 + |u(\tau)|_R^2]\,\mathrm{d}\tau\right] \tag{55}$$

$$\text{s.t. } \dot{a}_s(\tau) = A_s a_s(\tau) + B_s u(\tau)$$

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$$u_{\min} \leqslant u_i(\tau) \leqslant u_{\max}, \quad i = 1, 2 \tag{56}$$

$$\chi^{\min} \leqslant \int_0^{\pi} r(z) \left[\sum_{i=1}^{2} a_i(\tau)\phi_i(z) \right] \mathrm{d}z \leqslant \chi^{\max}, \quad \tau \in [t, t+T] \tag{57}$$

where $q_s = 8.79$, $R = rI$, with $r = 0.01$, and $T = 0.007$. To ensure stability, we also impose a terminal equality constraint of the form $a_s(t+T) = 0$ on the optimization problem. The resulting quadratic program is solved using the MATLAB subroutine QuadProg. The control action is then implemented on the 50th order model of Equation (53). Figure 1 shows the closed-loop state under MPC law of Equations (55)–(57) stabilizes the PDE state at the unstable zero steady-state starting from the initial condition $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 3.15\sin(3z) + 3.15\sin(4z)$. By examining Figure 2 (solid line), we observe that the integral constraint $R(z_c, t) = \int_0^{\pi} r(z)\bar{x}(z,t)\,\mathrm{d}z$ violates the lower constraint for some time. The violation of the state constraint is a consequence of neglecting the contribution of the a_f states to the state of the PDE in the MPC formulation.

We now revise the constraints in the previous MPC formulation, and consider the following objective function and constraints (analysed in Theorem 1):

$$\min_u \left[\int_t^{t+T} [q_s|a_s(\tau)|^2 + |u(\tau)|_R^2]\,\mathrm{d}\tau \right] \tag{58}$$

$$\text{s.t. } \dot{a}_s(\tau) = A_s a_s(\tau) + B_s u(\tau)$$

$$u_{\min} \leqslant u_i(\tau) \leqslant u_{\max}, \quad i = 1, 2 \tag{59}$$

$$S^{\min} \leqslant \int_0^{\pi} r(z) \left[\sum_{i=1}^{2} a_i(\tau)\phi_i(z) \right] \mathrm{d}z \leqslant S^{\max}, \quad \tau \in [t, t+T] \tag{60}$$

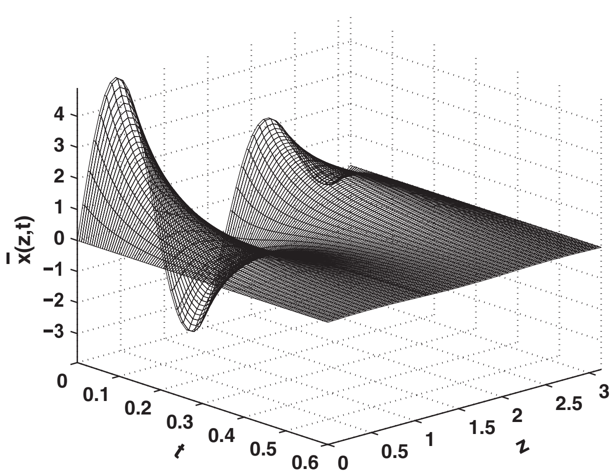

Figure 1. Closed-loop state profile under the MPC formulation of Equations (55)–(57) with $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 3.15\sin(3z) + 3.15\sin(4z)$.

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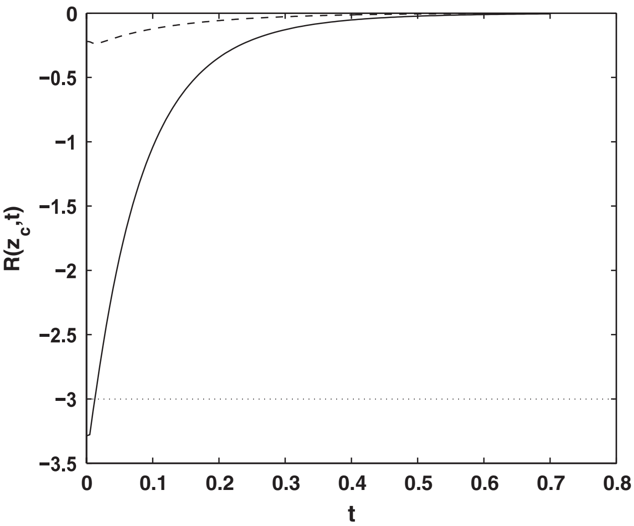

Figure 2. $R(z_c, t) = \int_0^{\pi} r(z)\bar{x}(z, t)\,\mathrm{d}z$ under the MPC formulation of Equations (55)–(57) (solid line) and under the MPC formulation of Equations (58)–(60) (dashed line). The dotted line represents the lower state constraint.

where q_s, R, r and T have the same values used in the first scenario, and $S^{\max} = 3 - \alpha$ and $S^{\min} = -3 + \alpha$. Following the result of Theorem 1, we have verified that $\delta = 1$ satisfies $\chi^{\max} - \chi^{\min} \geqslant 2(M_2^* \bar{u} + \delta)$ where $\bar{u} = 3$ and $M_2^* = M_2 M_3 = M_3 M_0 \|B_f\|_2/\gamma = 0.477$, and pick $\alpha = M_2^* \bar{u} + \delta = 2.43$ and $\beta = \delta/M_1^* = 1.0/1.003 = 0.99$.

Picking the initial condition $\bar{x}(z, 0) = 0.02\sin(z) + 0.01\sin(2z) + 0.95\sin(3z)$ which satisfies $\|x_f(0)\|_2 \leqslant 0.99$, the implementation of the predictive controller of Equations (58)–(60) results in the stabilization and satisfaction of the state constraint of the closed-loop system (Figure 3 and dashed lines in Figures 2 and 4). Note that, since the control action is computed on the basis of the slow states—and since the initial conditions for the slow states are the same as in the previous scenario—the controller implements the same control action as before (i.e. the solid and the dashed lines coincide in Figure 4). The closed-loop state profile, however, stays within the constraints because of appropriate initialization of the fast modes and modification of the state constraint in the predictive controller of Equations (58)–(60).

We now consider higher-order MPC formulations. First, we demonstrate the implementation of the MPC formulation of Equations (45)–(46) where the PDE state constraints are exactly accounted for in the controller design. In this case, the objective function and constraints are given by

$$\min_{u} \left[\int_t^{t+T} [q_s |a_s(\tau)|^2 + |u(\tau)|_R^2]\,\mathrm{d}\tau \right] \tag{61}$$

$$\text{s.t. } \dot{a}_s(\tau) = A_s a_s(\tau) + B_s u(\tau)$$

$$\dot{a}_f(\tau) = A_f a_f(\tau) + B_f u(\tau)$$

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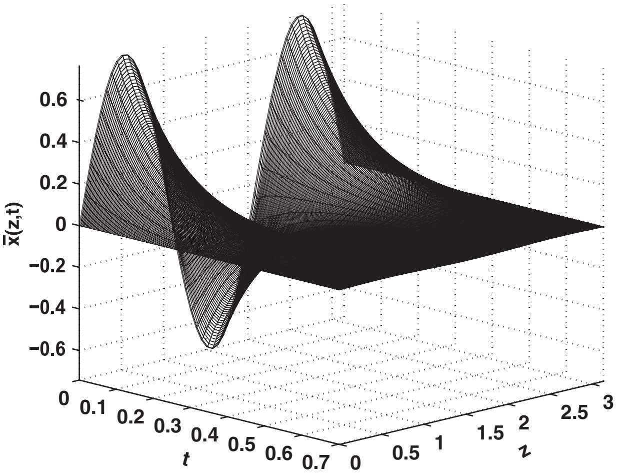

Figure 3. Closed-loop state profile under the MPC formulation of Equations (58)–(60) with $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 0.95\sin(3z)$.

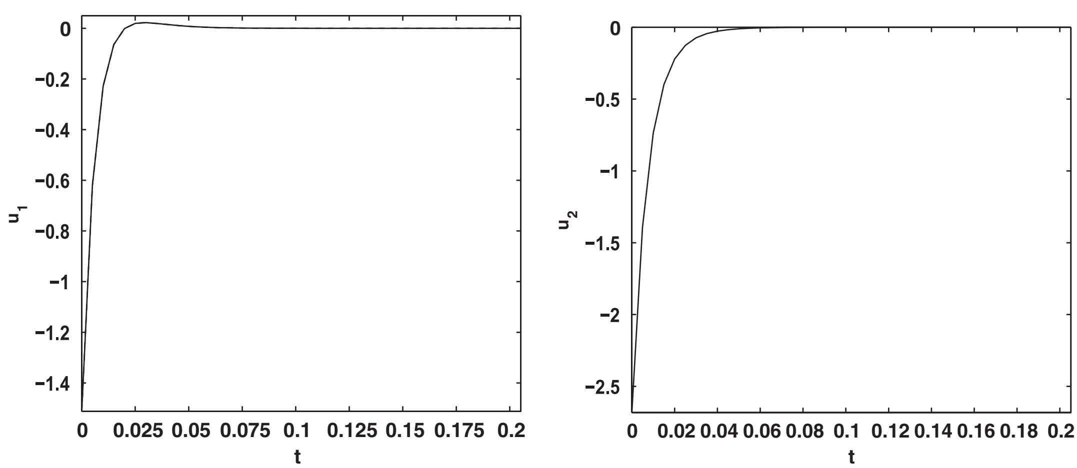

Figure 4. Manipulated input profiles for the first and second control actuators applied at $z_{a1} = \pi/3$ and $z_{a2} = 2\pi/3$ under the MPC formulation of Equations (55)–(57) (solid line) and under the MPC formulation of Equations (58)–(60) (dashed line); note that the dashed and solid line coincide because of the same initial conditions of the a_s-states.

$$u_{\min} \leqslant u_i(\tau) \leqslant u_{\max}, \quad i = 1, 2 \tag{62}$$

$$\chi^{\min} \leqslant \int_0^{\pi} r(z) \left[\sum_{i=1}^{2} a_i(\tau)\phi_i(z) + \sum_{i=3}^{l} a_i(\tau)\phi_i(z) \right] \mathrm{d}z \leqslant \chi^{\max} \tag{63}$$

where $\tau \in [t, t+T]$ and the MPC tuning parameters have the same values used in the previous two cases. The results are shown in Figures 5 and 7 (dashed lines), where we see that starting

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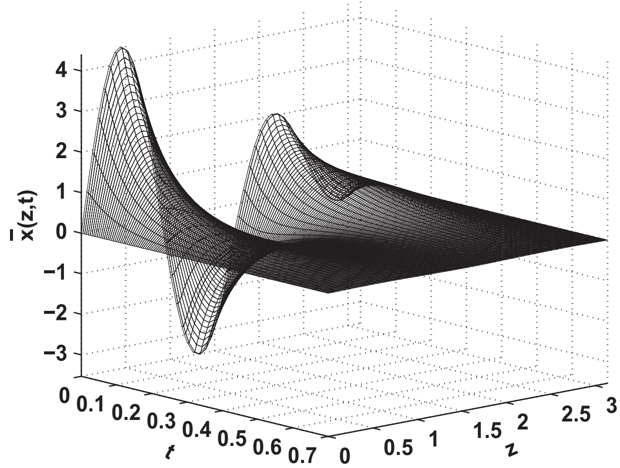

Figure 5. Closed-loop state profile under the MPC formulation of Equations (61)–(63) with $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 2.8\sin(3z) + 2.85\sin(4z)$.

from the initial condition $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 2.8\sin(3z) + 2.85\sin(4z)$ the predictive controller of Equations (61)–(63) successfully stabilizes the system at the zero steady-state and the PDE state constraint is satisfied for all times. The corresponding manipulated input profiles are given in Figure 8. Finally, we demonstrate the implementation of the MPC formulation of Equations (50)–(51) (analysed in Theorem 2). Using the approximation of Equation (49) in the formulation of Equations (61)–(63) yields the following objective function and constraints:

$$\min_u \left[\int_t^{t+T} [q_s |a_s(\tau)|^2 + |u(\tau)|_R^2] \, \mathrm{d}\tau \right] \tag{64}$$

$$\begin{aligned} \text{s.t. } & \dot{a}_s(\tau) = A_s a_s(\tau) + B_s u(\tau) \\ & u_{\min} \leqslant u_i(\tau) \leqslant u_{\max}, \quad i = 1, 2 \\ & \bar{S}^{\min} \leqslant \int_0^{\pi} r(z) \left[\sum_{i=1}^{2} a_i(\tau)\phi_i(z) + \sum_{i=3}^{l} \mathrm{e}^{\lambda_i(\tau - t)} a_i(t)\phi_i(z) \right] \mathrm{d}z \leqslant \bar{S}^{\max} \end{aligned} \tag{65}$$

where $\tau \in [t, t+T]$. $\bar{S}^{\max}$ and $\bar{S}^{\min}$ are calculated by using the value $M_2^* \bar{u} = 1.431$, as follows $\bar{S}^{\max} = \chi^{\max} - M_2^* \bar{u} = 1.569$ and $\bar{S}^{\min} = \chi^{\min} + M_2^* \bar{u} = -1.569$. The above formulation does not require solving the state evolution equation for the a_f-subsystem at each time step; instead it uses an explicit (approximate) expression, $a_f(\tau) = \mathrm{e}^{A_f(\tau - t)} a_f(t)$, to account for the dynamics of the fast subsystem which contribute to the PDE state constraints. The initial condition in this scenario is chosen as $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 1.45\sin(3z) + 1.5\sin(4z)$; note that for the initial condition $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 2.8\sin(3z) + 2.85\sin(4z)$, the formulation of Equations (64)–(65) is not feasible. The receding horizon implementation of the predictive controller implies that for subsequent computations, this expression is used with the

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updated value of $a_f(t)$. Note, however, that this does not imply that the optimization problem needs to solve the fast mode dynamics; it only means that at the new initial condition, the optimization problem in the predictive controller is solved starting from this updated system state, in line with the standard receding horizon implementation of predictive controllers. The resulting predictive controller, when implemented on the system of Equation (1) successfully stabilizes the zero steady-state and enforces PDE state constraints satisfaction (see Figure 6 and solid lines Figures 7 and 8).

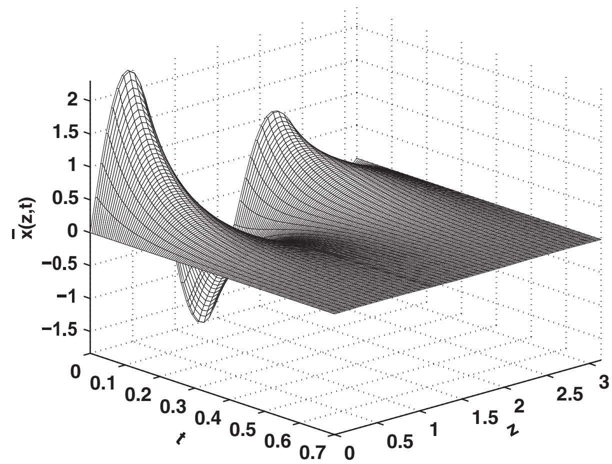

Figure 6. Closed-loop state profile under the MPC formulation of Equations (64)–(65) with $\bar{x}(z,0) = 0.02\sin(z) + 0.01\sin(2z) + 1.45\sin(3z) + 1.5\sin(4z)$.

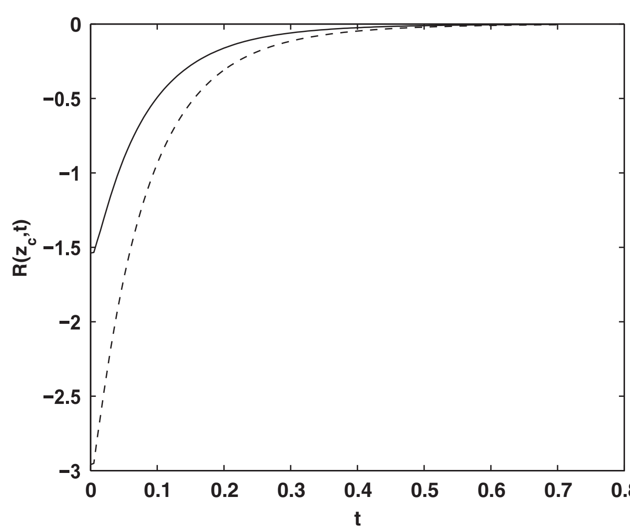

Figure 7. $R(z_c,t) = \int_0^{\pi} r(z)\bar{x}(z,t)\,\mathrm{d}z$ under the MPC formulation of Equations (61)–(63), (dashed line) and under the MPC formulation of Equations (64)–(65) (solid line).

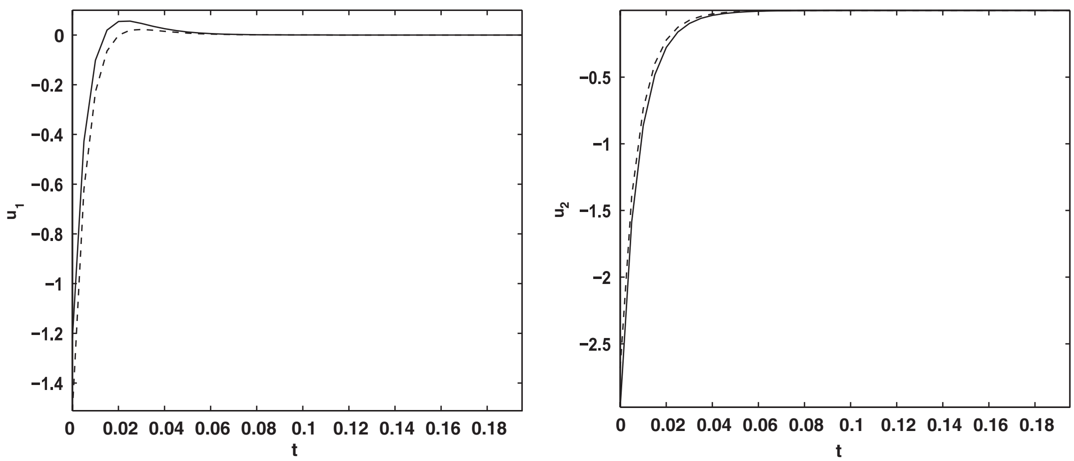

Figure 8. Manipulated input profiles for the first and second control actuators applied at $z_{a1} = \pi/3$ and $z_{a1} = 2\pi/3$ under the MPC formulation of Equations (61)–(63) (dashed line) and under the MPC formulation of Equations (64)–(65) (solid line).

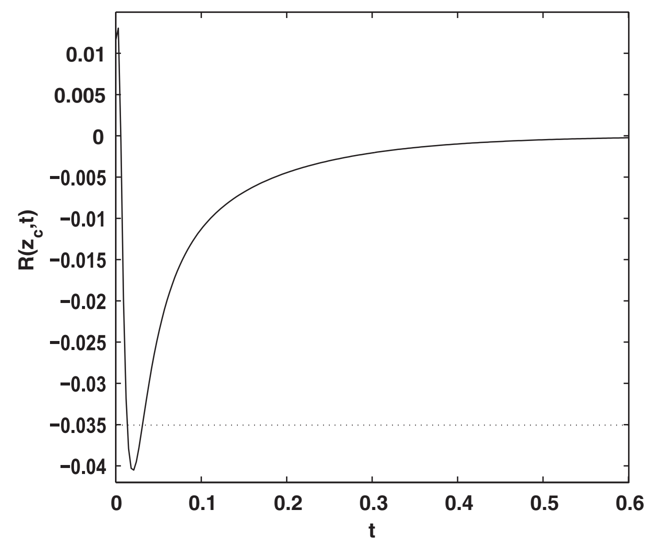

Figure 9. $R(z_c, t) = \int_0^{\pi} r(z)\bar{x}(z, t)\,\mathrm{d}z$ under the MPC formulation of Equations (55)–(57) with $\bar{x}(z, 0) = 0.04\sin(z) + 0.0005\sin(2z) + 0.07\sin(3z)$ and state constraints $-0.035 \leqslant R(z_c, t) \leqslant 2$ with $[u_{\max}, u_{\min}] = [-10, 10]$.

Remark 9

As a final note, we want to demonstrate that even if $\bar{x}(z, 0)$ does not violate the state constraints, these constraints can be violated for some time $t \geqslant 0$. To this end, we pick $\chi^{\min} = -0.035$ and $\chi^{\max} = 2$, and $u_i^{\min} = -10$, $u_i^{\max} = 10$, and use the a_s-subsystem in Equation (53) as the basis for the predictive controller design (the a_f-subsystem is neglected). For this case, we consider the

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predictive controller of Equations (55)–(57) where $q_s = 8.79$, $R = rI$, with $r = 0.01$, and $T = 0.007$. To ensure stability, we impose a terminal equality constraint of the form $a_s(t+T) = 0$ on the optimization problem. The control action is then implemented on the 50th order model of Equation (53). Figure 9 shows state constraint profile starting from the initial condition $\bar{x}(z,0) = 0.04\sin(z) + 0.0005\sin(2z) + 0.07\sin(3z)$, that does not violate state constraints. It is clear that the predictive controller successfully stabilizes the state at the zero steady-state and that the state violates the lower constraint for some time.

5. CONCLUSIONS

In this work we presented and compared a number of MPC formulations for control of linear parabolic PDEs with state and input constraints. Modal decomposition techniques were initially used to derive finite-dimensional systems that capture the dominant dynamics of the PDE, and express the infinite-dimensional state constraints as appropriate constraints on the finite-dimensional system states. The closed-loop stability properties of the infinite-dimensional system under the low order MPC designs were analysed and sufficient conditions, which guarantee stabilization and state constraints satisfaction for the infinite-dimensional system under the reduced order MPC formulations, were derived. We also presented other formulations which differed in the way the evolution of the fast eigenmodes is accounted for in the performance objective and state constraints. The comparison underscored the fact that the fast states, while unimportant in achieving closed-loop stability, are central to the predictive controller's ability to enforce the constraints in the closed-loop state of the infinite-dimensional system. Finally, the MPC formulations were applied, through simulations, to the problem of stabilizing an unstable steady-state of a linear parabolic PDE subject to state and control constraints.

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REFERENCES

1. Curtain RF, Pritchard AJ. In *Infinite Dimensional Linear Systems Theory*. Lecture Notes in Control and Information Sciences, vol. 9, Balakrishna AV, Thoma M (eds). Springer: Berlin, Germany, 1978.
2. Balas MJ. Feedback control of linear diffusion processes. *International Journal of Control* 1979; **29**:523–533.
3. Ray WH. *Advanced Process Control*. McGraw-Hill: New York, 1981.
4. Curtain RF. Finite-dimensional compensator design for parabolic distributed systems with point sensors and boundary input. *IEEE Transactions on Automatic Control* 1982; **27**:98–104.
5. Christofides PD. *Nonlinear and Robust Control of PDE Systems: Methods and Applications to Transport-Reaction Processes*. Birkhäuser: Boston, 2001.
6. Christofides PD, Daoutidis P. Finite-dimensional control of parabolic PDE systems using approximate inertial manifolds. *Journal of Mathematical Analysis and Applications* 1997; **216**:398–420.
7. Baker J, Christofides PD. Finite dimensional approximation and control of nonlinear parabolic PDE systems. *International Journal of Control* 2000; **73**:439–456.
8. Armaou A, Christofides PD. Dynamic optimization of dissipative PDE systems using nonlinear order reduction. *Chemical Engineering Science* 2002; **57**:5083–5114.

DOI: 10.1002/rnc

9. El-Farra NH, Armaou A, Christofides P. Analysis and control of parabolic PDE systems with input constraints. *Automatica* 2003; **39**:715–725.
10. Zheng D, Hoo KA, Piovoso MJ. Finite-dimensional modeling and control of distributed parameter systems. *Proceedings of the American Control Conference*, Anchorage, AK, 2002; 4377–4382.
11. Smyshlyaev A, Krstic M. Explicit formulae for boundary control of parabolic PDEs. *Lecture Notes in Control and Information Sciences*, vol. 301. Springer: New York, 2004; 231–249.
12. Demetriou MA, Kazantzis N. A new actuator activation policy for performance enhancement of controlled diffusion processes. *Automatica* 2004; **40**:415–421.
13. Garcia C, Prett D, Morari M. Model predictive control: theory and practice—a survey. *Automatica* 1989; **25**: 335–348.
14. Allgower F, Chen H. Nonlinear model predictive control schemes with guaranteed stability. In *NATO ASI on Nonlinear Model Based Process Control*, Berber R, Kravaris C (eds). Kluwer: Dordrecht, 1998; 465–494.
15. Rawlings JB. Tutorial overview of model predictive control. *IEEE Control Systems Magazine* 2000; **20**:38–52.
16. Mayne DQ, Rawlings JB, Rao CV, Scokaert POM. Constrained model predictive control: stability and optimality. *Automatica* 2000; **36**:789–814.
17. Ito K, Kunisch K. Receding horizon optimal control for infinite dimensional systems. *ESIAM*: *Control, Optimization and Calculus of Variations* 2002; **8**:741–760.
18. Dufour P, Touré Y, Blanc D, Laurent P. On nonlinear distributed parameter model predictive control strategy: on-line calculation time reduction and application to an experimental drying process. *Computers and Chemical Engineering* 2003; **27**:1533–1542.
19. Dubljevic S, Christofides PD. Predictive control of parabolic PDEs with boundary control actuation. *Chemical Engineering Science* 2006; **61**:6239–6248.
20. Dubljevic S, Christofides PD. Predictive output feedback control of parabolic PDEs. *Industrial and Engineering Chemistry Research*, in press.
21. Dubljevic S, Mhaskar P, El-Farra NH, Christofides PD. Predictive control of transport-reaction processes. *Computers and Chemical Engineering* 2005; **29**:2335–2345.
22. Riesz F, Sz-Nagy B. *Functional Analysis*. Dover Publications: New York, U.S.A., 1990.
23. El-Farra NH, Mhaskar P, Christofides PD. Uniting bounded control and MPC for stabilization of constrained linear systems. *Automatica* 2004; **40**:101–110.
24. Mhaskar P, El-Farra NH, Christofides PD. Predictive control of switched nonlinear systems with scheduled mode transitions. *IEEE Transactions on Automatic Control* 2005; **50**:1670–1680.
25. Curtain RF, Pritchard AJ. *Functional Analysis in Modern Applied Mathematics*, Mathematics in Science and Engineering, vol. 132, Bellman R (ed.). Academic Press: New York, NY, 1977.

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