Predictive Output Feedback Control of Parabolic Partial Differential Equations (PDEs)

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This work focuses on predictive output feedback control of linear parabolic partial differential equation (PDE) systems with state and control constraints. Under the assumption that a finite, yet sufficiently large, number of output measurements is available, two predictive output feedback controllers are constructed and sufficient conditions are derived under which these controllers enforce stability and constraint satisfaction in the infinite-dimensional closed-loop system, respectively. The controllers are applied to an example of a linear parabolic PDE with Dirichlet boundary conditions subject to state and control constraints, and the numerical simulations demonstrate their ability to enforce closed-loop system stability and constraint satisfaction.

1. Introduction

Transport−reaction processes (e.g., tubular reactors, chemical vapor deposition processes) exhibit significant spatial variations, because of the underlying diffusive and convective phenomena, and pose significant challenges from a control point of view. Specifically, the distinguishing feature of control problems arising in the context of transport−reaction processes is that they involve the regulation of distributed process variables, using spatially distributed control actuators and measurement sensors, and, thus, such control problems cannot be addressed on the basis of ordinary differential equation (ODE) models derived under the assumption of lumped process behavior. First-principle modeling of transport−reaction processes typically leads to systems of linear/nonlinear parabolic partial differential equations (PDEs) when the diffusive mode of transport is dominant (or as significant as convection).

From a mathematical point of view, the main feature of parabolic PDEs is that the spectrum of the spatial differential operator can be partitioned into a finite set of eigenvalues that are close to the imaginary axis ("slow" eigenvalues) and an infinite-dimensional complement which includes eigenvalues that are far left in the complex plane ("fast" eigenvalues).[1] This implies that the dominant dynamic behavior of parabolic PDEs can be approximately described by finite-dimensional systems. Therefore, the standard approach to control of parabolic PDEs utilizes eigenfunction expansion techniques to obtain an ODE system, which is then used for controller design (see, e.g., refs 2−4). The key disadvantage of this approach, especially for nonlinear parabolic PDEs, is that the number of modes that should be retained, to derive an ODE system that yields the desired degree of approximation, may be very large, leading to complex controller design and high dimensionality of the resulting controllers. These controller synthesis and implementation problems have motivated extensive research efforts on the problem of deriving low-order ODE systems that accurately reproduce the dynamics and solutions of nonlinear parabolic PDEs. The concept of inertial manifold (IM) has provided a natural framework for addressing this problem.[5] Unfortunately, even for PDE systems for which an IM is known to exist, the computation of the closed-form expression of the IM (and, therefore, the computation of accurate low-order ODE systems) is a formidable task. Motivated by this, a novel procedure based on singular perturbations was proposed in ref 6 for the construction of approximate inertial manifolds (AIMs), which are used to derive low-dimensional ODE systems that accurately reproduce the solutions of the parabolic PDE system (also see refs 7 and 8 for other approaches for the construction of AIMs). These ODE systems were used as the basis for the synthesis of nonlinear low-dimensional output feedback controllers that guarantee stability and enforce output tracking in the closed-loop system. More recently, control algorithms for diffusion−convection−reaction processes described by nonlinear parabolic PDEs that compensate for the effect of uncertain variables on process output were also developed,[9−11] as well as steady-state[12] and dynamic[13] optimization algorithms. The developed control methods were successfully applied to a rapid thermal chemical vapor deposition process[14] to achieve a spatially uniform deposition of a thin film and a Czochralski crystal growth process[15] to regulate crystal internal thermal gradients and were shown to outperform conventional control schemes. In addition to the aforementioned methods, other significant results on analysis and control of nonlinear PDE systems have been recently derived, including order reduction and control using wavelets as basis functions in Galerkin's method,[16] distributed control using generalized invariants[17,18] and concepts from passivity and thermodynamics,[9,19] techniques for monitoring and identification,[20,21] and techniques for optimal placement of actuators and sensors, and switching policies among actuators within an optimal control setting.[22−28]

Although the aforementioned research efforts have led to the development of several systematic approaches for controller design for broad classes of linear/nonlinear parabolic PDEs, they do not explicitly account for the presence of constraints in the manipulated inputs and states. Model predictive control (MPC), which is also known as receding horizon control, is a popular control method for handling constraints (both on manipulated inputs and state variables) within an optimal control setting. In MPC, the control action is obtained by solving repeatedly, on-line, a finite-horizon constrained open-loop optimal control problem (see refs 29−32 for surveys of results and references in this area). However, most of the research in the area of MPC has focused on lumped-parameter processes modeled by ODE systems. Compared with lumped-parameter systems, the problem of designing predictive controllers for distributed parameter

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systems has received much less attention (see refs 33 and 34 for recent results on MPC of first-order hyperbolic PDE systems and ref 35 for state feedback MPC of a diffusion reaction process on the basis of finite-dimensional approximations derived by the finite difference method). In a previous work,[36] we considered linear parabolic PDE systems and derived predictive controller formulations that systematically handle the objectives of state and input constraints satisfaction and stabilization of the infinite dimensional system; subsequently, we extended these results to quasi-linear parabolic PDEs.[37] However, in these works, we assumed that on-line measurements of the entire state of the PDE system are available and considered the full-state feedback predictive control problem.

To overcome this limitation, this work focuses on linear parabolic PDEs with state and control constraints and addresses the problem of designing predictive output feedback controllers that enforce stability and constraint satisfaction in the infinite-dimensional closed-loop system. The manuscript is structured as follows. First, the parabolic PDE is formulated as an abstract evolution equation in an appropriate Hilbert space. Modal decomposition techniques then are used to develop finite-dimensional systems that capture the dominant dynamics of the infinite-dimensional system. Subsequently, under the assumption that a finite, yet sufficiently large, number of output measurements is available, two predictive output feedback controllers are constructed and sufficient conditions are derived under which these controllers enforce stability and constraint satisfaction in the infinite-dimensional closed-loop system, respectively. The controllers are applied to an example of a linear parabolic PDE with Dirichlet boundary conditions subject to state and control constraints and the numerical simulations demonstrate their ability to enforce closed-loop system stability and constraint satisfaction.

2. Preliminaries

2.1. Parabolic Partial Differential Equations. This work focuses on predictive output feedback control of highly dissipative linear infinite-dimensional systems. To provide a PDE example that belongs in this class of infinite-dimensional systems, we begin by focusing on a linear parabolic PDE with distributed control of the form

$$\frac{\partial \bar{x}(z,t)}{\partial t} = \bar{b}\frac{\partial^2 \bar{x}(z,t)}{\partial z^2} + \bar{c}\bar{x}(z,t) + w\sum_{i=1}^{m} b_i(z)u_i(t) \tag{1a}$$

$$y_j(t) = \int_0^{\pi} s_j(z)\bar{x}(z, t)\,\mathrm{d}z \qquad (j = 1, ..., p) \tag{1b}$$

with the following boundary and initial conditions:

$$\bar{x}(0,t) = 0,\ \bar{x}(\pi,t) = 0,\ \bar{x}(z,0) = \bar{x}_0(z) \tag{2}$$

subject to the following input and state constraints:

$$u_i^{\min} \le u_i(t) \le u_i^{\max} \qquad (i = 1, ..., m) \tag{3}$$

$$\chi^{\min} \le \int_0^{\pi} r(z)\bar{x}(z,t)\,\mathrm{d}z \le \chi^{\max} \tag{4}$$

where $\bar{x}(z,t)$ denotes the state variable, $z \in [0,\pi]$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $y_j(t) \in \mathbb{R}$ is the jth measured output, $u_i(t) \in \mathbb{R}$ denotes the ith constrained manipulated input; $u_i^{\min}$ and $u_i^{\max}$ are real numbers representing the lower and upper bounds on the ith input, respectively, and $\chi^{\min}$ and $\chi^{\max}$ are real numbers representing the lower and upper state constraints, respectively. The term $\partial^2\bar{x}/\partial z^2$ denotes the second-order spatial derivative of $\bar{x}(z,t)$; $\bar{b}$, $\bar{c}$, and w are constant coefficients (with $\bar{b} > 0$), and $\bar{x}_0(z)$ is a sufficiently smooth function of z. The function $b_i(z) \in L_2(0,\pi)$ is a known square integrable function of z that describes how the control action, $u_i(t)$, is distributed in the spatial interval $[0,\pi]$; $L_2(0,\pi)$ denotes the space of square-integrable functions defined on the interval $[0,\pi]$. The function $s_j(z) \in L_2(0,\pi)$ is also a known square integrable function of z that captures the jth measurement sensor specifications. In eq 4, the function $r(z) \in L_2(0,\pi)$ is a "state constraint distribution" function that is square-integrable and describes how the state constraint is enforced in the spatial domain $[0,\pi]$. Whenever the control action is applied to the spatial domain at a single point z_{ai}, with $z_{ai} \in [0,\pi]$ (i.e., point actuation), the function $b_i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_{ai} - \mu, z_{ai} + \mu]$, where μ is a small positive real number, and zero elsewhere in $[0,\pi]$; a similar approach can be used to handle point sensing and point constraints. Throughout the paper, the notation $|\cdot|$ will be used to denote the standard Euclidian norm in $\mathbb{R}^n$, whereas the notation $|\cdot|_Q$ will be used to denote the weighted norm defined by $|x|_Q^2 = x'Qx$, where Q is a positive-definite matrix and x' denotes the transpose of x.

To proceed with the presentation of our results, we formulate the PDE of eqs 1, 2, 3, and 4 as an infinite-dimensional system in the state space $\mathscr{H} = L_2(0,\pi)$, with an inner product of

$$(\omega_1,\omega_2) = \int_0^{\pi} \omega_1(z)\omega_2(z)\,\mathrm{d}z \tag{5a}$$

and norm of

$$||\omega_1||_2 = (\omega_1,\omega_1)^{1/2} \tag{5b}$$

where ω_1 and ω_2 are any two elements of $L_2(0,\pi)$.

The state function $x(t)$ on the state-space $\mathscr{H} = L_2(0,\pi)$ is defined as

$$x(t) = \bar{x}(z,t) \qquad (\text{for } t > 0,\ 0 < z < \pi) \tag{6}$$

and the operator $\mathscr{A}$ is defined as

$$\mathscr{A}\phi = \bar{b}\left(\frac{d^2\phi}{dz^2}\right) + \bar{c}\phi \qquad (\text{for } 0 < z < \pi) \tag{7}$$

where $\phi(z)$ is a smooth function on $[0,\pi]$ with $\phi(0) = 0$ and $\phi(\pi) = 0$, with the following dense domain:

$$\mathscr{D}(\mathscr{A}) = \left\{\phi(z) \in L_2(0,\pi)\text{: } \phi(z) \text{ and } \frac{\mathrm{d}\phi(z)}{\mathrm{d}z} \text{ are absolutely continuous; } \mathscr{A}\phi \in L_2(0,\pi),\ \phi(0) = 0, \text{ and } \phi(\pi) = 0\right\} \tag{8}$$

The input operator is defined as

$$\mathscr{B}u(t) = w\sum_{i=1}^{m} b_i(\cdot)u_i(t) \tag{9}$$

the measured output operator is defined as

$$\mathscr{S}x(t) = [(s_1,x(t))\ (s_2,x(t)), ..., (s_j,x(t))] \tag{10}$$

and the state constraint is defined as

$$\chi^{\min} \le (r,x(t)) \le \chi^{\max} \tag{11}$$

Using the aforementioned definitions, the system of eqs 1, 2, 3, and 4 can be written as follows:

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)\ , x(0) = x_0 \tag{12}$$

$$y(t) = \mathcal{S}x(t) \tag{13}$$

$$u_i^{\min} \le u_i(t) \le u_i^{\max} \qquad (\text{for } i = 1, ..., m) \tag{14}$$

$$\chi^{\min} \le (r,x(t)) \le \chi^{\max} \tag{15}$$

on $\mathcal{H} = L_2(0,\pi)$. The spectrum of the operator $\mathcal{A}$ can be obtained by solving the following eigenvalue problem:

$$\mathcal{A}\phi_j = \bar{b}\left(\frac{d^2\phi_j}{dz^2}\right) + \bar{c}\phi_j = \lambda_j\phi_j \tag{16}$$

subject to

$$\phi_j(0) = 0, \phi_j(\pi) = 0 \tag{17}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction. A direct computation of the solution of the aforementioned eigenvalue problem yields

$$\lambda_j = \bar{c} - \bar{b}j^2, \phi_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz) \qquad (\text{for } j = 1, ..., \infty) \tag{18}$$

The spectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of $\mathcal{A}$, i.e., $\sigma(\mathcal{A}) = \{\lambda_1,\lambda_2,...\}$. All the eigenvalues of $\mathcal{A}$ are real, and for a given $\bar{b}$ and $\bar{c}$, only a finite number of unstable eigenvalues exists, and the distance between any two consecutive eigenvalues (i.e., λ_j and λ_{j+1}) increases as j increases. This implies that the dominant dynamics of the PDE can be described by a finite-dimensional system, and motivates the use of modal decomposition to derive a finite-dimensional system that captures the dominant (slow) dynamics of the PDE (see subsection 2.3 later in this paper).

From the properties of $\mathcal{A}$ and its spectrum, it follows that $\mathcal{A}$ generates a strongly continuous $\mathcal{C}_0$-semigroup, $\mathcal{T}(t)$.[38] Moreover, because $\mathcal{B}$ is a bounded operator, the system of eq 12 has a mild solution of the form

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-\tau)\mathcal{B}u(\tau)\, d\tau \tag{19}$$

Remark 1. Note that, although we use the PDE system of eqs 1−4 to motivate and illustrate the development of the infinite-dimensional system of eq 12, our subsequent results are not limited to single PDEs of the form of eqs 1−4. Specifically, the results developed in this work apply to parabolic PDEs, with other types of boundary conditions (for example, mixed boundary conditions) and systems of parabolic PDEs, as long as they possess an operator $\mathcal{A}$ for which the spectrum decomposition property holds. These conditions can be shown to hold for all linear parabolic PDEs with self-adjoint operators, which have finitely many eigenvalues with positive real part. In the domain of chemical process control, the class of linear parabolic PDEs with self-adjoint operators arises frequently from the linearization of first-principle models of transport−reaction processes. However, one must be cautious with the possible transformation applied in the case of nonhomogeneous boundary conditions, because the transformation applied might change the structure in the definition of operators and the inner product of the associated Hilbert state space (see examples in ref 38).

2.2. Model Predictive Control. Referring to the system of eq 12, we consider the problem of asymptotic stabilization of the origin, subject to the control constraints of eq 14 and the state constraint of eq 15. The problem will be addressed within the MPC framework (see ref 32 for a review of various MPC algorithms for finite-dimensional systems), where the control, at a state $x(t)$ and time t, is conventionally obtained by solving, on-line, a finite-horizon constrained optimal control problem of the form

$$P(x(t),t):\ \min\{J(x(t),t,u(\cdot))\ |\ u(\cdot)\in S\} \tag{20}$$

$$\text{s.t. } \dot{x}(\tau) = \mathcal{A}x(\tau) + \mathcal{B}u(\tau) \tag{21a}$$

$$u(\tau) \in \mathcal{U} \tag{21b}$$

$$\chi^{\min} \le (r,x(\tau)) \le \chi^{\max} \qquad (\text{for } \tau \in [t, t+T]) \tag{21c}$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t + T]$ into $\mathcal{U} = \{u(t) \in \mathbb{R}^m: u_i^{\min} \le u_i(t) \le u_i^{\max}$, for $i = 1, ..., m\}$, and T is the specified horizon. The control $u(\cdot)$ in S is characterized by the sequence $u[k]$, where $u[k] = u(k\Delta)$ and satisfies $u(t) = u[k]$ for all $t \in [k\Delta,(k + 1)\Delta)$. The performance index is given by

$$J(x,t,u(\cdot)) = \int_t^{t+T} [q||x^u(\tau; x,t)||_2^{\ 2} + |\ u(\tau)|_R^{\ 2}]\, d\tau + F(x(t+T)) \tag{22}$$

where q is a positive real number, and R is a positive number; $x^u(\tau; x(t),t)$ denotes the solution of eq 12 that is due to the control $u(t)$, with an initial state $x(t)$ at a time t, and $F(\cdot)$ denotes the terminal penalty. The minimizing control $u^0(\cdot) = \{u(k\Delta), u((k + 1)\Delta), u((k + 2)\Delta), ...\} \in S$ is then applied to the system over the interval $[k\Delta,(k + 1)\Delta]$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law,

$$u(t) = M(x(t)) = u^0(k\Delta; x(t),t) \tag{23}$$

in which $M(x(t))$ denotes the nonlinear map between the state and control. The predictive control problem of eqs 20 and 21 is formulated on the basis of an infinite-dimensional system, and therefore, it leads to a predictive controller that is of infinite order and cannot be realized in practice. To overcome this problem, in the next section, we develop computationally efficient predictive control formulations that achieve stabilization of the system of eq 12, subject to the control and state constraints of eqs 14 and 15. Throughout the manuscript, because we work with continuous time PDE systems in order to present the closed-loop stability results concisely, we will allow the hold time to have a value of $\Delta = 0$; our results can be extended to the case where Δ is greater than zero but sufficiently small (i.e., sample-and-hold implementation of the predictive controller) at the expense of proving practical stability of the closed-loop system (i.e., the state of the infinite-dimensional closed-loop system converges in finite-time in a small ball around the origin, whose size decreases as the hold time Δ decreases).

Remark 2. It is well-known that, even for the finite-dimensional systems, the control law that is defined by eqs 20−23 is not necessarily stabilizing. For finite-dimensional systems, the issue of closed-loop stability is usually addressed by means of imposing suitable penalties and constraints on the state at the end of the optimization horizon (e.g., see refs 30 and 32 for surveys of different approaches).

2.3. Modal Decomposition. Referring to the system of eq 12, let $\mathcal{H}_s$ and $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, which are defined as $\mathcal{H}_s = \text{span}\{\phi_1,\phi_2,...,\phi_m\}$ and $\mathcal{H}_f = \text{span}\{\phi_{m+1},\phi_{m+2},...\}$ (the existence of $\mathscr{H}_s$, $\mathscr{H}_f$ follows from the properties of $\mathscr{A}$, and m is chosen such that $\lambda_{m+1} < 0$). Defining the orthogonal projection operators, $\mathscr{P}_s$ and $\mathscr{P}_f$, such that $x_s(t) = \mathscr{P}_s x(t)$ and $x_f(t) = \mathscr{P}_f x(t)$, the state $x(t)$ of the system of eq 12 can be decomposed as

$$x(t) = \mathscr{P}_s x(t) + \mathscr{P}_f x(t) = x_s(t) + x_f(t) \tag{24}$$

Applying $\mathscr{P}_s$ and $\mathscr{P}_f$ to the system of eq 12 and using the aforementioned decomposition for $x(t)$, the system of eq 12 can be rewritten in the following equivalent form:

$$\frac{dx_s(t)}{dt} = \mathscr{A}_s x_s(t) + \mathscr{B}_s u(t),\ x_s(0) = \mathscr{P}_s x(0) = \mathscr{P}_s x_0 \tag{25a}$$

$$\frac{dx_f(t)}{dt} = \mathscr{A}_f x_f(t) + \mathscr{B}_f u(t),\ x_f(0) = \mathscr{P}_f x(0) = \mathscr{P}_f x_0 \tag{25b}$$

$$y(t) = \mathscr{S}_s x_s(t) + \mathscr{S}_f x_f(t) \tag{25c}$$

where $\mathscr{A}_s = \mathscr{P}_s\mathscr{A}$, $\mathscr{B}_s = \mathscr{P}_s\mathscr{B}$, $\mathscr{A}_f = \mathscr{P}_f\mathscr{A}$, $\mathscr{B}_f = \mathscr{P}_f\mathscr{B}$, $\mathscr{S}_s = \mathscr{S}\mathscr{P}_s$, and $\mathscr{S}_f = \mathscr{S}\mathscr{P}_f$. In the aforementioned system, $\mathscr{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathscr{A}_s = \text{diag}\{\lambda_j\}$ (note that some of the λ_j, $j = 1,\cdots, m$ eigenvalues may be unstable) and $\mathscr{A}_f$ is an infinite-dimensional operator that is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathscr{H}_s$,$\mathscr{H}_f$). From the properties of $\mathscr{A}_s$ and $\mathscr{A}_f$, it follows that there exist C_0 semigroups $\mathscr{T}_s(t)$ and $\mathscr{T}_f(t)$ for which the following characterization is given by the Hille−Yoshida theorem, $||\mathscr{T}_i(t)|| \le M_i e^{\omega_i t}$,$(i = s, f)$.[38] In addition, because of the fact that $\mathscr{B}_s$ and $\mathscr{B}_f$ are bounded operators, the $x_s(t)$ and $x_f(t)$ subsystems of eq 25 admit the following mild solutions:

$$x_s(t) = \mathscr{T}_s(t)x_s(0) + \int_0^t \mathscr{T}_s(t-\tau)\mathscr{B}_s u(\tau)\, d\tau \tag{26}$$

$$x_f(t) = \mathscr{T}_f(t)x_f(0) + \int_0^t \mathscr{T}_f(t-\tau)\mathscr{B}_f u(\tau)\, d\tau \tag{27a}$$

$$y(t) = \mathscr{S}_s x_s(t) + \mathscr{S}_f x_f(t) \tag{27b}$$

Furthermore, because $\mathscr{A}_f$ is a stable operator, the spectrum of $\mathscr{A}_f$ satisfies $\sup\{Re\ \{\sigma(\mathscr{A}_f)\}\} < \omega_f < 0$, for some $\omega_f > \bar{c} - \bar{b}(m + 1)^2$, and thus, $\mathscr{T}_f(t)$ satisfies

$$||\mathscr{T}_f(t)||_2 \le M_f e^{\omega_f t} \qquad (\text{for } t \ge 0) \tag{28}$$

where M_f is a positive constant. Furthermore, there exists a positive constant M such that

$$||\mathscr{T}(t)|| \le M e^{\omega t} \tag{29}$$

where $\mathscr{T}(t) = \text{diag}\{\mathscr{T}_s(t),\mathscr{T}_f(t)\}$ and $\omega = \max\{\omega_s,\omega_f\}$.

3. Predictive Output Feedback Control

3.1. State Estimation. The objective of this section is to propose predictive control formulations that make use of measurements of a finite number, p, of measured outputs y_j to enforce stability and constraint satisfaction in the closed-loop system. To this end, we follow[39] and use a state estimation scheme, based on direct inversion of the measured output map, to obtain estimates of the states $x_s(t)$ of the system of eq 25. To develop this estimation scheme, we must impose the following assumption on the number of measured outputs and the form of the sensor shape function.

Assumption 1: $p = m$ (i.e., the number of measured outputs is equal to the number of slow states) and $\mathscr{S}_s^{-1}$ exists. Assumption 1 ensures that the finite-dimensional $x_s(t)$ subsystem is stabilizable via static output feedback control (together with the assumption that the pair ($\mathscr{A}_s$,$\mathscr{B}_s$) is controllable) and can be satisfied by proper placement of the measurement sensors. Under assumption 1, an estimate of the state of the $x_s(t)$ subsystem can be obtained as follows:

$$\hat{x}_s(t) = \mathscr{S}_s^{-1} y(t) \tag{30}$$

where $\hat{x}_s(t)$ is an estimate of $x_s(t)$.

3.2. Low-Order Predictive Output Feedback Control. In this subsection, we present a predictive control formulation that is developed on the basis of the finite-dimensional $x_s(t)$ subsystem of eq 25. Specifically, the control action is computed by solving, in a receding horizon fashion, the following optimization problem:

$$\min_u \int_t^{t+T} [q||\hat{x}_s(\tau)||_2^{\,2} + |u(\tau)|_R^{\,2}]\, d\tau + F(\hat{x}_s(t+T)) \tag{31}$$

$$\text{s.t. } \dot{\hat{x}}_s(\tau) = \mathscr{A}_s\hat{x}_s(\tau) + \mathscr{B}_s u(\tau) \tag{32a}$$

$$u(\tau) \in \mathscr{U} \tag{32b}$$

$$\chi^{\min} \le (r,\hat{x}_s(\tau)) \le \chi^{\max} \qquad (\tau \in [t, t+T]) \tag{32c}$$

$$\hat{x}_s(t) = \mathscr{S}_s^{-1} y(t) \tag{32d}$$

where $F(\cdot)$ is the terminal cost. The predictive control law of eqs 31 and 32 is stabilizing for the set of initial conditions $x_s(0) \in \Omega_s$, where the set Ω_s is dependent on the constraints on the states and inputs, the system dynamics, the horizon length T, and the location where the output measurements are taken; the reader may refer to refs 40 and 41 for a Lyapunov-based approach for the construction of estimates of Ω_s.

Assumption 2 below states the properties that we assume that the model predictive controller of eqs 31 and 32 enforces in the closed-loop finite-dimensional $x_s(t)$ subsystem under the use of state feedback control (i.e., $\hat{x}_s(t) = x_s(t)$; the state feedback control $x_s(t)$ is assumed to be known).

Assumption 2: Consider the closed-loop system resulting from the finite-dimensional $x_s(t)$ subsystem of eq 25 and the predictive controller of eqs 31 and 32 with $\hat{x}_s(t) = x_s(t)$. There, then exists a compact set $\Omega_s \subset \mathscr{H}_s$ (Ω_s includes in its interior $x_s(t) = 0$) such that, for all $x_s(0) \in \Omega_s$, $x_s(t) \in \Omega_s$ for all $t \ge 0$ and the equilibrium $x_s(t) = 0$ of the closed-loop system is asymptotically stable and, locally, is exponentially stable.

Theorem 1 below states sufficient conditions under which the predictive output feedback controller of eqs 31 and 32 enforces stability in the infinite-dimensional closed-loop system.

Theorem 1: Consider the closed-loop infinite-dimensional system resulting from the application of the predictive controller of eqs 31 and 32 to the system of eq 25 and suppose that assumptions 1 and 2 hold. Then, given a compact set $\Omega_s' \subset \Omega_s$ and a positive constant b, there exists an ϵ^* such that if $\epsilon = |\lambda_1|/|\lambda_{m+1}| \in (0,\epsilon^*]$, and $x(0) = x_s(0) + x_f(0)$ is such that $x_s(0) \in \Omega_s'$ and $|x_f(0)| \le b$, the equilibrium $(x_s, x_f) = (0, 0)$ of the closed-loop infinite-dimensional system is asymptotically stable.

Proof: Let $u(t) = M(\hat{x}_s) = M(x_s(t),x_f(t))$ be the general expression of the control law corresponding to the predictive control formulation of eqs 31 and 32. Consider the closed-loop system under this control law:

$$\dot{x}_s(t) = \mathscr{A}_s x_s(t) + \mathscr{B}_s M(x_s(t),x_f(t)) \tag{33}$$

$$\dot{x}_f(t) = \mathscr{A}_f x_f(t) + \mathscr{B}_f M(x_s(t),x_f(t)) \tag{34}$$

Using $\epsilon = |\lambda_1|/|\lambda_{m+1}|$, we can rewrite the aforementioned system in the following standard singularly perturbed form:

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s M(x_s(t),x_f(t)) \tag{35}$$

$$\epsilon \dot{x}_f(t) = \mathcal{A}_{f_\epsilon} x_f(t) + \epsilon \mathcal{B}_f M(x_s(t),x_f(t)) \tag{36}$$

where $\mathcal{A}_{f_\epsilon} = \epsilon \mathcal{A}_f$ and the matrixes $\mathcal{A}_s$ and $\mathcal{A}_f$ have eigenvalues of the same order of magnitude. Performing a two-time scale decomposition of the above system, we can see that the slow subsystem takes the form

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s M(x_s(t),x_f(t)) \tag{37}$$

$$0 = \mathcal{A}_{f_\epsilon} x_f(t) \tag{38}$$

or

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s M(x_s(t)) \tag{39}$$

Under assumption 2, the origin of the system of eq 39 for every $x_s(0) \in \Omega_s$ is asymptotically stable and, locally, exponentially stable. Introducing the "stretched" time scale $\tau = t/\epsilon$, the system of eqs 35 and 36 can be written as

$$\frac{\mathrm{d}x_s(\tau)}{\mathrm{d}\tau} = \epsilon(\mathcal{A}_s x_s(\tau) + \mathcal{B}_s M(x_s(\tau),x_f(\tau))) \tag{40a}$$

$$\frac{\mathrm{d}x_f(\tau)}{\mathrm{d}\tau} = \mathcal{A}_{f_\epsilon} x_f(\tau) + \epsilon \mathcal{B}_f M(x_s(\tau), x_f(\tau)) \tag{40b}$$

and setting $\epsilon = 0$, we obtain the fast subsystem:

$$\frac{\mathrm{d}x_f(\tau)}{\mathrm{d}\tau} = \mathcal{A}_{f_\epsilon} x_f(\tau) \tag{41}$$

which is exponentially stable. From the stability properties of the subsystems of eqs 39−41, we can use arguments similar to those in ref 42 and the result of proposition 1 in ref 6 to conclude that, given a compact set $\Omega_s' \subset \Omega_s$ and a positive constant b, there exists an ϵ^* such that, if $\epsilon \in (0,\epsilon^*]$ and $x(0) = x_s(0) + x_f(0)$, such that $x_s(0) \in \Omega'$ and $|x_f(0)| \leq b$, the equilibrium $(x_s, x_f) = (0, 0)$ of the closed-loop infinite-dimensional system is asymptotically stable. This completes the proof of Theorem 1.

However, in the controller formulation described by eqs 31 and 32, the evolution of the fast states is not taken into account, neither in the cost functional nor in the state constraints. Therefore, a potential drawback of this formulation is the fact that the resulting predictive control law, when implemented on the infinite-dimensional system of eq 12, will enforce closed-loop stability but not necessarily full-state constraint satisfaction, because it neglects the evolution of the fast states. In particular, because the full state, $x(t)$, includes contributions from both $x_s(t)$ and $x_f(t)$ (recall that $x(t) = x_s(t) + x_f(t)$), it is possible that the $x_f(t)$ subsystem, which is affected by the control input, evolves in a way that causes the full-state constraints to be violated for some time. Therefore, although the stabilization objective can be achieved, the additional objective of state constraints satisfaction requires that the evolution of fast states be properly taken into account when designing the predictive controller.

3.3. Predictive Output Feedback Control Accounting for Fast Modes. In this subsection, we develop a predictive output feedback control formulation that explicitly accounts for the fast modes $x_f(t)$ and derive sufficient conditions under which the predictive controller enforces stability and constraint satisfaction in the infinite-dimensional closed-loop system. To proceed with this task, we must first devise an estimation scheme that will allow computing estimates of the $x_f(t)$ states from the output measurements. Because the number of output measurements is finite, the entire infinite-dimensional state $\bar{x}(z, t)$ (or $x(t)$) cannot be computed through inversion of the measurement equation. To this end, we propose to use a large, but finite-dimensional, approximation of the state $x_f(t)$ in the predictive controller. Therefore, we decompose $x(t)$ as

$$x(t) = x_s(t) + x_{f1}(t) + x_{f2}(t) \tag{42}$$

where $x_s(t) \in \mathcal{H}_s$ is the state of the slow subsystem, $x_{f1}(t) \in \mathcal{H}_{f1} = \text{span}\{\phi_{m+1},...,\phi_{m+q}\}$ is the state of a finite-dimensional system including modes from $(m + 1)$ to $(m + q)$, and $x_{f2}(t)$ is the state of an infinite-dimensional system including modes from $(m + q + 1)$ to infinity ($x_{f2}(t) \in \mathcal{H}_{f2} = \text{span}\{\phi_{m+q+1},\cdots,\infty\}$). Using this decomposition, the system of eq 25 can be written as

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s u(t) \tag{43a}$$

$$\dot{x}_{f1}(t) = \mathcal{A}_{f1} x_{f1}(t) + \mathcal{B}_{f1} u(t) \tag{43b}$$

$$\dot{x}_{f2}(t) = \mathcal{A}_{f2} x_{f2}(t) + \mathcal{B}_{f2} u(t) \tag{43c}$$

$$y(t) = \mathcal{S}_s x_s(t) + \mathcal{S}_{f1} x_{f1}(t) + \mathcal{S}_{f2} x_{f2}(t) \tag{43d}$$

where $\mathcal{A}_{f1} = \mathcal{P}_{f1}\mathcal{A}$, $\mathcal{B}_{f1} = \mathcal{P}_{f1}\mathcal{B}$, $\mathcal{A}_{f2} = \mathcal{P}_{f2}\mathcal{A}$, $\mathcal{B}_{f2} = \mathcal{P}_{f2}\mathcal{B}$, $\mathcal{S}_{f1} = \mathcal{S}\mathcal{P}_{f1}$, $\mathcal{S}_{f2} = \mathcal{S}\mathcal{P}_{f2}$ and $\mathcal{P}_{f1}$ and $\mathcal{P}_{f2}$ are orthogonal projection operators that project $x(t) \in \mathcal{H}$ onto $x_{f1}(t) \in \mathcal{H}_{f1}$ and $x_{f2}(t) \in \mathcal{H}_{f2}$ (i.e., $x_{f1}(t) = \mathcal{P}_{f1}x(t)$ and $x_{f2}(t) = \mathcal{P}_{f2}x(t)$), respectively. We introduce the following assumption that is important for the computation of estimates of the $x_s(t)$ and $x_{f1}(t)$ states.

Assumption 3: $p = m + q$ (i.e., the number of measurements is equal to the dimension of the model used for control design) and the inverse $\mathcal{S}_{sf1}^{-1}$ exists (where $\mathcal{S}_{sf1}$ is a $p \times p$ matrix whose (i, j) element is (s_j,ϕ_i), $i = 1,\cdots, p$; $j = 1,\cdots, p$), such that $[\hat{x}_s(t)\ \hat{x}_{f1}(t)]' = \mathcal{S}_{sf1}^{-1} y(t)$.

The predictive controller that uses fast states takes the form

$$\min_u \int_t^{t+T} [q||\hat{x}_s(\tau)||_2^{\,2} + |u(\tau)|_R^{\,2}]\, \mathrm{d}\tau + F(\hat{x}_s(t + T)) \tag{44}$$

$$\text{s.t. } \dot{\hat{x}}_s(\tau) = \mathcal{A}_s \hat{x}_s(\tau) + \mathcal{B}_s u(\tau) \tag{45a}$$

$$\dot{\hat{x}}_{f1}(\tau) = \mathcal{A}_{f1} \hat{x}_{f1}(\tau) + \mathcal{B}_{f1} u(\tau) \qquad (\tau \in [t, t + T]) \tag{45b}$$

$$u(\tau) \in \mathcal{U} \tag{45c}$$

$$\chi^{\min} - \alpha \leq (r,\hat{x}_s(\tau) + \hat{x}_{f1}(\tau)) \leq \chi^{\max} - \alpha \tag{45d}$$

$$[\hat{x}_s(t)\ \hat{x}_{f1}(t)]' = \mathcal{S}_{sf1}^{-1} y(t) \tag{45e}$$

where α is a positive constant. The important feature of the control law of eqs 44 and 45 is the inclusion of the fast states $x_{f1}(t)$ in the predictive control law. However, in eqs 44 and 45, the $(\hat{x}_s(t), \hat{x}_{f1}(t))$ in the state constraints is dependent on the accuracy of the state estimates, so it is necessary to develop some quantitative measure that would address the issue of the accuracy of the estimated modes, and therefore, the issue of the successful state and input constraints satisfaction. To address this issue in a quantitative manner, we introduce the "back-off" parameter α, which is dependent on the number of measurements p. In other words, this back-off correction decreases as the number of measurements increases, because, in this case, $\hat{x}_s(t)$ and $\hat{x}_{f1}(t)$ estimate better $x_s(t)$ and $x_{f1}(t)$. To provide the relation between the set of initial conditions, the controller parameters, and the back-off parameter α, we introduce the following assumption.

Assumption 4: Consider the closed-loop system

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s M(x_s(t),x_{f1}(t)) \quad (46)$$

$$\dot{x}_{f1}(t) = \mathcal{A}_{f1} x_{f1}(t) + \mathcal{B}_{f1} M(x_s(t),x_{f1}(t)) \quad (47)$$

There then exists a compact set Ω_{sf1} that includes $x_s = x_{f1} = 0$ in its interior, such that, for any $[x_s(0)\ x_{f1}(0)] \in \Omega_{sf1}$, the equilibrium $(x_s, x_{f1}) = (0, 0)$ of the system of eqs 46 and 47 is asymptotically stable and, locally, is exponentially stable.

Theorem 2 below states sufficient conditions under which the predictive output feedback controller of eqs 44 and 45 enforces stability and constraints satisfaction in the infinite-dimensional closed-loop system.

Theorem 2: Consider the infinite-dimensional system of eq 25 under the predictive output feedback controller of eqs 44 and 45 and suppose that assumptions 3 and 4 hold. Then, given a compact set $\mathbf{\Omega}'_{sf1} \subset \Omega_{sf1}$, there exist ϵ^* and b such that if $[x_s(0)\ x_{f1}(0)] \in \mathbf{\Omega}'_{sf1}$, $|x_f(0)| \le b$ and $\epsilon = |\lambda_1|/|\lambda_{m+1}| \in (0,\epsilon^*]$, the equilibrium $(x_s, x_f) = (0, 0)$ of the closed-loop infinite-dimensional system is asymptotically stable and the state constraint of eq 25 is satisfied in the closed-loop system for all $t \ge 0$.

Proof: Let the control law resulting from the predictive control formulation of eqs 44 and 45 be denoted by $u(t) = M_2(\hat{x}_s(t),\hat{x}_{f1}(t)) = M_2(x_s(t),x_{f1}(t),x_{f2}(t))$. Consider the closed-loop system under the predictive controller:

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s M_2(x_s(t),x_{f1}(t),x_{f2}(t)) \quad (48a)$$

$$\dot{x}_{f1}(t) = \mathcal{A}_{f1} x_{f1}(t) + \mathcal{B}_{f1} M_2(x_s(t),x_{f1}(t),x_{f2}(t)) \quad (48b)$$

$$\dot{x}_{f2}(t) = \mathcal{A}_{f2} x_{f2}(t) + \mathcal{B}_{f2} M_2(x_s(t),x_{f1}(t),x_{f2}(t)) \quad (48c)$$

Using $\epsilon = |\lambda_1|/|\lambda_{m+q+1}|$ and multiplying the $x_{f2}(t)$ subsystem by ϵ, we have

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s M_2(x_s(t),x_{f1}(t),x_{f2}(t)) \quad (49)$$

$$\dot{x}_{f1}(t) = \mathcal{A}_{f1} x_{f1}(t) + \mathcal{B}_{f1} M_2(x_s(t),x_{f1}(t),x_{f2}(t)) \quad (50)$$

$$\epsilon\dot{x}_{f2}(t) = \mathcal{A}_{f2_\epsilon} x_{f2}(t) + \epsilon\mathcal{B}_{f2} M_2(x_s(t),x_{f1}(t),x_{f2}(t)) \quad (51)$$

where $\mathcal{A}_{f2_\epsilon} = \epsilon\mathcal{A}_{f2}$. Performing a two-time scale decomposition of the aforementioned system, we obtain the slow subsystem:

$$\dot{x}_s(t) = \mathcal{A}_s x_s(t) + \mathcal{B}_s M_2(x_s(t),x_{f1}(t),0) \quad (52)$$

$$\dot{x}_{f1}(t) = \mathcal{A}_{f1} x_{f1}(t) + \mathcal{B}_{f1} M_2(x_s(t),x_{f1}(t),0) \quad (53)$$

Under assumption 4, the solution $(x_s, x_{f1}) = (0,0)$ of the aforementioned system is asymptotically stable and, locally, is exponentially stable for every $[x_s(0)\ x_{f1}(0)] \in \Omega_{sf1}$. Furthermore, the fast subsystem has the form

$$\frac{dx_{f2}(\tau)}{d\tau} = \mathcal{A}_{f2_\epsilon} x_{f2}(\tau) \quad (54)$$

which is exponentially stable. From the stability properties of the slow and fast subsystems of eqs 52, 53, and 54, we can use similar arguments to the proof of Theorem 1 to show that, given a compact set $\Omega'_{sf} \in \Omega_{sf1}$ and a positive real number b, there exists an ϵ^* such that if $[\hat{x}_s(0)\ \hat{x}_{f1}(0)] \in \Omega_{sf1}$ and $||x_f(0)|| \le b$ and $\epsilon \in (0,\epsilon_1^*]$, the equilibrium $(x_s,x_f) = (0,0)$ of the closed-loop infinite-dimensional system of eq 25 is asymptotically stable. Furthermore, there exist ϵ_2^* and b for which the estimation error is sufficiently small and $x_{f2}(\tau)$ is also sufficiently small (the fact that there exist ϵ_2^* and b such that $x_{f2}(\tau)$ is sufficiently small follows from the structure and stability property of the $x_{f2}(t)$ subsystem), and then the fact that $\chi^{\min} - \alpha \le (r,\hat{x}_s(\tau) + \hat{x}_{f1}(\tau)) \le \chi^{\max} - \alpha$ implies that $\chi^{\min} \le (r,\hat{x}_s(\tau) + \hat{x}_{f1}(\tau) + \hat{x}_{f2}(\tau)) \le \chi^{\max}$. Pick $\epsilon^* = \min\{\epsilon_1^*,\epsilon_2^*\}$ to complete the proof.

Remark 5. The back-off parameter α applied in eqs 44 and 45 can be seen as a conservative bound on the evolution of the x_{f2} states that guarantees state constraints satisfaction. This measure cannot exceed the absolute value of the difference between $\chi^{\max} - \chi^{\min}$, nor should it be such that it mutually excludes other constraints imposed, for example, a stability constraint of the form $x_s(t + T) = 0$. A good estimate of α can be determined by simulations.

Remark 6. The successful application of the predictive control law of eqs 44 and 45 hinges on the accuracy of the estimated states through the state reconstruction scheme that is used. To improve this accuracy, one can consider formulating and solving an optimal sensor placement problem. This has been addressed in ref 43, where the optimal sensor placement problem has been addressed via a min−max optimization formulation, which is solved through a guided search technique.

4. Simulation Example

In this section, we demonstrate and compare, through computer simulations, the implementation of the two MPC formulations discussed in the previous section. To this end, we consider the parabolic PDE of eq 1, with $\bar{b} = 1$, $\bar{c} = 1.66$, $w = 2$, and two control actuators ($m = 2$) with the following distribution functions: $b_i(z) = 1/\mu$ for $z \in [z_{ai} - \mu, z_{ai} + \mu]$ and $b_i(z) = 0$ elsewhere in $[0,\pi]$, where $\mu = 0.005$, $z_{a1} = \pi/3$, and $z_{a2} = 2\pi/3$. For these values, it was verified that the operating steady state, $\bar{x}(z,t) = 0$, is an unstable one. The control objective is to stabilize the state profile at the unstable zero steady state by manipulating $u_i(t)$ subject to the input and state constraints of eqs 3 and 4 with $u_i^{\min} = -2.5$, $u_i^{\max} = 2.5$, for $i = 1$ and 2, $\chi^{\min} = -0.035$, $\chi^{\max} = 2.0$. The state constraints distribution function, $r(z)$, is chosen to be $r(z) = 1/\mu$ for $z \in [z_c - \mu, z_c + \mu]$ and $r(z) = 0$ elsewhere in $[0,\pi]$, where $z_c = 1.047$ is the point where the state constraint is to be enforced and $\mu = 0.005$. The eigenvalue problem for the spatial differential operator of the PDE of eq 16 can be solved analytically, and its solution yields

$$\lambda_j = 1.66 - j^2,\ \phi_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz) \qquad (j = 1, ..., \infty) \quad (55)$$

For this system, we consider the first two eigenvalues to be the dominant ones and use two point control actuators ($m = 2$) to achieve the control objective, subject to the constraints of eqs 14 and 15. In particular, using modal decomposition, we can derive the following high-order ODE system that describes the temporal evolution of the first l modes:

$$\dot{a}_s(t) = A_s a_s(t) + B_s u(t) \quad (56a)$$

$$\dot{a}_f(t) = A_f a_f(t) + B_f u(t) \quad (56b)$$

$$y(t) = S_s a_s(t) + S_f a_f(t) \quad (56c)$$

where $a_s(t) = [a_1(t)\ a_2(t)]'$, $a_f(t) = [a_3(t)\ a_4(t) \cdots a_l(t)]'$, $a_i(t) \in \mathbb{R}$ is the ith mode, the notation $a_s(t)'$ denotes the transpose of $a_s(t)$, $u(t) = [u_1(t)\ u_2(t)]'$, the matrices A_s and A_f are diagonal matrixes, given by $A_s = \text{diag}\{\lambda_i\}$, for $i = 1$ and 2 and $A_f =$

diag$\{\lambda_i\}$, for $i = 3, ..., l$. B_s and B_f are a 2×2 matrix and an $(l - 2) \times 2$ matrix, respectively, the (i, j)th elements of which are given by $B_{ij} = (b_j(z),\phi_i(z))$. We now proceed with the design and implementation of the two predictive output feedback control formulations presented in the previous section. The following initial condition is considered in all simulation runs: $\bar{x}(z,0) = 0.04 \sin(z) + 0.015 \sin(2z)$ and l is chosen to be 35 (further increases of the value of l led to identical results for both the open-loop and closed-loop systems). In the first scenario, we use the predictive controller of Theorem 1. Specifically, we use the $a_s(t)$ subsystem of eq 56 as the basis for the predictive controller design (the $a_f(t)$ subsystem is neglected). According to Theorem 1, we consider an MPC formulation with the following objective function and constraints:

$$\min_u \int_t^{t+T} [q_s |\hat{a}_s(\tau)|_I^2 + |u(\tau)|_R^2]\, d\tau + F(\hat{a}_s(t+T)) \qquad (57)$$

$$\text{s.t. } \dot{\hat{a}}_s(\tau) = A_s\hat{a}_s(\tau) + B_s u(\tau) \qquad (58a)$$

$$u_i^{\min} \le u_i(\tau) \le u_i^{\max} \qquad (\text{for } i = 1, 2) \qquad (58b)$$

$$\chi^{\min} \le \int_0^\pi r(z)[\sum_{i=1}^{2} \hat{a}_i(\tau)\phi_i(z)]\, dz \le \chi^{\max} \qquad (58c)$$

$$\hat{a}_s(t) = S_s^{-1} y(t) \qquad (\tau \in [t, t+T]) \qquad (58d)$$

where $q_s = 60$, $R = rI$ (with $r = 0.01$), and $T = 0.0429$. To ensure stability, we impose a terminal equality constraint of the form $\hat{a}_s(t + T) = 0$ on the optimization problem and set $F(\hat{a}_s(t + T)) = 0$. According to assumption 1, the number of measurements needed to obtain estimates of the $\hat{a}_s(t)$ is $p = m = 2$ and the matrix S_s is a 2×2 matrix whose (i, j)th element is given by $(s_j(z),\phi_i(z))$, where $i = 1, 2$ and $j = 1, 2$; two point measurements are taken, at $\pi/3$ and $2\pi/3$. The resulting quadratic program is solved using the MATLAB subroutine QuadProg. The control action is then implemented on the 35th order model of eq 56. For this case, our calculations demonstrate that the use of two output measurements, although sufficient for the predictive controller of eqs 57 and 58 to achieve the closed-loop stability of the system of eq 56, is not sufficient to estimate the $\hat{a}_s(t)$ states accurately, so that PDE state constraint satisfaction is achieved. Furthermore, to demonstrate that, in addition to more measurements, we need to include fast states in the predictive controller to achieve constraint satisfaction, we assume that we have available 30 measurements placed equidistantly in the following way:

$$[\pi/31 \;\; 2\pi/31 \;\; \cdots \;\; 30\pi/31]$$

and use them to compute a better estimate of $\hat{a}_s(t)$, using the estimation scheme of assumption 3 (i.e., $[\hat{a}_s(t)\ \hat{a}_{f1}(t)]' = S_{sf1}^{-1}y(t)$), where S_{sf1} is a square matrix whose (i, j)th element is given by $(s_j(z),\phi_i(z))$ $i = 1, ..., 30$; $j = 1, ..., 30$. Figure 1, Figure 2, and Figures 3 and 4 (solid lines) show the closed-loop state, the closed-loop profile at the point where the state constraint is enforced, and manipulated input profiles, respectively, under the MPC controller of eqs 57 and 58, using 30 measurements. It is clear that the predictive controller successfully stabilizes the state at the zero steady state. However, by examining the solid line in Figure 2, we observe that the state at $z_{c1} = 1.047$ violates the lower constraint for some time. The violation of the state constraint comes from neglecting the contribution of the $\hat{a}_f(t)$ states in the predictive controller of eqs 57 and 58. To account for the evolution of the fast states in the optimization

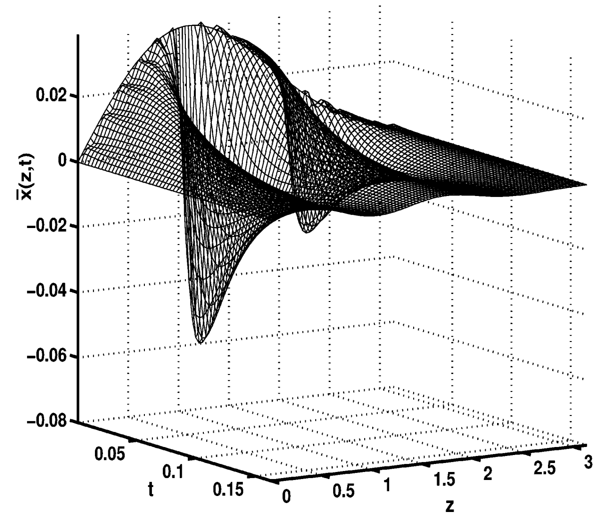

Figure 1. Closed-loop state profile under the MPC formulation of eqs 57 and 58, without accounting for the fast states in the state constraints, using 30 measurements.

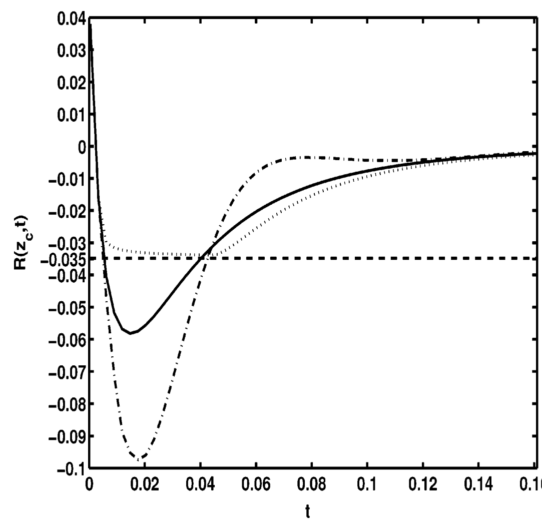

Figure 2. Closed-loop output profiles ($R(z_c,t) = \int_0^\pi \delta(z - z_c)\bar{x}(z, t)\, dz$) at $z_c = 1.047$, under the MPC formulation of eqs 57 and 58 (solid line), under the MPC formulation of eqs 59 and 60 with 30 measurement points (dotted line), and under the MPC formulation of eqs 59 and 60 with 5 measurement points (dashed−dotted line).

problem, we consider the predictive output feedback control formulation of Theorem 2, which takes the form

$$\min_u \int_t^{t+T} [q_s |\hat{a}_s(\tau)|_I^2 + |u(\tau)|_R^2]\, d\tau + F(\hat{a}_s(t+T)) \qquad (59)$$

$$\text{s.t. } \dot{\hat{a}}_s(\tau) = A_s\hat{a}_s(\tau) + B_s u(\tau) \qquad (60a)$$

$$\dot{\hat{a}}_{f1}(\tau) = A_{f1}\hat{a}_{f1}(\tau) + B_{f1}u(\tau) \qquad (60b)$$

$$u_i^{\min} \le u_i(\tau) \le u_i^{\max} \qquad (\text{for } i = 1, 2) \qquad (60c)$$

$$\chi^{\min} - \alpha \le \int_0^\pi r(z)[\sum_{i=1}^{30} \hat{a}_i(\tau)\phi_i(z)]\, dz \le \chi^{\max} - \alpha \qquad (60d)$$

$$[\hat{a}_s(t)\ \hat{a}_{f1}(t)]' = S_{sf1}^{-1} y(t) \qquad (\tau \in [t, t+T]) \qquad (60e)$$

where $\hat{a}_s(t) = [\hat{a}_1(t)\ \hat{a}_2(t)]'$ and $\hat{a}_{f1}(t) = [\hat{a}_3(t) \cdots \hat{a}_{30}(t)]'$. The control law tuning parameters have the same values as those used in the previous formulation. The back-off parameter is taken to be $\alpha = 0.0011$. The resulting quadratic program is solved using the MATLAB subroutine QuadProg. The control

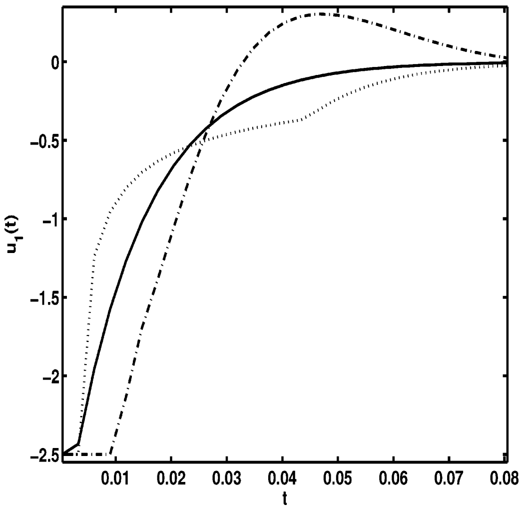

Figure 3. Manipulated input profiles for the second control actuator applied at $z_{a_1} = \pi/3$ under the MPC formulation of eqs 57 and 58 (solid line), under the MPC formulation of eqs 59 and 60 with 30 measurement points (dotted line), and under the MPC formulation of eqs 59 and 60 with 5 measurement points (dashed−dotted line).

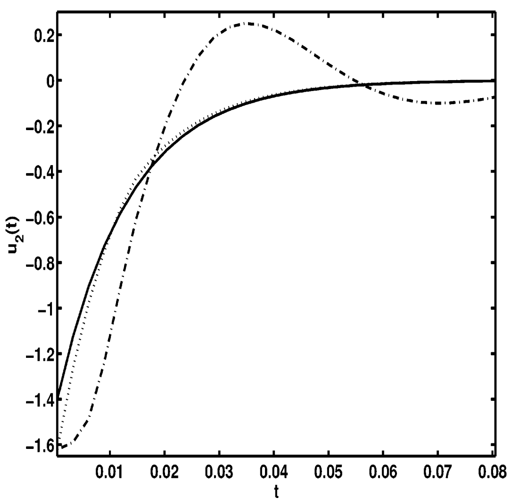

Figure 4. Manipulated input profiles for the second control actuator applied at $z_{a_2} = 2\pi/3$ under the MPC formulation of eqs 57 and 58 (solid line), under the MPC formulation of eqs 59 and 60 with 30 measurement points (dotted line), and under the MPC formulation of eqs 59 and 60 with 5 measurement points (dashed−dotted line).

action is then implemented on the 35th-order model of eq 56. Figure 5, Figure 2, and Figures 3 and 4 (dotted lines) show the closed-loop state, the closed-loop profile at the point where the state constraint is enforced, and manipulated input profiles, respectively, under the MPC controller of eqs 59 and 60, using 30 measurements. It can be seen that closed-loop stability and PDE state constraint satisfaction is achieved. Finally, to demonstrate that, in addition to including fast states in the predictive controller, we need to use a sufficiently large number of measurements to achieve state constraint satisfaction, we implemented the predictive controller of eqs 59 and 60 under the use of only 5 point measurements. In this case, the $\hat{a}_s(t) = [\hat{a}_1(t)\ \hat{a}_2(t)]'$ and $\hat{a}_{f1}(t) = [\hat{a}_3(t) \cdots \hat{a}_{30}(t)]'$ states were computed as

$$\hat{a}_i(t) = \frac{1}{\pi}\sum_{j=1}^{5} \bar{x}(z_j, t)\phi_i(z_j)\Delta z \qquad (\text{for } i = 1, 2, ..., 30) \tag{61}$$

where $\Delta z = \pi/5$ and $z_j = j(\pi/6)$ (for $j = 1, ..., 5$). The dashed−dotted lines in Figures 2, 3, and 4, and Figure 6, show that,

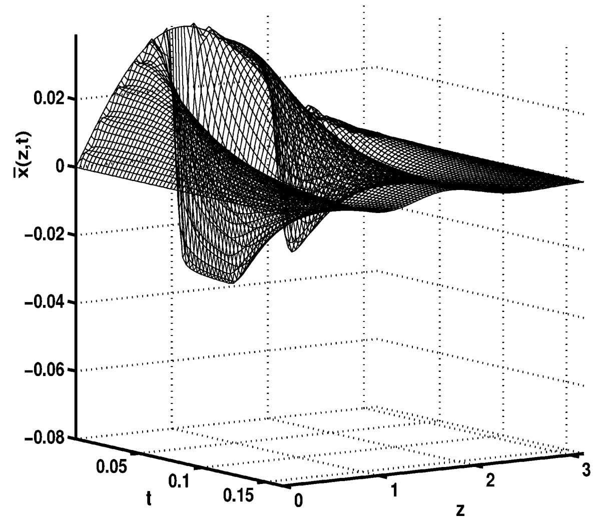

Figure 5. Closed-loop state profile under the MPC formulation of eqs 59 and 60, accounting for the fast states in the state constraints, using 30 measurements.

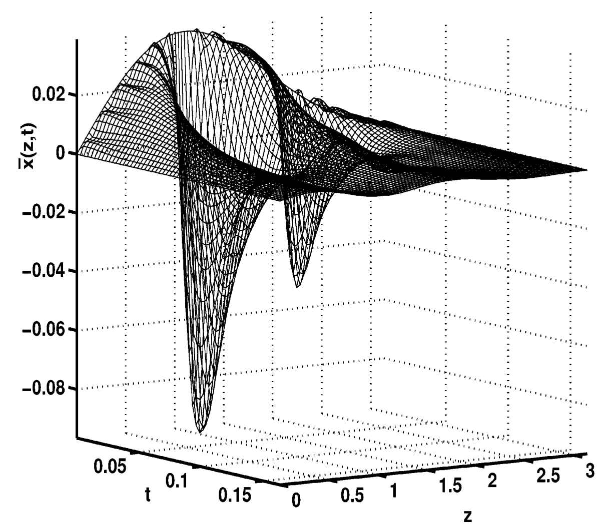

Figure 6. Closed-loop state profile under the MPC formulation of eqs 59 and 60, accounting for the fast states in the state constraints, using 5 measurements.

even though fast states estimates are used in the controller formulation of eqs 59 and 60 with $\alpha = 0$, the number of measurement points is not large enough so that the predictive controller can only achieve successful stabilization of the PDE system but not PDE state constraint satisfaction.

5. Conclusions

This work considered linear parabolic partial differential equation (PDE) systems with state and control constraints and focused on the problem of designing predictive output feedback controllers that enforce stability and constraint satisfaction in the infinite-dimensional closed-loop system. Under the assumption that a finite, yet sufficiently large, number of output measurements is available, two predictive output feedback controllers were constructed and sufficient conditions were derived under which these controllers enforce stability and constraint satisfaction in the infinite-dimensional closed-loop system, respectively. The performance of the proposed controllers was successfully tested using a linear parabolic PDE example.

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