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Distributed nonlinear control of diffusion–reaction processes

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SUMMARY

In this work, we focus on distributed control of quasi-linear parabolic partial differential equations (PDEs) and address the problem of enforcing a prespecified spatio-temporal behaviour in the closed-loop system using nonlinear feedback control and a sufficiently large number of actuators and sensors. Under the assumption that the desired spatio-temporal behaviour is described by a 'target parabolic PDE', we use a combination of Galerkin's method and nonlinear control techniques to design nonlinear state and static output feedback controllers to address this problem. We use examples of diffusion–reaction processes to demonstrate the formulation of the control problem and the effectiveness of our systematic approach to creating prespecified spatio-temporal behaviour. Using these illustrative examples, we demonstrate that both (a) a sufficiently large number of actuators/sensors, and (b) nonlinear control laws are needed to achieve this goal. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: distributed parameter systems; target parabolic PDE; Galerkin's method; nonlinear control; diffusion–reaction processes

INTRODUCTION

Parabolic PDE systems arise naturally in the modelling of transport-reaction processes in finite spatial domains and involve spatial differential operators whose spectrum can be partitioned into a finite (possibly unstable) slow part and an infinite stable fast complement. Therefore, the traditional approach to the control of parabolic PDEs involves the application of spatial discretization techniques (predominantly Galerkin's method) to the PDE system to derive large systems of ordinary differential equations (ODEs) that accurately describe the dynamics of the dominant (slow) modes of the PDE system. These are subsequently used as the basis for the synthesis of finite-dimensional controllers (e.g. [1–3]). A potential drawback of this approach is

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that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to complex controller design and high dimensionality of the resulting controllers.

Motivated by this, recent work has focused on the synthesis of linear and nonlinear low-order controllers on the basis of ODE models obtained through Galerkin's method with data-based construction of the basis functions (empirical eigenfunctions) and combination of Galerkin's method with techniques for the construction of approximate inertial manifolds (see, for example, [4–8] and the recent book [9] for results in this area and references). In addition to this work, other notable advances in control of PDE systems have been made, including controller design based on the infinite-dimensional system and subsequent use of approximation theory to design and compute low-order finite-dimensional compensators [10], results on distributed control using generalized invariants [11] and concepts from passivity and thermodynamics [12], and boundary backstepping control of a class of parabolic PDE systems [13].

While the above efforts have led to the development of a number of systematic approaches for distributed controller design, an underlying theme of the available approaches is to achieve stabilization of the process at a (possibly open-loop unstable) spatially non-uniform profile by using a finite (typically small) number of measurement sensors and control actuators, that are distributed along the spatial extent of the process. Significant recent advances in actuation and sensing technology make possible the use of large numbers of actuators and sensors to control spatially distributed processes. Examples include the manufacturing of arrays of micro-actuators/sensors for flow control (see, for example, the review article [14]) and the recent development of computer-controlled focused laser beams for temperature control of catalytic surfaces [15–17] (Figure 1). The advances in distributed actuation and sensing have motivated research on linear distributed control of linear spatially invariant distributed parameter systems (e.g. Reference [18]) and the discussion of the concept of inducing complex behaviour in spatially distributed systems using feedback (e.g. [19]). However, a systematic procedure for the design of nonlinear feedback laws that employ a large number of actuators and sensors to create a prespecified, yet complex, spatio-temporal behaviour is currently not available.

Motivated by the possibility of using such finely spatially resolved actuation/sensing, we focus on distributed control of quasi-linear parabolic PDEs and consider the problem of enforcing a

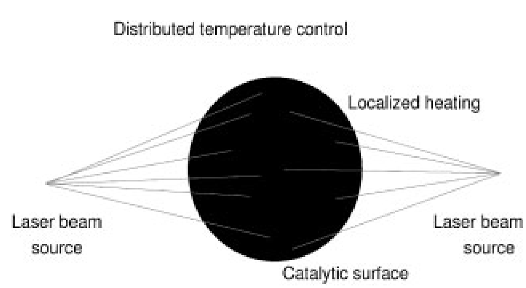

Figure 1. Example of distributed actuation for diffusion–reaction processes using a large number of computer-controlled focused laser beams.

prespecified spatio-temporal behaviour in the closed-loop system using distributed nonlinear feedback control and a large number of actuators and sensors. Our control objectives go beyond the stabilization of unstable profiles to feedback laws that completely change the nonlinear vector field of the underlying PDE model when the loop is closed. Under the assumption that the desired spatio-temporal behaviour is described by a 'target parabolic PDE', we use a combination of Galerkin's method and nonlinear control techniques to design nonlinear state and static output feedback controllers to address this problem. To simplify our development, we consider 'target parabolic PDEs' in which the structure of the spatial differential operator and boundary conditions is identical to the one of the original PDE model. We use examples of diffusion–reaction processes to demonstrate the formulation of the control problem and the effectiveness of the proposed systematic approach to creating prespecified spatio-temporal behaviour. Using these illustrative examples, we demonstrate that both (a) a sufficiently large number of actuators/sensors, and (b) nonlinear control laws are needed to achieve this goal.

PRELIMINARIES

In this work, we focus on quasi-linear parabolic PDE systems with the following state–space description:

$$\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= B\frac{\partial^2 \bar{x}}{\partial z^2} + w\sum_{i=1}^{m} b_i(z)u_i + f(\bar{x}) \\ y_m^{\kappa} &= \int_0^{\pi} s^{\kappa}(z)\omega\bar{x}(z,t)\,\mathrm{d}z, \quad \kappa = 1,\ldots,p \end{aligned} \tag{1}$$

subject to the boundary conditions

$$\bar{x}(0,t) = 0, \quad \bar{x}(\pi,t) = 0 \tag{2}$$

and the initial condition:

$$\bar{x}(z,0) = \bar{x}_0(z) \tag{3}$$

where $\bar{x}(z,t) = [\bar{x}_1(z,t)\cdots\bar{x}_n(z,t)]^{\mathrm{T}} \in \mathbb{R}^n$ denotes the vector of state variables, $z \in [0,\pi] \subset \mathbb{R}$ is the spatial co-ordinate, $t \in [0,\infty)$ is the time, $u_i \in \mathbb{R}$ denotes the ith manipulated input, and $y_m^{\kappa} \in \mathbb{R}$ denotes the κth measured output. $\partial^2\bar{x}/\partial z^2$ denotes the second-order spatial derivative of $\bar{x}$, $f(\bar{x})$ is a sufficiently smooth nonlinear vector function, w, ω are constant vectors, B is a positive-definite, constant matrix, and $\bar{x}_0(z)$ is the initial condition. $b_i(z)$ is a known smooth function of z which describes how the control action $u_i(t)$ is distributed in the finite interval $[0,\pi]$, and $s^{\kappa}(z)$ is a known smooth function of z which depends on the shape (point or distributed sensing) of the κth measurement sensors in the interval $[0,\pi]$. Whenever the control action enters the system at a single point z_0, with $z_0 \in [0,\pi]$ (i.e. point actuation), the function $b_i(z)$ is taken to be non-zero in a finite spatial interval of the form $[z_0-\mu, z_0+\mu]$, where μ is a small positive real number, and zero elsewhere in $[0,\pi]$.

To present the controller design method that we follow for enforcing complex spatio-temporal behaviour in the closed-loop system, we formulate Equation (1) as an infinite-dimensional system in the Hilbert space $\mathscr{H}([0,\pi];\mathbb{R}^n)$, with $\mathscr{H}$ being the space of measurable functions defined on $[0, \pi]$, with inner product and norm

$$(\omega_1, \omega_2) = \int_0^{\pi} (\omega_1(z), \omega_2(z))_{\mathbb{R}^n}\, \mathrm{d}z, \quad \|\omega_1\|_2 = (\omega_1, \omega_1)^{1/2} \tag{4}$$

where ω_1, ω_2 are two elements of $\mathscr{H}([0, \pi]; \mathbb{R}^n)$ and the notation $(\cdot, \cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function x on $\mathscr{H}([0, \pi]; \mathbb{R}^n)$ as

$$x(t) = \bar{x}(z, t), \quad t > 0, \ z \in [0, \pi] \tag{5}$$

the operator $\mathscr{A}$ in $\mathscr{H}([0, \pi]; \mathbb{R}^n)$ as

$$\begin{aligned} \mathscr{A}x = B\frac{\partial^2 \bar{x}}{\partial z^2}, \quad & x \in D(\mathscr{A}) = \{x \in \mathscr{H}([0, \pi]; \mathbb{R}^n) \\ \bar{x}(0, t) = 0, \quad & \bar{x}(\pi, t) = 0\} \end{aligned} \tag{6}$$

and the input and measured output operators as

$$\mathscr{B}u = \sum_{i=1}^{m} b_i u_i, \quad \mathscr{S}x = (s, x) \tag{7}$$

the system of Equations (1)–(3) takes the form

$$\begin{aligned} \dot{x} &= \mathscr{A}x + \mathscr{B}u + f(x), \quad x(0) = x_0 \\ y_m &= \mathscr{S}x \end{aligned} \tag{8}$$

where $f(x(t)) = f(\bar{x})$ and $x_0 = x_0(z)$.

PROBLEM FORMULATION AND SOLUTION METHOD

To formulate the control problem, we assume that the target spatio-temporal profile, to be enforced in the closed-loop system, can be accurately captured by a 'target parabolic PDE' system subject to boundary and initial conditions of the following form:

$$\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= \bar{B}\frac{\partial^2 \bar{x}}{\partial z^2} + \hat{f}(\bar{x}, z, t) \\ \bar{x}(0, t) = 0, \quad & \bar{x}(\pi, t) = 0, \quad \bar{x}(z, 0) = \bar{x}_0(z) \end{aligned} \tag{9}$$

where $\hat{f}(\bar{x}, z, t)$ is a nonlinear vector field and $\bar{B}$ is a positive-definite, constant matrix. It is assumed that $\hat{f}(\bar{x}, z, t)$ is a sufficiently smooth nonlinear vector function and satisfies $\hat{f}(0, 0, 0) = 0$. To simplify our development, the initial conditions of the original and 'target PDE' are chosen to be the same.

The control problem is formulated as the one of determining a sufficient number of control actuators and measurements sensors and computing the corresponding explicit form of the control law that make the closed-loop system to be close (with respect to an appropriate norm) to the target spatio-temporal profile (Equation (9)).

To address this problem, we employ the following methodology:

- Initially, Galerkin's method is used to derive nonlinear finite-dimensional approximations of the original (Equation (1)) and target (Equation (9)) parabolic PDE systems. The order of both finite-dimensional systems should be sufficiently high to ensure that the behaviour of the original PDE system and of the 'target PDE system' can be accurately captured by their finite-dimensional approximations (this is particularly important when the original and target PDEs involve complex spatio-temporal behaviour and can be established by careful numerical simulation). Furthermore, to simplify the controller design, the order of both finite-dimensional systems is chosen here to be the same and equal to the smallest order needed to derive finite-dimensional approximations for both the original and target PDE systems that yield the desired accuracy.

- Once the order of the finite-dimensional approximations is fixed, the number of control actuators and measurement sensors is chosen to be equal to the order of the discretization (this requirement simplifies the controller design and can be relaxed), and nonlinear inversion-based control techniques are used to design nonlinear state and output feedback control laws that enforce the target behaviour in the closed-loop finite-dimensional system. The design of the output feedback control laws involves a procedure proposed in [20] to obtain estimates for the states of the finite-dimensional system from the measurements.
- Finally, the closed-loop PDE system is analysed to confirm that the target behaviour is enforced in the infinite-dimensional system under appropriate convergence conditions of the finite-dimensional systems.

CONTROLLER DESIGN METHOD

Galerkin's method

We apply Galerkin's method to the system of Equation (8) to derive an approximate finite-dimensional system. Let $\mathscr{H}_s$, $\mathscr{H}_f$ be modal subspaces of $\mathscr{A}$, defined as $\mathscr{H}_s = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathscr{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \ldots,\}$ (the existence of $\mathscr{H}_s$, $\mathscr{H}_f$ follows from the properties of $\mathscr{A}$ and ϕ_j are eigenfunctions of $\mathscr{A}$). Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Equation (8) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x \tag{10}$$

Applying P_s and P_f to the system of Equation (8) and using the above decomposition for x, the system of Equation (8) can be equivalently written in the following form:

$$\begin{aligned}
\frac{\mathrm{d}x_s}{\mathrm{d}t} &= \mathscr{A}_s x_s + \mathscr{B}_s u + f_s(x_s, x_f) \\
\frac{\partial x_f}{\partial t} &= \mathscr{A}_f x_f + \mathscr{B}_f u + f_f(x_s, x_f) \\
y_m &= \mathscr{S} x_s + \mathscr{S} x_f
\end{aligned} \tag{11}$$

$$x_s(0) = P_s x(0) = P_s x_0, \quad x_f(0) = P_f x(0) = P_f x_0$$

where $\mathscr{A}_s = P_s\mathscr{A}$, $\mathscr{B}_s = P_s\mathscr{B}$, $f_s = P_s f$, $\mathscr{A}_f = P_f\mathscr{A}$, $\mathscr{B}_f = P_f\mathscr{B}$ and $f_f = P_f f$ and the notation $\partial x_f/\partial t$ is used to denote that the state x_f belongs in an infinite-dimensional space. Note that due to the choice of $\mathscr{H}_s$ and $\mathscr{H}_f$ to be modal subspaces of $\mathscr{A}$, $P_s\mathscr{A}x_f = 0$ and $P_f\mathscr{A}x_s = 0$. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = \text{diag}\{\lambda_j\}$ (λ_j are eigenvalues of $\mathcal{A}_s$), $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are Lipschitz vector functions, and $\mathcal{A}_f$ is an unbounded differential operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathcal{H}_s, \mathcal{H}_f$). Neglecting the fast and stable infinite-dimensional x_f-subsystem in the system of Equation (11), the following m-dimensional slow system is obtained:

$$\frac{\mathrm{d}\tilde{x}_s}{\mathrm{d}t} = \mathcal{A}_s\tilde{x}_s + \mathcal{B}_s u + f_s(\tilde{x}_s, 0)$$
$$\tilde{y}_m = \mathcal{S}\tilde{x}_s \tag{12}$$

where the bar symbol in $\tilde{x}_s$ and $\tilde{y}_m$ denotes that these variables are associated with a finite-dimensional system.

In our development, we will also need to use a finite-dimensional approximation of the target parabolic PDE system of Equation (9). To this end, Equation (9) is formulated as an infinite-dimensional system in the Hilbert space $\mathcal{H}([0, \pi]; \mathbb{R})$ and Galerkin's method is used to obtain the following target finite-dimensional system:

$$\frac{\mathrm{d}\tilde{x}_s}{\mathrm{d}t} = \hat{\mathcal{A}}_s\tilde{x}_s + \hat{f}_s(\tilde{x}_s, 0, t) \tag{13}$$

Referring to the finite-dimensional approximations of Equations (12)–(13), we make the following assumptions: (a) both approximations converge uniformly to the solution of their corresponding infinite-dimensional system as $m \to \infty$, and (b) the orders of both approximations are taken to be the same (equal to m) and sufficiently high to satisfy the desired accuracy with which the closed-loop behaviour needs to be enforced. From these assumptions, the first one is standard and holds for most parabolic PDEs arising in the modelling of diffusion–convection–reaction processes (see also the example in Section 4). The second assumption is made to simplify the controller design formulas and achieve the desired control objective, and it does not pose any limitations on the class of parabolic PDE systems for which the proposed method can be applied.

Nonlinear state feedback control

In this section, we consider the use of point control actuators and assume that measurements of the states of the system of Equation (12) are available. We first address the problem of synthesizing nonlinear static state feedback control laws of the general form

$$u = \mathcal{F}(\tilde{x}_s) \tag{14}$$

where $\mathcal{F}(\tilde{x}_s)$ is a nonlinear vector function that force the closed-loop finite-dimensional system to be identical to the target finite-dimensional system of Equation (13). To address this problem and simplify our development, we need to impose the following assumption.

Assumption 1

$l = m$ (i.e., the number of control actuators is equal to the order of the finite-dimensional approximations), and the inverse of the matrix $\mathcal{B}_s$ exists.

The requirement $l = m$ is sufficient and not necessary, and it is made to simplify the synthesis of the controller (see also discussion in Remark 2 below).

Proposition 1 that follows provides the explicit formula for the state feedback controller that achieves the control objective.

Proposition 1

Consider the finite-dimensional system of Equation (12) for which Assumption 1 holds. Then, under the state feedback controller:

$$u = \mathscr{B}_s^{-1}((\hat{\mathscr{A}}_s - \mathscr{A}_s)\tilde{x}_s + \hat{f}_s(\tilde{x}_s, 0, t) - f_s(\tilde{x}_s, 0)) \tag{15}$$

the closed-loop finite-dimensional system is identical to the one of Equation (13).

Remark 1

The proof of Proposition 1 can be obtained by substituting the controller of Equation (15) in Equation (12) and performing several algebraic manipulations to show that the resulting closed-loop finite-dimensional system is identical to the one of Equation (13).

Remark 2

It is important to note that the requirements $l = m$ and existence of $\mathscr{B}_s^{-1}$ are sufficient to design a state feedback law that enforces the behaviour of Equation (13) in the closed-loop finite-dimensional system. To ensure (approximate) controllability and invertibility of the matrix $\mathscr{B}_s$, we impose that $\text{rank}(\mathscr{B}_s) = m$ by appropriate selection of actuator locations z_j, this requirement excludes several point actuator locations for which rank $\mathscr{B}_s < m$; see [21] for discussion on this issue. For certain classes of systems, it may be possible to use co-ordinate changes and nonlinear feedback to achieve the desired control objective with a smaller number of control actuators.

Output feedback control

The nonlinear controller of Equation (15) was derived under the assumption that measurements of the states $\tilde{x}_s$ are available, which implies that measurements of the state variable, $x(z, t)$, are available at all positions and times. However, from a practical point of view, measurements of the state variables are only available at a finite number of spatial positions. Motivated by this, we address in this section the synthesis of nonlinear output feedback controllers that use measurements of the process outputs, y_m, that make the closed-loop finite-dimensional system to be identical to the 'target' finite-dimensional system of Equation (13).

Specifically, we consider output feedback control laws of the general form

$$u = \mathscr{F}(y_m) \tag{16}$$

where $\mathscr{F}(y_m)$ is a nonlinear vector function and y_m is the vector of measured outputs. The synthesis of the controller of Equation (16) will be achieved by combining the state feedback controller of Equation (15) with a procedure proposed in Reference [20] for obtaining estimates for the states of the approximate ODE model of Equation (12) from the measurements. To this end, we need to impose the following requirement on the number of measured outputs in order to obtain estimates of the states x_s of the finite-dimensional system of Equation (12), from the measurements y_m^κ, $\kappa = 1, \ldots, p$.

Assumption 2
$p = m$ (i.e., the number of measurements is equal to the order of the finite-dimensional approximations), and $\mathscr{S}^{-1}$ exists, so that $\hat{x}_s = \mathscr{S}^{-1} y_m$.

We note that the requirement that the inverse of the operator S exists can be achieved by appropriate choice of the location of the measurement sensors (i.e. functions $s^{\kappa}(z)$). When point measurement sensors are used, this requirement can be verified by checking the invertibility of a matrix (see example in the next section).

Theorem 1 presents the proposed output feedback controller which enforces the target behaviour in the closed-loop system.

Theorem 1
Consider the system of Equation (11), and the finite-dimensional system of Equation (12), for which Assumptions 1 and 2 hold, under the nonlinear output feedback controller:

$$\begin{aligned} \hat{x}_s &= \mathscr{S}^{-1} y_m \\ u &= \mathscr{B}_s^{-1}((\hat{\mathscr{A}}_s - \mathscr{A}_s)\hat{x}_s + \hat{f}_s(\hat{x}_s, 0, t) - f_s(\hat{x}_s, 0)) \end{aligned} \tag{17}$$

Then, the closed-loop infinite-dimensional system under the output feedback controller (Equations (1)–(17)) converges to the 'target' parabolic PDE system of Equation (9) as $m \to \infty$.

Proof
Under the output feedback controller of Equation (17), the infinite-dimensional closed-loop system takes the following form:

$$\begin{aligned} \hat{x}_s &= \mathscr{S}^{-1} y_m \\ \dot{x} &= \mathscr{A}x + f(x) + \mathscr{B}\mathscr{B}_s^{-1}((\hat{\mathscr{A}}_s - \mathscr{A}_s)\hat{x}_s + \hat{f}_s(\hat{x}_s, 0, t) - f_s(\hat{x}_s, 0)) \\ x(0) &= x_0 \\ y_m &= \mathscr{S}x \end{aligned} \tag{18}$$

or equivalently

$$\begin{aligned} \hat{x}_s &= \mathscr{S}^{-1}\mathscr{S}x \\ \dot{x} &= \mathscr{A}x + f(x) + \mathscr{B}\mathscr{B}_s^{-1}((\hat{\mathscr{A}}_s - \mathscr{A}_s)\hat{x}_s + \hat{f}_s(\hat{x}_s, 0, t) - f_s(\hat{x}_s, 0)), \\ x(0) &= x_0 \end{aligned} \tag{19}$$

Taking the limit as $m \to \infty$, we have that $\hat{x}_s \to x$, $\mathscr{B}_s \to \mathscr{B}$, $\hat{\mathscr{A}}_s \to \hat{\mathscr{A}}$, $\mathscr{A}_s \to \mathscr{A}$, $\hat{f}_s \to \hat{f}$ and $f_s \to f$, and thus Equation (19) takes the form

$$\begin{aligned} \dot{x} &= \mathscr{A}x + f(x) + \mathscr{B}\mathscr{B}^{-1}((\hat{\mathscr{A}} - \mathscr{A})x_s + \hat{f}(x, t) - f(x)) \\ x(0) &= x_0 \end{aligned} \tag{20}$$

or after some algebraic manipulations

$$\dot{x} = \hat{\mathscr{A}}x + \hat{f}(x, t), \quad x(0) = x_0 \tag{21}$$

which is the infinite-dimensional representation of the 'target' parabolic PDE system of Equation (19); this completes the proof. □

Remark 3

Even though static output feedback is more sensitive to measurement noise than dynamic output feedback, we prefer to use static feedback of y_m in the controller of Equation (17) because the use of a state observer to obtain estimates of the finite-dimensional state variables would significantly increase the computational demand for the computation of the control action, thereby impacting on the practical applicability of the proposed method. Note that in such a formulation, an mth order nonlinear ODE system (with m potentially a large number depending on the complexity of the spatio-temporal profile to be enforced in the closed-loop system) would have to be integrated in real time in order to compute the control action.

Remark 4

To understand the essence of the result of Theorem 1 and the structure of the controller, it is useful to put the control problem in question into perspective with respect to existing controller design methods whose objective is to make the process output to track a reference signal produced by an exogenous system. Under state feedback control, this problem is known as the state feedback regulator problem and was solved by Francis [22] for linear multivariable finite-dimensional systems. When only error measurements are available, this problem is known as the error (output) feedback regulator problem for which it was shown, in the context of linear finite-dimensional systems, by Francis and Wonham [23] that any controller (regulator) that solves this problem has to incorporate a model of the exogenous system generating the reference signal which is to be tracked (a property known as the internal model principle). More recently, the complete solution to the error (output) regulation problem for linear distributed parameter systems with bounded input/output operators was presented [24] and was shown that the resulting regulator incorporates a model of the exogenous system generating the tracking signal. With this in mind, it is easy to see why the controller of Equation (15) incorporates the entire vector field of the finite-dimensional approximation of the 'target parabolic PDE' (which captures the desired closed-loop behaviour); in this sense, the controller of Equation (15) obeys the internal model principle. Our result also shows that it is possible by using a large number of control actuators and sensors to force not only the output (tracking error in the regulation problem) but the entire distributed state to behave according to a 'target' behaviour; this is not unexpected since a large number of actuators and sensors provides large number of degrees of freedom to shape the process dynamics according to a known 'target PDE system'. We also note that the notion of 'target PDE system' was recently used in [13] in the context of backstepping boundary control of parabolic PDEs with the objective of stabilization.

Remark 5

Referring to the requirement that the initial condition of the original (Equation (1)) and 'target' (Equation (9)) parabolic PDEs are identical, it is important to note that this is sufficient to ensure that the 'target' spatio-temporal profile is enforced in the closed-loop system as $m \to \infty$ for all times. This requirement can be relaxed for initial conditions $\hat{x}_0(z)$ for which the solution of the ‘target’ parabolic PDE asymptotically converges to the target spatiotemporal profile. This can be seen from the proof of Theorem 1 since for $m \to \infty$, the resulting closed-loop system would take the form of Equation (21) with $x(0) = \hat{x}_0 \neq x_0$. So for $\hat{x}_0$, for which the solution of the ‘target’ parabolic PDE converges asymptotically ($t \to \infty$) to the ‘target’ spatio-temporal profile, the desired behaviour can be asymptotically ($t \to \infty$) enforced in the closed-loop system, provided that a sufficiently large number of control actuators and measurement sensors is employed.

SIMULATION STUDIES

Application to a 1D diffusion–reaction process

In this subsection, the proposed control methodology is applied to a one-dimensional diffusion–reaction process example to enforce desired spatio-temporal behaviour in the closed-loop system. The control is assumed to be implemented on the process by using spatially distributed arrays consisting of large numbers of control actuators and measurement sensors, which are placed at equidistant positions. Throughout the example, we consider point actuation and point sensing.

Specifically, we consider a diffusion–reaction process described by a quasi-linear parabolic PDE of the form

$$\frac{\partial \bar{x}}{\partial t} = \mathscr{D}\frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T e^{-\gamma/(1+\bar{x})} + \beta_U(b(z)u(t) - \bar{x}) - \beta_T e^{-\gamma} \tag{22}$$

subject to the boundary and initial conditions

$$\bar{x}(0,t) = 0, \quad \bar{x}(\pi,t) = 0, \quad \bar{x}(z,0) = \bar{x}_0(z) \tag{23}$$

where $\bar{x}$ denotes the state of the process, β_T denotes a dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, $\mathscr{D}$ denotes a dimensionless diffusion coefficient, $u(t)$ denotes the vector of manipulated inputs (control actions) and $b(z)$ is the vector of the actuator distribution functions. The following typical values are given to the process parameters: $\beta_T = 50.0$, $\beta_U = 2.0$, $\gamma = 4.0$, and $\mathscr{D} = 1$. For these, it was verified that the spatially uniform steady state $\bar{x}(z,t) = 0$ is an unstable one. Specifically, the linearization of the PDE of Equation (22) around $\bar{x}(z,t) = 0$ has the following form:

$$\frac{\partial \bar{x}}{\partial t} = \mathscr{D}\frac{\partial^2 \bar{x}}{\partial z^2} + (\beta_T e^{-\gamma}\gamma - \beta_U)\bar{x} + \beta_U b(z)u(t) \tag{24}$$

To demonstrate the applicability and investigate the effect of various process and controller parameters on the ability of the proposed method to enforce desired spatio-temporal behaviour in the closed-loop system, we will consider a set of three ‘target parabolic PDEs’. Note that in an engineering application, the desired closed-loop behaviour should be obtained from the solution of an optimization problem whose objective is to maximize process performance; in such a case the ‘target PDE’ can be subsequently constructed using a closed-form approximation of the optimal spatio-temporal profile. Specifically, we consider the following 'target parabolic PDEs': (1) a linear parabolic PDE with spatially dependent additive term of the form

$$\frac{\partial \bar{x}}{\partial t} = b\frac{\partial^2 \bar{x}}{\partial z^2} - 4\sin(2z)$$
$$\bar{x}(0,t) = 0, \quad \bar{x}(\pi,t) = 0, \quad \bar{x}(z,0) = \bar{x}_0(z) \tag{25}$$

where b is a positive constant parameter, with the value $b = 1.5$, (2) the so-called Chafee–Infante diffusion–reaction equation:

$$\frac{\partial \bar{x}}{\partial t} = b\frac{\partial^2 \bar{x}}{\partial z^2} + \alpha\bar{x} - \beta\bar{x}^3$$
$$\bar{x}(0,t) = 0, \quad \bar{x}(\pi,t) = 0, \quad \bar{x}(z,0) = \bar{x}_0(z) \tag{26}$$

where α, β and b are constant positive parameters, with the following values $\alpha = 2$, $\beta = 2$ and $b = 1.5$, and (3) a nonlinear time-varying parabolic PDE of the following form:

$$\frac{\partial \bar{x}}{\partial t} = b\frac{\partial^2 \bar{x}}{\partial z^2} + \hat{f}(\bar{x}, t) \tag{27}$$

where $\hat{f}(\bar{x}, t) = \alpha\bar{x} - \beta\bar{x}^3 + 2.5\sin(0.5t)\beta_T e^{-\gamma/(1+\bar{x})}$, and α, β, β_T, γ are constant positive parameters with the following values $\alpha = 2$, $\beta = 2$, $\beta_T = 50.0$ and $\gamma = 4.0$.

In the first set of simulation runs, we consider the linear PDE of Equation (24) (which results from the linearization of Equation (22) around $\bar{x}(z, t) = 0$, and thus, it has an unstable solution) and focus on linear state feedback control. The 'target PDE' is Equation (25) with $b = 1.5$. The reason for considering the linear PDE of Equation (24) as the starting point is to show that the desire to enforce a spatially varying profile necessitates the use of a large number of control actuators and cannot be enforced, with the desired accuracy, when a restricted number of control actuators is available. A 40th-order Galerkin discretization of Equation (25) is computed and used in the simulation (higher-order discretizations led to identical results). Figure 2 shows the target spatio-temporal behaviour described by Equation (25) with $b = 1.5$.

The control problem is to compute the sufficient number of actuators and the corresponding state feedback law so that the solution of a 40th-order discretization of the closed-loop PDE system (linear PDE of Equation (24) under the state feedback law) is very close to the solution

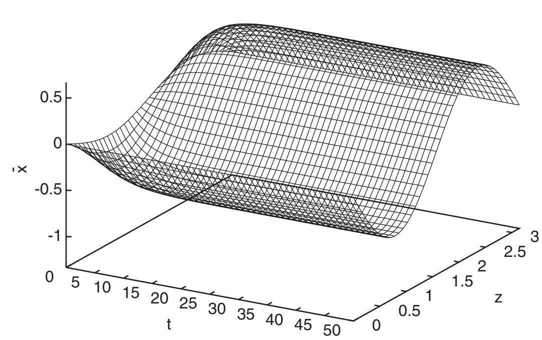

Figure 2. Spatio-temporal profile of linear parabolic PDE of Equation (25) with $b = 1.5$.

of the 40th-order discretization of the 'target PDE' of Equation (25). The closeness of the two solutions is achieved after a short time needed for the fast transients associated with the modes of the high-order discretization, which are not included in the model used for controller design, to die out (note that this is possible because the initial conditions of the original and 'target PDE' are chosen to be the same). To solve this problem, we derive several finite-dimensional approximations of Equation (24) and use them to design state feedback laws that make the vector fields of the corresponding closed-loop systems to be identical to the ones of the finite-dimensional approximations of the 'target PDE' of Equation (25) of the same dimension. At this point, it is important to note that (a) closeness of the solutions of the closed-loop system and of the target PDE can be defined by using various norms; in this work, the closeness is evaluated by checking the discrepancy between the two solutions at all positions and times, and (b) the fast decay of the modes which are not included in the model used for controller design is a consequence of the highly dissipative behaviour of parabolic PDEs.

Specifically, we construct second-, fifth- and tenth-order Galerkin approximations of the PDEs of Equations (24) and (25) and design three state feedback control laws that exactly enforce the target behaviour in the corresponding finite-dimensional closed-loop systems. Following our approach, the developed control laws require two, five and ten control actuators to be implemented. Figures 3–5 show the spatio-temporal profile of the closed-loop system (simulated by the 40th-order Galerkin approximation) in the case of using ten, five and two control actuators, respectively. The use of ten and five control actuators suffices to enforce the desired target behaviour in the closed-loop system after a short time needed for the transients associated with the higher-order modes to die out; the use of two control actuators is not adequate to enforce the desired behaviour with the desired accuracy, see Figure 6 for the profiles of the error between the solution of the 'target PDE' and the solution of the closed-loop system under ten, five and two control actuators.

In the second set of simulation runs, we consider the nonlinear parabolic PDE of Equation (22) as the starting point and the target PDE is Equation (25). The objective of this set of simulations is to study the ability of linear feedback control, which uses a large number of control actuators, to enforce (in the sense discussed above for the first set of simulation runs) the desired behaviour. A 40th-order Galerkin's discretization of Equation (25) with $b = 1.5$ is computed and used in the simulation (higher-order discretizations led to identical results).

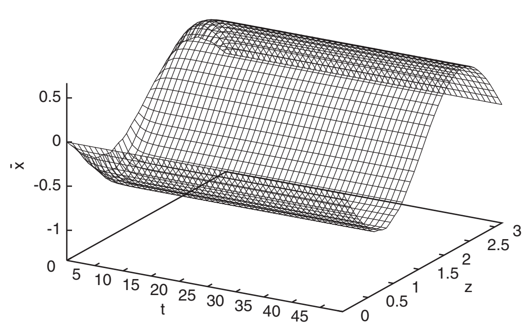

Figure 3. Closed-loop spatio-temporal profile of linearized diffusion–reaction equation (Equation (24)) under linear state feedback control with 10 equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

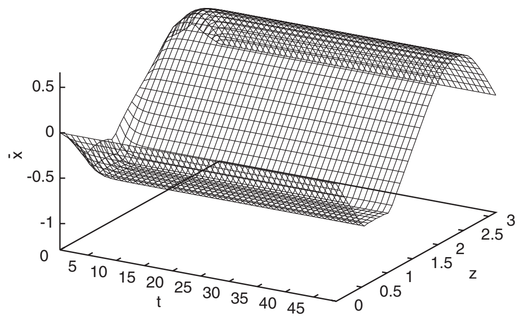

Figure 4. Closed-loop spatio-temporal profile of linearized diffusion–reaction equation (Equation (24)) under linear state feedback control with five equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

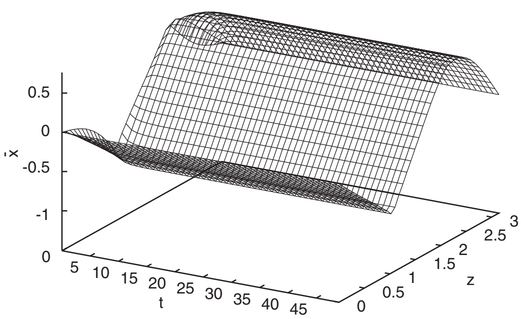

Figure 5. Closed-loop spatio-temporal profile of linearized diffusion–reaction equation (Equation (24)) under linear state feedback control with two equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

We apply the method of Section 4 to construct second order, fifth-order and twentieth-order Galerkin approximations of the PDEs of Equations (22) and (25) and to design three nonlinear state feedback control laws that exactly enforce the target behaviour to the finite-dimensional closed-loop systems. Then, the nonlinear state feedback laws are linearized around the steady state $\bar{x}(z,t) = 0$ to obtain three linear state feedback laws. The developed control laws are implemented on a 40th-order Galerkin approximation of the nonlinear parabolic PDE of Equation (22) using two, five and 20 control actuators, respectively. Figures 7–9 show the spatio-temporal profile of the closed-loop system in the case of using 20, five and two control actuators, respectively. It is clear that linear control cannot satisfactorily compensate for the presence of nonlinear terms in Equation (22), and thus, the behaviour of Equation (25) cannot be enforced in the closed-loop system. We note that when nonlinear control is implemented, the use of 10 control actuators suffices to enforce the desired closed-loop behaviour. This set of simulations demonstrates that linear feedback control may not be adequate to enforce a desired behaviour in a quasi-linear parabolic PDE system (even though a large number of control actuators is used).

In the third set of simulation runs, we consider the nonlinear parabolic PDE of Equation (22) as the starting point and the target PDE is the Chafee–Infante diffusion–reaction equation of

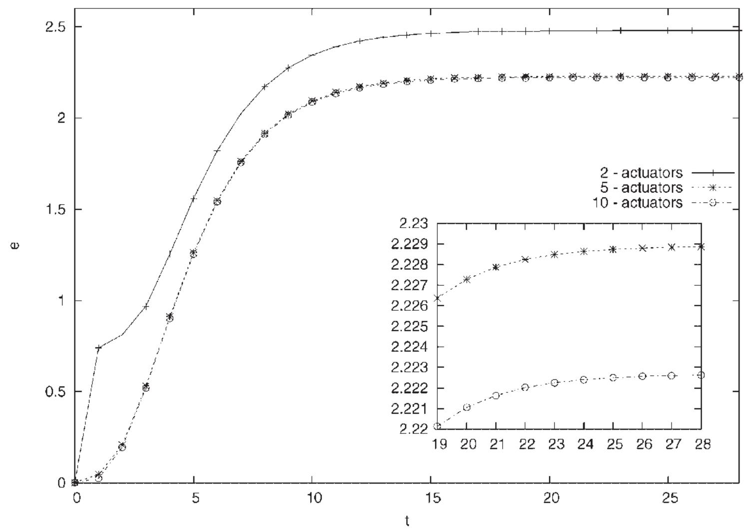

Figure 6. Time profiles of the error $e_{(t)}$ is defined as $e_{(t)} = \int_0^{\pi} (\bar{x} - x_{\text{target}})^2 \, dz$, between the solution of the closed-loop system under 2, 5 and 10 control actuators and the solution of the target PDE-original PDE-linearized diffusion–reaction equation (Equation (24))—target behaviour—linear parabolic PDE of Equation (25) with $b = 1.5$.

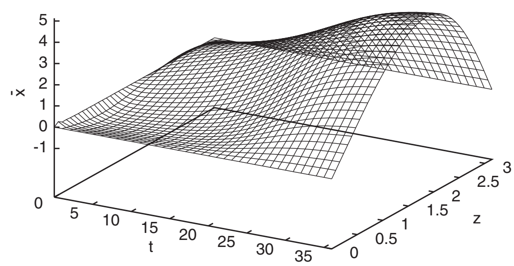

Figure 7. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under linear state feedback control with 20 equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

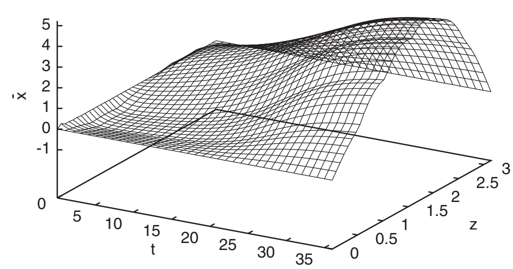

Figure 8. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under linear state feedback control with five equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

Equation (26). The objective of this set of simulations is to show that it is possible to enforce (in the sense discussed above for the first set of simulations) a desired behaviour in the closed loop which is described by a 'target nonlinear PDE'.

We use the methodology of Section 4 to construct second- and fifth-order Galerkin approximations of the PDEs of Equations (22) and (26) and to design two nonlinear state feedback control laws that exactly enforce the target behaviour to the finite-dimensional closed-loop systems. The developed control laws are implemented on a 40th-order Galerkin approximation of Equation (22) using two and five control actuators, respectively. Figures 10 and 11 show the spatio-temporal profile of the closed-loop system in the case of using two and five control actuators, respectively. Clearly, the use of five control actuators and nonlinear feedback enforces (in the sense discussed above for the first set of simulation runs) the desired closed-loop behaviour.

In the fourth set of simulation runs, we consider the nonlinear parabolic PDE of Equation (22) as the starting point and the target PDE is the nonlinear time-dependent PDE of Equation (27). The objective is to show that is possible to use the proposed method to enforce

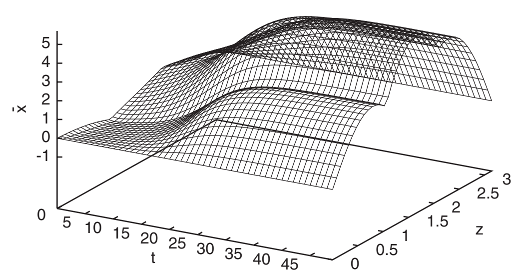

Figure 9. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under linear state feedback control with two equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

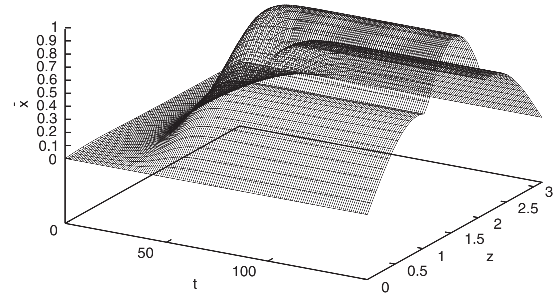

Figure 10. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear state feedback control with two equidistant control actuators—target behaviour: Chafee–Infante diffusion–reaction equation of Equation (26) with $\alpha = 2$, $b = 1.5$ and $\beta = 2$.

time-varying behaviour in the closed-loop system. We use the methodology of Section 4 to construct second- and tenth-order Galerkin approximations of the PDEs of Equations (22) and (27) and to design two nonlinear state feedback control laws that enforce the time-dependent behaviour described by the 'target' nonlinear PDE of Equation (27) in the closed-loop finite-dimensional system. The two control laws are implemented on the 40th-order Galerkin approximation of Equation (22) using two and ten control actuators, respectively. Figures 12 and 13 show the spatio-temporal profile of the closed-loop system in the case of using two and ten control actuators, respectively. It can be seen that while the use of two control actuators is not adequate to enforce the requested behaviour in the closed-loop system, the use of ten control actuators and nonlinear feedback suffices to enforce the desired time-varying behaviour in the closed-loop system (compare Figures 13 and 14).

In the fifth set of simulation runs, the objective is to investigate the effect of the coefficient of the diffusion term on the ability of the proposed method to enforce a desired spatio-temporal behaviour in the closed-loop system. We consider the nonlinear PDE of Equation (22) as the starting point and the target PDE is Equation (25). We also consider the following two values for the coefficient of the diffusion term in the original PDE (a) $\mathscr{D} = 0.4$ and (b) $\mathscr{D} = 0.06$. In the first simulation run $\mathscr{D} = 0.4$, we consider two control actuators and design a nonlinear state feedback control law; Figure 15 shows the state of the resulting closed-loop system.

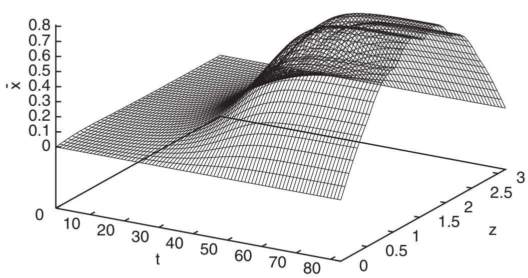

Figure 11. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear state feedback control with five equidistant control actuators—target behaviour: Chafee–Infante diffusion–reaction equation of Equation (26) with $\alpha = 2$, $b = 1.5$ and $\beta = 2$.

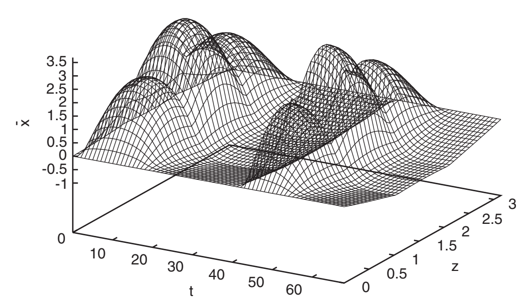

Figure 12. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear state feedback control with two equidistant control actuators—target behaviour: nonlinear parabolic time-varying PDE of Equation (27) with $\alpha = 2$, $\beta = 2$, $\beta_T = 50.0$ and $\gamma = 4.0$.

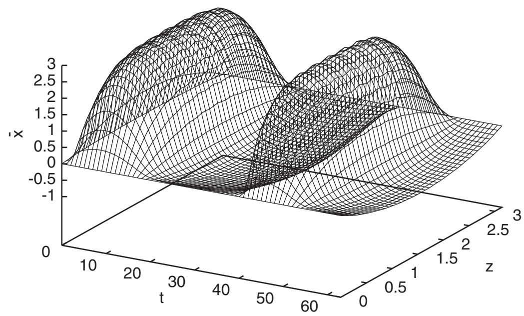

Figure 13. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear state feedback control with 10 equidistant control actuators—target behaviour: nonlinear parabolic time-varying PDE of Equation (27) with $\alpha = 2$, $\beta = 2$, $\beta_T = 50.0$ and $\gamma = 4.0$.

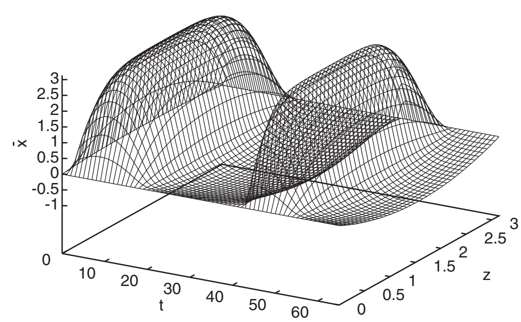

Figure 14. Evolution of the state of nonlinear time-varying parabolic PDE of Equation (27) with $\alpha = 2$, $\beta = 2$, $\beta_T = 50.0$ and $\gamma = 4.0$.

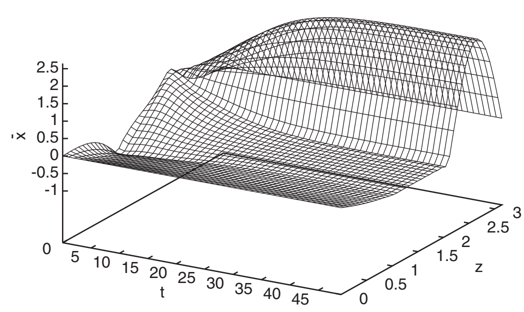

Figure 15. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) with $\mathscr{D} = 0.4$ under nonlinear state feedback control with two equidistant control actuators—linear parabolic PDE of Equation (25) with $b = 1.5$.

We can see that the use of two control actuators is not sufficient to enforce the behaviour of Equation (25) (note that when the value of diffusion coefficient $\mathscr{D} = 1$ the use of two control actuators is sufficient to achieve the same control objectives).

This result is expected since when $\mathscr{D} = 0.4$ the effect of the higher-order (residual) modes (modes which are not included in the model used for controller design) increases and leads to poor performance. Of course this effect can be suppressed by increasing the number of control actuators and number of modes used in the controller design model. To clearly demonstrate this point, we also considered the case of $\mathscr{D} = 0.06$ and used 20 control actuators and a nonlinear state feedback law to enforce the behaviour of Equation (25) in the closed-loop system (in order to preserve numerical stability, the order of the Galerkin approximation of the original PDE, where the controller is implemented is taken to be 80). Figure 16 shows the state of the closed-loop system. We can see that, even though the effect of the residual modes is stronger, the use of 20 control actuators and nonlinear feedback suffices to enforce the desirable behaviour in the closed-loop system.

In the sixth set of simulation runs, we consider relaxing the requirement that the initial conditions of the original (Equation (12)) and 'target' (Equation (13)) parabolic PDEs are identical. Specifically, the initial condition for the original PDE is taken to be $\hat{x}_0(z) = 0.9\sin(z) + 1.5\sin(2z)$. Figure 17 shows the profile of the closed-loop system under a nonlinear state feedback control law (designed using the proposed method) which uses ten control actuators. As expected, the 'target' spatio-temporal profile is asymptotically ($t \to \infty$) enforced in the closed-loop system.

Finally, we consider the case where state measurements are not available and we focus on output feedback controller design. Under the assumption that the number of measurements is equal to the number of control actuators (note that the number of control actuators should be chosen so that the target behaviour is enforced in the closed-loop system, with the desired accuracy, under state feedback control), we employ a procedure proposed in [20] for obtaining estimates for the states of the finite-dimensional system from the measurements. The nonlinear parabolic PDE of Equation (22) is taken to be as the starting point and the target PDE is the linear parabolic PDE of Equation (25). Two nonlinear output feedback control laws are derived on the basis of second- and fifth-order finite-dimensional approximations. Figure 18 shows the evolution of the state of Equation (22) under a nonlinear output feedback control law which

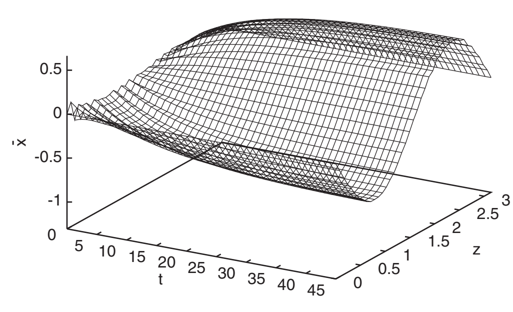

Figure 16. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) with $\mathscr{D} = 0.06$ under nonlinear state feedback control with 20 equidistant control actuators—linear parabolic PDE of Equation (25) with $b = 1.5$.

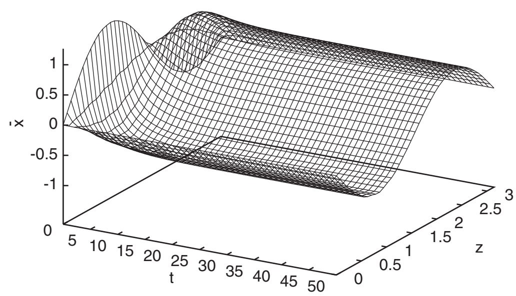

Figure 17. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear state feedback control with 10 equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$. Effect of variation on the initial condition.

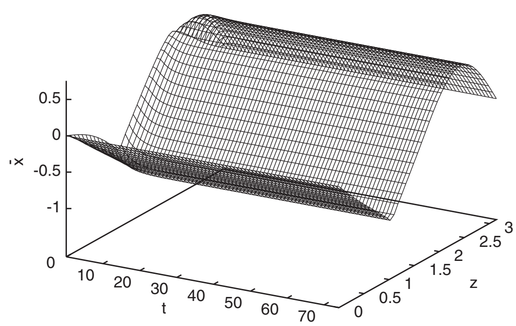

Figure 18. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear output feedback control with two equidistant control actuators and measurement sensors—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

uses two control actuators and two measurement sensors; Figure 19 shows the evolution of the state of Equation (22) under a nonlinear state feedback control law which uses two control actuators. Clearly, there is some discrepancy between the closed-loop systems under state and output feedback control which is due to the estimation error. Figure 20 shows the evolution of the state of Equation (22) under a nonlinear output feedback control law which uses five control actuators and five measurement sensors; the target behaviour has been enforced (in the sense discussed previously under state feedback control) which implies that five actuators/sensors suffice to achieve the desired control objective. We note that similar results have been obtained in the case of enforcing the spatio-temporal profiles of Equations (26)–(27) in the closed-loop system using output feedback control.

Application to a 2D diffusion–reaction process

In this subsection, the proposed control methodology is applied to a two-dimensional diffusion–reaction process example to enforce a desired spatio-temporal behaviour in the closed-loop system. The control is assumed to be implemented on the process by using spatially distributed

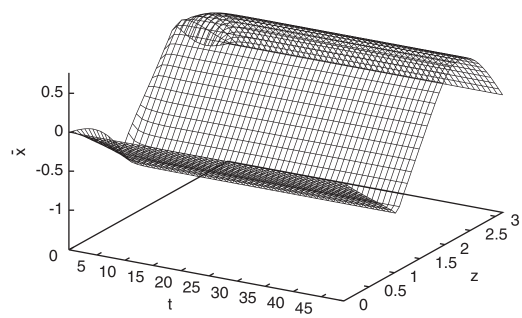

Figure 19. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear state feedback control with two equidistant control actuators—target behaviour: linear parabolic PDE of Equation (25) with $b = 1.5$.

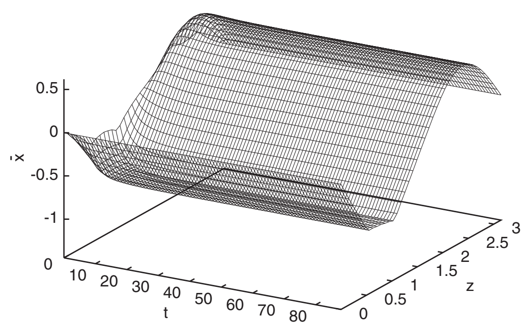

Figure 20. Closed-loop spatio-temporal profile of nonlinear diffusion–reaction equation (Equation (22)) under nonlinear output feedback control with five equidistant control actuators and measurement sensors—target behaviour: linear parabolic PDE of Equation (14) with $b = 1.5$.

arrays with large numbers of point control actuators, which are placed at equidistant positions. Specifically, we consider a parabolic PDE of the form

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial^2 \bar{x}}{\partial z^2} + \frac{\partial^2 \bar{x}}{\partial y^2} + \hat{f}(\bar{x}, t) \tag{28}$$

where $\hat{f}(\bar{x}, t)$ is a possibly nonlinear vector field, which is defined in a two-dimensional spatial domain $\Omega = [-\pi, \pi] \times [-\pi, \pi]$ (i.e. $z \in [-\pi, \pi]$ and $y \in [-\pi, \pi]$). Dirichlet boundary conditions are considered throughout the boundary of the domain, Γ, of the form $\bar{x}(z, y, t) = 0$ for all (z, y) on Γ.

To demonstrate the application of our method, we assume that the vector field of the 'original' PDE is given by $\hat{f}(\bar{x}, t) = 0$, whereas the vector field of the 'target PDE' is of the form $\hat{f}(\bar{x}, t) = \alpha\bar{x} - \beta\bar{x}^3$, where α, β are constant positive parameters with the following values $\alpha = 2$ and $\beta = 2$. A 1600th-order Galerkin discretization of Equation (28) is computed and used in the simulation (higher order discretizations led to identical results). Figure 21 shows the steady-state profile of the 'original' PDE.

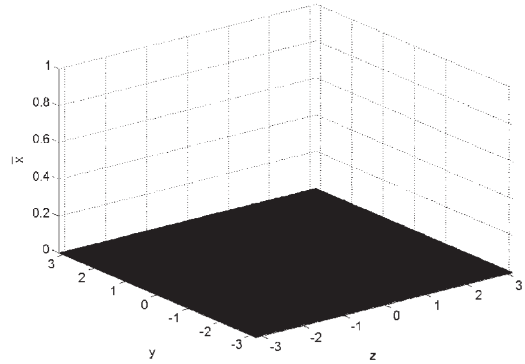

Figure 21. Steady-state of the two-dimensional parabolic PDE of Equation (28) with $\hat{f}(\bar{x}, t) = 0$.

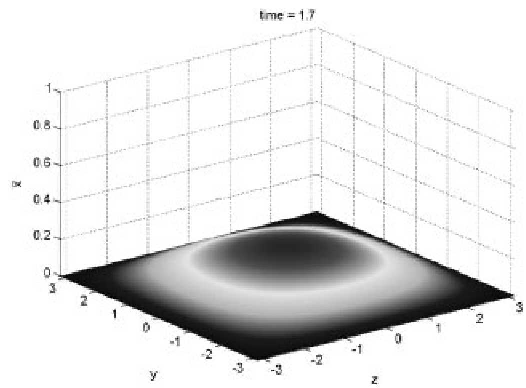

Figure 22. Closed-loop profile of the two-dimensional parabolic PDE under nonlinear state feedback control for $t = 1.7$—target behaviour: two-dimensional parabolic PDE of Equation (28) with $\hat{f}(\bar{x}, t) = 2\bar{x} - 2\bar{x}^3$.

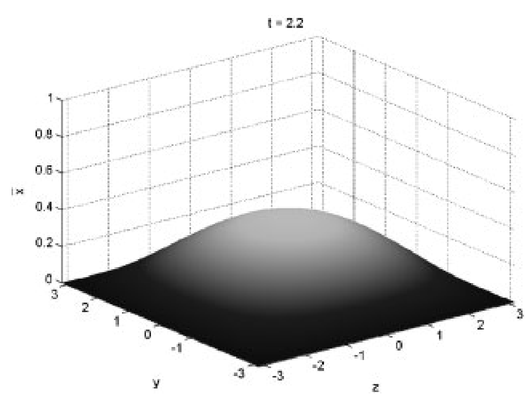

Figure 23. Closed-loop profile of the two-dimensional parabolic PDE under nonlinear state feedback control for $t = 2.2$—target behaviour: two-dimensional parabolic PDE of Equation (28) with $\hat{f}(\bar{x}, t) = 2\bar{x} - 2\bar{x}^3$.

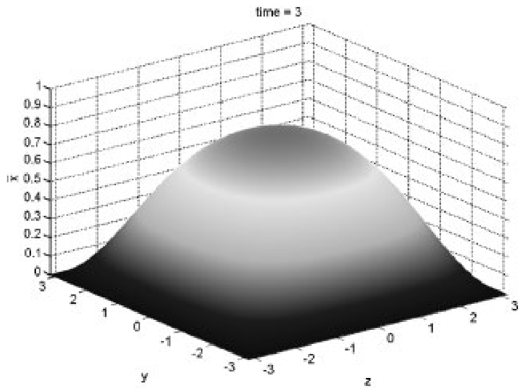

Figure 24. Closed-loop profile of the two-dimensional parabolic PDE under nonlinear state feedback control for $t = 3.0$—target behaviour: two-dimensional parabolic PDE of Equation (28) with $\hat{f}(\bar{x}, t) = 2\bar{x} - 2\bar{x}^3$.

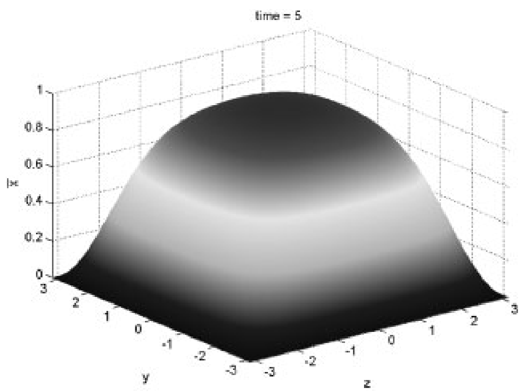

Figure 25. Closed-loop profile of the two-dimensional parabolic PDE under nonlinear state feedback control for $t = 5.0$—target behaviour: two-dimensional parabolic PDE of Equation (28) with $\hat{f}(\bar{x}, t) = 2\bar{x} - 2\bar{x}^3$.

Figures 22–25 show the profiles of the closed-loop *two*-dimensional parabolic PDE (simulated by the 1600th-order Galerkin approximation) under state feedback control, which employs one hundred control actuators, for $t = 1.7$, 2.2, 3.0 and 5.0, respectively. The target behaviour given by Equation (28) with $\hat{f}(\bar{x}, t) = 2\bar{x} - 2\bar{x}^3$ is asymptotically enforced in the closed-loop system.

CONCLUSIONS

This work focused on distributed control of quasi-linear parabolic PDEs and presented a systematic solution to the problem of enforcing a prespecified spatio-temporal behaviour in the closed-loop system using nonlinear feedback control and a sufficiently large number of actuators and sensors. Under the assumptions that (1) the desired spatio-temporal behaviour is described by a 'target parabolic PDE', and (2) a sufficiently large number of actuators and sensors is available, we designed nonlinear state and static output feedback controllers to address this problem on the basis of appropriate finite-dimensional approximations of the original and target PDEs. Illustrative examples of parabolic PDEs which model diffusion–reaction processes were used to demonstrate the formulation of the control problem and the effectiveness of our approach to enforcing a prespecified spatio-temporal behaviour, as well as to investigate the effect of various problem parameters. Using these illustrative examples, we demonstrated that both (a) a sufficiently large number of actuators/sensors, and (b) nonlinear control laws are needed to achieve this goal.

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