Optimal Actuator/Sensor Placement for Nonlinear Control of the Kuramoto-Sivashinsky Equation

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***Abstract*—In this paper, we use a methodology that was recently proposed by Antoniades and Christofides to compute the optimal actuator/sensor locations for the stabilization, via nonlinear static output feedback control, of the zero solution of the Kuramoto–Sivashinsky equation (KSE) for values of the instability parameter for which this solution is unstable. The theoretical results are illustrated through computer simulations of the closed-loop system using a high-order discretization of the KSE.**

***Index Terms*—Galerkin's method, highly dissipative partial differential equations, nonlinear control, optimal actuator/sensor placement, waves.**

I. INTRODUCTION

THE Kuramoto–Sivashinsky equation (KSE) is a nonlinear dissipative fourth-order partial differential equation (PDE) of the form

$$\frac{\partial U}{\partial t} = -\nu \frac{\partial^4 U}{\partial z^4} - \frac{\partial^2 U}{\partial z^2} - U\frac{\partial U}{\partial z} \tag{1}$$

where $\nu > 0$ is the so-called instability parameter, which describes incipient instabilities in a variety of physical and chemical systems. Examples include falling liquid films [6], unstable flame fronts [19], and interfacial instabilities between two viscous fluids [12]. Analytical and numerical studies of the dynamics of (1) with periodic boundary conditions (e.g., [6] and [20]) have revealed the existence of steady and periodic wave solutions, as well as chaotic behavior for very small values of ν.

In addition to the existence of complex solution patterns, the above studies have revealed that the dominant dynamics of the KSE can be adequately characterized by a small number of degrees of freedom (e.g., [20]). Motivated by this, research has focused on the design of linear/nonlinear finite-dimensional output feedback controllers [4], [5] for stabilization of the zero solution of the KSE on the basis of ordinary differential equation (ODE) approximations, obtained through linear [4] and nonlinear [5] Galerkin's method, that accurately describe the dominant dynamics of the KSE for a given value of the instability parameter. The global stabilization of the KSE has also been addressed via distributed static output feedback control [9]. A nonlinear boundary feedback controller was also proposed in [17] that enhances the rate of convergence to the spatially uniform steady-state of the KSE for values of ν for which this steady-state is open-loop stable. Even though the above works led to the systematic design of practically implementable feedback controllers for the KSE, they do not address the issue of optimal actuator/sensor placement so that the desired control objectives are achieved with minimal energy use.

The area of integration of feedback control design with optimal placement of control actuators and measurement sensors so that the desired control objectives are achieved with minimal energy use has received significant attention, especially in 1970s and early 1980s (see, for example, the review paper [14]), in the context of linear distributed parameter systems (DPS). Specifically, several results have been derived on the problem of integrating linear feedback control and optimal actuator placement for several classes of linear DPS including controllability measures and actuator placement in oscillatory systems [3], as well as optimal placement of actuators for linear feedback controllers in parabolic PDEs (see, e.g., [11]) and in actively controlled structures (see, e.g., [7]). Furthermore, the problem of selecting optimal locations for measurement sensors in linear distributed parameter systems has also received very significant attention (see, e.g., [15], [18]). Significant research efforts have also been made on the integrated optimal placement of actuators and sensors for various classes of linear DPS [14]. Recently, we initiated a line of work on the computation of optimal actuator/sensor locations of nonlinear controllers for spatially-distributed processes. In a previous work [1], we proposed a method for the integration of nonlinear output feedback control with optimal actuator/sensor placement for transport-reaction processes described by a broad class of quasilinear parabolic PDEs. The proposed method, based on Galerkin's method and the concept of full state-feedback linearization, allows simultaneously designing a stabilizing nonlinear output feedback controller and solving the optimal actuator/sensor placement problem in a highly efficient fashion.

In this work, we use the methodology proposed in [1] to compute optimal locations of point control actuators and measurement sensors for nonlinear output feedback control of the KSE with periodic boundary conditions. We initially synthesize stabilizing nonlinear state feedback controllers via geometric techniques on the basis of finite-dimensional approximations, obtained via Galerkin's method, that capture the dominant dynamics of the KSE. The optimal actuator location problem is subsequently formulated as the one of minimizing a meaningful cost functional that includes penalty on the response of the closed-loop system and the control action. Then, under the

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assumption that the number of measurement sensors is equal to the number of slow modes, we employ a procedure proposed in [10] for obtaining estimates for the states of the approximate finite-dimensional model from the measurements. The estimates are combined with the state feedback controllers to derive output feedback controllers. The optimal location of the measurement sensors is computed by minimizing a cost function of the estimation error in the closed-loop infinite-dimensional system. The theoretical results are successfully illustrated through computer simulations of the closed-loop system using a high-order discretization of the KSE.

II. Preliminaries

We consider the KSE in one spatial dimension with distributed control

$$\begin{aligned}\frac{\partial U}{\partial t} &= -\nu\frac{\partial^4 U}{\partial z^4} - \frac{\partial^2 U}{\partial z^2} - U\frac{\partial U}{\partial z} + \sum_{i=1}^{l} b_i u_i(t)\\ y_m^\kappa &= \int_{-\pi}^{\pi} s_\kappa(z) U dz, \qquad \kappa = 1,\ldots,p \end{aligned} \tag{2}$$

subject to the periodic boundary conditions

$$\frac{\partial^j U}{\partial z^j}(-\pi,t) = \frac{\partial^j U}{\partial z^j}(+\pi,t), \qquad j=0,\ldots,3 \tag{3}$$

and the initial condition

$$U(z,0) = U_0(z) \tag{4}$$

where $U(z,t)$ is the state of the PDE, $z \in [-\pi,\pi]$ is the spatial coordinate, t is the time and 2π is the length of the spatial domain, ν is the instability parameter, and $U_0(z)$ is the initial condition. $u_i \in \mathbb{R}$ denotes the ith manipulated input, l is the total number of manipulated inputs, $b_i(z)$ is the ith actuator distribution function (i.e., $b_i(z)$ determines how the control action computed by the ith control actuator, $u_i(t)$, is distributed (e.g., point or distributed actuation) in the spatial interval $[-\pi,\pi]$), $y_m^\kappa \in \mathbb{R}$ denotes a measured output and $s_\kappa(z)$ is a known smooth function of z which is determined by the location and type of the measurement sensors (e.g., point/distributed sensing). We note that whenever the control action (or sensing) is applied to the system at a single point z_0, with $z_0 \in [-\pi,\pi]$ (i.e., point actuation), the function $b_i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_0-\epsilon, z_0+\epsilon]$, where ϵ is a small positive real number, and zero elsewhere in $[-\pi,\pi]$. Throughout the manuscript, we will consider the problem of optimal actuator/sensor placement by considering point control actuators.

To present the method that we use for output feedback controller design and optimal actuator/sensor placement, we formulate (2) as an infinite dimensional system in the Hilbert space $\mathcal{H}([-\pi,\pi];\mathbb{R})$, with $\mathcal{H}$ being the space of measurable functions defined on $[-\pi,\pi]$, with inner product and norm

$$(\omega_1,\omega_2) = \int_{-\pi}^{\pi} (\omega_1(z),\omega_2(z))_{\mathbb{R}} dz, \|\omega_1\|_2 = (\omega_1,\omega_1)^{1/2} \tag{5}$$

where ω_1, ω_2 are two elements of $\mathcal{H}([-\pi,\pi];\mathbb{R})$ and the notation $(\cdot,\cdot)_{\mathbb{R}}$ denotes the standard inner product in $\mathbb{R}$. Defining the state function x on $\mathcal{H}([-\pi,\pi];\mathbb{R})$ as

$$x(t) = U(z,t), \qquad t>0; \quad z\in[-\pi,\pi] \tag{6}$$

the operator $\mathcal{A}$ in $\mathcal{H}([-\pi,\pi];\mathbb{R})$ as

$$\begin{aligned}\mathcal{A}x &= -\nu\frac{\partial^4 U}{\partial z^4} - \frac{\partial^2 U}{\partial z^2}\\ x\in D(\mathcal{A}) &= \left\{ x\in\mathcal{H}([-\pi,\pi];\mathbb{R}) : \frac{\partial^j U}{\partial z^j}(-\pi,t) = \frac{\partial^j U}{\partial z^j}(+\pi,t), \quad j=0,\ldots,3 \right\}\end{aligned} \tag{7}$$

and the input, controlled output, and measured output operators as

$$\mathcal{B}u = \sum_{i=1}^{m} b_i u_i \qquad \mathcal{S}x = (s,x) \tag{8}$$

the system of (2)–(4) takes the form

$$\begin{aligned}\dot{x} &= \mathcal{A}x + \mathcal{B}u + f(x), \qquad x(0)=x_0\\ y_m &= \mathcal{S}x\end{aligned} \tag{9}$$

where $f(x(t)) = -U(\partial U/\partial z)$ and $x_0 = U_0(z)$.

For $\mathcal{A}$, we can formulate the following eigenvalue problem:

$$\mathcal{A}\phi_n = -\nu\frac{\partial^4\phi_n}{\partial z^4} - \frac{\partial^2\phi_n}{\partial z^2} = \lambda_n\phi_n, \qquad n=1,\ldots,\infty \tag{10}$$

subject to

$$\frac{\partial^j\phi_n}{\partial z^j}(-\pi) = \frac{\partial^j\phi_n}{\partial z^j}(+\pi), \qquad j=0,\ldots,3 \tag{11}$$

where λ_n denotes an eigenvalue and ϕ_n denotes an eigenfunction. A direct computation of the solution of the above eigenvalue problem yields $\lambda_0 = 0$ with $\psi_0(z) = (1/\sqrt{2\pi})$, and $\lambda_n = -\nu n^4 + n^2$ (λ_n is an eigenvalue of multiplicity two) with eigenfunctions $\phi_n(z) = (1/\sqrt{\pi})\sin(nz)$ and $\psi_n(z) = (1/\sqrt{\pi})\cos(nz)$ for $n=1,\ldots,\infty$. We also define the eigenspectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, as the set of all eigenvalues of $\mathcal{A}$, i.e., $\sigma(\mathcal{A}) = \{\lambda_1,\lambda_2,\ldots,\}$. We note that the fact that $\mathcal{A}$ has a pure real point spectrum is a result of the fact that the spatial differential operator of the KSE with periodic boundary conditions is self-adjoint and the problem is considered in a bounded domain.

From the expression for the eigenvalues, it follows that for a fixed value of $\nu > 0$ the number of unstable eigenvalues of $\mathcal{A}$ is finite and the distance between two consecutive eigenvalues (i.e., λ_n and λ_{n+1}) increases as n increases. Furthermore, for a fixed value of $\nu > 0$, $\sigma(\mathcal{A})$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A})\bigcup\sigma_2(\mathcal{A})$, where $\sigma_1(\mathcal{A})$ contains the first m (with m finite) "slow" eigenvalues (i.e., $\sigma_1(\mathcal{A}) = \{\lambda_1,\ldots,\lambda_m\}$) and $\sigma_2(\mathcal{A})$ contains the remaining "fast" eigenvalues (i.e., $\sigma_2(\mathcal{A}) = \{\lambda_{m+1},\ldots,\}$ where $\lambda_{m+1} < 0$). To capture the separation between the "slow" and "fast" eigenvalues, we define the parameter $\epsilon = |\lambda_1|/|\lambda_{m+1}|$ (note that $\epsilon \to 0$ as $m\to\infty$). We note that the occurrence of a finite number of unstable eigenvalues and the separation of "slow" and "fast" eigenvalues is a common characteristic of the spectrum of spatial differential operators associated with diffusion-reaction processes and various classes of fluid dynamic systems [8].

The separation between the "slow" and "fast" eigenvalues suggests that the dominant dynamics of the KSE can be described by a finite-dimensional system. We apply standard Galerkin's method to the system of (9) to derive an approximate finite-dimensional system. Let $\mathcal{H}_s$, $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as $\mathcal{H}_s = span\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = span\{\phi_{m+1}, \phi_{m+2}, \ldots,\}$ (the existence of $\mathcal{H}_s$, $\mathcal{H}_f$ follows from the properties of $\mathcal{A}$). Defining the orthogonal projection operators P_s and P_f such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of (9) can be decomposed as

$$x = x_s + x_f = P_s x + P_f x. \tag{12}$$

Applying P_s and P_f to the system of (9) and using the above decomposition for x, the system of (9) can be equivalently written in the following form:

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u + f_s(x_s, x_f) \\ \frac{\partial x_f}{\partial t} &= \mathcal{A}_f x_f + \mathcal{B}_f u + f_f(x_s, x_f) \\ y_m &= \mathcal{S} x_s + \mathcal{S} x_f \\ x_s(0) &= P_s x(0) = P_s x_0, \quad x_f(0) = P_f x(0) = P_f x_0 \end{aligned} \tag{13}$$

where $\mathcal{A}_s = P_s \mathcal{A} P_s$, $\mathcal{B}_s = P_s \mathcal{B}$, $f_s = P_s f$, $\mathcal{A}_f = P_f \mathcal{A} P_f$, $\mathcal{B}_f = P_f \mathcal{B}$ and $f_f = P_f f$ and the notation $(\partial x_f / \partial t)$ is used to denote that the state x_f belongs in an infinite-dimensional space. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = \mathrm{diag}\{\lambda_j\}$, $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are Lipschitz vector functions, and $\mathcal{A}_f$ is an unbounded differential operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathcal{H}_s, \mathcal{H}_f$). Neglecting the fast and stable infinite-dimensional x_f-subsystem in the system of (13), the following m-dimensional slow system is obtained:

$$\begin{aligned} \frac{d\tilde{x}_s}{dt} &= \mathcal{A}_s \tilde{x}_s + \mathcal{B}_s u + f_s(\tilde{x}_s, 0) \\ \tilde{y}_m &= \mathcal{S} \tilde{x}_s \end{aligned} \tag{14}$$

where the bar symbol in $\tilde{x}_s$ and $\tilde{y}_m$ denotes that these variables are associated with a finite-dimensional system.

Remark 1: A physical system that can be described by the KSE is the motion of a liquid film falling down on a vertical wall [6]. In this case, $U(z,t)$ is the film height. Such a system has been found experimentally to exhibit wavy behavior of the type predicted by the KSE for values of ν smaller than one (see also Fig. 1). In many instances, it is desirable to suppress wavy behavior by using control actuators that add/remove fluid mass via blowing/suction. In this case, the control input enters directly into the partial differential equation and does not appear in the boundary conditions. This physical problem is consistent with the formulation of (2) since we consider distributed control actuation. Note that problems for which the inputs enters directly into the KSE but nonlinearly can be readily handled within our formulation by simply solving for u through the inversion of a nonlinear algebraic equation.

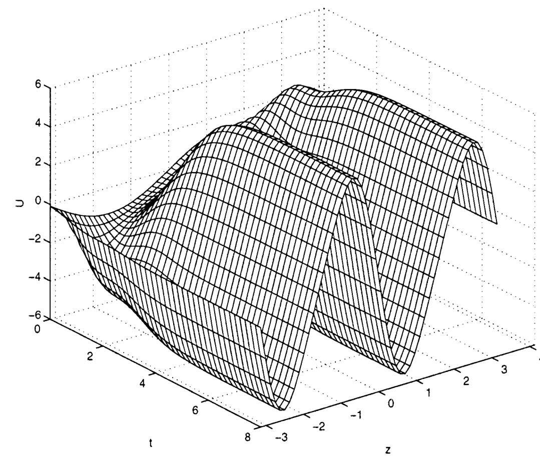

Fig. 1. Open-loop spatiotemporal profile of $U(z,t)$ for $\nu = 0.2$.

Remark 2: The consideration of approximate point control is motivated by the fact that most experimental point control actuators (including the ones that add/remove fluid mass via blowing/suction) have finite (but small) support and the fact that $b_i(z) = \delta(z - z_0)$ (where $\delta(\cdot)$ is the standard Dirac function) is not an element of $\mathcal{H}([-\pi, \pi]; \mathbb{R})$.

III. Integrating Nonlinear Control and Optimal Actuator Placement

A. Nonlinear State Feedback Controller Synthesis

In this section, we assume the use of point control actuators and assume that measurements of the states of the system of (14) are available. We first address the problem of synthesizing nonlinear static state feedback control laws of the general form

$$u = \mathcal{F}(z_a, \tilde{x}_s) \tag{15}$$

where $\mathcal{F}(z_a, \tilde{x}_s)$ is a nonlinear vector function and $z_a = [b_1(z_{a_1}) \cdots b_l(z_{a_l})]^T$ denotes the vector of the point actuator locations, that guarantee exponential stability of the closed-loop finite-dimensional system. To address this problem and simplify our development, we need to impose the following assumption.

Assumption 1: $l = m$ (i.e., the number of control actuators is equal to the number of slow modes), and the inverse of the matrix $\mathcal{B}_s$ exists.

The requirement $l = m$ is sufficient and not necessary, and it is made to simplify the synthesis of the controller and the solution of the optimal placement problem (see also discussion in Remark 4 below).

Proposition 1 that follows provides the explicit formula for the state feedback controller that achieves the control objective (the proof can be found in [1]).

Proposition 1: Consider the finite-dimensional system of (14) for which assumption 1 holds. Then, the state feedback controller

$$u = \mathcal{B}_s^{-1} \left((\Lambda_s - \mathcal{A}_s) \tilde{x}_s - f_s(\tilde{x}_s, 0) \right) \tag{16}$$

where Λ_s is a stable matrix, guarantees global exponential stability of the closed-loop finite-dimensional system.

Remark 3: The structure of the closed-loop finite-dimensional system under the controller of (16) has the following form:

$$\dot{\tilde{x}}_s = \Lambda_s \tilde{x}_s \tag{17}$$

and, thus, the response of this system depends only on the stable matrix Λ_s and the initial condition $x_s(0)$ and is independent of the actuator locations.

Remark 4: While the requirements $l = m$ and existence of $\mathcal{B}_s^{-1}$ are sufficient to design a state feedback law that fully linearizes the closed-loop finite dimensional system (17), it is important to note that full linearization of the closed-loop finite-dimensional system through coordinate change and nonlinear feedback can be achieved for any number of manipulated inputs (i.e., for any $l \in [1, m]$), provided that an appropriate set of involutivity conditions is satisfied by the corresponding vector fields of the system of (14) (see [13, p. 165] for details).

B. Computation of Optimal Actuator Locations

In this subsection, we compute the actuator locations so that the state feedback controller of (16) is near-optimal for the full KSE system of (13) with respect to a meaningful cost functional which is defined over the infinite time-interval and imposes penalty on the response of the closed-loop system and the control action. To this end, we initially focus on the ODE system of (14) and consider the following cost functional:

$$\begin{aligned} J_s = \int_0^\infty & ((\tilde{x}_s(x_s(0),t), Q_s \tilde{x}_s(x_s(0),t)) \\ & + u^T(\tilde{x}_s(x_s(0),t), z_a) R u(\tilde{x}_s(x_s(0),t), z_a)) dt \end{aligned} \tag{18}$$

where Q_s and R are positive definite matrices and the inner product in $\mathcal{H}$ notation of (5) is used in the first term of the integrand. The cost of (18) is well defined and meaningful since it imposes penalty on the response of the closed-loop finite-dimensional system and the control action. However, a potential problem of this cost is its dependence on the choice of a particular initial condition, $x_s(0)$, and thus, the solution to the optimal placement problem based on this cost may lead to actuator locations that perform very poorly for a large set of initial conditions. To reduce this dependence and obtain optimality over a broad set of initial conditions, we follow [16] and consider an average cost over a set of m linearly independent initial conditions $x_s^i(0)$ and $i = 1, \ldots, m$, of the following form:

$$\begin{aligned} \hat{J}_s = \frac{1}{m} \sum_{i=1}^{m} \int_0^\infty & ((\tilde{x}_s(x_s^i(0),t), Q_s \tilde{x}_s(x_s^i(0),t)) \\ & + u^T(\tilde{x}_s(x_s^i(0),t), z_a) R u(\tilde{x}_s(x_s^i(0),t), z_a)) dt. \end{aligned} \tag{19}$$

Referring to the above cost, we first note that the penalty on the response of the closed-loop system:

$$\hat{J}_{xs} = \frac{1}{m} \sum_{i=1}^{m} \int_0^\infty (\tilde{x}_s(x_s^i(0),t), Q_s \tilde{x}_s(x_s^i(0),t)) dt \tag{20}$$

is finite because the solution of the closed-loop system of (17) is exponentially stable by appropriate choice of Λ_s. Moreover, $\hat{J}_{xs}$ is independent of the actuator locations (Remark 3), and thus, the optimal actuator placement problem reduces to the one of minimizing the following cost which only includes penalty on the control action:

$$\hat{J}_{us} = \frac{1}{m} \sum_{i=1}^{m} \int_0^\infty u^T(\tilde{x}_s(x_s^i(0),t), z_a) R u(\tilde{x}_s(x_s^i(0),t), z_a) dt. \tag{21}$$

$\hat{J}_{us}$ is a function of multiple variables, $z_a = [z_{a1} \quad z_{a2} \quad \ldots \quad z_{al}]$, and thus, it obtains its local minimum values when its gradient with respect to the actuator locations is equal to zero, i.e.,

$$\frac{\partial \hat{J}_{us}}{\partial z_a} = \left[\frac{\partial \hat{J}_{us}}{\partial z_{a1}} \quad \ldots \quad \frac{\partial \hat{J}_{us}}{\partial z_{al}} \right]^T = [0 \quad \ldots \quad 0]^T \tag{22}$$

and $\nabla_{z_a z_a} \hat{J}_{us}(z_{am}) > 0$ where $\nabla_{z_a z_a} \hat{J}_{us}$ is the Hessian matrix of $\hat{J}_{us}$ and z_{am} is a solution of the system of nonlinear algebraic equations of (22) (which includes l equations with l unknowns). The solution z_{am} for which the above conditions are satisfied and $\hat{J}_{us}$ obtains its smallest value (global minimum) corresponds to the optimal actuator locations for the closed-loop finite-dimensional system.

Remark 5: Owing to the numerical complexity involved in computing the actuator locations that exactly minimize the cost of (21) (it involves search over an infinite number of locations), we initially assume a large number, say L, of equispaced locations along the length of the spatial domain in which the control actuators are possible to be placed (i.e., locations for which $\mathcal{B}_s^{-1}$ exists and the closed-loop finite-dimensional system is stabilizable). Then, we compute the value of the cost of (21) for all possible combinations of the actuator locations to calculate the optimal locations.

Remark 6: While the cost of (19) is meaningful in the sense that it imposes penalty both on the system response and the control effort and standard in the context of optimal actuator placement problems, it is possible to compute optimal actuator locations with respect to cost functionals that include penalty on the rate of change of the state and of the control action in order to enforce additional control objectives. The consideration of such costs can be addressed and studied in the context of the proposed framework however, it may lead to an increase in the magnitude of the computational demand needed to solve the optimization problem.

C. Output Feedback Control and Sensor Placement

The nonlinear controller of (16) was derived under the assumption that measurements of the states $\tilde{x}_s$ are available, which implies that measurements of the state variable, $U(z,t)$, are available at all positions and times. However, from a practical point of view, measurements of the state variables are only available at a finite number of spatial positions. Motivated by these practical problems, we address in this section: 1) the synthesis of nonlinear output feedback controllers that use measurements of the process outputs, y_m, to enforce stability in the closed-loop infinite-dimensional system and 2) the computation of optimal locations of the measurement sensors.

Specifically, we consider output feedback control laws of the general form

$$u = \mathcal{F}(y_m) \tag{23}$$

where $\mathcal{F}(y_m)$ is a nonlinear vector function and y_m is the vector of measured outputs. The synthesis of the controller of (23) will be achieved by combining the state feedback controller of (16) with a procedure proposed in [10] for obtaining estimates for the states of the approximate ODE model of (14) from the measurements. To this end, we need to impose the following requirement on the number of measured outputs in order to obtain estimates of the states x_s of the finite-dimensional system of (14), from the measurements y_m^κ, $\kappa = 1, \ldots, p$.

Assumption 2: $p = m$ (i.e., the number of measurements is equal to the number of slow modes), and the inverse of the operator $\mathcal{S}$ exist, so that $\hat{x}_s = \mathcal{S}^{-1} y_m$.

We note that the requirement that the inverse of the operator S exists can be achieved by appropriate choice of the location of the measurement sensors [i.e., functions $s^\kappa(z)$]. When point measurement sensors are used, this requirement can be verified by checking the invertibility of a matrix (see (33) in Section IV). The optimal locations for the measurement sensors can be then computed by minimizing an average cost function of the estimation error of the closed-loop infinite-dimensional system of the form:

$$\hat{J}(e) = \frac{1}{m}\sum_{i=1}^{m}\int_0^\infty (\|x_s(x_s^i(0),t) - \hat{x}_s(x_s^i(0),t)\|_2)dt \tag{24}$$

where x_s is the slow state of the closed-loop infinite-dimensional system of (13), $\hat{x}_s = \mathcal{S}^{-1} y_m$, and $e(t) = \|x_s - \hat{x}_s\|_2$ is the estimation error. In contrast to the solution of the optimal location problem for the control actuators, the solution to this optimization problem requires the solution of the closed-loop infinite-dimensional system in order to compute x_s, and $\hat{x}_s$ (from the measurements y_m^κ, $\kappa = 1, 2, \ldots, p$), and thus, it is more computationally demanding.

Theorem 1 that follows establishes that the proposed output feedback controller enforces stability in the closed-loop infinite-dimensional system and that the solution to the optimal actuator/sensor problem, which is obtained on the basis of the closed-loop finite-dimensional system, is near-optimal in the sense that it approaches the optimal solution for the infinite-dimensional system as the separation of the slow and fast eigenmodes increases. The proof can be found in [1].

Theorem 1: Consider the system of (13), and the finite-dimensional system of (14), for which assumptions 1 and 2 hold, under the nonlinear output feedback controller:

$$\begin{aligned}\hat{x}_s &= \mathcal{S}^{-1} y_m \\ u &= \mathcal{B}_s^{-1}\left((\Lambda_s - \mathcal{A}_s)\hat{x}_s - f_s(\hat{x}_s, 0)\right).\end{aligned} \tag{25}$$

Then, there exist positive real numbers μ_1, μ_2, and ϵ^* such that if $\|x_s(0)\|_2 \le \mu_1$, $\|x_f(0)\|_2 \le \mu_2$, and $\epsilon \in (0, \epsilon^*]$, then the controller of (25):

1) guarantees exponential stability of the infinite-dimensional closed-loop system
2) the locations of the point actuators and measurement sensors are near-optimal in the sense that the cost function associated with the controller of (25) and the system of (13) satisfies

$$\begin{aligned}\hat{J} =& \frac{1}{m}\sum_{i=1}^{m}\int_0^\infty ((x_s(x_s^i(0),t), Q_s x_s(x_s^i(0),t)) \\ &+ (x_f(x_s^i(0),t), Q_f x_f(x_f^i(0),t)) \\ &+ u^T(x_s(x_s^i(0)),t,z_a) R u(x_s(x_s^i(0)),t,z_a))dt \\ \longrightarrow& \hat{J}_s \text{ as } \epsilon \longrightarrow 0\end{aligned} \tag{26}$$

where $\hat{J}$ and $\hat{J}_s$ are the average cost functions of the infinite-dimensional system of (13) and the finite-dimensional system of (14), respectively, under the output feedback controller of (25).

Remark 7: Even though static output feedback is more sensitive to measurement noise than dynamic output feedback, we prefer to use static feedback of y_m in the controller of (25) because the use of a state observer to obtain estimates of the slow state variables would lead to the formulation of a very computationally-demanding optimization problem for the computation of the optimal sensor locations. Furthermore, the solution to the controller design and optimal actuator/sensor placement problems for parabolic PDE systems with uncertain variables can be found in [2].

Remark 8: From the definition of ϵ, it follows that $\epsilon \to 0$ as $m \to \infty$, which implies that the control design and optimal actuator/sensor placement problems for the KSE are solvable provided m is sufficiently large. While an estimate of ϵ^* can be obtained in principle from the proof of Theorem 1, such an estimate would be in general conservative, and thus, the number of slow modes is usually determined via computer simulations.

IV. Numerical Results

In this section, we present an application of optimal actuator/sensor placement to the KSE for $\nu = 0.2$ to achieve stabilization at the spatially-uniform steady state using a nonlinear static output feedback control law. For simplicity and in order to better present our theoretical results, we will consider the KSE in the space of odd functions with spatial zero mean. Introducing the Hilbert space $\mathcal{H}$ of sufficiently smooth odd functions that satisfy the boundary conditions of (3) and have spatial zero mean (i.e., $\forall\, \omega \in \mathcal{H}$, $\int_{-\pi}^{\pi}\omega(z)dz = 0$) and defining the state function $x \in \mathcal{H}$ as $x(t) = U(z,t)$, $\forall\, z \in [-\pi, \pi]$, the system of (2)–(4) can be written in the form of (9), where the domain of definition of the spatial differential operator $\mathcal{A}$ now takes the form

$$x \in D(\mathcal{A}) = \left\{x \in \mathcal{H}([-\pi,\pi]; \mathbb{R}); \frac{\partial^j U}{\partial z^j}(-\pi,t) = \frac{\partial^j U}{\partial z^j}(\pi,t), \qquad j = 0,\ldots,3\right\} \tag{27}$$

and the eigenvalue problem for $\mathcal{A}$ yields $\lambda_j = -\nu j^4 + j^2$, $\phi_j(z) = \sqrt{1/\pi}\sin(jz)$, $j = 1, \ldots, \infty$ (note that $\psi_0(z) = \sqrt{1/2\pi}$ and $\psi_j(z) = \sqrt{1/\pi}\cos(jz)$, $j = 1, \ldots, \infty$ are not considered here, since we focus only on odd functions with spatial zero mean).

All the simulation runs shown below were performed for $\nu = 0.2$, using a 30th-order nonlinear ordinary differential equation model obtained from the application of Galerkin's method to the system of (2) (the use of higher order Galerkin approximations led to identical numerical results, thereby implying that the following simulation runs are independent of the discretization). Linearizing the system around the spatially uniform steady-state for $\nu = 0.2$, we observe that the system of (9) possesses two unstable eigenvalues. The spatiotemporal evolution of $U(z,t)$ for $\nu = 0.2$ is shown in Fig. 1. It is clear that for $\nu = 0.2$, the spatially uniform steady-state $U(z,t) = 0$ is unstable. Therefore, we consider the first two Galerkin modes of the KSE as the slow modes and use Galerkin's method to construct a second-order ODE system which is used for the design of a state and an output feedback controllers and the optimal placement of two control actuators and measurement sensors (assumptions 1 and 2). The resulting second-order ODE system is of the form

$$\begin{bmatrix}(\dot{\tilde{x}}_{s1},\phi_1)\\(\dot{\tilde{x}}_{s2},\phi_2)\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}(\tilde{x}_{s1},\phi_1)\\(\tilde{x}_{s2},\phi_2)\end{bmatrix} + \begin{bmatrix}\phi_1(z_{a1}) & \phi_1(z_{a2})\\ \phi_2(z_{a1}) & \phi_2(z_{a2})\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} + \begin{bmatrix}(f_1(\tilde{x}_s,0),\phi_1)\\(f_2(\tilde{x}_s,0),\phi_2)\end{bmatrix} \quad (28)$$

where z_{a1} and z_{a2} are the locations of the two point actuators and the explicit forms of the terms $f_1(\tilde{x}_s,0)$ and $f_2(\tilde{x}_s,0)$ are omitted for brevity. The system (28) is derived by assuming that point actuation is applied to the system. When the point actuation is approximated by a control aciton applied to a small spatial interval, the optimal actuator/sensor placement results and the closed-loop simulation results are almost identical to those obtained by using the system of (28).

For the system of (28), the nonlinear state feedback controller of (16) takes the form

$$\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}\phi_1(z_{a1}) & \phi_1(z_{a2})\\ \phi_2(z_{a1}) & \phi_2(z_{a2})\end{bmatrix}^{-1}\left(\begin{bmatrix}-\alpha-\lambda_1 & 0\\ 0 & -\beta-\lambda_2\end{bmatrix} \times \begin{bmatrix}(\tilde{x}_{s1},\phi_1)\\(\tilde{x}_{s2},\phi_2)\end{bmatrix} - \begin{bmatrix}(f_1(\tilde{x}_s,0),\phi_1)\\(f_2(\tilde{x}_s,0),\phi_2)\end{bmatrix}\right). \quad (29)$$

Substituting the above controller into the system of (28), we obtain the following closed-loop ODE system

$$\begin{bmatrix}(\dot{\tilde{x}}_{s1},\phi_1)\\(\dot{\tilde{x}}_{s2},\phi_2)\end{bmatrix} = \begin{bmatrix}-\alpha & 0\\ 0 & -\beta\end{bmatrix}\begin{bmatrix}(\tilde{x}_{s1},\phi_1)\\(\tilde{x}_{s2},\phi_2)\end{bmatrix} \quad (30)$$

where α and β are positive real numbers. Since the response of the above system depends only on the parameters α, β and the initial condition x_s, and is independent of the actuator locations, we compute the optimal actuator locations by minimizing the following cost functional, which only includes penalty on the control action

$$\hat{J}_{us} = \frac{1}{2}\sum_{i=1}^{2}\int_0^\infty u^T(\tilde{x}_s(x_s^i(0),t),z_a)R \times u(\tilde{x}_s(x_s^i(0),t),z_a)dt. \quad (31)$$

Using the following values for the initial conditions ${x_s}^1(0) = [\phi_1\ 0]$ and ${x_s}^2(0) = [0\ \phi_2]$, and taking R, Q_s, Q_f to be unit matrices of appropriate dimensions and $\alpha = \beta = 1$, the optimal actuator locations were found, using the procedure discussed in remark 5, to be $z_{a1} = 0.31\pi$ and $z_{a2} = 0.69\pi$. We also compute the optimal sensor locations by minimizing the following cost functional of the estimation error:

$$\hat{J}(e) = \frac{1}{2}\sum_{i=1}^{2}\int_0^\infty (\|x_s(x_s^i(0),t) - \hat{x}_s(x_s^i(0),t)\|_2)dt \quad (32)$$

where x_s is obtained from the simulation of the full-order closed-loop system of (13), and $\hat{x}_s$ is obtained from the measured outputs of the full-order closed-loop system as follows:

$$\begin{bmatrix}\hat{x}_{s1}\\ \hat{x}_{s2}\end{bmatrix} = \begin{bmatrix}\phi_1 & 0\\ 0 & \phi_2\end{bmatrix}\begin{bmatrix}\phi_1(z_{s1}) & \phi_2(z_{s1})\\ \phi_1(z_{s2}) & \phi_2(z_{s2})\end{bmatrix}^{-1}\begin{bmatrix}y_{m1}(z_{s1},t)\\ y_{m2}(z_{s2},t)\end{bmatrix}. \quad (33)$$

We found the optimal location of measurement sensors to be: $z_{s1} = 0.35\pi$ and $z_{s2} = 0.64\pi$.

We performed several simulation runs to evaluate the performance of the proposed method for computing optimal locations of control actuators and measurement sensors. We initially apply the state feedback controller to the 30th-order Galerkin truncation of the system of (2) and investigate the influence of the different actuator locations on the various cost functions. Table I shows the values of the costs $\hat{J}_u$, $\hat{J}_x$, and $\hat{J}$ of the full-order closed-loop system under the state feedback controller, in the case of optimal actuator placement, and for the sake of comparison, the values of these costs in the case of alternative actuator placements. The cost for the control action used to stabilize the KSE at $U(z,t) = 0$ when the actuators are optimally placed, is clearly smaller than the case of actuator placement based on the case 2 (by 18.9%), case 3 (by 35.5%), and case 4 (by 34.0%). Figs. 2 and 3 show the norm of the control action, $\|u\|$, for $x_s(0) = [\phi_1\ 0]$ (Fig. 2) and $x_s(0) = [0\ \phi_2]$ (Fig. 3), for the optimal case (solid line), case 2 (long-dashed line), case 3 (short-dashed line), and case 4 (dotted line).

We also tested the computed optimal sensor locations $z_{s1} = 0.35\pi$ and $z_{s2} = 0.64\pi$. To this end, we implement

TABLE I
RESULTS FOR TWO CONTROL ACTUATORS

Case	Actuator Locations	$\hat{J}_u$	$\hat{J}_x$	$\hat{J}$
Optimal	$0.31\pi, 0.69\pi$	3.3705	0.5006	3.8711
2	$0.30\pi, 0.80\pi$	4.1536	0.5151	4.6687
3	$0.40\pi, 0.60\pi$	5.2238	0.5060	5.7298
4	$0.20\pi, 0.80\pi$	5.1089	0.5357	5.6445

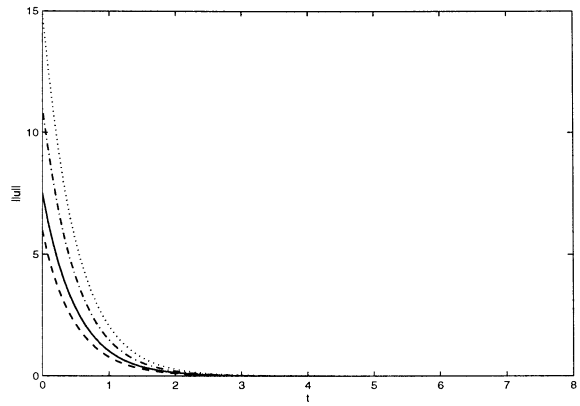

Fig. 2. Closed-loop norm of the control effort, $\|u\|$, for $x_s(0) = [\phi_1\ 0]$, for the optimal case (solid line), case 2 (dashed-dotted line), case 3 (dashed line), and case 4 (dotted line).

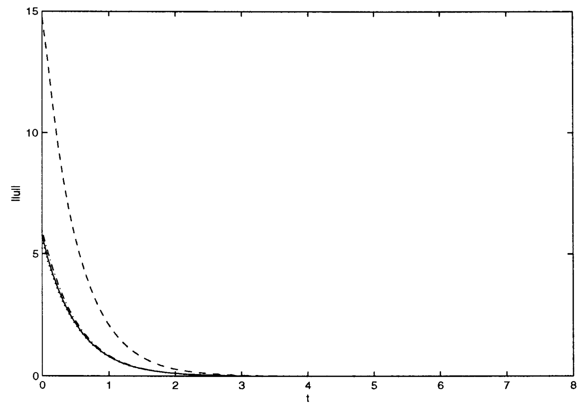

Fig. 3. Closed-loop norm of the control effort, $||u||$, for $x_s(0) = [0\ \phi_2]$, for the optimal case (solid line), case 2 (dashed-dotted line), case 3 (dashed line), and case 4 (dotted line).

TABLE II
RESULTS FOR TWO CONTROL ACTUATORS AND MEASUREMENT SENSORS

Case	Sensor locations	$\hat{J}(e)$	$\hat{J}$
Optimal	$0.35\pi, 0.64\pi$	2.860e-4	3.8748
2	$0.20\pi, 0.80\pi$	1.841e-3	3.8830
3	$0.20\pi, 0.50\pi$	5.020e-3	3.9273
4	$0.45\pi, 0.55\pi$	1.508e-3	3.8803

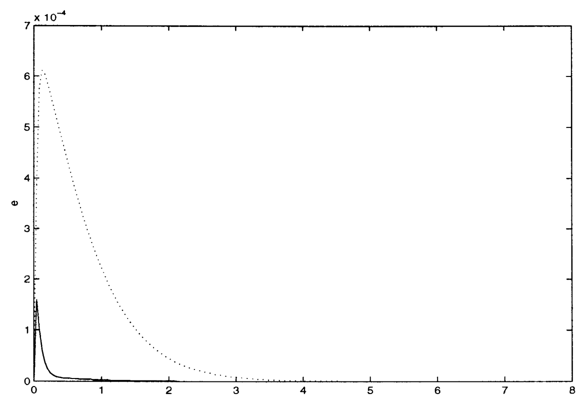

Fig. 4. Closed-loop estimation error e versus time, for the optimal actuator/sensor locations, for $x_s(0) = [\phi_1\ 0]$ (solid line) and $x_s(0) = [0\ \phi_2]$ (dotted line).

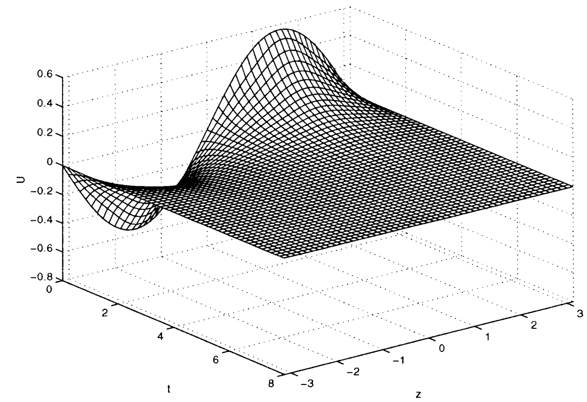

Fig. 5. Profile of KSE under output feedback control, for the optimal actuator/sensor locations, for $x_s(0) = [\phi_1\ 0]$.

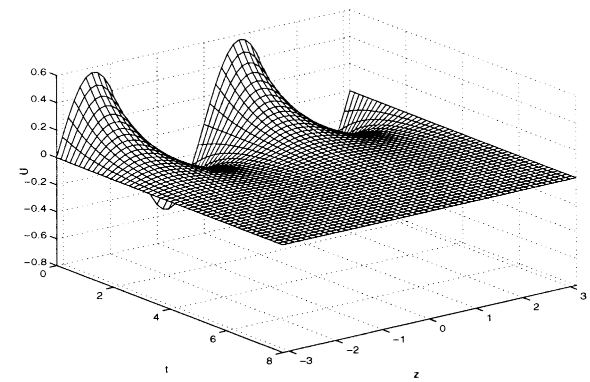

Fig. 6. Profile of KSE under output feedback control, for the optimal actuator/sensor locations, for $x_s(0) = [0\ \phi_2]$.

the nonlinear output feedback controller on the 30th-order Galerkin truncation of the system of (2) with actuator locations $z_{a1} = 0.31\pi$ and $z_{a2} = 0.69\pi$ and different sensor locations. Table II shows the values of the costs $\hat{J}(e)$ and $\hat{J}$ of the full-order closed-loop system under the output feedback controller, in the case of optimal sensor placement, and for the sake of comparison, the values of these costs in the case of two other sensor locations. The estimation error of the sensor locations of 0.35π and 0.64π computed by the proposed approach is smaller than the other two cases. In Fig. 4, we display the closed-loop estimation error ($e(t) = ||x_s - \hat{x}_s||_2$) versus time, for the optimal actuator/sensor locations, for $x_s(0) = [\phi_1\ 0]$ (solid line) and $x_s(0) = [0\ \phi_2]$ (dashed line). We can see that for both initial conditions the estimation error is very small. Finally, Figs. 5 and 6 show the profiles of the KSE system, under output feedback control, for the optimal actuator/sensor locations, for $x_s(0) = [\phi_1\ 0]$ (Fig. 5), and $x_s(0) = [0\ \phi_2]$ (Fig. 6). We can see that the proposed controller with optimal actuator/sensor locations, stabilizes the system to the spatially uniform operating steady state very quickly, for both cases.

Finally, we tested the robustness of the output feedback control law, using the optimal actuator/sensor location, for a 25% decrease in the value of the instability parameter (i.e., ν was taken to be equal to 0.15 in the high-order discretization of the KSE but it was used as 0.2 in the controller). Fig. 7 shows the profiles of the state (top plot) and of the manipulated inputs (bottom plot) for $x_s(0) = [\phi_1\ 0]$ and Fig. 8 shows the profiles of the state (top plot) and of the manipulated inputs (bottom plot) for $x_s(0) = [0\ \phi_2]$. Our simulations show that the controller is capable of stabilizing the KSE at the spatially uniform steady-state solution in the presence of significant uncertainty in ν.

Summarizing, we have computed the optimal [with respect to the costs of (31) and (32)] locations for two control actuators and two measurement sensors, associated with a nonlinear output feedback controller that achieves stabilization of the zero solution of the KSE for $\nu = 0.2$. We have compared the optimal actuator and sensor locations with respect to alternative placements (Tables I and II) and have verified the optimality of the computed locations and the robustness of the approach.

Remark 9: Referring to the effect of ν on the actuator/sensor locations, we note that ν determines how many of the eigenvalues

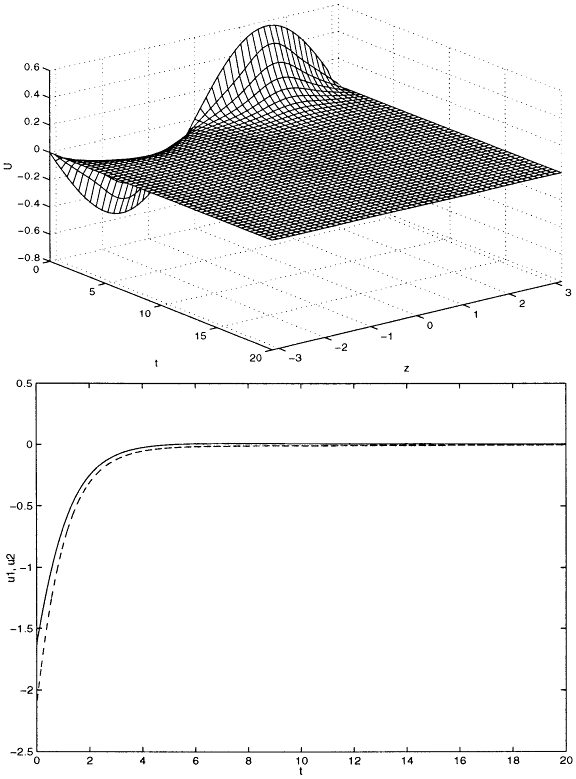

Fig. 7. Profile of KSE under output feedback control, for the optimal actuator/sensor locations, for $x_s(0) = [\phi_1 \ 0]$ and uncertainty in ν (top plot). Manipulated input profiles (bottom plot)—solid line u_1 and dashed line u_2.

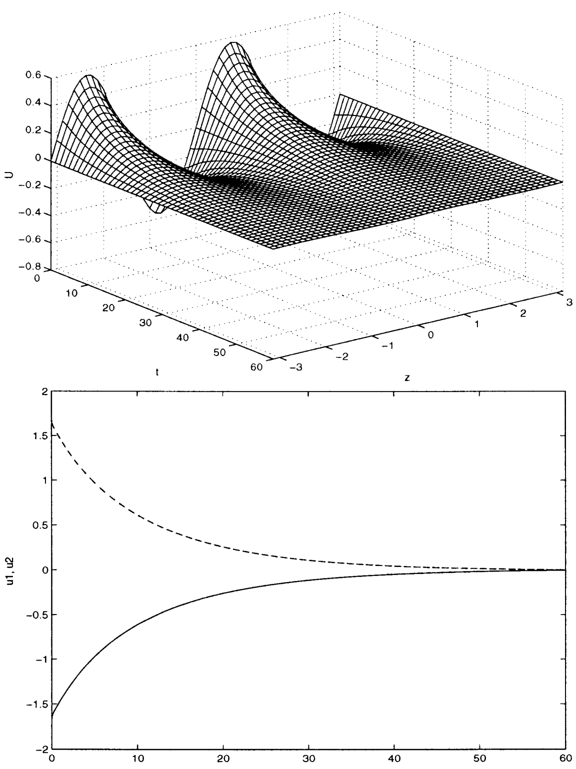

Fig. 8. Profile of KSE under output feedback control, for the optimal actuator/sensor locations, for $x_s(0) = [0 \ \phi_2]$ and uncertainty in ν (top plot). Manipulated input profiles (bottom plot)—solid line u_1 and dashed line u_2.

of the spatial differential operator are unstable, and thus, it affects the consideration of the number of slow modes. Since in our method the number of actuators and sensors is taken to be equal to the number of slow modes, it is clear that the value of the instability parameter influences the number of actuators and sensors. We also found through simulations that small variations of ν around the nominal value $\nu = 0.2$ considered in our calculations lead to slight changes in the optimal actuator locations.

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