Economic Model Predictive Control of Stochastic Nonlinear Systems

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This work focuses on the design of stochastic Lyapunov-based economic model predictive control (SLEMPC) systems for a broad class of stochastic nonlinear systems with input constraints. Under the assumption of stabilizability of the origin of the stochastic nonlinear system via a stochastic Lyapunov-based control law, an economic model predictive controller is proposed that utilizes suitable constraints based on the stochastic Lyapunov-based controller to ensure economic optimality, feasibility and stability in probability in a well-characterized region of the state-space surrounding the origin. A chemical process example is used to illustrate the application of the approach and demonstrate its economic benefits with respect to an EMPC scheme that treats the disturbances in a deterministic, bounded manner.

Keywords: economic MPC, stochastic nonlinear systems, probabilistic closed-loop stability

Introduction

While tracking model predictive control, in which the cost function has its minimum at the steady-state (typically taken to be the origin), has been extensively researched and practiced in industry (see, for example, the review papers[1–3]) since the mid-70s, economic model predictive control (EMPC), in which the cost function does not have in general its minimum at the steady-state and penalizes directly process economics, has been a relatively recent development (over the last decade) and continues to pose challenging control problems.[4] Part of the appeal of EMPC is that it provides a direct way for integrating in a single layer the process economic optimization layer (responsible for calculating economically optimal operating steady-states) and the process feedback control layer (where tracking MPC is typically employed) that are traditionally utilized in industrial chemical process control to optimize process economics by allowing for dynamic (off steady-state) feedback control-based operating policies. This allows for the opportunity to shift chemical manufacturing operations from steady-state operation to dynamic operation to optimize the process economic performance and thus, effectively combine dynamic economic process optimization and feedback control into one layer. Several control designs have been proposed within this new EMPC paradigm (e.g., Refs. 5–9).

One important issue that requires further study is the handling of process model uncertainty within EMPC. In Ref. 10, a multi-stage scenario-based nonlinear model predictive controller was utilized to deal with uncertainties in economic nonlinear MPC. In Ref. 8, the way to address uncertainty is to assume that the uncertain process variables are bounded, utilize a nominal process model within EMPC, and then establish robustness of the EMPC with respect to the worst-case values (typically the bounds) of the uncertain variables such that the state of the closed-loop system stays within a well-characterized region of the state-space as long as the uncertain variables are within their bounds (which must be sufficiently small). This robustness treatment of the uncertain variables is particularly useful when no information is provided for the uncertain variables other than the bounds. However, if the uncertain process variables are unbounded (or if the bounds are very conservative such that they are not sufficiently small), the traditional Lyapunov-based economic model predictive

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control (LEMPC) framework above may be unable to deal with such disturbances and lead to controller infeasibility, particularly in the presence of input constraints.

Therefore, an alternative way to handle process model uncertainty within EMPC is to model the disturbances in a probabilistic manner and consider EMPC of stochastic nonlinear systems with the goal of deriving closed-loop stability results in probability, taking advantage of the probability distribution of the uncertain variables and targeting less conservative results for closed-loop operating regions than the EMPC that treats the disturbances in a deterministic, bounded manner.

In this direction, MPC of stochastic nonlinear systems has received some attention recently. In Refs. 11 and 12, min-max and tube-based economic MPC methods were developed for nonlinear uncertain systems to achieve desired closed-loop performance and reduced computational time. The works[13,14] utilized the Markov-chain Monte Carlo technique to solve constrained stochastic optimization problems to guarantee the convergence to a near-optimal solution in a probabilistic sense. In Ref. 15, a Lyapunov-based model predictive control (LMPC) method was proposed for stochastic nonlinear systems, which guaranteed probabilistic stability (in the sense of driving the state to a steady-state) and feasibility from an explicitly characterized region of attraction. At this stage, even though the EMPC problem has attracted considerable research interest, the EMPC problem for nonlinear stochastic systems has not been investigated sufficiently. Specifically, computationally-efficient EMPC designs for handling disturbances that are not required to be small are needed with theoretical results regarding their ability to maintain the process in a well-defined region of state-space without being overly conservative such that the benefits of EMPC are reduced.

Motivated by these considerations, this work focuses on the design of stochastic Lyapunov-based economic model predictive control (SLEMPC) designs for a broad class of stochastic nonlinear systems with input constraints. Under the assumption of stabilizability of the origin of the stochastic nonlinear system via a stochastic Lyapunov-based control law, an economic model predictive control method is proposed that utilizes suitable constraints based on the stochastic Lyapunov-based controller to ensure economic optimality, feasibility and stability in probability in a well-characterized region of the state-space surrounding the origin.

The rest of the manuscript is organized as follows: in Preliminaries, the notations, the class of systems considered, and the stabilizability assumptions are given. In Main Results, we first introduce the LEMPC and establish its robustness for nonlinear systems with bounded disturbances. Subsequently, we develop a SLEMPC for nonlinear systems subject to stochastic disturbances with unbounded variation, and establish the probabilistic stability and feasibility for the closed-loop system under the sample-and-hold implementation of the SLEMPC within an explicitly characterized set of initial conditions. Finally, a nonlinear chemical process example is used to demonstrate the application of the proposed SLEMPC and also its advantages through the comparison of its economic benefits with the LEMPC.

Preliminaries

Notation

Throughout the paper, $(\Omega, \mathcal{F}, \mathbf{P})$ denotes a probability space, where Ω is the set of all possible outcomes, $\mathcal{F}$ is a σ-algebra of events, and $\mathbf{P}$ is the probability measure function of the event. Consider a stochastic process $x(t,w):[0,\infty)\times\Omega \to \mathbf{R}^n$ on $(\Omega, \mathcal{F}, \mathbf{P})$. For each $w \in \Omega$, $x(\cdot, w)$ is a realization or trajectory of the stochastic process, and we abbreviate $x(t, w)$ as $x_w(t)$. Given an event A, $\mathbf{E}(A)$, $\mathbf{P}(A)$, $\mathbf{E}(A\,|\,\cdot)$, and $\mathbf{P}(A\,|\,\cdot)$ are the expectation, the probability, the conditional expectation, and the conditional probability of the occurrence of A, respectively. The hitting time (or first hit time) τ_X of a set X is defined as the first time that the state trajectory hits the boundary of X. Based on this, we also define $\tau_X(t)=\min\{\tau_X, t\}$ and $\tau_{X,T}(t)=\min\{\tau_X, T, t\}$, where T is defined as the operation time. The notation $|\cdot|$ is used to denote the Euclidean norm of a vector, and the notation $|\cdot|_Q$ denotes the weighted Euclidean norm of a vector (i.e., $|x|_Q=x^TQ\,x$ where Q is a positive definite matrix). x^T denotes the transpose of x. $\mathbf{R}_+$ denotes the set $[0,\infty)$. The notation $L_fV(x)$ denotes the standard Lie derivative $L_fV(x):=\frac{\partial V(x)}{\partial x}f(x)$. Given a set $\mathcal{D}$, we denote the boundary of $\mathcal{D}$ by $\partial\mathcal{D}$, the closure of $\mathcal{D}$ by $\overline{\mathcal{D}}$, and the interior of $\mathcal{D}$ by $\mathcal{D}^{o}$. Set subtraction is denoted by "\", i.e., $A\backslash B:=\{x\in\mathbf{R}^n\,|\,x\in A, x\notin B\}$. A continuous function $\alpha:[0,a)\to[0,\infty)$ is said to be a class $\mathcal{K}$ function if $\alpha(0)=0$ and it is strictly increasing. The function $f(x)$ is said to be a class C^k function if the i^{th} derivative of f exists and is continuous for all $i=1,2,\ldots,k$.

Class of systems

Consider a class of continuous-time stochastic nonlinear systems described by the following stochastic differential equation (SDE):

$$dx(t)=f(x(t))dt+g(x(t))u(t)dt+h(x(t))dw(t) \tag{1}$$

where $x\in\mathbf{R}^n$ is the stochastic state vector and $u\in\mathbf{R}^m$ is the input vector. The available control action is defined by $U:=\{u\in\mathbf{R}^m\,|\,u_{\min}\le u\le u_{\max}\}\subset\mathbf{R}^m$. The disturbance $w(t)$ is a standard q-dimensional independent Wiener process defined on the probability space $(\Omega,\mathcal{F},\mathbf{P})$. $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are sufficiently smooth vector and matrix functions of dimensions $n\times1$, $n\times m$, $n\times q$, respectively. For the sake of brevity, it is assumed that the steady-state of the nominal system with $w(t)\equiv0$ is $(x_s^*, u_s^*)=(0,0)$, and the initial time t_0 is taken to be zero $(t_0=0)$. In Eq. 1, $f(x(t))+g(x(t))u(t)$ is the deterministic drift and $h(x(t))$ is the diffusion matrix. We also assume that $h(0)=0$ so that the disturbance term of Eq. 1 vanishes at the origin.

Definition 1. [16,17] *Consider an n-dimensional Itô process $dx(t)=f(x(t))dt+h(x(t))dw(t)$, where f and h are an $n\times1$ vector and $n\times q$ matrix, respectively. Let $V(t,x(t))$ be a C^2 map from $[0,\infty)\times\mathbf{R}^n\to\mathbf{R}$. Then, the process $Y(t)=V(t,x(t))$ is also an Itô process which satisfies the following equation*:

$$dY(t)=\Big[\frac{\partial V(t,x(t))}{\partial t}+\frac{\partial V(t,x(t))}{\partial x}f(x(t))+\frac{1}{2}Tr\{h(x(t))^T \times\frac{\partial^2V(t,x(t))}{\partial x^2}h(x(t))\}\Big]dt+\frac{\partial V(t,x(t))}{\partial x}h(x(t))dw(t) \tag{2}$$

Definition 2. *Given a C^2 Lyapunov function $V:\mathbf{R}^n\to\mathbf{R}_+$, the infinitesimal generator (denoted by the operator $\mathcal{L}$) of the system of Eq. 1 is defined as follows:*

$$\mathcal{L}V(x)=\quad L_fV(x)+L_gV(x)u+\frac{1}{2}Tr\{h(x)^T\frac{\partial^2V}{\partial x^2}h(x)\} \tag{3}$$

where $f=[f_1\cdots f_n]^T$, $g=[g_1\cdots g_m]$, and $g_i=[g_{i1}\cdots g_{in}]^T$, $(i=1,2,\cdots,m)$. Throughout the work, we assume that $L_fV(x)$, $L_gV(x)$, and $h(x)^T\frac{\partial^2V(x)}{\partial x^2}h(x)$ are locally Lipschitz.

Definition 3. [16] *Consider the unforced stochastic nonlinear system of Eq. 1 as follows:*

$$dx(t)=f(x(t))dt+h(x(t))dw(t) \tag{4}$$

Assuming that the equilibrium of the system of Eq. 4 is at the origin, then the origin is said to be asymptotically stable in probability, if for any $\epsilon>0$, the following conditions hold:

$$\lim_{x(0)\to 0}\mathbf{P}(\lim_{t\to\infty}|x(t)|=0)=1, \qquad \lim_{x(0)\to 0}\mathbf{P}(\sup_{t\geq 0}|x(t)|>\epsilon)=0 \tag{5}$$

Proposition 1. (Dynkin's Formula[17]) Assuming $x(0)\in\mathcal{Z}\subset\mathbf{R}^n$, $T>0$ and the solution *x(t)* of Eq. 4 exists for all time, then *x(t)* satisfies the following condition for $t\in[0,\tau_{\mathcal{Z},T}(t)]$:

$$\mathbf{E}(V(x(\tau_{\mathcal{Z},T}(t))))-V(x(0))=\mathbf{E}(\int_0^{\tau_{\mathcal{Z},T}(t)}\mathcal{L}V(x(s))ds) \tag{6}$$

Proposition 2. [16] Given the system of Eq. 4, if for all $x\in D_0\subset\mathbf{R}^n$, where D_0 is an open neighborhood of the origin, $\mathcal{L}V<0$ holds $\forall t\in(0,\infty)$, then $\mathbf{E}(V(x(t)))<V(x(0))$, $\forall t\in(0,\infty)$ and the origin of the system of Eq. 4 is asymptotically stable in probability.

Stabilizability assumptions

Consider the nominal system of Eq. 1 with $w(t)\equiv 0$ described by the following differential equation:

$$\dot{x}=f(x(t))+g(x(t))u(t) \tag{7}$$

We first assume that there exists a stabilizing feedback control law $u=\Phi_n(x)\in U$ such that the origin of the deterministic system of Eq. 7 can be rendered asymptotically stable for all $x\in D_1\subset\mathbf{R}^n$, where D_1 is an open neighborhood of the origin when a small control property is satisfied, in the sense that there exist a positive definite C^1 control Lyapunov function V and a class $\mathcal{K}$ function $\alpha_0(\cdot)$ that satisfy the following inequality:

$$\dot{V}=L_fV(x)+L_gV(x)\Phi_n(x)\leq-\alpha_0(|x|) \tag{8}$$

An example of a feedback control law that is continuous for all x in a neighborhood of the origin and renders the origin asymptotically stable is the following control law[18]:

$$\varphi_i(x)=\begin{cases}-\dfrac{p+\sqrt{p^2+|q|^4}}{|q|^2}q, & \text{if} \quad q\neq 0\\ 0, & \text{if} \quad q=0\end{cases} \tag{9a}$$

$$\Phi_{n,i}(x)=\begin{cases}u_i^{\min}, & \text{if} \quad \varphi_i(x)<u_i^{\min}\\ \varphi_i(x), & \text{if} \quad u_i^{\min}\leq\varphi_i(x)\leq u_i^{\max}\\ u_i^{\max}, & \text{if} \quad \varphi_i(x)>u_i^{\max}\end{cases} \tag{9b}$$

where p denotes $L_fV(x)$, q denotes $(L_gV(x))^T=[L_{g_1}V(x)\cdots L_{g_m}V(x)]^T$, $f=[f_1\cdots f_n]^T$, and $g_i=[g_{i1}\cdots g_{in}]^T, i=1,2,\cdots,m$. $\varphi_i(x)$ of Eq. 9a represents the i^{th} component of the control law $\Phi_n(x)$ before considering saturation of the control action at the input bounds. $\Phi_{n,i}(x)$ of Eq. 9b represents the i^{th} component of the saturated control law $\Phi_n(x)$ that accounts for the input constraint $u\in U$. Based on the above controller $\Phi_n(x)$ that satisfies Eq. 8, the set of initial conditions from which the controller $\Phi_n(x)$ can stabilize the origin of the input-constrained system of Eq. 7 is characterized as: $\phi_n=\{x\in\mathbf{R}^n\,|\,\dot{V}+\kappa V(x)\leq 0, u=\Phi_n(x)\in U, \kappa>0\}$. Additionally, we define a level set of $V(x)$ inside ϕ_n as $\Omega_{\rho'}:=\{x\in\phi_n\,|\,V(x)\leq\rho'\}$, which represents the stability region in the next LEMPC subsection.

We also assume that there exists a stochastic stabilizing feedback control law $u=\Phi_s(x)\in U$ such that the origin of the system of Eq. 1 can be rendered asymptotically stable in probability for all $x\in D_2\subset\mathbf{R}^n$, where D_2 is an open neighborhood of the origin, in the sense that there exists a positive definite C^2 stochastic control Lyapunov function V that satisfies the following inequalities:

$$\mathcal{L}V(x)=L_fV(x)+L_gV(x)\Phi_s(x)+\frac{1}{2}Tr\{h^T\frac{\partial^2V}{\partial x^2}h\}\leq-\alpha_1(|x|) \tag{10}$$

$$h(x)^T\frac{\partial^2V}{\partial x^2}h(x)\geq 0 \tag{11}$$

where $\alpha_1(\cdot)$ is a class $\mathcal{K}$ function. One of the candidate controllers that can render the origin of the stochastic system of Eq. 1 asymptotically stable in probability is given in the form of Eq. 9[19,20] where p denotes $L_fV(x)+\frac{1}{2}Tr\{h(x)^T\frac{\partial^2V}{\partial x^2}h(x)\}$, q denotes $(L_gV(x))^T$. Again, the controller $\Phi_s(x)$ is the saturated control law that accounts for the input constraints $u\in U$.

Similarly, based on the controller $\Phi_s(x)$, we define the set of initial conditions for the stochastic system of Eq. 1 from which the origin is rendered asymptotically stable in probability: $\phi_d=\{x\in\mathbf{R}^n\,|\,\mathcal{L}V+\kappa V(x)\leq 0, u=\Phi_s(x)\in U, \kappa>0\}$. It should be noted that $u=\Phi_s(x)$ is not the only controller that can be used to characterize ϕ_d. For example, the controller $\Phi_n(x)$ that is designed for the nominal system of Eq. 1 can still be applied to characterize a set of initial conditions for the stochastic system of Eq. 1. However, the resulting set of initial conditions is more conservative than the set of initial conditions ϕ_n for the nominal system of Eq. 1 due to the positive semi-definite Hessian term in Eq. 10. It may also be more conservative than the set of initial conditions ϕ_d characterized for the stochastic system of Eq. 1 using the controller $\Phi_s(x)$ since $\Phi_n(x)$ of Eq. 9 does not account for the disturbance in its formulation. Furthermore, if the diffusion term vanishes, the feedback control law that satisfies Eqs. 10 and 11 is identical to the one for deterministic problems.

Similarly, we define a level set of $V(x)$ inside ϕ_d as $\Omega_\rho:=\{x\in\phi_d\,|\,V(x)\leq\rho\}$. Although the level set $\Omega_{\rho'}$ of $V(x)$ in ϕ_n is an invariant set for the nominal system of Eq. 7 under $\Phi_n(x)$, the level set Ω_ρ in ϕ_d is not invariant for the stochastic system of Eq. 1 under $\Phi_s(x)$ due to the unbounded variation of $w(t)$. However, based on Proposition 2, the origin of the system of Eq. 1 is still rendered asymptotically stable in probability within Ω_ρ under $\Phi_s(x)$.

Main Results

In this section, the optimization problems of Lyapunov-based EMPC (LEMPC) and stochastic LEMPC (SLEMPC) are first presented. Then, stability and feasibility in probability of the closed-loop system of Eq. 1 are investigated under the sample-and-hold implementation of the SLEMPC.

Lyapunov-based EMPC for nominal systems

LEMPC optimizes an economic cost function $L_e(\cdot,\cdot)$ and maintains the closed-loop states of the nominal system of Eq. 7 in a stability region $\Omega_{\rho'}$, which is characterized as a level set of V inside ϕ_n. The LEMPC design[8] is given by the following optimization problem:

$$\max_{u\in S(\Delta)} \int_{t_k}^{t_k+\tau_P\Delta} L_e(\tilde{x}(t),u(t))\,dt \tag{12a}$$

$$\text{s.t. } \dot{\tilde{x}}(t)=f(\tilde{x}(t))+g(\tilde{x}(t))u(t) \tag{12b}$$

$$\tilde{x}(t_k)=x(t_k) \tag{12c}$$

$$u(t)\in U, \quad \forall\, t\in[t_k,t_k+\tau_P\Delta) \tag{12d}$$

$$V(\tilde{x}(t))<\rho'_e,\ \forall t\in[t_k,t_k+\tau_P\Delta),\ \text{if } x(t_k)\in\Omega^{o}_{\rho'_e} \tag{12e}$$

$$\dot{V}(x(t_k),u(t_k))\le\dot{V}(x(t_k),\Phi_n(x(t_k))),\ \text{if } x(t_k)\in\Omega_{\rho'}\backslash\Omega^{o}_{\rho'_e} \tag{12f}$$

where $\tilde{x}$ is the nominal predicted state trajectory, $S(\Delta)$ is the set of piecewise constant functions with period Δ, τ_P is the number of sampling periods of the prediction horizon, and $\dot{V}(x(t_k),u(t_k))$ represents $\frac{\partial V(x(t_k))}{\partial x}(f(x(t_k))+g(x(t_k))u(t_k))$. $\Omega_{\rho'_e}$ is a level set of V inside ϕ_n: $\Omega_{\rho'_e}:=\{x\in\mathbf{R}^n\,|\,V(x)\le\rho'_e\}$ where $0<\rho'_e<\rho'$ such that $\Omega_{\rho'_e}\subset\Omega_{\rho'}$. It is chosen to make the region $\Omega_{\rho'}:=\{x\in\phi_n\,|\,V(x)\le\rho'\}$ a forward invariant set under the LEMPC of Eq. 12.

In the optimization problem of Eq. 12, the objective function of Eq. 12a is the integral of $L_e(\tilde{x}(t),u(t))$ over the prediction horizon. The constraint of Eq. 12b is the nominal system of Eq. 7 that is used to predict the states of the closed-loop system. Eq. 12c defines the initial condition $\tilde{x}(t_k)$ of the nominal system determined from a state measurement $x(t_k)$ at $t=t_k$. Eq. 12d represents the input constraints applied over the entire prediction horizon. The constraint of Eq. 12e maintains the predicted states in the interior $\Omega^{o}_{\rho'_e}$ of $\Omega_{\rho'_e}$ when the current state $x(t_k)\in\Omega^{o}_{\rho'_e}$. However, if $x(t_k)\in\Omega_{\rho'}\backslash\Omega^{o}_{\rho'_e}$, the constraint of Eq. 12f is activated to decrease $V(x)$ at the first sampling time in the prediction horizon such that $x(t)$ for the nominal system of Eq. 7 will move toward the origin. Since $V(x)$ is required by Eq. 12f to decrease in a sampling period at least at the worst-case rate that it would decrease under the Lyapunov-based controller $\Phi_n(x)$ implemented in a sample-and-hold fashion, it is guaranteed that within finite sampling steps, $x(t)$ for the nominal system of Eq. 7 will enter $\Omega^{o}_{\rho'_e}$ again. It is notable that due to the robustness properties of Φ_n,[8] the LEMPC of Eq. 12 can maintain the closed-loop state within $\Omega_{\rho'}$ even in the presence of small bounded disturbances when ρ'_e and Δ are sufficiently small.

Lyapunov-based EMPC for uncertain systems

Consider that a bounded disturbance $w(t)$, rather than a disturbance of Wiener process $w(t)$, is introduced into the nominal system of Eq. 7 (i.e., the uncertain system can be written by the following differential equation: $\dot{x}(t)=f(x(t))+g(x(t))u(t)+h(x(t))w(t)$). Consider that the bound on this disturbance is not necessarily sufficiently small. A straightforward method to eliminate the impact of the bounded disturbances is to take the worst case scenario (i.e., the bound of $w(t)$) into consideration in the control design. An LEMPC with a form similar to that in Eq. 12 is therefore developed to account for the disturbances bounded by $|w(t)|\le\theta$, and is given as follows:

$$\max_{u\in S(\Delta)} \int_{t_k}^{t_k+\tau_P\Delta} L_e(\tilde{x}(t),u(t))\,dt \tag{13a}$$

$$\text{s.t. } \dot{\tilde{x}}(t)=f(\tilde{x}(t))+g(\tilde{x}(t))u(t) \tag{13b}$$

$$\tilde{x}(t_k)=x(t_k) \tag{13c}$$

$$u(t)\in U, \quad \forall\, t\in[t_k,t_k+\tau_P\Delta) \tag{13d}$$

$$V(\tilde{x}(t))<\tilde{\rho}'_e,\ \forall t\in[t_k,t_k+\tau_P\Delta),\ \text{if } x(t_k)\in\Omega^{o}_{\tilde{\rho}'_e} \tag{13e}$$

$$\hat{\dot{V}}(x(t_k),u(t_k))\le\hat{\dot{V}}(x(t_k),\Phi_n(x(t_k))),\ \text{if } x(t_k)\in\Omega_{\tilde{\rho}'}\backslash\Omega^{o}_{\tilde{\rho}'_e} \tag{13f}$$

where the notations and constraints of Eqs. 13a–13f follow the notations and constraints of Eqs. 12a–12f except that $\dot{V}$ in Eq. 12f is replaced by $\hat{\dot{V}}$ in Eq. 13f which includes the disturbance term $|L_hV(x)\theta|$ to account for the worst case of the bounded disturbances: $\hat{\dot{V}}=L_fV(x)+L_gV(x)u+|L_hV(x)\theta|$. Similarly, we can characterize the set of initial conditions from which the controller $\Phi_n(x)$ can stabilize the origin of the input-constrained uncertain system of Eq. 7 with $|w(t)|\le\theta$ as: $\phi'_d=\{x\in\mathbf{R}^n\,|\,\hat{\dot{V}}+\kappa V(x)=L_fV(x)+L_gV(x)u+|L_hV(x)\theta|+\kappa V(x)\le 0,\ u=\Phi_n(x)\in U,\kappa>0\}$. The regions $\Omega_{\tilde{\rho}'}$ and $\Omega_{\tilde{\rho}'_e}$ are level sets of V within ϕ'_d. Though a controller Φ_n is required that meets this condition to guarantee that a control law exists for the system in the presence of bounded disturbances that can asymptotically stabilize the origin of the closed-loop system, if θ is very big, since u is bounded, it is not guaranteed that there is any u that can cause $\hat{\dot{V}}+\kappa V(x)\le 0$. Therefore, obtaining deterministic results for control of systems with unbounded disturbances may be difficult, even if L_hV and θ are known. In other words, the set of initial conditions ϕ'_d becomes more conservative than ϕ_n, or may not even exist in the presence of large bounded disturbances.

Stochastic Lyapunov-based EMPC (SLEMPC)

Inspired by the LEMPC design, the SLEMPC design is given by the following optimization problem:

$$\max_{u\in S(\Delta)} \int_{t_k}^{t_k+\tau_P\Delta} L_e(\tilde{x}(t),u(t))\,dt \tag{14a}$$

$$\text{s.t. } \dot{\tilde{x}}(t)=f(\tilde{x}(t))+g(\tilde{x}(t))u(t) \tag{14b}$$

$$\tilde{x}(t_k)=x(t_k) \tag{14c}$$

$$u(t)\in U, \quad \forall\, t\in[t_k,t_k+\tau_P\Delta) \tag{14d}$$

$$V(\tilde{x}(t))<\rho_e,\ \forall t\in[t_k,t_k+\tau_P\Delta),\ \text{if } x(t_k)\in\Omega^{o}_{\rho_e} \tag{14e}$$

$$\mathcal{L}V(x(t_k),u(t_k))\le\mathcal{L}V(x(t_k),\Phi_s(x(t_k))),\ \text{if } x(t_k)\in\Omega_{\rho}\backslash\Omega^{o}_{\rho_e} \tag{14f}$$

where the notations and constraints of Eqs. 14a–14f follow the notations and constraints of Eqs. 12a–12f except that we use ρ, ρ_e, $\Phi_s(x)$ and $\mathcal{L}V$ to replace ρ', ρ'_e, $\Phi_n(x)$ and $\dot{V}$ respectively to indicate that $\Phi_s(x)$ is used in developing Eq. 14 rather than Φ_n. Ω_{ρ_e} is a level set inside ϕ_d: $\Omega_{\rho_e}:=\{x\in\mathbf{R}^n\,|\,V(x)\le\rho_e\}$ where $0<\rho_e<\rho$ such that $\Omega_{\rho_e}\subset\Omega_\rho$. The optimal input trajectory determined by the optimization problem of the SLEMPC is denoted by $u^*(t)$, which is calculated over the entire prediction horizon $t\in[t_k,\ t_k+\tau_P\Delta)$. The control action computed for the first sampling period of the prediction horizon $u^*(t_k)$ is sent to the actuators to be applied over the sampling period and the SLEMPC is re-solved at the next sampling time.

In the optimization problem of Eq. 14, the constraints of Eqs. 14a–14d, like the constraints of Eqs. 12a–12d, define the objective function, the prediction model, the initial condition of the prediction model, and the input constraints of the optimization problem, respectively. If the current state $x(t_k)$ belongs to $\Omega^{o}_{\rho_e}$, the constraint of Eq. 14e maintains the predicted states in $\Omega^{o}_{\rho_e}$. However, if $x(t_k) \in \Omega_\rho \backslash \Omega^{o}_{\rho_e}$, the constraint of Eq. 14f is activated to decrease $V(x)$ at the first sampling time at least at the rate that it would decrease under the stochastic Lyapunov-based controller $\Phi_s(x)$ such that $x(t)$ will move toward the origin within this sampling period in probability. Therefore, if the constraint of Eq. 14f is applied recursively, $x(t)$ will enter $\Omega^{o}_{\rho_e}$ within finite sampling steps in probability, which will be proved later.

Remark 1. There exist differences between the feedback control law $\Phi_n(x)$ applied in the LEMPC of Eq. 12 and $\Phi_s(x)$ applied in the SLEMPC of Eq. 14. Specifically, $\Phi_n(x)$ satisfies Eq. 8 and forces the states of the closed-loop nominal system of Eq. 7 to converge to the origin. However, under the control law $\Phi_s(x)$ that satisfies Eq. 10, the origin of the system of Eq. 1 subject to stochastic disturbances is rendered asymptotically stable in probability. In our previous work,[8] it is shown that if the constraint of Eq. 12f is recursively applied in a sample-and-hold fashion, the states of the closed-loop nominal system of Eq. 7 are guaranteed to be bounded in a small region around the origin ultimately. Similarly, it will be shown in subsequent sections that if the constraint of Eq. 14f is recursively applied in a sample-and-hold fashion for any $x(0) \in \Omega_\rho$, the states of the closed-loop stochastic system of Eq. 1 still converge to a small region around the origin, but in probability.

Remark 2. All the controllers of Eqs. 12–14 use the nominal system model of Eq. 7 in Eqs. 12b, 13b, and 14b which allows the state predictions of the LEMPCs of Eqs. 12 and 13, and the SLEMPC of Eq. 14 to be deterministic. One of the advantages of applying the deterministic prediction model is that it can offer significant savings in computation time compared to existing methods for calculating the propagation of stochastic uncertainties through the dynamics of the nonlinear system of Eq. 1.[21] Also, based on the SLEMPC of Eq. 14 with a deterministic model and appropriate constraints, we are still able to obtain probabilistic results for a process that accounts for the distributional information of the stochastic disturbances (i.e., the diffusion term of the system of Eq. 1), which will be proved in the next sections.

Sample-and-hold implementation

Since the control actions are implemented in a sample-and-hold fashion in the SLEMPC, in this subsection, we investigate the impact of the sample-and-hold implementation on the stability of the closed-loop system of Eq. 1 following similar arguments to those in Refs. 15 and 22. The results given in Theorem 1 will then be used in the next subsection "Stability in Probability" to derive the probabilities of closed-loop stability of the system of Eq. 1 under the SLEMPC of Eq. 14. Specifically, the probability of the set Ω_ρ remaining invariant under the sample-and-hold implementation of the SLEMPC of Eq. 14 with a sampling period Δ is given as follows.

Theorem 1. Consider the system of Eq. 1 under the SLEMPC of Eq. 14 applied in a sample-and-hold fashion (i.e., $u(t)=u(i\Delta)$, $\forall\, i\Delta \le t < (i+1)\Delta$, $i=0,1,2,\ldots$). Let $t_k=i\Delta$, $i \ge 0$, then given any probability $\lambda \in (0,1]$, there exists $\rho_s < \rho_{\min} < \rho_e < \rho$ where $\Omega_{\rho_s} := \{x \in \mathbf{R}^n \mid V(x) \le \rho_s\}$ is a small level set around the origin where $\mathcal{L}V$ is not guaranteed to remain negative for the nominal system of Eq. 7 under the sample-and-hold implementation of $u(t)$, and a sampling period $\Delta^* := \Delta^*(\lambda)$ such that if $\Delta \in (0, \Delta^*]$, then

$$\mathbf{P}\Big(\sup_{t\in[t_k,t_k+\Delta)} V(x(t)) < \rho\Big) \ge 1-\lambda,\ \forall x(t_k) \in \Omega^{o}_{\rho_e} \quad (15)$$

$$\mathbf{P}\Big(\sup_{t\in[t_k,t_k+\Delta)} V(x(t)) < \rho_{\min}\Big) \ge 1-\lambda,\ \forall x(t_k) \in \Omega^{o}_{\rho_s} \quad (16)$$

$$\mathbf{P}\Big(\sup_{t\in[t_k,t_k+\Delta)} \mathcal{L}V(x(t)) < -\epsilon < 0\Big) \ge 1-\lambda,\ \forall x(t_k) \in \Omega_\rho \backslash \Omega^{o}_{\rho_s} \quad (17)$$

Proof. Let $A_B := \{w \in \mathbf{R}^q \mid \sup_{t\in[t_k,t_k+\Delta^*)} |w(t)| \le B\}$. Using the results for standard Brownian motion,[23] given any probability $\lambda \in (0,1]$, there exists a sufficiently small B, such that $P(A_B)=1-\lambda$. Also, because x is almost surely local Hölder-continuous[16] with exponent $r<1/2$, for each realization $x_w(t)$ with $x(t_k) \in \Omega_\rho$ and $w \in A_B$, there almost surely exists a positive real number k_1, such that $\sup_{t\in[t_k,t_k+\Delta^*)} |x_w(t)-x(t_k)| \le k_1(\Delta^*)^r$, where $r<1/2$. Therefore, for all $w \in \mathbf{R}^q$, the probability of the event $A_W := \{\sup_{t\in[t_k,t_k+\Delta^*)} |x(t)-x(t_k)| \le k_1(\Delta^*)^r\}$ is: $\mathbf{P}(A_W) \ge 1-\lambda$. Based on the sample-and-hold implementation of the control actions under the SLEMPC of Eq. 14, i.e., $u(t)=u(t_k), \forall t \in [t_k, t_k+\Delta^*)$, we first prove the probability of Eq. 15. Since $V(x)$ satisfies the local Lipschitz condition, there exists a positive real number k_2, such that $|V(x(t))-V(x(t_k))| \le k_2|x(t)-x(t_k)|,\ \forall x \in \phi_d$.

Therefore, for any w that satisfies A_W, if $\Delta^* < \Delta_1 = (\frac{\rho-\rho_e}{k_2 k_1})^{(\frac{1}{r})}$, it follows that $|V(x(t))-V(x(t_k))| < \rho-\rho_e, \forall t \in [t_k, t_k+\Delta^*)$. Furthermore, $\forall x(t_k) \in \Omega^{o}_{\rho_e}$, it is obtained that $V(x(t)) < \rho$, $\forall t \in [t_k, t_k+\Delta^*)$ since $-(\rho-\rho_e) < V(x(t))-V(x(t_k)) < \rho-\rho_e$ and $\sup_{x(t_k)\in\Omega^{o}_{\rho_e}} V(x(t_k))=\rho_e$. Therefore, $\forall w \in \mathbf{R}^q$, if $x(t_k) \in \Omega^{o}_{\rho_e}$, the probability of $x(t)$ staying inside Ω_ρ within one sampling period is $\mathbf{P}(\sup_{t\in[t_k,t_k+\Delta^*)} V(x(t)) < \rho) \ge 1-\lambda$. Similarly, if $\Delta^* < \Delta_2=(\frac{\rho_{\min}-\rho_s}{k_2 k_1})^{(\frac{1}{r})}$, for any $x(t_k) \in \Omega^{o}_{\rho_s}$, the probability of Eq. 16 that $x(t)$ stays inside $\Omega_{\rho_{\min}}$ within one sampling period is $\mathbf{P}(\sup_{t\in[t_k,t_k+\Delta^*)} V(x(t)) < \rho_{\min}) \ge 1-\lambda$. Next, we prove the probability of Eq. 17 by first deriving the following equation for all $t \in [t_k, t_k+\Delta^*)$:

$$\begin{aligned}\mathcal{L}V(x(t)) &= \mathcal{L}V(x(t_k)) + (\mathcal{L}V(x(t)) - \mathcal{L}V(x(t_k)))\\ &= \mathcal{L}V(x(t_k)) + (L_fV(x(t)) - L_fV(x(t_k))) + (L_gV(x(t))\\ &\quad - L_gV(x(t_k)))u(t_k) + \frac{1}{2}Tr\{h(x(t))^T \frac{\partial^2 V(x(t))}{\partial x^2} h(x(t))\}\\ &\quad - \frac{1}{2}Tr\{h(x(t_k))^T \frac{\partial^2 V(x(t_k))}{\partial x^2} h(x(t_k))\}\end{aligned} \quad (18)$$

For any $x(t_k) \in \Omega_\rho \backslash \Omega^{o}_{\rho_s}$, it holds that $V(x(t_k)) \ge \rho_s$, which implies $\mathcal{L}V(x(t_k)) \le -\kappa V(x(t_k)) \le -\kappa\rho_s$ by the definition of ϕ_d. Since $L_fV(x)$, $L_gV(x)$, and $h(x(t))^T \frac{\partial^2 V(x)}{\partial x^2} h(x(t))$ are locally Lipschitz, there exist positive real numbers k_3, k_4, k_5, such that $|L_fV(x(t))-L_fV(x(t_k))| \le k_3|x(t)-x(t_k)|$, $|L_gV(x(t))u(t_k) - L_gV(x(t_k))u(t_k)| \le k_4|x(t)-x(t_k)|$, and $|\frac{1}{2}Tr\{h(x(t))^T \frac{\partial^2 V(x(t))}{\partial x^2} h(x(t))\} - \frac{1}{2}Tr\{h(x(t_k))^T \frac{\partial^2 V(x(t_k))}{\partial x^2} h(x(t_k))\}| \le k_5|x(t)-x(t_k)|, \forall x \in \phi_d$. Let $0 < \epsilon < \kappa\rho_s$ and $\Delta^* < \Delta_3 = (\frac{\kappa\rho_s-\epsilon}{k_1(k_3+k_4+k_5)})^{(\frac{1}{r})}$. It follows from these conditions and Eq. 18 that $\forall w \in A_W$, $\mathcal{L}V(x(t)) < -\epsilon < 0$, $\forall t \in [t_k, t_k+\Delta^*)$ holds. Therefore, by choosing the sampling period $\Delta \in (0, \Delta^*]$, given any initial condition $x(t_k) \in \Omega_\rho \backslash \Omega^{o}_{\rho_s}$, the probability that $\mathcal{L}V(x(t)) < -\epsilon$ is as follows: $\mathbf{P}(\sup_{t\in[t_k,t_k+\Delta^*)} \mathcal{L}V(x(t)) < -\epsilon) \ge 1-\lambda$. Finally, let $\Delta^* \le \min\{\Delta_1, \Delta_2, \Delta_3\}$, then the probabilities of Eqs. 15–17 are all satisfied for $\Delta \in (0, \Delta^*]$. ■

Remark 3. Theorem 1 gives the conditions under which the closed-loop state under the SLEMPC of Eq. 14 does not leave Ω_ρ in probability within a sampling period when the initial condition is within $\Omega^o_{\rho_e}$, and also gives the conditions under which, under repeated application of the constraint of Eq. 14f at subsequent sampling periods regardless of the location of the state measurement at t_k, the closed-loop state moves toward a neighborhood of the origin in a sampling period or once it enters it, does not leave that neighborhood in a sampling period, in probability. The conditions developed require that the sampling period be sufficiently small for the probability desired for the results, and also that the level sets of the Lyapunov function bounded by ρ_s, ρ_{min}, ρ_e, and ρ be sized sufficiently with respect to one another and Δ so that the results hold. For example, a critical step in the proof requires that $|V(x(t))-V(x(t_k))| < k_2k_1\Delta^r$, where Δ is a direct function of upper bounds on level sets of V. Consider Δ_1, for example. If $\rho-\rho_e$ is larger (i.e., Ω_{ρ_e} is more conservative), then there is a larger difference between $V(x(t))$ and ρ for $t\in[t_k, t_k+\Delta^*)$, which gives greater conservatism to the probabilistic results. This concept of creating a larger gap between ρ_e and ρ to reduce the likelihood that the closed-loop state will exit the stability region during a sampling period will be demonstrated in the section "Application to a Chemical Process Example." Furthermore, to achieve a certain probability of staying inside either Ω_ρ or $\Omega_{\rho_{\min}}$, Δ has to be small enough. Since Δ is dependent on the difference between the upper bounds on two level sets of V in the proof of Theorem 1, there is a tradeoff between making Δ smaller and making the smaller level set (Ω_{ρ_e} or Ω_{ρ_s}) less conservative compared to the larger region (Ω_ρ or $\Omega_{\rho_{\min}}$) while achieving the same probability of maintaining the closed-loop state within this larger region. This indicates that there are parallels at a theoretical level between the conditions required in Ref. 8 for guaranteed closed-loop stability in the presence of sufficiently small bounded disturbances (i.e., the disturbances, sampling period, and upper bounds on level sets of the Lyapunov function must be sufficiently small) and the conditions which have been derived in Theorem 1 (where to achieve certain probabilities of closed-loop stability, the bounds on the disturbances, sampling period, and upper bounds on level sets of the Lyapunov function are all again required to be sufficiently small, as defined with respect to one another, to achieve the theoretical results).

Stability in probability

In this subsection, the probabilistic stability of the SLEMPC of Eq. 14 applied in a sample-and-hold fashion is established through three aspects, which are the probability of the closed-loop states $x(t)$ staying in Ω_ρ when initialized within Ω_{ρ_e}, the probability of $x(t)$ moving to Ω_{ρ_e} when it is outside that region, and the probability of $x(t)$ converging to a small ball around the origin $\Omega_{\rho_{\min}}$ if at some point it is required to operate the system of Eq. 1 at the steady-state by applying the constraint of Eq. 14f all the time. Theorem 2 below provides the probabilities with respect to the above three events. Example trajectories exemplifying each of these events are also shown as the realizations starting from x^1, x^2 and x^3, respectively in Figure 1. In the following theorem, we will make use of Eq. 6. Though $x(0)$ in that equation represents a general initial condition and could therefore conceptually be replaced with t_k, for ease of presentation in the following, we refer to $x(t_k)$ as $x(0)$, and therefore represent the time within each prediction horizon as extending from 0 to Δ.

Theorem 2. Consider the system of Eq. 1 under the SLEMPC of Eq. 14 applied in a sample-and-hold fashion, i.e., $u(t)=u(0)$, $\forall t\in[0,\Delta)$. Then, given the initial condition $x(0)\in\Omega_\rho$, positive real numbers $\rho>\rho_e>\rho_{\min}>\rho_s$ and $\rho_c\in[\rho_e,\rho]$ and probability $\lambda\in(0,1]$, there exist a sampling period $\Delta\in(0,\Delta^*(\lambda)]$ and probabilities $\beta,\beta',\gamma,\gamma'\in[0,1]$ such that

$$\mathbf{P}(\sup_{t\in[0,\Delta)} V(x(t))<\rho)\geq(1-\beta)(1-\lambda),\quad \forall x(0)\in\Omega_{\rho_e} \tag{19}$$

$$\mathbf{P}(\tau_{\mathbf{R}^n\backslash\Omega^o_{\rho_e}}(\Delta)\leq\tau_{\Omega_\rho}(\Delta))\geq(1-\gamma')(1-\lambda),\quad \forall x(0)\in\Omega_{\rho_c}\backslash\Omega^o_{\rho_e} \tag{20}$$

$$\begin{aligned}&\mathbf{P}(\sup_{t\in[0,\Delta)} V(x(t))<\rho,\ \tau_{\mathbf{R}^n\backslash\Omega^o_{\rho_s}}<\infty,\\ &\sup{}_{t\in[0,\Delta)} V(x(t+\tau_{\mathbf{R}^n\backslash\Omega^o_{\rho_s}}))<\rho_{\min})\\ &\geq(1-\beta')(1-\gamma)(1-\lambda)^2,\qquad \forall x(0)\in\Omega_{\rho_e}\backslash\Omega^o_{\rho_s}\end{aligned} \tag{21}$$

where

$$\frac{\sup_{x\in\partial\Omega_{\rho_e}}V(x)}{\inf_{x\in\mathbf{R}^n\backslash\Omega_\rho}V(x)}\leq\beta \tag{22a}$$

$$\sup_{x\in\Omega_{\rho_e}\backslash\Omega^o_{\rho_s}}\frac{V(x)}{\rho}\leq\gamma \tag{22b}$$

$$\sup_{x\in\Omega_{\rho_c}\backslash\Omega^o_{\rho_e}}\frac{V(x)}{\rho}\leq\gamma' \tag{22c}$$

$$\frac{\sup_{x\in\partial\Omega_{\rho_s}}V(x)}{\inf_{x\in\mathbf{R}^n\backslash\Omega_{\rho_{\min}}}V(x)}\leq\beta' \tag{22d}$$

Proof. The proof consists of four parts. In the first part, we show that under the SLEMPC of Eq. 14, any state trajectory initiated from $x(0)\in\Omega_{\rho_e}$ has the probability defined by Eq. 19 of staying in Ω_ρ throughout a sampling period. However, if $x(0)\in\Omega_{\rho_c}\backslash\Omega^o_{\rho_e}$, where $\rho_c\in[\rho_e,\rho]$, we prove that under the SLEMPC of Eq. 14, there exists the probability of Eq. 20 for the state of the closed-loop system to move back into Ω_{ρ_e} before it leaves Ω_ρ in a sampling period. Additionally, we show that under the SLEMPC of Eq. 14, if the contractive constraint of Eq. 14f is applied all the time to operate the system of Eq. 1 at its steady-state, the state of the closed-loop system will ultimately enter a small ball $\Omega_{\rho_{\min}}$ around the origin while always remaining in Ω_ρ with the probability of Eq. 21. Lastly, we show that the above results that are derived for one sampling period can be generalized to overall probabilities over the entire operating period. For the sake of simplicity, we denote the probabilities and expectations conditional on the event of A_W as $\mathbf{P}^*(\cdot)$ and $\mathbf{E}^*(\cdot)$.

Part 1 : Because $(1-\lambda)\geq(1-\beta)(1-\lambda)$, for $\beta,\lambda\in(0,1]$, the result of Eq. 15 causes Eq. 19 to hold for $x(0)\in\Omega^o_{\rho_e}$. Therefore, to show that Eq. 19 holds for all $x(0)\in\Omega_{\rho_e}$, it is necessary to show that the extreme case $x(0)\in\partial\Omega_{\rho_e}$ satisfies Eq. 19. Assuming $x(0)\in\partial\Omega_{\rho_e}$, under the constraint of Eq. 14f, the optimization problem of Eq. 14 is solved such that $\mathcal{L}V$ is forced to be negative at $t=0$, which implies that Eq. 17 holds with the probability of the event A_W.

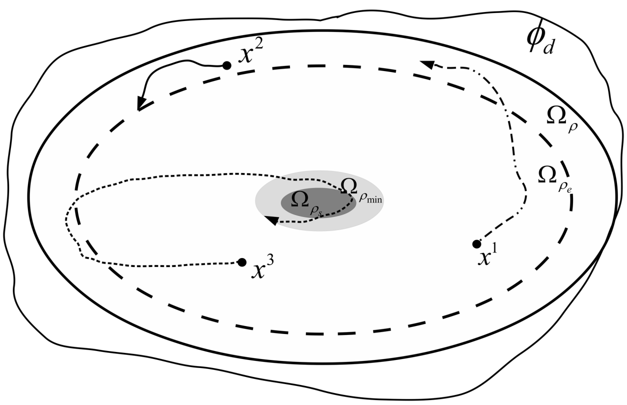

Figure 1. A schematic representing the characterized set of initial conditions ϕ_d, the level sets $\Omega_\rho, \Omega_{\rho_e}, \Omega_{\rho_{\min}}$, and Ω_{ρ_s}, together with three example realizations starting from x^1, x^2, and x^3, which correspond to the events listed in Theorem 2.

Using Eq. 6, the following equation can be derived with $\mathcal{Z}=\Omega_\rho\setminus\Omega^o_{\rho_e}, t\in[0,\Delta)$ and $T=\infty$:

$$\mathbf{E}^*(V(x(\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_e}}(t))))=V(x(0))+\mathbf{E}^*\Big(\int_0^{\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_e}}(t)}\mathcal{L}V(x(s))ds\Big) \tag{23}$$

We can also derive the following probability as in Ref. 24, for all $x(0)\in\partial\Omega_{\rho_e}$:

$$\mathbf{E}^*(V(x(\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_e}}(t))))=\int_{V\geq\tilde{\lambda}}V(x(\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_e}}(t)))d\mathbf{P}^* + \int_{V<\tilde{\lambda}}V(x(\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_e}}(t)))d\mathbf{P}^* \geq \tilde{\lambda}\mathbf{P}^*(V(x(\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_e}}(t)))\geq\tilde{\lambda}) \tag{24}$$

Using Eq. 23 and setting $\tilde{\lambda}=\inf_{x\in\mathbf{R}^n\setminus\Omega_\rho}V(x)$, the following inequality is derived $\forall x(0)\in\partial\Omega_{\rho_e}$:

$$\mathbf{P}^*(V(x(t))\geq\rho,\ \text{for some}\ t\in[0,\Delta)) \leq \frac{V(x(0))+\mathbf{E}^*\Big(\int_0^{\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_e}}(t)}\mathcal{L}V(x(s))ds\Big)}{\inf_{x\in\mathbf{R}^n\setminus\Omega_\rho}V(x)} \leq \frac{V(x(0))}{\inf_{x\in\mathbf{R}^n\setminus\Omega_\rho}V(x)} \tag{25}$$

where the last inequality follows from Eq. 23 and the fact that the value of $\mathcal{L}V$ is negative, conditioned on A_W. Bounding Eq. 25 with Eq. 22a and taking the complementary events, the following probability is obtained: $\inf_{x(0)\in\partial\Omega_{\rho_e}}\mathbf{P}^*(V(x(t))<\rho,\forall t\in[0,\Delta))\geq(1-\beta)$, from which the probability of Eq. 19 is obtained via the definition of conditional probability.

Part 2 : If $x(0)\in\Omega_\rho\setminus\Omega^o_{\rho_e}$, we consider the event that the closed-loop realization of the system of Eq. 1 moves back to Ω_{ρ_e} before reaching the boundary of Ω_ρ in a sampling period, of which the probability is obtained in this part. Consider that these hitting times occur within a sampling period. Assuming $x(0)\in\partial\Omega_{\rho_c}$, where $\Omega_{\rho_c}:=\{x\in\phi_d\,|\,V(x)\leq\rho_c,\ \rho_c\in[\rho_e,\rho]\}$, we can show that $\mathbf{P}^*(\tau_{\Omega_\rho\setminus\Omega^o_{\rho_e}}<\infty)=1$ using similar arguments as in Ref. 24. Then, consider the event $A_T=\{\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_e}}>\tau_{\Omega_\rho}\}$, which implies that the state of the closed-loop system of Eq. 1 reaches the boundary of Ω_ρ before it reaches the boundary of Ω_{ρ_e}. The probability of A_T is determined via Eq. 24 and the fact that the event $\{\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_e}}>\tau_{\Omega_\rho}\}$ belongs to the event $\Big\{\frac{V(x(\tau_{\Omega_\rho\setminus\Omega^o_{\rho_e}}))}{\rho}\geq 1\Big\}$, which is shown as follows:

$$\mathbf{P}^*(\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_e}}>\tau_{\Omega_\rho})\leq\mathbf{P}^*\Big(\frac{V(x(\tau_{\Omega_\rho\setminus\Omega^o_{\rho_e}}))}{\rho}\geq 1\Big)\leq\frac{V(x(0))}{\rho} \tag{26}$$

Bounding Eq. 26 by Eq. 22c, it follows that $\sup_{x(0)\in\Omega_{\rho_c}\setminus\Omega^o_{\rho_e}}\mathbf{P}^*(\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_e}}>\tau_{\Omega_\rho})\leq\gamma'$. Therefore, taking the complementary event of A_T and using the definition of conditional probability, the probability of Eq. 20 is obtained. If we consider that neither hitting time $\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_e}}$ nor τ_{Ω_ρ} occurs within the sampling period, then $\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_e}}(\Delta)=\tau_{\Omega_\rho}(\Delta)=\Delta$, which is always a true statement, and therefore the inequality of Eq. 20 is again obtained.

Part 3 : Next, we consider the scenario where the contractive constraint of Eq. 14f is implemented recursively to drive the state of the system of Eq. 1 to its equilibrium, rather than causing it to remain in Ω_ρ only, and calculate the probability of the closed-loop trajectory entering $\Omega_{\rho_{\min}}$ in finite time. Since we assume that the constraint of Eq. 14f is applied for all $x(0)\in\Omega_\rho\setminus\Omega^o_{\rho_s}$, $\mathcal{L}V<-\epsilon$ holds from Eq. 17 with the probability of A_W. As a result, the following equations can be derived using the same steps for Eqs. 23 and 26 when $\mathcal{L}V<-\epsilon$ holds $\forall x(0)\in\Omega_\rho\setminus\Omega^o_{\rho_s}$:

$$\mathbf{E}^*(V(x(\tau_{T,\Omega_\rho\setminus\Omega^o_{\rho_s}}(t))))\leq V(x(0)) \tag{27a}$$

$$\mathbf{P}^*(\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_s}}>\tau_{\Omega_\rho})\leq\mathbf{P}^*\Big(\frac{V(x(\tau_{\Omega_\rho\setminus\Omega^o_{\rho_s}}))}{\rho}\geq 1\Big)\leq\frac{V(x(0))}{\rho} \tag{27b}$$

Due to the fact that $\mathbf{P}^*(\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_s}}=\tau_{\Omega_\rho})=0$, the following probability is obtained by bounding Eq. 27b with Eq. 22b and taking the complementary events:

$$\inf_{x(0)\in\Omega_{\rho_e}\setminus\Omega^o_{\rho_s}}\mathbf{P}^*(\tau_{\mathbf{R}^n\setminus\Omega^o_{\rho_s}}<\tau_{\Omega_\rho})\geq 1-\gamma \tag{28}$$

It remains to show that for all $x(0)\in\Omega_{\rho_s}$, there exists a characterizable probability that the states of the closed-loop system will stay in $\Omega_{\rho_{\min}}$. The results of Eq. 16 cover the case that $x(0)\in\Omega^o_{\rho_s}$ and will be used in the proof of Eq. 21 below. First, however, we will cover the proof of Eq. 21 when $x(0)\in\partial\Omega_{\rho_s}$ and obtain the following probability using the same steps as performed in Eq. 25:

$$\mathbf{P}^*(V(x(t))\geq\rho_{\min},\ \text{for some}\ t\in[0,\Delta))\leq\frac{V(x(0))}{\inf_{x\in\mathbf{R}^n\setminus\Omega_{\rho_{\min}}}V(x)} \tag{29}$$

Using Eq. 22d, and taking the complementary event, we can derive the following probability:

$$\inf_{x(0)\in\partial\Omega_{\rho_s}}\mathbf{P}^*(V(x(t))<\rho_{\min},\ \forall t\in[0,\Delta))\geq(1-\beta') \tag{30}$$

Therefore, the probability of Eq. 21 is obtained from Eqs. 28 and 30, the strong Markov property and the definition of conditional probability. When $x(0)\in\Omega^o_{\rho_s}$, Eq. 16 holds. Because $(1-\lambda)\geq(1-\lambda)(1-\beta')$, we again have that Eq. 21 holds even if $x(0)\in\Omega^o_{\rho_s}$.

Part 4 : The probabilities of Eqs. 19–21 are derived to quantify the probabilistic closed-loop stability of the system of Eq. 1 under the SLEMPC of Eq. 14 during one sampling period. Moreover, we can further calculate the probabilities of Eqs. 19–21 over the entire operation period $[0, \tau_N\Delta)$ for the SLEMPC of Eq. 14. For example, assuming the initial condition $x(0) \in \Omega_{\rho_e}$, the probability that the states of the closed-loop system stay within Ω_ρ with $t \in [0, \tau_N\Delta)$ is the product of each probability within one sampling period because the events corresponding to the state of the closed-loop system of Eq. 1 remaining inside Ω_ρ during each sampling period starting from different given initial conditions are independent. Specifically, $\forall x(0) \in \Omega_{\rho_e}$, let $V(x(t+i\Delta))=\rho_i < \rho$, $i=0,1,\ldots,\tau_N-1$, and let $A_S=\{\sup_{t\in[0,\tau_N\Delta)} V(x(t)) < \rho\}$ represent the event that under the SLEMPC of Eq. 14, the stability region Ω_ρ remains invariant over the operation period $t \in [0, \tau_N\Delta)$. Then the probability of A_S extended from Eq. 19 can be calculated as follows:

$$\mathbf{P}(A_S) \geq (1-\lambda)^{\tau_N} \prod_{i=0,1,\ldots,\tau_N-1} (1-\beta_i) \tag{31}$$

where β_i follows a similar definition of β to that of Eq. 22a

$$\frac{\sup_{x\in\partial\Omega_{\rho_i}} V(x)}{\inf_{x\in\mathbf{R}^n\setminus\Omega_\rho} V(x)} \leq \beta_i', \quad \beta_i=\max\{\beta, \beta_i'\} \tag{32}$$

From Eq. 32, β_i takes the maximum value of β and β_i' for the reason that if $x(t) \in \Omega_{\rho_i} \subset \Omega_{\rho_e}$, the probability is already given in Eq. 19, while the probability for the case that $x(t) \in \Omega_{\rho_i} \supset \Omega_{\rho_e}$ is, however, dependent on the value of ρ_i (i.e., Eq. 32 should be used to recover Eq. 22c). ■

Remark 4. The sampling period $[t_k, t_k+\Delta)$ is used in Theorem 1 to obtain the general results of sample-and-hold implementation. In Theorem 2, the sampling period $[0, \Delta)$ is used to obtain the probability of closed-loop stability under the SLEMPC. However, it should be noted that without loss of generality, $[0, \Delta)$ can be replaced with the general form $[t_k, t_k+\Delta)$ in Theorem 2 because the result of Theorem 2 does not depend on $t=0$ or $t=t_k$.

Remark 5. It should be noted that the prediction horizon length τ_P will affect the control action $u(t)$. However, since all the $u(t)$ have to satisfy the constraint of the SLEMPC of Eq. 14, the corresponding probability has to be greater than the lower bound (i.e., Eqs. 19–21). But the probability itself may be different. Additionally, from Eq. 31, it is observed that as the operation time $\tau_N \to \infty$, the lower bound of the probability that the trajectory is contained in Ω_ρ tends to zero. It should be noted that while the length of the prediction horizon influences the probability of closed-loop stability (but not the lower bound of the probability of closed-loop stability), this dependence is not unique to the MPC, but to all control designs that try to keep the process state within a specific region in state-space when it is influenced by disturbances of unbounded variation (i.e., the probability to keep the process state within Ω_ρ goes to zero as the process operation time $\tau_N \to \infty$). In the practical application of SLEMPC, the operation time τ_N is considered as a finite number. Based on this practical consideration, the probabilities derived in this work can still provide the probability that the trajectory is contained in Ω_ρ within finite operation time.

Remark 6. Under repeated application of the constraint of Eq. 14f, the same result (i.e., the probability of Eq. 21) as in the stochastic LMPC work,[15] where the closed-loop state is driven, in probability, to a neighborhood of the origin under the Lyapunov-based controller, is achieved. However, under the regular operation of the SLEMPC of Eq. 14, where the system of Eq. 1 is operated in a bounded region of state-space that is not necessarily a neighborhood of the origin, the other two probabilities of Eqs. 19 and 20 are established to derive the probabilistic closed-loop stability. Note that the closed-loop stability under the SLEMPC of Eq. 14 is conceptually different from the one in Ref. 15 (i.e., it allows for time-varying operation to maximize profit in a predefined region around the steady-state, rather than enforcing steady-state operation). Therefore, the contribution of the SLEMPC of Eq. 14 lies in its potential to be economically beneficial while still providing probabilistic closed-loop stability in the sense of maintaining (with probability) the closed-loop states in a well-characterized stability region.

Remark 7. As for Eqs. 15–17, the proof of Eqs. 19–21 reveals that the probabilities of the closed-loop stability results in these equations holding are impacted by the desired value of λ, the sampling period, ρ_e, ρ, ρ_s, and $\rho_{\min}$. First, because the results of Eqs. 15–17 are used in deriving the results in Eqs. 19–21, the principles for selecting λ, Δ, ρ_e, ρ, ρ_s, and $\rho_{\min}$ described in Remark 3 hold in this case. In addition, to manipulate the probabilities in Eqs. 19–21 further, ρ_e, ρ, ρ_s, and $\rho_{\min}$ can be adjusted to manipulate the values of β, γ, γ', and β' in Eq. 22.

Remark 8. In Theorem 2, it is established that under the proposed SLEMPC of Eq. 14, the probabilistic closed-loop stability results for the stochastic nonlinear system of Eq. 1 can be achieved via the known distributional information of the stochastic disturbances. Also, based on the previous work on LEMPC,[8] it has been established that closed-loop stability for the system of Eq. 1 with sufficiently small disturbances can be guaranteed under the LEMPC of Eq. 12. If we compare the SLEMPC of Eq. 14 with the LEMPC of Eq. 12, the advantage of the SLEMPC is its ability to cope with stochastic disturbances of which the variation is unbounded, which is a more general case in practice compared to the LEMPC applied to the system with bounded disturbances. However, it should be noted that because of stochastic disturbances with unbounded variation, there is no way to stabilize the steady-state with certainty for both SLEMPC and LEMPC, and therefore backup controllers and safety systems of Eq. 14 should be designed that are able to handle the process state exiting the stability region.

Remark 9. It is noted that the probability of maintaining the closed-loop state within the stability region throughout the prediction horizon changes at each sampling period as measurements occur. For example, at t_0, the probability of the closed-loop state remaining within the stability region throughout the remainder of the prediction horizon may be as in Eq. 31. However, if at t_1, the closed-loop state is still within the stability region, then there is certainty that at t_1 the state was within this region. Therefore, the probability that the closed-loop state remains within the stability region becomes Eq. 31, but with τ_N replaced by τ_N-1. Therefore, though the probability of maintaining the closed-loop state within the prediction horizon throughout the entire period of operation may look low at t_0, this is not the probability that will continue to hold at each sampling period, but it will become higher at each sampling period as the closed-loop state does not exit Ω_ρ.

Feasibility in probability

Due to the consideration of stochastic disturbances with unbounded variation, no guarantee can be made that the optimization problem of Eq. 14 is recursively feasible. Therefore, Theorem 3 below provides the probability that the SLEMPC of Eq. 14 is recursively feasible for time $t \in [0, \tau_N \Delta)$.

Theorem 3. Consider the system of Eq. 1 under the SLEMPC of Eq. 14 applied in a sample-and-hold fashion with $\Phi_s(x)$ meeting Eq. 10. Then, if $x(0) \in \Omega_\rho$, the probability of the event A_F which represents that the SLEMPC of Eq. 14 is solved with satisfaction of recursive feasibility for time $t \in [0, \tau_N \Delta)$ is as follows:

$$\mathbf{P}(A_F) \geq (1-\lambda)^{\tau_N} \prod_{i=0,1,\ldots,\tau_N-1} (1-\beta_i) \tag{33}$$

Proof. To calculate the probability of A_F, we first show that $\mathbf{P}(A_F \mid A_S)=1$. Based on the probability of the event A_S associated with the operation period $[0, \tau_N \Delta)$ of Eq. 31, we can develop the following proof for probabilistic recursive feasibility. If $x(t_k) \in \Omega^{\text{o}}_{\rho_e}$ at time $t=t_k=k\Delta$ where $k \in [0, \tau_N]$, then the constraints of Eqs. 14d and 14e over the prediction horizon are guaranteed to be satisfied by $u(t)=\Phi_s(\tilde{x}(t_q)) \in U$, $\forall t \in [t_q, t_{q+1})$, $q=k,\ldots,k+\tau_P-1$ since $\Phi_s(\tilde{x}(t_q)) \in U$ by definition can force $V(\tilde{x}(t_q)) \leq V(x(t_k)) < \rho_e$, $q=k,\ldots,k+\tau_P-1$ if using the deterministic prediction model of Eq. 14b.[8] Additionally, if $x(t_k) \in \Omega_\rho \backslash \Omega^{\text{o}}_{\rho_e}$, again let $u(t_k)=\Phi_s(x(t_k)) \in U$, and this again satisfies Eq. 14d and it trivially satisfies the constraint of Eq. 14f. Therefore, as long as $x(t_k) \in \Omega_\rho$, $\forall t \in [0, \tau_N \Delta)$, according to the above two scenarios, the optimization problem of Eq. 14 can be solved recursively while satisfying all the constraints, which implies the probability of recursive feasibility is equal to the probability of closed-loop stability (i.e., $\mathbf{P}(A_F \mid A_S)=1$). Combined with Eq. 31, $\mathbf{P}(A_F)$ is obtained via the definition of conditional probability. ■

Remark 10. The SLEMPC of Eq. 14 cannot guarantee closed-loop stability or recursive feasibility with certainty (only in probability). As a result, it is possible that the state may leave the stability region when the process of Eq. 1 is operated under the SLEMPC of Eq. 14. Once the state of the closed-loop system of Eq. 1 leaves the stability region Ω_ρ, there is no systematic way to design a control law to drive the state back into Ω_ρ. A potential approach that could be tried is to find a new control law $\Phi_s(x)$, under which a larger set of initial conditions (i.e., ϕ_d) can be characterized, and thus, the state may be driven back toward the origin for states outside the current Ω_ρ. However, the problem caused by stochastic disturbances with unbounded variation is essentially a trade-off between the higher economic benefits and the closed-loop stability in larger probability. Specifically, if we want to add more conservatism to the SLEMPC design of Eq. 14 (i.e., choosing a smaller ρ_e), then we should be sacrificing economic performance. Therefore, in order to achieve a balanced solution for both process economics and closed-loop stability, it is suggested to conduct numerical simulations to determine the optimal controller parameters.

Application to a Chemical Process Example

In this section, a chemical process example is used to illustrate the application of the proposed SLEMPC and how the performance of a nonlinear process under this controller compares with that of the process under LEMPC. Specifically, a nonisothermal continuous stirred tank reactor (CSTR) where an irreversible second-order exothermic reaction takes place is considered. In the reactor, the reactant A is converted to the product B via the chemical reaction $A \rightarrow B$. The CSTR is coated with a heating jacket that supplies or removes heat from the reactor. Based on material and energy balances, the CSTR dynamic model is of the following form:

Table 1. Parameter Values of the CSTR

$T_0=300$ K	$F=5$ m^3/hr
$V=1$ m^3	$E=5\times10^4$ kJ/kmol
$k_0=8.46\times10^6$ m^3/kmol hr	$\Delta H=-1.15\times10^4$ kJ/kmol
$C_p=0.231$ kJ/kg K	$R=8.314$ kJ/kmol K
$\rho=1000$ kg/m^3	$C_{A0_s}=4$ kmol/m^3
$Q_s=0.0$ kJ/hr	$C_{A_s}=1.22$ kmol/m^3
$T_s=438$ K	

$$dC_A=\frac{F}{V}(C_{A0}-C_A)dt-k_0e^{-E/RT}C_A^2dt+\sigma_1(C_A-C_{As})dw_1(t) \tag{34a}$$

$$\begin{aligned} dT=&\frac{F}{V}(T_0-T)dt-\frac{\Delta H k_0}{\rho C_p}e^{-E/RT}C_A^2dt \\ &+\frac{Q}{\rho C_p V}dt+\sigma_2(T-T_s)dw_2(t) \end{aligned} \tag{34b}$$

where C_A is the concentration of reactant A in the reactor, V is the volume of the reacting liquid in the reactor, T is the temperature of the reactor and Q denotes the heat input rate. The concentration of reactant A in the feed is C_{A0}. The feed temperature and the volumetric flow rate are T_0 and F, respectively. The liquid has a constant density of ρ and a heat capacity of C_p. k_0, E, and ΔH are the reaction preexponential factor, activation energy and the enthalpy of the reaction, respectively. Process parameter values are listed in Table 1.

The CSTR is initially operated at the steady-state $x_s=(C_{As}, T_s)=(1.22$ kmol/m^3, 438 K$)$ and $u_s=(C_{A0_s}\ Q_s)=(4$ kmol/m^3, 0 kJ/hr$)$. The manipulated inputs are the inlet concentration of species A and the heat input rate, which are represented by the deviation variables $\Delta C_{A0}=C_{A0}-C_{A0_s}$, $\Delta Q=Q-Q_s$, respectively. The manipulated inputs are bounded by: $|\Delta C_{A0}| \leq 3.5$ kmol/m^3 and $|\Delta Q| \leq 5\times10^5$ kJ/hr. Therefore, the states and the inputs in deviation variable form for the closed-loop system are $x^T=[C_A-C_{As}\ T-T_s]$ and $u^T=[\Delta C_{A0}\ \Delta Q]$, respectively, such that the equilibrium point of the system is at the origin of the state-space. The disturbance terms dw_1 and dw_2 in Eq. 34 are independent standard Gaussian white noise with the standard deviations $\sigma_1=2.5\times10^{-3}$ and $\sigma_2=0.15$, respectively. It is noted that the disturbance terms of Eq. 34 vanish at the steady-state. Also, the disturbances become larger as the closed-loop states of Eq. 34 deviate from the steady-state (normal operating conditions), which is consistent with the fact that it is more likely to introduce the noise into the system under off steady-state operating conditions.

The control objective of the SLEMPC of Eq. 14 is to maximize the economic cost of the CSTR process of Eq. 34 while keeping the closed-loop state trajectories in the stability region Ω_ρ. Thus, the objective function of Eq. 14a that is maximized is the production rate of B: $L_e(\tilde{x},u)=k_0e^{-E/RT}C_A^2$. The Lyapunov function is designed using the standard quadratic form $V(x)=x^TPx$, where the positive definite matrix $P=\begin{bmatrix}1060 & 22 \\ 22 & 0.52\end{bmatrix}$ is chosen to characterize the set of initial conditions ϕ_d for the stochastic system of Eq. 34. The stability region Ω_ρ is a level set inside ϕ_d, which is chosen as $\rho=368$. Additionally, the explicit Euler method with an integration time step of $h_c=10^{-4}$ hr is applied to numerically simulate the dynamic model of Eq. 34. The nonlinear optimization

Table 2. Experimental Probability for Different Values of ρ_e

ρ_e/ρ	$\mathbf{P}(A_V)$
0.99	48.6%
0.95	67.2%
0.92	77.8%
0.90	87.5%
0.87	97.2%

problem of the SLEMPC of Eq. 14 is solved using the IPOPT software package[25] with the sampling period $\Delta=10^{-2}$ hr. Based on the results in the "Sample-and-hold Implementation" section, we can obtain a larger probability of closed-loop stability for the system of Eq. 34 as the sampling period Δ becomes smaller. However, considering the practical application of MPC, and the time scale of the dynamics of the CSTR process, we choose $\Delta=10^{-2}$ hr and mainly focus on the impact of ρ_e on the probabilistic closed-loop stability in the simulations below.

We first demonstrate that the choice of the difference between ρ_e and ρ, for a fixed Δ, impacts the likelihood that the closed-loop state is maintained within Ω_ρ throughout a sampling period, as implied by the theoretically derived probability bounds of Theorems 1 and 2. We derived the experimental probabilities via 500 simulation runs. Let A_V denote the event that the maximum value of $V(x)$ in each realization is less than ρ with the operating time $t_s=1$ hr and the initial condition $x^T=[0\ 0]$. The results are reported in Table 2. Also, the plots of the maximum value of $V(x)$ in each realization with respect to the number of simulation runs are shown in Figure 2 for $\rho_e=0.99\rho$ (the top one) and $\rho_e=0.87\rho$ (the bottom one), respectively. From Table 2, it is observed that $\mathbf{P}(A_V)$ increases in an approximately linear fashion as ρ_e decreases, and the probability that the states of the closed-loop system of Eq. 34 remain in Ω_ρ reaches 97.2% as ρ_e decreases to $\rho_e=320$. Therefore, letting $\rho_e=320$, it is observed from the top plot of Figure 2 that almost all the points fall below ρ, which implies that the closed-loop system under the SLEMPC can be regarded as a system with the closed-loop states bounded within Ω_ρ with relatively high probability in this case.

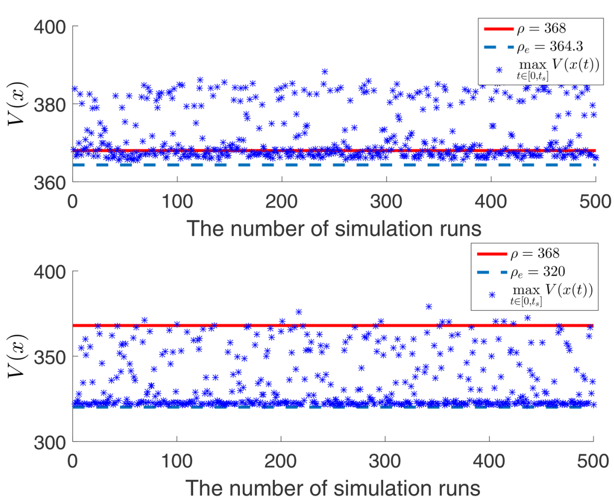

Figure 2. The maximum value of $V(x(t))$ in each realization originating from (0, 0) for 500 simulation runs, in which $\rho=368$ and $\rho_e=364.3$ (top), and $\rho=368$ and $\rho_e=320$ (bottom).

[Color figure can be viewed at wileyonlinelibrary.com]

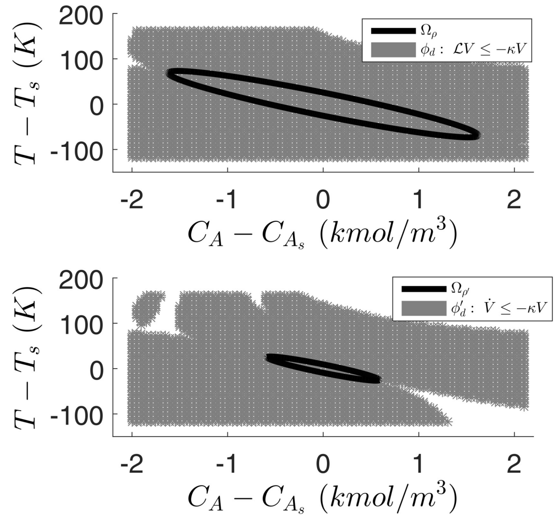

Figure 3. The stability region Ω_ρ for the closed-loop CSTR under the stochastic Lyapunov-based controller $\Phi_s(x)$ (top), and the stability region $\Omega_{\rho'}$ for the closed-loop CSTR under the Lyapunov-based controller $\Phi_n(x)$ (bottom).

Next, we will show the application of the LEMPC of Eq. 13 to the uncertain system of Eq. 34 subject to stochastic disturbances. In general, LEMPC is designed using the nominal system model and applied to the uncertain system with bounded disturbances. However, because unbounded variations in w are included in the process model of Eq. 34, we do not have a case with bounded disturbances, but we can approximate a reasonable bound so that most of the disturbances are included within the bounds such that the LEMPC of Eq. 13 can be designed. For example, since the disturbances $dw(t)$ are of standard normal distributions, let the disturbances be bounded by $\theta_1=2.2\sigma_1$ and $\theta_2=2.2\sigma_2$, respectively; this implies that 97.2% of the disturbance values will fall within the above interval. Under the LEMPC of Eq. 13, the set of initial conditions ϕ_d' and the level set inside ϕ_d' can be characterized appropriately. Figure 3 displays the level sets Ω_ρ and $\Omega_{\rho'}$ inside the sets ϕ_d and ϕ_d', respectively. It is observed from Figure 3 that ϕ_d' is more conservative than ϕ_d to achieve robustness of the LEMPC to the disturbances within the 2.2σ intervals. As a result, a larger level set of V is chosen as the stability region under the stochastic Lyapunov-based controller $\Phi_s(x)$ than the standard Lyapunov-based controller $\Phi_n(x)$. Specifically, $\Omega_{\rho'} \subset \phi_d'$ with $\rho'=46$ and $\Omega_\rho \subset \phi_d$ with $\rho=368$ are chosen as the stability regions for the LEMPC of Eq. 13 and the SLEMPC of Eq. 14, respectively. Furthermore, we let $\rho_e'=40$, such that for all $|w(t)| \le \theta$, the closed-loop trajectory of the system of Eq. 34 does not reach the boundary of $\Omega_{\rho'}$ within one sampling period for any $x(0)$ originating on the boundary of $\Omega_{\rho_e'}$ under any control action.

Again, through 500 simulation runs, the stability region $\Omega_{\rho'}$ remained invariant under the LEMPC of Eq. 13. Additionally, the average of the total economic cost $L_E=\int_0^{t_s} L_e(x,u)dt$ under the SLEMPC of Eq. 14 with Ω_ρ and the LEMPC of Eq. 13 with $\Omega_{\rho'}$ are calculated via 500 simulations runs with the same initial condition $x^T=[0\ 0]$. The averaged total economic cost over the operation time $t_s=1\ hr$ is $L_E=27.2$ under the SLEMPC, which represents an improvement of approximately 63% compared to $L_E=16.7$ under the LEMPC of Eq. 13, and 103% compared to $L_E=13.4$ under steady-state operation.

Therefore, under the SLEMPC of Eq. 14, the stability region Ω_ρ that accounts for the distributional information $h(x)$ of the uncertainty term leads to higher economic benefits for the closed-loop system of Eq. 34 than the LEMPC with a conservative stability region $\Omega_{\rho'}$, and steady-state operation. In addition, it has been shown through 500 simulation runs that by choosing $\rho_e=320$, the closed-loop states of the system of Eq. 34 can be bounded in Ω_ρ under the SLEMPC of Eq. 14 with the same probability of 97.2% as the one under the LEMPC of Eq. 13. Therefore, it is concluded that the closed-loop system of Eq. 34 under the SLEMPC with $\rho=368$ and $\rho_e=320$ may achieve an acceptable probability of closed-loop stability and a satisfactory process economic performance simultaneously.

Conclusion

In this work, a stochastic Lyapunov-based EMPC method was developed for stochastic nonlinear systems with input constraints. We first characterized a closed-loop stability region for which the probability that the closed-loop state trajectory would remain within could be quantified. We then reviewed a Lyapunov-based EMPC method for nonlinear systems subject to bounded disturbances, and presented the optimization-based control strategy of the SLEMPC and the probabilities that the SLEMPC would remain feasible at each sampling time and would maintain the closed-loop state within the stability region. The application of the proposed SLEMPC method was demonstrated through a chemical process example, from which it was demonstrated that under the SLEMPC, the system operating in the stability region in probability outperformed the one under the LEMPC in terms of economic benefits.

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Literature Cited

1. García CE, Prett DM, Morari M. Model predictive control: theory and practice–a survey. *Automatica.* 1989;25:335–348.
2. Mayne DQ, Rawlings JB, Rao CV, Scokaert POM. Constrained model predictive control: Stability and optimality. *Automatica.* 2000;36:789–814.
3. Qin SJ, Badgwell TA. A survey of industrial model predictive control technology. *Control Eng Pract.* 2003;11(7):733–764.
4. Ellis M, Durand H, Christofides PD. A tutorial review of economic model predictive control methods. *J Process Control.* 2014;24(8):1156–1178.
5. Diehl M, Amrit R, Rawlings JB. A Lyapunov function for economic optimizing model predictive control. *IEEE Trans Automat Control.* 2011;56(3):703–707.
6. Huang R, Harinath E, Biegler LT. Lyapunov stability of economically oriented NMPC for cyclic processes. *J Process Control.* 2011; 21(4):501–509.
7. Angeli D, Amrit R, Rawlings JB. On average performance and stability of economic model predictive control. *IEEE Trans Automat Control.* 2012;57(7):1615–1626.
8. Heidarinejad M, Liu J, Christofides PD. Economic model predictive control of nonlinear process systems using Lyapunov techniques. *AIChE J.* 2012;58(3):855–870.
9. Rawlings JB, Angeli D, Bates CN. Fundamentals of economic model predictive control. In: *Proceedings of the 51st IEEE Conference on Decision and Control.* Maui, Hawaii, 2012:3851–3861.
10. Lucia S, Andersson JA, Brandt H, Diehl M, Engell S. Handling uncertainty in economic nonlinear model predictive control: A comparative case study. *J Process Control.* 2014;24:1247–1259.
11. Bayer FA, Müller MA, Allgöwer F. Min-max economic model predictive control approaches with guaranteed performance. In: *Proceedings of the 55th IEEE Conference on Decision and Control.* Las Vegas, NV, 2016:3210–3215.
12. Bayer FA, Müller MA, Allgöwer F. Tube-based robust economic model predictive control. *J Process Control.* 2014;24(8):1237–1246.
13. Visintini AL, Glover W, Lygeros J, Maciejowski J. Monte Carlo optimization for conflict resolution in air traffic control. *IEEE Trans Intell Transp Syst.* 2006;7:470–482.
14. Maciejowski JM, Visintini AL, Lygeros J. NMPC for complex stochastic systems using a Markov chain Monte Carlo approach. In: *Assessment and Future Directions of Nonlinear Model Predictive Control.* Berlin, Germany: Springer, 2007:269–281.
15. Mahmood M, Mhaskar P. Lyapunov-based model predictive control of stochastic nonlinear systems. *Automatica.* 2012;48(9):2271–2276.
16. Khasminskii R. *Stochastic Stability of Differential Equations.* Springer Science & Business Media, 2011.
17. Øksendal B. *Stochastic Differential Equations.* Berlin, Germany: Springer, 2003.
18. Lin Y, Sontag ED. A universal formula for stabilization with bounded controls. *Syst Control Lett.* 1991;16(6):393–397.
19. Deng H, Krstic M, Williams R. Stabilization of stochastic nonlinear systems driven by noise of unknown covariance. *IEEE Trans Automat Control.* 2001;46:1237–1253.
20. Florchinger P. A universal formula for the stabilization of control stochastic differential equations. *Stochastic Anal Appl.* 1993;11(2): 155–162.
21. Weissel F, Huber MF, Hanebeck UD. Stochastic nonlinear model predictive control based on Gaussian mixture approximations. *In: Informatics in Control, Automation and Robotics.* Berlin, Germany: Springer, 2009:239–252.
22. Homer T, Mhaskar P. Output-feedback Lyapunov-based predictive control of stochastic nonlinear systems. *IEEE Trans Automat Control.* 2018;63(2):571–577.
23. Ciesielski Z, Taylor SJ. First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans Am Math Soc.* 1962;103(3):434–450.
24. Battilotti S, Santis AD. Stabilization in probability of nonlinear stochastic systems with guaranteed region of attraction and target set. *IEEE Trans Automat Control.* 2003;48(9):1585–1599.
25. Wächter A, Biegler LT. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math Program.* 2006;106(1):25–57.

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