Chapter 3

Lyapunov Stability

Stability theory plays a central role in systems theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems. This chapter is concerned with stability of equilibrium points. In later chapters we shall see other kinds of stability, such as stability of periodic orbits and input-output stability. Stability of equilibrium points is usually characterized in the sense of Lyapunov, a Russian mathematician and engineer who laid the foundation of the theory which now carries his name. An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity. These notions are made precise in Section 3.1, where the basic theorems of Lyapunov's method for autonomous systems are given. An extension of the basic theory, due to LaSalle, is given in Section 3.2. For a linear time-invariant system $\dot{x}(t) = Ax(t)$, the stability of the equilibrium point $x = 0$ can be completely characterized by the location of the eigenvalues of A. This is discussed in Section 3.3. In the same section, it is shown when and how the stability of an equilibrium point can be determined by linearization about that point. In Sections 3.4 and 3.5, we extend Lyapunov's method to nonautonomous systems. In Section 3.4, we define the concepts of uniform stability, uniform asymptotic stability, and exponential stability of the equilibrium of a nonautonomous system. We give Lyapunov's method for testing uniform asymptotic stability, as well as exponential stability. In Section 3.5, we study linear time-varying systems and linearization.

Lyapunov stability theorems give sufficient conditions for stability, asymptotic stability, and so on. They do not say whether the given conditions are also necessary. There are theorems which establish, at least conceptually, that for many of Lyapunov stability theorems the given conditions are indeed necessary. Such theorems are usually called converse theorems. We present two converse theorems converse theorem for e.s.

in Section 3.6. Moreover, we use the converse theorem for exponential stability to show that an equilibrium point of a nonlinear system is exponentially stable if and only if the linearization of the system about that point has an exponentially stable equilibrium at the origin.

3.1 Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{3.1}$$

where $f : D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n. Suppose $\bar{x} \in D$ is an equilibrium point of (3.1); that is,

$$\dot{X} = f(\bar{x}) = 0$$

Our goal is to characterize and study stability of $\bar{x}$. For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\bar{x} \neq 0$, and consider the change of variables $y = x - \bar{x}$. The derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

In the new variable y, the system has equilibrium at the origin. Therefore, without loss of generality, we shall always assume that $f(x)$ satisfies $f(0) = 0$, and study stability of the origin $x = 0$.

Definition 3.1 *The equilibrium point $x = 0$ of (3.1) is*

- *stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\, t \geq 0$$

- *unstable if not stable.*
- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

The ϵ-δ requirement for stability takes a challenge-answer form. To demonstrate that the origin is stable, then, for any value of ϵ that a challenger may care to designate, we must produce a value of δ, possibly dependent on ϵ, such that a trajectory

starting in a δ neighborhood of the origin will never leave the ϵ neighborhood. The three types of stability properties can be illustrated by the pendulum example of Section 1.1.1. The pendulum equation

Pendulum Equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \end{aligned}$$

has two equilibrium points at $(x_1 = 0,\ x_2 = 0)$ and $(x_1 = \pi,\ x_2 = 0)$. Neglecting friction, by setting $k = 0$, we have seen in Example 1.1 that trajectories in the neighborhood of the first equilibrium are closed orbits. Therefore, by starting sufficiently close to the equilibrium point, trajectories can be guaranteed to stay within any specified ball centered at the equilibrium point. Therefore, the ϵ-δ requirement for stability is satisfied. The equilibrium point, however, is not asymptotically stable since trajectories starting off the equilibrium point do not tend to it eventually. Instead, they remain in their closed orbits. When friction is taken into consideration $(k > 0)$, the equilibrium point at the origin becomes a stable focus. Inspection of the phase portrait of a stable focus shows that the ϵ-δ requirement for stability is satisfied. In addition, trajectories starting close to the equilibrium point tend to it as t tends to ∞. The second equilibrium point at $x_1 = \pi$ is a saddle point. Clearly the ϵ-δ requirement cannot be satisfied since, for any $\epsilon > 0$, there is always a trajectory that will leave the ball $\{x \in R^n \mid \|x - \bar{x}\| \leq \epsilon\}$ even when $x(0)$ is arbitrarily close to the equilibrium point.

Implicit in Definition 3.1 is a requirement that solutions of (3.1) be defined for all $t \geq 0$.[1] Such global existence of the solution is not guaranteed by the local Lipschitz property of f. It will be shown, however, that the additional conditions needed in Lyapunov's theorem will ensure global existence of the solution. This will come as an application of Theorem 2.4.

Having defined stability and asymptotic stability of equilibrium points, our task now is to find ways to determine stability. The approach we used in the pendulum example relied on our knowledge of the phase portrait of the pendulum equation. Trying to generalize that approach amounts to actually finding all solutions of (3.1), which may be difficult or even impossible. However, the conclusions we reached about the stable equilibrium of the pendulum can also be reached by using energy concepts. Let us define the energy of the pendulum $E(x)$ as the sum of its potential and kinetic energies, with the reference of the potential energy chosen such that $E(0) = 0$; that is,

Energy concept

$$E(x) = \int_0^{x_1} \left(\frac{g}{l}\right)\sin y\, dy + \tfrac{1}{2}x_2^2 = \left(\frac{g}{l}\right)(1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

[1] It is possible to change the definition to alleviate the implication of global existence of the solution. In [137], stability is defined on the maximal interval of existence $[0, t_1)$, without assuming that $t_1 = \infty$.

$$V(x) = \frac{g}{l}(1 - \cos x_1) + \frac{1}{2}x_2^2$$

Pendulum Energy.

When friction is neglected ($k = 0$), the system is conservative; that is, there is no dissipation of energy. Hence, $E =$ constant during the motion of the system or, in other words, $dE/dt = 0$ along the trajectories of the system. Since $E(x) = c$ forms a closed contour around $x = 0$ for small c, we can again arrive at the conclusion that $x = 0$ is a stable equilibrium point. When friction is accounted for ($k > 0$), energy will dissipate during the motion of the system, that is, $dE/dt \leq 0$ along the trajectories of the system. Due to friction, E cannot remain constant indefinitely while the system is in motion. Hence, it keeps decreasing until it eventually reaches zero, showing that the trajectory tends to $x = 0$ as t tends to ∞. Thus, by examining the derivative of E along the trajectories of the system, it is possible to determine the stability of the equilibrium point. In 1892, Lyapunov showed that certain other functions could be used instead of energy to determine stability of an equilibrium point. Let $V : D \to R$ be a continuously differentiable function defined in a domain $D \subset R^n$ that contains the origin. The derivative of V along the trajectories of (3.1), denoted by $\dot{V}(x)$, is given by

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$$

$$= \begin{bmatrix} \frac{\partial V}{\partial x_1}, & \frac{\partial V}{\partial x_2}, & \ldots, & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x)$$

The derivative of V along the trajectories of a system is dependent on the system's equation. Hence, $\dot{V}(x)$ will be different for different systems. If $\phi(t; x)$ is the solution of (3.1) that starts at initial state x at time $t = 0$, then

$$\dot{V}(x) = \frac{d}{dt} V(\phi(t; x))\bigg|_{t=0}$$

Therefore, if $\dot{V}(x)$ is negative, V will decrease along the solution of (3.1). We are now ready to state Lyapunov's stability theorem.

Theorem 3.1 *Let $x = 0$ be an equilibrium point for (3.1) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function, such that*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \tag{3.2}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{3.3}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{3.4}$$

then $x = 0$ is asymptotically stable. ◇

Lyapunov's Stability Theorem

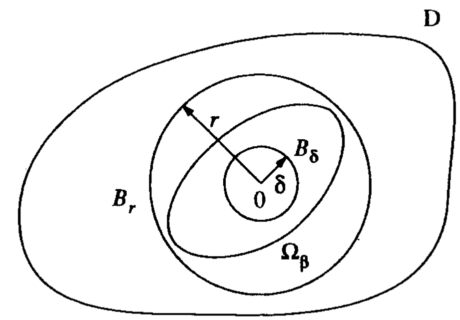


Figure 3.1: Geometric representation of sets in the proof of Theorem 3.1.

Proof: Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \leq r\} \subset D$$

Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ by (3.2). Take $\beta \in (0, \alpha)$, and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

Then, Ω_β is in the interior of B_r;[2] see Figure 3.1. The set Ω_β has the property that any trajectory starting in Ω_β at $t = 0$ stays in Ω_β for all $t \geq 0$. This follows from (3.3) since

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \ \forall\, t \geq 0$$

Since Ω_β is a compact set,[3] we conclude from Theorem 2.4 that (3.1) has a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$. Since $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$\|x\| \leq \delta \Rightarrow V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

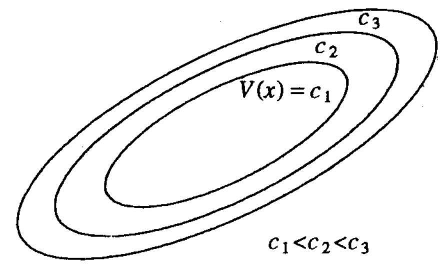
$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \epsilon, \quad \forall\, t \geq 0$$

[2] This fact can be shown by contradiction. Suppose Ω_β is not in the interior of B_r, then there is a point $p \in \Omega_\beta$ that lies on the boundary of B_r. At this point, $V(p) \geq \alpha > \beta$ but, for all $x \in \Omega_\beta$, $V(x) \leq \beta$, which is a contradiction.

[3] Ω_β is closed by definition, and bounded since it is contained in B_r.



Level surface

Figure 3.2: Level surfaces of a Lyapunov function.

which shows that the equilibrium point $x = 0$ is stable. Now, assume that (3.4) holds as well. To show asymptotic stability, we need to show that $x(t) \to 0$ as $t \to \infty$; that is, for every $a > 0$, there is $T > 0$ such that $\|x(t)\| < a$, for all $t > T$. By repetition of previous arguments, we know that for every $a > 0$, we can choose $b > 0$ such that $\Omega_b \subset B_a$. Therefore, it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since $V(x(t))$ is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \to c \geq 0 \quad \text{as} \quad t \to \infty$$

To show that $c = 0$, we use a contradiction argument. Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. The limit $V(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set $\{d \leq \|x\| \leq r\}$.[4] By (3.4), $-\gamma < 0$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leq V(x(0)) - \gamma t$$

Since the right-hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$. □

A continuously differentiable function $V(x)$ satisfying (3.2) and (3.3) is called a *Lyapunov function*. The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a *level surface*. Using Lyapunov surfaces, Figure 3.2 makes the theorem intuitively clear. It shows Lyapunov surfaces for decreasing values of c. The condition $\dot{V} \leq 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ and can never come out again. When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov

[4] See [7, Theorem 4-20].

surface with a smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin, showing that the trajectory approaches the origin as time progresses. If we only know that $\dot{V} \leq 0$, we cannot be sure that the trajectory will approach the origin,[5] but we can conclude that the origin is stable since the trajectory can be contained inside any ball B_ϵ by requiring the initial state $x(0)$ to lie inside a Lyapunov surface contained in that ball.

A function $V(x)$ satisfying condition (3.2), that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, is said to be *positive definite*. If it satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, it is said to be *positive semidefinite*. A function $V(x)$ is said to be *negative definite* or *negative semidefinite* if $-V(x)$ is positive definite or positive semidefinite, respectively. If $V(x)$ does not have a definite sign as per one of these four cases, it is said to be indefinite. With this terminology, we can rephrase Lyapunov's theorem to say that *the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.*

Lyapunov theorem

A class of scalar functions $V(x)$ for which sign definiteness can be easily checked is the class of functions of the quadratic form

eig(P) positive ⇒ V = pd or psd.

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

where P is a real symmetric matrix. In this case, $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative), which is true if and only if all the leading principal minors of P are positive (all principal minors of P are nonnegative).[6] If $V(x) = x^T P x$ is positive definite (positive semidefinite), we say that the matrix P is positive definite (positive semidefinite) and write $P > 0$ ($P \geq 0$).

Example 3.1 Consider

$$\begin{aligned} V(x) &= a x_1^2 + 2 x_1 x_3 + a x_2^2 + 4 x_2 x_3 + a x_3^2 \\ &= [x_1\ x_2\ x_3] \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x \end{aligned}$$

The leading principal minors of P are a, a^2, and $a(a^2-5)$. Therefore, $V(x)$ is positive definite if $a > \sqrt{5}$. For negative definiteness, the leading principal minors of $-P$ should be positive; that is, the leading principal minors of P should have alternating signs, with the odd-numbered minors being negative and the even-numbered minors being positive. Therefore, $V(x)$ is negative definite if $a < -\sqrt{5}$. By calculating all

[5] See, however, LaSalle's theorem in Section 3.2

[6] This is a well-known fact in matrix theory. Its proof can be found in [16] or [53].

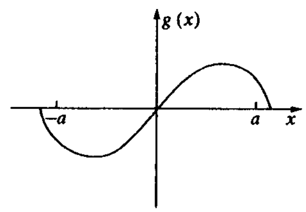


Figure 3.3: A possible nonlinearity in Example 3.2.

principal minors, it can be seen that $V(x)$ is positive semidefinite if $a \geq \sqrt{5}$ and negative semidefinite if $a \leq -\sqrt{5}$. For $a \in (-\sqrt{5}, \sqrt{5})$, $V(x)$ is indefinite. △

Lyapunov's theorem can be applied without solving the differential equation (3.1). On the other hand, there is no systematic method for finding Lyapunov functions. In some cases, there are natural Lyapunov function candidates like energy functions in electrical or mechanical systems. In other cases, it is basically a matter of trial and error. The situation, however, is not as bad as it might seem. As we go over various examples and applications throughout the book, some ideas and approaches for searching for Lyapunov functions will be delineated.

Example 3.2 Consider the first-order differential equation

$$\dot{x} = -g(x)$$

where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\ x \neq 0, \ x \in (-a, a)$$

A sketch of a possible $g(x)$ is shown in Figure 3.3. The system has an isolated equilibrium point at the origin. It is not difficult in this simple example to see that the origin is asymptotically stable, because solutions starting on either side of the origin will have to move toward the origin due to the sign of the derivative $\dot{x}$. To arrive at the same conclusion using Lyapunov's theorem, consider the function

$$V(x) = \int_0^x g(y)\, dy$$

Over the domain $D = (-a, a)$, $V(x)$ is continuously differentiable, $V(0) = 0$, and $V(x) > 0$ for all $x \neq 0$. Thus, $V(x)$ is a valid Lyapunov function candidate. To see whether or not $V(x)$ is indeed a Lyapunov function, we calculate its derivative along the trajectories of the system.

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \ \forall\ x \in D - \{0\}$$

Thus, by Theorem 3.1 we conclude that the origin is asymptotically stable. △

Example 3.3 Consider the pendulum equation without friction:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 \end{aligned}$$

since $\dot{V}$ is always 0. so (0,0) is a stable eq point.

and let us study the stability of the equilibrium point at the origin. A natural Lyapunov function candidate is the energy function

$$V(x) = \left(\frac{g}{l}\right)(1 - \cos x_1) + \tfrac{1}{2}x_2^2$$

Clearly, $V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = \left(\frac{g}{l}\right)\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = \left(\frac{g}{l}\right)x_2 \sin x_1 - \left(\frac{g}{l}\right)x_2 \sin x_1 = 0$$

Thus, conditions (3.2) and (3.3) of Theorem 3.1 are satisfied and we conclude that the origin is stable. Since $\dot{V}(x) \equiv 0$, we can also conclude that the origin is not asymptotically stable; for trajectories starting on a Lyapunov surface $V(x) = c$ remain on the same surface for all future time. △

Example 3.4 Consider again the pendulum equation, but this time with friction:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \end{aligned}$$

Let us try again $V(x) = (g/l)(1 - \cos x_1) + \frac{1}{2}x_2^2$ as a Lyapunov function candidate.

$$\dot{V}(x) = \left(\frac{g}{l}\right)\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -\left(\frac{k}{m}\right)x_2^2$$

$\dot{V} \leq 0 \Rightarrow$ stable.

$\dot{V}(x)$ is negative semidefinite. It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis. Therefore, we can only conclude that the origin is stable. However, using the phase portrait of the pendulum equation, we have seen that when $k > 0$, the origin is asymptotically stable. The energy Lyapunov function fails to show this fact. We shall see later in Section 3.2 that a theorem due to LaSalle will enable us to arrive at a different conclusion using the energy Lyapunov function. For now, let us look for a Lyapunov function $V(x)$ that would have a negative definite $\dot{V}(x)$. Starting from the energy

In the pendulum with friction example, the Lyapunov eq can only show that (0,0) is stable. From the phase portrait, (0,0) is a.s. ⇒ we can pick a new Lyapunov function or use LaSalle's to show (0,0) is a.s.

Lyapunov function, let us replace the term $\frac{1}{2}x_2^2$ by the more general quadratic form $\frac{1}{2}x^T P x$ for some 2×2 positive definite matrix P.

$$V(x) = \tfrac{1}{2}x^T P x + \left(\frac{g}{l}\right)(1 - \cos x_1) = \tfrac{1}{2}[x_1\ x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} + \left(\frac{g}{l}\right)(1 - \cos x_1)$$

For the quadratic form $\frac{1}{2}x^T P x$ to be positive definite, the elements of the matrix P must satisfy

$$p_{11} > 0;\ p_{22} > 0;\ p_{11}p_{22} - p_{12}^2 > 0$$

The derivative $\dot{V}(x)$ is given by

$$\begin{aligned}\dot{V}(x) &= \left[p_{11}x_1 + p_{12}x_2 + \left(\frac{g}{l}\right)\sin x_1\right] x_2 \\ &\quad + (p_{12}x_1 + p_{22}x_2)\left[-\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\right] \\ &= \left(\frac{g}{l}\right)(1 - p_{22})x_2 \sin x_1 - \left(\frac{g}{l}\right)p_{12}x_1 \sin x_1 \\ &\quad + \left[p_{11} - p_{12}\left(\frac{k}{m}\right)\right]x_1x_2 + \left[p_{12} - p_{22}\left(\frac{k}{m}\right)\right]x_2^2\end{aligned}$$

Now we want to choose p_{11}, p_{12}, and p_{22} such that $\dot{V}(x)$ is negative definite. Since the cross product terms $x_2 \sin x_1$ and x_1x_2 are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = (k/m)p_{12}$. With these choices, p_{12} must satisfy $0 < p_{12} < (k/m)$ for $V(x)$ to be positive definite. Let us take $p_{12} = 0.5(k/m)$. Then, $\dot{V}(x)$ is given by

$$\dot{V}(x) = -\tfrac{1}{2}\left(\frac{g}{l}\right)\left(\frac{k}{m}\right)x_1 \sin x_1 - \tfrac{1}{2}\left(\frac{k}{m}\right)x_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Taking $D = \{x \in R^2 \mid |x_1| < \pi\}$, we see that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D. Thus, by Theorem 3.1 we conclude that the origin is asymptotically stable. △

This example emphasizes an important feature of Lyapunov's stability theorem; namely, *the theorem's conditions are only sufficient*. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate. Whether the equilibrium point is stable (asymptotically stable) or not can be determined only by further investigation.

In searching for a Lyapunov function in Example 3.4, we approached the problem in a backward manner. We investigated an expression for $\dot{V}(x)$ and went back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ negative definite. This is a useful idea in searching for a Lyapunov function. A procedure that exploits this idea is known as the *variable gradient method*. To describe this procedure, let $V(x)$ be a scalar function of x and $g(x) = \nabla V = (\partial V/\partial x)^T$. The derivative $\dot{V}(x)$ along the trajectories of (3.1) is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x}f(x) = g^T(x)f(x)$$

The idea now is to try to choose $g(x)$ such that it would be the gradient of a positive definite function $V(x)$ and, at the same time, $\dot{V}(x)$ would be negative definite. It is not difficult (Exercise 3.5) to verify that $g(x)$ is the gradient of a scalar function if and only if the Jacobian matrix $[\partial g/\partial x]$ is symmetric; that is,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, \ldots, n$$

Under this constraint, we start by choosing $g(x)$ such that $g^T(x)f(x)$ is negative definite. The function $V(x)$ is then computed from the integral

$$V(x) = \int_0^x g^T(y)\ dy = \int_0^x \sum_{i=1}^n g_i(y)\ dy_i$$

The integration is taken over any path joining the origin to x.[7] Usually, this is done along the axes; that is,

$$\begin{aligned}V(x) &= \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\ dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\ dy_2 \\ &\quad + \cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\ dy_n\end{aligned}$$

By leaving some parameters of $g(x)$ undetermined, one would try to choose them to ensure that $V(x)$ is positive definite. The variable gradient method can be used to arrive at the Lyapunov function of Example 3.4. Instead of repeating the example, we illustrate the method on a slightly more general system.

Example 3.5 Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2\end{aligned}$$

[7] The line integral of a gradient vector is independent of the path; see [7, Theorem 10-37].

where $a > 0$, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$ and $yh(y) > 0$ for all $y \neq 0$, $y \in (-b, c)$ for some positive constants b and c. The pendulum equation is a special case of this system. To apply the variable gradient method, we want to choose a second-order vector $g(x)$ that satisfies

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + ax_2] < 0, \quad \text{for } x \neq 0$$

and

$$V(x) = \int_0^x g^T(y)\, dy > 0, \quad \text{for } x \neq 0$$

Let us try

$$g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$$

where the scalar functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, and $\delta(\cdot)$ are to be determined. To satisfy the symmetry requirement, we must have

$$\beta(x) + \frac{\partial \alpha}{\partial x_2}x_1 + \frac{\partial \beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1}x_1 + \frac{\partial \delta}{\partial x_1}x_2$$

The derivative $\dot{V}(x)$ is given by

$$\dot{V}(x) = \alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 - a\delta(x)x_2^2 - \delta(x)x_2h(x_1) - \gamma(x)x_1h(x_1)$$

To cancel the cross product terms, we choose

$$\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$$

so that

$$\dot{V}(x) = -[a\delta(x) - \beta(x)]x_2^2 - \gamma(x)x_1h(x_1)$$

To simplify our choices, let us take $\delta(x) = \delta =$ constant, $\gamma(x) = \gamma =$ constant, and $\beta(x) = \beta =$ constant. Then, $\alpha(x)$ depends only on x_1, and the symmetry requirement is satisfied by choosing $\beta = \gamma$. The expression for $g(x)$ reduces to

$$g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

By integration, we obtain

$$\begin{aligned} V(x) &= \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\, dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\, dy_2 \\ &= \tfrac{1}{2}a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\, dy + \gamma x_1x_2 + \tfrac{1}{2}\delta x_2^2 = \tfrac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)\, dy \end{aligned}$$

where

$$P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$$

Choosing $\delta > 0$ and $0 < \gamma < a\delta$ ensures that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. For example, taking $\gamma = ak\delta$ for $0 < k < 1$ yields the Lyapunov function

$$V(x) = \frac{\delta}{2}x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\, dy$$

which satisfies conditions (3.2) and (3.4) of Theorem 3.1 over the domain $D = \{x \in R^2 \mid -b < x_1 < c\}$. Therefore, the origin is asymptotically stable. △

When the origin $x = 0$ is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be and still converge to the origin as t approaches ∞. This gives rise to the definition of the *region of attraction* (also called *region of asymptotic stability*, *domain of attraction*, or *basin*). Let $\phi(t; x)$ be the solution of (3.1) that starts at initial state x at time $t = 0$. Then, the region of attraction is defined as the set of all points x such that $\lim_{t\to\infty} \phi(t; x) = 0$. Finding the exact region of attraction analytically might be difficult or even impossible. However, Lyapunov functions can be used to estimate the region of attraction, that is, to find sets contained in the region of attraction. From the proof of Theorem 3.1 we see that if there is a Lyapunov function that satisfies the conditions of asymptotic stability over a domain D, and if $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is bounded and contained in D, then every trajectory starting in Ω_c remains in Ω_c and approaches the origin as $t \to \infty$. Thus, Ω_c is an estimate of the region of attraction. This estimate, however, may be conservative; that is, it may be much smaller than the actual region of attraction. In Section 4.2, we shall solve examples on estimating the region of attraction and see some ideas to enlarge the estimates. Here, we want to pursue another question: under what conditions will the region of attraction be the whole space R^n? This will be the case if we can show that for any initial state x, the trajectory $\phi(t; x)$ approaches the origin as $t \to \infty$, no matter how large $\|x\|$ is. If an asymptotically stable equilibrium point at the origin has this property, it is said to be ***globally asymptotically stable***. Recalling again the proof of Theorem 3.1, we can see that global asymptotic stability can be established if any point $x \in R^n$ can be included in the interior of a bounded set Ω_c. It is obvious that for this to hold, the conditions of the theorem must hold globally, that is, $D = R^n$; but, is this enough? It turns out that we need more conditions to ensure that any point in R^n can be included in a bounded set Ω_c. The problem is that for large c, the set Ω_c need not be bounded. Consider, for example, the function

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

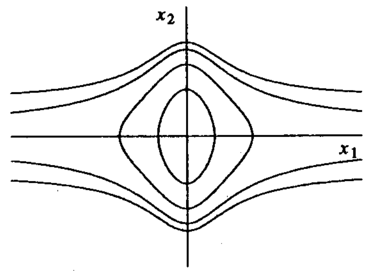


Figure 3.4: Lyapunov surfaces for $V(x) = x_1^2/(1+x_1^2) + x_2^2$.

Figure 3.4 shows the surfaces $V(x) = c$ for various positive values of c. For small c, the surface $V(x) = c$ is closed; hence, Ω_c is bounded since it is contained in a closed ball B_r for some $r > 0$. This is a consequence of the continuity and positive definiteness of $V(x)$. As c increases, a value is reached after which the surface $V(x) = c$ is open and Ω_c is unbounded. For Ω_c to be in the interior of a ball B_r, c must satisfy $c < \inf_{\|x\|\geq r} V(x)$. If

$$l = \lim_{r\to\infty} \inf_{\|x\|\geq r} V(x) < \infty$$

then Ω_c will be bounded if $c < l$. In the preceding example,

$$l = \lim_{r\to\infty} \min_{\|x\|=r} \left[\frac{x_1^2}{1+x_1^2} + x_2^2\right] = \lim_{|x_1|\to\infty} \frac{x_1^2}{1+x_1^2} = 1$$

Hence, Ω_c is bounded only for $c < 1$. An extra condition that ensures that Ω_c is bounded for all values of $c > 0$ is

$$V(x) \to \infty \text{ as } \|x\| \to \infty$$

A function satisfying this condition is said to be *radially unbounded.*

Barbashin - Krasovskii theorem.

Theorem 3.2 *Let $x = 0$ be an equilibrium point for* (3.1). *Let $V : R^n \to R$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \neq 0 \tag{3.5}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{3.6}$$

$$\dot{V}(x) < 0, \quad \forall\, x \neq 0 \tag{3.7}$$

then $x = 0$ is globally asymptotically stable. ◇

R.U (radially unbounded).

Proof: Given any point $p \in R^n$, let $c = V(p)$. Condition (3.6) implies that for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus $\Omega_c \subset B_r$, which implies that Ω_c is bounded. The rest of the proof is similar to that of Theorem 3.1. □

Theorem 3.2 is known as Barbashin-Krasovskii theorem. Exercise 3.7 gives a counterexample to show that the radial unboundedness condition of the theorem is indeed needed.

Example 3.6 Consider again the second-order system of Example 3.5, but this time assume that the condition $yh(y) > 0$ holds for all $y \neq 0$. The Lyapunov function

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\, dy$$

is positive definite for all $x \in R^2$ and radially unbounded. The derivative

$$\dot{V}(x) = -a\delta(1-k)x_2^2 - a\delta k x_1 h(x_1)$$

is negative definite for all $x \in R^2$ since $0 < k < 1$. Therefore, the origin is globally asymptotically stable. △

If the origin $x = 0$ is a globally asymptotically stable equilibrium point of a system, then it must be the unique equilibrium point of the system. For if there was another equilibrium point $\bar{x}$, the trajectory starting at $\bar{x}$ would remain at $\bar{x}$ for all $t \geq 0$; hence, it would not approach the origin which contradicts the claim that the origin is globally asymptotically stable. Therefore, global asymptotic stability is not studied for multiple equilibria systems like the pendulum equation.

Theorems 3.1 and 3.2 are concerned with establishing the stability or asymptotic stability of an equilibrium point. There are also instability theorems for establishing that an equilibrium point is unstable. The most powerful of these theorems is Chetaev's theorem, which will be stated as Theorem 3.3. Before we state the theorem, let us introduce some terminology that will be used in the theorem's statement. Let $V : D \to R$ be a continuously differentiable function on a domain $D \subset R^n$ that contains the origin $x = 0$. Suppose $V(0) = 0$ and there is a point x_0 arbitrarily close to the origin such that $V(x_0) > 0$. Choose $r > 0$ such that the ball $B_r = \{x \in R^n \mid \|x\| \leq r\}$ is contained in D, and let

$$U = \{x \in B_r \mid V(x) > 0\} \tag{3.8}$$

The set U is a nonempty set contained in B_r. Its boundary is given by the surface $V(x) = 0$ and the sphere $\|x\| = r$. Since $V(0) = 0$, the origin lies on the boundary of U inside B_r. Notice that U may contain more than one component. For example, Figure 3.5 shows the set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. The set U can be always constructed provided that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 arbitrarily close to the origin.

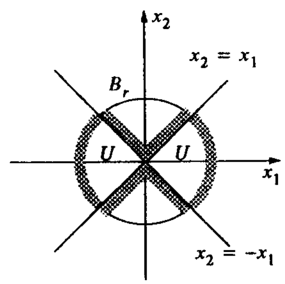


Figure 3.5: The set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$.

Chetaev's theorem

Theorem 3.3 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set U as in (3.8) and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.* ◇

Proof: The point x_0 is in the interior of U and $V(x_0) = a > 0$. The trajectory $x(t)$ starting at $x(0) = x_0$ must leave the set U. To see this point, notice that as long as $x(t)$ is inside U, $V(x(t)) \geq a$ since $\dot{V}(x) > 0$ in U. Let

$$\gamma = \min\{\dot{V}(x) \mid x \in U \text{ and } V(x) \geq a\}$$

which exists since the continuous function $\dot{V}(x)$ has a minimum over the compact set $\{x \in U \text{ and } V(x) \geq a\} = \{x \in B_r \text{ and } V(x) \geq a\}$.[8] Then, $\gamma > 0$ and

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\, ds \geq a + \int_0^t \gamma\, ds = a + \gamma t$$

This inequality shows that $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U. Now, $x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) \geq a$. Hence, it must leave U through the sphere $\|x\| = r$. Since this can happen for an arbitrarily small $\|x_0\|$, the origin is unstable. □

There are other instability theorems which were proved before Chetaev's theorem, but they are corollaries of the theorem; see Exercises 3.10 and 3.11.

Example 3.7 Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

[8] See [7, Theorem 4-20].

where $g_1(\cdot)$ and $g_2(\cdot)$ satisfy

$$|g_i(x)| \leq k\|x\|_2^2$$

in a neighborhood D of the origin. This inequality implies that $g_i(0) = 0$. Hence, the origin is an equilibrium point. Consider the function

$$V(x) = \tfrac{1}{2}(x_1^2 - x_2^2)$$

On the line $x_2 = 0$, $V(x) > 0$ at points arbitrarily close to the origin. The set U is shown in Figure 3.5. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

The magnitude of the term $x_1 g_1(x) - x_2 g_2(x)$ satisfies the inequality

$$|x_1 g_1(x) - x_2 g_2(x)| \leq \sum_{i=1}^{2} |x_i| \cdot |g_i(x)| \leq 2k\|x\|_2^3$$

Hence,

$$\dot{V}(x) \geq \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2(1 - 2k\|x\|_2)$$

Choosing r such that $B_r \subset D$ and $r < 1/2k$, it is seen that all the conditions of Theorem 3.3 are satisfied. Hence, the origin is unstable. △

3.2 The Invariance Principle

In our study of the pendulum equation with friction (Example 3.4), we saw that the energy Lyapunov function fails to satisfy the asymptotic stability condition of Theorem 3.1 because $\dot{V}(x) = -(k/m)x_2^2$ is only negative semidefinite. Notice, however, that $\dot{V}(x)$ is negative everywhere except on the line $x_2 = 0$, where $\dot{V}(x) = 0$. For the system to maintain the $\dot{V}(x) = 0$ condition, the trajectory of the system must be confined to the line $x_2 = 0$. Unless $x_1 = 0$, this is impossible because from the pendulum equation

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; \sin x_1(t) \equiv 0$$

Hence, on the segment $-\pi < x_1 < \pi$ of the $x_2 = 0$ line, the system can maintain the $\dot{V}(x) = 0$ condition only at the origin $x = 0$. Therefore, $V(x(t))$ must decrease toward 0, and consequently $x(t) \to 0$ as $t \to \infty$. This is consistent with the physical understanding that, due to friction, energy cannot remain constant while the system is in motion.

The above argument shows, formally, that if in a domain about the origin we can find a Lyapunov function whose derivative along the trajectories of the system is negative semidefinite, and if we can establish that no trajectory can stay identically at points where $\dot{V}(x) = 0$ except at the origin, then the origin is asymptotically stable. This idea follows from LaSalle's ***invariance principle***, which is the subject of this section. To state and prove LaSalle's invariance theorem, we need to introduce a few definitions. Let $x(t)$ be a solution of (3.1). A point p is said to be a ***positive limit point*** of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the ***positive limit set*** of $x(t)$. A set M is said to be an ***invariant set*** with respect to (3.1) if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\ t \in R$$

That is, if a solution belongs to M at some time instant, then it belongs to M for all future and past time. A set M is said to be a ***positively invariant set*** if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\ t \geq 0$$

We also say that $x(t)$ approaches a set M as t approaches infinity, if for each $\epsilon > 0$ there is $T > 0$ such that

$$\text{dist}(x(t), M) < \epsilon, \quad \forall\ t > T$$

where $\text{dist}(p, M)$ denotes the distance from a point p to a set M, that is, the smallest distance from p to any point in M. More precisely,

$$\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

These few concepts can be illustrated by examining an asymptotically stable equilibrium point and a stable limit cycle in the plane. The asymptotically stable equilibrium is the positive limit set of every solution starting sufficiently near the equilibrium point. The stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle. The solution approaches the limit cycle as $t \to \infty$. Notice, however, that the solution does not approach any specific point on the limit cycle. In other words, the statement $x(t) \to M$ as $t \to \infty$ does not imply that the limit $\lim_{t\to\infty} x(t)$ exists. The equilibrium point and the limit cycle are invariant sets, since any solution starting in either set remains in the set for all $t \in R$. The set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ for all $x \in \Omega_c$ is a positively invariant set since, as we saw in the proof of Theorem 3.1, a solution starting in Ω_c remains in Ω_c for all $t \geq 0$.

A fundamental property of limit sets is stated in the following lemma, whose proof is given in Appendix A.2.

Lemma 3.1 *If a solution $x(t)$ of (3.1) is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover,*

$$x(t) \to L^+ \text{ as } t \to \infty$$

◇

We are now ready to state LaSalle's theorem.

Theorem 3.4 *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (3.1). Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.* ◇

Proof: Let $x(t)$ be a solution of (3.1) starting in Ω. Since $\dot{V}(x) \leq 0$ in Ω, $V(x(t))$ is a decreasing function of t. Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω. Therefore, $V(x(t))$ has a limit a as $t \to \infty$. Note also that the positive limit set L^+ is in Ω because Ω is a closed set. For any $p \in L^+$, there is a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. By continuity of $V(x)$, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$. Hence, $V(x) = a$ on L^+. Since L^+ is an invariant set (by Lemma 3.1), $\dot{V}(x) = 0$ on L^+. Thus,

$$L^+ \subset M \subset E \subset \Omega$$

Since $x(t)$ is bounded, $x(t)$ approaches L^+ as $t \to \infty$ (by Lemma 3.1). Hence, $x(t)$ approaches M as $t \to \infty$. □

Unlike Lyapunov's theorem, Theorem 3.4 does not require the function $V(x)$ to be positive definite. Note also that the construction of the set Ω does not have to be tied in with the construction of the function $V(x)$. However, in many applications the construction of $V(x)$ will itself guarantee the existence of a set Ω. In particular, if $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is bounded and $\dot{V}(x) \leq 0$ in Ω_c, then we can take $\Omega = \Omega_c$. When $V(x)$ is positive definite, Ω_c is bounded for sufficiently small $c > 0$. This is not necessarily true when $V(x)$ is not positive definite. For example, if $V(x) = (x_1 - x_2)^2$, the set Ω_c is not bounded no matter how small c is. If $V(x)$ is radially unbounded, that is, $V(x) \to \infty$ as $\|x\| \to \infty$, the set Ω_c is bounded for all values of c. This is true whether or not $V(x)$ is positive definite. However, checking radial unboundedness is easier for positive definite functions since it is enough to let x approach ∞ along the principal axes. This may not be sufficient if the function is not positive definite, as can be seen from $V(x) = (x_1 - x_2)^2$. Here, $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ and $x_2 = 0$, but not when $\|x\| \to \infty$ along the line $x_1 = x_2$.

When our interest is in showing that $x(t) \to 0$ as $t \to \infty$, we need to establish that the largest invariant set in E is the origin. This is done by showing that no solution can stay identically in E, other than the trivial solution $x(t) = 0$. Specializing Theorem 3.4 to this case and taking $V(x)$ to be positive definite, we obtain the following two corollaries which extend Theorems 3.1 and 3.2.[9]

[9] **Corollaries 3.1 and 3.2 are known as the theorems of Barbashin and Krasovskii, who proved them before the introduction of LaSalle's invariance principle.**

Corollary 3.1 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a continuously differentiable positive definite function on a domain D containing the origin $x = 0$, such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is asymptotically stable.* ◇

Corollary 3.2 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : R^n \to R$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \leq 0$ for all $x \in R^n$. Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is globally asymptotically stable.* ◇

When $\dot{V}(x)$ is negative definite, $S = \{0\}$. Then, Corollaries 3.1 and 3.2 coincide with Theorems 3.1 and 3.2, respectively.

Example 3.8 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -g(x_1) - h(x_2)\end{aligned}$$

where $g(\cdot)$ and $h(\cdot)$ are locally Lipschitz and satisfy

$$g(0) = 0, \quad yg(y) > 0, \quad \forall\, y \neq 0, \; y \in (-a, a)$$

$$h(0) = 0, \quad yh(y) > 0, \quad \forall\, y \neq 0, \; y \in (-a, a)$$

The system has an isolated equilibrium point at the origin. Depending upon the functions $g(\cdot)$ and $h(\cdot)$, it might have other equilibrium points. The equation of this system can be viewed as a generalized pendulum equation with $h(x_2)$ as the friction term. Therefore, a Lyapunov function candidate may be taken as the energy-like function

$$V(x) = \int_0^{x_1} g(y)\, dy \; + \; \tfrac{1}{2}x_2^2$$

Let $D = \{x \in R^2 \mid -a < x_i < a\}$. $V(x)$ is positive definite in D. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = g(x_1)x_2 + x_2[-g(x_1) - h(x_2)] = -x_2 h(x_2) \leq 0$$

Thus, $\dot{V}(x)$ is negative semidefinite. To characterize the set $S = \{x \in D \mid \dot{V}(x) = 0\}$, note that

$$\dot{V}(x) = 0 \;\Rightarrow\; x_2 h(x_2) = 0 \;\Rightarrow\; x_2 = 0, \quad \text{since } -a < x_2 < a$$

Hence, $S = \{x \in D \mid x_2 = 0\}$. Suppose $x(t)$ is a trajectory that belongs identically to S.

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow g(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Therefore, the only solution that can stay identically in S is the trivial solution $x(t) = 0$. Thus, the origin is asymptotically stable. △

Example 3.9 Consider again the system of Example 3.8, but this time let $a = \infty$ and assume that $g(\cdot)$ satisfies the additional condition:

$$\int_0^y g(z)\, dz \;\to\; \infty \;\text{ as }\; |y| \to \infty$$

The Lyapunov function

$$V(x) = \int_0^{x_1} g(y)\, dy \; + \; \tfrac{1}{2}x_2^2$$

is radially unbounded. Similar to the previous example, it can be shown that $\dot{V}(x) \leq 0$ in R^2, and the set

$$S = \{x \in R^2 \mid \dot{V}(x) = 0\} = \{x \in R^2 \mid x_2 = 0\}$$

contains no solutions other than the trivial solution. Hence, the origin is globally asymptotically stable. △

Not only does LaSalle's theorem relax the negative definiteness requirement of Lyapunov's theorem, but it also extends Lyapunov's theorem in three different directions. First, it gives an estimate of the region of attraction which is not necessarily of the form $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$. The set Ω of Theorem 3.4 can be any compact positively invariant set. We shall use this feature in Section 4.2 to obtain less conservative estimates of the region of attraction. Second, LaSalle's theorem can be used in cases where the system has an equilibrium set rather than an isolated equilibrium point. This will be illustrated by an application to a simple adaptive control example from Section 1.1.6. Third, the function $V(x)$ does not have to be positive definite. The utility of this feature will be illustrated by an application to the neural network example of Section 1.1.5.

Example 3.10 Consider the first-order system

$$\dot{y} = ay + u$$

together with the adaptive control law

$$u = -ky, \qquad \dot{k} = \gamma y^2, \quad \gamma > 0$$

Taking $x_1 = y$ and $x_2 = k$, the closed-loop system is represented by

$$\begin{aligned}\dot{x}_1 &= -(x_2 - a)x_1\\ \dot{x}_2 &= \gamma x_1^2\end{aligned}$$

The line $x_1 = 0$ is an equilibrium set for this system. We want to show that the trajectory of the system approaches this equilibrium set as $t \to \infty$, which means that the adaptive controller succeeds in regulating y to zero. Consider the Lyapunov function candidate

$$V(x) = \tfrac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$$

where $b > a$. The derivative of V along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b)\dot{x}_2 \\ &= -x_1^2(x_2 - a) + x_1^2(x_2 - b) = -x_1^2(b - a) \le 0\end{aligned}$$

Hence, $\dot{V}(x) \le 0$. Since $V(x)$ is radially unbounded, the set $\Omega_c = \{x \in R^2 \mid V(x) \le c\}$ is a compact, positively invariant set. Thus, taking $\Omega = \Omega_c$, all the conditions of Theorem 3.4 are satisfied. The set E is given by

$$E = \{x \in \Omega_c \mid x_1 = 0\}$$

Since any point on the line $x_1 = 0$ is an equilibrium point, E is an invariant set. Therefore, in this example $M = E$. From Theorem 3.4, we conclude that every trajectory starting in Ω_c approaches E as $t \to \infty$; that is, $x_1(t) \to 0$ as $t \to \infty$. Moreover, since $V(x)$ is radially unbounded, this conclusion is global; that is, it holds for all initial conditions $x(0)$ because for any $x(0)$ the constant c can be chosen large enough that $x(0) \in \Omega_c$. △

Note that the Lyapunov function in Example 3.10 is dependent on a constant b which is required to satisfy $b > a$. Since in this adaptive control problem the constant a is not known, we may not know the constant b explicitly but we know that it always exists. This highlights another feature of Lyapunov's method, which we have not seen before; namely, in some situations we may be able to assert the existence of a Lyapunov function that satisfies the conditions of a certain theorem even though we may not explicitly know that function. In Example 3.10 we can determine the Lyapunov function explicitly if we know some bound on a. For example, if we know that $|a| \le \alpha$, where the bound α is known, we can choose $b > \alpha$.

Example 3.11 The neural network of Section 1.1.5 is represented by

$$\dot{x}_i = \frac{1}{C_i}h_i(x_i)\left[\sum_j T_{ij}x_j - \frac{1}{R_i}g_i^{-1}(x_i) + I_i\right]$$

for $i = 1, 2, \ldots, n$, where the state variables x_i are the voltages at the amplifier outputs. They can only take values in the set

$$H = \{x \in R^n \mid -V_M < x_i < V_M\}$$

The functions $g_i : R \to (-V_M, V_M)$ are sigmoid functions,

$$h_i(x_i) = \left.\frac{dg_i}{du_i}\right|_{u_i = g_i^{-1}(x_i)} > 0, \quad \forall\ x_i \in (-V_M, V_M)$$

I_i are constant current inputs, $R_i > 0$, and $C_i > 0$. Assume that the symmetry condition $T_{ij} = T_{ji}$ is satisfied. The system may have several equilibrium points in H. We assume that all equilibrium points in H are isolated. Due to the symmetry property $T_{ij} = T_{ji}$, the vector whose ith component is

$$-\left[\sum_j T_{ij}x_j - \frac{1}{R_i}g_i^{-1}(x_i) + I_i\right]$$

is a gradient vector of a scalar function. By integration, similar to what we have done in the variable gradient method, it can be shown that this scalar function is given by

$$V(x) = -\tfrac{1}{2}\sum_i\sum_j T_{ij}x_ix_j + \sum_i \frac{1}{R_i}\int_0^{x_i} g_i^{-1}(y)\ dy - \sum_i I_ix_i$$

This function is continuously differentiable, but (typically) not positive definite. We rewrite the state equations as

$$\dot{x}_i = -\frac{1}{C_i}h_i(x_i)\frac{\partial V}{\partial x_i}$$

Let us now apply Theorem 3.4 with $V(x)$ as a candidate function. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = \sum_{i=1}^n \frac{\partial V}{\partial x_i}\dot{x}_i = -\sum_{i=1}^n \frac{1}{C_i}h_i(x_i)\left(\frac{\partial V}{\partial x_i}\right)^2 \le 0$$

Moreover,

$$\dot{V}(x) = 0 \Rightarrow \frac{\partial V}{\partial x_i} = 0 \Rightarrow \dot{x}_i = 0, \quad \forall\ i$$

Hence, $\dot{V}(x) = 0$ only at equilibrium points. To apply Theorem 3.4, we need to construct a set Ω. Let

$$\Omega(\epsilon) = \{x \in R^n \mid -(V_M - \epsilon) \le x_i \le (V_M - \epsilon)\}$$

where $\epsilon > 0$ is arbitrarily small. This set $\Omega(\epsilon)$ is closed and bounded, and $\dot{V}(x) \le 0$ in $\Omega(\epsilon)$. It remains to show that $\Omega(\epsilon)$ is a positively invariant set; that is, every trajectory starting in $\Omega(\epsilon)$ stays for all future time in $\Omega(\epsilon)$. To simplify this task, we assume a specific form for the sigmoid function $g_i(\cdot)$. Let

$$g_i(u_i) = \frac{2V_M}{\pi}\tan^{-1}\frac{\lambda\pi u_i}{2V_M}, \quad \lambda > 0$$

Then

$$\dot{x}_i = \frac{1}{C_i}h_i(x_i)\left[\sum_j T_{ij}x_j - \frac{2V_M}{\lambda\pi R_i}\tan\frac{\pi x_i}{2V_M} + I_i\right]$$

For $|x_i| \geq V_M - \epsilon$,

$$\left|\tan\frac{\pi x_i}{2V_M}\right| \geq \tan\frac{\pi(V_M-\epsilon)}{2V_M} \to \infty \text{ as } \epsilon \to 0$$

Since x_i and I_i are bounded, ϵ can be chosen small enough to ensure that

$$x_i\sum_j T_{ij}x_j - \frac{2V_M x_i}{\lambda\pi R_i}\tan\frac{\pi x_i}{2V_M} + x_iI_i < 0, \quad \text{for } V_M - \epsilon \leq |x_i| < V_M$$

Hence,

$$\frac{d}{dt}\left(x_i^2\right) = 2x_i\dot{x}_i < 0, \quad \text{for } V_M - \epsilon \leq |x_i| < V_M, \ \forall\ i$$

Consequently, trajectories starting in $\Omega(\epsilon)$ will stay in $\Omega(\epsilon)$ for all future time. In fact, trajectories starting in $H - \Omega(\epsilon)$ will converge to $\Omega(\epsilon)$. This implies that all the equilibrium points of the system lie in the compact set $\Omega(\epsilon)$. Hence, there can be only a finite number of isolated equilibrium points. In $\Omega(\epsilon)$, $E = M =$ the set of equilibrium points inside $\Omega(\epsilon)$. By Theorem 3.4, we know that every trajectory starting inside $\Omega(\epsilon)$ approaches M as t approaches infinity. Since M consists of isolated equilibrium points, it can be shown (Exercise 3.22) that a trajectory approaching M must approach one of these equilibria. Thus, for all possible initial conditions the trajectory of the system will always converge to one of the equilibrium points. This ensures that the system will not oscillate. △

3.3 Linear Systems and Linearization

The linear time-invariant system

$$\dot{x} = Ax \tag{3.9}$$

has an equilibrium point at the origin. The equilibrium point is isolated if and only if $\det(A) \neq 0$. If $\det(A) = 0$, the matrix A has a nontrivial null space. Every point in the null space of A is an equilibrium point for the system (3.9). In other words, if $\det(A) = 0$, the system has an equilibrium subspace. Notice that a linear system cannot have multiple isolated equilibrium points. For, if $\bar{x}_1$ and $\bar{x}_2$ are two equilibrium points for (3.9), then by linearity every point on the line connecting $\bar{x}_1$ and $\bar{x}_2$ is an equilibrium point for the system. Stability properties of the origin can be characterized by the locations of the eigenvalues of the matrix A. Recall from linear system theory[10] that the solution of (3.9) for a given initial state $x(0)$ is given by

$$x(t) = \exp(At)x(0) \tag{3.10}$$

and that for any matrix A there is a nonsingular matrix P (possibly complex) that transforms A into its Jordan form; that is,

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

where J_i is the Jordan block associated with the eigenvalue λ_i of A. A Jordan block of order m takes the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m\times m}$$

Therefore,

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\lambda_i t)R_{ik} \tag{3.11}$$

where m_i is the order of the Jordan block associated with the eigenvalue λ_i.[11] The following theorem characterizes the stability properties of the origin.

Theorem 3.5 *The equilibrium point $x = 0$ of (3.9) is stable if and only if all eigenvalues of A satisfy $Re\lambda_i \leq 0$ and every eigenvalue with $Re\lambda_i = 0$ has an associated Jordan block of order one. The equilibrium point $x = 0$ is (globally) asymptotically stable if and only if all eigenvalues of A satisfy $Re\lambda_i < 0$.* ◇

Proof: From (3.10) we can see that the origin is stable if and only if $\exp(At)$ is a bounded function of t for all $t \geq 0$. If one of the eigenvalues of A is in the open right-half complex plane, the corresponding exponential term $\exp(\lambda_i t)$ in (3.11) will grow unbounded as $t \to \infty$. Therefore, we must restrict the eigenvalues to be in the closed left-half complex plane. However, those eigenvalues on the imaginary axis

[10] See, for example, [29], [41], [70], [86], or [142].

[11] Equivalently, m_i is the multiplicity of λ_i as a zero of the minimal polynomial of A.

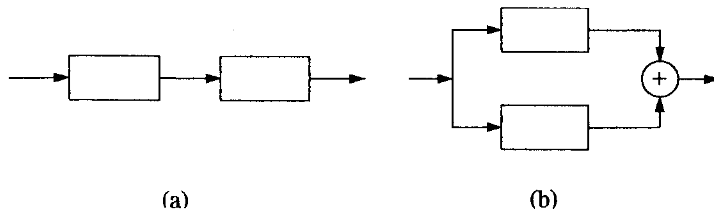


Figure 3.6: (a) Series connection; (b) parallel connection.

(if any) could give rise to unbounded terms if the order of the associated Jordan block is higher than one, due to the term t^{k-1} in (3.11). Therefore, we must restrict eigenvalues on the imaginary axis to have Jordan blocks of order one. Thus, we conclude that the condition for stability is a necessary one. It is clear that the condition is also sufficient to ensure that $\exp(At)$ is bounded. For asymptotic stability of the origin, $\exp(At)$ must approach 0 as $t \to \infty$. From (3.11), this is the case if and only if $Re\lambda_i < 0$, $\forall\ i$. Since $x(t)$ depends linearly on the initial state $x(0)$, asymptotic stability of the origin is global. □

The proof shows, mathematically, why eigenvalues on the imaginary axis must have Jordan blocks of order one to ensure stability. The following example may shed some light on the physical meaning of this requirement.

Example 3.12 Figure 3.6 shows a series connection and a parallel connection of two identical systems. Each system is represented by the state-space model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

where u and y are the input and output of the system. Let A_s and A_p be the matrices of the series and parallel connections, when modeled in the form (3.9) (no driving inputs). Let J_s and J_p be the corresponding Jordan blocks. Then

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad J_p = \begin{bmatrix} j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & -j \end{bmatrix}$$

and

$$A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad J_s = \begin{bmatrix} j & 1 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & -j & 1 \\ 0 & 0 & 0 & -j \end{bmatrix}$$

where $j = \sqrt{-1}$. The matrices A_p and A_s have the same eigenvalues on the imaginary axis, but the associated Jordan blocks for the parallel connection are of order one while those of the series connection are of order two. Thus, by Theorem 3.5, the origin of the parallel connection is stable while the origin of the series connection is unstable. To physically see the difference between the two connections, notice that in the parallel connection nonzero initial conditions produce sinusoidal oscillations of frequency 1 rad/sec, which are bounded functions of time. The sum of these sinusoidal signals remains bounded. On the other hand, nonzero initial conditions in the first component of the series connection produce a sinusoidal oscillation of frequency 1 rad/sec, which acts as a driving force for the second component. Since the second component has an undamped natural frequency of 1 rad/sec, the driving force causes "resonance" and the response grows unbounded. △

When all eigenvalues of A satisfy $Re\lambda_i < 0$, A is called a *stability matrix* or a *Hurwitz matrix*. The origin of (3.9) is asymptotically stable if and only if A is a stability matrix. Asymptotic stability of the origin can be also investigated using Lyapunov's method. Consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x$$

where P is a real symmetric positive definite matrix. The derivative of V along the trajectories of the linear system (3.9) is given by

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q \tag{3.12}$$

If Q is positive definite, we can conclude by Theorem 3.1 that the origin is asymptotically stable; that is, $Re\lambda_i < 0$ for all eigenvalues of A. Here we follow the usual procedure of Lyapunov's method, where we choose $V(x)$ to be positive definite and then check the negative definiteness of $\dot{V}(x)$. In the case of linear systems, we can reverse the order of these two steps. Suppose we start by choosing Q as a real symmetric positive definite matrix, and then solve (3.12) for P. If (3.12) has a positive definite solution, then we can again conclude that the origin is asymptotically stable. Equation (3.12) is called the *Lyapunov equation*. The following theorem characterizes asymptotic stability of the origin in terms of the solution of the Lyapunov equation.

Theorem 3.6 *A matrix A is a stability matrix; that is, $Re\lambda_i < 0$ for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the Lyapunov equation* (3.12). *Moreover, if A is a stability matrix, then P is the unique solution of* (3.12). ◇

Proof: Sufficiency follows from Theorem 3.1 with the Lyapunov function $V(x) = x^T P x$, as we have already shown. To prove necessity, assume that all eigenvalues of A satisfy $Re\lambda_i < 0$ and consider the matrix P, defined by

$$P = \int_0^\infty \exp(A^T t) Q \exp(At) \, dt \tag{3.13}$$

The integrand is a sum of terms of the form $t^{k-1}\exp(\lambda_i t)$, where $Re\lambda_i < 0$. Therefore, the integral exists. The matrix P is symmetric and positive definite. The fact that it is positive definite can be shown as follows. Supposing it is not so, there is a vector $x \neq 0$ such that $x^T P x = 0$. However,

$$\begin{aligned} x^T P x = 0 \;\; &\Rightarrow \;\; \int_0^\infty x^T \exp(A^T t) Q \exp(At) x \, dt = 0 \\ &\Rightarrow \;\; \exp(At)x \equiv 0, \; \forall \, t \geq 0 \;\Rightarrow\; x = 0 \end{aligned}$$

since $\exp(At)$ is nonsingular for all t. This contradiction shows that P is positive definite. Now, substitution of (3.13) in the left-hand side of (3.12) yields

$$\begin{aligned} PA + A^T P &= \int_0^\infty \exp(A^T t) Q \exp(At) A \, dt + \int_0^\infty A^T \exp(A^T t) Q \exp(At) \, dt \\ &= \int_0^\infty \frac{d}{dt} \exp(A^T t) Q \exp(At) \, dt \;=\; \exp(A^T t) Q \exp(At)\Big|_0^\infty \;=\; -Q \end{aligned}$$

which shows that P is indeed a solution of (3.12). To show that it is the unique solution, suppose there is another solution $\tilde{P} \neq P$. Then,

$$(P - \tilde{P})A + A^T (P - \tilde{P}) = 0$$

Premultiplying by $\exp(A^T t)$ and postmultiplying by $\exp(At)$, we obtain

$$0 = \exp(A^T t)[(P - \tilde{P})A + A^T(P - \tilde{P})]\exp(At) = \frac{d}{dt}\exp(A^T t)(P - \tilde{P})\exp(At)$$

Hence,

$$\exp(A^T t)(P - \tilde{P})\exp(At) \equiv \text{a constant } \forall \, t$$

In particular, since $\exp(A0) = I$, we have

$$(P - \tilde{P}) = \exp(A^T t)(P - \tilde{P})\exp(At) \to 0 \text{ as } t \to \infty$$

Therefore, $\tilde{P} = P$. □

The positive definiteness requirement on Q can be relaxed. It is left as an exercise to the reader (Exercise 3.24) to verify that Q can be taken as a positive semidefinite matrix of the form $Q = C^T C$, where the pair (A, C) is observable.

Equation (3.12) is a linear algebraic equation which can be solved by rearranging it in the form $Mx = y$ where x and y are defined by stacking the elements of P and Q in vectors, as will be illustrated in the next example. There are numerically efficient methods for solving such equations.[12]

Example 3.13 Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

where, due to symmetry, $p_{12} = p_{21}$. The Lyapunov equation (3.12) can be rewritten as

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

The unique solution of this equation is given by

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.0 \end{bmatrix} \;\Rightarrow\; P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$

The matrix P is positive definite since its leading principal minors (1.5 and 1.25) are positive. Hence, all eigenvalues of A are in the open left-half complex plane. △

The Lyapunov equation can be used to test whether or not a matrix A is a stability matrix, as an alternative to calculating the eigenvalues of A. One starts by choosing a positive definite matrix Q (for example, $Q = I$) and solves the Lyapunov equation (3.12) for P. If the equation has a positive definite solution, we conclude that A is a stability matrix; otherwise, it is not so. However, there is no computational advantage in solving the Lyapunov equation over calculating the eigenvalues of A.[13] Besides, the eigenvalues provide more direct information about the response of the linear system. The interest in the Lyapunov equation is not in its use as a stability test for linear systems;[14] rather, it is in the fact that it provides

[12] Consult [57] on numerical methods for solving linear algebraic equations. The Lyapunov equation can also be solved by viewing it as a special case of the Sylvester equation $PA + BP + C = 0$, which is treated in [57]. Almost all commercial software programs for control systems include commands for solving the Lyapunov equation.

[13] A typical procedure for solving the Lyapunov equation, the Bartels-Stewart algorithm [57], starts by transforming A into its real Schur form which gives the eigenvalues of A. Hence, the computational effort for solving the Lyapunov equation is more than calculating the eigenvalues of A. Other algorithms for solving the Lyapunov equation take an amount of computations comparable to the Bartels-Stewart algorithm.

[14] It might be of interest, however, to know that one can use the Lyapunov equation to derive the classical Routh-Hurwitz criterion; see [29, pp. 417–419].

a procedure for finding a Lyapunov function for any linear system (3.9) when A is a stability matrix. The mere existence of a Lyapunov function will allow us to draw conclusions about the system when the right-hand side Ax is perturbed, whether such perturbation is a linear perturbation in the coefficients of A or a nonlinear perturbation. This advantage will unfold as we continue our study of Lyapunov's method.

Let us go back to the nonlinear system (3.1)

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is a continuously differentiable map from a domain $D \subset R^n$ into R^n. Suppose that the origin $x = 0$ is in the interior of D and is an equilibrium point for the system; that is, $f(0) = 0$. By the mean value theorem,

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)\, x$$

where z_i is a point on the line segment connecting x to the origin. The above equality is valid for any point $x \in D$ such that the line segment connecting x to the origin lies entirely in D. Since $f(0) = 0$, we can write $f_i(x)$ as

$$f_i(x) = \frac{\partial f_i}{\partial x}(z_i)x = \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

Hence,

$$f(x) = Ax + g(x)$$

where

$$A = \frac{\partial f}{\partial x}(0), \quad \text{and} \quad g_i(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x$$

The function $g_i(x)$ satisfies

$$|g_i(x)| \le \left\| \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0) \right\| \|x\|$$

By continuity of $[\partial f / \partial x]$, we see that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \quad \text{as} \quad \|x\| \to 0$$

This suggests that in a small neighborhood of the origin we can approximate the nonlinear system (3.1) by its linearization about the origin

$$\dot{x} = Ax, \quad \text{where} \quad A = \frac{\partial f}{\partial x}(0)$$

The following theorem spells out conditions under which we can draw conclusions about stability of the origin as an equilibrium point for the nonlinear system by investigating its stability as an equilibrium point for the linear system. The theorem is known as *Lyapunov's indirect method.*

Theorem 3.7 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \left. \frac{\partial f}{\partial x}(x) \right|_{x=0}$$

Then,

1. *The origin is asymptotically stable if $Re\lambda_i < 0$ for all eigenvalues of A.*

2. *The origin is unstable if $Re\lambda_i > 0$ for one or more of the eigenvalues of A.*

◇

Proof: To prove the first part, let A be a stability matrix. Then, by Theorem 3.6 we know that for any positive definite symmetric matrix Q, the solution P of the Lyapunov equation (3.12) is positive definite. We use

$$V(x) = x^T P x$$

as a Lyapunov function candidate for the nonlinear system. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\ &= x^T(PA + A^T P)x + 2x^T P g(x) \\ &= -x^T Q x + 2x^T P g(x) \end{aligned}$$

The first term on the right-hand side is negative definite, while the second term is (in general) indefinite. The function $g(x)$ satisfies

$$\frac{\|g(x)\|_2}{\|x\|_2} \to 0 \quad \text{as} \quad \|x\|_2 \to 0$$

Therefore, for any $\gamma > 0$ there exists $r > 0$ such that

$$\|g(x)\|_2 < \gamma \|x\|_2, \quad \forall\ \|x\|_2 < r$$

Hence,

$$\dot{V}(x) < -x^T Q x + 2\gamma \|P\|_2 \|x\|_2^2, \quad \forall \ \|x\|_2 < r$$

but

$$x^T Q x \geq \lambda_{\min}(Q) \|x\|_2^2$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. Note that $\lambda_{\min}(Q)$ is real and positive since Q is symmetric and positive definite. Thus

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma \|P\|_2] \|x\|_2^2, \quad \forall \ \|x\|_2 < r$$

Choosing $\gamma < \lambda_{\min}(Q)/2\|P\|_2$ ensures that $\dot{V}(x)$ is negative definite. By Theorem 3.1, we conclude that the origin is asymptotically stable. To prove the second part of the theorem, let us consider first the special case when A has no eigenvalues on the imaginary axis. If the eigenvalues of A cluster into a group of eigenvalues in the open right-half plane and a group of eigenvalues in the open left-half plane, then there is a nonsingular matrix T such that[15]

$$TAT^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1 and A_2 are stability matrices. Let

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where the partition of z is compatible with the dimensions of A_1 and A_2. The change of variables $z = Tx$ transforms the system

$$\dot{x} = Ax + g(x)$$

into the form

$$\begin{aligned} \dot{z}_1 &= -A_1 z_1 + g_1(z) \\ \dot{z}_2 &= A_2 z_2 + g_2(z) \end{aligned}$$

where the functions $g_i(z)$ have the property that for any $\gamma > 0$, there exists $r > 0$ such that

$$\|g_i(z)\|_2 < \gamma \|z\|_2, \quad \forall \ \|z\|_2 \leq r, \ i = 1, 2$$

The origin $z = 0$ is an equilibrium point for the system in the z-coordinates. Clearly, any conclusion we arrive at concerning stability properties of $z = 0$ carries over to the equilibrium point $x = 0$ in the x-coordinates, since T is nonsingular.[16] To show

[15] There are several methods for finding the matrix T, one of which is to transform the matrix A into its real Jordan form [57].

[16] See Exercise 3.27 for a general discussion of stability-preserving maps.

that the origin is unstable, we apply Theorem 3.3. The construction of a function $V(z)$ will be done basically as in Example 3.7, except for working with vectors instead of scalars. Let Q_1 and Q_2 be positive definite symmetric matrices of the dimensions of A_1 and A_2, respectively. Since A_1 and A_2 are stability matrices, we know from Theorem 3.6 that the Lyapunov equations

$$P_i A_i + A_i^T P_i = -Q_i, \ i = 1, 2$$

have unique positive definite solutions P_1 and P_2. Let

$$V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2 = z^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} z$$

In the subspace $z_2 = 0$, $V(z) > 0$ at points arbitrarily close to the origin. Let

$$U = \{z \in R^n \mid \|z\|_2 \leq r \text{ and } V(z) > 0\}$$

In U,

$$\begin{aligned} \dot{V}(z) &= -z_1^T (P_1 A_1 + A_1^T P_1) z_1 + 2 z_1^T P_1 g_1(z) \\ &\quad - z_2^T (P_2 A_2 + A_2^T P_2) z_2 - 2 z_2^T P_2 g_2(z) \\ &= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + 2 z^T \begin{bmatrix} P_1 g_1(z) \\ -P_2 g_2(z) \end{bmatrix} \\ &\geq \lambda_{\min}(Q_1) \|z_1\|_2^2 + \lambda_{\min}(Q_2) \|z_2\|_2^2 \\ &\quad - 2\|z\|_2 \sqrt{\|P_1\|_2^2 \|g_1(z)\|_2^2 + \|P_2\|_2^2 \|g_2(z)\|_2^2} \\ &> (\alpha - 2\sqrt{2}\beta\gamma) \|z\|_2^2 \end{aligned}$$

where

$$\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}, \quad \beta = \max\{\|P_1\|_2, \|P_2\|_2\}$$

Thus, choosing $\gamma < \alpha / 2\sqrt{2}\beta$ ensures that $\dot{V}(z) > 0$ in U. Therefore, by Theorem 3.3, the origin is unstable. Notice that we could have applied Theorem 3.3 in the original coordinates by defining the matrices

$$P = T^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} T; \quad Q = T^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} T$$

which satisfy the equation

$$PA + A^T P = Q$$

The matrix Q is positive definite, and $V(x) = x^T P x$ is positive for points arbitrarily close to the origin $x = 0$. Let us consider now the general case when A may have eigenvalues on the imaginary axis, in addition to eigenvalues in the open right-half complex plane. We can reduce this case to the special case we have just studied by a simple trick of shifting the imaginary axis. Suppose A has m eigenvalues with $Re\lambda_i > \delta > 0$. Then, the matrix $[A - (\delta/2)I]$ has m eigenvalues in the open right-half plane, but no eigenvalues on the imaginary axis. By previous arguments, there exist matrices $P = P^T$ and $Q = Q^T > 0$ such that

$$P\left[A - \frac{\delta}{2}I\right] + \left[A - \frac{\delta}{2}I\right]^T P = Q$$

where $V(x) = x^T P x$ is positive for points arbitrarily close to the origin. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= x^T(PA + A^TP)x + 2x^TPg(x)\\ &= x^T\left[P\left(A - \frac{\delta}{2}I\right) + \left(A - \frac{\delta}{2}I\right)^T P\right]x + \delta x^TPx + 2x^TPg(x)\\ &= x^TQx + \delta V(x) + 2x^TPg(x)\end{aligned}$$

In the set

$$\{x \in R^n \mid \|x\|_2 \le r \text{ and } V(x) > 0\}$$

where r is chosen such that $\|g(x)\|_2 \le \gamma\|x\|_2$ for $\|x\|_2 < r$, $\dot{V}(x)$ satisfies

$$\dot{V}(x) \ge \lambda_{\min}(Q)\|x\|_2^2 - 2\|P\|_2\|x\|_2\|g(x)\|_2 \ge (\lambda_{\min}(Q) - 2\gamma\|P\|_2)\,\|x\|_2^2$$

which is positive for $\gamma < \lambda_{\min}(Q)/2\|P\|_2$. Application of Theorem 3.3 concludes the proof. □

Theorem 3.7 provides us with a simple procedure for determining stability of an equilibrium point at the origin. We calculate the *Jacobian matrix*

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

and test its eigenvalues. If $Re\lambda_i < 0$ for all i or $Re\lambda_i > 0$ for some i, we conclude that the origin is asymptotically stable or unstable, respectively. Moreover, the proof of the theorem shows that when $Re\lambda_i < 0$ for all i, we can also find a Lyapunov function for the system that will work locally in some neighborhood of the origin. The Lyapunov function is the quadratic form $V(x) = x^TPx$, where P is the solution of the Lyapunov equation (3.12) for any positive definite symmetric matrix Q. Note that Theorem 3.7 does not say anything about the case when $Re\lambda_i \le 0$ for all i, with $Re\lambda_i = 0$ for some i. In this case, linearization fails to determine stability of the equilibrium point.[17]

[17] See Section 4.1 for further investigation of the critical case when linearization fails.

Example 3.14 Consider the scalar system

$$\dot{x} = ax^3$$

Linearization of the system about the origin $x = 0$ yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\big|_{x=0} = 0$$

There is one eigenvalue which lies on the imaginary axis. Hence, linearization fails to determine stability of the origin. This failure is genuine in the sense that the origin could be asymptotically stable, stable, or unstable, depending on the value of the parameter a. If $a < 0$, the origin is asymptotically stable as can be seen from the Lyapunov function $V(x) = x^4$ whose derivative along the trajectories of the system is given by $\dot{V}(x) = 4ax^6 < 0$ when $a < 0$. If $a = 0$, the system is linear and the origin is stable according to Theorem 3.5. If $a > 0$, the origin is unstable as can be seen from Theorem 3.3 and the function $V(x) = x^4$, whose derivative along the trajectories of the system is given by $\dot{V}(x) = 4ax^6 > 0$ when $a > 0$. △

Example 3.15 The pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\end{aligned}$$

has two equilibrium points at $(x_1 = 0,\ x_2 = 0)$ and $(x_1 = \pi,\ x_2 = 0)$. Let us investigate the stability of both points using linearization. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(\frac{g}{l}\right)\cos x_1 & -\left(\frac{k}{m}\right)\end{bmatrix}$$

To determine stability of the origin, we evaluate the Jacobian at $x = 0$.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -\left(\frac{g}{l}\right) & -\left(\frac{k}{m}\right)\end{bmatrix}$$

The eigenvalues of A are

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 - \frac{4g}{l}}$$

For all $g,\ l,\ m,\ k > 0$, the eigenvalues satisfy $Re\lambda_i < 0$. Hence, the equilibrium point at the origin is asymptotically stable. In the absence of friction ($k = 0$), both eigenvalues are on the imaginary axis. In this case, we cannot determine stability of the origin through linearization. We have seen in Example 3.3 that, in this case, the origin is a stable equilibrium point as determined by an energy Lyapunov function. To determine stability of the equilibrium point at $(x_1 = \pi,\ x_2 = 0)$, we evaluate the Jacobian at this point. This is equivalent to performing a change of variables $z_1 = x_1 - \pi,\ z_2 = x_2$ to shift the equilibrium point to the origin, and evaluating the Jacobian $[\partial f/\partial z]$ at $z = 0$.

$$\tilde{A} = \left.\frac{\partial f}{\partial x}\right|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ \left(\frac{g}{l}\right) & -\left(\frac{k}{m}\right) \end{bmatrix}$$

The eigenvalues of $\tilde{A}$ are

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + \frac{4g}{l}}$$

For all $g,\ l,\ m > 0$ and all $k \geq 0$, there is one eigenvalue in the open right-half plane. Hence, the equilibrium point at $(x_1 = \pi,\ x_2 = 0)$ is unstable. △

3.4 Nonautonomous Systems

Consider the nonautonomous system

$$\dot{x} = f(t, x) \tag{3.14}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point for (3.14) at $t = 0$ if

$$f(t, 0) = 0, \quad \forall\ t \geq 0$$

An equilibrium at the origin could be a translation of a nonzero equilibrium point or, more generally, a translation of a nonzero solution of the system. To see the latter point, suppose that $\bar{y}(\tau)$ is a solution of the system

$$\frac{dy}{d\tau} = g(\tau, y)$$

defined for all $\tau \geq a$. The change of variables

$$x = y - \bar{y}(\tau); \quad t = \tau - a$$

transforms the system into the form

$$\dot{x} = g(\tau, y) - \dot{\bar{y}}(\tau) = g(t + a, x + \bar{y}(t + a)) - \dot{\bar{y}}(t + a) \stackrel{\text{def}}{=} f(t, x)$$

Since

$$\dot{\bar{y}}(t + a) = g(t + a, \bar{y}(t + a)), \quad \forall\ t \geq 0$$

the origin $x = 0$ is an equilibrium point of the transformed system at $t = 0$. Therefore, by examining the stability behavior of the origin as an equilibrium point for the transformed system, we determine the stability behavior of the solution $\bar{y}(\tau)$ of the original system. Notice that if $\bar{y}(\tau)$ is not constant, the transformed system will be nonautonomous even when the original system is autonomous, that is, even when $g(\tau, y) = g(y)$. That is why studying the stability behavior of solutions in the sense of Lyapunov can only be done in the context of studying the stability behavior of the equilibria of nonautonomous systems.

The notions of stability and asymptotic stability of the equilibrium of a nonautonomous system are basically the same as we introduced in Definition 3.1 for autonomous systems. The new element here is that, while the solution of an autonomous system depends only on $(t - t_0)$, the solution of a nonautonomous system may depend on both t and t_0. Therefore, the stability behavior of the equilibrium point will, in general, be dependent on t_0. The origin $x = 0$ is a stable equilibrium point for (3.14) if for each $\epsilon > 0$ and any $t_0 \geq 0$ there is $\delta = \delta(\epsilon, t_0) > 0$ such that

auto vs. non-auto

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq t_0$$

The constant δ is, in general, dependent on the initial time t_0. The existence of δ for every t_0 does not necessarily guarantee that there is one constant δ, dependent only on ϵ, that would work for all t_0, as illustrated by the following example.

Example 3.16 The linear first-order system

$$\dot{x} = (6t \sin t - 2t)x$$ → non-autonomous.

has the closed-form solution $\frac{dx}{x} = (6t \sin t - 2t)\,dt$

$$\begin{aligned} x(t) &= x(t_0) \exp\left[\int_{t_0}^{t} (6\tau \sin \tau - 2\tau)\, d\tau\right] \\ &= x(t_0) \exp\left[6 \sin t - 6t \cos t - t^2 - 6 \sin t_0 + 6t_0 \cos t_0 + t_0^2\right] \end{aligned}$$

For any t_0, the term $-t^2$ will eventually dominate, which shows that the exponential term is bounded for all $t \geq t_0$ by a constant $c(t_0)$ dependent on t_0. Hence,

$$|x(t)| < |x(t_0)| c(t_0), \quad \forall\ t \geq t_0$$

For any $\epsilon > 0$, the choice $\delta = \epsilon/c(t_0)$ shows that the origin is stable. Now, suppose t_0 takes on the successive values $t_0 = 2n\pi$ for $n = 0, 1, 2, \ldots$, and suppose that $x(t)$ is evaluated π seconds later in each case. Then,

$$x(t_0 + \pi) = x(t_0) \exp\left[(4n + 1)(6 - \pi)\pi\right]$$

This implies that, for $x(t_0) \neq 0$,

$$\frac{x(t_0+\pi)}{x(t_0)} \to \infty \text{ as } n \to \infty$$

Thus, given $\epsilon > 0$, there is no δ independent of t_0 that would satisfy the stability requirement uniformly in t_0. △

Nonuniformity with respect to t_0 could also appear in studying asymptotic stability of the origin, as the following example shows.

Example 3.17 The linear first-order system

$$\dot{x} = -\frac{x}{1+t}$$

has the closed-form solution

$$x(t) = x(t_0)\exp\left(\int_{t_0}^{t} \frac{-1}{1+\tau}\, d\tau\right) = x(t_0)\frac{1+t_0}{1+t}$$

Since $|x(t)| \leq |x(t_0)|$, $\forall\ t \geq t_0$, the origin is clearly stable. Actually, given any $\epsilon > 0$, we can choose δ independent of t_0. It is also clear that

$$x(t) \to 0 \text{ as } t \to \infty$$

Hence, according to Definition 3.1, the origin is asymptotically stable. Notice, however, that the convergence of $x(t)$ to the origin is not uniform with respect to the initial time t_0. Recall that convergence of $x(t)$ to the origin is equivalent to saying that, given any $\epsilon > 0$, there is $T = T(\epsilon, t_0) > 0$ such that $|x(t)| < \epsilon$ for all $t \geq t_0 + T$. Although this is true for every t_0, the constant T cannot be chosen independent of t_0. △

As a consequence, we need to refine Definition 3.1 to emphasize the dependence of the stability behavior of the origin on the initial time t_0. We are interested in a refinement that defines stability and asymptotic stability of the origin as uniform properties with respect to the initial time.[18]

Non-Autonomous :

Definition 3.2 *The equilibrium point $x = 0$ of (3.14) is*

- *stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq t_0 \geq 0 \tag{3.15}$$

[18] See [62] or [87] for other refinements of Definition 3.1.

- *uniformly stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of t_0, such that (3.15) is satisfied.*
- *unstable if not stable.*
- *asymptotically stable if it is stable and there is $c = c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$.*
- *uniformly asymptotically stable if it is uniformly stable and there is $c > 0$, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\epsilon > 0$, there is $T = T(\epsilon) > 0$ such that*

$$\|x(t)\| < \epsilon, \quad \forall\ t \geq t_0 + T(\epsilon),\ \forall\ \|x(t_0)\| < c$$

- *globally uniformly asymptotically stable if it is uniformly stable and, for each pair of positive numbers ϵ and c, there is $T = T(\epsilon, c) > 0$ such that*

$$\|x(t)\| < \epsilon, \quad \forall\ t \geq t_0 + T(\epsilon, c),\ \forall\ \|x(t_0)\| < c$$

Uniform stability and asymptotic stability can be characterized in terms of special scalar functions, known as class $\mathcal{K}$ and class $\mathcal{KL}$ functions. CLASS K & Class KL.

Definition 3.3 *A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 3.4 *A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

- $\beta(r,s)$ is class K w.r.t. r for each fixed s
- $\beta(r,s)$ is decreasing in s, for each fixed r, and $\beta(r,s) \to 0$ as $s \to \infty$

Example 3.18

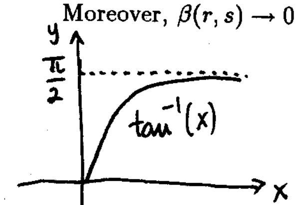
- $\alpha(r) = \tan^{-1} r$ is strictly increasing since $\alpha'(r) = 1/(1 + r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$.
- $\alpha(r) = r^c$, for any positive real number c, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$.
- $\beta(r, s) = r/(ksr + 1)$, for any positive real number k, is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr + 1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr + 1)^2} < 0$$

Moreover, $\beta(r, s) \to 0$ as $s \to \infty$. Hence, it belongs to class $\mathcal{KL}$.



- $\alpha(r) = kr \to$ class K_∞
- $\beta(r,s) = re^{-s} \to$ class KL

- $\beta(r,s) = r^c e^{-s}$, for any positive real number c, belongs to class $\mathcal{KL}$. △

The following lemma states some obvious properties of class $\mathcal{K}$ and class $\mathcal{KL}$ functions, which will be needed later on. The proof of the lemma is left as an exercise for the reader (Exercise 3.32).

Lemma 3.2 *Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be class $\mathcal{K}$ functions on $[0,a)$, $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ be class $\mathcal{K}_\infty$ functions, and $\beta(\cdot,\cdot)$ be a class $\mathcal{KL}$ function. Denote the inverse of $\alpha_i(\cdot)$ by $\alpha_i^{-1}(\cdot)$. Then,*

- *α_1^{-1} is defined on $[0,\alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- *α_3^{-1} is defined on $[0,\infty)$ and belongs to class $\mathcal{K}_\infty$.*
- *$\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$.*
- *$\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$.*
- *$\sigma(r,s) = \alpha_1(\beta(\alpha_2(r),s))$ belongs to class $\mathcal{KL}$.* ◇

The following lemma gives equivalent definitions of uniform stability and uniform asymptotic stability using class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

Lemma 3.3 *The equilibrium point $x = 0$ of (3.14) is*

- *uniformly stable if and only if there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \le \alpha(\|x(t_0)\|), \quad \forall\, t \ge t_0 \ge 0, \ \forall\, \|x(t_0)\| < c \tag{3.16}$$

- *uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \ge t_0 \ge 0, \ \forall\, \|x(t_0)\| < c \tag{3.17}$$

- *globally uniformly asymptotically stable if and only if inequality (3.17) is satisfied for any initial state $x(t_0)$.* ◇

Proof: Appendix A.3.

As a consequence of Lemma 3.3, we see that in the case of autonomous systems stability and asymptotic stability per Definition 3.1 imply the existence of class $\mathcal{K}$ and class $\mathcal{KL}$ functions that satisfy inequalities (3.16) and (3.17). This is the case because, for autonomous systems, stability and asymptotic stability of the origin are uniform with respect to the initial time t_0.

A special case of uniform asymptotic stability arises when the class $\mathcal{KL}$ function β in (3.17) takes the form $\beta(r,s) = kre^{-\gamma s}$. This case is very important and will be designated as a distinct stability property of equilibrium points.

Definition 3.5 *The equilibrium point $x = 0$ of (3.14) is exponentially stable if inequality (3.17) is satisfied with*

$$\beta(r,s) = kre^{-\gamma s}, \quad k > 0, \quad \gamma > 0$$

and is globally exponentially stable if this condition is satisfied for any initial state.

Lyapunov theory for autonomous systems can be extended to nonautonomous systems. For each of Theorems 3.1–3.4, one can state various extensions to nonautonomous systems. We shall not document all these extensions here.[19] Instead, we concentrate on uniform asymptotic stability.[20] This is the case we encounter in most nonautonomous applications of Lyapunov's method. To establish uniform asymptotic stability of the origin, we need to verify inequality (3.17). The class $\mathcal{KL}$ function in this inequality will appear in our analysis through the solution of an autonomous scalar differential equation. We start with two preliminary lemmas; the first one states properties of the solution of this special equation and the second one gives upper and lower bounds on a positive definite function in terms of class $\mathcal{K}$ functions.

Lemma 3.4 *Consider the scalar autonomous differential equation*

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

where $\alpha(\cdot)$ is a locally Lipschitz class $\mathcal{K}$ function defined on $[0,a)$. For all $0 \le y_0 < a$, this equation has a unique solution $y(t)$ defined for all $t \ge t_0$. Moreover,

$$y(t) = \sigma(y_0, t - t_0)$$

where $\sigma(r,s)$ is a class $\mathcal{KL}$ function defined on $[0,a) \times [0,\infty)$. ◇

Proof: Appendix A.4.

We can see that the claim of this lemma is true by examining specific examples where a closed-form solution of the scalar equation can be found. For example, if $\dot{y} = -ky$, $k > 0$, then the solution is

$$y(t) = y_0 \exp[-k(t - t_0)] \ \Rightarrow\ \sigma(r,s) = r\exp(-ks)$$

As another example, if $\dot{y} = -ky^2$, $k > 0$, then the solution is

$$y(t) = \frac{y_0}{ky_0(t - t_0) + 1} \ \Rightarrow\ \sigma(r,s) = \frac{r}{krs + 1}$$

[19] Lyapunov theory for nonautonomous systems is well documented in the literature. Good references on the subject include [62] and [137], while good introductions can be found in [181] and [118].

[20] See Exercise 3.35 for a uniform stability theorem.

Lemma 3.5 *Let $V(x) : D \to R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

for all $x \in B_r$. Moreover, if $D = R^n$ and $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$ and the foregoing inequality holds for all $x \in R^n$. ◇

Proof: Appendix A.5.

For a quadratic positive definite function $V(x) = x^T P x$, Lemma 3.5 follows from the inequalities

$$\lambda_{\min}(P)\|x\|_2^2 \le x^T P x \le \lambda_{\max}(P)\|x\|_2^2$$

We are now ready to state and prove the main result of this section.

Theorem 3.8 *Let $x = 0$ be an equilibrium point for (3.14) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \to R$ be a continuously differentiable function such that*

$$W_1(x) \le V(t, x) \le W_2(x) \tag{3.18}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x) \tag{3.19}$$

$\forall\ t \ge 0$, $\forall\ x \in D$ where $W_1(x)$, $W_2(x)$, and $W_3(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable. ◇

Proof: The derivative of V along the trajectories of (3.14) is given by

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x)$$

Choose $r > 0$ and $\rho > 0$ such that $B_r \subset D$ and $\rho < \min_{\|x\|=r} W_1(x)$. Then, $\{x \in B_r \mid W_1(x) \le \rho\}$ is in the interior of B_r. Define a time-dependent set $\Omega_{t,\rho}$ by

$$\Omega_{t,\rho} = \{x \in B_r \mid V(t, x) \le \rho\}$$

The set $\Omega_{t,\rho}$ contains $\{x \in B_r \mid W_2(x) \le \rho\}$ since

$$W_2(x) \le \rho \Rightarrow V(t, x) \le \rho$$

On the other hand, $\Omega_{t,\rho}$ is a subset of $\{x \in B_r \mid W_1(x) \le \rho\}$ since

$$V(t, x) \le \rho \Rightarrow W_1(x) \le \rho$$

Thus,

$$\{x \in B_r \mid W_2(x) \le \rho\} \subset \Omega_{t,\rho} \subset \{x \in B_r \mid W_1(x) \le \rho\} \subset B_r$$

for all $t \ge 0$. For any $t_0 \ge 0$ and any $x_0 \in \Omega_{t_0,\rho}$, the solution starting at (t_0, x_0) stays in $\Omega_{t,\rho}$ for all $t \ge t_0$. This follows from the fact that $\dot{V}(t, x)$ is negative on $D - \{0\}$; hence, $V(t, x)$ is decreasing. Therefore, the solution starting at (t_0, x_0) is defined for all $t \ge t_0$ and $x(t) \in B_r$. For the rest of the proof, we will assume that $x_0 \in \{x \in B_r \mid W_2(x) \le \rho\}$. By Lemma 3.5, there exist class $\mathcal{K}$ functions α_1, α_2, and α_3, defined on $[0, r]$, such that

$$W_1(x) \ge \alpha_1(\|x\|), \quad W_2(x) \le \alpha_2(\|x\|), \quad W_3(x) \ge \alpha_3(\|x\|)$$

Hence, V and $\dot{V}$ satisfy the inequalities

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\dot{V}(t, x) \le -\alpha_3(\|x\|)$$

Consequently,

$$\dot{V} \le -\alpha_3(\|x\|) \le -\alpha_3\left(\alpha_2^{-1}(V)\right) \stackrel{\text{def}}{=} -\alpha(V)$$

The function $\alpha(\cdot)$ is a class $\mathcal{K}$ function defined on $[0, r]$ (see Lemma 3.2). Assume, without loss of generality,[21] that $\alpha(\cdot)$ is locally Lipschitz. Let $y(t)$ satisfy the autonomous first-order differential equation

$$\dot{y} = -\alpha(y), \quad y(t_0) = V(t_0, x(t_0)) \ge 0$$

By the comparison lemma (Lemma 2.5),

$$V(t, x(t)) \le y(t), \quad \forall\ t \ge t_0$$

By Lemma 3.4, there exists a class $\mathcal{KL}$ function $\sigma(r, s)$ defined on $[0, r] \times [0, \infty)$ such that

$$V(t, x(t)) \le \sigma(V(t_0, x(t_0)), t - t_0), \quad \forall\ V(t_0, x(t_0)) \in [0, \rho]$$

Therefore, any solution starting in $\Omega_{t_0,\rho}$ satisfies the inequality

$$\begin{aligned} \|x(t)\| &\le \alpha_1^{-1}(V(t, x(t))) \le \alpha_1^{-1}(\sigma(V(t_0, x(t_0)), t - t_0)) \\ &\le \alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|), t - t_0)) \stackrel{\text{def}}{=} \beta(\|x(t_0)\|, t - t_0) \end{aligned}$$

[21] If α is not locally Lipschitz, we can choose a locally Lipschitz class $\mathcal{K}$ function β such that $\alpha(r) \ge \beta(r)$ over the domain of interest. Then, $\dot{V} \le -\beta(V)$ and we can continue the proof with β instead of α. For example, suppose $\alpha(r) = \sqrt{r}$. The function $\sqrt{r}$ is a class $\mathcal{K}$ function, but not locally Lipschitz at $r = 0$. Define β as $\beta(r) = r$ for $r < 1$ and $\beta(r) = \sqrt{r}$ for $r \ge 1$. The function β is class $\mathcal{K}$ and locally Lipschitz. Moreover, $\alpha(r) \ge \beta(r)$ for all $r \ge 0$.

Lemma 3.2 shows that $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}$ function. Thus, inequality (3.17) is satisfied for all $x(t_0) \in \{x \in B_r \mid W_2(x) \le \rho\}$, which implies that $x = 0$ is uniformly asymptotically stable. □

A function $V(t,x)$ satisfying the left inequality of (3.18) is said to be *positive definite*. A function satisfying the right inequality of (3.18) is said to be *decrescent*. A function $V(t,x)$ is said to be *negative definite* if $-V(t,x)$ is positive definite. Therefore, Theorem 3.8 says that the origin is uniformly asymptotically stable if there is a continuously differentiable, positive definite, decrescent function $V(t,x)$ whose derivative along the trajectories of the system is negative definite. In this case, $V(t,x)$ is called a Lyapunov function.

The proof of Theorem 3.8 estimates the region of attraction of the origin by the set

$$\{x \in B_r \mid W_2(x) \le \rho\}$$

This estimate allows us to obtain a global version of Theorem 3.8.

Corollary 3.3 *Suppose that all the assumptions of Theorem 3.8 are satisfied globally (for all $x \in R^n$) and $W_1(x)$ is radially unbounded. Then, $x = 0$ is globally uniformly asymptotically stable.* ◇

Proof: Since $W_1(x)$ is radially unbounded, so is $W_2(x)$. Therefore, the set $\{x \in R^n \mid W_2(x) \le \rho\}$ is bounded for any $\rho > 0$. For any $x_0 \in R^n$, we can choose ρ large enough so that $x_0 \in \{x \in R^n \mid W_2(x) \le \rho\}$. The rest of the proof is the same as that of Theorem 3.8. □

A Lyapunov function $V(t,x)$ satisfying (3.18) with radially unbounded $W_1(x)$ is said to be *radially unbounded*.

The manipulation of class $\mathcal{K}$ functions in the proof of Theorem 3.8 simplifies when the class $\mathcal{K}$ functions take the special form $\alpha_i(r) = k_i r^c$. In this case, we can actually show that the origin is exponentially stable.

Corollary 3.4 *Suppose all the assumptions of Theorem 3.8 are satisfied with*

$$W_1(x) \ge k_1\|x\|^c, \quad W_2(x) \le k_2\|x\|^c, \quad W_3(x) \ge k_3\|x\|^c$$

for some positive constants k_1, k_2, k_3, and c. Then, $x = 0$ is exponentially stable. Moreover, if the assumptions hold globally, then $x = 0$ is globally exponentially stable. ◇

Proof: V and $\dot{V}$ satisfy the inequalities

$$k_1\|x\|^c \le V(t,x) \le k_2\|x\|^c$$

$$\dot{V}(t,x) \le -k_3\|x\|^c \le -\frac{k_3}{k_2}V(t,x)$$

By the comparison lemma (Lemma 2.5),

$$V(t,x(t)) \le V(t_0,x(t_0))e^{-(k_3/k_2)(t-t_0)}$$

Hence,

$$\begin{aligned}\|x(t)\| &\le \left[\frac{V(t,x(t))}{k_1}\right]^{1/c} \le \left[\frac{V(t_0,x(t_0))e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/c}\\ &\le \left[\frac{k_2\|x(t_0)\|^c e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/c} = \left(\frac{k_2}{k_1}\right)^{1/c}\|x(t_0)\|e^{-(k_3/k_2c)(t-t_0)}\end{aligned}$$

Hence, the origin is exponentially stable. If all the assumptions hold globally, the foregoing inequality holds for all $x(t_0) \in R^n$. □

Example 3.19 Consider the scalar system

$$\dot{x} = -[1+g(t)]x^3$$

where $g(t)$ is continuous and $g(t) \ge 0$ for all $t \ge 0$. Using the Lyapunov function candidate $V(x) = \frac{1}{2}x^2$, we obtain

$$\dot{V} = -[1+g(t)]x^4 \le -x^4, \quad \forall\ x \in R,\ \forall\ t \ge 0$$

The assumptions of Theorem 3.8 are satisfied globally with $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$. Hence, the origin is globally uniformly asymptotically stable. △

Example 3.20 Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 - g(t)x_2\\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

where $g(t)$ is continuously differentiable and satisfies

$$0 \le g(t) \le k \ \text{ and } \ \dot{g}(t) \le g(t), \quad \forall\ t \ge 0$$

Taking $V(t,x) = x_1^2 + [1+g(t)]x_2^2$ as a Lyapunov function candidate, it can be easily seen that

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2, \quad \forall\ x \in R^2$$

Hence, $V(t,x)$ is positive definite, decrescent, and radially unbounded. The derivative of V along the trajectories of the system is given by

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

Using the inequality

$$2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2$$

we obtain

$$\dot{V}(t,x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \stackrel{\text{def}}{=} -x^T Q x$$

where Q is positive definite; hence, $\dot{V}(t,x)$ is negative definite. Thus, all the assumptions of Theorem 3.8 are satisfied globally with quadratic positive definite functions W_1, W_2, and W_3. From Corollary 3.4, we conclude that the origin is globally exponentially stable. △

Example 3.21 The linear time-varying system

$$\dot{x} = A(t)x \tag{3.20}$$

has an equilibrium point at $x = 0$. Let $A(t)$ be continuous for all $t \geq 0$. Suppose there is a continuously differentiable, symmetric, bounded, positive definite matrix $P(t)$, that is,

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall\ t \geq 0$$

which satisfies the matrix differential equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \tag{3.21}$$

where $Q(t)$ is continuous, symmetric, and positive definite; that is,

$$Q(t) \geq c_3 I > 0, \quad \forall\ t \geq 0$$

Consider a Lyapunov function candidate

$$V(t,x) = x^T P(t) x$$

The function $V(t,x)$ is positive definite, decrescent, and radially unbounded since

$$c_1 \|x\|_2^2 \leq V(t,x) \leq c_2 \|x\|_2^2$$

The derivative of $V(t,x)$ along the trajectories of the system (3.20) is given by

$$\begin{aligned} \dot{V}(t,x) &= x^T \dot{P}(t) x + x^T P(t)\dot{x} + \dot{x}^T P(t) x \\ &= x^T[\dot{P}(t) + P(t)A(t) + A^T(t)P(t)]x \;=\; -x^T Q(t) x \;\leq\; -c_3\|x\|_2^2 \end{aligned}$$

Hence, $\dot{V}(t,x)$ is negative definite. All the assumptions of Corollary 3.4 are satisfied globally with $c = 2$. Therefore, the origin is globally exponentially stable. △

3.5 Linear Time-Varying Systems and Linearization

The stability behavior of the origin as an equilibrium point for the linear time-varying system (3.20):

$$\dot{x}(t) = A(t)x$$

can be completely characterized in terms of the state transition matrix of the system. From linear system theory,[22] we know that the solution of (3.20) is given by

$$x(t) = \Phi(t,t_0)x(t_0)$$

where $\Phi(t,t_0)$ is the state transition matrix. The following theorem characterizes uniform asymptotic stability in terms of $\Phi(t,t_0)$.

Theorem 3.9 *The equilibrium point $x = 0$ of (3.20) is (globally) uniformly asymptotically stable if and only if the state transition matrix satisfies the inequality*

$$\|\Phi(t,t_0)\| \leq k e^{-\gamma(t-t_0)}, \quad \forall\ t \geq t_0 \geq 0 \tag{3.22}$$

for some positive constants k and γ. ◇

Proof: Due to the linear dependence of $x(t)$ on $x(t_0)$, if the origin is uniformly asymptotically stable it is globally so. Sufficiency of (3.22) is obvious since

$$\|x(t)\| \leq \|\Phi(t,t_0)\|\,\|x(t_0)\| \leq k\|x(t_0)\|e^{-\gamma(t-t_0)}$$

To prove necessity, suppose the origin is uniformly asymptotically stable. Then, there is a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0), \quad \forall\ t \geq t_0, \ \forall\ x(t_0) \in R^n$$

From the definition of an induced matrix norm (Section 2.1), we have

$$\|\Phi(t,t_0)\| = \max_{\|x\|=1} \|\Phi(t,t_0)x\| \leq \max_{\|x\|=1} \beta(\|x\|, t-t_0) = \beta(1, t-t_0)$$

Since

$$\beta(1,s) \to 0 \ \text{ as } \ s \to \infty$$

there exists $T > 0$ such that $\beta(1,T) \leq 1/e$. For any $t \geq t_0$, let N be the smallest positive integer such that $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T each. Using the transition property of $\Phi(t,t_0)$, we can write

$$\Phi(t,t_0) = \Phi(t, t_0 + (N-1)T)\Phi(t_0 + (N-1)T, t_0 + (N-2)T)\cdots\Phi(t_0+T, t_0)$$

[22] See, for example, [29], [41], [86], or [142].

Hence,

$$\begin{aligned}\|\Phi(t,t_0)\| &\le \|\Phi(t,t_0+(N-1)T)\| \prod_{k=1}^{k=N-1} \|\Phi(t_0+kT,t_0+(k-1)T)\| \\ &\le \beta(1,0)\prod_{k=1}^{k=N-1}\frac{1}{e} = e\beta(1,0)e^{-N} \\ &\le e\beta(1,0)e^{-(t-t_0)/T} = ke^{-\gamma(t-t_0)}\end{aligned}$$

where $k = e\beta(1,0)$ and $\gamma = 1/T$. □

Theorem 3.9 shows that, for linear systems, uniform asymptotic stability of the origin is equivalent to exponential stability. Although inequality (3.22) characterizes uniform asymptotic stability of the origin without the need to search for a Lyapunov function, it is not as useful as the eigenvalue criterion we have for linear time-invariant systems because knowledge of the state transition matrix $\Phi(t,t_0)$ requires solving the state equation (3.20). Note that, for linear time-varying systems, uniform asymptotic stability cannot be characterized by the location of the eigenvalues of the matrix A [23] as the following example shows.

Example 3.22 Consider a second-order linear system with

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1-1.5\sin t\cos t & -1+1.5\sin^2 t\end{bmatrix}$$

For each t, the eigenvalues of $A(t)$ are given by $-0.25 \pm 0.25\sqrt{7}j$. Thus, the eigenvalues are independent of t and lie in the open left-half plane. Yet, the origin is unstable. It can be verified that

$$\Phi(t,0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t\end{bmatrix}$$

which shows that there are initial states $x(0)$, arbitrarily close to the origin, for which the solution is unbounded and escapes to infinity. △

Although Theorem 3.9 may not be very helpful as a stability test, we shall see that it guarantees the existence of a Lyapunov function for the linear system (3.20). We saw in Example 3.21 that if we can find a positive definite, bounded matrix $P(t)$ that satisfies the differential equation (3.21) for some positive definite $Q(t)$,

[23] There are special cases where uniform asymptotic stability of the origin as equilibrium for (3.20) is equivalent to an eigenvalue condition. One case is periodic systems; see Exercise 3.40 and Example 8.8. Another case is slowly-varying systems; see Example 5.13.

then $V(t,x) = x^T P(t)x$ is a Lyapunov function for the system. If the matrix $Q(t)$ is chosen to be bounded in addition to being positive definite, that is,

$$0 < c_3 I \le Q(t) \le c_4 I, \quad \forall\, t \ge 0$$

and if $A(t)$ is continuous and bounded, then it can be shown that when the origin is uniformly asymptotically stable, there is a solution of (3.21) that possesses the desired properties.

Theorem 3.10 *Let $x = 0$ be the uniformly asymptotically stable equilibrium of (3.20). Suppose $A(t)$ is continuous and bounded. Let $Q(t)$ be a continuous, bounded, positive definite, symmetric matrix. Then, there is a continuously differentiable, bounded, positive definite, symmetric matrix $P(t)$ which satisfies (3.21). Hence, $V(t,x) = x^T P(t)x$ is a Lyapunov function for the system that satisfies the conditions of Theorem 3.8.* ◇

Proof: Let

$$P(t) = \int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\, d\tau$$

and $\phi(\tau,t,x)$ be the solution of (3.20) that starts at (t,x). Due to linearity, $\phi(\tau,t,x) = \Phi(\tau,t)x$. In view of the definition of $P(t)$, we have

$$x^T P(t)x = \int_t^\infty \phi^T(\tau,t,x)Q(\tau)\phi(\tau,t,x)\, d\tau$$

Use of (3.22) yields

$$\begin{aligned} x^T P(t)x &\le \int_t^\infty c_4\|\Phi(\tau,t)\|_2^2\, \|x\|_2^2 d\tau \\ &\le \int_t^\infty k^2 e^{-2\gamma(\tau-t)}\, d\tau\; c_4\|x\|_2^2 = \frac{k^2 c_4}{2\gamma}\|x\|_2^2 \stackrel{\text{def}}{=} c_2\|x\|_2^2\end{aligned}$$

On the other hand, since

$$\|A(t)\|_2 \le L, \quad \forall\, t \ge 0$$

the linear system (3.20) satisfies a global Lipschitz condition with a Lipschitz constant L. Therefore,[24] the solution $\phi(\tau,t,x)$ satisfies the lower bound

$$\|\phi(\tau,t,x)\|_2^2 \ge \|x\|_2^2 e^{-2L(\tau-t)}$$

Hence,

$$\begin{aligned} x^T P(t)x &\ge \int_t^\infty c_3\|\phi(\tau,t,x)\|_2^2\, d\tau \\ &\ge \int_t^\infty e^{-2L(\tau-t)}\, d\tau\; c_3\|x\|_2^2 = \frac{c_3}{2L}\|x\|_2^2 \stackrel{\text{def}}{=} c_1\|x\|_2^2\end{aligned}$$

[24] See Exercise 2.22.

Thus,

$$c_1\|x\|_2^2 \le x^T P(t) x \le c_2\|x\|_2^2$$

which shows that $P(t)$ is positive definite and bounded. The definition of $P(t)$ shows that it is symmetric and continuously differentiable. The fact that $P(t)$ satisfies (3.21) can be shown by differentiation of $P(t)$ using the property

$$\frac{\partial}{\partial t}\Phi(\tau,t) = -\Phi(\tau,t)A(t)$$

In particular,

$$\begin{aligned}\dot{P}(t) &= \int_t^\infty \Phi^T(\tau,t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau,t)\,d\tau \\ &\quad + \int_t^\infty \frac{\partial}{\partial t}\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau \;-\; Q(t) \\ &= -\int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau\,A(t) \\ &\quad -A^T(t)\int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau \;-\; Q(t) \\ &= -P(t)A(t) - A^T(t)P(t) - Q(t)\end{aligned}$$

The fact that $V(t,x) = x^T P(t) x$ is a Lyapunov function is shown in Example 3.21. □

When the linear system (3.20) is time invariant, that is, when A is constant, the Lyapunov function $V(t,x)$ of Theorem 3.10 can be chosen to be independent of t. Recall that, for linear time-invariant systems,

$$\Phi(\tau,t) = \exp[(\tau - t)A]$$

which satisfies (3.22) when A is a stability matrix. Choosing Q to be a positive definite, symmetric (constant) matrix, the matrix $P(t)$ is given by

$$P = \int_t^\infty \exp[(\tau-t)A^T]Q\exp[(\tau-t)A]\,d\tau = \int_0^\infty \exp[A^T s]Q\exp[As]\,ds$$

which is independent of t. Comparison of this expression for P with (3.13) shows that P is the unique solution of the Lyapunov equation (3.12). Thus, the Lyapunov function of Theorem 3.10 reduces to the one we used in Section 3.3.

The existence of Lyapunov functions for linear systems per Theorem 3.10 will now be used to prove a linearization result that extends Theorem 3.7 to the nonautonomous case. Consider the nonlinear nonautonomous system (3.14)

$$\dot{x} = f(t,x)$$

where $f : [0,\infty)\times D \to R^n$ is continuously differentiable and $D = \{x \in R^n \mid \|x\|_2 < r\}$. Suppose the origin $x = 0$ is an equilibrium point for the system at $t = 0$; that is, $f(t,0) = 0$ for all $t \ge 0$. Furthermore, suppose the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t; thus,

$$\left\|\frac{\partial f_i}{\partial x}(t,x_1) - \frac{\partial f_i}{\partial x}(t,x_2)\right\|_2 \le L_1\|x_1 - x_2\|_2, \quad \forall\; x_1, x_2 \in D, \;\forall\; t \ge 0$$

for all $1 \le i \le n$. By the mean value theorem,

$$f_i(t,x) = f_i(t,0) + \frac{\partial f_i}{\partial x}(t,z_i)\;x$$

where z_i is a point on the line segment connecting x to the origin. Since $f(t,0) = 0$, we can write $f_i(t,x)$ as

$$f_i(t,x) = \frac{\partial f_i}{\partial x}(t,z_i)\;x = \frac{\partial f_i}{\partial x}(t,0)\;x + \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right]x$$

Hence,

$$f(t,x) = A(t)x + g(t,x)$$

where

$$A(t) = \frac{\partial f}{\partial x}(t,0) \quad\text{and}\quad g_i(t,x) = \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right]x$$

The function $g(t,x)$ satisfies

$$\|g(t,x)\|_2 \le \left(\sum_{i=1}^n \left\|\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right\|_2^2\right)^{1/2} \|x\|_2 \le L\|x\|_2^2$$

where $L = \sqrt{n}L_1$. Therefore, in a small neighborhood of the origin, we may approximate the nonlinear system (3.14) by its linearization about the origin. The following theorem states Lyapunov's indirect method for showing uniform asymptotic stability of the origin in the nonautonomous case.

Theorem 3.11 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t,x)$$

where $f : [0,\infty)\times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left.\frac{\partial f}{\partial x}(t,x)\right|_{x=0}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

◇

Proof: Since the linear system has an exponentially stable equilibrium point at the origin and $A(t)$ is continuous and bounded, Theorem 3.10 ensures the existence of a continuously differentiable, bounded, positive definite symmetric matrix $P(t)$ that satisfies (3.21) where $Q(t)$ is continuous, positive definite, and symmetric. We use $V(t,x) = x^T P(t) x$ as a Lyapunov function candidate for the nonlinear system. The derivative of $V(t,x)$ along the trajectories of the system is given by

$$\begin{aligned}
\dot{V}(t,x) &= x^T P(t) f(t,x) + f^T(t,x) P(t) x + x^T \dot{P}(t) x \\
&= x^T [P(t)A(t) + A^T(t)P(t) + \dot{P}(t)]x + 2x^T P(t) g(t,x) \\
&= -x^T Q(t) x + 2x^T P(t) g(t,x) \\
&\le -c_3\|x\|_2^2 + 2c_2 L \|x\|_2^3 \\
&\le -(c_3 - 2c_2 L\rho)\|x\|_2^2, \quad \forall\ \|x\|_2 < \rho
\end{aligned}$$

Choosing $\rho < \min\{r, c_3/2c_2L\}$ ensures that $\dot{V}(t,x)$ is negative definite in $\|x\|_2 < \rho$. Hence, all the conditions of Corollary 3.4 are satisfied in $\|x\|_2 < \rho$, and we conclude that the origin is exponentially stable. □

converse theorem for e.s to show that an eq point is e.s IFF the linearization about eq point has an e.s eq at the origin.

3.6 Converse Theorems

Theorem 3.8 and Corollaries 3.3 and 3.4 establish uniform asymptotic stability (or exponential stability) of the origin by requiring the existence of a Lyapunov function $V(t,x)$ that satisfies certain conditions. Requiring the existence of an auxiliary function $V(t,x)$ that satisfies certain conditions is typical in many theorems of Lyapunov's method. The conditions of these theorems cannot be checked directly on the data of the problem. Instead, one has to search for the auxiliary function. Faced with this searching problem, two questions come to mind. First, is there a function that satisfies the conditions of the theorem? Second, how can we search for such a function? In many cases, Lyapunov theory provides an affirmative answer to the first question. The answer takes the form of a converse Lyapunov theorem, which is the inverse of one of Lyapunov's theorems. For example, a converse theorem for uniform asymptotic stability would confirm that if the origin is uniformly asymptotically stable, then there is a Lyapunov function that satisfies the conditions of Theorem 3.8. Most of these converse theorems are proven by actually constructing auxiliary functions that satisfy the conditions of the respective theorems. Unfortunately, this construction almost always assumes the knowledge of the solutions

Converse Lyapunov theorem:
- ∃ origin is e.s. eq point
- ∃ origin is u.a.s eq point.

of the differential equation. Therefore, these theorems do not help in the practical search for an auxiliary function. The mere knowledge that a function exists is, however, better than nothing. At least we know that our search is not hopeless. The theorems are also useful in using Lyapunov theory to draw conceptual conclusions about the behavior of dynamical systems. Theorem 3.13 is an example of such use. Other examples will appear in the following chapters. In this section, we give two converse Lyapunov theorems.[25] The first one is a converse Lyapunov theorem when the origin is an exponentially stable equilibrium, and the second one is when the origin is uniformly asymptotically stable.

The idea of constructing a converse Lyapunov function is not new to us. We have done it for linear systems in the proof of Theorem 3.10. A careful reading of that proof shows that linearity of the system does not play a crucial role in the proof, except for showing that $V(t,x)$ is quadratic in x. This observation leads to the first of our two converse theorems, whose proof is a simple extension of the proof of Theorem 3.10.

Theorem 3.12 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t,x)$$

where $f : [0,\infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let k, γ, and r_0 be positive constants with $r_0 < r/k$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|x(t)\| \le k\|x(t_0)\| e^{-\gamma(t-t_0)}, \quad \forall\ x(t_0) \in D_0,\ \forall\ t \ge t_0 \ge 0$$

Then, there is a function $V : [0,\infty) \times D_0 \to R$ that satisfies the inequalities

$$c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4\|x\|$$

for some positive constants c_1, c_2, c_3, and c_4. Moreover, if $r = \infty$ and the origin is globally exponentially stable, then $V(t,x)$ is defined and satisfies the above inequalities on R^n. Furthermore, if the system is autonomous, V can be chosen independent of t. ◇

[25] See [62] or [96] for a comprehensive treatment of converse Lyapunov theorems.

Proof: Due to the equivalence of norms, it is sufficient to prove the theorem for the 2-norm. Let $\phi(\tau, t, x)$ denote the solution of the system that starts at (t, x); that is, $\phi(t, t, x) = x$. For all $x \in D_0$, $\phi(\tau, t, x) \in D$ for all $\tau \geq t$. Let

$$V(t,x) = \int_t^{t+T} \phi^T(\tau,t,x)\phi(\tau,t,x)\; d\tau$$

where T is a positive constant to be chosen later. Due to the exponentially decaying bound on the trajectories, we have

$$\begin{aligned} V(t,x) &= \int_t^{t+T} \|\phi(\tau,t,x)\|_2^2 \; d\tau \\ &\leq \int_t^{t+T} k^2 e^{-2\gamma(\tau-t)}\; d\tau\; \|x\|_2^2 \;=\; \frac{k^2}{2\gamma}(1-e^{-2\gamma T})\|x\|_2^2 \end{aligned}$$

On the other hand, the Jacobian matrix $[\partial f/\partial x]$ is bounded on D. Let

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\|_2 \leq L, \quad \forall\; x \in D$$

The function $f(t,x)$ is Lipschitz on D with a Lipschitz constant L. Therefore, the solution $\phi(\tau, t, x)$ satisfies the lower bound[26]

$$\|\phi(\tau,t,x)\|_2^2 \geq \|x\|_2^2 e^{-2L(\tau-t)}$$

Hence,

$$V(t,x) \geq \int_t^{t+T} e^{-2L(\tau-t)}\; d\tau\; \|x\|_2^2 = \frac{1}{2L}(1-e^{-2LT})\|x\|_2^2$$

Thus, $V(t,x)$ satisfies the first inequality of the theorem with

$$c_1 = \frac{(1-e^{-2LT})}{2L} \quad \text{and} \quad c_2 = \frac{k^2(1-e^{-2\gamma T})}{2\gamma}$$

To calculate the derivative of V along the trajectories of the system, define the sensitivity functions

$$\phi_t(\tau,t,x) = \frac{\partial}{\partial t}\phi(\tau,t,x); \quad \phi_x(\tau,t,x) = \frac{\partial}{\partial x}\phi(\tau,t,x)$$

Then,

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &= \phi^T(t+T,t,x)\phi(t+T,t,x) - \phi^T(t,t,x)\phi(t,t,x) \\ &\quad + \int_t^{t+T} 2\phi^T(\tau,t,x)\phi_t(\tau,t,x)\; d\tau \\ &\quad + \int_t^{t+T} 2\phi^T(\tau,t,x)\phi_x(\tau,t,x)\; d\tau f(t,x) \\ &= \phi^T(t+T,t,x)\phi(t+T,t,x) - \|x\|_2^2 \\ &\quad + \int_t^{t+T} 2\phi^T(\tau,t,x)[\phi_t(\tau,t,x) + \phi_x(\tau,t,x)f(t,x)]\; d\tau \end{aligned}$$

[26]See Exercise 2.22.

It is not difficult to show that[27]

$$\phi_t(\tau,t,x) + \phi_x(\tau,t,x)f(t,x) \equiv 0, \quad \forall\; \tau \geq t$$

Therefore,

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &= \phi^T(t+T,t,x)\phi(t+T,t,x) - \|x\|_2^2 \\ &\leq -(1-k^2e^{-2\gamma T})\|x\|_2^2 \end{aligned}$$

Choosing $T = \ln(2k^2)/2\gamma$, the second inequality of the theorem is satisfied with $c_3 = 1/2$. To show the last inequality, let us note that $\phi_x(\tau,t,x)$ satisfies the sensitivity equation

$$\frac{\partial}{\partial \tau}\phi_x = \frac{\partial f}{\partial x}(\tau, \phi(\tau,t,x))\; \phi_x, \quad \phi_x(t,t,x) = I$$

Since

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\|_2 \leq L$$

on D, ϕ_x satisfies the bound[28]

$$\|\phi_x(\tau,t,x)\|_2 \leq e^{L(\tau-t)}$$

Hence,

$$\begin{aligned} \left\| \frac{\partial V}{\partial x} \right\|_2 &= \left\| \int_t^{t+T} 2\phi^T(\tau,t,x)\phi_x(\tau,t,x)\; d\tau \right\|_2 \\ &\leq \int_t^{t+T} 2\|\phi(\tau,t,x)\|_2\; \|\phi_x(\tau,t,x)\|_2\; d\tau \\ &\leq \int_t^{t+T} 2ke^{-\gamma(\tau-t)}\; e^{L(\tau-t)}\; d\tau\; \|x\|_2 \\ &= \frac{2k}{(\gamma-L)}[1-e^{-(\gamma-L)T}]\|x\|_2 \end{aligned}$$

[27]See Exercise 2.46.
[28]See Exercise 2.22.

Thus, the last inequality of the theorem is satisfied with

$$c_4 = \frac{2k}{(\gamma - L)}[1 - e^{-(\gamma-L)T}]$$

If all the assumptions hold globally, then clearly r_0 can be chosen arbitrarily large. If the system is autonomous, then $\phi(\tau, t, x)$ depends only on $(\tau - t)$; that is,

$$\phi(\tau, t, x) = \psi(\tau - t, x)$$

Then,

$$V(t, x) = \int_t^{t+T} \psi^T(\tau - t, x)\psi(\tau - t, x)\, d\tau = \int_0^T \psi^T(s, x)\psi(s, x)\, ds$$

which is independent of t. □

In Theorem 3.11, we saw that if the linearization of a nonlinear system about the origin has an exponentially stable equilibrium, then the origin is an exponentially stable equilibrium for the nonlinear system. We will use Theorem 3.12 to prove that exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the origin.

Theorem 3.13 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left.\frac{\partial f}{\partial x}(t, x)\right|_{x=0}$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if and only if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

◇

Proof: The "if" part follows from Theorem 3.11. To prove the "only if" part, write the linear system as

$$\dot{x} = f(t, x) - [f(t, x) - A(t)x] = f(t, x) - g(t, x)$$

Recalling the argument preceding Theorem 3.11, we know that

$$\|g(t, x)\|_2 \le L\|x\|_2^2, \quad \forall\ x \in D,\ \forall\ t \ge 0$$

Since the origin is an exponentially stable equilibrium of the nonlinear system, there are positive constants k, γ, and c such that

$$\|x(t)\|_2 \le k\|x(t_0)\|_2 e^{-\gamma(t-t_0)}, \quad \forall\ t \ge t_0 \ge 0,\ \forall\ \|x(t_0)\|_2 < c$$

Choosing $r_0 < \min\{c, r/k\}$, all the conditions of Theorem 3.12 are satisfied. Let $V(t, x)$ be the function provided by Theorem 3.12, and use it as a Lyapunov function candidate for the linear system. Then,

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}A(t)x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t, x) - \frac{\partial V}{\partial x}g(t, x) \\ &\le -c_3\|x\|_2^2 + c_4 L\|x\|_2^3 \\ &\le -(c_3 - c_4 L\rho)\|x\|_2^2, \quad \forall\ \|x\|_2 < \rho \end{aligned}$$

The choice $\rho < \min\{r_0, c_3/c_4 L\}$ ensures that $\dot{V}(t, x)$ is negative definite in $\|x\|_2 < \rho$. Hence, all the conditions of Corollary 3.4 are satisfied in $\|x\|_2 < \rho$, and we conclude that the origin is an exponentially stable equilibrium point for the linear system. □

Example 3.23 Consider the first-order system

$$\dot{x} = -x^3$$

We saw in Example 3.14 that the origin is asymptotically stable, but linearization about the origin results in the linear system

$$\dot{x} = 0$$

whose A matrix is not Hurwitz. Using Theorem 3.13, we conclude that the origin is not exponentially stable. △

We conclude this section by stating another converse theorem which applies to the more general case of uniformly asymptotically stable equilibria. The proof of the theorem is more involved than that of Theorem 3.13.

Theorem 3.14 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let $\beta(\cdot, \cdot)$ be *a class $\mathcal{KL}$ function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the trajectory of the system satisfies*

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\ x(t_0) \in D_0,\ \forall\ t \ge t_0 \ge 0$$

Then, there is a continuously differentiable function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -\alpha_3(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le \alpha_4(\|x\|)$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, and $\alpha_4(\cdot)$ are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t. ◇

Proof: Appendix A.6.

3.7 Exercises

Exercise 3.1 Consider a second-order autonomous system $\dot{x} = f(x)$. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable. Justify your answer using phase portraits.
(1) stable node (2) unstable node (3) stable focus
(4) unstable focus (5) center (6) saddle

Exercise 3.2 Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \le k|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

Exercise 3.3 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

(1) $$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

(2) $$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

(3) $$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 + x_1 \end{aligned}$$

(4) $$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{aligned}$$

Exercise 3.4 Using $V(x) = x_1^2 + x_2^2$, study stability of the origin of the system

$$\begin{aligned} \dot{x}_1 &= x_1(k^2 - x_1^2 - x_2^2) + x_2(x_1^2 + x_2^2 + k^2) \\ \dot{x}_2 &= -x_1(k^2 + x_1^2 + x_2^2) + x_2(k^2 - x_1^2 - x_2^2) \end{aligned}$$

when (a) $k = 0$ and (b) $k \neq 0$.

Exercise 3.5 Let $g(x)$ be a map from R^n into R^n. Show that $g(x)$ is the gradient vector of a scalar function $V : R^n \to R$ if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, 2, \ldots, n$$

Exercise 3.6 Using the variable gradient method, find a Lyapunov function $V(x)$ that shows asymptotic stability of the origin of the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 + x_2) - \sin(x_1 + x_2) \end{aligned}$$

Exercise 3.7 ([62]) Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= \frac{-6x_1}{u^2} + 2x_2 \\ \dot{x}_2 &= \frac{-2(x_1 + x_2)}{u^2} \end{aligned}$$

where $u = 1 + x_1^2$. Let $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$.

(b) Consider the hyperbola $x_2 = 2/(x_1 - \sqrt{2})$. Show, by investigating the vector field on the boundary of this hyperbola, that trajectories to the right of the branch in the first quadrant cannot cross that branch.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), show that $\dot{x}_2/\dot{x}_1 = -1/(1 + 2\sqrt{2}x_1 + 2x_1^2)$ on the hyperbola, and compare with the slope of the tangents to the hyperbola.

Exercise 3.8 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \text{sat}(2x_1 + x_2) \end{aligned}$$

(a) Show that the origin is asymptotically stable.

(b) Show that all trajectories starting in the first quadrant to the right of the curve $x_1x_2 = c$ (with sufficiently large $c > 0$) cannot reach the origin.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), consider $V(x) = x_1x_2$; calculate $\dot{V}(x)$ and show that on the curve $V(x) = c$ the derivative $\dot{V}(x) > 0$ when c is large enough.

Exercise 3.9 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \le -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x\; d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \le -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x)Pf(x)$ is positive definite for all $x \in R^n$.

(c) Show that $V(x)$ is radially unbounded.

(d) Using $V(x)$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable.

Exercise 3.10 Using Theorem 3.3, prove Lyapunov's first instability theorem: For the system (3.1), if a continuous function $V_1(x)$ with continuous first partial derivatives can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Exercise 3.11 Using Theorem 3.3, prove Lyapunov's second instability theorem: For the system (3.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \ge 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Exercise 3.12 Show that the origin of the following system is unstable.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^6 \\ \dot{x}_2 &= x_2^3 + x_1^6 \end{aligned}$$

Exercise 3.13 Show that the origin of the following system is unstable.

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= x_1^6 - x_2^3 \end{aligned}$$

Hint: Show that the set $\Gamma = \{0 \le x_1 \le 1\} \cap \{x_2 \ge x_1^3\} \cap \{x_2 \le x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

Exercise 3.14 ([23]) Consider the system

$$\begin{aligned} \dot{z} &= -\sum_{i=1}^m a_i y_i \\ \dot{y}_i &= -h(z,y)y_i + b_i g(z), \quad i = 1, 2, \ldots, m \end{aligned}$$

where z is a scalar, $y^T = (y_1, \ldots, y_m)$. The functions $h(\cdot,\cdot)$ and $g(\cdot)$ are continuously differentiable for all (z, y) and satisfy $zg(z) > 0$, $\forall\, z \ne 0$, $h(z,y) > 0$, $\forall\, (z,y) \ne 0$, and $\int_0^z g(\sigma)\, d\sigma \to \infty$ as $|z| \to \infty$. The constants a_i and b_i satisfy $b_i \ne 0$ and $a_i/b_i > 0$, $\forall\, i = 1, 2, \ldots, m$. Show that the origin is an equilibrium point, and investigate its stability using a Lyapunov function candidate of the form

$$V(z,y) = \alpha \int_0^z g(\sigma)\, d\sigma + \sum_{i=1}^m \beta_i y_i^2$$

Exercise 3.15 ([143]) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2\, \text{sat}(x_2^2 - x_3^2) \\ \dot{x}_3 &= x_3\, \text{sat}(x_2^2 - x_3^2) \end{aligned}$$

where sat$(\cdot)$ is the saturation function. Show that the origin is the unique equilibrium point, and use $V(x) = x^T x$ to show that it is globally asymptotically stable.

Exercise 3.16 The origin $x = 0$ is an equilibrium point of the system

$$\begin{aligned} \dot{x}_1 &= -kh(x)x_1 + x_2 \\ \dot{x}_2 &= -h(x)x_2 - x_1^3 \end{aligned}$$

Let $D = \{x \in R^2 \mid \|x\|_2 < 1\}$. Using $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, investigate stability of the origin in each of the following cases.
(1) $k > 0$, $h(x) > 0$, $\forall\, x \in D$; (2) $k > 0$, $h(x) > 0$, $\forall\, x \in R^2$;
(3) $k > 0$, $h(x) < 0$, $\forall\, x \in D$; (4) $k > 0$, $h(x) = 0$, $\forall\, x \in D$;
(5) $k = 0$, $h(x) > 0$, $\forall\, x \in D$; (6) $k = 0$, $h(x) > 0$, $\forall\, x \in R^2$.

Exercise 3.17 Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + g(x_3)\\ \dot{x}_2 &= -g(x_3)\\ \dot{x}_3 &= -ax_1 + bx_2 - cg(x_3)\end{aligned}$$

where a, b, and c are positive constants and $g(\cdot)$ satisfies

$$g(0) = 0 \text{ and } yg(y) > 0, \ \forall\ 0 < |y| < k, \quad k > 0$$

(a) Show that the origin is an isolated equilibrium point.

(b) With $V(x) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2 + \int_0^{x_3} g(y)\ dy$ as a Lyapunov function candidate, show that the origin is asymptotically stable.

(c) Suppose $yg(y) > 0\ \forall\ y \in R - \{0\}$. Is the origin globally asymptotically stable?

Exercise 3.18 ([67]) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\ dy + \frac{1}{2}x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat (b) using $V(x) = \frac{1}{2}\left[x_2 + \int_0^{x_1} h(y)\ dy\right]^2 + \int_0^{x_1} g(y)\ dy$.

Exercise 3.19 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 - kx_1 - dx_2 - cx_3\\ \dot{x}_3 &= -x_3 + x_2\end{aligned}$$

where all coefficients are positive and $k > a$. Using

$$V(x) = 2a\int_0^{x_1} \sin y\ dy + kx_1^2 + x_2^2 + px_3^2$$

with some $p > 0$, show that the origin is globally asymptotically stable.

Exercise 3.20 The mass-spring system of Exercise 1.11 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

Exercise 3.21 Consider the equations of motion of an m-link robot, described in Exercise 1.3.

(a) With $u = 0$, use the total energy $V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin ($q = 0$, $\dot{q} = 0$) is stable.

(b) With $u = -K_d\dot{q}$, where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q^*) - K_d\dot{q}$, where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^m, show that the point ($q = q^*$, $\dot{q} = 0$) is an asymptotically stable equilibrium point.

Exercise 3.22 Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

Exercise 3.23 ([70]) A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\partial V/\partial x]^T$ and $V : D \subset R^n \to R$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \le 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Suppose the set $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is compact for every $c \in R$. Show that every solution of the system is defined for all $t \ge 0$.

(c) Continuing with part (b), suppose $\nabla V(x) \ne 0$ except for a finite number of points $p_1, \ldots, p_r$. Show that for every solution $x(t)$, the limit $\lim_{t\to\infty} x(t)$ exists and equals one of the points $p_1, \ldots, p_r$.

Exercise 3.24 Consider the Lyapunov equation $PA + A^TP = -C^TC$, where the pair (A, C) is observable. Show that A is Hurwitz if and only if there exists $P = P^T > 0$ that satisfies the equation. Furthermore, show that if A is Hurwitz, the Lyapunov equation will have a unique solution.
Hint: Apply LaSalle's theorem and recall that for an observable pair (A, C), the vector $C\exp(At)x \equiv 0\ \forall\ t$ if and only if $x = 0$.

Exercise 3.25 Consider the linear system $\dot{x} = (A - BR^{-1}B^TP)x$, where (A, B) is controllable, $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0$$

$R = R^T > 0$, and $Q = Q^T \geq 0$. Using $V(x) = x^TPx$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when

(1) $Q > 0$

(2) $Q = C^TC$ and (A, C) is observable; see the hint of the previous exercise.

Exercise 3.26 Consider the linear system $\dot{x} = Ax + Bu$, where (A, B) is controllable. Let $W = \int_0^\tau e^{-At}BB^Te^{-A^Tt}\,dt$ for some $\tau > 0$. Let $K = -B^TW^{-1}$.

(a) Show that W^{-1} is positive definite.

(b) Using $V(x) = x^TW^{-1}x$ as a Lyapunov function candidate for the system $\dot{x} = (A + BK)x$, show that $(A + BK)$ is Hurwitz.

Exercise 3.27 Let $\dot{x} = f(x)$, where $f : R^n \to R^n$. Consider the change of variables $z = T(x)$, where $T(0) = 0$ and $T : R^n \to R^n$ is a diffeomorphism in the neighborhood of the origin; that is, the inverse map $T^{-1}(\cdot)$ exists, and both $T(\cdot)$ and $T^{-1}(\cdot)$ are continuously differentiable. The transformed system is

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=T^{-1}(z)}$$

(a) Show that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if $z = 0$ is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

(b) Show that $x = 0$ is stable (asymptotically stable/unstable) if and only if $z = 0$ is stable (asymptotically stable/unstable).

Exercise 3.28 Show that the system

$$\begin{aligned} \dot{x}_1 &= \frac{1}{1 + x_3} - x_1 \\ \dot{x}_2 &= x_1 - 2x_2 \\ \dot{x}_3 &= x_2 - 3x_3 \end{aligned}$$

has a unique equilibrium point in the region $x_i \geq 0$, $i = 1, 2, 3$, and investigate stability of this point using linearization.

Exercise 3.29 Consider the system

$$\begin{aligned} \dot{x}_1 &= (x_1x_2 - 1)x_1^3 + (x_1x_2 - 1 + x_2^2)x_1 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

(a) Show that $x = 0$ is the unique equilibrium point.

(b) Show, using linearization, that $x = 0$ is asymptotically stable.

(c) Show that $\Gamma = \{x \in R^2 \mid x_1x_2 \geq 2\}$ is a positively invariant set.

(d) Is $x = 0$ globally asymptotically stable? Justify your answer.

Exercise 3.30 For each of the following systems, use linearization to show that the origin is asymptotically stable. Then, show that the origin is globally asymptotically stable.

(1) $$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= (x_1 + x_2)\sin x_1 - 3x_2 \end{aligned}$$

(2) $$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -ax_1 - bx_2, \quad a, b > 0 \end{aligned}$$

Exercise 3.31 For each for the following systems, investigate stability of the origin.

(1) $$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 \\ \dot{x}_2 &= -x_2 + x_3^2 \\ \dot{x}_3 &= x_3 - x_1^2 \end{aligned}$$

(2) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_3 + x_1[-2x_3 - \text{sat}(y)]^2 \\ \dot{x}_3 &= -2x_3 - \text{sat}(y) \end{aligned}$$
where $y = -2x_1 - 5x_2 + 2x_3$

(3) $$\begin{aligned} \dot{x}_1 &= -2x_1 + x_1^3 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= -x_3 \end{aligned}$$

(4) $$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_1 - x_2 - x_3 - x_1x_3 \\ \dot{x}_3 &= (x_1 + 1)x_2 \end{aligned}$$

Exercise 3.32 Prove Lemma 3.2.

Exercise 3.33 Let α be a class $\mathcal{K}$ function on $[0, a)$. Show that

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2), \quad \forall\, r_1, r_2 \in [0, a)$$

Exercise 3.34 Is the origin of the scalar system $\dot{x} = -x/t$, $t \geq 1$, uniformly asymptotically stable?

Exercise 3.35 Suppose the conditions of Theorem 3.8 are satisfied except that $\dot{V}(t, x) \leq -W_3(x)$ where $W_3(x)$ is positive semidefinite. Show that the origin is uniformly stable.

Exercise 3.36 Consider the linear time-varying system

$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -1 \end{bmatrix} x$$

where $\alpha(t)$ is continuous for all $t \geq 0$. Show that the origin is exponentially stable.

Exercise 3.37 ([87]) An RLC circuit with time-varying elements is represented by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{L(t)} x_2 \\ \dot{x}_2 &= -\frac{1}{C(t)} x_1 - \frac{R(t)}{L(t)} x_2 \end{aligned}$$

Suppose that $L(t)$, $C(t)$, and $R(t)$ are continuously differentiable and satisfy the inequalities $k_1 \leq L(t) \leq k_2$, $k_3 \leq C(t) \leq k_4$, and $k_5 \leq R(t) \leq k_6$ for all $t \geq 0$, where k_1, k_2, and k_3 are positive. Consider a Lyapunov function candidate

$$V(t,x) = \left[R(t) + \frac{2L(t)}{R(t)C(t)}\right] x_1^2 + 2x_1x_2 + \frac{2}{R(t)} x_2^2$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Find conditions on $\dot{L}(t)$, $\dot{C}(t)$, and $\dot{R}(t)$ that will ensure exponential stability of the origin.

Exercise 3.38 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + \alpha(t) x_2 \\ \dot{x}_2 &= -\alpha(t) x_1 - x_2^3 \end{aligned}$$

where $\alpha(t)$ is a continuous, bounded function. Show that the origin is globally uniformly asymptotically stable. Is it exponentially stable?

Exercise 3.39 ([137]) A pendulum with time-varying friction is represented by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - g(t) x_2 \end{aligned}$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty; \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t,x) = \tfrac{1}{2}(a \sin x_1 + x_2)^2 + [1 + a g(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3)$, where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

Exercise 3.40 (Floquet theory) [29] Consider the linear system $\dot{x} = A(t)x$, where $A(t) = A(t+T)$. Let $\Phi(\cdot,\cdot)$ be the state transition matrix. Define a constant matrix B via the equation $\exp(BT) = \Phi(T,0)$, and let $P(t) = \exp(Bt)\Phi(0,t)$. Show that

(a) $P(t+T) = P(t)$.

(b) $\Phi(t,\tau) = P^{-1}(t) \exp[(t-\tau)B] P(\tau)$.

(c) the origin of $\dot{x} = A(t)x$ is exponentially stable if and only if B is Hurwitz.

Exercise 3.41 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2 \end{aligned}$$

(a) Verify that $x_1(t) = t$, $x_2(t) = 1$ is a solution.

(b) Show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $x(t)$ approaches $\begin{bmatrix} t \\ 1 \end{bmatrix}$ as $t \to \infty$.

Exercise 3.42 Consider the system

$$\begin{aligned} \dot{x}_1 &= -2x_1 + g(t) x_2 \\ \dot{x}_2 &= g(t) x_1 - 2x_2 \end{aligned}$$

where $g(t)$ is continuously differentiable and $|g(t)| \leq 1$ for all $t \geq 0$. Show that the origin is uniformly asymptotically stable.

Exercise 3.43 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - (1 + b\cos t) x_2 \end{aligned}$$

Find $b^* > 0$ such that the origin is exponentially stable for all $|b| < b^*$.

[29] See [142] for a comprehensive treatment of Floquet theory.

Exercise 3.44 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 - g(t)x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -x_1 - g(t)x_2(x_1^2 + x_2^2)\end{aligned}$$

where $g(t)$ is continuously differentiable, bounded, and $g(t) \ge k > 0$ for all $t \ge 0$. Is the origin uniformly asymptotically stable? Is it exponentially stable?

✱**Exercise 3.45** Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + (x_1^2 + x_2^2)\sin t\\ \dot{x}_2 &= -x_2 + (x_1^2 + x_2^2)\cos t\end{aligned}$$

Show that the origin is exponentially stable and estimate the region of attraction.

Exercise 3.46 Consider two systems represented by

$$\dot{x} = f(x) \qquad (3.23)$$

$$\dot{x} = h(x)f(x) \qquad (3.24)$$

where $f : R^n \to R^n$ and $h : R^n \to R$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of (3.23) is exponentially stable if and only if the origin of (3.24) is exponentially stable.

Exercise 3.47 Show that the system

$$\begin{aligned}\dot{x}_1 &= -ax_1 + b\\ \dot{x}_2 &= -cx_2 + x_1(\alpha - \beta x_1 x_2)\end{aligned}$$

where all coefficients are positive, has a globally exponentially stable equilibrium point.
Hint: Shift the equilibrium point to the origin and use V of the form $V = k_1y_1^2 + k_2y_2^2 + k_3y_1^4$, where (y_1, y_2) are the new coordinates.

Exercise 3.48 Consider the system

$$\dot{x} = f(t, x); \quad f(t, 0) = 0$$

where $[\partial f/\partial x]$ is bounded and Lipschitz in x in a neighborhood of the origin, uniformly in t for all $t \ge t_0 \ge 0$. Suppose that the origin of the linearization at $x = 0$ is exponentially stable, and the solutions of the system satisfy

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \ \ \forall\, t \ge t_0 \ge 0, \ \ \forall\, \|x(t_0)\| < c \qquad (3.25)$$

for some class $\mathcal{KL}$ function β and some positive constant c.

(a) Show that there is a class $\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant γ such that

$$\|x(t)\| \le \alpha(\|x(t_0)\|)\exp[-\gamma(t - t_0)], \quad \forall\, t \ge t_0, \ \ \forall\, \|x(t_0)\| < c$$

(b) Show that there is a positive constant M, possibly dependent on c, such that

$$\|x(t)\| \le M\|x(t_0)\|\exp[-\gamma(t - t_0)], \quad \forall\, t \ge t_0, \ \ \forall\, \|x(t_0)\| < c \qquad (3.26)$$

(c) If inequality (3.25) holds globally, can you state inequality (3.26) globally?

In the next few exercises, we deal with the discrete-time dynamical system[30]

$$x(k+1) = f(x(k)), \quad f(0) = 0 \qquad (3.27)$$

The rate of change of a scalar function $V(x)$ along the motion of (3.27) is defined by

$$\Delta V(x) = V(f(x)) - V(x)$$

Exercise 3.49 Restate Definition 3.1 for the origin of the discrete-time system (3.27).

Exercise 3.50 Show that the origin of (3.27) is stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite. Show that it is asymptotically stable if, in addition, $\Delta V(x)$ is negative definite. Finally, show that the origin is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded.

Exercise 3.51 Show that the origin of (3.27) is asymptotically stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite and $\Delta V(x)$ does not vanish identically for any $x \ne 0$.

Exercise 3.52 Consider the linear system $x(k+1) = Ax(k)$. Show that the following statements are equivalent:

(1) $x = 0$ is asymptotically stable.

(2) $|\lambda_i| < 1$ for all eigenvalues of A.

(3) Given any $Q = Q^T > 0$ there exists $P = P^T > 0$, which is the unique solution of the linear equation

$$A^T PA - P = -Q$$

[30] See [87] for a detailed treatment of Lyapunov stability for discrete-time dynamical systems.

Exercise 3.53 Let A be the linearization of (3.27) at the origin; that is,

$$A = \frac{\partial f}{\partial x}(0)$$

Show that the origin is asymptotically stable if all the eigenvalues of A have magnitudes less than one.

Exercise 3.54 Let $x = 0$ be an equilibrium point for the nonlinear discrete-time system

$$x(k+1) = f(x(k))$$

where $f : D \to R^n$ is continuously differentiable and $D = \{x \in R^n \mid \|x\| < r\}$. Let C, $\gamma < 1$, and r_0 be positive constants with $r_0 < r/C$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the solutions of the system satisfy

$$\|x(k)\| \le C\|x(0)\|\gamma^k, \quad \forall\ x(0) \in D_0, \ \forall\ k \ge 0$$

Show that there is a function $V : D_0 \to R$ that satisfies

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\Delta V(x) = V(f(x)) - V(x) \le -c_3\|x\|^2$$

$$|V(x) - V(y)| \le c_4\|x-y\|\,(\|x\| + \|y\|)$$

for all $x, y \in D_0$ for some positive constants c_1, c_2, c_3, and c_4.